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AN APPLICATION OF  
A LINEAR PROGRAMING TECHNIQUE  
TO NONLINEAR MINIMAX PROBLEMS

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# AN APPLICATION OF A LINEAR PROGRAMING TECHNIQUE TO NONLINEAR MINIMAX PROBLEMS

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## SUMMARY

A differential correction technique for solving nonlinear minimax problems is presented. The basis of the technique is a linear programing algorithm which solves the linear minimax problem. By linearizing the original nonlinear equations about a nominal solution, both nonlinear approximation and estimation problems using the minimax norm may be solved iteratively. Some consideration is also given to improving convergence and to the treatment of problems with more than one measured quantity. A sample problem is treated with this technique and with the least-squares differential correction method to illustrate the properties of the minimax solution. The results indicate that for the sample approximation problem, the minimax technique provides better estimates than the least-squares method if a sufficient amount of data is used. For the sample estimation problem, the minimax estimates are better if the mathematical model is incomplete.

## INTRODUCTION

The purpose of this paper is to present a viable approach to solving nonlinear approximation and estimation problems by using the minimax norm. The linear approximation of functions using the minimax (Chebyshev) norm has been thoroughly studied; the resulting approximation errors are uniformly bounded, which for many applications is highly desirable. Numerous methods for solving this problem can be found in the literature (refs. 1 to 5). In particular, Barrodale and Young (ref. 4) treat the problem as a linear programing problem and apply a modified simplex algorithm to solve the corresponding dual problem (ref. 6). In this paper, the nonlinear minimax approximation problem is solved by linearizing the function about an assumed solution and iteratively applying the Barrodale-Young algorithm to improve the solution.

Approximation problems involve the fitting of a functional form to a known continuous function so that a specified norm of the errors is minimized (e.g., fitting a cubic polynomial to  $\sin x$  by using the least-squares norm). If the known function is replaced by measurements corrupted by noise and available at discrete values of the independent variable, the problem can be classified as an estimation problem. Therefore, by using the technique

developed for solving nonlinear minimax approximation problems, the problem of estimating the parameters of a system of nonlinear equations in the presence of noisy measurements is also treated.

Consideration is also given to accelerating the convergence of the technique and to the treatment of problems involving more than one measurement type. The accelerated convergence method was originally presented by Osborne and Watson (ref. 5) who proved the necessary conditions for the method to converge. Then, in order to treat problems using two or more measurement types, a vector approach to minimax problems is presented. Coincident with this presentation, the manner in which data are weighted for estimation problems is discussed.

Finally, a sample problem is presented to demonstrate the application of the Barrodale-Young algorithm to nonlinear problems. The minimax results for both approximation and estimation problems are presented and are compared with the results obtained from the least-squares differential correction method.

#### SYMBOLS

$A$   $n \times 1$  vector of unknown coefficients

$\Delta A$   $n \times 1$  vector of corrections to  $A$

$a_j$   $j$ th component of  $A$

$F(A, t)$  real-valued linear approximating function,

$$\sum_{j=1}^n a_j g_j(t)$$

$f(t)$  real-valued function to be approximated; for the estimation problem, the uncorrupted measurement

$g_j(t)$   $j$ th real-valued function in linear approximating function set

$I$  a closed interval of the real line

$J_2, J_3, J_4$  oblateness coefficients in sample problem

$\ell$  number of measurement types at each measurement time

$N$	number of measurements
$n$	number of unknown coefficients
$T$	finite subset of points in $I$
$t$	independent variable
$W$	maximum absolute error
$x_0, y_0, z_0$	initial position coordinates of vehicle in sample problem, km
$\dot{x}_0, \dot{y}_0, \dot{z}_0$	initial velocity coordinates of vehicle in sample problem, km/sec
$\alpha$	nonnegative unknown parameters in primal problem
$\epsilon_i$	noise on the $i$ th measurement
$\dot{\rho}(t)$	range rate of vehicle, km/sec

### LINEAR MINIMAX APPROXIMATION

Before considering the nonlinear problem a brief discussion of the linear problem will be presented for later reference.

Let  $f(t)$  be a known real-valued function defined on a finite subset  $T = \{t_1, t_2, \dots, t_N\}$  of an interval  $I$  of the real line, and let  $\{g_i(t)\}_{i=1}^n$  be a set of  $n$  ( $n < N$ ) real-valued continuous functions defined on the same interval. Let  $F(A, t) = \sum_{j=1}^n a_j g_j(t)$  for the real vector  $A = [a_1, a_2, \dots, a_n]^T$ . Then, the linear minimax approximation problem consists of finding the vector  $A$  which minimizes

$$\text{Max}_{1 \leq i \leq N} |f(t_i) - F(A, t_i)| \quad (1)$$

Let  $A^*$  be the minimizing vector and  $r^*$  be the corresponding minimum value of expression (1). An approximation  $F(A^*, t)$  that minimizes expression (1) will be called a best minimax approximation to  $f(t)$ .

An obvious advantageous characteristic of the minimax approximation is that once  $A^*$  and, therefore,  $r^*$  have been determined, then the error given by

$$r(t_i) = f(t_i) - F(A^*, t_i)$$

at every point  $t_i$  does not exceed  $r^*$  in absolute value. In other words, the minimax norm provides a uniform approximation to the function. In contrast to this, the standard least-squares technique gives an approximation which fulfills a specified criterion on the entire set  $T$  without indicating the goodness of the approximation at any individual point.

Under certain conditions, minimax approximations possess a characterization which is often used to derive methods for obtaining the approximation. First, it is necessary to define the Haar condition.

