APPENDIX C

NUMERICAL SIMULATION OF WEATHER AND CLIMATE

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INTERACTION OF A CUMULUS CLOUD ENSEMBLE

WITH THE LARGE-SCALE ENVIRONMENT

by

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2. Modification of the large-scale environment by cumulus clouds

Consider a horizontal area at some level between cloud base and the highest cloud top. This horizontal area, shown schematically in Fig. 1, which we designate as our unit horizontal area, must be large enough to contain an ensemble of cumulus clouds but small enough to cover only a fraction of a large-scale disturbance. The existence of such an area is one of the basic assumptions of this paper.

Because we are not concerned here with acoustic waves, the mass continuity equation can be simplified to its quasi-Boussinesq form,

$$\nabla \cdot (\rho \underset{\sim}{V}) + \frac{\partial}{\partial z}(\rho w) = 0, \tag{1}$$

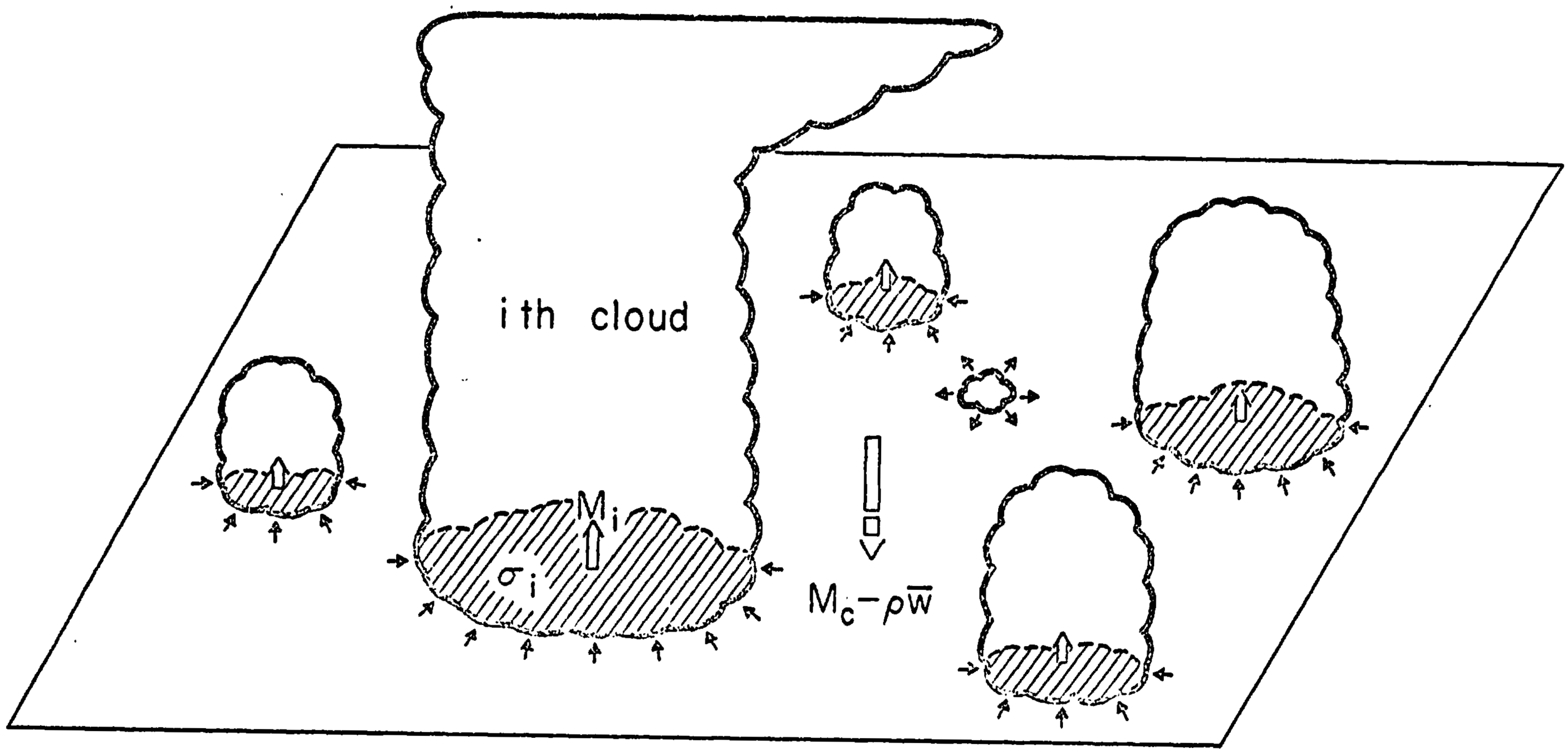
where the density ρ is a function of height only, $\underset{\sim}{V}$ is the horizontal velocity, ∇ is the horizontal del operator, w is the vertical velocity and z is the vertical coordinate.

Let $\sigma_i(z,t)$ be the fractional area covered by the ith cloud, in a horizontal cross-section at level z and time t. The vertical mass flux through σ_i is

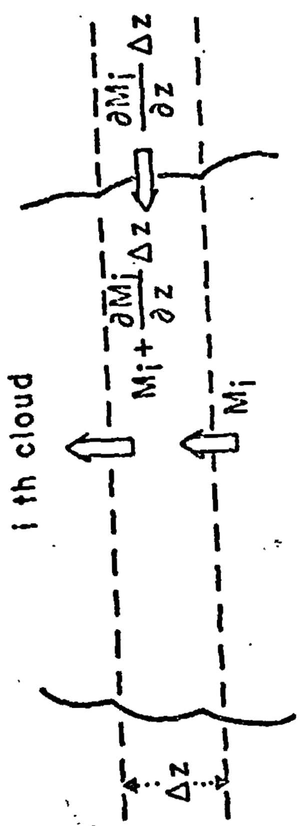
$$M_i = \int_{\sigma_i} \rho w \, d\sigma = \rho \sigma_i w_i , \tag{2}$$

where $\int_{\sigma_i} d\sigma$ is the area integral over the area σ_i and w_i is the average vertical velocity of the ith cloud at this level.

The inward mass flux per unit height, normal to the boundary of the ith cloud, is given by $\partial M_i/\partial z$ from the mass continuity equation (1) (See Fig. 2.)



i th cloud
M_i
σ_i
$M_c - \rho\overline{w}$


i th cloud
M_i
$M_i + \frac{\partial M_i}{\partial z} \Delta z$
$\frac{\partial M_i}{\partial z} \Delta z$
Δz

2

Here the boundary is not necessarily vertical. Then the mass added to the cloud, which may be horizontally expanding or shrinking, is $\partial M_i/\partial z + \rho\,\partial\sigma_i/\partial t$ per unit height and unit time. The entrainment and detrainment of mass are given by

$$\text{entrainment:} \qquad E_i = \left(\frac{\partial M_i}{\partial z} + \rho\frac{\partial \sigma_i}{\partial t}\right), \quad \text{when} \quad \frac{\partial M_i}{\partial z} + \rho\frac{\partial \sigma_i}{\partial t} > 0; \tag{3}$$

$$\text{detrainment:} \qquad D_i = -\left(\frac{\partial M_i}{\partial z} + \rho\frac{\partial \sigma_i}{\partial z}\right), \quad \text{when} \quad \frac{\partial M_i}{\partial z} + \rho\frac{\partial \sigma_i}{\partial t} < 0. \tag{4}$$

E_i can be rewritten as $\sigma_i \partial(\rho w_i)/\partial z + \rho(\partial/\partial t + w_i \partial/\partial z)\sigma_i$. Thus, an entrainment of mass, which is originally caused by turbulent mixing at the cloud boundary, appears either as a vertical divergence of the mass flux within the cloud, or as a horizontal expansion of the cloud as it rises, or as a combination of both, depending on the dynamics of the cloud.

