Final Report

and

Semiannual Progress Report No. 18

A STUDY OF SELECTED RADIATION AND
PROPAGATION PROBLEMS RELATED TO ANTENNAS
AND PROBES IN MAGNETO-IONIC MEDIA

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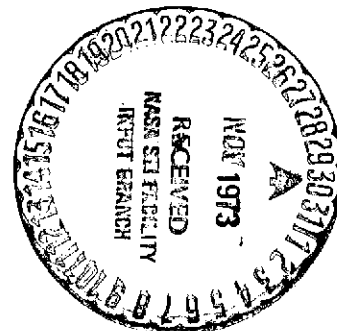University of Illinois
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I.   Summary of Research Under NASA Grant NGR14-005-009

I.1.   Brief Review

The research on the NASA Grant grew out of a rocket experiment on the cross-modulation between two RF signals in the ionosphere, the so-called "Luxemburg effect." In the original experiment over ten years ago, a pulsed signal near the gyroresonance frequency was produced to heat the electrons in the E layer, and a sensing signal was sent through the heated region for detection of the heating effect. The local disturbance of the ionosphere was observed. However, it was found that an accurate quantitative interpretation of the experimental results required a thorough understanding of the behavior of antennas in anisotropic media. The problem of the antenna in magnetoplasma has been a subject of great interest in the past decade because of its intimate relevance to space communication. During the past few years our laboratory has contributed significantly in this area as shown in the list of publications in I.2. Some of the highlights of this research is now reviewed.

(1)   A theory for a short dipole antenna in a magnetoplasma was advanced[4*] and a formula for the antenna impedance was derived using the quasi-static approximation. This formula has been widely used and experimentally tested with some success both in the ionosphere and in the laboratory.[4,36]

(2)   The computation of Green's function in the magnetoplasma has been a problem because of its complexity. A significant simplification was achieved with a new expression which separates the singular algebraic terms from some proper integrals.[1] More recently this expression has been

rearranged so that it can be handled more easily by some numerical means (see (6) below).

(3) For arbitrary electric current sources, the far field can be expressed in terms of the geometrical properties of the medium dispersion surface.[5,6] These results were later extended to compressible magnetoplasma[15] and to sources of magnetic currents, mechanical forces, and actual injection of charged particles. The possible significance of these sources in the case of compressible plasma was demonstrated,[17,25] using a variational method.

(4) Rigorous and numerical solutions to the current distribution on an infinitely long cylindrical antenna parallel to the magnetizing field were obtained for various cases.[23] These solutions differ from others in that the boundary condition on the conducting cylinder was satisfied.

(5) Dipole impedance in a plasma-filled circular guide has been investigated both numerically and experimentally with good agreement.[24] It was shown that the interaction between the dipole and the plasma could lead to a useful diagnostic technique for the plasma.

(6) In the past the investigations were made largely on the assumptions that the medium was idealized; the sheath was neglected or represented by a dielectric layer, and the current distribution was known (except (4) where the antenna was assumed to be infinitely long). In the past year we have addressed ourselves to the problem of current distribution.

The method relies heavily on the computer. It would have been unthinkable a few years ago, but with the present availability of larger and faster computers we have found this approach feasible with a reasonable computing effort.

An integral equation was solved numerically for the antenna current distribution in some cases. Because of the extreme complexity of the Green's function, the numerical procedure is very involved. The program has been successfully tested for simple known cases.

(7) In the past year, an experimental program was also initiated for the study of the nonlinear effect of the plasma on the antenna. It was found that even for input power on the order of milliwatts there was significant nonlinear effect on the antenna impedance. This should be kept in mind in evaluating the results of many existing experimental works. The nonlinear effect of the plasma depends on its relaxation time in comparison with the periods corresponding to the signal and modulation frequencies. At microwave frequencies, it is an extremely complex problem, and rigorously speaking the antenna impedance as well as the so-called ei-characteristics in such a system become rather meaningless. Therefore a more complete description is needed. Since at microwave frequencies it is usually the measurable reflection coefficient that is of interest, its dependence on the level of incident wave in the case of a simple nonlinear plasma has been evaluated.

More recently other nonlinear effects, such as harmonic generation and cross-modulation between two or more signals, have been demonstrated in the laboratory plasma. (See Part II of the report).

I. 2.   List of Publications and Scientific Reports under NASA Grant
        NSG 395 and NGR 14-005-009

1. R. Mittra and G. A. Deschamps, "Field Solution for a Dipole in an
    Anisotropic Medium," *Proceedings of the International Symposium on
    Electromagnetic Theory*, Pergamon Press, pp. 495-512, 1963.

2. G. A. Deschamps, "Dispersion Surfaces and Characteristic Plane Waves
    in a Gyrotropic Medium," *Proceedings of the Application Forum on
    Antenna Research*, ed. P. E. Mayes, University of Illinois, pp. 290-367,
    January 1964.

3. R. Mittra, "Antennas in Anisotropic Media," *Applications Forum on
    Antenna Research*, ed. P. E. Mayes, pp. 424-452, January 1964.

4. K. G. Balmain, "The Impedance of a Short Dipole Antenna in a Magneto-
    plasma," *IEEE Transactions*, Vol. AP-12, pp. 605-617, September 1964.

5. G. A. Deschamps and O. B. Kesler, "Radiation Field of an Arbitrary
    Antenna in a Magnetoplasma," *IEEE Transactions*, Vol. AP-12, pp. 783-
    785, November 1964.

6. G. A. Deschamps, "Angular Dependence of the Refractive Index in the
    Ionosphere," *Radio Science*, 69D, pp. 395-400, March 1965.

7. K. G. Balmain, "Impedance of a Short Dipole in a Compressible Plasma,"
    Radio Science, 69D, No. 4, pp. 559-566, April 1965.

8. R. Mittra and G. L. Duff, "A Systematic Study of the Radiation Patterns
    of a Dipole in a Magnetoplasma Based on a Classification of the
    Associated Dispersion Surfaces," *Radio Science*, 69D, No. 5, pp. 681-692,
    May 1965.

9. C. P. Bates and R. Mittra, "Plane-Wave Spectrum Representation of a
    Field in a Magnetoionic Medium Due to a Plane Distribution of Source
    Currents," *Electronics Letters*, Vol. 1, No. 3, p. 74, May 1965.

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    of Plasma Models in a Bounded Region," *Electronics Letters of the IEEE*,
    Vol. 1, No. 5, p. 128, July 1965.

11. S. W. Lee and Y. T. Lo, "Radiation Resistance of an Elliptical Loop
    Antenna with Constant Current in Compressible Plasma," *Electronics
    Letters of the IEEE*, Vol. 1, No. 5, pp. 132-133, July 1965.

12. G. L. Duff and R. Mittra, "Input Impedance of a Small Loop of Uniform
    Electric Current in an Anisotropic Cold Plasma," *Electronics Letters*,
    Vol. 1, No. 5, pp. 117-118, July 1965.

13. S. W. Lee, C. Liang and Y. T. Lo, "Further Remarks on Boundary Conditions
    of Plasma Models in a Bounded Region," *Electronics Letters of the IEEE*,
    Vol. 1, No. 7, September 1965.

14. S.W. Lee, Y.T. Lo and R. Mittra, "Finite and Infinite H-plane Difurcation Waveguide with Anisotropic Plasma Medium," *Proceedings of the International Symposia of Antenna and Propagation*, Washington, D. C., August 1965; also *Canadian Journal of Physics*, Vol. 43, pp. 2123-2135, December 1965.

15. G. A. Deschamps, "Rayonnement d'une Antenne dans un milieu anisostrope," *L'Onde Electrique 45*, pp. 1379-1385, December 1965.

16. S. W. Lee and Y. T. Lo, "Radiation in a Moving Anisotropic Medium," *Radio Science*, Vol. 1, (New Series), No. 3, pp. 313-324, March 1966.

17. J. W. Carlin and R. Mittra, "Terminal Admittance of a Thin Biconical Antenna in an Isotropic Compressible Plasma," *Electronics Letters*, Vol. 2, No. 6, pp. 199-201, June 1966.

18. S. W. Lee, C. Liang and Y. T. Lo, "Wave Propagation in Plasma with Very Strong Magnetic Field," *Radio Science*, Vol. 1 (New Series), No. 7, pp. 815-824, July 1966.

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20. G. A. Deschamps, "Impedance of Antennas in Ionized Media," *Electromagnetics and Antennas*, ed. P. E. Mayes, University of Illinois, pp. 498-512, January 1967.

21. S. W. Lee, "Propagation in Ionosphere-earth Waveguide," *Appl. Science Research*, Vol. 17, No. 2, pp. 115-132, 1967.

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23. S. W. Lee and Y. T. Lo, "Current Distribution and Input Admittance of a Cylindrical Antenna in Anisotropic Plasma," presented at Spring URSI Meeting, Washington, D. C., April 1966; also, *IEEE Transactions*, Vol. AP-15, No.2, pp. 244-252, March 1967.

24. S. W. Lee and Y. T. Lo, "On the Coupling of Modal Waves in a Plasma-filled Parallel-plate Waveguide," *Radio Science*, Vol. 2, No. 4, pp. 401-406, April 1967.

25. J. W. Carlin and R. Mittra, "Acoustic Waves and Their Effects on Antenna Impedance," *IEEE Intl. Antennas and Propagation Symposium Digest*, pp. 102-105, December 1966, also, *Canadian Journal of Physics*, Vol. 45, pp. 1251-1269, 1967.

26. S. W. Lee and R. Mittra, "Group Velocity of a Uniaxial Plasma," *IEEE Transactions*, Vol. AP-15, No. 4, pp. 588-589, July 1967.

27. G. A. Deschamps and O. B. Kesler, "Radiation of an Antenna in a Compressible Magnetoplasma," *Radio Science*, August 1967.

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30. R. Mittra and S. W. Lee, "Discontinuity in an Anisotropic Waveguide," *Journal of Applied Physics*, Vol. 38, pp. 178-186, 1967.

31. S. W. Lee and R. Mittra, "Edge Condition and 'Intrinsic Loss' in Uniaxial Plasma," *Canadian Journal of Physics*, Vol. 46, pp. 111-120, 1968.

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33. J. Carlin, "Variational Formula for Antenna Impedance in a Warm Plasma," *Electronics Letters*, Vol. 3, No. 7, 1967.

34. D. K. Wainco and R. Mittra, "Short Dipole in a Uniaxial Medium," *Applied Science Research*, Vol. 18, 1968.

35. I. Akkaya, "A Comparative Study of Several Models of the Magnetoplasma," *Radio Science*, Vol. 4, No. 1, pp. 83-90, January 1969.

36. D. Snyder and R. Mittra, "Experimental Study of the Impedance of a Short Dipole in a Plasma for Parallel and Perpendicular Orientation with Respect to the D.C. Magnetic Field," Sci. Tech. Rept. No. 11, Antenna Laboratory, University of Illinois, Urbana, Illinois, December 1968. This work was also presented at IEEE PGAP Symposium, Boston, Massachusetts, December 1968.

37. S. W. Lee, "Cylindrical Antenna in Uniaxial Resonant Plasma," *Radio Science*, Vol. 4, No. 2, pp. 179-189, February 1969.

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39. G. L. Duff and R. Mittra, "Loop Impedance in a Magnetoplasma: Theory and Experiment," *Radio Science*, Vol. 5, pp. 81-94, January 1970.

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43. R. Mittra, D. H. Schaubert and M. Mostafavi, "Some Methods for Determining the Profile Functions of Inhomogeneous Media," to be published in the *Proceedings of the Workshop on the Mathematics of Profile Inversion*, 1971.

44. G. A. Deschamps, K. G. Balmain and S. W. Lee, "Antennas in Ionized Media," to be published in *Advances in Electronics and Electron Physics*.

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45. K. G. Balmain, "The Impedance of a Short Dipole Antenna in a Magneto-plasma," Scientific Report No. 1, University of Illinois, May 1963.

46. R. Mittra and G. Duff, "A Classification of the Dispersion Surfaces in a Magneto-Ionic Medium and a Study of the Associated Radiation Patterns," Scientific Report No. 2, University of Illinois, October 1964.

47. R. Mittra, "Solution of a Ferrite Boundary Value Problem and Re-solution of Lewin's Paradox," Scientific Report No. 3, Antenna Lab Report No. 65-7, University of Illinois, March 1965.

48. S. W. Lee, Y. T. Lo and R. Mittra, "Finite and Infinite H-Plane Bi-furcation of Waveguide with Anisotropic Plasma Medium," Scientific Report No. 4, Antenna Lab Report No. 65-8, University of Illinois, March 1965.

49. O. B. Kesler, "Propagation of EM Waves in Linear, Passive, Generalized Media," Scientific Report No. 5, Antenna Lab. Report No. 65-9, University of Illinois, October 1965.

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53. J. W. Carlin and R. Mittra, "Antenna Impedance in a Warm Plasma," Scientific Report No. 9, Antenna Lab. Report. No. 66-18, University of Illinois, December 1966.

54. I. Akkaya, "Waveguides Filled with Magnetoplasmas of Various Types," Scientific Report No. 10, Antenna Lab. Report No. 67-4, University of Illinois, May 1967.

55. D. E. Snyder and R. Mittra, "Experimental Study of the Impedance of a Short Dipole in a Plasma for Parallel and Perpendicular Orientation with Respect to the D.C. Magnetic Field," Scientific Report No. 11, Antenna Lab. Report No. 68-4, University of Illinois, December 1968.

56. R. J. Kostelnicek and R. Mittra, "Radiation from an Open-Ended Waveguide into an Inhomogeneously Filled Space," Scientific Report No. 12, Antenna Lab. Report No. 69-10, University of Illinois, July 1969.

57. S. W. Lee and M. J. Al-Hakkak, "Impedance of Cylindrical Antennas in Plasma -- A Review," Scientific Report No. 13, Antenna Lab. Report No. 70-3, University of Illinois, January 1970.

58. M. Mostafavi and R. Mittra, "Remote Probing of Inhomogeneous Media Using Parameter Optimization Techniques," Scientific Report No. 14, Antenna Lab. Report No. 70-6, University of Illinois, April 1970.

59. D. Schaubert and R. Mittra, "Remote Probing Methods for the Determination of the Profile of Inhomogeneous Media," Scientific Report No. 15, Antenna Lab. Report No. 70-7, University of Illinois, May 1970.

60. H. Sievering and R. Mittra, "The Effect of Coherence and Multiple Scattering on Laser Radar Air Pollution Measurements," Scientific Report No. 16, Antenna Lab. Report No. 71-5, University of Illinois, June 1971.

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62. S. W. Lee and G. A. Deschamps, "Singular Perturbation Methods and the Warm Plasma Model," Scientific Report No. 18, Antenna Lab Report No. 71-10, University of Illinois, August 1971.

63. C. A. Klein, P. W. Klock and G. A. Deschamps, "Current Distribution on a Cylindrical Antenna with Parallel Orientation in a Lossy Magnetoplasma," Scientific Report No. 19, Antenna Lab. Report No. 72-3, July 1972.

64. M. J. Johnson and R. Mittra, "A Matrix Equation Solution by an Optimization Technique," Scientific Report No. 20, Antenna Lab. Report No. 72-11, University of Illinois, December 1972.

II.   SEMIANNUAL PROGRESS REPORT

II.1.   Summary

The research during the last period consisted of computations toward the solution of the problem of the current distribution on a cylindrical antenna in a magneto plasma. The case of an antenna parallel to the applied magnetic field was treated (reference 63 of Part I). A systematic method of asymptotic expansion was found which simplifies the solution in the general case by giving the field of a dipole even at relatively short range (II.2). Some useful properties of the dispersion surfaces in a lossy medium have also been found (II.2). The laboratory experiment was directed toward evaluating nonlinear effects, such as those due to power level, bias voltage and electron heating (II.3). The problem of reflection and transmission of waves in an electron heated plasma was treated theoretically (II.4). [This has been accepted for publication]. The profile inversion problem has been pursued. Some results are very encouraging. The general question of stability of the solution remains unsolved.

II.2. Cylindrical Dipole in a Magnetoplasma (Numerical Analysis)

1. Introduction

The input impedance to a dipole in a magnetoplasma has been computed using the method of moments.[1] This paper discusses asymptotic approximations to integrals representation of fields. By applying these approximations the computation of fields from a dipole source in a plasma can be simplified and still give good results at "reasonable" distance from the source. In this manner, the asymptotic solution can be applied to the problem of obtaining the input impedance of a dipole in a magnetoplasma. Such an application could increase the accuracy of the computation.

This paper discusses methods for obtaining the first few terms of an asymptotic expansion. Both one and two dimensional integrals are considered of the type

$$\int_c (\text{weighting function}) \ (\text{plane wave}) \ d(\text{parameter}) \tag{1}$$

The field solution in the free space case is examined as a preliminary step to the eventual application to a plasma. Also some geometrical considerations about the characteristic fields in an anisotropic region are presented.

2. Saddle Point Integration

The saddle-point method of integration,[2] also known as the method of steepest descent, is an asymptotic integration technique which may have some usefulness in integrating the equation

$$I = \int f(x) \ e^{i\nu\phi(x)} \ dx \tag{2}$$

where ν is a large number and f, ϕ are functions of x that vary "slowly" with x – x being a point in a space of arbitrary dimension. The particular application of (2) used here is the consideration of the field described by a spectrum of plane waves. Then the variable x is the vector k, the phase $\nu\phi(x)$ becomes $k \cdot r$ (where r is the observation point), and F(x) becomes a weighing function F(k). The integral considered thereby becomes

$$\int_c F(k)e^{ik \cdot r} dk \quad .$$ (3)

The path of integration c is shifted to a steepest descent path that is one along which $Im(k \cdot r)$ varies at a maximum rate of change. This can be done without changing the value of the integral only if there are no singularities between the true path. Otherwise the effect of the poles or branch points on the new integration path must be considered.