Definition of Haar condition (ref. 3): A set of functions  $\{g_i(t)\}_{i=1}^n$  defined on the interval  $[a, b]$  satisfy the Haar condition if each function is continuous and if, for any value of  $t \in [a, b]$ , every set of  $n$  vectors of the following form is independent:

$$[g_1(t), g_2(t), \dots, g_n(t)]^T$$

That is, the determinant

$$\begin{vmatrix} g_1(t_1) & g_2(t_1) & \dots & g_n(t_1) \\ g_1(t_2) & g_2(t_2) & \dots & g_n(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(t_n) & g_2(t_n) & \dots & g_n(t_n) \end{vmatrix}$$

is nonzero for all distinct values of  $t_i$ .

The set of functions  $\{1, x, x^2, \dots, x^n\}$ , which are used in polynomial approximations, is an example of a set fulfilling the Haar condition. The nonzero determinant for these functions is the well-known Vandermonde determinant.

The characterization of linear minimax approximations is stated in the Alternation Theorem as follows:

Alternation Theorem (ref. 3): Let  $\{g_i\}_{i=1}^n$  be a set of functions defined on  $[a, b]$  and satisfying the Haar condition; let  $T$  be a closed subset of  $[a, b]$ . A necessary and sufficient condition that

$$F(A, t) = \sum_{j=1}^n a_j g_j(t)$$

be a best minimax approximation on  $T$  to a continuous function  $f(t)$  is that the error function

$$r(t) = f(t) - F(A, t)$$

exhibit at least  $n + 1$  alternations on  $T$ . That is,

$$r(t_i) = -r(t_{i+1}) = \pm r^*$$

for  $t_0 < t_1 < \dots < t_n, t_i \in T$ .

In other words, under the hypotheses of the theorem, the error function attains the maximum value at least  $n + 1$  times with alternating signs if the best approximation has been obtained. By using the converse of this theorem, several algorithms, such as the Remes single-exchange algorithm (ref. 3), construct the best minimax approximation by solving the  $n + 1$  alternant equations for the coefficient set  $A^*$  and the maximum error  $r^*$ . These algorithms, however, are hampered by the fact that the proper selection of the points  $\{t_0, t_1, \dots, t_n\}$  is unknown. Therefore, the major effort of the algorithms is to determine this set of points.

It should be observed that it may not be practical to determine if the Haar condition holds for a particular set of functions. Because of this difficulty, other approaches for obtaining minimax approximations are generally more useful.

#### BARRODALE-YOUNG ALGORITHM

In the Barrodale-Young algorithm (ref. 4), the linear minimax problem is rewritten so that it may be treated as a linear programming problem. The algorithm is not based

on the Alternation Theorem or on any other special characteristics of the minimax problem. The dual problem to the original linear programming problem is then solved by using the simplex algorithm.

First, in order to insure the nonnegativity of the parameters being sought, define the set  $\{a_j\}_{j=1}^{n+1}$  by

$$\alpha_{n+1} = \text{Max} \left( 0, -\text{Min}_{1 \leq j \leq n} a_j \right)$$

$$\alpha_j = a_j + \alpha_{n+1} \quad (1 \leq j \leq n)$$

Further, let

$$e_i = e(t_i) = F(A, t_i) - f(t_i)$$

and

$$W = \text{Max} |e_i|$$

By using these definitions,  $e_i$  becomes

$$\begin{aligned} e_i &= F(A, t_i) - f(t_i) \\ &= \sum_{j=1}^n a_j g_j(t_i) - f(t_i) \\ &= \sum_{j=1}^n (\alpha_j - \alpha_{n+1}) g_j(t_i) - f(t_i) \\ &= \sum_{j=1}^n \alpha_j g_j(t_i) - \alpha_{n+1} \sum_{j=1}^n g_j(t_i) - f(t_i) \\ &= \sum_{j=1}^{n+1} \alpha_j g_j(t_i) - f(t_i) \end{aligned}$$

where

$$g_{n+1}(t_i) = - \sum_{j=1}^n g_j(t_i)$$

This allows the minimax problem to be written as the linear programming problem. For example, minimize  $W$  with respect to the nonnegative parameters  $\alpha_j$  where  $1 \leq j \leq (n+1)$  and subject to the  $2N$  constraint expressions

$$\left. \begin{aligned} \sum_{j=1}^{n+1} \alpha_j g_j(t_i) + W &\geq f(t_i) \\ - \sum_{j=1}^{n+1} \alpha_j g_j(t_i) + W &\geq -f(t_i) \end{aligned} \right\} \quad (1 \leq i \leq N)$$

Since in linear programming it is easier to solve a problem with many parameters and few constraints than one with few parameters and many constraints (ref. 6), the dual problem of the aforementioned is solved.

First, the primal problem may be written in vector notation as follows: Minimize

$$W = D^T \bar{\alpha}$$

subject to

$$B\bar{\alpha} \geq C$$

where  $D$  is the  $(n+2) \times 1$  vector  $D = [0, 0, \dots, 0, 1]^T$ ,  $\bar{\alpha}$  is the  $(n+2) \times 1$  vector  $\bar{\alpha} = [\alpha_1, \dots, \alpha_{n+1}, W]^T$ ,  $B$  is the  $2N \times (n+2)$  matrix, or

$$B = \begin{bmatrix} g_1(t_1) & \dots & g_{n+1}(t_1) & 1 \\ \vdots & & \vdots & \vdots \\ g_1(t_N) & \dots & g_{n+1}(t_N) & 1 \\ -g_1(t_1) & \dots & -g_{n+1}(t_1) & 1 \\ \vdots & & \vdots & \vdots \\ -g_1(t_N) & \dots & -g_{n+1}(t_N) & 1 \end{bmatrix}$$

and  $C$  is the  $2N \times 1$  vector, or

$$C = [f(t_1), \dots, f(t_N), -f(t_1), \dots, -f(t_N)]^T$$

In the aforementioned expressions, the symbol " $\geq$ " indicates that each vector component fulfills this condition.