The total vertical mass flux by all of the clouds in the ensemble is

$$M_c = \sum_i M_i, \tag{5}$$

where $\sum$ denotes the summation over all clouds which are penetrating the level being considered.

Let $\rho\bar{w}$ be the net vertical mass flux for the large-scale horizontal unit area. It satisfies the continuity equation

$$\overline{\nabla\cdot(\rho \mathbf{V})} + \frac{\partial}{\partial z}(\rho\bar{w}) = 0, \tag{6}$$

where the bar denotes the average over the unit horizontal area. In general, the total vertical mass flux in the clouds, M_c , is not the same as the large-scale net vertical mass flux, $\rho\bar{w}$. The difference between M_c and $\rho\bar{w}$ is equal to the downward mass flux between the clouds,

$$-\widetilde{M} = M_c - \rho\overline{w}. \tag{7}$$

With sufficiently intense cumulus activity, M_c can exceed $\rho\overline{w}$ and subsidence, $\widetilde{M} < 0$, appears in the environment, as in Fig. 1.

At a given height, some clouds may be detraining while other clouds are entraining (see Fig. 1). We define total entrainment E and total detrainment D, at each level, by

$$E = \sum_{e.c.} E_i \tag{8}$$

and

$$D = \sum_{d.c.} D_i, \tag{9}$$

respectively. Here $\sum_{e.c.}$ denotes the summation over all clouds which are entraining at that level, and $\sum_{d.c.}$ denotes the summation over all clouds which are detraining at that level. E and D, as well as M_c, are functions of z. From (3), (4), (5), (8) and (9), we obtain

$$E - D = \frac{\partial M_c}{\partial z} + \rho\frac{\partial \sigma_c}{\partial t}, \tag{10}$$

where

$$\sigma_c \equiv \sum \sigma_i \tag{11}$$

is the total fractional area covered by all the clouds of the ensemble.

We define the static energy by

$$s = c_p T + gz, \tag{12}$$

where c_p is the specific heat of air under constant pressure, T is the temperature and g is gravity. $c_p T$ is the specific enthalpy of the air, and gz is its geopotential per unit mass. The static energy s is approximately conserved by the individual air parcel during dry adiabatic processes. Hydrostatic balance, which we assume for the environment, gives

$$\frac{\partial s}{\partial z} = c_p \left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} \frac{\partial \theta}{\partial z}, \tag{13}$$

where p is the pressure, p_0 is a standard pressure, R is the gas constant, and $\theta \equiv (p_0/p)^{\frac{R}{c_p}} T$ is the potential temperature.

From the budgets of static energy and water vapor in the environment we obtain

$$\frac{\partial}{\partial t}\left[(1-\sigma_c)\rho \tilde{s}\right] = -\overline{\nabla \cdot (\rho \underset{\sim}{v} s)} - E\tilde{s} + \sum_{d.c.} D_i s_{Di} - \frac{\partial}{\partial z}(\tilde{M}\tilde{s}) - L\mathcal{E} + \tilde{Q}_R, \tag{14}$$

$$\frac{\partial}{\partial t}\left[(1-\sigma_c)\rho \tilde{q}\right] = -\overline{\nabla \cdot (\rho \underset{\sim}{v} q)} - E\tilde{q} + \sum_{d.c.} D_i q_{Di} - \frac{\partial}{\partial z}(\tilde{M}\tilde{q}) + \mathcal{E}, \tag{15}$$

where the bar denotes the horizontal average over the large-scale unit area, the tilda denotes the value in the environment (which is assumed to be horizontally homogeneous for all of the variables) and the suffix Di the value in the detraining air from the ith cloud. L is the latent heat per unit mass of water vapor. $\mathcal{E}$ is the evaporation of the liquid water detrained from the clouds per unit height, and $\tilde{Q}_R$ is the radiational heating of the environment per unit height. q is the mixing ratio of water vapor. The first three terms on the right of (14) and (15) come from the horizontal area integrations of $-\nabla\cdot(\rho \underset{\sim}{v} s)$ and $-\nabla\cdot(\rho \underset{\sim}{v} q)$ over the environment. $-\partial(\tilde{M}\tilde{s})/\partial z$ and $-\partial(\tilde{M}\tilde{q})/\partial z$ represent the convergence of vertical fluxes in the environment.

Using (10), (9), (7) and (6), Eqs. (14) and (15) may be rewritten as

$$(1-\sigma_c)\rho\frac{\partial \tilde{s}}{\partial t} = \sum_{d.c.} D_i(s_{Di}-\tilde{s}) - L\mathcal{E} - \tilde{M}\frac{\partial \tilde{s}}{\partial z} - \left[\overline{\nabla\cdot(\rho \underset{\sim}{v} s)} - \overline{\nabla\cdot(\rho \underset{\sim}{v})}\,\tilde{s}\right] + \tilde{Q}_R, \tag{16}$$

$$(1-\sigma_c)\rho\frac{\partial \tilde{q}}{\partial t} = \sum_{d.c.} D_i(q_{Di}-\tilde{q}) + \mathcal{E} - \tilde{M}\frac{\partial \tilde{q}}{\partial z} - \left[\nabla\cdot(\rho \underset{\sim}{v} q) - \nabla\cdot(\rho \underset{\sim}{v})\,\tilde{q}\right]. \tag{17}$$

To simplify Eqs. (16) and (17), we use the empirical fact that the total fractional area covered by active clouds is small compared to unity. (See, for example, Malkus et al., 1961.) This is consistent with the theoretical finding first obtained by Bjerkı (1938), that conditional instability favors the smallest possible horizontal cross section for rising motion and the largest possible horizontal cross section for sinking motion, if there is no friction nor entrainment. Asai and Kasahara (1967) found that cumulus convection most efficiently transports heat upwards, and therefore most efficiently releases kinetic energy, when the fractional horizontal area of the rising motion is of the order of several per cent. It seems that

$$\sigma_c \ll 1 \tag{18}$$

is an acceptable first approximation when we are concerned with only active cumulus clouds.

By definition, s and q averaged horizontally over the large-scale unit area are given by

$$\bar{s} = (1-\sigma_c)\tilde{s} + \sum \sigma_i s_i, \tag{19}$$

$$\bar{q} = (1-\sigma_c)\tilde{q} + \sum \sigma_i q_i, \tag{20}$$

or using (11),

$$\bar{s} = \tilde{s} + \sum (s_i - \tilde{s})\sigma_i, \tag{21}$$

$$\bar{q} = \tilde{q} + \sum (q_i - \tilde{q})\sigma_i . \tag{22}$$

$(s_i - \tilde{s})$ and $(q_i - \tilde{q})$ are the excess static energy and the excess mixing ratio of water vapor in the ith cloud over the environmental values. Using the assumption (18) and the empirical fact that the temperature difference between the cloud and the environment is small, or $max(s_i - \tilde{s}) \ll \tilde{s}$, we obtain, from (21)

$$\bar{s} \fallingdotseq \tilde{s}. \tag{23}$$

However, the corresponding inequality, $max(q_i - \tilde{q}) \ll \tilde{q}$, does not hold when the environment is not near saturation. Therefore, we use instead

