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## DECOUPLING IN LINEAR TIME-VARYING MULTIVARIABLE SYSTEMS

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# DECOUPLING IN LINEAR TIME-VARYING MULTIVARIABLE SYSTEMS

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## SUMMARY

This report deals with the problem of decoupling in linear multivariable systems which has received much attention recently. The necessary and sufficient condition for the decoupling of an  $m$ -input,  $m$ -output, linear time-varying dynamical system by state-variable feedback is described. The class of feedback matrices which decouple the system is characterized. Systems which do not satisfy these conditions are treated by partitioning the system. Finally, necessary and sufficient conditions for decoupling a system which has both plant and observation noise are illustrated.

## INTRODUCTION

Multivariable systems are defined as systems which have a multiplicity of inputs and outputs. Most of the problems associated with modern technology including aircraft and aerospace problems can be cast into such a framework.

Multivariable systems are characterized by coupling or interaction which usually occurs between the variables of the system. Thus, in general, any given output is affected by variation of any of the input variables of the system. Similarly, variation of each input will affect all the output variables. Considerable effort has been directed recently toward restructuring the system so that the new system is noninteracting or decoupled. Since coupling is usually inherent in any plant, decoupling is a condition which is a part of the design objective. Thus, from an input-output point of view the new system possesses non-interaction, whereas from the point of view of original plant input and output, it is still interacting.

Noninteracting multivariable systems can be conceived as consisting of a collection of individual subsystems, each of which has a single input and a single output. Dealing with single-input single-output subsystems has two advantages: (1) the problem of specifying performance requirements is simplified, and (2) each subsystem can be treated separately. With these advantages, the problem of designing to meet the needs is more tractable, because of the many design techniques available for single input, single output systems.

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Earlier work in formulating design procedures in which decoupling is required was reported by Voznesenskiy (ref. 1), Piven (ref. 2), Freeman (ref. 3), and Kavanagh (refs. 4 and 5). Horowitz (ref. 6) has discussed a practical method of synthesizing controllers to reduce the effect of plant parameter variations. Meerov (ref. 7) and Mesarovic (ref. 8) studied the structure of multivariable systems. All these contributions use the transfer-function description of the dynamic systems.

Morgan (ref. 9) in 1964 approached the problem based on the state variable representation. His main result was that the time invariant linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

can be decoupled if the matrix  $CB$  is nonsingular. Rekasius (ref. 10) extended Morgan's result and outlined an essentially trial-and-error procedure for specifying a certain number of system poles while the system is being decoupled. Falb and Wolovich (refs. 11 and 12) in 1967 established the necessary and sufficient conditions for decoupling of multivariable systems by state variable feedback. They also made definite contributions to the synthesis problem. Gilbert (ref. 13) considered the complete structure of the solution of the system.

Porter (ref. 14), Viswanadham (ref. 15), Sankaran (ref. 16), and Majumdar and Choudhury (ref. 17) extended Falb and Wolovich's result to the time-varying case. Wonham and Morse (ref. 18) considered the problem in a geometric framework and they also proposed dynamic compensators (ref. 19) to achieve a desired pole distribution for the closed-loop-system transfer matrix. Extension of the problems with disturbances and problems with discrete time systems are considered by Sankaran and Srinath (refs. 16, 20 to 22). Decoupling using output feedback has been considered by Wang (ref. 23), Howze (ref. 24), and Singh and Rugh (ref. 25). The results for a class of nonlinear systems have been reported by Nazar and Rekasius (ref. 26), Majumdar and Choudhury (ref. 27), and Tokumaru and Iwai (ref. 28). Application of decoupling theory to aircraft control problems can be seen in Cliff and Lutze (refs. 29 and 30) and in Yore (ref. 31).

The purpose of this paper is to document the thesis work presented in reference 16 which relates the requirements for decoupling multivariable systems by state variable feedback (refs. 11 and 12) to the time-varying case and includes a treatment of disturbances in the systems. The emphasis is to give reasonable in-depth treatment of the modern problems facing the decoupling techniques and to present the results to date, but certainly not all the existing techniques for decoupling multivariable systems. The

results are based on Rozonoér's paper (ref. 32) on "A Variational Approach to the Problem of Invariance of Automatic Control Systems." However, this concept is extended to the multivariable decoupling problem. This report is organized in the following manner:

First, the main results on the decoupling problem are presented. With the preliminary statement of the problem, the necessary and sufficient conditions for decoupling a linear dynamical system are given. The design approach for the feedback and feed-forward matrices is discussed and illustrated with an example problem.

Secondly, systems which do not satisfy the given conditions for decoupling are treated and necessary and sufficient conditions to partially decouple these systems are given. An example is presented to clarify the results.

Finally, systems with stochastic disturbances are considered. A concept of decoupling is introduced and definitions and theorems applicable to such problems are presented.

A bibliography of contemporary research works which concern decoupling is also given. In order to improve usability, the bibliography is arranged alphabetically.

#### SYMBOLS

$A(t)$	system matrix of dimension $n \times n$
$A^c(t)$	compensated system matrix of dimension $n \times n$
$B(t)$	input matrix of dimension $n \times m$
$B^c(t)$	compensated input matrix of dimension $n \times m$
$B_j(t)$	$j$ th column of matrix $B(t)$
$C(t)$	output matrix of dimension $m \times n$
$C_i(t)$	$i$ th row of matrix $C(t)$
$D(t)$	decoupling matrix of dimension $m \times m$
$D_i(t)$	$i$ th row of matrix $D(t)$
$d_i$	index parameter

$E\{ \}$	expected value operation on $\{ \}$
$e_i$	unit column vector with one in the $i$ th element and zero elsewhere
$F(t)$	feedback matrix of dimension $m \times n$
$G(t)$	feedforward matrix of dimension $m \times m$
$H(t)$	matrix of dimension $m \times n$
$I$	identity matrix
$\left. \begin{matrix} i,j,k, \\ m,n,s \end{matrix} \right\}$	indices
$J_i(t)$	scalar function obtained as product $C_i(t) x(t)$
$\Delta J(t)$	matrix with elements $\Delta J_{i,j}(t)$
$\Delta J_{i,j}(t)$	increment in $J_i(t)$ due to any variation $\Delta \omega_j$ in $\omega_j(\tau)$ $(t_0 \leq \tau \leq t)$
$L(t)$	matrix of dimension $m \times n$
$m_j(t)$	arbitrary functions
$P^i(t)$	matrix of dimension $n \times m$
$Q(t)$	matrix operator $\left( I \frac{d}{dt} - A - BF \right)$
$Q^k(t)$	operation of $Q$ , $k$ times
$S_{k,i}(t)$	$i$ th row of matrix $S_k(t)$ of dimension $m \times n$
$S_{k,i}^c(t)$	$i$ th row of matrix $S_k^c(t)$ of dimension $m \times n$
$t, t^*, \tau$	variable times
$t_0$	initial time

