STREAMLINE CURVATURE DESIGN PROCEDURE
FOR SUBSONIC AND TRANSONIC DUCTS

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SUMMARY

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INTRODUCTION

When the design parameters (length, maximum diameter, and velocity variation) of a duct permit a very gradual area variation, the one-dimensional approximation (ref. 1) can be used for the design. Such is not the case, however, for many applications, such as wind-tunnel contraction sections, nozzles, and engine inlet diffusers for which there are severe restraints on allowable dimensions. For ducts of this type the radial variation of the flow parameters is often not negligible, and so the one-dimensional approximation does not suffice.

If a duct is intended to be used for low-speed applications, it can be designed by a generalization (see ref. 2 and its references) of an incompressible flow method originated by Tsien (ref. 3). The usual procedure for the compressible case is to utilize a series expansion in powers of some parameter (refs. 4 to 7), but such solutions rapidly increase in complexity with increasing number of terms, and therefore must be truncated after a small number of terms. For example, the solution of reference 5, which utilizes an expansion in powers of $r^2$, becomes unmanageably complicated even for the first five terms.

The problems associated with series expansion solutions are avoided by the method of reference 8, which can be used as a design procedure. That method is a relatively complicated direct numerical attack on the problem, involving a continuous transformation of coordinates and a finite-difference solution of the differential equations working outward from the center line.

The present analysis, in comparison, makes use of the streamline curvature equations, which represent a natural formulation of the equations for this type of problem.
inasmuch as any streamline can be taken to be a wall contour. The streamline curvature equations have also been used for the design of centrifugal impellers (ref. 9). The present method differs from that of reference 9 in several respects:

1. The design velocity distribution is specified on the axis rather than on an inner hub profile
2. The equations are formulated directly in terms of the streamline curvature rather than the radius of curvature, which becomes singular at an inflection point of a streamline
3. The graphical method of reference 9 is replaced by a mathematical model which can be programmed for machine calculation

The streamline curvature equations, as used in the present analysis, represent a form of the exact, compressible, inviscid flow equations. The method is therefore applicable from low subsonic to supersonic speeds, but no treatment of shocks is included. The equations, and the computer program in the appendix, have been written for the axisymmetric case, but the analysis can be applied to problems of two-dimensional design simply by replacing the factor $2\pi r$ in the continuity equation with the width of the duct. Then a symmetric duct can be designed by specifying a velocity distribution along a center line, or one wall can be specified together with a velocity distribution on the wall.

**SYMBOLS**

The International System of Units (SI) is used in this study.

- $a$: streamline curvature, $d\varphi/ds$
- $C$: contraction ratio
- $c$: speed of sound
- $M$: Mach number
- $\dot{m}$: mass flow rate
- $n$: distance normal to streamlines
- $q$: total velocity
- $s$: distance along streamline
- $2$
V  center-line design velocity

X,R  cylindrical coordinates nondimensionalized with respect to estimated entrance diameter

x,r  cylindrical coordinates

γ  ratio of specific heats

ρ  density

φ  \( \tan^{-1} \frac{dr}{dx} \) for a curve \( r = r(x) \)

Subscripts:

c  center line

i  ith point on a given streamline

k  kth streamline

t  total conditions

wall  at wall

A bar over a symbol indicates average value: \( \frac{(\cdot)_k + (\cdot)_{k+1}}{2} \).

ANALYSIS

Flow Equations

When the circumferential component of velocity is neglected, the compressible inviscid streamline curvature equations for axisymmetric flow consist of the continuity equation in integrated form,

\[
\dot{m} = 2\pi \int_0^\infty \rho q r \, dn
\]

(1)
and the velocity equation,

\[ q = q_c \exp\left( \int_{0}^{n} a \, dn \right) \]  

(2)

where the density is given by

\[ \rho = \rho_t \left[ 1 - \frac{\gamma - 1}{2} \frac{q}{c_t} \right]^{2/(\gamma - 1)} \]  

(3)

and the curvature by

\[ a = \frac{d\varphi}{ds} \]  

(4)

(See ref. 9.) The streamline curvature is the rate of change of flow angle with distance along the streamline; it is the reciprocal of the radius of curvature.