$max(q_i - \tilde{q}^*) \ll \tilde{q}^*$, where $\tilde{q}^*$ is the saturation mixing ratio of water vapor in the environment. Using this in (22), we obtain

$$\bar{q} \doteqdot \tilde{q} + (\tilde{q}^* - \tilde{q})\,\sigma_c$$
$$= \tilde{q}\left[1 + \frac{1-r}{r}\sigma_c\right], \tag{24}$$

where r is the relative humidity of the environment, $\tilde{q}/\tilde{q}^*$. When $r \gg \sigma_c/(1+\sigma_c) \doteqdot \sigma_c$, that is, when the environment is not very dry, the second term in the bracket of (24) can be neglected. Then,

$$\bar{q} \doteqdot \tilde{q}. \tag{25}$$

We further assume that

$$\overline{\nabla\cdot(\rho \underset{\sim}{V})} \doteqdot \nabla\cdot(\rho \bar{\underset{\sim}{V}}), \tag{26}$$

$$\overline{\nabla\cdot(\rho \underset{\sim}{V} s)} \doteqdot \nabla\cdot(\rho \bar{\underset{\sim}{V}}\,\bar{s}), \tag{27}$$

$$\overline{\nabla\cdot(\rho \underset{\sim}{V} q)} \doteqdot \nabla\cdot(\rho \bar{\underset{\sim}{V}}\,\bar{q}). \tag{28}$$

Here the bar on the right hand sides denotes a running horizontal space average, on the scale of the unit area, and not just the average within the fixed area. The approximations (26) through (28) are valid when the net lateral horizontal transports across the boundary of the fixed large-scale area by cumulus convection (the horizontal cumulus eddy transports) are negligible compared to the horizontal transports by the large-scale motion.

Using the approximations (18), (23), and (25) through (28), Eqs. and (17) may be rewritten as

$$\rho \frac{\partial \bar{s}}{\partial t} = \sum_{d.c.} D_i (s_{Di} - \bar{s}) - L\mathcal{E} - \tilde{M} \frac{\partial \bar{s}}{\partial z} - \rho \bar{\underset{\sim}{V}} \cdot \nabla \bar{s} + \tilde{Q}_R , \tag{29}$$

$$\rho \frac{\partial \bar{q}}{\partial t} = \sum_{d.c.} D_i (q_{Di} - \bar{q}) + \mathcal{E} - \tilde{M} \frac{\partial \bar{q}}{\partial z} - \rho \bar{\underset{\sim}{V}} \cdot \nabla \bar{q} . \tag{30}$$

$\partial \bar{s}/\partial z$ is a measure of the static stability of the environment (see (13) and (23)), which is usually positive. $-\tilde{M}\partial \bar{s}/\partial z$ represents adiabatic warming of the environment when $-\tilde{M} = M_c - \rho\bar{w} > 0$ (or cooling when $M_c - \rho\bar{w} < 0$) due to the vertical motion. For $\partial \bar{q}/\partial z < 0$ (the normal condition) $-\tilde{M}\partial \bar{q}/\partial z$ represents drying of the environment when $M_c - \rho\bar{w} > 0$ (or moistening when $M_c - \rho\bar{w} < 0$), due to the vertical motion.

In addition to the detrainment and evaporation terms, cumulus clouds modify the environment through the cumulus-induced subsidence in the environment. The latent heat released within the clouds does not directly warm the environment, but it maintains the buoyancy of the clouds against the adiabatic cooling due to the upward motion and the cooling produced by the entrainment of drier and colder air from the environment. Thus, the latent heat released within the clouds maintains the vertical mass flux of the clouds and thereby, the cumulus-induced subsidence in the environment. The drying and warming of the environment, by the cumulus induced subsidence, are the indirect effects of condensation and release of latent heat, but their vertical distributions can be very different from the vertical distribution of the condensation within the clouds. This important role of the cumulus induced subsidence in the environment was not explicitly made use of in parameterizing cumulus convection until Arakawa (1969)[1].

Eqs. (29) and (30) were derived from budgets for the environment only. But they approximately govern the time changes of $\bar{s}$ and $\bar{q}$, which are averages over the total area. This means that the prediction of the large-scale averaged field is practically the same as the prediction of the cloud environment, insofar as the thermodynamic variables are concerned. This important simplif[illegible], which was used also in the earlier parameterizations by Arakawa (1969) and by Ooyama (1971), comes from the neglect of accumulative storage of the static energy and water vapor in the ensemble of clouds. In fact, we can rederive the right hand sides of (29) and (30), as was done by Yanai (1971b, 1973), from budgets for the total area. These budgets give

$$\rho\frac{\partial\bar{s}}{\partial t} = -\overline{\nabla\cdot(\rho \mathbf{v} s)} - \frac{\partial}{\partial z}(\rho\overline{w s}) + L\left(\sum C_i - \mathcal{E}\right) + \left(\sum Q_{Ri} + \tilde{Q}_R\right), \tag{31}$$

$$\rho\frac{\partial\bar{q}}{\partial t} = -\overline{\nabla\cdot(\rho \mathbf{v} q)} - \frac{\partial}{\partial z}(\rho\overline{w q}) - \left(\sum C_i - \mathcal{E}\right), \tag{32}$$

(32)

[1] A description of this parameterization was given by Haltiner (1970, pp. .)

where C_i and Q_{Ri} are the rates of condensation of water vapor and radiational heating in the ith cloud per unit height. Using the approximations (27), (28) and (26) with Eq. (6), (31) and (32) can be rewritten as

$$\rho\frac{\partial\bar{s}}{\partial t} = -\frac{\partial}{\partial z}\left[\rho\overline{ws} - \rho\bar{w}\bar{s}\right] + L(\sum C_i - \mathcal{E}) + \sum Q_{Ri} - \rho\bar{\mathbf{v}}\cdot\nabla\bar{s} - \rho\bar{w}\frac{\partial\bar{s}}{\partial z} + \tilde{Q}_R, \tag{33}$$

$$\rho\frac{\partial\bar{q}}{\partial z} = -\frac{\partial}{\partial z}\left[\rho\overline{wq} - \rho\bar{w}\bar{q}\right] - (\sum C_i - \mathcal{E}) - \rho\bar{\mathbf{v}}\cdot\nabla\bar{q} - \rho\bar{w}\frac{\partial\bar{q}}{\partial z}. \tag{34}$$

The quantities inside the brackets are the <u>eddy</u> vertical transports by cumulus convection. The eddy transport of s may be written as

$$\rho\overline{ws} - \rho\bar{w}\bar{s} = \left(\sum M_i s_i + \tilde{M}\tilde{s}\right) - \rho\bar{w}\bar{s}.$$

Using (7), (2), (19) and (11), the eddy transport can be rewritten as

$$\rho\overline{ws} - \rho\bar{w}\bar{s} = \sum\left[M_i(s_i - \bar{s}) - \sigma_i\tilde{M}(s_i - \tilde{s})\right].$$

The second term in the bracket is negligible compared to the first, when $w_i \gg \tilde{w}$, where $\tilde{w}$ is the vertical velocity in the environment, and $s_i - \bar{s} \sim s_i - \tilde{s}$. Then we have

$$\rho\overline{ws} - \rho\bar{w}\bar{s} \doteqdot \sum M_i(s_i - \bar{s}). \tag{35}$$

Similarly, the eddy transport of q may be rewritten as

$$\rho\overline{wq} - \rho\bar{w}\bar{q} \doteqdot \sum M_i(q_i - \bar{q}). \tag{36}$$

When there is no accumulative storages of mass, static energy and water vapor in the cloud ensemble,

$$E - D - \frac{\partial M_c}{\partial z} = 0, \tag{37}$$

$$E\tilde{s} - \sum_{d.c.} D_i s_{Di} - \frac{\partial}{\partial z}\sum M_i s_i + L\sum C_i + \sum Q_{Ri} = 0, \tag{38}$$

$$E\tilde{q} - \sum_{d.c.} D_i q_{Di} - \frac{\partial}{\partial z}\sum M_i q_i - \sum C_i = 0. \tag{39}$$

Using (35) through (39), we can easily show that (33) and (34) are identical to (29) and (30), where the heat of condensation and the radiational heating in the clouds do not explicitly appear.