The dual problem may immediately be written with this notation and is as follows:

Maximize

$$C^T Y$$

subject to

$$B^T Y \leq D$$

where  $Y$  is a  $2N \times 1$  vector.

In order to solve the dual problem, nonnegative slack variables are added to the constraint equations to change the inequalities to equalities. A modified simplex algorithm is then used to solve this problem; the resulting coefficients of the slack variables in the solution of the dual problem are the unknown values of  $\alpha_i$  of the primal problem.

### NONLINEAR MINIMAX PROBLEMS

The approach taken here to solve nonlinear minimax problems is the one usually taken in solving a nonlinear problem. That is, the nonlinear problem is linearized about an assumed nominal solution and an appropriate linear technique is applied.

Let  $F(A, t)$  now be a nonlinear function of the parameter vector  $A$ . Let  $A_0$  be a particular "guess" of the vector which provides a nominal evaluation of  $F(A, t)$ ; that is, in some sense,  $F(A_0, t)$  approximates  $f(t)$  "closely." Expanding  $F(A, t)$  in a Taylor series about  $A_0$  yields

$$F(A, t) = F(A_0, t) + \left[ \frac{\partial F}{\partial A} (A, t) \right]_{A=A_0} \Delta A + (\text{Higher order terms})$$

where  $\Delta A$  is the  $n \times 1$  vector, or

$$\Delta A = [\Delta a_1, \Delta a_2, \dots, \Delta a_n]^T$$

If the nominal  $F(A_0, t)$  is sufficiently "close" to the true solution to assume that the the higher order terms are small, then this series may be truncated after the linear term.

By using the linearization of  $F(A, t)$ , the minimax problem is to minimize

$$\begin{aligned} \text{Max}_{1 \leq i \leq N} |f(t_i) - F(A, t_i)| &= \text{Max}_{1 \leq i \leq N} \left| f(t_i) - \left\{ F(A_0, t_i) + \left[ \frac{\partial F}{\partial A} (A, t_i) \right]_{A=A_0} \Delta A \right\} \right| \\ &= \text{Max}_{1 \leq i \leq N} \left| \left[ f(t_i) - F(A_0, t_i) \right] - \left[ \frac{\partial F}{\partial A} (A, t_i) \right]_{A=A_0} \Delta A \right| \quad (2) \end{aligned}$$

with respect to the difference vector  $\Delta A$ . This is simply the linear minimax problem in which the function  $f(t_i) - F(A_0, t_i)$  is being approximated by the function

$$\left[ \frac{\partial F}{\partial A} (A, t_i) \right]_{A=A_0} \Delta A$$

Therefore, the Barrodale-Young method can be used to obtain  $\Delta A$ , and the nominal  $A_0$  can be corrected to give  $A_1 = A_0 + \Delta A$ . Since the function  $F(A, t)$  is linearized, the vector  $A_1$  does not minimize expression (1) as desired; therefore, with the estimate of  $A_1$  instead of  $A_0$ , equation (2) is again minimized with respect to  $\Delta A$  to obtain a new correction  $\Delta A$  and a new estimate  $A_2 = A_1 + \Delta A$ . The entire process is iterated until the Chebyshev norm has been sufficiently minimized. Note that, as often occurs in the application of the least-squares technique to a linearized problem, the process may diverge if the initial estimate  $A_0$  is not "close" to the best estimate of  $A$ .

Osborne and Watson (ref. 5) have proposed a modification to this scheme and have proven convergence conditions for the modified scheme. In the modification, after  $\Delta A$  has been obtained in the  $j$ th iteration, Osborne and Watson suggest that

$$\text{Max}_{1 \leq i \leq N} |f(t_i) - F(A_{j-1} + \beta \Delta A, t_i)|$$

be minimized with respect to the scalar  $\beta$ . By using the resulting value of  $\beta$ , the estimate of  $A$  becomes  $A_j = A_{j-1} + \beta \Delta A$  at the end of the iteration. Since  $\beta$  is a scalar, the minimization is readily obtained by using a one-dimensional technique. For the modified scheme, the convergence criteria are given in the following theorem:

Theorem (ref. 5): If the matrix  $\nabla F(A)$  is of rank  $n$  and satisfies the Haar condition for  $A$  in the region of interest, then the sequence  $A_0, A_1, A_2, \dots$  converges.

Remark 1:  $\nabla F(A)$  is the matrix with the  $N$  rows

$$\left[ \frac{\partial F}{\partial A} (A, t_i) \right] \quad (\text{where } i = 1, 2, \dots, N)$$

Remark 2: If for a matrix of rank  $n$  every  $n \times n$  submatrix is nonsingular, then the matrix satisfies the Haar condition.

First, although the modified scheme is guaranteed to converge under the stated conditions, it should be noted that insuring that the hypotheses are true is impractical. Specifically, for a matrix of rank  $n$  on the set of  $N$  points, proof of the Haar condition for one vector  $A$  requires checking the singularity of  $\binom{N}{n}$  submatrices. For a reasonable problem such as  $N = 40$  points and  $n = 6$  parameters, this is 3,838,380 submatrices. For this reason, it is easier simply to assume that the hypotheses hold.

Second, it should be observed that the second step of the modified scheme can be computationally expensive, depending on the particular function  $F(A, t)$  and the technique used to obtain  $\beta$ . For example, if a search technique requiring four additional function evaluations were applied to the sample problem presented later, this would require numerically integrating the differential equations four additional times during each iteration. For the current study, the unmodified algorithm without the one-dimensional search is considered in order to provide a valid comparison with the standard least-squares differential correction method.