$t_1$	final time
$u(t)$	$m$ -vector representing the system input
$v(t)$	$m$ -vector representing observation noise
$w(t)$	$n$ -vector representing plant noise
$x(t)$	$n$ -vector representing the state of the system
$\hat{x}(t)$	minimum variance unbiased estimate of $x(t)$
$\tilde{x}(t)$	error in estimating $x$ as $\hat{x}$
$y(t)$	$m$ -vector representing the system output
$Z_{i,j}(t)$	scalar function obtained as product $\phi_i^T(t) B_j^c(t)$
$z(t)$	observation vector of order $m$
$\alpha_1^i(t) \dots \alpha_n^i(t)$	elements of $P^i(t)$ matrix
$\delta(\ )$	Dirac delta function
$\Lambda(t)$	arbitrary diagonal matrix of dimension $m \times m$
$\lambda_1(t) \dots \lambda_m(t)$	elements of $\Lambda(t)$ matrix
$\mu(t)$	scalar function which does not vanish simultaneously for any $t$
$\phi(t)$	adjoint system
$\omega(t)$	$m$ -vector representing reference input

A superscript  $T$  denotes the transpose of the quantity. A bar over a symbol used in partial decoupling represents partitioning of the quantity corresponding to the rank  $p$  of  $D(t)$ . A tilde over a symbol in partial decoupling represents partitioning of the quantity corresponding to  $(m - p)$  terms. Dots over symbols denote derivatives with respect to time. The symbol  $\forall$  denotes "for all values."

**PROBLEM STATEMENT AND RESULTS IN DECOUPLING  
BY STATE VARIABLE FEEDBACK**

Consider the continuous time linear dynamical system

$$\dot{x}(t) = A(t) x(t) + B(t) u(t) \tag{1}$$

$$y(t) = C(t) x(t) \tag{2}$$

where  $x(t)$  is a real  $n$ -vector representing the state of the system, and  $y(t)$  and  $u(t)$  are real  $m$ -vectors representing system output and input, respectively. The matrices  $A(t)$ ,  $B(t)$ , and  $C(t)$  are  $n \times n$ ,  $n \times m$ , and  $m \times n$  matrices, respectively. Let  $F(t)$  and  $G(t)$  be  $m \times n$  feedback and  $m \times m$  feedforward matrices for all  $t \in [t_0, t_1]$ .

A compensated system suitable for decoupling investigations using these feedback and feedforward matrices may be constructed by defining

$$u(t) = F(t) x(t) + G(t) \omega(t) \tag{3}$$

The  $m$ -component  $\omega$  vector is an external reference input to be used in controlling the  $m$ -component,  $y$ -output vector. (See fig. 1.) Different degrees of decoupling between the input and the output are defined later.

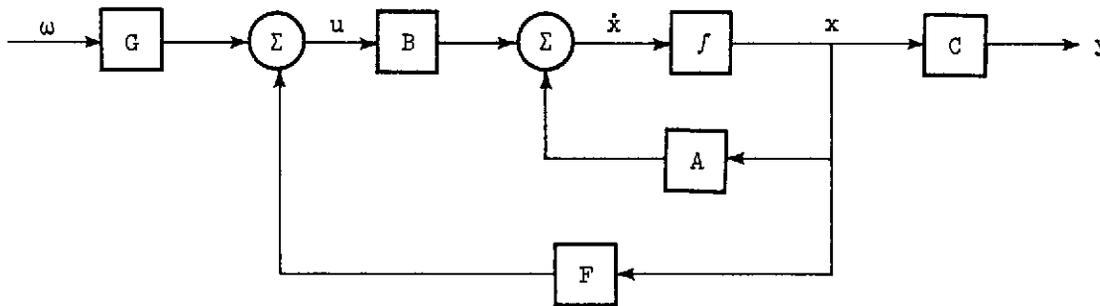


Figure 1.- Block diagram for decoupled system.

By using the state variable feedback, the compensated system can be written as

$$\dot{x}(t) = A^c(t) x(t) + B^c(t) \omega(t) \tag{4}$$

$$y(t) = C(t) x(t) \tag{5}$$

where

$$A^c(t) = A(t) + B(t) F(t) \quad (6)$$

$$B^c(t) = B(t) G(t) \quad (7)$$

It is assumed that the elements of  $A^c(t)$ ,  $B^c(t)$ , and  $C(t)$  together with their first  $(n - 2)$ ,  $(n - 1)$ , and  $(n - 2)$  derivatives, respectively, are continuous functions.

### Problem Statement

Consider the function

$$J_i(t) = C_i(t) x(t) \quad (i = 1, 2, \dots, m) \quad (8)$$

where  $C_i(t)$  is the  $i$ th row of  $C(t)$ . It follows that

$$J_i(t) = y_i(t) \quad (9)$$

Let  $\Delta J_{i,j}(t)$  be the increment in  $J_i(t)$  due to any variation  $\Delta \omega_j$  in  $\omega_j(\tau)$  ( $t_0 \leq \tau \leq t$ ). Also let  $\Delta J$  represent a matrix with elements  $\Delta J_{i,j}$ .

**Definition 1:** The system given by equations (4) and (5) is said to be uniformly decoupled for all  $t \in [t_0, t_1]$  if (1) the variation in  $y_i(t)$  due to any variation in  $\omega_j(\tau)$  ( $t_0 \leq \tau \leq t$ ) whenever  $i \neq j$  is identically zero  $\forall t \in [t_0, t_1]$ , and (2) the variation in  $y_i(t)$  due to any variation in  $\omega_i(\tau)$  ( $t_0 \leq \tau \leq t$ ) is not zero  $\forall t \in [t_0, t_1]$ . From this definition, it follows that

$$\Delta J_{i,j}(t) = 0 \quad \left( \forall t \in [t_0, t_1]; i \neq j \right) \quad (10)$$

$$\Delta J_{i,i}(t) \neq 0 \quad \left( \forall t \in [t_0, t_1]; i, j = 1, 2, \dots, m \right) \quad (11)$$

for uniformly decoupled systems.

## Main Results for Decoupling by State Variable Feedback

**Theorem 1:** For  $\Delta J_{i,j}(t)$  to be identically zero on  $[t_0, t_1]$ ,  $i, j = 1, 2, \dots, m$ ;  $i \neq j$ , it is necessary and sufficient that

$$C_i(t) Q^k(t) B_j^c(t) = 0 \quad \left( \begin{array}{l} \forall t \in [t_0, t_1]; \quad i \neq j; \\ i, j = 1, 2, \dots, m; \\ k = 0, 1, \dots, n-1 \end{array} \right) \quad (12)$$

where the matrix operator  $Q$  is given by

$$Q = I \frac{d}{dt} - A^c \quad (13)$$

$Q^k$  denotes the operation of  $Q$   $k$  times and  $B_j^c(t)$  is the  $j$ th column of  $B^c(t)$  (ref. 32).