If the evaporation of the detrained liquid water takes place at nearly the same level where the water is detrained from the clouds,

$$\mathcal{E} = \sum_{d.c.} D_i \ell_{Di}, \tag{40}$$

where ℓ_{Di} is the mixing ratio of liquid water in the air detrained from the ith cloud. The above assumption is probably justifiable for the evaporation of detrained cloud droplets, but would not hold for rapidly falling raindrops. With the assumption (40) and Eq. (7), (29) and (30) can be rewritten as

$$\rho\frac{\partial\bar{s}}{\partial t} = \sum_{d.c.} D_i(s_{Di} - L\ell_{Di} - \bar{s}) + M_c\frac{\partial\bar{s}}{\partial z} - \rho\bar{\underset{\sim}{V}}\cdot\nabla\bar{s} - \rho\bar{w}\frac{\partial\bar{s}}{\partial z} + \widetilde{Q}_R, \tag{41}$$

$$\rho\frac{\partial\bar{q}}{\partial t} = \sum_{d.c.} D_i(q_{Di} + \ell_{Di} - \bar{q}) + M_c\frac{\partial\bar{q}}{\partial z} - \rho\bar{\underset{\sim}{V}}\cdot\nabla\bar{q} - \rho\bar{w}\frac{\partial\bar{q}}{\partial z}. \tag{42}$$

The detrainment terms will be further simplified in the next section.

3. Budget equations for an individual cloud and assumptions for the detrainment

We first consider the budgets of mass, static energy, water vapor and liquid water for an individual cloud. In the entrainment layer of the i th cloud, the budget equations can be written as

mass:
$$\frac{\partial}{\partial t}(\rho\sigma_i) = E_i - \frac{\partial}{\partial z} M_i , \tag{43}$$

static energy:
$$\frac{\partial}{\partial t}(\rho\sigma_i s_i) = E_i \bar{s} - \frac{\partial}{\partial z}(M_i s_i) + L C_i + Q_{Ri} , \tag{44}$$

water vapor:
$$\frac{\partial}{\partial t}(\rho\sigma_i q_i) = E_i \bar{q} - \frac{\partial}{\partial z}(M_i q_i) - C_i , \tag{45}$$

liquid water:
$$\frac{\partial}{\partial t}(\rho\sigma_i \ell_i) = - \frac{\partial}{\partial z}(M_i \ell_i) + C_i - R_i , \tag{46}$$

where ℓ_i is the mixing ratio of liquid water in the form of cloud droplets and R_i the rate of conversion of the liquid water to precipitation per unit height. The approximations (23) and (25) have already been used. The budget equations in the detrainment layer are

mass:
$$\frac{\partial}{\partial t}(\rho\sigma_i) = - D_i - \frac{\partial}{\partial z} M_i , \tag{47}$$

static energy:
$$\frac{\partial}{\partial t}(\rho\sigma_i s_i) = - D_i s_{Di} - \frac{\partial}{\partial z}(M_i s_i) + L C_i + Q_{Ri} , \tag{48}$$

water vapor:
$$\frac{\partial}{\partial t}(\rho\sigma_i q_i) = - D_i q_{Di} - \frac{\partial}{\partial z}(M_i q_i) + C_i , \tag{49}$$

liquid water:
$$\frac{\partial}{\partial t}(\rho\sigma_i \ell_i) = - D_i \ell_{Di} - \frac{\partial}{\partial z}(M_i \ell_i) + C_i - R_i . \tag{50}$$

Equations (37), (38) and (39) are obtained by summation of (43), (44) and (45) over all entraining clouds and of (47), (48) and (49) over all detraining clouds and subsequently dropping the time derivative terms.

We define the moist static energy by

$$h \equiv s + Lq \equiv c_pT + gz + Lq, \tag{51}$$

where Lq is the latent heat per unit mass of the air. h is approximately conserved by the individual air parcel during moist adiabatic processes. We also define the saturation moist static energy by

$$h^* \equiv s + Lq^* \equiv c_pT + gz + Lq^*, \tag{52}$$

where the * denotes the saturation value of the variable. $\partial h^*/\partial z \gtreqless 0$ defines the moist adiabatically stable, neutral and unstable lapse rates, respectively.

We assume that air is saturated in the clouds. Then

$$\begin{aligned} q_i &= q^*(T_i, p_i) \\ &\doteqdot q^*(T_i, \bar{p})^{1} \\ &\doteqdot \overline{q^*} + \left(\frac{\partial \overline{q^*}}{\partial \bar{T}}\right)_{\bar{p}} (T_i - \bar{T}) \\ &= \overline{q^*} + \frac{1}{c_p}\left(\frac{\partial \overline{q^*}}{\partial \bar{T}}\right)_{\bar{p}} (s_i - \bar{s}), \end{aligned} \tag{53}$$

where $\overline{q^*} \equiv q^*(\bar{T}, \bar{p})$. Since $h_i = s_i + Lq_i$ and $\overline{h^*} = \bar{s} + L\overline{q^*}$,

$$s_i - \bar{s} = \frac{1}{1+\gamma}(h_i - \overline{h^*}), \tag{54}$$

$$q_i - \overline{q^*} = \frac{\gamma}{1+\gamma}\frac{1}{L}(h_i - \overline{h^*}), \tag{55}$$

[1] Here, we neglect the effect on q^* of a pressure difference between the cloud and the environment. Although the scale analysis by Ogura and Phillips (1962) did not justify this approximation, the recent numerical integration by Wilhelmson and Ogura (1972) indicates that the pressure difference was overestimated by the scale analysis.

where

$$\gamma \equiv \frac{L}{c_p}\left(\frac{\partial \bar{q}^*}{\partial \bar{T}}\right)_{\bar{p}}. \tag{56}$$

For simplicity, we neglect in this paper the difference between temperature and virtual temperature, except for turbulent thermal convection in the subcloud mixed layer. Then, the sign of the buoyancy is given by

$$\text{buoyancy} \gtreqless 0 \qquad \text{according to} \quad h_i \gtreqless \bar{h}^*. \tag{57}$$

Eliminating C_i from (44) and (45) and from (45) and (46), we obtain

$$\frac{\partial}{\partial t}(\rho\sigma_i h_i) = E_i \bar{h}_i - \frac{\partial}{\partial z}(M_i h_i) + Q_{Ri}, \tag{58}$$

$$\frac{\partial}{\partial t}\left[\rho\sigma_i(q_i + l_i)\right] = E_i \bar{q} - \frac{\partial}{\partial z}\left[M_i(q_i + l_i)\right] - R_i. \tag{59}$$