For nonlinear estimation problems, the preceding linearization technique can also be applied. The major difference between the approximation and estimation problems is that the function  $f(t)$  in the estimation problem is a measured quantity corrupted by noise. Thus, if the actual measurements are given by

$$f_M(t_i) = f(t_i) + \epsilon_i$$

where  $\epsilon_i$  is the measurement noise and  $M$  denotes a measured quantity, then the problem is to minimize

$$\text{Max}_{1 \leq i \leq N} \left| \left[ f_M(t_i) - F(A, t_i) \right] - \left[ \frac{\partial F}{\partial A} (A, t_i) \right]_{A=A_0} \Delta A \right| \quad (3)$$

with respect to  $\Delta A$ . By assuming that  $A_0$  is a good initial estimate of  $A$  and that  $F(\hat{A}, t)$  is a perfect model of  $f(t)$ , then

$$\text{Max}_{1 \leq i \leq N} |f_M(t_i) - F(\hat{A}, t_i)| = \text{Max}_{1 \leq i \leq N} |\epsilon_i|$$

In other words, if the function  $f(t)$  is perfectly modeled, the Chebyshev norm is bounded below by the maximum value of the measurement noise. This also indicates the major difficulty of minimax estimation: an extraneous data point ("wild point") drastically affects the norm value and consequently the estimates. Therefore, before applying a minimax estimation procedure, it may be advisable to edit the data judiciously.

A limited amount of research has been devoted to the statistical aspects of minimax estimates. However, Howard and Kaufman (refs. 7 and 8) have derived sufficient conditions for the estimated measurements to be unbiased. In particular, if the noise samples  $\epsilon_i$  are independent random variables with

$$\text{Max}_{1 \leq i \leq N} \epsilon_i = - \text{Min}_{1 \leq i \leq N} \epsilon_i \quad (4)$$

then the predicted measurements  $F(A, t)$  obtained from minimizing

$$\text{Max}_{1 \leq i \leq N} |f_M(t_i) - F(A, t_i)|$$

will be unbiased. Generally, this result is not very useful since the conditions on the noise given in equation (4) will not hold. Nevertheless, if the data do not contain extraneous points and if a large number of points are used, it is expected that the noise will approach the conditions in equation (4).

#### FURTHER ESTIMATION CONSIDERATIONS

Many estimation problems involve the processing of several measurement types at each measurement time; depending on knowledge of the measurements, it is often advisable or even necessary to weight the measurements in order to account for confidence in the data. Although the application discussions will be centered on minimax estimation based on unweighted scalar measurements, the more realistic problem of estimating with weighted vector measurements will be briefly considered in this section.

Consider the general linear problem of estimating the  $n \times 1$  vector  $A$  based on a sequence of  $l \times 1$  vector measurements  $Y_1, Y_2, \dots, Y_m$  where

$$Y_i = G_i A + \epsilon_i \quad (i = 1, 2, \dots, m) \quad (5)$$

In equation (5),  $G_i$  is an  $\ell \times n$  matrix and  $\epsilon_i$  is the  $\ell \times 1$  measurement noise vector. Assuming that weight may be attached to the various components of  $Y_i$ , the weighted least-squares solution of equation (5) is obtained by minimizing with respect to  $A$  the function

$$\sum_{i=1}^m \epsilon_i^T W \epsilon_i = \sum_{i=1}^m (Y_i - G_i A)^T W (Y_i - G_i A) \quad (6)$$

In equation (6),  $W$  is usually an  $\ell \times \ell$  diagonal matrix, or  $W = \text{Diag}(w_1^2, w_2^2, \dots, w_\ell^2)$ , where  $w_i$  ( $1 \leq i \leq \ell$ ) is the weight associated with the  $i$ th measurement. For this reason,  $W$  can be written as the product  $W = V^T V$ , where  $V = \text{Diag}(w_1, w_2, \dots, w_\ell)$ , and equation (6) can be written as

$$\sum_{i=1}^m \epsilon_i^T W \epsilon_i = \sum_{i=1}^m [V(Y_i - G_i A)]^T [V(Y_i - G_i A)]$$

The original problem of equation (5) can then be restated as

$$VY_i = (VG_i)A + V\epsilon_i \quad (i = 1, 2, \dots, m) \quad (7)$$

In order to estimate  $A$  of equation (5) by using a "weighted minimax" approach, consider instead the minimax estimate of  $A$  based on equation (7). The description of a general linear minimax problem stated previously requires that the function being approximated be real-valued. Since in equations (5) and (7) the function  $Y_i$  is vector-valued, a straightforward approach (see, for example, ref. 8) is to treat the  $\ell$  components of  $Y_i$  at the  $m$  points of the data set as the  $\ell m$  values of a single real-valued function. If  $Y_i^j$  indicates the  $j$ th component of  $Y_i$  and  $G_i^j$  the  $j$ th row of  $G_i$ , then the weighted minimax solution of equation (5) is obtained by minimizing

$$\text{Max}_{\substack{1 \leq i \leq m \\ 1 \leq j \leq \ell}} |w_j(Y_i^j - G_i^j A)|$$

with respect to  $A$ , which gives the minimax solution of equation (7):

For weighted least-squares problems, the weight  $w_j$  is usually chosen to be the reciprocal of the standard deviation of the noise of the corresponding measurement. Such a selection not only provides a relative weighting of the measurements but also means that

the components of equation (7) are dimensionless. For a weighted minimax approach, the weights may be chosen in the same fashion initially. However, since different measurement types are represented in equation (7), some adjustment of the weights may be required for a particular problem in order to insure that the various components of equation (7) are of the same magnitude. In other words, although the components of equation (7) are dimensionless, they may vary considerably in magnitude due to the nature of the specific measurement types; since the minimax estimates would be influenced mainly by the component of largest magnitude, all of the components will affect the solution only if they are all of the same magnitude.