**Proof:** The quantity  $\Delta J_{i,j}(t)$ , the variation in  $J_i$  ( $i = 1, 2, \dots, m$ ) due to a variation in  $\omega_j(\tau)$  ( $t_0 \leq \tau \leq t$ ;  $i \neq j$ ) may be shown to be given by

$$\Delta J_{i,j}(t) = - \int_{t_0}^t \phi_i^T(\tau) B_j^c(\tau) \Delta \omega_j(\tau) d\tau \quad (14)$$

where  $\phi_i(\tau)$  satisfies the adjoint equations

$$\frac{d}{d\tau} \phi_i(\tau) = -A^{cT}(\tau) \phi_i(\tau) \quad (15)$$

and

$$\phi_i(t) = -C_i^T(t) \quad (i = 1, 2, \dots, m) \quad (16)$$

where the superscript  $T$  denotes the transpose of the quantity.

Necessity: From equation (14), it is stated that equation (10) is satisfied if and only if

$$Z_{i,j}(\tau) = \phi_i^T(\tau) B_j^c(\tau) = 0 \quad \left( \forall \tau \in [t_0, t] \right) \quad (17)$$

If equation (17) is fulfilled, then the fulfillment of equation (10) is clear by virtue of equation (14). The converse is true, namely, equation (10) implies equation (17). To prove this statement, assume that equation (17) is not fulfilled at some single point  $\tau \in [t_0, t]$ , for example suppose that  $\phi_i^T(\tau) B_j^c(\tau) > 0$ . Then the continuity of the function  $Z_{i,j}(\tau)$  implies the existence of an entire segment including the point  $\tau$  on which  $\phi_i^T(\tau) B_j^c(\tau) > 0$ . Now by choosing the increment  $\Delta\omega_j(\tau)$  equal to zero outside this segment and equal, for example, to unity inside it, it can be seen that by virtue of equation (14),  $\Delta J_{i,j}(t) < 0$ , that is, equation (10) is not satisfied which is a contradiction. In a similar argument, it can be shown that equation (10) is not satisfied for  $\phi_i^T(\tau) B_j^c(\tau) < 0$ . Thus the statement has been proved.

Differentiating equation (17) and making use of equation (15) and the definition for  $Q$  result in

$$\frac{d^k Z_{i,j}(\tau)}{d\tau^k} = \phi_i^T(\tau) Q^k(\tau) B_j^c(\tau) = 0 \quad \left( \begin{array}{l} i \neq j; \quad i, j = 1, 2, \dots, m; \\ k = 0, 1, \dots, n-1 \end{array} \right) \quad (18)$$

Substitution of equation (16) into equation (18) yields

$$\left. \frac{d^k Z_{i,j}(\tau)}{d\tau^k} \right|_{\tau=t} = -C_i(t) \left[ Q^k(\tau) B_j^c(\tau) \right]_{\tau=t} = 0 \quad \left( \begin{array}{l} i \neq j, i; \quad j = 1, 2, \dots, m; \\ k = 0, 1, \dots, n-1 \end{array} \right) \quad (19)$$

Since  $t$  is arbitrary, equation (12) follows and hence the necessity is proved.

Sufficiency: From equation (12), because  $C_i(t)$  is not identically zero, it follows that the vectors  $B_j^c(t)$ ,  $QB_j^c(t)$ ,  $\dots$ ,  $Q^{n-1}B_j^c(t)$  are linearly dependent; that is, there exist functions  $\mu_0(t)$ ,  $\mu_1(t)$ ,  $\dots$ ,  $\mu_{n-1}(t)$  which do not vanish simultaneously for any  $t$  for which

$$\sum_{s=0}^{n-1} \mu_s(t) Q^s B_j^c(t) = 0 \quad (20)$$

From equations (18) and (20) it follows that the function  $Z_{i,j}(t)$  satisfies the differential equation

$$\sum_{s=0}^{n-1} \mu_s(t) \frac{d^s}{dt^s} Z_{i,j}(t) = 0$$

with the boundary condition given by equation (19). The unique function which satisfies this relation is

$$Z_{i,j}(t) = \phi_i^T(t) B_j^c(t) = 0 \quad \left( \forall t \in [t_0, t_1] \right)$$

and hence the sufficiency follows.

To apply the results of theorem 1 to the synthesis problem, it is convenient to express equation (12) in terms of the row vectors  $S_{k,i}^c(t)$  ( $k = 0, 1, \dots, n-1$ ;  $i = 1, 2, \dots, m$ ) defined by

$$\left. \begin{aligned} S_{0,i}^c(t) &= C_i(t) \\ S_{k+1,i}^c(t) &= \frac{d}{dt} S_{k,i}^c(t) + S_{k,i}^c(t) A^c(t) \end{aligned} \right\} \quad (21)$$

**Lemma 1:** Theorem 1 holds if equation (12) in the statement of the theorem is replaced by

$$S_{k,i}^c(t) B_j^c(t) = 0 \quad \left( \begin{array}{l} \forall t \in [t_0, t_1]; \quad i \neq j; \\ i, j = 1, 2, \dots, m; \\ k = 0, 1, \dots, n-1 \end{array} \right) \quad (22)$$

**Proof:** From equation (21),

$$C_i(t) Q^k(t) B_j^c(t) = \sum_{m=0}^k (-1)^m \binom{k}{m} \frac{d^{k-m}}{dt^{k-m}} \left[ S_{m,i}^c(t) B_j^c(t) \right] \quad (23)$$

From the results of the theorem 1 and equation (23), the lemma is proved.

To obtain the inequality condition of equation (11), the concepts of uniform output controllability or uniform input observability are used. By use of the dual relation presented by Silverman and Meadows (ref. 33), the following comments can be made:

Comment 1: The  $i$ th subsystem is uniformly input observable on  $[t_0, t_1]$  if and only if the rank of the vector

$$\begin{bmatrix} S_{0,i}^c(t) B_i^c(t) & S_{1,i}^c(t) B_i^c(t) & \dots & S_{n-1,i}^c(t) B_i^c(t) \end{bmatrix} \quad (24)$$

is one for all  $t \in [t_0, t_1]$ .

From the definition of uniform decoupling, if the system given by equations (4) and (5) satisfies simultaneously equation (22) and comment 1, then it is uniformly decoupled.

From the preceding discussion and theorem 1, it follows that if the  $n \times m$  matrix

$$P^i(t) = \begin{bmatrix} S_{0,i}^c(t) B_i^c(t) \\ S_{1,i}^c(t) B_i^c(t) \\ \cdot \\ \cdot \\ S_{n-1,i}^c(t) B_i^c(t) \end{bmatrix} \quad (i = 1, 2, \dots, m) \quad (25)$$

has all the columns identically zero except the  $i$ th column, that is, it has at least one non-zero element

$$P^i(t) = \begin{bmatrix} \alpha_1^i(t) \\ \alpha_2^i(t) \\ \cdot \\ \cdot \\ \alpha_n^i(t) \\ 0 & 0 & \cdot & 0 & 0 \end{bmatrix} \quad (i = 1, 2, \dots, m) \quad (26)$$

then it is uniformly decoupled.

Comment 2: The system is uniformly decoupled for all  $t \in [t_0, t_1]$ , if and only if for all  $i = 1, 2, \dots, m$ ,  $P^i(t)$  is of the form of equation (26) so that the rank of its  $i$ th column is 1 for all  $t \in [t_0, t_1]$ .