Eqs. (58) and (59) describe the budgets of the moist static energy and water substance for the entrainment layer of the ith cloud. Combining (58) and (59) with (43), we may write

$$\left(\frac{\partial}{\partial t} + w_i \frac{\partial}{\partial z}\right) h_i = -\mu_i(h_i - \bar{h}) + Q_{Ri}/M_i, \tag{60}$$

$$\left(\frac{\partial}{\partial t} + w_i \frac{\partial}{\partial z}\right)(q_i + l_i) = -\mu_i(q_i + l_i - \bar{q}) - R_i/M_i, \tag{61}$$

where μ_i is the fractional rate of entrainment, defined by

$$\mu_i \equiv E_i / M_i. \tag{62}$$

The ith cloud may be in its growing stage, with a rising cloud top. We then assume that there is entrainment into the cloud at all levels including the cloud top. Only after the cloud top has lost positive buoyancy and has stopped rising does detrainment take place in a thin layer at the cloud top.[1] From (57), the level, $z = \hat{z}_i$ at which the cloud top loses positive buoyancy, is given by

$$(h_i - \overline{h^*})_{z=\hat{z}_i} = 0. \tag{63}$$

Also, from (54) and (55), we have

$$(s_i - \bar{s})_{z=\hat{z}_i} = 0, \tag{64}$$

$$(q_i - \overline{q^*})_{z=\hat{z}_i} = 0. \tag{65}$$

Eqs. (63), (64) and (65) show that all clouds which lose buoyancy at the same level z share common values, $\overline{h^*}(z)$, $\bar{s}(z)$ and $\overline{q^*}(z)$, for h_i, s_i and q_i at that level. We therefore assume that these clouds are of the same type and have a common value $\hat{\ell}$ for ℓ_i at that level. That is,

$$(\ell_i - \hat{\ell})_{z=\hat{z}_i} = 0. \tag{66}$$

$\hat{\ell}$ is the liquid water mixing ratio at the level of vanishing buoyancy and is not necessarily equal to the liquid water mixing ratio of the air which spreads into the environment, because an additional condensation (or evaporation) may take place nea. .he cloud top due to the concentrated Q_{Ri} there. We let $\hat{\ell}$ be a function of z, but $\hat{\ell}$ for different levels refer to different types of clouds.

For the detrainment layer, it is convenient to eliminate C_i from (48) and (50). Then we have

$$\frac{\partial}{\partial t}\left[\rho\sigma_i(s_i - L\ell_i)\right] = -D_i(s_{Di} - L\ell_{Di}) - \frac{\partial}{\partial z}\left[M_i(s_i - L\ell_{Di})\right] + LR_i + Q_{Ri}. \tag{67}$$

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[1] Simpson et al. (1965) pointed out that their model supports the rule of thumb that the clouds cease vertical growth when their temperature soundings recross the environment curve.

We also have the budget equation for water substance,

$$\frac{\partial}{\partial t}\left[\rho\sigma_i(q_i+\ell_i)\right] = -D_i(q_{Di}+\ell_{Di}) - \frac{\partial}{\partial z}\left[M_i(q_i+\ell_i)\right] - R_i \; . \tag{68}$$

Because the thickness of the detrainment layer, Δz_{Di}, is assumed to be small, the mass budget equation for the detrainment layer, (47), may be approximated by

$$D_i \Delta z_{Di} = (M_i)_{z=\hat{z}_i} \; . \tag{69}$$

M_i at $z = \hat{z}_i$ is the mass flux of the cloud entering the detrainment layer from below. Similar simplifications of (65) and (66), and use of (67), (64), (65) and (66) give

$$\begin{aligned} D_i(s_{Di} - L\ell_{Di}) &= D_i(s_i - L\ell_i)_{z=\hat{z}_i} + Q_{Ri} \\ &= D_i(\bar{s} - L\hat{\ell}) + Q_{Ri} \end{aligned} \tag{70}$$

and

$$\begin{aligned} D_i(q_{Di} + \ell_{Di}) &= D_i(q_i + \ell_i)_{z=\hat{z}_i} \\ &= D_i(\bar{q}^* + \hat{\ell})_{z=\hat{z}_i} \; . \end{aligned} \tag{71}$$

The radiation term in (68) is retained because the radiation flux at the cloud top is almost discontinuous, whereas $Q_{Ri}\Delta z_{Di}$ is finite even when Δz_{Di} is infinitesimally small.

Summations of (70) and (71) over all clouds which are detraining in the same thin layer give

$$\sum_{d.c.} D_i(s_{Di} - L\ell_{Di}) = D(\bar{s} - L\hat{\ell}) + \sum_{d.c.} Q_{Ri} , \tag{72}$$

$$\sum_{d.c.} D_i(q_{Di} + \ell_{Di}) = D(\bar{q}^* + \hat{\ell}) , \tag{73}$$

where D is the total detrainment, $\sum_{d.c.} D_i$. Substituting (72) and (73) into (41) and (42), we obtain

$$\rho \frac{\partial \bar{s}}{\partial t} = - L D \hat{\ell} + M_c \frac{\partial \bar{s}}{\partial z} + \sum_{d.c.} Q_{Ri}$$
$$- \rho \bar{\underset{\sim}{V}} \cdot \nabla \bar{s} - \rho \bar{w} \frac{\partial \bar{s}}{\partial z} + \widetilde{Q}_R , \tag{74}$$

$$\rho \frac{\partial \bar{q}}{\partial t} = D(\bar{q}^* + \hat{\ell} - \bar{q}) + M_c \frac{\partial \bar{q}}{\partial z}$$
$$- \rho \bar{\underset{\sim}{V}} \cdot \nabla \bar{q} - \rho \bar{w} \frac{\partial \bar{q}}{\partial z} . \tag{75}$$

These are the basic equations we use to describe the time changes of the large-scale temperature and moisture fields. However, these equations are valid only above the cloud base. The subcloud layer and its interaction with the cloud layer will be discussed in a later section.

Equations similar to (74) and (75), for a three-level model of the large-scale temperature and moisture fields, were derived and used by Arakawa (1969). Equations which are almost identical to (74) and (75), except for the radiation terms, were derived by Ooyama (1971), Arakawa (1971) and Yanai (1971b). Yanai et al. (1973) used these equations to determine the bulk properties of tropical cumulus cloud clusters from the observed large-scale budgets of heat and moisture. In that study, however, the large-scale tendency terms were obtained from observations, while our problem is prognostic and the large-scale tendency terms are exactly what we want to find through the parameterization of cumulus convection.

4. Spectral representation of cumulus ensemble

Eqs. (74) and (75) clearly show which properties of the cumulus ensemble must be parameterized to predict the large-scale temperature and moisture fields. The modification of the large-scale fields by cumulus convection depends on:

(i) the total mass flux in the clouds, $M_c(z)$;

(ii) the total detrainment from the clouds into the environment, $D(z)$;

(iii) the mixing ratio of liquid water at the vanishing buoyancy level, $\hat{\ell}(z)$.

(In addition, cumulus clouds modify the large-scale temperature through their effect on radiation.) The problem of parameterization of cumulus convection is now reduced to relating these three properties of the cumulus ensemble to the large-scale temperature, moisture and velocity fields.

The total detrainment, $D(z)$, at different levels refer to different types of clouds. When the thickness of the detrainment layer, Δz_{Di}, is infinitesimally small, the total detrainment in the layer between z and $z + dz$, $D(z)dz$, is equal to the total mass flux, at level z, of the clouds which lose buoyancy within that layer. It is now clear that finding the total detrainment, $D(z)$, as a function of height, is equivalent to finding the distribution of the mass flux in the different types of clouds which lose buoyancy at the different levels. This suggests that we represent the cloud ensemble in spectral form, by dividing the ensemble into sub-ensembles, each of which has a characteristic cloud type.

