Relativistic Particle Motion in Nonuniform Electromagnetic Waves

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Abstract

It is shown that a charged particle moving in a strong nonuniform electromagnetic wave suffers a net acceleration in the direction of the negative intensity gradient of the wave. Electrons will be expelled perpendicularly from narrow laser beams and various instabilities can result.
Particles moving in electromagnetic wave fields strong enough to drive them relativistic have been extensively investigated in recent years, both with regard to cosmic ray production in pulsars\textsuperscript{1} and laser particle interaction. Linearly polarized plane waves can lead to particle acceleration in the forward direction while particles in a circularly polarized wave are thought to produce d.c. magnetic fields\textsuperscript{2}. Plasma effects\textsuperscript{3} including instabilities\textsuperscript{4} have also been studied in some cases. Some problems in which the nonuniformity of the wave plays a role have recently been investigated by Kaw and Kulsrud\textsuperscript{5} and Vittitoe and Wright\textsuperscript{6}.

Here a general treatment is presented of particles moving in weakly nonuniform waves where the wave intensity varies slowly on the space and time scale of the particle oscillation period. It has long been known that nonrelativistic particles suffer a net acceleration in such a wave in the direction of the negative intensity gradient\textsuperscript{7}. This behavior is characterized by a "ponderomotive force" and leads to mode coupling and instabilities in plasmas\textsuperscript{8}. While the lowest order particle motion in the nonrelativistic case is that of a harmonic oscillator the relativistic case is more involved, but it will be shown that field nonuniformities lead to a generalized ponderomotive acceleration tending to expell particles from regions of strong field intensity.

The equation of motion of an electron in an electromagnetic field may be written as

\[
\frac{dU}{dt} = \frac{e}{m} \left( \frac{\partial A}{\partial x^\nu} - \frac{\partial A^\mu}{\partial x^\nu} \right) \frac{dx^\nu}{dt}
\]

where \(U^\mu\) is the four velocity and \(A^\mu\) the vector potential. This equation may be cast in the more convenient form
Consider first in the lowest order the motion of the particle in a uniform plane wave. Adopting the Coulomb gauge we have \( A_\mu = (a \hat{A}_1, 0, 0) \); \( a = \text{constant} \); \( \hat{A}_1(x, ct) = \hat{A}_1(n) \). Eq. (2) reads now in components

\[
\frac{d}{dt} (U_\mu - \frac{e}{m} A_\mu) = -\frac{e}{m} \frac{\partial A_\nu}{\partial x_\mu} U_\nu \tag{2}
\]

One may integrate Eqs. (3a) and (4) to obtain constants of motion.

Now we let \( A_\mu = (a \hat{A}_1, bB(n), 0) \) where \( a \) and \( b \) vary slowly on scale lengths much larger than the particle and wave oscillation periods. From \( \nabla \cdot \hat{A} = 0 \)

\[
\hat{A}_1 \cdot \frac{\partial a}{\partial x_1} + \frac{\partial b}{\partial x_n} B + b \frac{\partial B}{\partial n} = 0 \tag{5}
\]

The two dominant terms are the first and the last one giving \( bB \sim a \hat{A}_1 \frac{\lambda}{L} \) where \( \lambda \) is the wave length and \( L \) the scale length of variation of the slowly varying \( a \) and \( b \).

Equation (2) yields now

\[
\frac{d}{dt} (\hat{U}_1 - \frac{e}{m} a \hat{A}_1) = -\frac{e}{m} \hat{A}_1 \cdot \hat{U}_1 \frac{\partial a}{\partial x_1} \tag{6}
\]

\[
\frac{d}{dt} (U_n - c\gamma) = -\frac{e}{m} \hat{A}_1 \cdot \hat{U}_1 \left( \frac{\partial}{\partial x_n} + \frac{1}{c} \frac{\partial}{\partial t} \right) a \tag{7}
\]
where terms on the right hand sides represent small perturbations due to field nonuniformities and the longitudinal bB terms have been neglected, being of order \( \lambda/L \) times the perturbation terms arising from \( a^2 \). These two equations plus the exact \( \gamma = \sqrt{c^2 + U''^2 + U''^2} \) equation (arising from \( U'' U'' = \text{constant} \)) are the complete set of equations to be solved.

Substituting now the lowest order solution for \( U'' \) from the integral Eq. (3a) to the right hand side of Eqs. (6) and (7) and integrating over a period of particle oscillation yields

\[
\Delta(U'' - \gamma) = -\frac{e^2}{2m^2} \int A^2 \frac{\partial a}{\partial x^2} \tag{9}
\]

where \( \Delta \) represents the change of a quantity over an oscillation period. This period can be characterized by the time it takes for the particle to complete an oscillation between equal values of \( \Delta_x \). The integral in Eqs. (8) and (9) is easily evaluated

\[
\int A^2 \frac{\partial a}{\partial x} = \int A^2 \frac{\partial n}{\partial \tau} - \int A^2 \frac{\partial n}{\partial \tau} = \frac{\lambda}{K} <A^2> \tag{10}
\]

where \( dn/\tau = U'' - \gamma = -K \) is constant to lowest order from Eq. (4), and \( <A^2> \) is half the amplitude square for linear polarization and the amplitude square for circular polarization.