### Feedback and Feedforward Matrices

In order to obtain the requisite  $F(t)$  and  $G(t)$  consider the indices  $d_i$  defined as

$$d_i \triangleq \text{Min} \left\{ j: S_{j,i}(t) B(t) \neq 0; \forall t \in [t_0, t_1]; j = 1, 2, \dots, n-1 \right\} \quad (27)$$

where the row vectors  $S_{j,i}(t)$  ( $j = 0, 1, \dots, n-1$ ;  $i = 1, 2, \dots, m$ ) are defined by

$$\left. \begin{aligned} S_{0,i}(t) &= C_i(t) \\ S_{j+1,i}(t) &= \frac{d}{dt} S_{j,i}(t) + S_{j,i}(t) A(t) \end{aligned} \right\} \quad (28)$$

The  $d_i$  values are, in general, time dependent. But if  $S_{j,i}(t) B(t) \neq 0$  for some  $j$  and some  $t = t^*$ , then the continuity and differentiability assumptions on  $A(t)$ ,  $B(t)$ , and  $C(t)$  imply that  $S_{j,i}(t) B(t) \neq 0$  on some subinterval including the point  $t^*$ . Hence, when  $d_i$  values exist, they are constants over some subinterval of  $[t_0, t_1]$ . In the material that follows, it is assumed that the  $d_i$  values are constants over  $[t_0, t_1]$ .

Lemma 2:  $S_{k,i}^c(t) = S_{k,i}(t)$  for  $k \leq d_i$  and for all  $t \in [t_0, t_1]$ .

Proof: From equation (21) and the definition of  $S_{0,i}(t)$

$$S_{0,i}^c(t) = C_i(t) = S_{0,i}(t) \quad (29)$$

the proof can be completed by induction. Assume for some  $p < d_i$

$$S_{d_i-p,i}^c(t) = S_{d_i-p,i}(t) \quad \left( \forall t \in [t_0, t_1] \right) \quad (30)$$

Then for  $d_i - p + 1$  by equation (21),

$$S_{d_i-p+1,i}^c(t) = \frac{d}{dt} \left[ S_{d_i-p,i}^c(t) \right] + S_{d_i-p,i}^c(t) A^c(t) \quad (31)$$

From equations (30) and (6),

$$S_{d_i-p+1,i}^c(t) = \frac{d}{dt} \left[ S_{d_i-p,i}^c(t) \right] + S_{d_i-p,i}^c(t) A(t) + S_{d_i-p,i}^c(t) B(t) F(t) \quad (32)$$

$$S_{d_i-p+1,i}^c(t) = S_{d_i-p+1,i}^c(t) + S_{d_i-p,i}^c(t) B(t) F(t) \quad (33)$$

In view of equation (27) and the fact that  $d_i - p < d_i$ ,

$$S_{d_i-p+1,i}^c(t) = S_{d_i-p+1,i}^c(t) \quad (34)$$

Thus, by induction it follows that

$$S_{k,i}^c(t) = S_{k,i}^c(t) \quad \left( \forall t \in [t_0, t_1]; k \leq d_i \right) \quad (35)$$

Hence the lemma is proved.

Let

$$D_i(t) = S_{d_i,i}^c(t) B(t) \quad (i = 1, 2, \dots, m) \quad (36)$$

and let  $D(t)$  be the matrix whose  $i$ th row is given by equation (36); that is,

$$D(t) = \begin{bmatrix} S_{d_1,1}^c(t) B(t) \\ S_{d_2,2}^c(t) B(t) \\ \cdot \\ \cdot \\ S_{d_m,m}^c(t) B(t) \end{bmatrix} \quad (37)$$

Let  $\Lambda(t)$  be an arbitrary diagonal matrix, nonsingular for all  $t \in [t_0, t_1]$ , namely,

$$\Lambda(t) = \begin{bmatrix} \lambda_1(t) & & & & & \\ & \lambda_2(t) & & & & \\ & & \lambda_3(t) & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \lambda_m(t) \end{bmatrix} \quad (38)$$

and let

$$H(t) = \begin{bmatrix} S_{d_1+1,1}(t) \\ S_{d_2+1,2}(t) \\ \cdot \\ \cdot \\ S_{d_m+1,m}(t) \end{bmatrix} \quad (39)$$

**Theorem 2:** The system given by equations (4) and (5) is uniformly decoupled if and only if  $D(t)$  is nonsingular for all  $t \in [t_0, t_1]$ : Then the class of decoupling feedback matrices is given by

$$F(t) = -D^{-1}(t) H(t) \quad (40)$$

$$G(t) = D^{-1}(t) \Lambda(t) \quad (41)$$

**Proof:**

**Sufficiency:** In view of equations (27) and (35), equation (25) can be written as

$$P^i(t) = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ S_{d_i,i}^c(t) B(t) G(t) \\ \cdot \\ \cdot \\ \cdot \\ S_{n-1,i}^c(t) B(t) G(t) \end{bmatrix} \quad (i = 1, 2, \dots, m) \quad (42)$$

Now

$$S_{d_i,i}^c(t) B(t) G(t) = S_{d_i,i}(t) B(t) G(t) = D_i(t) D^{-1}(t) \Lambda(t) = e_i^T \Lambda(t) \quad \left( \forall t \in [t_0, t_1] \right) \quad (43)$$

$$S_{d_i,i}^c(t) B(t) G(t) = \lambda_i e_i^T \quad \left( \forall t \in [t_0, t_1] \right) \quad (44)$$

where  $e_i^T$  is a unit row vector with one in the  $i$ th place and zero elsewhere. From equations (36) and (40)

$$S_{d_{i+1},i}^c(t) = S_{d_{i+1},i}(t) + S_{d_i,i}(t) B(t) F(t) = S_{d_{i+1},i}(t) - D_i(t) D^{-1}(t) H(t) = 0 \quad \left( \forall t \in [t_0, t_1]; i = 1, 2, \dots, m \right) \quad (45)$$

From equations (45) and (21), it follows that

$$S_{d_{i+k},i}^c(t) = 0 \quad \left( \forall t \in [t_0, t_1]; k \geq 1 \right)$$

Hence,

$$P^i(t) = \begin{bmatrix} & & 0 & & & \\ & & \cdot & & & \\ & & \cdot & & & \\ & & \cdot & & & \\ 0 & 0 & \lambda_i(t) & 0 & \dots & 0 \\ & & \cdot & & & \\ & & \cdot & & & \\ & & \cdot & & & \\ & & 0 & & & \end{bmatrix} \quad (46)$$

Since  $\lambda_i(t) \neq 0$  for all  $t \in [t_0, t_1]$ ,  $P^i(t)$  has the form of equation (26) with the rank of its  $i$ th column 1 for all  $t$ . Hence by comment 2 the system is uniformly decoupled.

Necessity: Suppose the system is uniformly decoupled, then by comment 2,  $P^i(t)$  is of the form of equation (26), so that for each  $t \in [t_0, t_1]$ , at least one member of its  $i$ th column is nonzero. From equations (27) and (41), it follows that  $D_i(t) G(t) = \lambda_i(t) e_i^T$  where  $\lambda_i(t) \neq 0$  for all  $t \in [t_0, t_1]$  ( $i = 1, \dots, m$ ).