It is easy to show[3] that forcing $Im(k \cdot r)$ to vary at a maximum rate of change implies that $Re(k \cdot r)$ is a constant.

A complex analytic function cannot have a maximum or minimum, so the stationary phase point is actually a saddle point. There is both a steepest descent path, along which $Re(k \cdot r)$ has its maximum at the saddle point, and also a steepest ascent path along which $Re(k \cdot r)$ has its minimum at the saddle point.

If one can change the integration path of (3) to the steepest descent path and if F(k) varies "slowly" one now has a means for approximating the solution to (3). Specifically, if one integrates along the steepest descent path then the main contribution to the integration comes from the region of the saddle point. In this manner expanding the integral in terms of a Taylor's series about the saddle point for "well-behaved" functions should produce reasonable approximations to the solution of (3).

At a saddle point $d(kr) = 0$ or for a given r, $dk \cdot r \doteq 0$. This

means that r is perpendicular to the dispersion surface. At point k,

in trying to find an approximation to the integral solutions to a field

at an observation point r one must find the saddle points. This intro-

duces a geometric problem necessary for the solution of the integral (3)

of finding all those k's for which $dk \cdot r = 0$. This problem will be

discussed later.

Consider now a one-dimensional problem with an integral of the form

$$I = \int_{c_{sdp}} F(z) e^{-af(z)} \, dz \tag{4}$$

Here c_{sdp} is the steepest descent path. It is assumed that $F(z)$ varies

"slowly" and a is "reasonably" large. Also one assumes that $F(z)$ is

well-behaved at the saddle-point, having no singularities there. The

approximation to (4) about one saddle point is now examined. Two methods

are mentioned here.

In approximation (4) define $s^2 = f - f(z_o)$ where z_o is the saddle

point. Making a change of variables from z to s results in

$$I = e^{-a \, f(z_o)} \int_{c_{sdp}} \phi(s) \, e^{-as^2} \, ds$$

$$\phi(s) = \frac{2F(z)}{f'(z)} \tag{5}$$

To evaluate (5), $\phi(s)$ is expanded about the point $s = 0$ (or $z = z_o$).

Let the nth derivative of $\phi(s)$ and $f(s)$ evaluated at the saddle point

be denoted by ϕ_n and f_n respectively. The Taylor's expansion for $\phi(s)$

becomes

$$\phi(s) = \phi_o + s\phi_1 + \frac{s^2}{2} \phi_2 + \ldots \tag{6}$$

and $\phi_1 = 0$ at the saddle point. In order to have the Im $(-as^2)$ a constant, s^2 must be real. So, appropriate approximations to the steepest descent path valid about the saddle point for $\phi(s)$ varying "slowly" are limits of integration $-\infty$ to $+\infty$. Putting the expansion (6) into (5) yields an approximate solution to (5) of

$$I = \sqrt{\frac{\pi}{a}} \; e^{-af_o} \; [\phi_o + \frac{1}{4a} \phi_2 + \frac{3}{8a^2} \phi_4 \; ...] \tag{7}$$

This computation has been performed to the first two significant terms using powers of f_n and F_n with the following results

$$I = e^{-af_o} \left(\frac{2\pi}{af_2}\right)^{1/2} \{F - \frac{2}{af_2} \left(\frac{f_4}{f_2} - \frac{5}{3} \frac{f_3^3}{f_2^3}\right) + \frac{1}{4} \left(F_1 \frac{f_3}{f_2} - F_2\right)\}$$

$$+ e^{-af_o} \left(-\left(\frac{2}{af_2}\right) \{F \frac{f_6}{4f_3} \frac{1}{48} - \left(\frac{f_4}{f_2}\right)^2 \frac{35}{1536}\right.$$

$$- \frac{f_3 f_5}{f_2^2} \frac{7}{192} + \frac{f_3 f_4}{f_2^2} \frac{7}{192} + \frac{f_3^2 f_4}{f_2^3} \frac{35}{256}$$

$$+ F_1 \left[\frac{f_5}{f_2} \frac{1}{32} - \frac{f_3 f_4}{f_2^2} \frac{35}{192} + \left(\frac{f_3}{f_2}\right)^3 \frac{35}{192}\right] +$$

$$+ \frac{F_2}{2} \left[\frac{f_4}{f_2} \frac{5}{32} - \frac{385}{(64)(72)} \left(\frac{f_3}{f_2}\right)^4 - \left(\frac{f_3}{f_2}\right)^2 \frac{35}{96}\right]$$

$$- F_3 \left[\frac{f_3}{f_2} \frac{5}{48} + F_4 \frac{1}{32}\right]\} \tag{8}$$

An equivalent expansion for the one-dimensional problem has been found by a slightly different method which is more readily extended to the nth dimensional case. In equation (4), $F(z)$ and $f(z)$ are expanded

in a Taylor's series about a saddle point z. Let $\zeta = z - z_o$. Then

$$I = \int_{c_{sdp}} (F_o + F_1 \zeta + \frac{F_2}{2} \zeta^2 + \ldots)$$

$$\exp(-a\{f_o + f_1 \zeta + \frac{f_2}{2} \zeta^2 + \ldots\}) \, d\zeta \qquad (9)$$

The exponential term in (9) is now expanded.

$$\exp(-a\ f(z)) = \exp(-af)\ \exp(-\frac{af_2}{2} \zeta^2)$$

$$\{1 - a(\frac{f_3}{3!} \zeta^3 + \frac{f_4}{4!} \zeta^4 + \frac{f_5 - \zeta_5}{5!} \ldots)$$

$$+ \frac{a^2}{2!} (\frac{f_3}{3!} \zeta^3 + \frac{f_4}{4!} \zeta^4 + \frac{f_5}{5!} \zeta^5 \ldots)^2 -$$

$$\frac{a^3}{3!} (\ldots)^3 \ldots)\} \qquad (10)$$

substituting (10) into (9) reduces the integral equation to an infinite sum of terms of the form

$$e^{-af_o} \int_{c_{sdp}} C_n \zeta^n e^{-\frac{af_2}{2} \zeta^2} \, d\zeta \qquad (11)$$

where C_n is a constant. The limits of integration are taken from $-\infty$ to ∞ as an approximation around the saddle point.

With this simplification, it is known that

$$\int_{-\infty}^{\infty} \zeta^n e^{-\rho^2 \zeta^2} \, d\zeta = \begin{cases} \dfrac{\Gamma[(n+1)/2]}{\rho^{(n+1)/2}} & ; \quad n \text{ even} \\[2ex] 0 & ; \quad n \text{ odd} \end{cases} \qquad (12)$$

When a sum of these factors is examined in terms of $\frac{1}{a}^{1/2}$ and $\frac{1}{a}$, a series equivalent to (5) has been obtained. While this method is more

cumbersome because of the number of terms that must be combined, it is a straight forward technique that may easily be applied to the two-dimensional problem.

3. Two-Dimensional Problem

In a two-dimensional problem, one considers integrals of the type

$$I_2 = \iint\limits_{c_{sdp}} F(u,v)\, e^{-\rho f(u,v)}\, du\, dv \tag{13}$$

Again it is assumed that $F(x,y)$ varies "slowly" and is well-behaved at the saddle point.

To simplify the manipulation of (13) a convenient notation is first introduced. Let $-J$ be an ordered pair of indices so that $J = j_1 j_2$. Also, let y be a vector in C^2 so that $y_1 = u$ and $y_2 = v$. The following notation will now be used.

$$y^J = y_1^{j_1}\, y_2^{j_2}$$

$$\partial^{-J} f\Big|_{y = y_o} = \partial_{y_1}^{j_1}\, \partial_{y_2}^{j_2}\, f\Big|_{y = y_o}$$

$$j! = j_1!\, j_2!$$

$$|J| = j_1 + j_2$$

$$f_L = \sum_{|J| = L} \frac{\partial^J F}{J!}\, (y - y_o)^J \tag{14}$$

With this abbreviation, (18) takes on the form

$$I_2 = \iint\limits_{c_{sdp}} F(y)\, e^{-\rho f(y)}\, dy \tag{15}$$

To obtain an asymptotic expansion, $F(y)$ and $f(y)$ are rewritten in terms of a Taylor's expansion about the saddle point y_o.

$$F(y) = \sum_{K=1}^{n} F$$

$$f(y) = \sum_{K=1}^{n} f \tag{16}$$

The exponential term is expanded so that

$$e^{-\rho f(y)} = e^{-\rho f_o} e^{-\rho/z} f_2 \{1 - \rho[f_3 + f_4 + \ldots] +$$

$$\frac{\rho^2}{2}[f_3 + f_4 + \ldots]^2 + \ldots + \frac{(-\rho)^n}{n!}[f_3 + f_4 + \ldots]^n + \ldots] \tag{17}$$

If one lets $u - u_o = \xi$ and $v - v_o = \eta$, this manipulation yields a solution to (15) in terms of a summation of integrals of the type

$$I_{nm} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi^m \eta^n e^{-\rho/2 [f_{uu}\xi^2 + 2f_{uv}\xi\eta + f_{vv}\eta^2]} d\xi d\eta \tag{18}$$

where the limits $-\infty$ to $+\infty$ have been taken as an approximation to c_{sdp}. A solution to integrals of the type in (18) has been found to be

$$I_{nm} = \frac{C_{n+m,m}}{\rho^{(n+m)/2 + 1}}$$

where

$$C_{n+m,m} = \begin{cases} 2^{(\frac{m+n}{2} + 1)} f_{uu}^{(\frac{m-n}{2})} \sum_{k=0}^{n} R(n,k) & ; \text{ for } n+m \text{ even} \\ 0 & ; \text{ for } n+m \text{ odd} \end{cases}$$

and

$$R(n,k) = \begin{cases} \binom{n}{k} \dfrac{(-f_{uv})^k}{(f_{uu}f_{vv} - f_{uv}^2)^{(n+k)/2}} \Gamma(\frac{m+k}{2}+1) \Gamma(\frac{m-k}{2}+1) \\[2mm] \qquad ; \text{ for } m + k \text{ and } n - k \text{ even} \\[2mm] 0 \qquad ; \text{ for } m + k \text{ or } n - k \text{ odd} \end{cases} \tag{19}$$

To obtain an asymptotic series, the expansion of integral (13) is combined in terms of increasing powers of $1/\rho$.

If there are polynomials of ξ^{n_1} and η^{m_1} associated with the Taylor's expansion of $F(y)$ and polynomials of ξ^{n_2} and η^{m_2} associated with the expansion of $\rho^{f(y)}$, from the form of (19) it can be seen that these polynomials have a contribution of $(\frac{1}{\rho})^{(n_1 + n_2 + m_1 + m_2)/2 + 1}$. The subscript $J = n_1 + m_1$ is now associated with the Jth term of the Taylor's expansion, and a value $\Sigma\ell = n_z + m_z$ is associated by Table 1 with products of f_L in the expansion of $e^{f(y)}$.

There is also a contribution of $(\rho)^k$ directly from the expansion of $e^{-\rho f(y)}$ in (17). A rule can now be formulated for generating the nth term of the asymptotic expansion. Specifically, one relates the nth term of the asymptotic expansion through terms of f_L's and F_K's for which

$$(\frac{J + \Sigma\ell}{z} + 1 - k) = n \tag{20}$$

Those terms associated with $n = 1, 2, 3$ are summarized in Table 2. Furthermore, the contribution by these combinations can be written

$$\frac{e^{-\rho f_o}}{\rho^n} \frac{(-1)^k}{k!} F_J f_{L1} \cdots f_{Ln} \, C_{J + \Sigma\ell, \, m_1 + m_2} \tag{21}$$

where $\Sigma\ell = L_1 + \ldots + L_n$, $J + \Sigma\ell = m_1 + m_2 + n_1 + n_2$ and by $C_{J + \Sigma\ell, \, m_1 + m_2}$ is meant, the integration constant in (19) associated with the polynominals

TABLE 1. ORGANIZATION OF TERMS FROM $e^{-\rho f(y)}$

I · J ... N Corresponds to $f_I\ f_J\ ...\ f_N$

0	1	2	3	4	5	6	7	8	9	...	Σℓ / k
0											0
			3	4	5	6	7	8	9		1
						33	34	35	36		2
							43	44	45		
								53	54		
									55		
									333		3

TABLE 2. VALUES OF J, Σℓ, k FOR
FIRST 3 TERMS OF ASYMPTOTIC EXPANSION

J	Σℓ	k	n		J	Σℓ	k	n
0	0	0	1		1	5	1	3
2	0	0	2		0	6	1	3
1	3	1	2		2	6	2	3
0	4	1	2		1	7	2	3
0	6	2	2		0	8	2	3
4	0	0	3		1	9	3	3
3	3	1	3		0	10	3	3
2	4	1	3		0	12	4	3

$\xi^{n_1 + n_2} \eta^{m_1 + m_2}$. This value of C, though being different for the various combinations involved in F_J, f_L, will be abbreviated to $C_{J + \Sigma\ell}$. The first three terms of the asymptotic expansion then becomes:

$$\frac{1}{\rho} e^{-\rho f_o} F_o C_o$$

$$\frac{1}{\rho^2} e^{-\rho f_o} \{F_2 \, C_2 - F_1 f_3 C_4 - F_o \, f_4 C_4$$

$$\frac{1}{2!} F_o f_3 f_3 C_6\}$$

$$\frac{1}{\rho^3} e^{-\rho f_o} \{F_4 C_4 - F_3 f_3 C_6 - F_2 f_4 C_6$$

$$-F_1 f_5 C_6 - F_o f_6 C_6 + \frac{1}{2!} F_2 f_3 f_3 C_8$$

$$+ \frac{1}{2!} \, 2 \, F_1 f_4 f_3 C_8 + \frac{1}{2!} \, 2 \, F_o f_3 f_5 C_8$$

$$+ \frac{1}{2!} F_o f_4 f_4 C_8 - \frac{1}{3!} F_1 f_3 f_3 f_3 C_{10}$$

$$- \frac{1}{3!} \, 3 \, F_o f_3 f_3 f_4 C_{10} + \frac{1}{4!} F_o f_3 f_3 f_3 f_3 C_{12}\} \tag{22}$$

This form of the asymptotic expansion for the first three terms has been programmed on a computer. Considerable effort must be spent obtaining the many partial derivatives needed. In fact, since this is a significant difficulty in using higher order terms, some thought should be given to developing a computer program to perform this algebra.

4. Applications to the Free Space Problem

The asymptotic techniques developed in the proceeding sections were applied to a dipole current source in free space as a test to the approximation.

From Maxwell's equations

$$\text{curl } E(r) = -j\omega\mu H(r)$$
$$\text{curl } H(r) = j\omega\epsilon E + J(r)$$

(23)

One obtains

$$\text{curl curl } E(\sim) = \omega^2\mu\epsilon E - j\omega\mu J(r)$$

E is now considered to be a plane wave and $J(r)$ a source in the z direction only. In the transform domain, then

$$[k^2 - (\xi^2 + \eta^2 + \zeta^2)] \tilde{E} = \frac{1}{j\omega\epsilon} \tilde{J}(k^2 - \zeta^2)$$

where

$$\tilde{E}(k)' = \int\int\int_{-\infty}^{\infty} E e^{+j(\xi x + \eta y + \zeta z)} dxdydz$$

and $\tilde{J}$ is the Fourier transform of $J(z)$. It follows then that

$$E = \frac{1}{8\pi^3} \frac{1}{j\omega\epsilon} \int\int\int_{-\infty}^{\infty} \frac{(k^2 - \zeta^2)\tilde{J} e^{-jk \cdot r}}{k^2 - (\xi^2 + \eta^2 + \zeta^2)} d^3k$$

(24)

A change of variables is now performed

$$\zeta = \Gamma \cos \beta \qquad\qquad x = r \cos \theta$$
$$\eta = \Gamma \sin \beta \sin \alpha \qquad\qquad y = r \sin \theta \sin \phi$$
$$\xi = \Gamma \sin \beta \cos \alpha \qquad\qquad z = r \sin \theta \cos \phi$$

With this change

$$E = \frac{1}{8\pi} \frac{1}{j\omega\epsilon} \int_{-\infty}^{\infty} \int_{C(\alpha)} \int_{C(\beta)} \tilde{J} \frac{k^2 - \Gamma^2 \cos^2\beta}{k^2 - \Gamma^2} \Gamma^2 \sin \beta \, e^{-j\Gamma r \cos \gamma} d\Gamma \, d\alpha \, d\beta \quad (25)$$

where $\cos \gamma = \cos \theta \cos \beta + \sin \theta \sin \beta (\cos(\alpha - \phi))$ and $C(\alpha)$, $C(\beta)$ are paths in the complex α and β plane.

The Γ integration is done by a method of contour integration resulting is

$$E(z) = \frac{k^3}{\omega\epsilon \pi^2} \int_{C(\alpha)} \int_{C(\beta)} \tilde{J} \sin^3 \beta \, e^{-jkr \cos \gamma} d\alpha d\beta$$

(26)

An asymptotic expansion of the integral in equation (26) is now used. We have, using previous notation,

$$F(\alpha, \beta) = \tilde{J} \sin^3 \beta$$

$$f(\alpha, \beta) = \cos \gamma \tag{27}$$

$$\text{and} \quad \rho = -jkr$$

And the saddle point is at $\alpha = \phi$, $\beta = \theta$.