### AN APPLICATION OF THE DIFFERENTIAL CORRECTION TECHNIQUE

The problem chosen for demonstration in this study is the determination of the initial conditions and one of the parameters of the equations of motion of a space vehicle. The equations of motion, which represent the forces acting on the vehicle due to the presence of an oblate earth, are given by (ref. 9)

$$\ddot{\bar{x}}(t) = -\frac{\mu\bar{x}(t)}{r^3} + J_2g_1(\bar{x}(t)) + J_3g_2(\bar{x}(t)) + J_4g_3(\bar{x}(t)) \quad (8)$$

In equation (8),  $\bar{x}(t)$  and  $\ddot{\bar{x}}(t)$  are the position and acceleration vectors, respectively, of the vehicle relative to a Cartesian coordinate system centered at the earth, and  $r$  is the magnitude of  $\bar{x}(t)$ . The parameters  $\mu$  and  $J_2, J_3,$  and  $J_4$  are, respectively, the gravitational constant and oblateness coefficients. The vectors  $g_1(\bar{x}(t)), g_2(\bar{x}(t)),$  and  $g_3(\bar{x}(t))$  are nonlinear functions of the position.

For this particular example, the measurement function was chosen to be the range rate  $\dot{\rho}(t)$  which is a nonlinear function of the vehicle and tracking-station positions and velocities (ref. 9). The tracking station was assumed to be based on a rotating earth. Therefore, when the minimax procedure is applied to the linearized problem, equation (2) becomes

$$\text{Max}_{1 \leq i \leq N} \left| \left[ \dot{\rho}_M(t_i) - \dot{\rho}_c(t_i) \right] - \left[ \frac{\partial \dot{\rho}}{\partial P}(t_i) \right]_c^T \Delta P \right| \quad (9)$$

where  $c$  denotes values obtained from evaluations along the calculated nominal trajectory. The vectors  $P$  and  $\Delta P$  are, respectively, the unknown parameters and corrections to the parameters. For the cases considered here, the parameters are the initial position  $(x_0, y_0, z_0)$ , initial velocity  $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ , and the first oblateness coefficient  $J_2$ . The vector of partials in expression (9) is obtained from the chain rule

$$\left[ \frac{\partial \dot{\rho}}{\partial \mathbf{P}}(t_i) \right]_{\mathbf{c}}^{\mathbf{T}} = \begin{bmatrix} \frac{\partial \dot{\rho}(t_i)}{\partial \bar{\mathbf{x}}(t_i)} & \frac{\partial \dot{\rho}(t_i)}{\partial \dot{\bar{\mathbf{x}}}(t_i)} \end{bmatrix}^{\mathbf{T}} \phi(t_i)$$

where the first factor on the right is a  $1 \times 6$  row vector of partials of the range rate with respect to the position and velocity at  $t_i$  and  $\phi(t)$  is the  $6 \times 7$  matrix of partials of the current position and velocity with respect to the parameters. The matrix  $\phi(t)$  is obtained by numerically integrating along the nominal trajectory the partials of equation (8) with respect to the parameters. This is accomplished by assuming that the derivatives are continuous in  $t$  and in the variables, so that the order of differentiation may be interchanged (ref. 10); for example,

$$\frac{d}{dt} \left[ \frac{\partial \mathbf{x}}{\partial \mathbf{y}_0} \right] = \frac{\partial}{\partial \mathbf{y}_0} \left[ \frac{d\mathbf{x}}{dt} \right]$$

The classical technique for solving nonlinear approximation and estimation problems is the least-squares differential correction method (also called modified Newton-Raphson or quasilinearization). Since this is still the standard approach used on many problems, it was chosen to provide a comparative illustration of the results which can be expected from a standard technique and the minimax technique.

Briefly described, the least-squares differential correction technique (hereinafter referred to simply as "least squares") uses the same linearization about an assumed nominal solution as that of the minimax technique; for the present example, the linearized problem is

$$\Delta \dot{\rho}_i = \dot{\rho}_M(t_i) - \dot{\rho}_c(t_i) = \left[ \frac{\partial \dot{\rho}}{\partial \rho}(t_i) \right]_{\mathbf{c}}^{\mathbf{T}} \Delta \mathbf{P} \quad (10)$$

Letting

$$\epsilon_i = \Delta \dot{\rho}_i - \left[ \frac{\partial \dot{\rho}}{\partial \rho}(t_i) \right]_{\mathbf{c}}^{\mathbf{T}} \Delta \mathbf{P} \quad (11)$$

in equation (6), the least-squares criterion requires that equation (6) now be minimized with respect to  $\Delta \bar{\mathbf{P}}$ . The well-known solution to this linear problem (ref. 11) is

$$\Delta P = \left( \sum_{i=1}^m \left[ \frac{\partial \dot{\rho}(t_i)}{\partial P} \right]_c W \left[ \frac{\partial \dot{\rho}(t_i)}{\partial P} \right]_c^T \right)^{-1} \left( \sum_{i=1}^m \left[ \frac{\partial \dot{\rho}(t_i)}{\partial P} \right]_c W \Delta \dot{\rho}_i \right) \quad (12)$$

The new value of  $P$  is obtained by adding  $\Delta P$  to the initial value of  $P$ . Since the new  $P$  was obtained by solving the linearized problem, it does not generally minimize the sum of the squares of the measurement residuals. Therefore, the new value of  $P$  is used to obtain a second correction from equation (12). In general, the procedure must be iterated until a satisfactory solution is obtained. Thus, the application of the two linear techniques (classical least squares and linear minimax) to a nonlinear problem is identical except for the method (i.e., norm) used to obtain the parameter corrections.