Thus,

$$D(t) G(t) = \text{diag} \left\{ \lambda_1(t), \lambda_2(t), \dots, \lambda_m(t) \right\} = \Lambda \quad (47)$$

The matrix  $G(t)$  is desired to be nonsingular for all  $t \in [t_0, t_1]$  in order that all the inputs influence all the outputs. Hence it follows from equation (47) that  $D(t)$  is nonsingular and hence the theorem.

The matrix  $F(t)$  obtained in equation (40) is a class of matrix to insure that  $P^i(t)$  has rank one for  $i = 1, 2, \dots, m$  and for all  $t \in [t_0, t_1]$ . As is evident, there is no choice in determining  $F(t)$ . However, by assigning arbitrary coefficients, a particular form of  $F(t)$  still insuring that the rank of  $P^i(t)$  is one is given by the following comment:

Comment 3: The rank of  $P^i(t)$  is one for all  $t \in [t_0, t_1]$  ( $i = 1, \dots, m$ ) if  $F(t) = -D^{-1}(t) L(t)$ , where

$$L(t) \triangleq \begin{bmatrix} S_{d_1+1,1}(t) + \sum_{j=0}^{d_1} m_j^1(t) S_{j,1}(t) \\ \cdot \\ \cdot \\ S_{d_m+1,m}(t) + \sum_{j=0}^{d_m} m_j^m(t) S_{j,m}(t) \end{bmatrix} \quad (48)$$

where the  $m_j^i(t)$  coefficients are arbitrary coefficients available to the analyst to insure that the compensated system is stable.

As is evident  $H(t)$  of equation (39) is a particular case of equation (48).

#### An Example Problem

Consider the system given by equations (1) and (2) where

$$A(t) = \begin{bmatrix} 1 & 4 + \cos t & -1 \\ 4 + \cos t & \sin t & 0 \\ -1 & 0 & 2 \end{bmatrix} \quad (49)$$

$$B(t) = \begin{bmatrix} e^{-t} & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 4 + \cos t & 1 \end{bmatrix} \quad (50)$$

and

$$C(t) = \begin{bmatrix} 4 + \cos t & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 4 + \cos t & 1 \end{bmatrix} \quad (51)$$

Then from equation (37)

$$D(t) = \begin{bmatrix} S_{0,1}(t) B(t) \\ S_{0,2}(t) B(t) \\ S_{0,3}(t) B(t) \end{bmatrix}$$

$$D(t) = \begin{bmatrix} (4 + \cos t)e^{-t} & 0 & 4 + \cos t \\ 0 & 4 + \cos t & 1 \\ 4 + \cos t & 4 + \cos t & 1 \end{bmatrix} \quad (52)$$

Now

$$D^{-1}(t) = \frac{1}{4 + \cos t} \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 - \frac{e^{-t}}{4 + \cos t} & \frac{e^{-t}}{4 + \cos t} \\ 1 & e^{-t} & -e^{-t} \end{bmatrix} \quad (53)$$

Because  $D$  is nonsingular, the system can be decoupled and the corresponding feedback and feedforward matrices are given by equations (40) and (41); that is,

$$F(t) = \frac{-1}{4 + \cos t} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \quad (54)$$

and

$$G(t) = \frac{1}{4 + \cos t} \begin{bmatrix} 0 & -\lambda_2 & \lambda_3 \\ -\lambda_1 & \lambda_2 - \frac{\lambda_2 e^{-t}}{4 + \cos t} & \frac{\lambda_3 e^{-t}}{4 + \cos t} \\ \lambda_1 & \lambda_2 e^{-t} & -\lambda_3 e^{-t} \end{bmatrix} \quad (55)$$

where

$$f_{11} = (4 + \cos t)^2$$

$$f_{12} = (4 + \cos t) \left( \sin t + m_0^3 \right) - \sin t$$

$$f_{13} = 4 + m_0^3 - m_0^2$$

$$f_{21} = - \left( 1 + e^{-t} + m_0^1 \right) + \frac{\sin t}{4 + \cos t} - 1 + e^{-t} (4 + \cos t)$$

$$f_{22} = -e^{-t} \frac{\left( m_0^1 + 2 \sin t \right)}{4 + \cos t} - (4 + \cos t) + \left( \sin t + m_0^3 \right)$$

$$f_{23} = -1 + \frac{m_0^2 e^{-t}}{4 + \cos t} + \frac{m_0^3 e^{-t}}{4 + \cos t}$$

$$f_{31} = -\sin t + (4 + \cos t) \left( 1 + e^{-t} + m_0^1 + (4 + \cos t) e^{-t} \right)$$

$$f_{32} = e^{-t} \left[ m_0^1 + 2 \sin t - (4 + \cos t) \left( \sin t + m_0^3 \right) \right] + (4 + \cos t)^2$$

$$f_{33} = -(4 + \cos t) + e^{-t} \left( -4 + m_0^2 - m_0^3 \right)$$

and the  $m_j^i$  and  $\lambda_j$  terms are arbitrary and may be used to insure stability of the compensated system. The compensated matrices can be obtained by using equations (6) and (7) and these relations; that is,

$$B^c(t) = \begin{bmatrix} \frac{\lambda_1}{4 + \cos t} & 0 & 0 \\ 0 & \frac{-\lambda_2}{4 + \cos t} & \frac{\lambda_3}{4 + \cos t} \\ 0 & \lambda_2 & 0 \end{bmatrix} \quad (56)$$

and

$$A^c(t) = A(t) + B(t) F(t)$$

#### Time-Invariant System

The preceding results are available for time-invariant systems (ref. 12). The necessary and sufficient condition for decoupling by state variable feedback is that the matrix  $D$  whose  $i$ th row is given by

$$D_i = C_i A^{d_i} B \quad (57)$$

be nonsingular, where

$$d_i = \text{Min} \left\{ j: C_i A^j B \neq 0; j = 0, 1, \dots, n-1 \right\} = n - 1 \quad \left( C_i A^j B = 0 \text{ for all } j \right) \quad (58)$$

The decoupling feedback and feedforward matrices  $F$  and  $G$  are given by

$$F = -D^{-1} A^* \quad (59)$$

and

$$G = D^{-1} \quad (60)$$

where

$$A^* = \begin{bmatrix} c_1 A^{d_1+1} \\ c_2 A^{d_2+1} \\ \cdot \\ \cdot \\ \cdot \\ c_m A^{d_m+1} \end{bmatrix} \quad (61)$$

### PARTIAL DECOUPLING BY STATE VARIABLE FEEDBACK

In the previous sections it was shown that for decoupling by state variable feedback, it is necessary and sufficient that the decoupling matrix  $D(t)$  is nonsingular. When such is not the case, it is desirable to decouple at least part of the output from part of the reference input.