**Method of Solution**

The calculation is initiated by prescribing a velocity distribution along the center line. For purposes of calculation the velocity on the kth streamline is determined by the approximation

\[ q_k = q_{k-1} e^{a \Delta n} \]  

(5)

The mass flow between streamlines is estimated by

\[ \Delta \dot{m} = 2\pi r \rho \dot{q} \Delta n \]

In order to compute the curvature \( a \), \( \Delta \varphi \) is determined as the change in angle of slope of the straight lines joining the ith point to the i - 1 point and to the i + 1 point and then \( \Delta \varphi \) is divided by the corresponding \( \Delta s \).

The equations were solved by an iterative numerical technique. First, the shape of a stream tube having a small maximum diameter is estimated from the one-dimensional flow relations using the assumed center-line velocity distribution. The normals to this
streamline are determined, as well as its slope and curvature distributions. These functions permit the velocity and density distributions to be computed on this streamline by means of equations (5) and (3). Next, the mass flow is computed through each normal and compared with the required mass flow for this stream tube. The streamline coordinates are then adjusted slightly in the direction that will reduce the error in mass flow, the error is computed, and the coordinates are readjusted, and so on until the desired mass flow is obtained within a reasonable tolerance. This procedure is carried out at each point on the streamline. Then the curvature of this streamline is computed and used to obtain a new estimate for \( \bar{\alpha} \) to be used in equation (5). With this new estimate the entire procedure is repeated to obtain a better estimate for the streamline. The velocity along this streamline is then taken as input data to compute the next streamline, which is first estimated from the one-dimensional equations, in the same manner as for the previous streamline.

The successive streamlines are computed similarly until a specified mass flow is obtained, unless the solution becomes unstable before the mass flow condition is met. In such case the last stable streamline can be taken as the wall contour, unless the solution displays an unacceptable pressure gradient along the wall.

Stability of Convergence

One drawback to the use of the stream curvature equations is the tendency of the solutions to become unstable relatively easily. This tendency is a direct result of the use of the streamline curvature. A small change in \( r \) can result in a relatively large change in the curvature. Since the curvature is in the exponent in equation (2), the velocity is sensitive to the curvature and thus to small changes in \( r \).

This sensitivity can be counteracted to a considerable extent by the manner in which the calculation is performed. In the present method, a smooth distribution of curvature is used in the initial estimate for each streamline. Then when the actual curvature is computed, it is not used directly but is averaged with the curvature of the previous streamline for use in equation (5). This procedure tends to stabilize the convergence and has been used in the calculations of this paper.

There is an alternate procedure which is mathematically more stable, but it requires somewhat more computing time. If many streamlines are computed, the following approximation will apply in equation (2):

\[
\int_0^n a \, dn \approx \sum_{k=1}^{k_n} a_k(\Delta n)_k \approx \sum_{k=1}^{k_n-1} a_k(\Delta n)_k
\]
Thus the coordinates of each streamline can be computed from the distributions along the previous streamline, and it is not necessary to determine the coordinates of a given streamline from its own curvature by iteration.

**Design Velocity Distributions**

Some of the existing design analyses (e.g., refs. 3 and 5) prescribe the design axial velocity distribution in terms of a particular analytic expression. Where high-order derivatives are required, this analytic expression must be selected so that these derivatives do not become unmanageably complicated.

The present method, on the other hand, is not restricted to a particular type of design velocity distribution inasmuch as the input data for the calculation can be in the form of values specified at discrete x-locations on the axis. This system facilitates "experimenting" with various designs by making small local variations in the design distribution. However, the input data should be reasonably smooth in order to obtain wall shapes without excessive curvature, as well as to insure stable convergence of the calculation. For some designs requiring large values of the stream function and rapid flow changes, the design velocity must be extremely smooth. This requirement can be met by inputting a smooth second derivative and integrating it twice.