For simplicity, we assume that a single positive parameter λ can fully characterize a cloud type. Then the detrainment level, z_D, which is the maximum height of the cloud top, becomes a function of λ. By choosing λ properly, we let $z_D(\lambda)$ decrease as λ increases, as shown schematically in Fig. 3. A more specific choice of λ will be made later. Let $\lambda_D(z)$ be λ of the clouds which are detraining at level z. $\lambda_D(z)$ is the inverse function of $z_D(\lambda)$, which satisfies

$$z \equiv z_D(\lambda_D(z)) \tag{76}$$

identically. $\lambda_D(z)$ is also the maximum value of λ at level z, because the clouds which have larger λ than λ_D have detrainment levels lower than z.

The total mass flux in the clouds, M_c, can be expressed as

$$M_c(z) = \int_0^{\lambda_D(z)} m(z,\lambda)\, d\lambda, \tag{77}$$

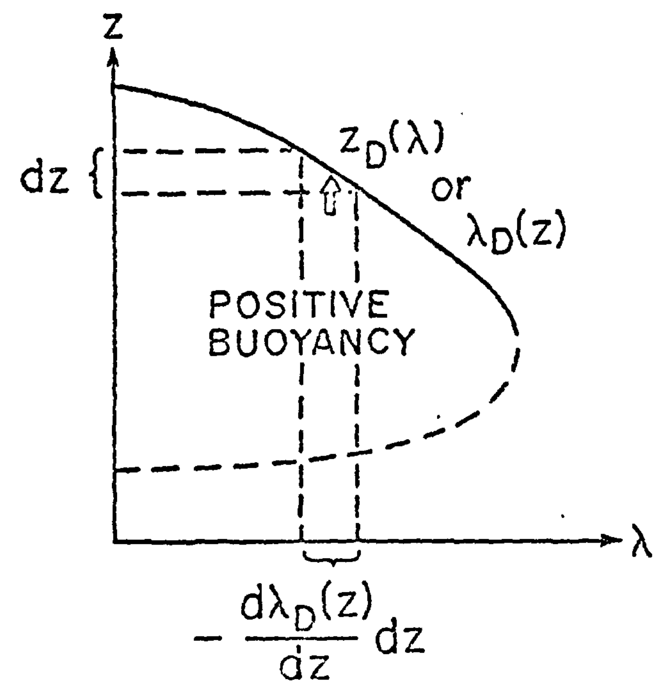
where

$$m(z,\lambda)d\lambda = \sum_{\lambda_i \in (\lambda, \lambda+d\lambda)} M_i(z) \tag{78}$$

is the sub-ensemble mass flux due to the clouds which have the parameter λ_i in the interval $(\lambda, \lambda+d\lambda)$.

The total detrainment, $D(z)dz$, in the layer between z and $z+dz$ is equal to the sub-ensemble mass flux, at level z, due to the clouds which have parameter λ_i in the interval $\lambda_D(z)-(-d\lambda_D(z)/dz)\,dz$ to $\lambda_D(z)$, as is shown in Fig. 3. Then we have

$$D(z) = -m(z,\lambda_D(z))\frac{d\lambda_D(z)}{dz}. \tag{79}$$


z
dz
$z_D(\lambda)$
or
$\lambda_D(z)$
POSITIVE
BUOYANCY
λ
$-\frac{d\lambda_D(z)}{dz}dz$

3

It is convenient to normalize $m(z,\lambda)$ by

$$m(z,\lambda) \equiv m_B(\lambda)\,\eta(z,\lambda), \tag{80}$$

$$m_B(\lambda) \equiv m(z_B,\lambda), \tag{81}$$

where z_B is a properly chosen base of the updrafts associated with the clouds. Obviously we have

$$\eta(z_B,\lambda) = 1. \tag{82}$$

We shall find it convenient to choose the top of the subcloud mixed layer as the base, z_B, as shown in Fig. 4. Observations, which we will refer to in the next section, show that the top of the mixed layer is located somewhat below the cloud base, z_c, which is approximately the lifting condensation level. The vertical mass flux below the cloud base, then, should be interpreted as the mass flux of the updrafts associated with the clouds but not in the clouds.

Next we consider the budgets of mass, moist static energy and water substance for the sub-ensemble. The summation of (43) over all members of the sub-ensemble and subsequent dropping of the time derivative term give

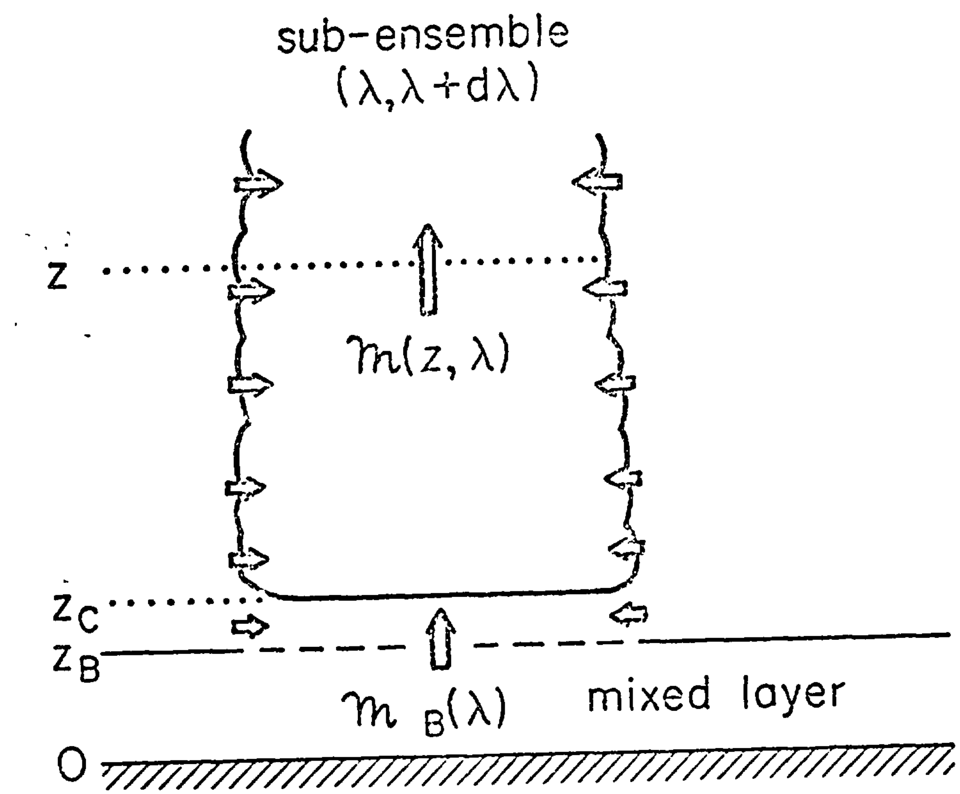
$$\frac{\partial \eta(z,\lambda)}{\partial z} = \sum_{\lambda_i \in (\lambda,\lambda+d\lambda)} E_i(z) \Big/ m_B(\lambda)\,d\lambda. \tag{83}$$

Here (78) and (80) have been used. Similarly, (58) and (59) give

$$\frac{\partial}{\partial z}\left[\eta(z,\lambda)\,h_c(z,\lambda)\right] = \frac{\partial \eta(z,\lambda)}{\partial z}\,\bar{h}(z), \tag{84}$$

$$\frac{\partial}{\partial z}\left[\eta(z,\lambda)\left(q_c(z,\lambda)+\ell(z,\lambda)\right)\right] = \frac{\partial \eta(z,\lambda)}{\partial z}\,\bar{q}(z) - \eta(z,\lambda)\,r(z,\lambda). \tag{85}$$

Here (78), (80) and (83) have been used. $h_c(z,\lambda)$, $q_c(z,\lambda)$ and $\ell(z,\lambda)$ are h,


sub-ensemble
(λ,λ+dλ)
z
m(z, λ)
z_C
z_B
m_B(λ)
mixed layer
O

q and ℓ, respectively, in the clouds which are members of the sub-ensemble. $r(z,\lambda)$ is defined by

$$\mathcal{m}(z,\lambda)\, r(z,\lambda)\, d\lambda \equiv \sum_{\lambda_i \in (\lambda, \lambda+d\lambda)} R_i \,. \tag{86}$$

The radiational heating in the entrainment layer is dropped for simplicity.