For the cases considered, the "true" initial conditions are given in table I. The "true" solution generated with these initial conditions was used for both the approximation and the estimation problems. The measurements (range rate) were generated every 30 seconds along the "true" solution. For the estimation problem, normally distributed zero mean noise having a standard deviation of 0.1 meter per second (0.002 percent of the maximum range rate) was added to the measurements. For all the cases presented, the initial estimates of the position and velocity components were perturbed 0.1 kilometer and 0.1 km/sec, respectively, from the "true" values. Each technique was iterated until the corresponding norm of the error attained a constant minimum value (i.e., the first six digits were unchanged); the technique was then considered to have converged.

By using the measurements without any noise added, the nonlinear function represented by the range rate evaluated along the described orbit presents an excellent test of the outlined approach to solving a nonlinear minimax approximation problem. The first approximating function used in these experiments consists of the equations of motion (8) with  $J_2$ ,  $J_3$ , and  $J_4$  fixed at zero; the initial conditions become the parameters available for adjustment.

The results obtained by using the minimax and, as a comparison, the least-squares norm on the approximation problem are given in table II. The five cases presented differ by the number of data points used (which is equivalent to the length of the interval of the independent variable  $t$ ). The entries in the table are the errors in the six initial conditions. The table suggests that, for this particular problem at least, the initial-condition errors are less for the minimax approximation if a sufficient number of data points (about 70 points) are used. As an example of the function approximation errors to be encountered, figure 1 presents a plot of the range-rate residuals plotted against time for both norms applied to the 61-point case. The minimax errors all lie within the band  $\pm 2.5833 \times 10^{-5}$  km/sec; in contrast to this, the least-squares errors generally increase, with the largest error in absolute value at the final point.

The second approximation problem consisted of fixing  $J_3$  and  $J_4$  at zero and seeking the values of the initial conditions and  $J_2$  that minimize the corresponding norm. The initial estimate of  $J_2$  was zero. Since  $J_3$  and  $J_4$  are three orders of magnitude smaller than  $J_2$ , this approximation should be very close to the original function. Table III presents the results of this approximation for the same five sets of data used previously. However, since seven parameters are required, the 41-point approximation diverged for both norms due to an inadequate amount of information. As in the previous approximation, the minimax initial condition and  $J_2$  errors are smaller than the least-squares errors if a sufficient amount of data is used. It should also be noted that for both norms, almost the same number of iterations were required on the same cases. For the state-only approximation problem, five iterations were required on all cases except in the 71-point minimax approximation which required three iterations. On the state-plus- $J_2$  problem, four iterations were required by both norms except for the 51-point case for which both procedures used five iterations.

Because the present problem required an extensive amount of computer time, only a limited number of Monte Carlo simulations were performed to test the minimax algorithm as an estimation procedure. The "true" trajectory and the initial estimates of the initial state and  $J_2$  that were used for the approximation problems were also used for the estimation problem. However, for the estimation study, ten different sets of normally distributed random numbers, each having a mean of zero and a standard deviation of 0.1 meter per second, were added to the "true" measurements to produce ten different sets of noise-corrupted measurements. For all of the estimation problems discussed later (e.g., estimation of initial state only), each of the ten sets of noisy measurements was used by the minimax and least-squares procedures to obtain a set of estimates. The root-mean-square (rms) errors of these ten sets of estimates then provided a comparison of the two estimation techniques.

Since noisy measurements cause an estimation problem to be more difficult than an approximation problem, a larger number of data points must be used for the estimation problem. For the problems considered here, 61- and 81-point data sets were used. Neither are a sufficient number of measurements to yield accurate estimates; however, the estimation procedures did converge for these cases and, therefore, provide comparisons between the two estimators.

Normally, a weighted least-squares procedure is applied to an estimation problem. For the present problem, no weighting was used since only one measurement type was considered and the constant weight which would have been applied to these measurements can be factored out. Thus, this study reduces to a comparison of standard (unweighted) least-squares and minimax techniques.

The first estimation problem illustrates state estimation with an incomplete model. ("Incomplete model" refers to state equations which do not completely represent all the dynamics present in the measurements.) The parameters  $J_2$ ,  $J_3$ , and  $J_4$  were fixed at zero and only the initial state was estimated. The rms errors given in the first section of table IV illustrate the resulting estimates. For both estimators, the estimates are generally improved when the number of data points is increased; however, for both data sets, more of the minimax estimates are better than the least-squares estimates.

The second problem consisted of fixing the parameters at the true values and again estimating only the initial state. The entries in table IV show that the least-squares estimates are consistently better than the minimax estimates. In addition, all of the least-squares estimates have improved with an increase in the number of data points; however, most of the minimax estimates have degraded for the larger data set.

A comparison of the results of the first two estimation problems suggests the following conclusions: First, for estimation problems based on an incomplete mathematical model, the minimax estimator may provide better estimates than the least-squares estimator. This result is derived from the fact that with an incomplete model most of the error may be caused by the model errors rather than by the measurement noise; thus, the effects of random noise would be negligible. Second, for problems using a complete model, the least-squares estimates are better than the minimax errors because the only error source (aside from computational errors) is the random-measurement noise; under these conditions the least-squares estimates are identical to the minimum variance estimates (ref. 11). As indicated previously, the minimax norm is bounded below by the maximum noise value; thus, one measurement containing noise at the three-sigma level has considerably more influence on the minimax estimates than on the least-squares estimates. This leads to the third conclusion: the minimax estimates may degrade as the number of measurements is increased. The degradation of the estimates is caused by the fact that when more measurements are processed the noise approaches the maximum level more closely (say, three sigma). To an extent, this degradation may be offset by the fact that the data interval has also increased. Thus, a trade-off occurs between the effects induced by increasing error bounds and longer measurement intervals.