The selection of a design velocity distribution need not be simply a cut-and-try process. The one-dimensional equations are often useful to provide an estimate of the flow relations. Moreover, in those cases where both the upstream and downstream flows are uniform, the one-dimensional equations give correct relationships between the flow values in these two regions, even though the flow between these regions is nonuniform (provided, of course, that no separation takes place).

For example, suppose that a wind-tunnel contraction cone is to be designed for an exit Mach number of 1 and a contraction ratio of C. Then the uniform entering velocity would be determined by the equation

\[
\frac{1}{C} = \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{2}} \left(\frac{V}{c_t}\right)^2 \left[1 - \frac{\gamma - 1}{\gamma - 1} \left(\frac{V}{c_t}\right)^2\right]^{1/(\gamma - 1)}
\]

(See ref. 10, eq. (82).)

Since in the entering flow \( V/c_t \) is generally much less than 1, the nonlinear term can be neglected in an approximation, and so, for \( \gamma = 1.4 \), \( V/c_t = 125/216C \). The velocity at the exit would be given by \( V/c_t = \sqrt{5/6} \) (ref. 10, eq. (68)).
Examples

Three sample cases have been computed: a wind-tunnel contraction section, a nozzle, and a diffuser. The contraction section is shown in figure 1 together with several streamlines. The most critical part of such a duct is that near the exit, where the flow becomes parallel and, correspondingly, the wall curvature becomes zero. A wall which is otherwise smooth but has a rapid decrease in curvature incurs an adverse pressure gradient at the wall, with the possibility of flow separation and the certainty of a severe radial velocity gradient. For the example shown, the curvature variation is gradual, so that the flow is fairly uniform radially, as indicated.

The other critical region for a contraction section is near the wall at the large end, where the slow-moving flow enters a region of negative curvature. This adverse gradient may separate the flow locally. However, for the sake of obtaining a short contraction section, some separation might be tolerated in this region with the expectation that the subsequent favorable acceleration section will reduce flow nonuniformities.

A nozzle design is shown in figure 2. Observe that where the wall curvature is slight, the variation of Mach number along a streamline normal is slight, although the duct may be sharply contracting. On the other hand, the radial variation is relatively large in the throat, where the curvature is greater. The supersonic Mach number distribution at the wall is smooth because of the smooth variation of wall curvature. This result may be compared with that obtained with a cone—circular-arc—cone nozzle, which displays a drop in wall Mach number at the curvature discontinuity in the expanding section of the nozzle (see ref. 11, fig. 1).

Figure 3 shows a design for a subsonic diffuser. This case is more sensitive than that of the contraction section because the negative curvature at the wall occurs where the flow is already decelerating. Since this critical region is near the terminal end of the diffuser, the results of any flow separation will be seen at the end of the diffuser.

To illustrate the influence of the curvature on the solution, the wall Mach number distribution for the nozzle is plotted in figure 4, as calculated from both one-dimensional and streamline curvature equations. The differences are larger in the vicinity of the throat, where the Mach number is relatively sensitive to flow geometry. The maximum difference in Mach number is about 0.06.

CONCLUDING REMARKS

A computerized procedure that makes use of the inviscid streamline curvature equations has been developed for designing compressible-flow ducts from an assumed
center-line velocity distribution. Three examples are given: a wind-tunnel contraction section, a nozzle, and a diffuser.