Eqs. (84) and (85) can be rewritten as

$$\frac{\partial h_c(z,\lambda)}{\partial z} = -\mu(z,\lambda)\left[h_c(z,\lambda) - \bar{h}(z) \right], \tag{87}$$

$$\frac{\partial}{\partial z}\left[q_c(z,\lambda) + \ell(z,\lambda) \right] = -\mu(z,\lambda)\left[q_c(z,\lambda) + \ell(z,\lambda) - \bar{q}(z) \right] - r(z,\lambda), \tag{88}$$

where $\mu(z,\lambda)$ is the fractional rate of entrainment for the sub-ensemble, given by

$$\mu(z,\lambda) \equiv \frac{1}{\eta(z,\lambda)} \frac{\partial \eta(z,\lambda)}{\partial z} \tag{89}$$

Except for the cloud microphysical and dynamical processes which determine $r(z,\lambda)$ and the subcloud layer processes which determine $h(z_B,\lambda)$ and $q(z_B,\lambda)$, the problem of parameterizing cumulus convection has now been reduced to finding $\eta(z,\lambda)$, the normalized vertical profile of the sub-ensemble mass flux, and $\mathcal{m}_B(\lambda)$, the mass flux distribution function at the top of the mixed layer.

When $\eta(z,\lambda)$ (and, therefore, $\mu(z,\lambda)$) is known, $h_c(z,\lambda)$ can be readily obtained by integrating (84) or (87) with respect to height under given $\bar{h}(z)$ and $h(z_B,\lambda)$, and the detrainment level, $z_D(\lambda)$, and its inverse function, $\lambda_D(z)$, can be found from the condition of vanishing buoyancy

$$h_c(z_D(\lambda), \lambda) = \bar{h}^*(z_D(\lambda)), \tag{90}$$

and

$$z = z_D\left(\lambda_D(z)\right). \tag{91}$$

Also, $q_c(z,\lambda)+\ell(z,\lambda)$ can then be obtained by integrating (85) or (88), with a parameterized $r(z,\lambda)$, under given $\bar{q}(z)$ and $q(z_B,\lambda)$. $\ell(z_B,\lambda)$ is zero because level z_B is below the cloud base. From the known

$q(z,\lambda)+\ell(z,\lambda)$ and the vanishing buoyancy condition in the form

$$q\left(z,\lambda_D(z)\right) = \bar{q}^*\left(\lambda_D(z)\right), \tag{92}$$

we can find

$$\hat{\ell}(z) = \ell\left(z,\lambda_D(z)\right), \tag{93}$$

which is the liquid water mixing ratio at the vanishing buoyancy level. Then, only the mass flux distribution function, $m_B(\lambda)$, which is needed for computing $M(z)$ and $D(z)$ from (77) and (79), remains unknown.

Although our knowledge of the dynamics of clouds is far from sufficient, the determination of the normalized vertical profile of the sub-ensemble mass flux, $\eta(z,\lambda)$, is logically more straightforward than the determination of the mass flux distribution function, $m_B(\lambda)$. We may assume that the members of a sub-ensemble are at random phases in their life cycle and, therefore, the summation of the mass flux over all members of the sub-ensemble, as in (78), is proportional to the mass flux of a single cloud averaged over its entire lifetime. The constant of proportionality is the number of clouds. But the constant of proportionality does not matter for $\eta(z,\lambda)$, since $\eta(z,\lambda)$ is normalized. A dynamical model which governs the life cycle of a single cloud should be able to determine the vertical profile of the time-averaged mass flux of that cloud, and, therefore, $\eta(z,\lambda)$, for each cloud type characterized by parameter λ. However, we are assuming that a single scalar parameter λ is sufficient to characterize the cloud type. For this assumption to be at least approximately valid, we must choose λ properly.

The one-dimensional model of the cumulus tower, developed by (Simpson et al. (1965) and Simpson and Wiggert (1969, 1971), has been extensively tested against observations. This model specifies the fractional rate of entrainment by

$$\mu = \frac{2\alpha}{R} \tag{94}$$

where R is the radius of the rising cumulus tower and α is the entrainment constant (see also Simpson, 1971). R is either measured or assumed at the cloud base and given to the model as an input. The assumption that R is constant with height, in the Lagrangian sense, leads to better agreement with observations than the alternative assumptions of horizontally expanding thermals or starting plumes (Simpson et al. 1965).

In our model also, we assume that R is constant with height in the Lagrangian sense. We are not assuming that the cloud is a column-like steady jet, and it may consist of a sequence of active elements which have a negligible horizontal expansion rate below the level of vanishing buoyancy. But we do assume that the fractional rate of entrainment for the time averaged mass flux of the cloud is approximately cons[illegible] ht. We now choose this constant fractional rate of entrainment as the parameter λ which characterizes the cloud type. Although the dependence of the entrainment on the radius, as given by (94), is not used in this paper explicitly, we may interpret the larger λ as representing the smaller clouds and the smaller λ as representing the larger clouds.

The asumption of constant fractional rate of entrainment greatly simplifies the determination of $\eta(z,\lambda)$. This assumption decouples the determination of $\eta(z,\lambda)$ from the solution of the entire system of equations which govern the life cycle of a cloud. An assumption about the geometry, like the expanding spherical bubbles used by Ooyama (1971), gives a similar simplification.

Replacing $\mu(z,\lambda)$ in (89) by λ, we obtain

$$\frac{\partial \eta(z,\lambda)}{\partial z} = \lambda\, \eta(z,\lambda). \tag{95}$$