For the third problem, the parameters  $J_3$  and  $J_4$  were fixed at the true values,  $J_2$  was initially estimated as zero, and both the state and  $J_2$  were estimated. The results in table IV show that the least-squares estimator again provides better estimates for a complete model. However, both estimators provided improved estimates when the data set was increased. This problem indicates the smaller data set (61 points) is approximately the minimum number of measurements required to obtain fairly reasonable estimates of the seven parameters. For the larger data set, the estimate improvement due to using the 33-percent longer interval overrides any degradation of the minimax estimates

induced by an increased maximum noise level. Thus, problems 2 and 3 illustrate that minimax estimates may improve or degrade with an increase in the number of measurements, depending on the number of parameters being estimated and the number of measurements being used.

The results of this study were obtained by using FORTRAN programs written for the CDC 6000 series computers. Both techniques required approximately the same number of memory locations; depending on the number of parameters being estimated, the minimax technique required 10 to 18 percent more computer time.

### CONCLUDING REMARKS

By linearizing the original nonlinear function, nonlinear approximation and estimation problems using the minimax norm may be treated by using the Barrodale-Young algorithm. This approach has been applied to a simple dynamical problem. For the cases in which the sample problem is treated as an approximation problem, the minimax-parameter estimates are better than those obtained by using the least-squares technique when a sufficient amount of data is used.

By adding random noise to the measurements, the sample problem has been treated in a limited Monte Carlo simulation as a nonlinear estimation problem. The results suggest that for an incomplete model the minimax estimates are better than the least-squares estimates; however, for a complete model the minimax estimates are worse. Because of this behavior, selection of one of these norms for a particular problem requires deciding whether accurate parameter estimates or uniformly bounded measurement errors is more important.

Langley Research Center,  
National Aeronautics and Space Administration,  
Hampton, Va., June 29, 1973.

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TABLE I  
TRUE VALUES OF INITIAL CONDITIONS AND PARAMETERS

Initial conditions and parameters:	Values:
$x_0$ , km . . . . .	7920.0
$y_0$ , km . . . . .	0
$z_0$ , km . . . . .	0
$\dot{x}_0$ , km/sec . . . . .	0
$\dot{y}_0$ , km/sec . . . . .	3.67204354
$\dot{z}_0$ , km/sec . . . . .	6.11130673
$J_2$ . . . . .	$1.0823 \times 10^{-3}$
$J_3$ . . . . .	$-2.3 \times 10^{-6}$
$J_4$ . . . . .	$-1.8 \times 10^{-6}$

TABLE II  
LEAST-SQUARES AND MINIMAX ERRORS FOR TWO-BODY  
APPROXIMATION PROBLEM

Technique	$x_0$ , km	$y_0$ , km	$z_0$ , km	$\dot{x}_0$ , km/sec	$\dot{y}_0$ , km/sec	$\dot{z}_0$ , km/sec
41 data points						
Least squares. .	5.4773	-3.9365	15.2882	-0.011570	0.056829	-0.020330
Minimax . . . . .	6.6781	-1.1443	13.1722	-0.011453	0.053413	-0.028319
51 data points						
Least squares. .	1.6789	-12.0558	21.1387	-0.011516	0.065399	-0.035579
Minimax . . . . .	1.2914	-12.8184	21.6812	-0.011497	0.066123	-0.036330
61 data points						
Least squares. .	-0.8422	-15.5293	22.3462	-0.010049	0.063877	-0.035245
Minimax . . . . .	-1.1187	-15.6605	22.1912	-0.009754	0.062966	-0.035125
71 data points						
Least squares. .	-1.9417	-14.2756	18.9720	-0.007300	0.052732	-0.029565
Minimax . . . . .	-1.7123	-11.9056	15.8533	-0.005729	0.044871	-0.025543
81 data points						
Least squares. .	-1.7844	-8.8373	11.5633	-0.003372	0.033131	-0.019207
Minimax . . . . .	-1.2839	-5.3534	7.2817	-0.001341	0.022690	-0.013782

TABLE III  
LEAST-SQUARES AND MINIMAX ERRORS FOR STATE AND  $J_2$  APPROXIMATION PROBLEM

Technique	$x_0'$ km	$y_0'$ km	$z_0'$ km	$\dot{x}_0'$ km/sec	$\dot{y}_0'$ km/sec	$\dot{z}_0'$ km/sec	$J_2$
41 data points							
Least squares . .	Diverged						
Minimax . . . . .	Diverged						
51 data points							
Least squares . .	0.0215	0.3346	-0.4526	0.000230	-0.001216	0.000645	$0.552 \times 10^{-5}$
Minimax . . . . .	0.0778	-0.0193	-0.5532	0.000070	-0.000853	0.001088	$0.32 \times 10^{-6}$
61 data points							
Least squares . .	0.0312	0.2417	-0.2980	0.000146	-0.000743	0.000391	$0.181 \times 10^{-5}$
Minimax . . . . .	0.0350	0.2614	-0.3315	0.000158	-0.000830	0.000445	$0.16 \times 10^{-6}$
71 data points							
Least squares . .	0.0274	0.1738	-0.2023	0.000103	-0.000470	0.000243	$0.610 \times 10^{-5}$
Minimax . . . . .	0.0270	0.1684	-0.1958	0.000100	-0.000452	0.000234	$0.628 \times 10^{-5}$
81 data points							
Least squares . .	0.0230	0.1390	-0.1566	0.000084	-0.000343	0.000173	$0.833 \times 10^{-5}$
Minimax . . . . .	0.0226	0.1359	-0.1529	0.000082	-0.000335	0.000169	$0.810 \times 10^{-5}$