Let us consider a sinusoidal current distribution

$J = (x,y,z) = I_m \sin (k(H - |z|) \zeta_{(x)} \zeta_{(y)}$ for which the field is already known. This is the form for the current distribution on a thin dipole of length 2H. The solution[4] for E_z is

$$E(z) = -j30 I_m \left(\frac{e^{-j\beta k_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} - \frac{2\cos(\beta H)}{2} e^{-j\beta r} \right) \tag{28}$$

The geometry of this dipole is shown in Figure 1.

For this current an asymptotic approximation to (26) can be found, and the results compared with (28). Moreover the exact solution shown (28) can be expanded in terms of a series of $(\frac{1}{r})$ and compared with the asymptotic expansion, which is also written in terms of a $\frac{1}{r}$ series.

It should be noted that the calculation of the first three terms of the asymptotic expansion is facilitated by the condition $f(\phi, \theta) = 0$.

The approximate and general solutions were generated for $\frac{2H}{\lambda} = \frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{10}$ and the radius of the dipole was taken much smaller than a wavelength. The first three terms of the asymptotic approximation agree exactly with the first three terms of the expansion of the exact solution (28) in terms of a series of $(\frac{1}{r})$. At a distance of .4 wavelengths, the asymptotic solution was within 1% of the exact solution, and at .2λ the asymptotic solution was within 10%.

Within .2λ the value for the approximate solution quickly diverges from the exact solution. Thus, it appears, and this was expected, that for very small values of z, E(z) cannot be evaluted using an asymptotic approximation. However, the asymptotic approximation using the first three terms is quite good for intermediate and far distances.

5. Dispersion Surfaces

As mentioned earlier, to find the approximation to the field at an observation point r one expands about the saddle points, which correspond to those points for which dk · r = 0. In physical terms, one is finding the k vector of plane waves which contribute mostly to the far-field in a particular direction. These correspond to those points k for which the perpendicular to the dispersion relation is in the r direction.

The geometric problem associated with saddle point integration can be stated, therefore, as given an observation direction r, generate the k vectors for dk · r = 0 and the dispersion relation being satisfied.

The dispersion relation can be described[3] using Maxwell's equations for a plane wave

$$\begin{Bmatrix} E \\ H \end{Bmatrix} = \begin{Bmatrix} E_o \\ H_o \end{Bmatrix} e^{-jkr} \tag{29}$$

So Maxwell's equation becomes

$$kxE = \omega B$$
$$kxH = -\omega D \tag{30}$$

Letting $K = \omega^2 \mu_o \varepsilon_o \tilde{\tilde{\varepsilon}} = K$ one has

$$kxkxE = -KE \tag{31}$$

or

$$(k\, k^T - k^2 + K)E = 0 \tag{32}$$

For non-trival solutions, therefore,

$$\det (kk^T - k^2 + K) = 0 \tag{33}$$

which describes the dispersion relation for the media. This dispersion relation behaves differently for different types of media.

For an isotropic space (33) implies that

$$k^2 = \omega^2 \mu_o \varepsilon \tag{34}$$

From which it follows that

$$2k^T dk = 0 \tag{35}$$

One conclusion of (35) is that in isotropic regions k and the observation angle are in the same direction at the saddle point.

Consider now the uniaxial plasma. This type of region corresponds to a plasma with a large external magnetic field with an $\tilde{\tilde{\varepsilon}}$ matrix of the type

$$\tilde{\tilde{\varepsilon}} = \begin{bmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{bmatrix} \tag{36}$$

One solution to the dispersion relation is that

$$k^2 = \omega^2 \mu_o \varepsilon_\perp \tag{37}$$

And for this value of k

$$2k^T dk = 0 \tag{38}$$

so that k again is in the same direction as the observation angle at the saddle point. However, other solutions to the dispersion relation are available.

Letting $P = (k^2 - k)^{-1} kk^T$ (33) becomes

$$\det (I - P) E = 0 \qquad (39)$$

Because the image of the operator P is always d, P has rank one. Thus,

$$\det (I - P) = 1 - \text{trace } P = 0 \qquad (40)$$

Or

$$\text{trace } P = 1 \qquad (41)$$
$$k^T(k^2 - K)^{-1} k = 1$$

So, recalling that $dk \cdot r = 0$ at the saddle point, (41) can be utilized to obtain the r direction by taking the differential with respect to k. Sepcifically,

$$0 = dk^T(k^2 - K)^{-1} k + k^T(k^2 - K)^{-1} dk \qquad (42)$$
$$-2 \, k^T(k^2 - K)^{-1} 2k \, k^T \, (k^2 - K)^{-1}$$

$$0 = dk^T[(k^2 - K)^{-1} k + (k^2 - K^T)^{-1} k - 2 \, k^T(k^2 - K)^{-1} (k^2 - K)^{-1} k \, k] \qquad (43)$$

Thus, the observation point is in the direction of a vector k where

$$R = (k^2 - K)^{-1} k + (k^2 - K^T)^{-1} k - 2k^T(k^2 - K)^{-1}(k^2 - K)^{-1} k \, k \qquad (44)$$

Equation (44) holds for the general lossy anisotropic plasma also.

It is necessary that an inverse relationship be found. That is, given a radial direction r, a direction of k must be found, k may then be scaled to fit the dispersion relation (33).

6. Uniaxial Case

The problem of determining the direction of k at the saddle point has been solved exactly for the case $K = K^T$. Equation (44) reduces to

$$\frac{1}{2} R = (k^2 - K)^{-1} k - k^T (k^2 - K)^{-1T} (k^2 - K)^{-1} k \, k \qquad (45)$$

Some physical interpretation can be given to (45). Recall that $(k\, k^T - k^2 + K)\, E = 0$ so that for $(k^2 - K)$ nonsingular

$$E = (k^2 - K)^{-1}\, k\, k^T E \tag{46}$$

or

$$E = (k \cdot E)\, (k^2 - K)^{-1}\, k \tag{47}$$

Thus $\frac{1}{2}R = CE - 2C^2(E \cdot E)\, k$ for $C = \frac{1}{k \cdot E}$, and $\frac{1}{2}R = CE \times (k \times CE)$

$$= C^2 \omega\mu\ (ExH) \tag{48}$$

Equation (48) has an interesting physical significance. It implies that the normal to the dispersion surface is not in the direction of the Poynting vector, the direction of energy flow.

Now some dual relations can be constructed to solve the geometric problem. Let

$$S = \frac{\omega}{(R \cdot k)}\, R \tag{49}$$

Then

$$S \times B = \frac{1}{\omega}\, SX\ (k \times E)$$

$$S \times B = \frac{1}{\omega}\, [(S \cdot E)\, k - (S \cdot k)\, E] \tag{50}$$

$$S \times B = -E$$

Similarly

$$S \times D = H \tag{51}$$

A dispersion relation for S may now be derived.

$$SX\ (S \times D) = - \frac{1}{\mu\epsilon_o}\, \tilde{\tilde{\epsilon}}^{-1}\, D \tag{52}$$

Letting $V = \frac{1}{\mu\epsilon_o}\, \tilde{\tilde{\epsilon}}^{-1}$, $V = V^T$ for the uniaxial case. So,

$$(SS^T - S^2 + V)\quad D = 0\ . \tag{53}$$

This implies that for non-trivial solution

$$\det (SS^T - S^2 + V) = 0 \tag{54}$$

so that

$$\det (1 - (S^2 - V)^{-1} SS^T) = 0 \tag{55}$$

and

$$S^T (S^2 - V)^{-1} S = 1 \tag{56}$$

Taking the differential of (56) with respect to S results in

$$[dS^T((S^2 - V)^{-1} S - S^T(S^2 - V)^{-1} (S^2 - V)^{-1} S S)] = 0 \tag{57}$$

The results of (57) must now be examined. From (53) we have that

$$D = (S \cdot D) (S^2 - V)^{-1} S \tag{58}$$

From which it can be shown that $k' = (S^2 - V)^{-1}S - S^T(S^2 - V)^{-1}(S^2 - V)^{-1} S S$

is in the D X H or k direction.

Thus for a medium in which $\tilde{\tilde{\varepsilon}}^T = \tilde{\tilde{\varepsilon}}$ one is able to obtain the direction

of k at the saddle point given the direction of the observation point.

This value of k is then scaled to fit the dispersion relation (41).

7. General Magnetoplasma Region

For the general anisotropic region, $\tilde{\tilde{\varepsilon}}$ has the form

$$\tilde{\tilde{\varepsilon}} = \begin{vmatrix} \varepsilon_\perp & -j\varepsilon_x & 0 \\ +j\varepsilon_x & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{vmatrix} \tag{59}$$

One has not been able to find an analytical solution to the geometric

problem of obtaining the k vector for which $dk \cdot r = 0$.

Optimization techniques have been tried in the anisotropic region. However, the general lossy anisotropic region will have many multiple solutions for which a searching scheme must be devised.

8. Conclusions

Using asymptotic techniques an approximation to the solution of the integral $\int_C F(k)\, e^{ik \cdot r}\, dk$ can be made for $F(k)$ varying "slowly" at a reasonable magnitude of $(k \cdot r)$. This approximation, expands this integral about a saddle point. This has been done for the one- and two-dimensional problem. Preliminary results indicate that in free space, the estimate is with 10% of the exact solution at $r = .2$ wavelengths and within 1% beyond $r = .4$ wavelengths.

The problem of finding the saddle points corresponds to finding the k vectors which contribute to the field in a given direction. An analytic solution to this problem has been presented for the special case in which the dielectric matrix has the symmetry $\tilde{\tilde{\varepsilon}} = \tilde{\tilde{\varepsilon}}^T$. Otherwise, optimization techniques are suggested.

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3. G. A. Deschamps, Private communication.

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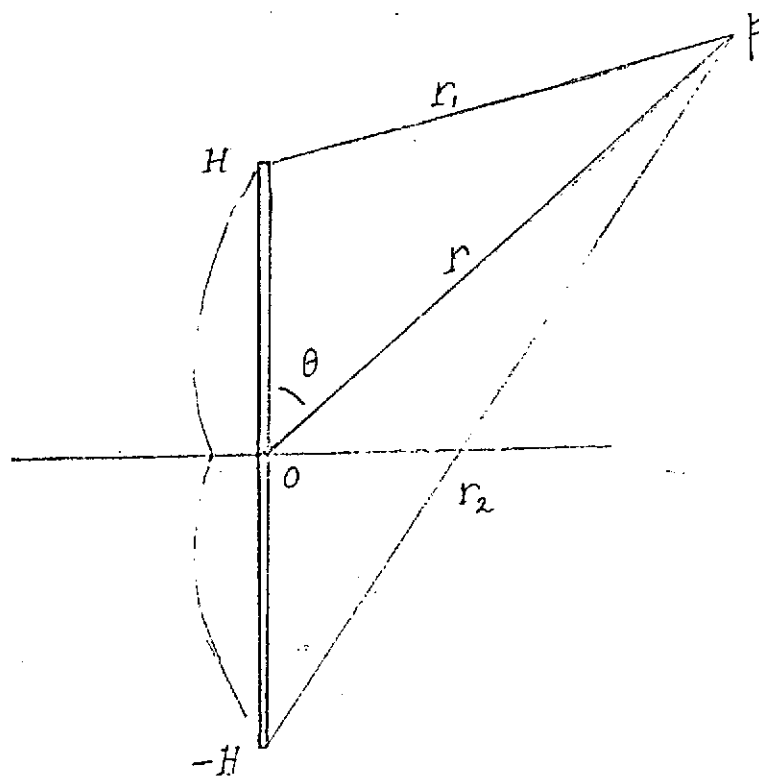
Figure 1.   Geometry of Dipole Antenna.

II. 3. Laboratory Measurements

a. Impedance Measurements - Effect of Power Level and Voltage Bias

In measuring the variations of antenna impedance with the input power level some difficulties with the vacuum system were observed. The following modifications improved the consistency of the results considerably. First the vacuum system was thoroughly cleaned and new gaskets were installed. The diffusion pump jet assembly was found damaged due to oxidation of the diffusion pump oil. A new jet was then installed and silicon oil was used to prevent further accidental oxidation. The helium flask was replaced also. A second mechanical pump was added to the original system and connected to the backfill port of the pumping station. This pump provides the capability to continuously exhaust the plasma tube without the fear of damaging the diffusion pump by accidental breakage of vacuum or other contaminations. It also keeps the plasma tube at low pressure even when the tube is not in use. This reduces the pump down time considerably. An electronic discharge current regulator was built to stablize the current and it worked beautifully. The RF system was revised also. When operating at 500 MHz a RF amplifier was added to boost the available power to about 2 watts. With the addition of power attenuators and a directional coupler to a power meter it is possible to vary the power level from 2 watts to less than 5 milliwatts and to continuously monitor the input power level.

At 800 MHz and above, the oscillator/amplifier combination is replaced by a powerful oscillator at our disposal, while the remainder of the RF system remains the same. Due to the high power level involved in the measurements it was found necessary to place most RF equipments inside an anechoic chamber in order to avoid erratic readings on VSWR meters.

Many efforts have been expended to determine the impedance variation of the antenna when the input power level was varied. Two frequencies were used, 800 MHz and 1 GHz. Several typical results are given in Figures 1 to 3. It is interesting to note that all the impedance variations as the power level increases follow the same general pattern. This behavior is not completely understood at the present moment, but the results are very consistent. The dipole started to glow when the input power level reached few hundred milliwatts. The glowing is an indication of induced ionization by strong RF electric field near the dipole. We expect that the electron density increases under this circumstance. This is probably the reason why the impedance curve bends backwards near this region as it does when the electron density is raised by increasing the discharge current. At low power we noticed that even for a level of a few milliwatts the impedance may change drastically. In Figure 1, points 1 and 2 correspond to power level of 1.6 mW and 7.5 mW respectively, but there is a substantial difference in the impedance. Similarly in Figure 2 points 4 and 5 correspond to the power level of 5.0 mW and 10 mW respectively. However, the variation of impedance from point 1 to point 3 may be considered as small. Their power levels are 79 μW and 1.3 μW respectively. Therefore from these measurements it seems that 1 mW could be considered as the threshold for small signal approximation. Since in practice measurements are commonly performed at a power level of several milliwatts or higher, this casts a great doubt in the validity of many existing results. Therefore it is concluded that any antenna impedance measurement without reference to the input power level may be meaningless. Another interesting feature is that when the external magnetic field is zero, the impedance of the antenna shows little changes as seen in Figure 3.

b. Observation of Nonlinear Effects

Harmonics Generation

The generation of second harmonic in plasma has been observed with a spectrum analyzer to monitor the reflected power from the antenna and also by tuning a receiver to the second harmonic frequency. The second harmonic power level was measured against the magnetizing field for various bias voltages. The results are shown in Figure 4. In all cases, a local maximum is shown at the cyclotron resonance. When the bias voltages were varied, the more positive the bias is, the higher is the second harmonic level as might be expected. However, the second harmonic level eventually became saturated when the bias reached several tens of volts. Under certain circumstances, the third harmonic has also been observed but no attempt was made to measure the third harmonic level. When modulated signals were fed into the antenna, the modulations were detected by a receiver at the second harmonic.

Luxemburg Effect

In this experiment two transmitting antennas in the plasma were used. One was fed with a modulated signal with carrier frequency f_1 at a fairly high power level and the other with a continuous wave at frequency f_2. A receiving antenna was located several feet away from the plasma tube and the receiver was tuned to f_2. It was found that the modulation or audio signal was detected through a speaker, indicating that part of the modulation was transferred to the wave at f_2. When one of the two transmitting antennas was placed outside of the plasma chamber. A similar phenomenon was observed but with weaker modulation. This is the so called Luxemburg effect or crossmodulation effect. In ionosphere it is believed that the electron heating is the main cause of crossmodulation. In a laboratory plasma many

other nonlinear effects could produce crossmodulation between signals.

It is not clear which nonlinear effect is the dominant contributor.