Consider first a light pulse broad in the direction perpendicular to its propagation such that \( \frac{\partial a}{\partial x} \gg 0 \). Since for such a short pulse \( (\frac{\partial a}{\partial x} + \frac{1}{c} \frac{\partial a}{\partial \tau})a^2 = 0 \) all perturbation quantities vanish. As the pulse moves through an originally stationary particle, after the pulse passed the particle is left stationary \( (\Delta \Delta = 0, \Delta \dot{U} = 0, \Delta U'' = 0) \).

In the following we will focus our attention to time independent fields \( \partial a/\partial t = 0 \). Since \( \Delta(\gamma) = (\gamma)^{-1}(U''\Delta U'' + U''\Delta U'') \) one may combine Eqs. (8), (9) and (10) to find the change of particle energy during an oscillation period
\[ \Delta(c) = -\frac{\lambda}{K^2} \frac{e^2}{2m^2} <A^2_1 \cdot \frac{dA^2}{dT} + \frac{e^2}{2mK} A^2_1 \Delta a^2 \] (11)

For \( K = \text{const} (\partial a/\partial x_1, = 0) \) one may integrate Eq. (11) over an oscillation period to find \( <\Delta c> = 0 \) where \( < > \) signifies the average of a quantity over an oscillation period. The integration of Eq. (7) shows that \( K = k_0 + 0(\lambda \partial a^2 \partial x_1) \), hence corrections in Eq. (11) due to the variation of \( K \) give higher order terms. Hence one finds in general the interesting result that the average particle energy is an adiabatic invariant.

Introduce now the (proper) time of an oscillation \( \Delta \tau = \int (d\eta/d\tau)^{-1} d\eta = \lambda K^{-1} \) and use Eqs. (8) and (9) to calculate the particle acceleration over the slow time scale.

\[ \left< \frac{\Delta U^1}{\Delta \tau} \right> = -\frac{e^2}{2m^2} <A^2_1 > \frac{\partial a^2}{\partial x_1} \] (12)
and

\[ \left< \frac{\Delta U^2}{\Delta \tau} \right> = -\frac{e^2}{2m^2} <A^2_1 > \frac{\partial a^2}{\partial \gamma} \] (13)

or

\[ \frac{d\hat{U}^d}{d\tau} = -\frac{e^2}{2m^2} \nabla <a^2 A^2_1 > \] (14)

where \( \hat{U}^d \) is the drift velocity of the particle, whose acceleration is due to the intensity gradient of the wave. Hence a particle oscillating in a strong wave field suffers an acceleration toward the weaker field region and is ultimately ejected from the beam. In the process the oscillatory particle energy turns into directed energy keeping \( \gamma \) constant on the average. One may multiply Eq. (14) by \( \hat{U}^d \) to find

\[ U^2_d + \frac{e^2}{m^2} <a^2 A^2_1 > = \text{constant} \] (15)

which expresses again the constancy of the sum of oscillatory and drift energies. Eq. (15) may be regarded as an energy equation with \( \phi = e^2/m^2 <a^2 A^2_1 > \)
to play the role of a potential. A particle when injected from outside the beam will be reflected as from a potential barrier.

Finally we wish to point out some consequences of this acceleration. Particles in a laser beam will be accelerated and ejected sideways. The presence of a plasma will modify our equations (e.g. the phase velocity of waves is no longer $c$), but the basic effect of ponderomotive acceleration is still there. The ejection of particles leads now to a change of dielectric function along the beam path and results in self focusing and filamentation as in the nonrelativistic case.

For a circularly polarized wave the lowest order particle motion is gyration with $\vec{A}_1 \cdot \vec{U}_1 = \text{constant}$, with the electric force providing the centripetal acceleration and $\vec{v} \times \vec{B} = 0$. In the presence of $\partial a/\partial x^+$ the guiding center will be accelerated toward the weaker field, as can be seen from Eq. (6) with the right hand side providing a constant acceleration. The physical reason for this acceleration is easily seen; the electric force acting on the particle is stronger along part of its orbit where the intensity is larger than on the weaker field side, providing a net accelerating force. The presence of a background plasma is known to lead to an axial d.c. magnetic field. This may be easily incorporated in the calculation and one finds that the outward ponderomotive force leads to an azimuthal drift in a cylindrical beam. A perturbation of such an equilibrium can lead to flute type instabilities. Details of this problem will be published elsewhere.

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