#### Problem Statement

Consider for the system given by equations (1) and (2). Let the rank of  $D(t) = p < m$ . Partition the output vector as

$$y = \begin{bmatrix} \bar{y} \\ \tilde{y} \end{bmatrix}$$

where  $\bar{y}$  is a  $p$ -vector corresponding to the rank of  $D(t)$  and  $\tilde{y}$ , a  $m - p$  vector. Similarly, partition the input vector as

$$u = \begin{bmatrix} \bar{u} \\ \tilde{u} \end{bmatrix}$$

where  $\bar{u}$  is a  $p$ -vector and  $\tilde{u}$ , a  $m - p$  vector.

Then equations (1) and (2) can be written as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t) \mathbf{x}(t) + \begin{bmatrix} \bar{\mathbf{B}}(t) & \tilde{\mathbf{B}}(t) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}}(t) \\ \tilde{\mathbf{u}}(t) \end{bmatrix} \quad (62)$$

$$\begin{bmatrix} \bar{\mathbf{y}}(t) \\ \tilde{\mathbf{y}}(t) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{C}}(t) \\ \tilde{\mathbf{C}}(t) \end{bmatrix} \mathbf{x}(t) \quad (63)$$

where  $\bar{\mathbf{B}}$ ,  $\tilde{\mathbf{B}}$ ,  $\bar{\mathbf{C}}$ , and  $\tilde{\mathbf{C}}$  are matrices of compatible order. Let the reference input vector be  $\bar{\mathbf{w}}$  corresponding to the  $p$  inputs  $\bar{\mathbf{u}}$ . Let  $\bar{\mathbf{F}}(t)$  and  $\bar{\mathbf{G}}(t)$  be  $p \times n$  and  $p \times p$  matrices for all  $t \in [t_0, t_1]$ . Then let the input  $\bar{\mathbf{u}}$  to the system using state variable feedback be (fig. 2)

$$\bar{\mathbf{u}}(t) = \bar{\mathbf{F}}(t) \mathbf{x}(t) + \bar{\mathbf{G}}(t) \bar{\mathbf{w}}(t) \quad (64)$$

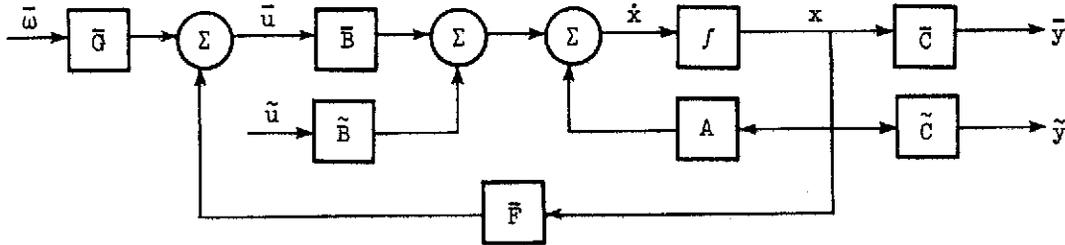


Figure 2.- Block diagram for partially decoupled system.

Consider the function

$$\bar{\mathbf{J}}_i(t) = \bar{\mathbf{C}}_i(t) \mathbf{x}(t) \quad (i = 1, 2, \dots, p) \quad (65)$$

$$\tilde{\mathbf{J}}_i(t) = \tilde{\mathbf{C}}_i(t) \mathbf{x}(t) \quad (i = 1, 2, \dots, m-p) \quad (66)$$

where  $\bar{\mathbf{C}}_i(t)$  is the  $i$ th row of  $\bar{\mathbf{C}}(t)$  and  $\tilde{\mathbf{C}}_i(t)$  is the  $i$ th row of  $\tilde{\mathbf{C}}(t)$ . From equation (52) it follows that  $\bar{\mathbf{J}}_i(t) = \bar{\mathbf{y}}_i(t)$  and  $\tilde{\mathbf{J}}_i(t) = \tilde{\mathbf{y}}_i(t)$ .

Let  $\overline{\Delta J}_{i,j}(t)$  be the increment in  $\overline{J}_1(t)$  due to any variation  $\overline{\Delta \omega}_j$  in  $\overline{\omega}_j(\tau)$  ( $t_0 \cong \tau \cong t$ ) and  $\Delta \tilde{J}_{i,j}(t)$  be the increment in  $\tilde{J}_1(t)$  due to any variation  $\Delta \tilde{\omega}_j$  in  $\tilde{\omega}_j(\tau)$  ( $t_0 \cong \tau \cong t$ ). Also let  $\overline{\Delta J}$  and  $\Delta \tilde{J}$  represent matrices with elements  $\overline{\Delta J}_{i,j}$  and  $\Delta \tilde{J}_{i,j}$ , respectively.

**Definition 2:** The system given by equations (62) and (63) is said to be uniformly partially decoupled for all  $t \in [t_0, t_1]$  if (1) the variation in  $\bar{y}_1(t)$  due to any variation in  $\bar{\omega}_j(\tau)$  ( $t_0 \cong \tau \cong t$ ;  $j \neq i$ ) is identically zero  $\forall t \in [t_0, t_1]$ , (2) the variation in  $\bar{y}_1(t)$  due to any variation in  $\bar{\omega}_1(\tau)$  ( $t_0 \cong \tau \cong t$ ) is not zero  $\forall t \in [t_0, t_1]$ , and (3)  $\|\Delta \tilde{J}(t)\|$  is minimized.

From this definition, it follows that

$$\overline{\Delta J}_{i,j}(t) = 0 \quad \left( i \neq j; \forall t \in [t_0, t_1] \right) \quad (67)$$

$$\overline{\Delta J}_{i,i}(t) \neq 0 \quad \left( \forall t \in [t_0, t_1]; i, j = 1, 2, \dots, p \right) \quad (68)$$

and

$$\|\Delta \tilde{J}(t)\| = \sup_{t \in [t_0, t_1]} \sum_{i=1}^{m-p} \sum_{j=1}^p |\Delta \tilde{J}_{i,j}(t)| \quad (69)$$

is minimized.

**Theorem 3:** For the system given by equations (67) and (68) to be uniformly partially decoupled in the sense of the preceding definitions,

(1) It is necessary and sufficient that

$$\overline{C}_1(t) Q^k(t) \overline{B}_j(t) = 0 \quad \left( \begin{array}{l} \forall t \in [t_0, t_1]; i \neq j; \\ i, j = 1, 2, \dots, p; \\ k = 0, 1, \dots, n-1 \end{array} \right) \quad (70)$$

where  $Q$  is the matrix operator  $\left[ I \frac{d}{dt} - A - \overline{BF} \right]$  and

(2) To minimize the cross-coupling effect, it is sufficient that

$$\text{Sup}_{t \in [t_0, t_1]} \sum_{i=1}^{m-p} \sum_{j=1}^p \left| \tilde{C}_i(t) Q^k(t) \overline{B}_j(t) \right| \quad (k = 0, 1, \dots, n-1) \quad (71)$$

is minimized.