NAME	TITLE TAPED LINE	DWG. NO. # 74
	Impedance of Dipole #3 in Plasma ↔ Incident Power Level	DATE June 21, 1972
SMITH CHART Form 5301-7560-N	GENERAL RADIO COMPANY, WEST CONCORD, MASSACHUSETTS	

IMPEDANCE OR ADMITTANCE COORDINATES

DIPOLE ∥ TO AXIS OF TUBE

B = 160 Gauss

	Point #	dbm at dipole
1.6 k.w.	1	~ 2
7.5 m.w.	2	8.7
	3	12.4
	4	17.0
	5	21.0
	6	25.8
0.8 w	7	29.0

Fig. 1

RADIALLY SCALED PARAMETERS

NAME	TITLE *TAPED LINE* *Impedance of Dipole #5 in Plasma* vs. *Incident Power Level*	DWG. NO. #106
SMITH CHART FORM 5301-7560 N	GENERAL RADIO COMPANY, WEST CONCORD, MASSACHUSETTS	DATE *July 18, 1972*

Dipole ∥ to axis of tube   IMPEDANCE OR ADMITTANCE COORDINATES

$I_D = 10$ ma

$B = 320$ gauss

KEY TO INCIDENT POWER LEVELS

POINT	POWER	POINT	POWER
1	79 μw	5	10 mw
1a	79 μw	6	16 mw
1b	79 μw	7	50 mw
2	320 μw	8	130 mw
3	1.3 mw	9	360 mw
4	5.0 mw	10	910 mw

Fig. 2

RADIALLY SCALED PARAMETERS

TOWARD GENERATOR →   ← TOWARD LOAD

CENTER

NAME .	TITLE TAPED LINE	DWG. NO. # 56
	Impedance of Diode #3 in Plasma Vs. Incident Power Level	DATE June 9, 1972
SMITH CHART FORM 5301-7550-N	GENERAL RADIO COMPANY, WEST CONCORD, MASSACHUSETTS	

IMPEDANCE OR ADMITTANCE COORDINATES

DIPOLE ‖ TO AXIS OF TUBE

No B-field

Point #	Incident Level at Dipole
1	11.7 d.
2	15.6 db
3	19.3 db
4	23.7 d
5	27.3 d
6	28.7 d
7	32.5 d

Fig. 3

RADIALLY SCALED PARAMETERS

TOWARD GENERATOR ⟶   ⟵ TOWARD LOAD

CENTER
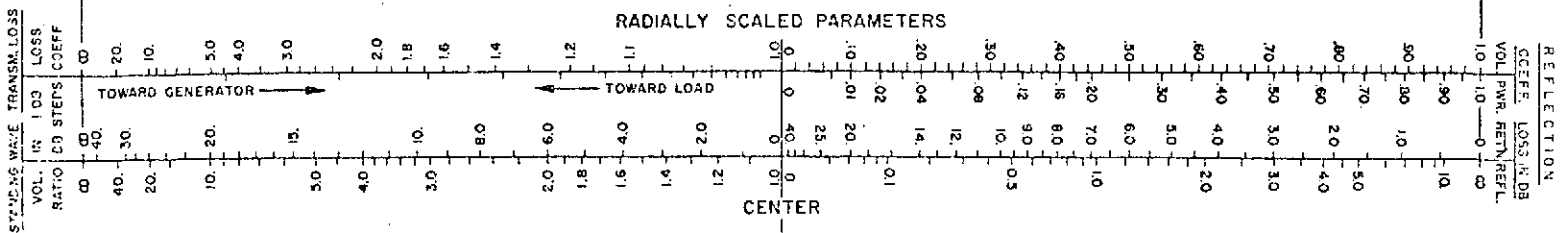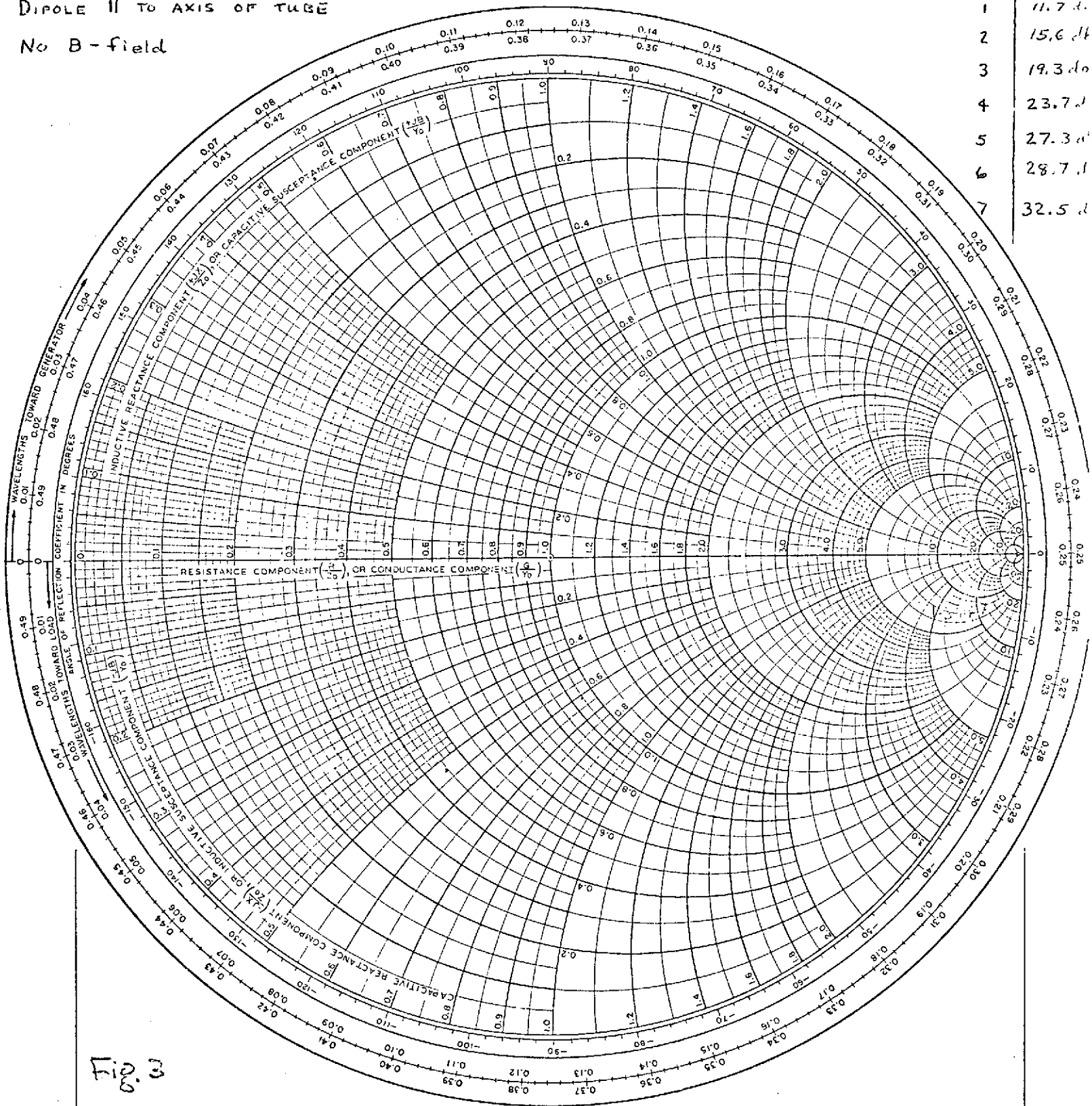
2nd Harmonic
Level
(10^{db}/inch)

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NO. 341-10 DIETZGEN GRAPH PAPER
10 X 10 PER INCH

0 db for
+15.96
volts

0 db for
-15.96 volts

0 db for
-30
volts

0 db for
-50 volts

0 db for
-70 volts

+15.96 volts

-15.96 volts

-30.00 volts

-50.0 volts

-70.0 volts

$\frac{\omega_c}{\omega}=0$ $\frac{\omega_c}{\omega}=.304$ $\frac{\omega_c}{\omega}=.609$ $\frac{\omega_c}{\omega}=.913$ $\frac{\omega_c}{\omega}=1.22$ $\frac{\omega_c}{\omega}=1.52$ $\frac{\omega_c}{\omega}=1.83$ $\frac{\omega_c}{\omega}=2.$

0   50   100   150 ω-ω 200   250   300   350
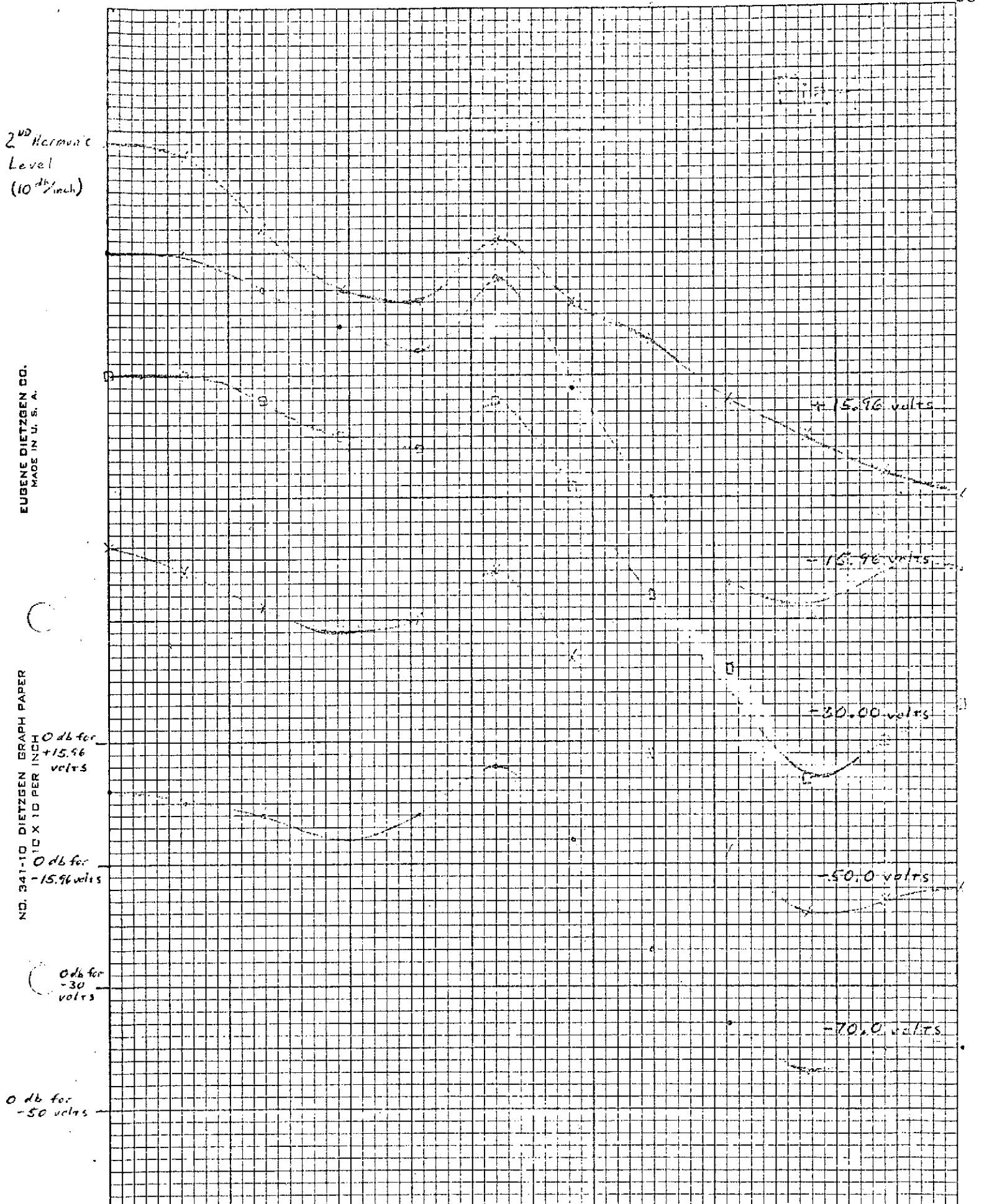
I.   Introduction

There are various kinds of nonlinear phenomena in a plasma. Tsytovich
(1) has made an extensive and detailed discussion on the nonlinearity due
to the $(\vec{v} \cdot \nabla) \vec{v}$ and $\vec{v} \times \vec{B}$ terms in fluid equations whereas Whitmer and
Barrett (2-5) considered the same set of equations and calculated the re-
flected and transmitted waves, including harmonics, through a plasma
layer. Keller and Millman (6) assumed a very general constitutive re-
lation with no particular reference to physical processes and solved the
Maxwell's equations for several propagation problems by using perturbation
method. Ginzburg (7,8) discussed the nonlinearity due to electron heating,
in particular its application to the cross modulation of waves in ionosphere,
the so-called Luxemburg effect. In his elementary theory the state of
the plasma was characterized by two variables, the average electron velocity
$\vec{v}$ and the effective electron temperature T_e. Ginzburg showed that if the
applied field intensity is smaller than the "plasma field" E_p the state
of the plasma changes slightly under the influence of the field. However
when the applied field intensity becomes larger then E_p, the parameters
of the plasma change significantly. Since in commonly encountered plasmas
the value of E_p is small, significant nonlinear effects may result even
for moderately large applied fields. In this paper, the nonlinear effect on
the reflection coefficient for a plane wave incident on air-plasma interface is
investigated. The reflection from a semi-infinite nonlinear plasma is
considered first and from a conducting plate backed plasma layer next.
The reflection coefficient as a function of incident field strength is
solved for both cases.

II. General Formulation

In a nonlinear plasma, the medium property depends on the applied field strength. Ginzburg (7) has shown that if the nonlinearity of the plasma is mainly due to electron heating processes, the local effective temperature of the plasma T_e is a function of the magnitude of the field strength E_o. At low frequencies the properties of the plasma are modulated by the applied field and the problem becomes rather complicated. Therefore we will confine our discussion to high frequency limit such that $\omega^2 \gg \nu^2$. In the high frequency limit, the field varies much faster than energy relaxation time, therefore we can replace E by its rms value $\bar{E}$ or aside from an unimportant constant, by its maximum value E_o. From Ginzburg's elementary theory the relation between T_e and E_o can be expressed as follows

$$\tau^2 \equiv \frac{T_e}{T} = 1 + (\frac{E_o}{E_p})^2 \frac{\omega^2 + \nu_o^2}{\omega^2 + \nu^2(T_e)} \approx 1 + (\frac{E_o}{E_p})^2 \quad \text{if } \omega^2 \gg \nu^2 \tag{1}$$

where T is the equilibrium electron temperature of the plasma in the absence of external field, ν_o and ν are effective collision frequencies in the absence and in the presence of external field respectively. ν can be considered as a function of T_e or E_o. E_p is the so called "characteristic plasma field".

$$E_p = \sqrt{3kT \frac{m}{e^2} \delta (\omega^2 + \nu_o^2)} \tag{2}$$

where δ is the average relative fraction of energy transfered during a collision between electrons and heavy particles. It is seen that when $E_o \ll E_p$ the heating effect will be small and $T_e \approx T$, but when $E_o \gtrsim E_p$ the electron heating processes becomes significant. It is convenient to introduce the concept of dielectric permittivity and conductivity for a plasma. It is well known that

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 + \nu^2} \quad , \quad \sigma = \frac{\nu\omega_p^2}{4\pi(\omega^2 + \nu^2)} \tag{3}$$

where ω_p is the plasma frequency. Since ν is a function of E_o, therefore ε and σ are also functions of E_o.

Consider a one dimensional case, the electric field inside the plasma satisfies the simple wave equation

$$\frac{d^2E}{dz^2} + \omega^2\varepsilon' \frac{E}{c^2} = 0 \tag{4}$$

and here $\varepsilon' \equiv \varepsilon - i \frac{4\pi\sigma}{\omega} = (n - i\chi)^2$ (5)

In plasma, usually $\varepsilon^2 \gg (\frac{4\pi\sigma}{\omega})^2$ is satisfied

$$\therefore \quad n = \sqrt{\varepsilon} \quad , \quad \chi = \frac{2\pi\sigma}{\omega\sqrt{\varepsilon}} \tag{6}$$

We will also assume that collisions are mainly between molecules and electrons. The collision cross-section is approximately a constant independent of veloicty such that $\nu = \tau\nu_o$. In high frequency limit $\chi = \tau\chi_c$ where

$$\chi_o = \frac{2\pi e^2 N\nu_o}{m\omega^2 \sqrt{1 - \frac{4\pi Ne^2}{m\omega^2}}} \quad ,$$

and as a first approximation n is a constant. In this case we can use the geometrical optics approximation. A solution for the wave equation (4) can be written as

$$E = A \exp[i\omega t + i \frac{\omega}{c} \int \sqrt{\varepsilon'} \, dz] = E_o \exp[i(\omega t - \frac{\omega}{c} \int_o^z n \, dz)] \tag{7}$$

For simplicity we denote E_o as the magnitude of the field. Sometimes it is easier to use a differential equation for E_o instead of (7). Differentiation of E_o with respect to z results in

$$\frac{dE_o}{dz} = - \frac{\omega}{c} \chi(E_o) E_o \tag{8}$$

Now equations (1) and (8) are to be solved simultaneously subject to suitable boundary conditions.

III. Reflection from a Semi-Infinite Plasma

First consider a simple problem, a plane wave incident normally on a semi-infinite plasma which is assumed to be uniform and isotropic. Let the interface between free space and plasma be the $z = o$ plane, and the electric field is in the x direction as shown in Figure 1.

Substituting χ into equation (8) and make use of equation (1)

$$\frac{dE_o}{dz} = - \frac{\omega}{c} \tau \chi_o E_o = - \frac{E_p^2}{E_o} \frac{\omega}{c} \chi_o (\tau^2 - 1) \tau \tag{9}$$

$$\left. \frac{dE_o}{dz} \right|_{z=0} = - \frac{E_p^2}{E_o(0)} \frac{\omega}{c} \chi_o \tau_o (\tau_o^2 - 1) \quad , \quad \tau_o = \tau(z=0) \tag{10}$$

From Maxwell's equations the magnetic field can be computed

$$- \frac{i\omega}{c} B_y (z = 0) = \left. \frac{dE}{dz} \right|_{z=0} = \left. \frac{dE_o}{dz} \right|_{z=0} \left. e^{i(\omega t - \frac{\omega}{c} \int_o^z n dz)} \right|_{z=0}$$

$$+ E_o(z=0) \left[-i \frac{\omega}{c} \frac{d}{dz} \int_o^z n dz \right]_{z=0} \left. e^{i(\omega t - \frac{\omega}{c} \int_o^z n dz)} \right|_{z=0} \tag{11}$$

$$B_y(0) = \sqrt{\varepsilon} \, E_o(0) \, [1 + (\frac{2\pi\sigma}{\omega\varepsilon} \tau_o)^2]^{1/2} \, e^{i\phi} \, e^{i\omega t} \tag{12}$$

$$= C \, E_o(0) \, e^{i\phi} \, e^{i\omega t}$$

where $\phi = - \tan^{-1} (\frac{2\pi\sigma}{\omega\varepsilon} \tau_o)$ \hfill (13)

The reflection coefficient R and transmission coefficient T then can be calculated easily by matching the boundary conditions at $z = 0$. It can be shown that

$$|R| \doteq \left| \frac{1 - Ce^{i\phi}}{1 + Ce^{i\phi}} \right|$$

and as usual $1 + R = T$.