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APPENDIX

COMPUTER PROGRAM

The program language is FORTRAN 4. The input, by the NAMELIST format, consists of the following quantities:

- **X**: an array of axial positions for which velocities are specified
- **Q**: an array of the design velocity values
- **RMX**: the desired radius at \( X(1) \)
- **ROI**: \( \rho_t \)
- **CT**: \( c_t \)
- **KMX**: number of streamlines to be computed
- **N**: number of \( X \) and \( Q \) values input

Before beginning the calculation the program prints the design center-line velocity distribution with the headings:

- **J**: point index
- **X**: axial coordinate
- **Q**: velocity

After computing each streamline the program prints the results for that streamline with the following headings:

- **PT.**: point index
- **ITER.**: number of iterations required to satisfy continuity at Jth point
- **X**: axial coordinate
APPENDIX – Continued

R radial coordinate

SLOPE streamline slope

CURV. streamline curvature

VEL. velocity along streamline

P/PT. local pressure divided by total pressure

M local Mach number

Computing times for the three sample cases ranged from 2 to 5 seconds.

```plaintext
PROGRAM MAKADUC_INPUT, OUTPUT, TAPE 1=INPUT
DIMENSION X(50), Q(50), S(50), P(50), R(J), S(50), R(J), S(50), SK(50)
1 READ/X, R(J), S(50), P(50), R(J), S(50)
READ INPUT AND PRINT VELOCITY DISTRIBUTION. NONDIEMENSIONALIZE VEL.
1 READ(N, INPUT) \$F(EOF, 1272)
2 PT=3.14159 \$K=1
PRINT 40
40 FORMAT(1H1, *4X1HJ, 11X1X, 11X1H/)
PRINT 41(JX(J), Q(J), J=1,N)
NO X J=1,N \$O(J)=X(J)/CT \$SK(J)=0 \&O(J) \#0(J)=0 \&SK(J)=0
41 FORMAT(110, 2F12.6)
6 \$C(J)=0
FOR KTH STREAMLINE, COMPUTE PARAMETERS AND INITIAL ESTIMATES FOR
COORDINATES AND SLOPES
7 K=FLOAT(K)*RM/FLOAT(KM)*AI=PT*(R(J)*R(J)-R(J)*R(J)) & O(J) J=1,N \$O(J)=O(J) \&0(J)=0(J)
R(J)=R(J)*R(J)+R(J)*R(J) & A=Q(J)*AI/RA(J)*RA(J)
8 R(J)=SORT(A/PT+R(J)*R(J)) \$SL(J)=0 \$O(J)=0 \&J=2,N
C COMPUTE NORMALS TO KTH STREAMLINE
9 SL(J)=(R(J)-R(J-1))/(X(J)-X(J-1)) & DO 14 J=2,N
IF(J.NE.1) GO TO 13
IF(SL(J).*G.T.0) \$AND SL(J+1).*L.T.0) GO TO 10
IF(SL(J).*G.T.0) \$AND SL(J+1).*G.T.0) GO TO 11
IF(SL(J).*L.T.0) \$L(J+1).*G.T.0) GO TO 13
10 SM=SL(J) \$K=R(J-1) \$X=X(J-1) \$RL=R(J) \$XL=X(J) \$GO TO 12
11 SM=SL(J+1) \$R=R(J) \$X=X(J) \$XL=X(J) \$RL=R(J+1)
12 IF(SM.*G.T.0) \$X(J)=SM/(SM*SM+1)*R(J)-R(J)*SM*X(J)/SM
IF(SM.*L.T.0) \$X(J)=R(J)+SM/(SM+1)*X(J)
X(J)=R(J)+SM/(SM+1) \$GO TO 14
13 H(J)=R(J) \$X(J)=X(J)
14 CONTINUE \$X(J)=0 \$R(J)=R(J) \$SK(J)=0
```

