A GEOMETRICAL INTERPRETATION OF THE
2n-th CENTRAL DIFFERENCE

by

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ABSTRACT

Many algorithms used for data smoothing, data classification and error detection require the calculation of the distance from a point to the polynomial interpolating its 2n neighbors (n on each side) as is clearly the case in [1]. This computation, if performed naively, would require the solution of a system of equations and could create numerical problems. It is the purpose of this note to show that if the data is equally spaced, then this calculation can be performed using a simple recursion formula.

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1. Introduction. Many algorithms used for data smoothing, data classification and error detection require the calculation of the distance from a point to the polynomial interpolating its $2n$ neighbors ($n$ on each side) as is clearly the case in [1]. This computation, if performed naïvely, would require the solution of a system of equations and could create numerical problems. It is the purpose of this note to show that if the data is equally spaced, then this calculation can be performed using a simple recursion formula.

2. The $2n$-th Central Difference. Given $\{w_i: i = 0, \pm 1, \pm 2, \ldots\} \subseteq \mathbb{R}^2$ and a scalar $h$ we may define the second difference operator

$$D(w_i; h) = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}.$$ 

Consider $Y = \{(x_i, y_i) \subseteq \mathbb{R}^2: i = 0, \pm 1, \pm 2, \ldots\}$ with $x_{i+1} - x_i = h$ for all $i$. Recall that by the $2n$-th central difference of $Y$ at the $i$-th point we mean

$$r_i^{(2n)} = D(r_i^{(2n-2)}; h), \quad n = 1, 2, 3, \ldots$$

with $r_i^{(0)} = y_i$ for all $i$.

By the $2n$-th canonical central difference $r_i^{(2n)}$ we mean the $2n$-th central difference of the set

$$\{(jh, \delta_0): j = 0, \pm 1, \pm 2, \ldots\}$$

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at \((0,1)\) \(\delta_{ij}\) is the Kronecker delta).

A well-known property of the \(2n\)-th central difference is that it is a good approximation to the \(2n\)-th derivative for small \(h\), i.e., if \(y'_j = f(x_i)\) and \(f\) is \(2n\) times differentiable, then \(r^{(2n)}_i \to f^{(2n)}(x_i)\) as \(h \to 0\). We offer the following alternate and very useful interpretation of the \(2n\)-th central difference.

**Theorem.** Let \(P(x)\) be the unique polynomial of degree \(2n - 1\) interpolating the points

\[\{(x_{i+j}, y_{i+j}); \ j = \pm1, \pm2, \ldots, \pm n\} \ (n \geq 1).\]

Then

\[y_j - P(x_i) = \frac{r^{(2n)}_i}{r^{(2n)}_n} \text{ for any } h \neq 0.\]

**Proof.** We merely sketch a proof. Consider the following table.

**TABLE 1**

<table>
<thead>
<tr>
<th>(n)</th>
<th>(y_{1-4})</th>
<th>(y_{1-3})</th>
<th>(y_{1-2})</th>
<th>(y_{1-1})</th>
<th>(y_1)</th>
<th>(y_{1+1})</th>
<th>(y_{1+2})</th>
<th>(y_{1+3})</th>
<th>(y_{1+4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>-4</td>
<td>6</td>
<td>-4</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>-6</td>
<td>15</td>
<td>-20</td>
<td>15</td>
<td>-6</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-8</td>
<td>28</td>
<td>56</td>
<td>70</td>
<td>-56</td>
<td>28</td>
<td>-8</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1 is constructed by taking for the \(n\)-th row the second difference of the \((n-1)\)-th row with \(h = 1\), for \(n = 1, 2, \ldots\). Let \(a_{nj}\) denote the element in Table 1 in the \(n\)-th row and in the column corresponding to \(y_j\).
From Table 1 we obtain \( r_{1}^{(2n)} = \sum_{j} a_{nj} y_j \). Moreover \( r^{(2n)} = a_{n1} \).

We now construct Table 2 from Table 1 by dividing by the column corresponding to \( y_i \), i.e., by dividing the \( n \)-th row by \( r^{(2n)} \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( y_{i-4} )</th>
<th>( y_{i-3} )</th>
<th>( y_{i-2} )</th>
<th>( y_{i-1} )</th>
<th>( y_i )</th>
<th>( y_{i+1} )</th>
<th>( y_{i+2} )</th>
<th>( y_{i+3} )</th>
<th>( y_{i+4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1/2</td>
<td>1</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>1/6</td>
<td>-2/3</td>
<td>1</td>
<td>-2/3</td>
<td>1/6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
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<td>-1/20</td>
<td>3/10</td>
<td>-3/4</td>
<td>1</td>
<td>-3/4</td>
<td>3/10</td>
<td>-1/20</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1/70</td>
<td>-4/35</td>
<td>4/10</td>
<td>-4/5</td>
<td>1</td>
<td>-4/5</td>
<td>4/10</td>
<td>-4/35</td>
<td>1/70</td>
</tr>
</tbody>
</table>

As in Table 1 let \( b_{nj} \) denote a typical element of Table 2. It can be shown that \( b_{nj} \) for \(-n \leq j \leq n\) is the value at \( x_j \) of the Lagrangian polynomial of degree \( 2n - 1 \) which interpolates the \( n \) points on each side of \( x_i \) and is equal to 1 at \( x_{i+j} \) and 0 at the remaining points. Once this has been established, using Lagrange's formula it follows that

\[
y_i - P(x_i) = \sum_{j} b_{nj} y_j ;
\]

however \( \frac{1}{r^{(2n)}} = \sum_{j} b_{nj} y_j \). This proves the theorem.

REFERENCES