Figure 2 shows the reflection coefficient and transmission coefficient as functions of incident field intensity. The horizontal axis are incident field intensity normalized with respect to E_p. The parameters chosen were typical for a laboratory plasma and the characteristic plasma fields E_p are of the order of 100 volts/cm.

IV. Wave Reflection from a Nonlinear Plasma Layer Backed by Conducting Plane

Now consider a plane wave incident normally onto a nonlinear plasma slab with thickness d and backed by a perfect conducting plate as shown in Figure 3.

It has been shown that when $\omega^2 \gg \nu^2$, n can be considered as a constant. In the present case the incident field E_1 and the reflected field E_2 in the plasma can then be expressed in the following form

$$\begin{cases} E_1 = ae^{-ikz} \, e^{- \frac{\omega}{c} \int_o^z \chi \, dz} \\ \\ E_2 = be^{ikz} \, e^{\frac{\omega}{c} \int_d^z \chi \, dz} \end{cases} \tag{14}$$

where $k = \frac{\omega}{c} n$ and for simplicity the time factor $e^{i\omega t}$ has been suppressed. One of the difficulties of solving a nonlinear problem is the in applicability

of the superposition principle. For the present problem the absorption
coefficient χ depends on the total electric field strength $\tilde{E}_o$ which is the
sum of E_1 and E_2; therefore the unknown functions E_1, E_2 and χ have to
be solved simultaneously. The equations are

$$\frac{d|E_1|}{dz} + \frac{\omega}{c} \chi (E_o) |E_1| = 0 \tag{15}$$

$$\frac{d|E_2|}{dz} - \frac{\omega}{c} \chi (E_o) |E_2| = 0 \tag{16}$$

$$\left(\frac{E_o}{E_p} \right)^2 = \tau^2 - 1 \tag{17}$$

where $E_o = |\tilde{E}_o|$

From (14) and the boundary condition for $\tilde{E}_o$ at $z = d$, the magnitude of $\tilde{E}_o$
can be written as

$$E_o = |E_1 + E_2| = E (z) [1 + e^{4 \frac{\omega}{c} \int_d^z \chi dz} - 2 \cos 2k(z - d) \cdot e^{2 \frac{\omega}{c} \int_d^z \chi dz}]^{1/2} \tag{18}$$

where $E (z) = |E_1(z)|$, χ is a function of E_o.

Introduce a new function $U(z) \equiv 2 \frac{\omega}{c} \chi_o \int_d^z \tau dz$ then substitution of
equation (18) into (17) yields

$$E_d^2 e^{-U} = \frac{E_p^2 (\tau^2 - 1)}{[1 + e^{2U} - 2e^U \cos 2k(z-d)]} \tag{19}$$

where $E_d = E (z=d)$. Making use of $\frac{dU}{dz} = 2 \frac{\omega}{c} \chi_o \tau$, equation (19) can be
converted into a nonlinear differential equation

$$\left(\frac{dU}{dz} \right)^2 = \frac{8E_d^2 \omega^2 \chi_o^2}{E_p^2 c^2} [\cosh U(z) - \cos 2k(z-d)] + \frac{4\omega^2 \chi_o^2}{c^2} \tag{20}$$

It appears that this nonlinear differential equation cannot be solved in closed form, and therefore numerical techniques are used to determine U and $\frac{dU}{dz}$. One set of calculated results are given in Figure 4 for $E_{inc}/E_p = 0.18$, 0.47, 1.16, 3.29, 13.0 in which the total field strength E_o is plotted against z. The plasma parameters chosen in these calculations are the same as before and the slab is five wavelengths thick. When the fields is weak as in 4(A) the solution is essentially the same as that for a linear plasma slab; i.e., the heating effect is insignificant. However as the field intensity increases the absorption also increases and the field distribution inside the plasma alsb changes considerably, becoming strongest near the interface. In contrast, for a linear plasma the shape of the field distribution remain unaltered, regardless of incident field strength. The reflection coefficient $|R|$ is plotted against the normalized field intensity E_{inc} in Figure 5. It is interesting to note the existence of a minimum reflection for a particular value of E_{inc}.

V. Conclusion

The reflection coefficients $|R|$ from a semi-infinite plasma and a plasma layer backed by a conducting plane are determined as a function of incident field strength. In the first case it is found that $|R|$ is increasing monotonically and approaches 1 as the incident field increases toward infinity (within the range of validity of our theory). For the second case, the reflection coefficient has been found to have a minimum for a certain value of incident field. For large value of incident field the reflection coefficient increases steadily and approaches 1 as in case one. The total field strength inside the plasma layer is evaluated also. It is expected that this particular nonlinear effect can be observed in the laboratory by measuring the reflection coefficient from a discharge. For ionosphere the effect may be too small to be significant.

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Figure 1. Wave reflection from a semi-infinite plasma.

Figure 2. Reflection and transmission coefficients as functions of

incident field intensity.

$f = 6 \times 10^{10}$ Hz $\nu_o = 1 \times 10^9$/sec

$N = 1 \times 10^{12}$/cm^3 $\delta = 1 \times 10^{-3}$

$T = 1 \times 10^4$ °K

Figure 3. Reflection from a plasma layer backed by conducting plane.

Figure 4(A)-4(E). The variation of field intensities inside the plasma

layer as the incident field increases. At the air-plasma

interface E_{inc}/E_p equals:

(A) $E_{inc}/E_p = 0.18$

(B) $E_{inc}/E_p = 0.47$

(C) $E_{inc}/E_p = 1.16$

(D) $E_{inc}/E_p = 3.29$

(E) $E_{inc}/E_p = 13.0$

Figure 5. Reflection coefficients as functions of E_{inc}/E_p.

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FREE SPACE   PLASMA

E_x

H_y

E

H

z

z=0

Fig. 1

Reflection from a Semi-infinite Plasma.

FREE SPACE   PLASMA   Perfect Conductor

E', B' E_1, B_1

E'', B'' E_2, B_2

z

z=0   z=d
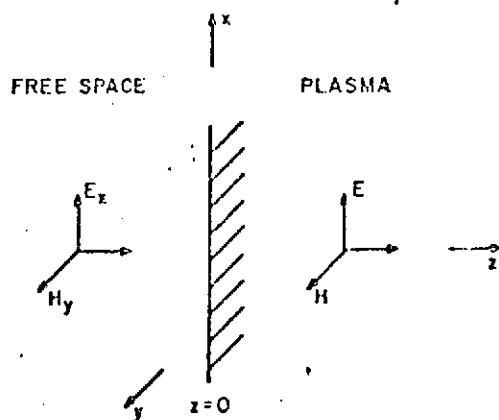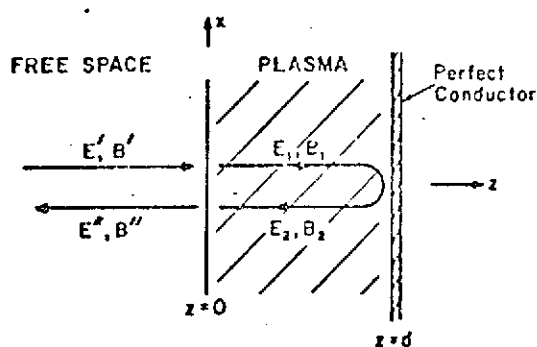
Fig. 3

Reflection from Plasma Layer
Backed by Conducting Plane

Reflection Coefficient

Transmission Coefficient

———— REFLECTION
- - - - TRANSMISSION

E_{inc}/E_p

Fig. 2

(A)

$f = 6 \times 10^{10}$ Hz
$N = 1 \times 10^{12} / cm^3$
$T = 1 \times 10^{4}$ K°
$\gamma_0 = 1 \times 10^{9} / sec$
$\delta = 1 \times 10^{-3}$

$E_0/E_p \longrightarrow$

$Z/\lambda \longrightarrow$

Fig. 4A

(B)

$E_0/E_p \longrightarrow$

$Z/\lambda \longrightarrow$

Fig. 4B

(C)

$E_0/E_p \longrightarrow$

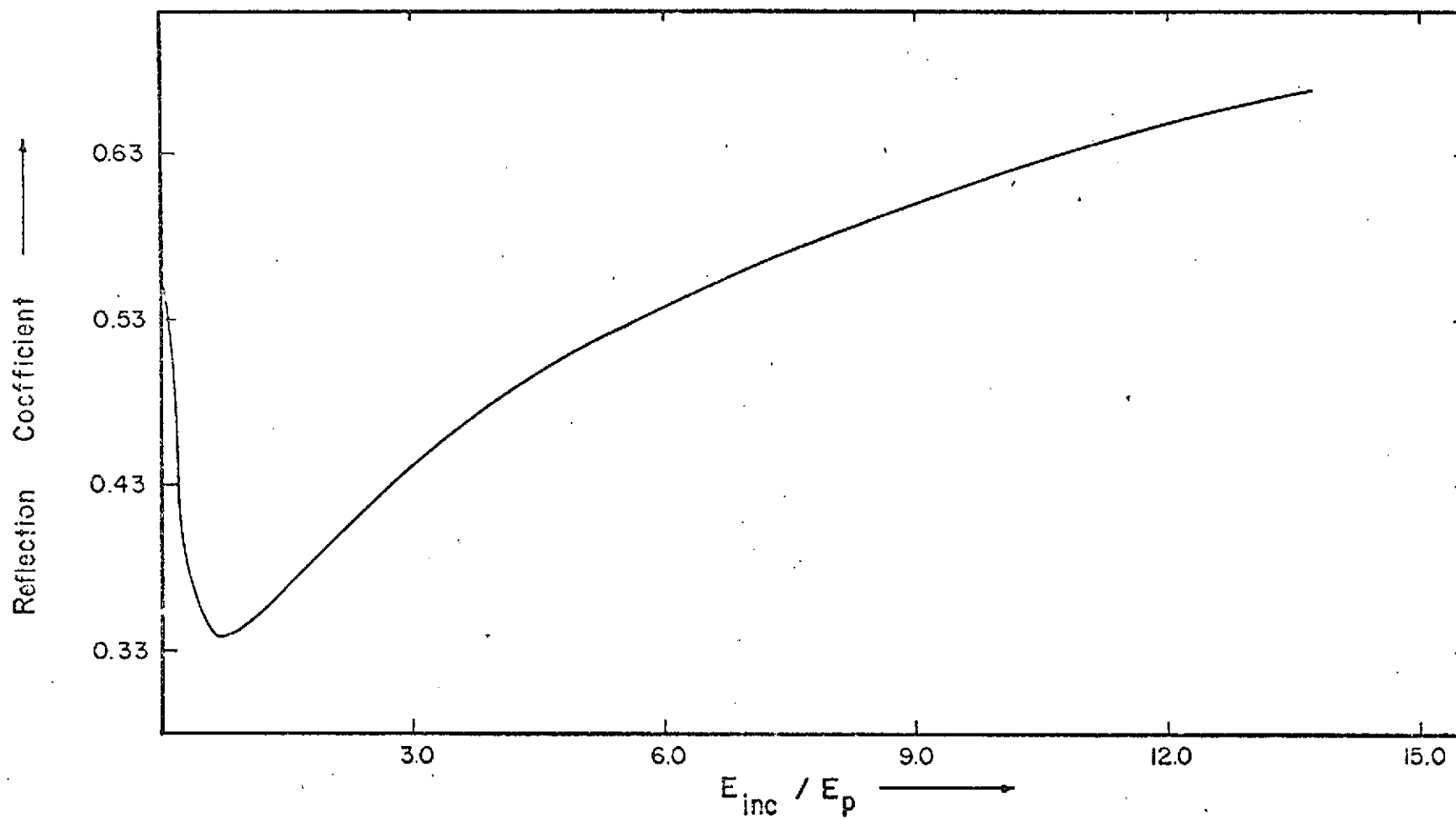$Z/\lambda \longrightarrow$
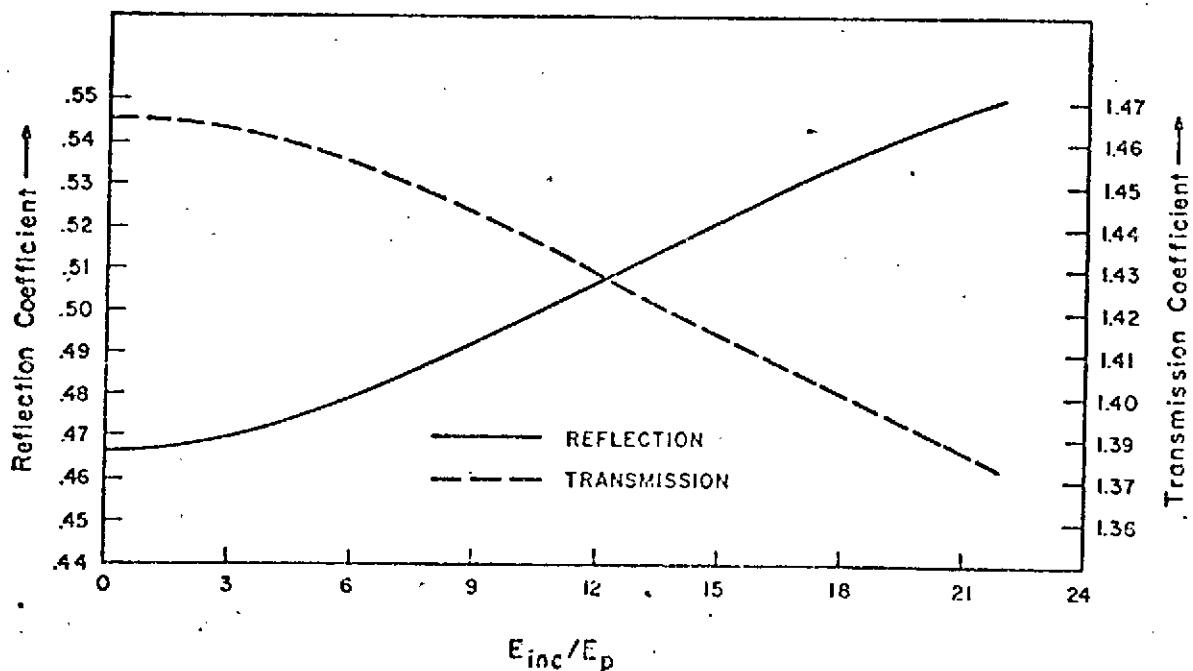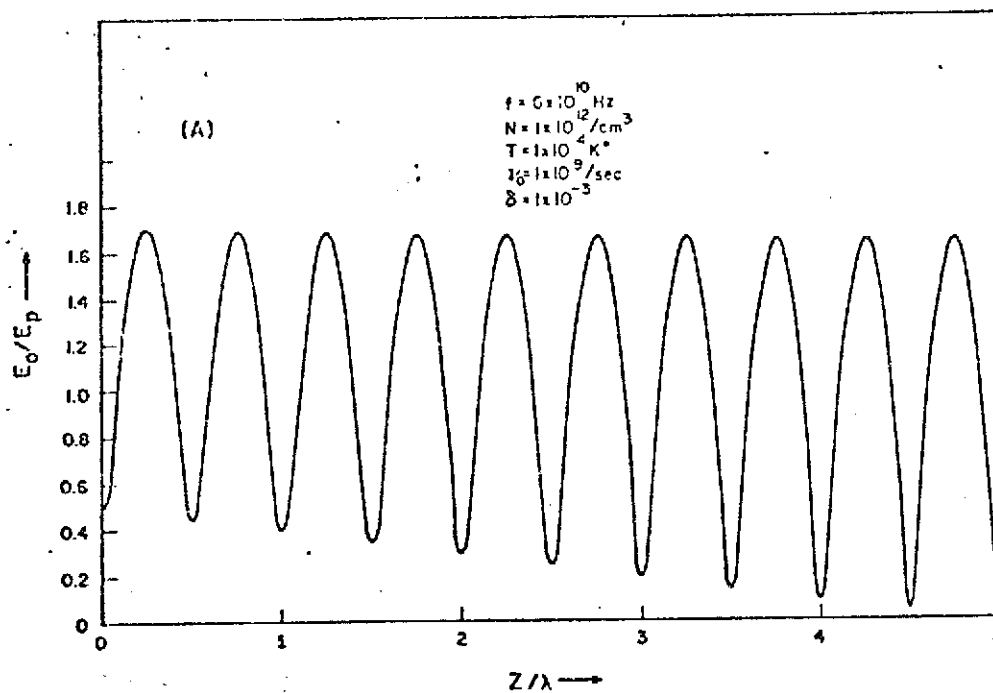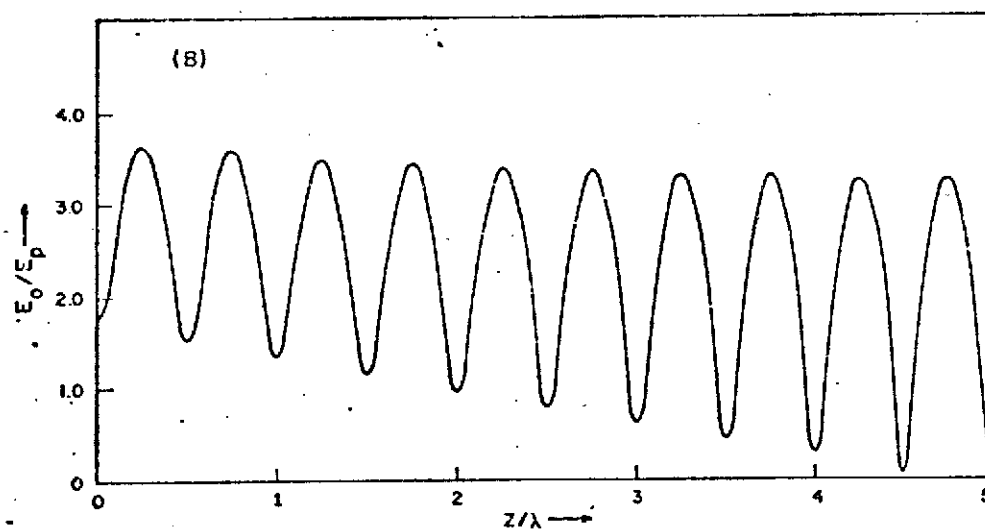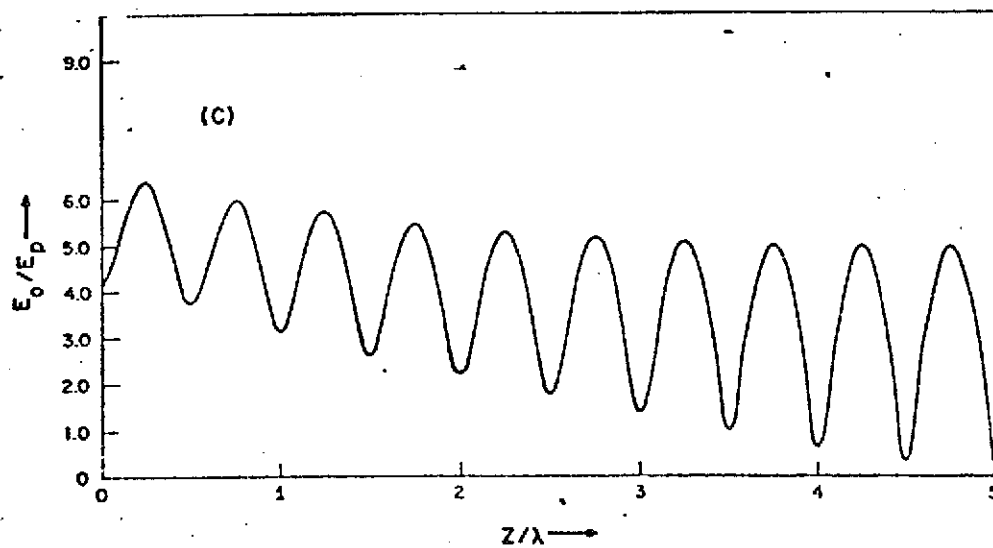
Fig. 4C
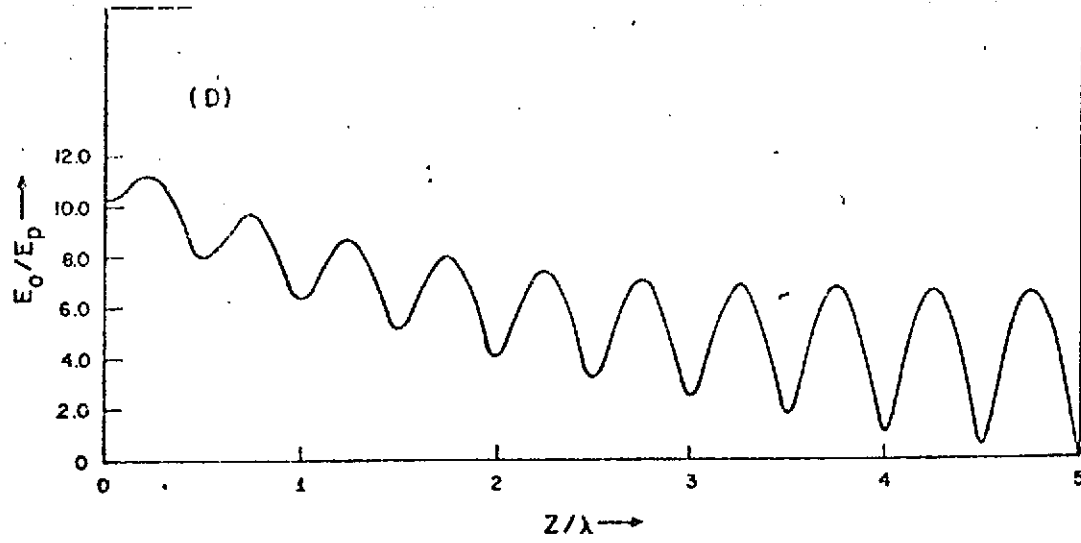
Fig. 4D
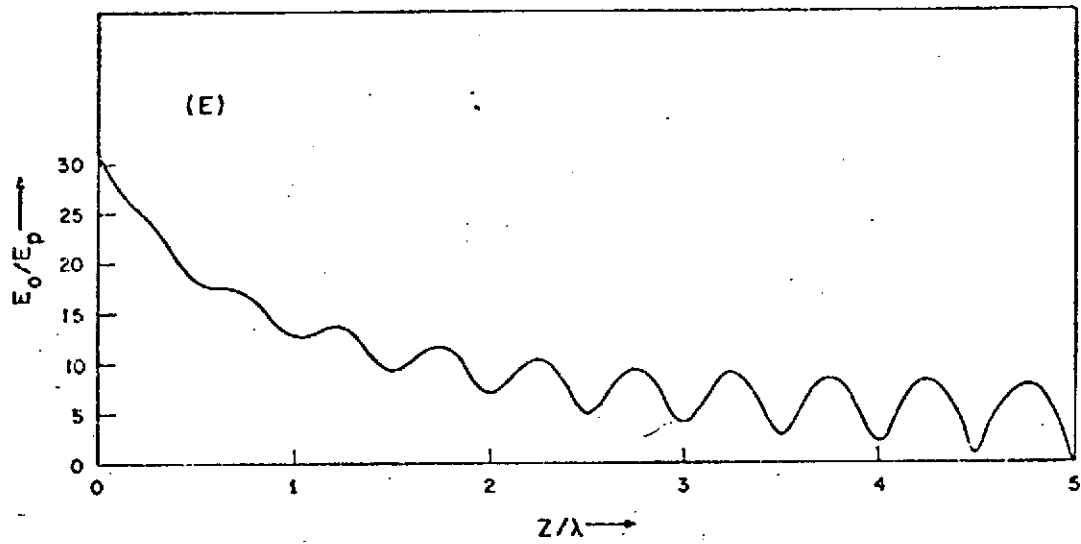
Fig. 4E