**Proof:** The proof follows along the lines of theorem 1 and can be found in reference 16.

**Comment 4:** Equivalently, for the system to be uniformly partially decoupled, (1) it is necessary and sufficient that  $\overline{D}(t)$  whose  $i$ th row

$$\overline{D}_i(t) = S_{d_{i,i}}(t) \overline{B}(t) \quad (i = 1, 2, \dots, p) \quad (72)$$

is nonsingular  $\left( \forall t \in [t_0, t_1] \right)$  and (2) to minimize the cross-coupling effect it is sufficient that

$$\left\| \overline{\tilde{D}}(t) \right\| = \left\| S_{d_1}(t) \overline{B}(t) \right\| \quad (i = p + 1, \dots, m) \quad (73)$$

is minimum where the norm is selected as given by equation (69).

In order to relate these results to the  $D(t)$  of the original system (1) and (2), partition  $D(t)$  as follows:

$$D(t) = \begin{bmatrix} \overline{D}(t) & \tilde{\tilde{D}}(t) \\ \tilde{\tilde{D}}(t) & \tilde{D}(t) \end{bmatrix} \quad (74)$$

where  $\overline{D}(t)$  is a  $p \times p$  matrix of rank  $p$   $\left( \forall t \in [t_0, t_1] \right)$ ,  $\tilde{D}(t)$  is  $m - p \times m - p$  matrix,  $\tilde{\tilde{D}}(t)$  is  $m - p \times p$  matrix, and  $\tilde{\tilde{D}}(t)$  is  $p \times m - p$  matrix. The matrix  $\overline{D}(t)$

is invertible and hence the design procedure for  $\bar{F}(t)$  and  $\bar{G}(t)$  is identical to ordinary decoupling.

It may be noted that the rows of  $D(t)$  correspond to the output  $y(t)$  and the columns of  $D(t)$  correspond to the input  $u(t)$ . This fact gives us an idea of what outputs and inputs to choose for decoupling purposes.

Comment 5: The classes of decoupling matrices of equation (64) are given by

$$\bar{F}(t) = -\bar{D}^{-1}(t) \bar{L}(t) \quad (75)$$

$$\bar{G}(t) = \bar{D}^{-1}(t) \bar{\Lambda}(t) \quad (76)$$

where

$$\bar{L}(t) = \begin{bmatrix} s_{d_1+1,1}(t) + \sum_{j=0}^{d_1} \bar{m}_j^1(t) s_{j,1}(t) \\ \vdots \\ s_{d_p+1,p}(t) + \sum_{j=0}^{d_p} \bar{m}_j^p(t) s_{j,p}(t) \end{bmatrix} \quad (77)$$

where  $\bar{m}_j^i$  terms are arbitrary coefficients and  $\bar{\Lambda}(t)$  is a diagonal matrix with arbitrary coefficients and is nonsingular ( $\forall t \in [t_0, t_1]$ ).

#### An Example Problem

As an example, consider the system given by

$$A(t) = \begin{bmatrix} 1 & 4 + \cos t & -1 \\ 4 + \cos t & \sin t & 0 \\ -1 & 0 & 2 \end{bmatrix} \quad (78)$$

$$B(t) = \begin{bmatrix} e^{-t} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 4 + \cos t & 1 \end{bmatrix} \quad (79)$$

$$C(t) = \begin{bmatrix} 4 + \cos t & e^{-t} & 0 \\ 0 & 0 & 1 \\ 0 & 4 + \cos t & 1 \end{bmatrix} \quad (80)$$

Then

$$D(t) = \begin{bmatrix} (4 + \cos t)e^{-t} & 0 & 4 + \cos t \\ 0 & 4 + \cos t & 1 \\ 0 & 4 + \cos t & 1 \end{bmatrix} \quad (81)$$

and  $D(t)$  is singular for all  $t \in [0, \infty)$ . Hence the system cannot be decoupled. However, for  $t \in [0, \infty)$ , the rank of  $D(t)$  is two and so two of the outputs can be decoupled from the reference inputs and the cross-coupling effects minimized by using these results.

Without loss of generality, consider the outputs to be decoupled to be  $y_1$  and  $y_2$ . Then,

$$\bar{C}(t) = \begin{bmatrix} 4 + \cos t & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (82)$$

and

$$\tilde{C}(t) = [0 \quad 4 + \cos t \quad 1]$$

In order to minimize  $\bar{D}$  in the sense of equation (69), that is, the cross coupling between the output  $y_3$  and the reference inputs  $\omega_1$  and  $\omega_2$ , assign the inputs  $u_1$  and  $u_3$  to  $\omega_1$  and  $\omega_2$ ; that is,

$$\bar{D}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad (83)$$

Hence

$$\bar{D}(t) = \begin{bmatrix} (4 + \cos t)e^{-t} & 4 + \cos t \\ 0 & 1 \end{bmatrix} \quad (84)$$

$$\bar{E}(t) = \begin{bmatrix} e^{-t} & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (85)$$

and

$$\bar{B}(t) = \begin{bmatrix} 0 \\ 0 \\ 4 + \cos t \end{bmatrix} \quad (86)$$

From this discussion

$$\bar{u}(t) = \bar{F}(t) x(t) + \bar{G}(t) \omega(t) \quad (87)$$

where

$$\bar{u}(t) = \begin{bmatrix} u_1(t) \\ u_3(t) \end{bmatrix} \quad (88)$$

$$F(t) = -\frac{1}{4 + \cos t} \begin{bmatrix} \bar{f}_{11} & \bar{f}_{12} & \bar{f}_{13} \\ \bar{f}_{21} & \bar{f}_{22} & \bar{f}_{22} \end{bmatrix} \quad (89)$$

$$\bar{G}(t) = \begin{bmatrix} \frac{\bar{\lambda}_1(t)}{(4 + \cos t)e^{-t}} & -\frac{\bar{\lambda}_2(t)}{e^{-t}} \\ 0 & \bar{\lambda}_2(t) \end{bmatrix} \quad (90)$$

where

$$\left. \begin{aligned} \bar{f}_{11} &= \left\{ (4 + \cos t) \left[ 2 + e^{-t} + \bar{m}_0^1(t) \right] - \sin t \right\} e^t \\ \bar{f}_{12} &= \left[ \bar{m}_0^1(t) + \sin t - 1 \right] + e^t (4 + \cos t)^2 \\ \bar{f}_{13} &= -e^t (4 + \cos t) \left[ 3 + \bar{m}_0^2(t) \right] \\ \bar{f}_{21} &= -(4 + \cos t) \\ \bar{f}_{22} &= 0 \\ \bar{f}_{23} &= \left[ 2 + \bar{m}_0^2(t) \right] (4 + \cos t) \end{aligned} \right\} \quad (91)$$

and the  $m_j^i$  and  $\lambda_j$  terms are arbitrary and may be used to insure stability of the partially compensated system. The corresponding compensated system matrices are given by

$$A^c(t) = \begin{bmatrix} a_{11}^c & a_{12}^c & 0 \\ a_{21}^c & a_{22}^c & 0 \\ 0 & 0 & a_{33}^c \end{bmatrix} \quad (92)$$

where

$$a_{11}^c = -\left(\overline{m}_0^1 + e^{-t}\right) + \frac{\sin t}{4 + \cos t}$$

$$a_{12}^c = \frac{-\left(\overline{m}_0^1 + \sin t - 1\right)e^{-t}}{4 + \cos t}$$

$$a_{21}^c = 4 + \cos t$$

$$a_{22}^c = \sin t$$

$$a_{23}^c = \overline{m}_0^2$$

and

$$\overline{B}^c(t) = \begin{bmatrix} \frac{\bar{\lambda}_1(t)}{4 + \cos t} & 0 \\ 0 & 0 \\ 0 & \bar{\lambda}_2(t) \end{bmatrix} \quad (93)$$