10
APPENDIX — Concluded

C COMPUTE CURVATURE, VELOCITY, DENSITY, AND MASS FLOW
LL=1
15 00 17 J=2,N SL=1
16 IF(LL.EQ.1.AND.K.*GT.1) SK(J)=SKT(J)*FLOAT(K)/FLOAT(K-1)
   ON=SORT((X(J)-X(J))**2+*PR(JJ)-RT(JJ))**2) $SKM=SK(J)+SKT(J)**5
   0LJ=01(J)*EXP(SKM*DN)
   GM=01(J)+(J)**5 $ROM=(1,-2*ROM*ROM)**2,5
   PM=5*R(J/J)+R(J) $FMM=2.*PI*ROM*ROM*ROM*ROM=FM-FMM/FM
   IF(ABS(FR).LT.0.00001)GO TO 17
C ADJUST COORDINATES OF KTH STREAMLINE ALONG NORMAL TO SATISFY
C CONTINUITY
R(J)=-SQRT((X(J)-X(J-1))**2+*PR(JJ)-RT(JJ))**2)
IF(R(J).LT.0.00001)RT(J)=P(J(J-1))
IF(J.EQ.2)X(J(J)-1)=R(J(J)
SL(J)=(R(J)-X(J(J)-1))/(X(J(J))-X(J(J-1))
17 CONTINUE
44 FORMAT(7,4P1.5)
C RECOMPUTE CURVATURE FOR ADJUSTED COORDINATES AND REPEAT
C CALCULATION FOR KTH STREAMLINE
LL=LL+1 IF(LL.GT.99)GO TO 27
GO TO 16
18 SL(J)=R(J)-X(J(J)-1)/(X(J(J))-X(J(J-1))
91=N-1 90 10 J=2,N
   IF((R(J(J)-X(J(J)-1))*PR(J(J)-1)-RT(J(J)))*GE.0)GO TO 19
   R(J(J))=SK(J(J)+1)+R(J(J-1))
   SL(J)=(R(J(J)-1)-R(J/J))/X(J(J))-X(J(J-1))
19 IF(J.LT.2)SL(J)=L
20 CONTINUE $SK(1)=.5 $SK(N)=.0
20 23 J=2,N
   SS=5*|SORT((R(J(J)-X(J(J)-1))**2+(X(J(J))-X(J(J-1)))*0)+SORT((R(J(J))**2)
   I=SK(J(J)+1)+X(J(J)-1)-X(J(J))**2)
   DFL=ATAN(SL(J(J)))-ATAN(SL(J(J)))
   SK(J(J))=DFL+SK(J(J))**5
23 CONTINUE 24 J=2,N1 $TF(SK(J(J))$SK(J(J))-6.AND.SK(J(J))$SK(J(J))
   LT.0)SK(J(J)=SK(J(J)+1)$SK(J(J))=5*SK(J(J)+1)*SK(J(J)
24 CONTINUE $SK(1)=.0 $SK(2)=SK(J(J))**5 $GO TO 15
C NONDIMENSIONALIZE AND PRINT RESULTS FOR KTH STREAMLINE, SHIFT
C INDEX, AND PROCEED TO NEXT STREAMLINE
25 PRINT 42
42 FORMAT(11H1,5XSHITLEX,5X3HT,5X1HX,12X1H0,9XSHSLOP1,10XSHCURV,1
   110X4HVL,9X4HP/PT,12X1HM)
26 29 J=2,N
   Q(J)=X(J,J) $XJ(J)=X(J(J)) $SK(J(J)=SK(J)
   $P=1-2*X(J(J))$XM=X(J(J)/SORT(ROD) $SPRT=ROD**3.5
   SK=SK(J(J)+1)**5 $SX=X(J(J)/(2.*ROD)
23 FORMAT(7PA7,6)
   PRINT 43 $LSX=SL(J(J),SK(J(J))$Q(J+1)+ROD*XM
   IF(K.EQ.KMX)0$J(J)=0
26 $HT(J)=R(J(J)+1)+R(J(J)+1)$SK=K+1 $TF(K,LF,KMX)GO TO 7 $GO TO 1
27 STOP $END
REFERENCES


Figure 1.- Wind-tunnel contraction section with streamlines and design velocity distribution.
Figure 2.- Nozzle with streamlines and design velocity distributions.
Figure 3.- Diffuser with streamlines and design velocity distribution.
Figure 4.- Mach number distribution along wall of nozzle shown in figure 2, as obtained by streamline curvature calculation and by one-dimensional theory.
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