Fig.5

II. 5. Profile Inversion of Inhomogeneous Media

1. Background and General Description of the Problem

We have investigated the problem of determining the electrical permittivity of a medium which is uniform in the x and y directions and varies according to the profile function, $\kappa(z)$, in the z direction. The mathematical problem associated with this profile inversion problem is typical of many other remote sensing experiments.

Using the techniques of Gelfand and Levitan[1] and of Marchenko,[2] many authors, e.g., Becker and Sharpe,[3] have reported the solution of this type of problem. All of their work, however, requires that the response of the medium be known as a function of the temporal frequency. It is also required that the medium itself be independent of frequency, i.e., nondispersive. Often, however, it is desirable to make measurements at a single frequency. Therefore, in this paper we consider the remote probing problem in the angular spectrum domain. Here the independent variable, β, is the spatial frequency which can be associated with $k \sin \theta$ where k is the free space wave number and θ is the angle of incidence of a probing plane wave.

The technique by which the response of the medium is measured is to illuminate the medium using a spatially confined source. The total field at the interface is then measured. The information is then converted to the angular spectrum domain by taking the Fourier transform with respect to the x direction.

In the next section, the formulation of the problem is presented. The remaining sections are devoted to techniques for solving the problem.

2. Formulating the Problem

The geometry of the problem to be considered is shown in Figure 1. Note that there are no variations with respect to the y direction. The dielectric is also uniform in the x direction and varies in the z direction as $\kappa(z)\varepsilon_o$ where ε_o is the permittivity of free space. The permeability is assumed to be constant and equal to that of free space. It is assumed that a known electromagnetic wave of a single frequency is incident from the left on the air-dielectric interface at $z = 0$ and that the tangential fields at the surface of the dielectric can be measured for all values of x. Of course, the actual measurement could be performed at some plane $z = \alpha$, $\alpha < 0$, and then the fields at the interface could be computed since the propagation in free space is completely known.

If the incident field is polarized such that the components E_x, E_z, and H_y are all zero, then propagation in the medium is completely described by the scalar equation

$$\nabla^2 v(x,z) + \omega^2 \mu_o \kappa(z) \varepsilon_o v(x,z) = 0 \qquad (1)$$

(ω is the angular frequency of the incident wave). The function $v(x,z)$ may be identified with E_y or H_z, which ever is more convenient for the problem at hand.

To obtain the differential equation in the angular spectrum domain, we introduce the Fourier transform relations:

$$u(\beta,z) = \int_{-\infty}^{\infty} v(x,z) \exp(-i\beta x) \, dx \qquad (2a)$$

$$v(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\beta,z) \exp(i\beta x) \, d\beta \qquad (2b)$$

Then Equation (1) becomes

$$\frac{d^2 u(\beta,z)}{dz^2} + [\omega^2 \mu_o \kappa(z) \varepsilon_o - \beta^2] \, u(\beta,z) = 0 \qquad (3)$$

This is one form of the differential equation in the angular spectrum domain. Recall that ω, μ_o, and ϵ_o are constants, β is the independent (spectral) variable, and $\kappa(z)$ is the unknown profile function.

For the work which follows, it is convenient to introduce a different, but equivalent, form of (3). That is,

$$\frac{d^2 u(\gamma,z)}{dz^2} + \gamma^2 u(\gamma,z) = Q(z)u(\gamma,z) \tag{4}$$

where

$$\gamma^2 = k^2 - \beta^2$$

$$Q(z) = -k^2[\kappa(z) - 1]$$

$$k^2 = \omega^2 \mu_o \epsilon_o$$

Note that if we determine $Q(z)$, $\kappa(z)$ can easily be found. Furthermore, note that we intend to apply (4) in the source-free region $0 \leq z < \infty$.

3. The Solutions $\phi(\gamma,z)$ and $f(\gamma,z)$

Equation (4) turns out to be of the same form as the one-dimensional Schroedinger wave equation. This equation received considerable attention in the 1950's from physicists who were interested in scattering by potentials. Most of the work on the inverse scattering problem has been based on the techniques of Gelfand-Levitan[1] and Marchenko[2]. An extensive review of work on the inverse scattering problem has been presented by Faddeyev.[4]

We will assume that the profile function $Q(z)$ is real and bounded and that

$$\int_0^\infty z|Q(z)| \, dz < \infty \tag{5}$$

We then consider two independent solutions of (4), $\phi(\gamma,z)$ and $f(\gamma,z)$, satisfying the boundary conditions.

$$\phi(\gamma,0) = 0 \tag{6a}$$

$$\frac{d\phi(\gamma,0)}{dz} = 1 \tag{6b}$$

and

$$\lim_{z \to \infty} [f(\gamma,z) \ e^{-i\gamma z}] = 1 \tag{7}$$

The solution $f(\gamma,z)$ corresponds to the situation shown in Figure 1. The solution $\phi(\gamma,z)$ corresponds to situation where a perfectly conducting wall is placed at $z = 0$ and the incident wave comes from $z = +\infty$. It is demonstrated in Faddeyev's paper that

$$\lim_{z \to \infty} \phi(\gamma,z) = \frac{A(\gamma)}{\gamma} \sin[\gamma z - \psi(\gamma)] \tag{8}$$

where

$$f(\gamma,0) = A(\gamma) \ e^{i\psi(\gamma)} \tag{9}$$

4. The Inversion Technique

Mittra, et al.,[5] have introduced the following representation of $f(\gamma,z)$.

$$f(\gamma,z) = \exp(i\gamma z) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\eta,z) \frac{\exp[i(\eta + \gamma)z]}{\eta + \gamma} P(\eta)d\eta \tag{10}$$

In this case, the profile function $Q(z)$ is given by

$$Q(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} [i \frac{df(\eta,z)}{dz} - \eta \ f(\eta,z)]\exp(i\eta z)P(\eta) \ d\eta \tag{11}$$

The function $P(\eta)$ can be determined from $f(\gamma,0)$ by considering (10) at $z = 0$,

$$f(\gamma,0) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\eta,0) \frac{P(\eta)}{\eta + \gamma} \ d\eta \tag{12}$$

These three equations provide a technique for solving the inverse scattering problem in the spectral domain. That is, a technique for obtaining $Q(z)$ from a knowledge of $f(\gamma,o)$.

It is common for inverse problems in mathematical physics to be ill-conditioned. This means that small errors in the data result in large errors in the computed profile function. Mathematically, ill-conditioning is usually associated with the solution of a first kind Fredholm equation. We note that (12) is indeed a first kind equation for the unknown function $P(\eta)$. Thus we would expect to have trouble solving (12). On the other hand, once $P(\eta)$ is specified, (10) becomes a second kind equation for $f(\gamma,z)$——z is fixed and γ is the independent variable. We, therefore, would not expect (10) to be ill-conditioned. Thus, we are led to the conclusion that if we can obtain $P(\eta)$ accurately from $f(\gamma,o)$, we will have a procedure for obtaining $Q(z)$ which is numerically stable.

We have been able to show an analogy between (10) and the representation which Marchenko uses for $f(\gamma,z)$. This has allowed us to determine that $P(\eta)$ is given by

$$P(\eta) = 1 - e^{-i2\psi(\eta)} \tag{13}$$

Using (13), we then solve (10) for $f(\gamma,z_j)$ for several values z_j. The profile function $Q(z)$ is then obtained from (11).

It should be noted that (13) is valid only if $f(\gamma,0)$ has no poles or zeros in the upper half plane. This condition implies the absence of surface wave solutions to (4). If surface waves are present in the problem, the inversion must be carried out in two steps. The procedure is exactly analogous to the method described in Faddeyev's paper.

One further point to note about this inversion technique is that $P(\eta)$ as given by (13) utilizes only the phase function $\psi(\eta)$. The condition under which (13) is valid imply that $f(o,\eta)$ is an analytic function of η in the upper half plane. Thus a knowledge of $\psi(\eta)$ is sufficient to determine $A(\eta)$, and vice-versa. Hence, it is sufficient to specify either the magnitude or the phase for this case.

.5. Numerical Solution of Equation (10) and Examples

Using the technique described in the previous section, several examples of profile inversion have been solved. To do this, (10) must be converted to a matrix equation which can be handled numerically. Before proceeding, with the numerical formulation it is convenient to multiply (10) by $P(\gamma)$ and then, defining $\psi(\gamma,z) \equiv P(\gamma) \cdot f(\gamma,z)$, we have

$$\psi(\gamma,z) - \frac{P(\gamma)}{2\pi i} \int_{-\infty}^{\infty} \psi(\eta,z) \frac{\exp[i(\eta + \gamma)z]}{\eta + \gamma} \, d\eta = P(\gamma)e^{i\gamma z} \tag{14}$$

Investigations of the asymptotic behavior of $f(\gamma,z)$ and $P(\gamma)$ indicate that as $|\gamma| \to \infty$

$$P(\gamma) \sim 0(\frac{1}{\gamma}) \tag{15}$$

$$f(\gamma,z) \sim 0(e^{i\gamma z}) \tag{16}$$

Also, as $z \to \infty$,

$$f(\gamma,z) \sim 0(e^{i\gamma z}) \tag{17}$$

Furthermore, if the medium consists of several homogeneous layers, then $f(\gamma,0)$ will not have any branch singularities. It can also be shown that

$$f(-\gamma,z) = \overline{f(\gamma,z)} \tag{18}$$

where the bar denotes complex conjugate. These criteria have led to an approximation of $\psi(\gamma,z)$ of the form

$$\psi(\gamma,z) = e^{i\gamma z} \sum_{j=1}^{n} \frac{iA_j(z)\gamma^3 + B_j(z)\gamma^2 + iC_j(z)\gamma + D_j(z)}{\gamma^4 + P_j^4} \qquad (19)$$

(This form is not the most general rational expression that could be used, but the results obtained using this form have been quite acceptable.)

We are now ready to convert (14) to a matrix equation. First, we take the inner product of (14) with the functions $\{\delta(\gamma - \gamma_k)\}_{k=1}^{m}$. This generates m equations of the form

$$\psi(\gamma_k,z) - \frac{P(\gamma_k)}{2\pi i} \int_{-\infty}^{\infty} \psi(\eta,z) \frac{\exp[i(\eta + \gamma_k)z]}{\eta + \gamma_k} d\eta = P(\gamma_k)e^{i\gamma_k z} \quad k = 1,2,\ldots,m \qquad (20)$$

(Recall that z is a fixed parameter in (20)). Next, we substitute (19) into (20) to obtain

$$iA_j(z)[H_{kj}^3 - I_{kj}^3] + B_j(z)[H_{kj}^2 - I_{kj}^2]$$

$$+ iC_j(z)[H_{kj}^1 - I_{kj}^1] + D_j(z)[H_{kj}^0 - I_{kj}^0] \qquad (21)$$

$$= P(\gamma_k)e^{i\gamma_k z}$$

$$k = 1, 2, \ldots, m$$

where

$$H_{kj}^\ell \equiv e^{i\gamma_k z} \frac{\gamma^\ell}{\gamma^4 + P_j^4} \qquad \ell = 0, 1, 2, 3$$

$$I_{kj}^\ell \equiv \frac{P(\gamma_k)}{2\pi i} \int_{\infty}^{\infty} \left\{\frac{\eta^\ell}{\eta + \gamma_k}\right\} \left\{\frac{\exp[i(2\eta + \gamma_k)z]}{\eta^4 + P_j^4}\right\} d\eta \qquad \ell = 0,1,2,3$$

Equation (21) can be cast into the form of a matrix equation which is easily solved by numerical methods.