#### STOCHASTIC DECOUPLING BY STATE ESTIMATOR VARIABLE FEEDBACK

In the previous sections decoupling of linear multivariable system by state feedback and a transformation of the input is considered. Even though output feedback can be used for decoupling, some of the flexibility obtained with state variable feedback as in specifying closed loop poles, and so forth, will, in general, be lost. If the states are not directly accessible, a possible approach is to construct an observer to reconstruct the inaccessible states to generate the control law. In this section stochastic decoupling in linear multivariable system with known plant and observation noise is considered. A definition of decoupling applicable to this and a procedure for achieving such decoupling are presented.

Consider the linear plant and observation model

$$\dot{x}(t) = A(t) x(t) + B(t) u(t) + w(t) \quad (94)$$

$$y(t) = C(t) x(t) \quad (95)$$

$$z(t) = y(t) + v(t) \quad (96)$$

where  $x(t)$  is a real  $n$ -vector representing the state of the plant,  $y(t)$ ,  $u(t)$ , and  $z(t)$  are real  $m$ -vectors representing the plant output and input and observation, respectively.  $A(t)$ ,  $B(t)$ , and  $C(t)$  are matrices of compatible dimensions. The vectors  $w(t)$  and  $v(t)$  are plant and observation noise vectors that are assumed to be zero mean and Gaussian with

$$\left. \begin{aligned} \text{Cov}\{w(t), w(\tau)\} &= Q(t) \delta(t - \tau) \\ \text{Cov}\{v(t), v(\tau)\} &= R(t) \delta(t - \tau) \\ \text{Cov}\{w(t), v(\tau)\} &= S(t) \delta(t - \tau) \end{aligned} \right\} \quad (97)$$

and  $x(t_0)$  is a Gaussian random variable with known mean and covariance which is independent of  $w(t)$  and  $v(t)$  for all  $t \geq t_0$ . Let  $\hat{x}(t)$  be the minimum variance estimate of  $x(t)$ , and let the control law be given by

$$u(t) = F(t) \hat{x}(t) + G(t) \omega(t) \quad (98)$$

where the  $m$ -vector  $\omega(t)$  is the reference input and  $F(t)$  and  $G(t)$  are matrices of compatible dimensions.

Consider the random functions,

$$J_i(t) = C_i(t) x(t) \quad (i = 1, 2, \dots, m) \quad (99)$$

where  $C_i$  is the  $i$ th row of  $C(t)$ . Let  $\Delta J_{i,j}(t)$  be the increment in  $J_i(t)$  due to any variation  $\Delta \omega_j$  in  $\omega_j(\tau)$  ( $t_0 \leq \tau \leq t$ ) and let  $\Delta J$  represent the matrix with elements  $\Delta J_{i,j}$ .

**Definition 3:** The system given by equations (94) to (96) is said to be stochastically decoupled, if (1) the expected variation in  $y_i(t)$  due to any variation in  $\omega_j(\tau)$  ( $t_0 \leq \tau \leq t$ ) if  $i \neq j$  is zero; and (2) the expected variation in  $y_i(t)$  due to any variation in  $\omega_i(\tau)$  ( $t_0 \leq \tau \leq t$ ) is not zero.

Mathematically,

$$E \left\{ \Delta J_{i,j}(t) \right\} = 0 \quad (i \neq j) \quad (100)$$

and

$$E \left\{ \Delta J_{1,i}(t) \right\} \neq 0 \quad \left( i, j = 1, 2, \dots, m; \forall t \in [t_0, t_1] \right) \quad (101)$$

Theorem 4: For the stochastic control problem to satisfy the equality relation of equation (88), it is necessary and sufficient that

$$C_i(t) Q^k(t) B_j(t) = 0 \quad \left( \begin{array}{l} i \neq j; \quad i, j = 1, 2, \dots, m; \\ k = 0, 1, \dots, n-1; \\ \forall t \in [t_0, t_1] \end{array} \right) \quad (102)$$

where  $Q$  is the matrix operator  $\left( I \frac{d}{dt} - A - BF \right)$  and  $B_j(t)$  is the  $j$ th column of  $B(t)$ .

Proof: Because

$$\hat{x} = x - \tilde{x} \quad (103)$$

where  $\tilde{x}$  is the error in estimating  $x$  as  $\hat{x}$ , then

$$\dot{\tilde{x}}(t) = (A + BF)x + BG\omega + w - BF\tilde{x} \quad (104)$$

If  $\hat{x}$  represents an unbiased estimate,

$$E \left\{ \tilde{x}(t) \right\} = 0$$

The rest of the proof follows along the lines of theorem 1.

Comment 6: For the stochastic control problem to be decoupled, it is necessary and sufficient that  $D(t)$  whose  $i$ th row

$$D_i(t) = S_{d_i,i}(t) B(t) \quad (i = 1, 2, \dots, m) \quad (105)$$

is nonsingular  $\left( \forall t \in [t_0, t_1] \right)$ .

The classes of feedback and feedforward matrices are the same as for ordinary decoupling. Also the results can be extended in a straightforward manner to the case when the  $D$  matrix is singular to obtain a partially decoupled system.

### CONCLUDING REMARKS

The necessary and sufficient conditions for decoupling, time-varying, continuous-time, linear-multivariable systems have been presented in this paper. A constructive method for determining the required feedback and feedforward matrices was described and illustrated in an example problem.

For systems which do not satisfy the necessary and sufficient conditions, a method for accomplishing partial decoupling was presented and illustrated through an example problem. Finally, a definition of decoupling applicable to linear systems with known plant and observation noise and a procedure for decoupling such systems were given.

Although the procedures discussed in this report apply to linear continuous-time systems, it is obvious that with minor modification they would apply equally well to discrete-time systems.

In spite of the generality of the present theory, there remain a number of important problems for future investigation. Probably the most challenging of these is to extend the present theory to a broader class of systems. Although efforts have been made to develop a decoupling theory for nonlinear systems and distributed systems, the considerable difficulties of these generalizations of the problem so far have inhibited substantial progress.

Another important endeavor would be to develop meaningful questions regarding the sensitivity of decoupled systems. Efforts in this direction were made by the author and presented in the Proceedings IEEE Trans., Feb. 1973 (pp. 241-242). In fact, it is clear that in practice no physical system can be decoupled exactly. Obviously, what is needed is a useful and precise definition for an "approximately" decoupled system. Quantitative considerations such as this one are of considerable practical importance and almost nothing is known about them.

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