TABLE IV  
 LEAST-SQUARES AND MINIMAX ESTIMATION RESULTS  
 (a) Problem 1; state estimation; incomplete model

Technique	Root-mean-square errors for -						$J_2$
	$x_0$ , km	$y_0$ , km	$z_0$ , km	$\dot{x}_0$ , km/sec	$\dot{y}_0$ , km/sec	$\dot{z}_0$ , km/sec	
61 data points							
Least squares..	0.99208	15.7914	22.7019	0.01021	0.06458	0.03575	
Minimax .....	1.10573	11.5311	22.4604	0.01028	0.06420	0.03570	
81 data points							
Least squares..	1.86367	9.4162	12.3973	0.00385	0.03515	0.02039	
Minimax .....	1.76134	8.7720	11.5656	0.00421	0.03271	0.01902	

(b) Problem 2; state estimation; complete model

61 data points							
Least squares..	0.36982	2.3658	3.0586	0.00164	0.00776	0.00425	
Minimax .....	1.32146	3.4590	4.1687	0.00213	0.01053	0.00579	
81 data points							
Least squares..	0.30125	1.7741	2.3222	0.00113	0.00579	0.00326	
Minimax .....	0.76948	4.4074	5.3765	0.00251	0.01311	0.00726	

(c) Problem 3; state plus  $J_2$  estimation; complete model

61 data points							
Least squares..	0.59785	12.7094	17.9681	0.00838	0.05199	0.02876	0.00188
Minimax .....	1.9473	19.0346	26.4148	0.01295	0.07497	0.04020	0.00108
81 data points							
Least squares..	0.29727	1.7074	2.2634	0.00108	0.00585	0.00333	0.00014
Minimax .....	1.10238	6.3002	8.0801	0.00342	0.02113	0.01194	0.00046

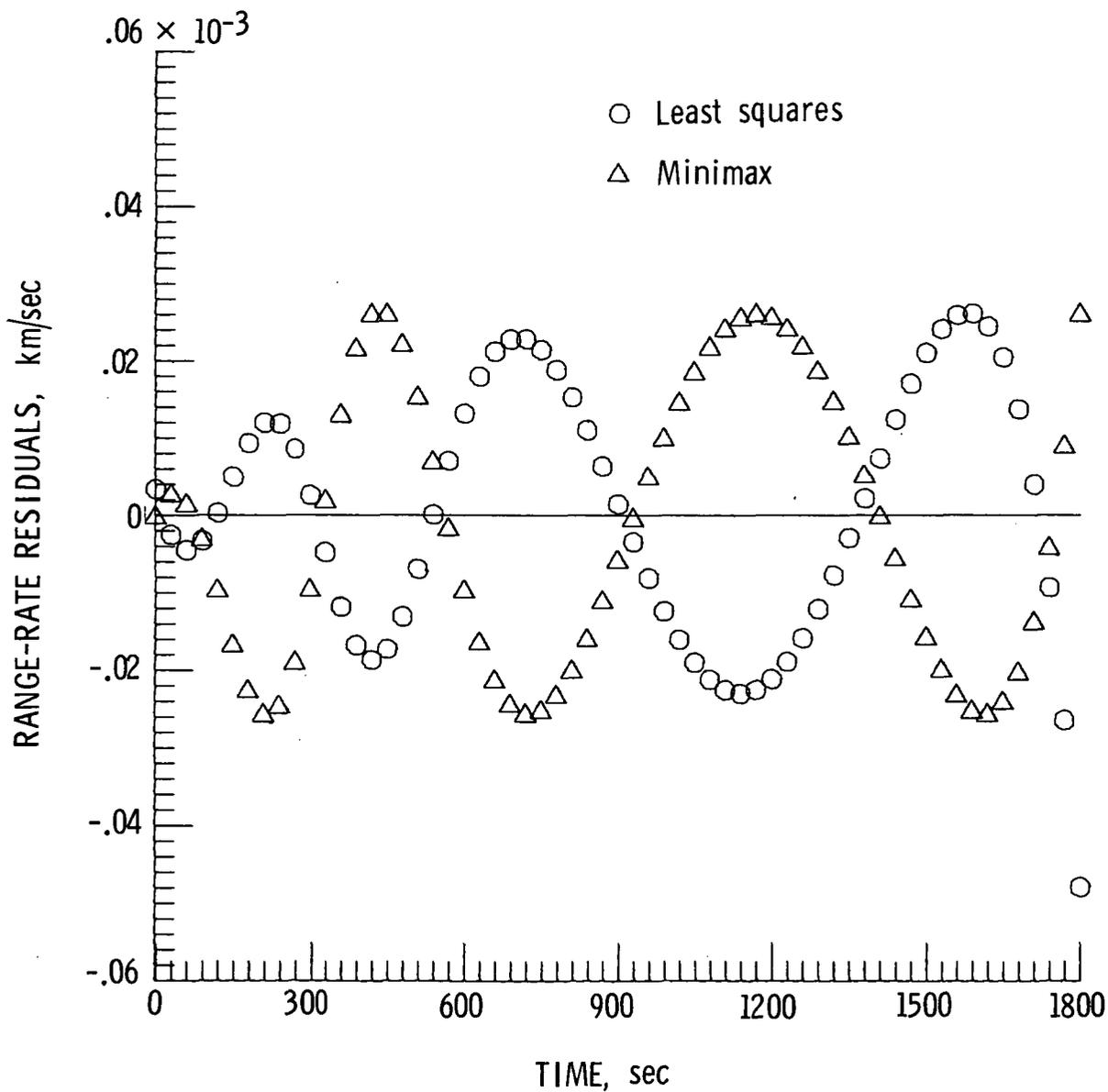


Figure 1.- Range-rate errors for 61-point two-body approximation.



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