Several comments should be made concerning this formulation of the problem. First, (13) is used to obtain $P(\gamma)$ so that the required data are the values of the phase function $\psi(\gamma)$ at the points γ_k, $k = 1, 2, \ldots, m$. Second, the integrals defining the coefficients I_{kj}^{ℓ} can be evaluated by closing the contour in the upper half plane and using Cauchy's Residue Theorem. (The pole at $\eta = -\gamma_k$ is outside the contour and does not contribute to this evaluation). The ability to represent $\psi(\gamma, z)$ by functions for which the integrals can be analytically evaluated is important to the success of the numerical solution of this problem. Most of the examples we have solved required a matrix of order 16. The computer time needed to generate the matrix elements and to solve the matrix equation is about one second on a machine such as the IBM 360.

Another point to note about this formulation is that the substitution of (19) into (11) yields a summation of terms similar to the I_{kj}^{ℓ}. These can again be evaluated using the Residue Theorem. Finally, note that the choice of the parameters p_j used in the representation of $\psi(\gamma, z)$ is fairly easy. We simply note that $\psi(\gamma, 0)$ is known from the data about $f(\gamma, 0)$.

$$\psi(\gamma, 0) = P(\gamma) f(\gamma, 0)$$

$$= A(\gamma) e^{i\psi(\gamma)} - A(\gamma) e^{-i\psi(\gamma)} \tag{22}$$

$$= i 2 \text{Im}\{f(\gamma, 0)\}.$$

Thus, we solve the equation (21) corresponding to $z = 0$ for a few choices of the p_j's and choose the set which gives the best fit to $\psi(\gamma, 0)$.

Figure 2 indicates the type of results which have been obtained. All of these examples correspond to cases where no surface wave exists. Note that several of the inverted profiles show large deviations near $z = 0$. This difficulty seems to be linked to the numerical behavior

of the equation (21). The location and the number of poles used in the representation (19) seems to effect the behavior of the profile more near $z = 0$ than for larger z. The best results are obtained when one uses the minimum number of poles which still gives a good approximation to the specific surface fields. Figures 2c, 2e, and 2f show results obtained when the data was polluted by noise. The noise did not greatly deteriorate the results of Figures 2c and 2f, but the inversion of Figure 2c shows large deviations in the region $z < 0.5$. This result is not surprising because the profiles in Figures 2d and 2c were somewhat difficult to obtain even when no noise was present. This difficulty appears to be related to the fact that the true profile "oscillates." Mittra and Mostafavi[5] also found that surface field data could not be used to distinguish the true profile from an erroreous profile when the true profile has oscillations.

6. Another Inversion Technique

The concept of transformation operators which was introduced by Friedrichs[6] has been used as a basis for developing inversion schemes (See, for example, Faddeyev). The idea behind this process is that under appropriate conditions an operator U exists which transforms a given operator L_1 into another operator L_2. In our case, L_1 and L_2 are differential operators and the transformation operator U can be determined from $f(\gamma,0)$ in such a way that

$$L_2 \equiv U L_1 U^{-1}$$
(23)

is the operator corresponding to the profile function $Q(z)$. In other words, if ϕ_1 and ϕ_2 are eigenfunctions of L_1 and L_2, respectively, then

$$\phi_2 = U\phi_1$$
(24)

Considering the technique introduced in section II, we might say that (10) relates the eigenfunctions, $f(\gamma,z)$ to the free space eigenfunctions, $\exp(i\gamma z)$. Using techniques similar to those used to derive (10), we have developed the following relationship between $\phi_1(\gamma,z)$ and $\phi_2(\gamma,z)$ which are solutions of (4) corresponding to profile functions $Q_1(z)$ and $Q_2(z)$ and satisfying conditions (6).

$$\phi_2(\gamma,z) = \phi_1(\gamma,z) + \int_0^\infty \phi_2(\eta,z) \; [W_1(\eta) - W_2(\eta)]\psi(z,\gamma,\eta)d\eta \tag{25}$$

where

$$\psi(z,\gamma,\eta) \equiv \int_0^x \phi_1(\eta,t)\phi_1(\gamma,t)dt \tag{26}$$

$$W_i(\eta) = A_i(\eta)^{-2} = \frac{1}{|f_i(\eta,o)|^2} \qquad i = 1, \, 2. \tag{27}$$

Recall that we assume the operator L_1 to be known. This means that $Q_1(z)$, $\phi_1(\gamma,z)$ and $W_1(\eta)$ are all known. Also, $W_2(\eta)$ is known from the given value of $f_2(\eta,o)$. Thus, for fixed z, (25) is a second kind Fredholm equation for $\phi_2(\gamma,z)$. The equation (27) defining $W_i(\eta)$ was derived by establishing an analogy between (25) and the Gelfand-Levitan formulation of the inverse problem. As was the case with the method presented earlier, we could choose some fixed z, say z_0, where $\phi_2(\gamma,z_0)$ is known and not identically zero. Then (25) becomes a first kind Fredholm equation for $W_2(\eta)$. As stated in section 4, regarding (12) this procedure would be less desirable than making use of (27). Again however, (27) is valid only when no surface wave solutions satisfying (6) exist. If such surface waves are present, a two-step process is required. One further point to note is that an equation analogous to (11) can be derived to relate $Q_2(z)$ to $Q_1(z)$.

...ke the above comments more precise, we will present the equations

...lt if L_1 is the free space operator, i.e., $Q_1(z) = 0$. In this

$$\phi_1(\gamma,z) = \sqrt{\frac{2}{\pi}} \frac{\sin \gamma z}{\gamma} \tag{28}$$

$$W_1(\gamma) = 1 \tag{29}$$

...e have

$$= \sqrt{\frac{2}{\pi}} \frac{\sin \gamma z}{\gamma} +$$

$$\frac{}{} \int_0^\infty \eta\phi(\eta,z)\{1 - W(\eta)\} \{\frac{\sin(\eta - \gamma)z}{\eta - \gamma} - \frac{\sin(\eta + \gamma)z}{\eta + \gamma}\} \, d\eta \tag{30}$$

$$ 4 \sqrt{\frac{2}{\pi}} \int_0^\infty \{\frac{d\phi(\eta,z)}{dz} \sin \eta x + \eta \, \phi(\eta,z) \cos \eta z\} \{1 - W(\eta)\}\eta d\eta \tag{31}$$

...$(\eta) = A(\eta)^{-2}$. It is important to note that the integrands in (30)

...1) can be shown to be even functions of η on the real line. There-

the integrals may be taken from $-\infty$ to ∞ and the result halfed. Once

...done, the contour can be closed and the Residue Theorem can again

...to evaluate the integral.

...before, the conditions under which (27) is valid imply $f(o,\eta)$ is

...tic function of η in the upper half plane. Hence, it is sufficient

...fy either the magnitude or the phase.

...ary

...problem of remotely probing a stratified medium has been con-

...One formulation of the problem has been investigated and some

...results obtained. A second formulation is also presented.

...both of these techniques are valid at arbitrary frequency

...only as a parameter in the formulation), probing at frequencies

...face waves are present requires added computation in order to

...problem.

However, oscillatory errors do occur in the region near $z = 0$ for many cases. It should be noted, though, that the surface fields generated by these erroreous profiles do not differ significantly from those generated by the true profile. Thus, some of the difficulty is the result of the poor conditioning of this type of inverse problem.

An important feature of this method is that the numerical formulation requires a modest amount of computer storage so that these techniques might be suitable for use with measuring set-ups which have a small computer. This could be very helpful for diagnostic work in physics, biomedical applications, pollution studies and agricultural surveys.

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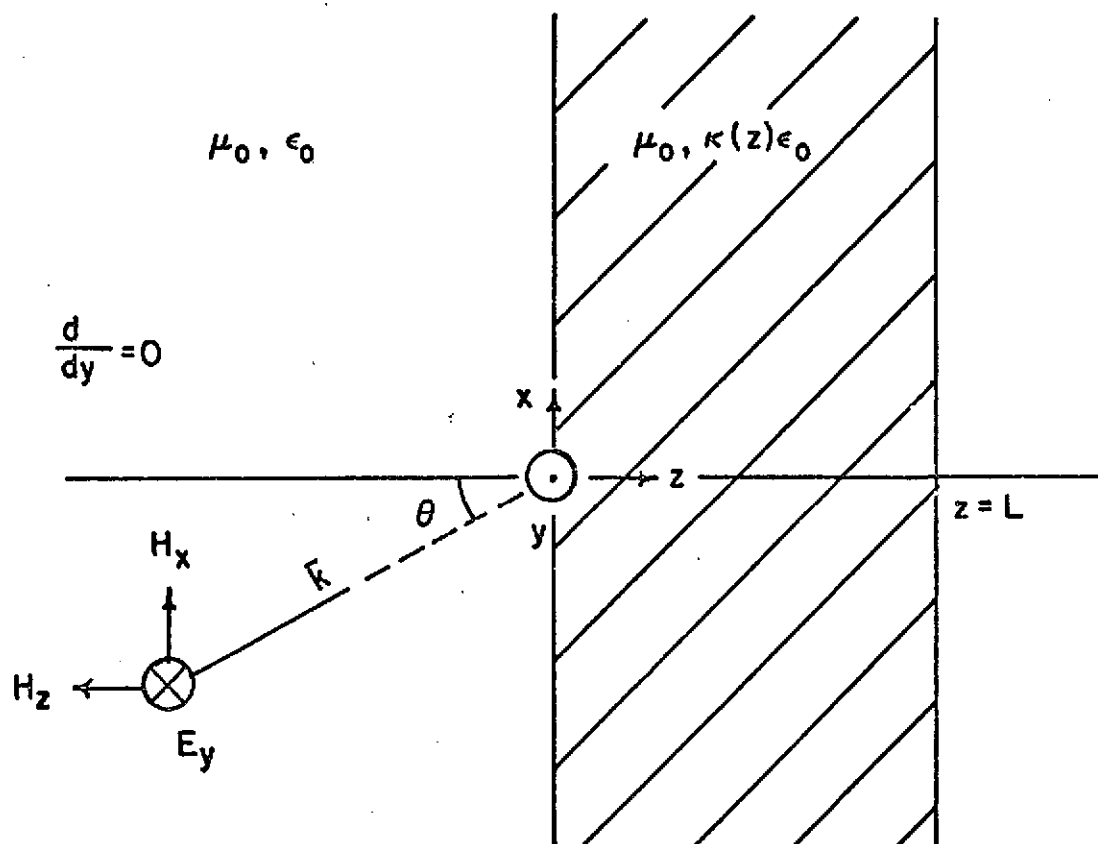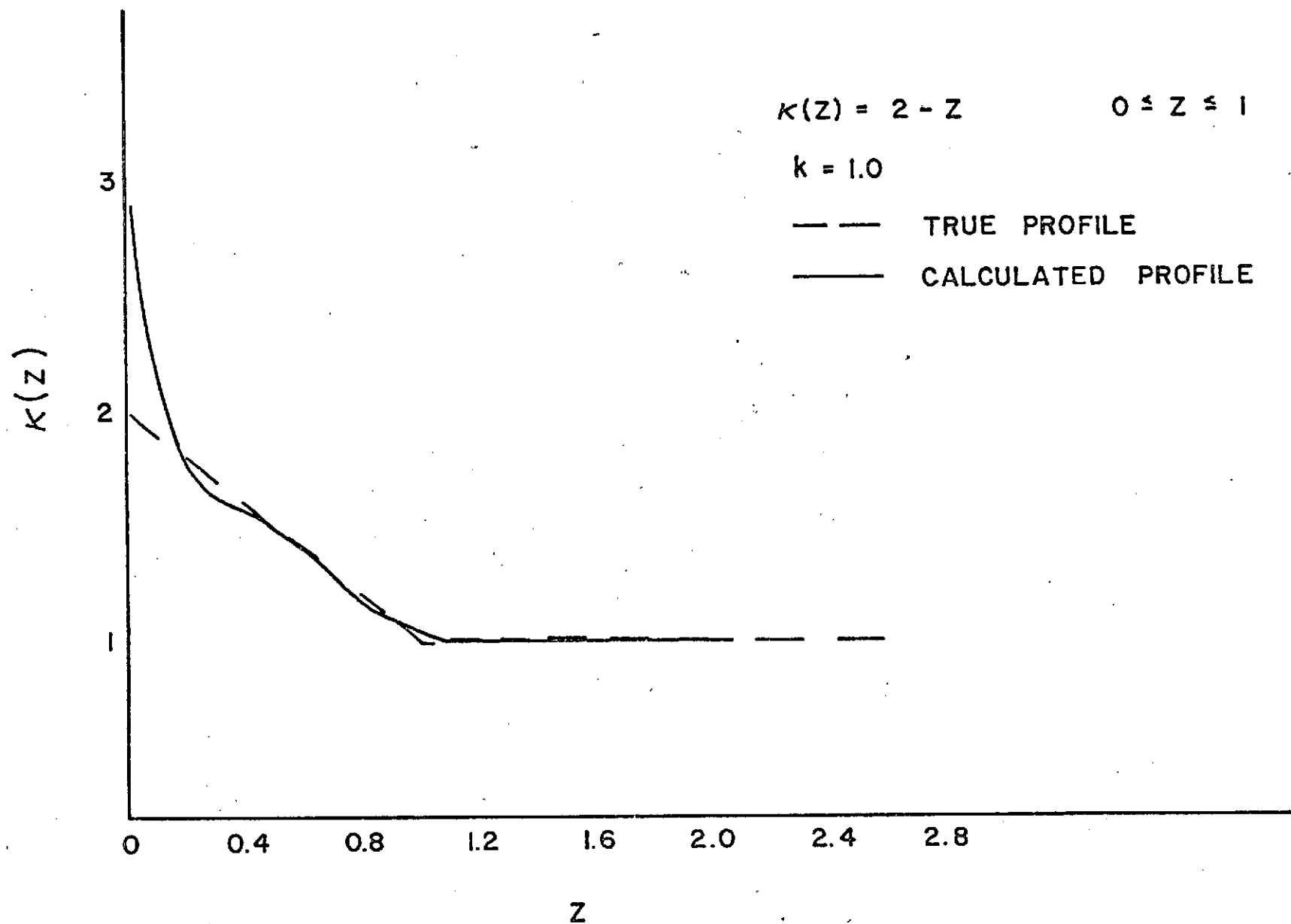
Figure 1.   Remote probing in the angular spectrum domain.

$K(Z) = 2 - Z \qquad 0 \leq Z \leq 1$

$k = 1.0$

— — TRUE PROFILE

——— CALCULATED PROFILE

Figure 2a. Inversion obtained for a linear profile.

$K(Z) = 3 - 2Z \qquad 0 \le Z \le 1$
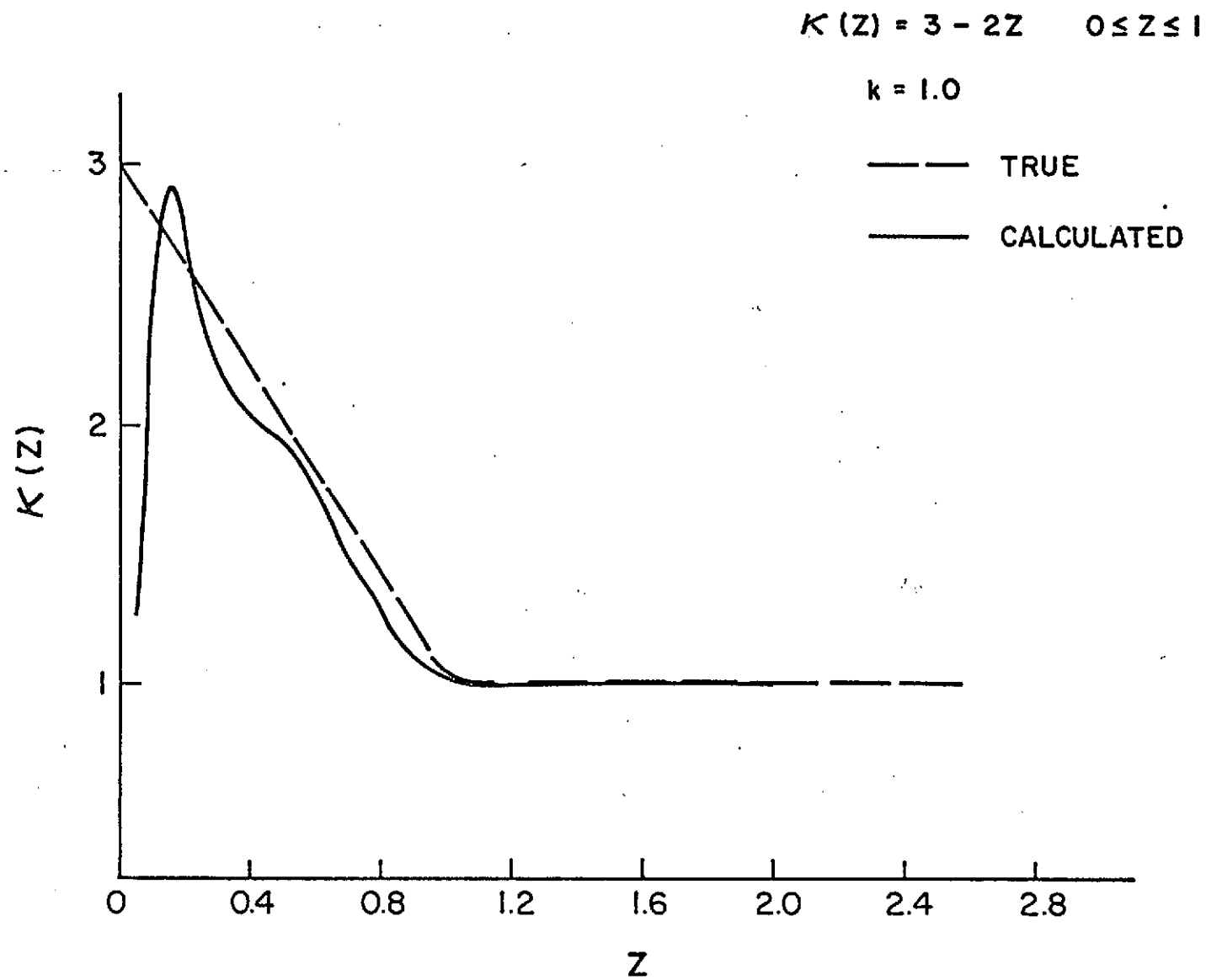
$k = 1.0$

— — TRUE

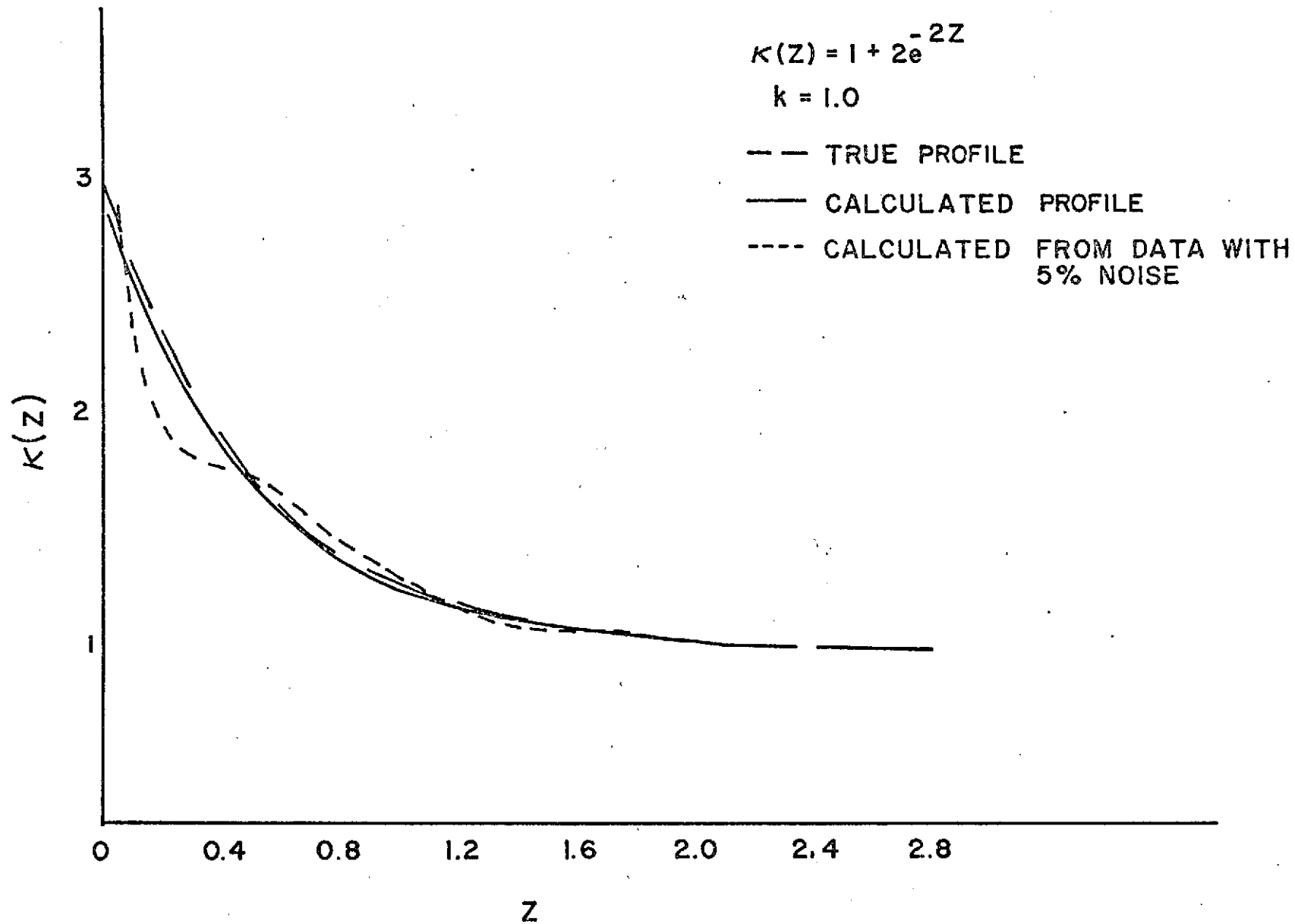——— CALCULATED

Figure 2b.   Inversion obtained for another linear profile.

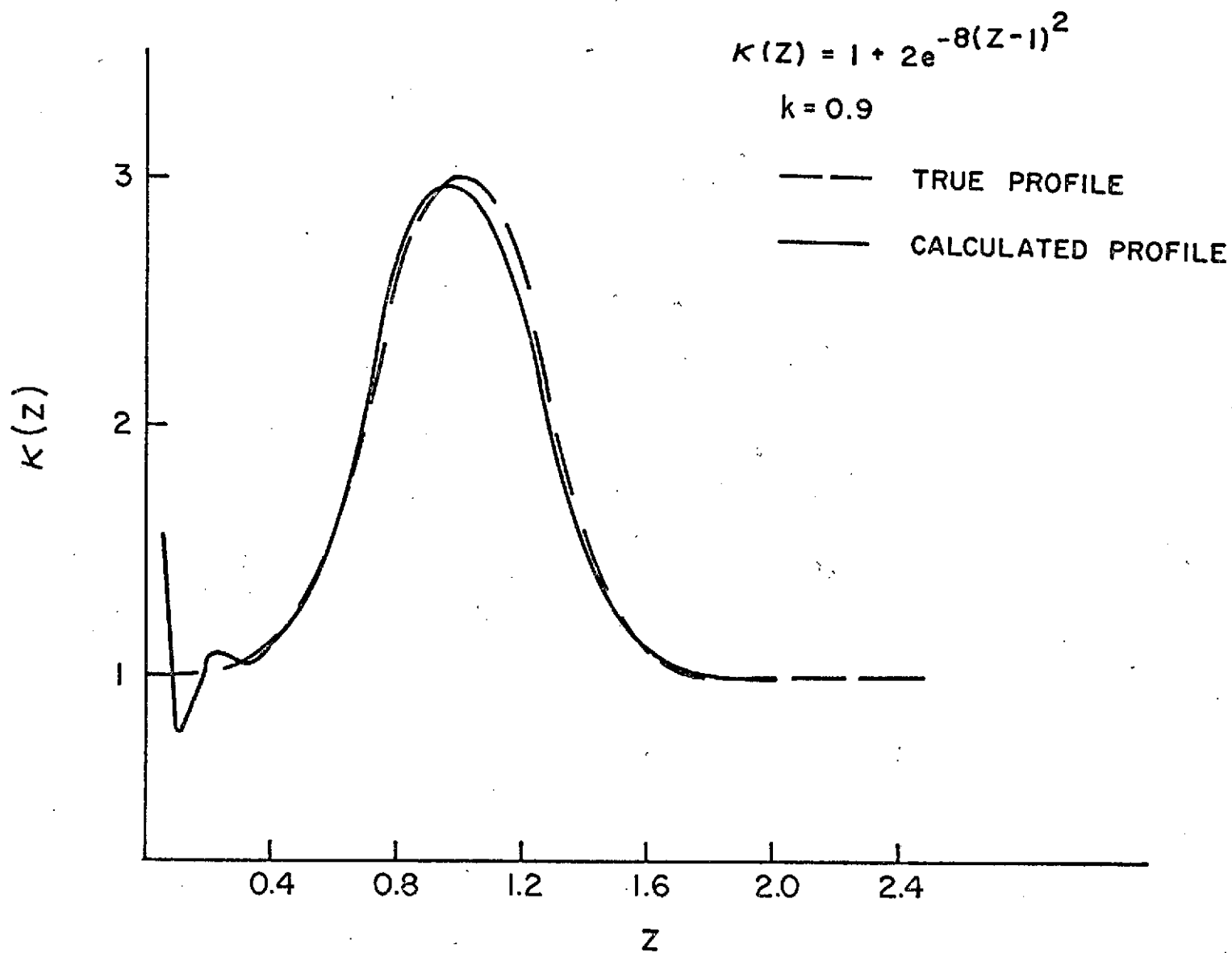Figure 2c.   Inversion obtained for an exponential profile, including case with 5% noise.

Figure 2d. Inversion obtained for another Gaussian profile.

$$\kappa(Z) = 1 + 2e^{-4(Z-1)^2}$$

$$k = 0.7$$

— — — TRUE PROFILE

———— CALCULATED PROFILE
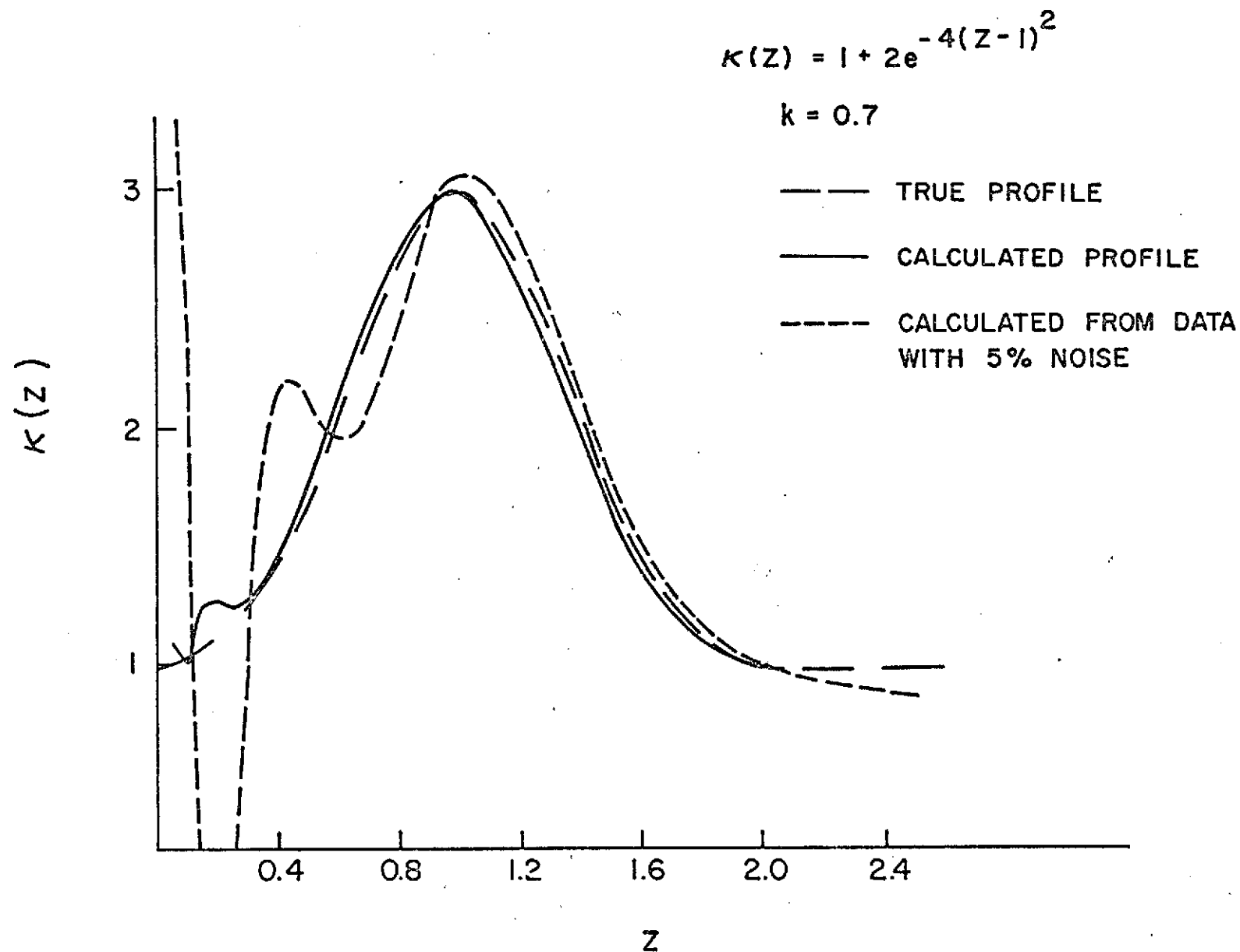
- - - - - CALCULATED FROM DATA WITH 5% NOISE

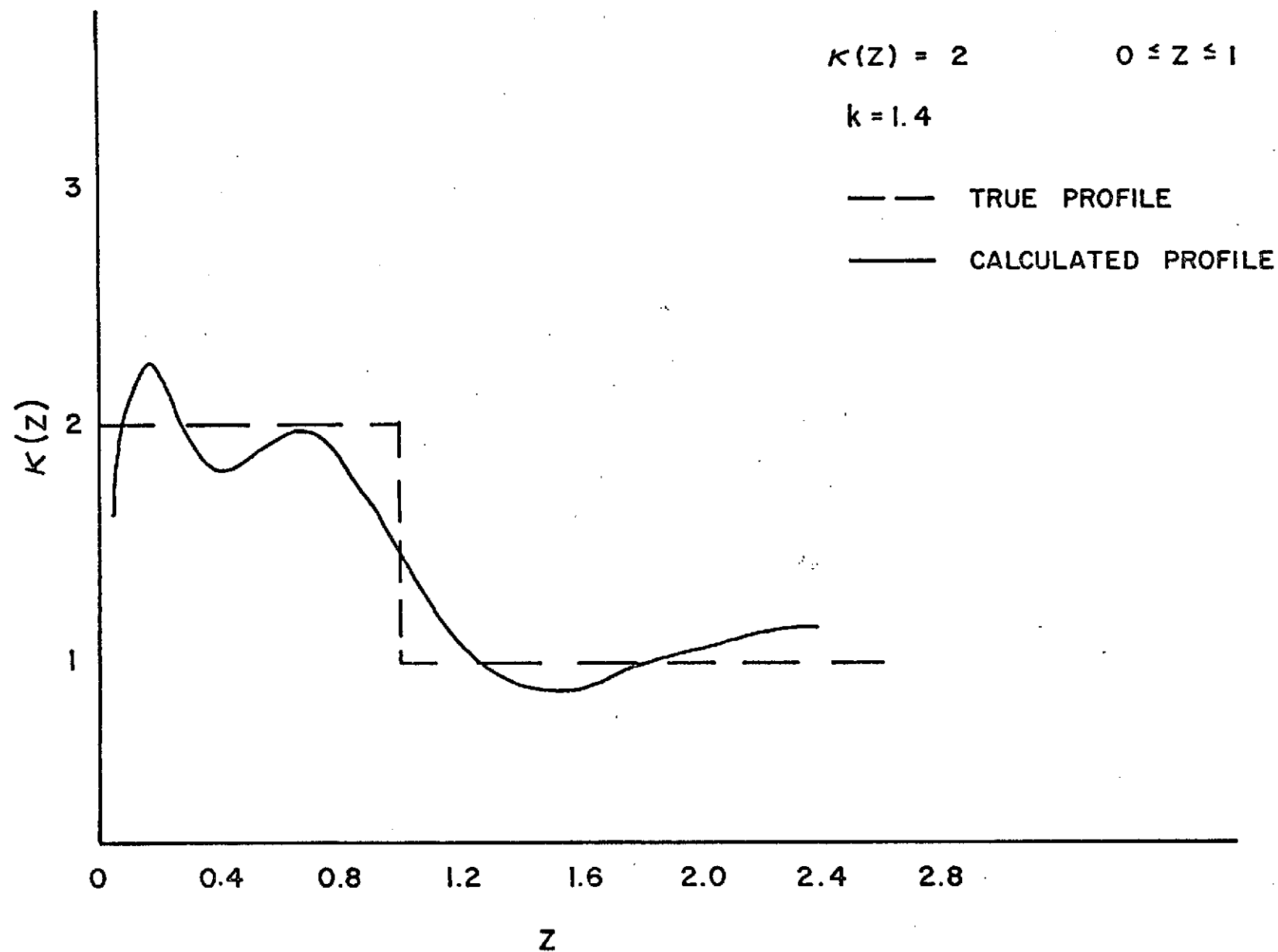Figure 2e.   Inversion obtained for a Gaussian profile.

Figure 2f. Inversion obtained for a discontinuous profile.