(95), (82) and the definition of $z_D(\lambda)$ immediately give

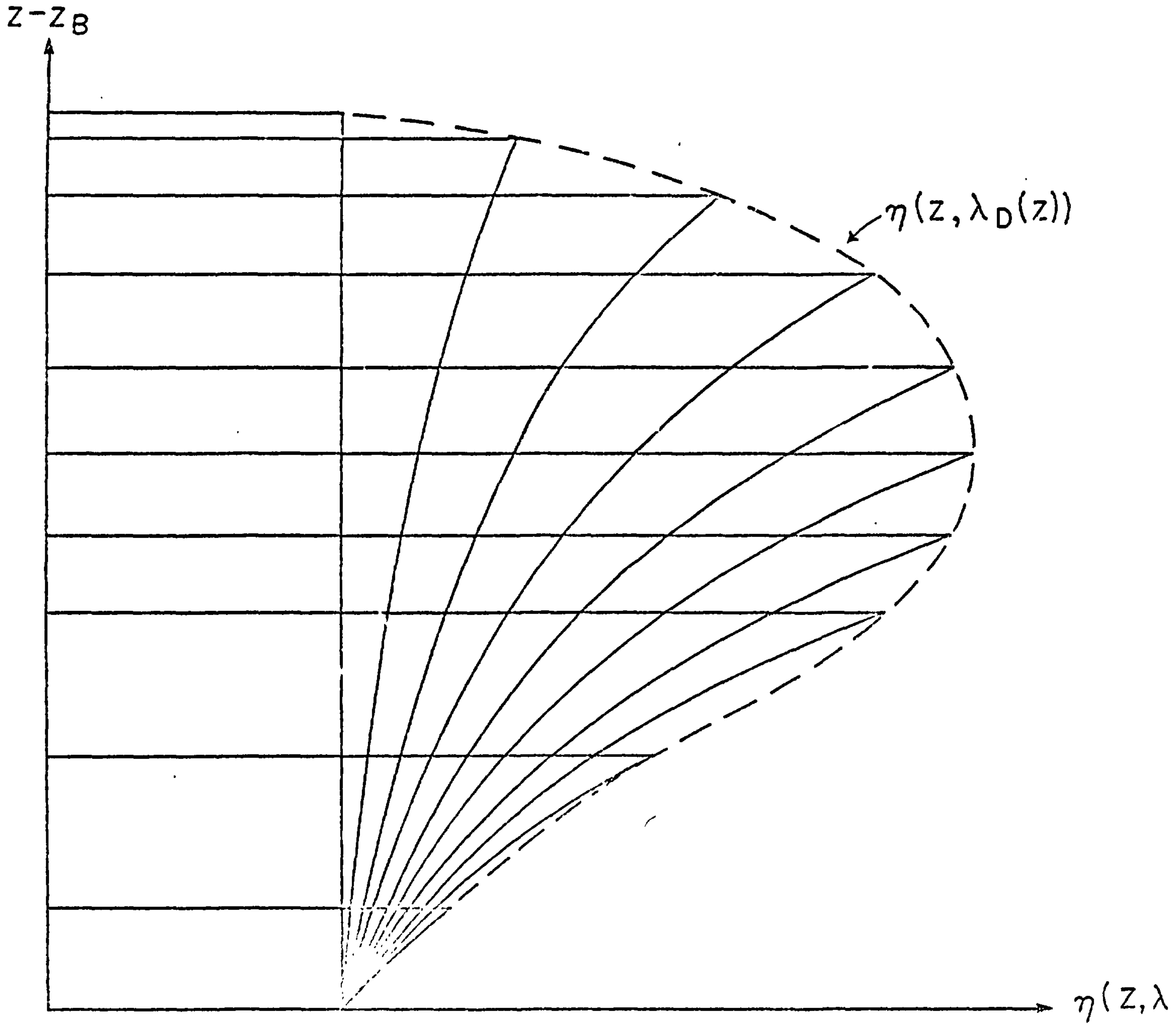
$$\eta(z,\lambda) = \begin{cases} e^{\lambda(z-z_B)} & z_B \leqq z \leqq z_D(\lambda), \\ 0 & z_D(\lambda) \leqq z. \end{cases} \tag{96}$$

Thus the sub-ensemble vertical mass flux increases exponentially with height due to the entrainment. Above the detrainment level, $z_D(\lambda)$, the mass flux becomes zero. Fig. 5 schematically shows $\eta(z,\lambda)$ for various λ.

To determine $z_D(\lambda)$, we must find $h_c(z,\lambda)$. The solution of (84) is given by

$$h_c(z,\lambda) = \frac{1}{\eta(z,\lambda)}\left[h_c(z_B,\lambda) + \lambda \int_{z_B}^{z} \eta(z',\lambda)\, \bar{h}(z')\, dz' \right]. \tag{97}$$

Here, (95) has been used. (54) gives



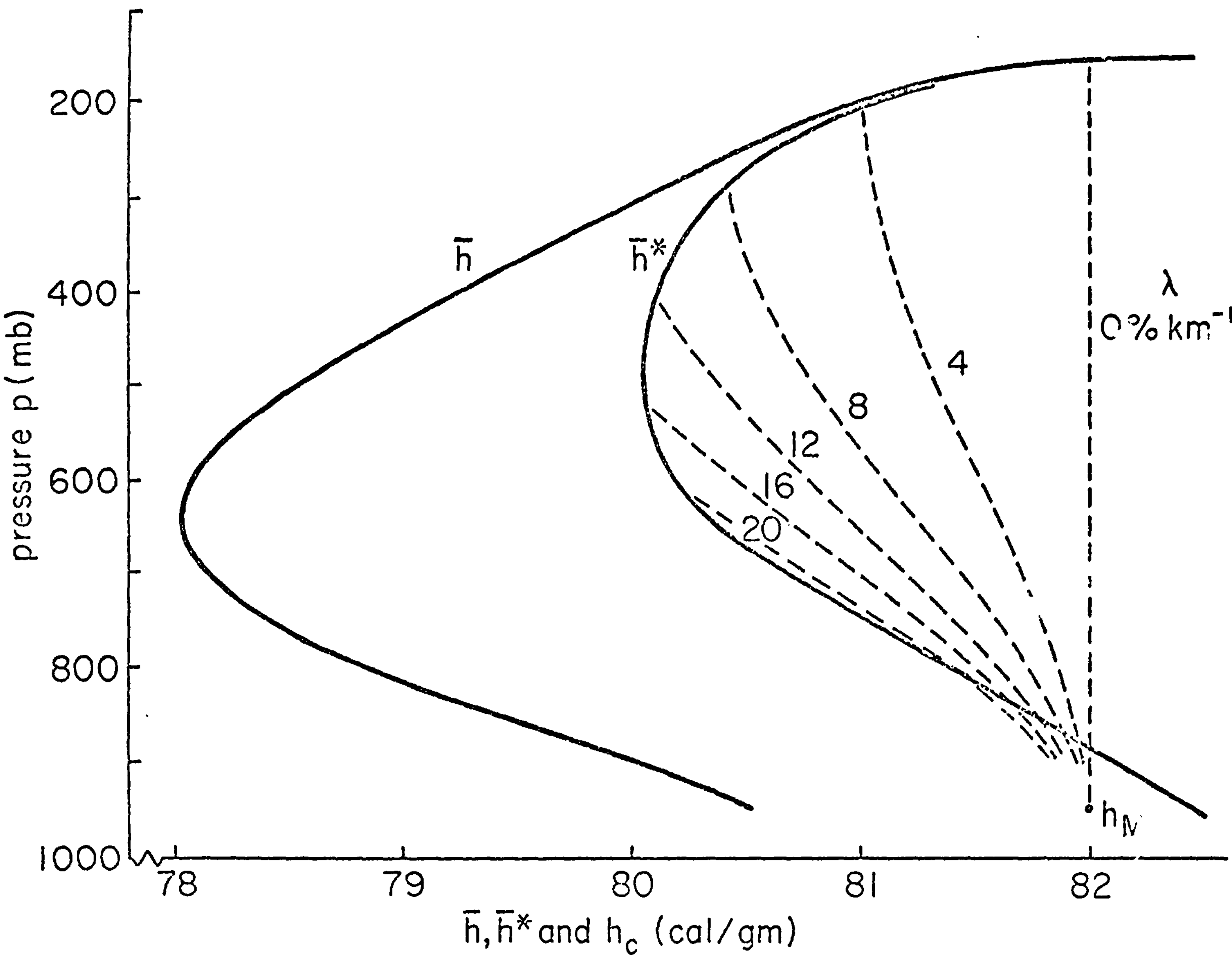
5

$$S_c(z,\lambda) - \bar{S}(z) = \frac{1}{1+\gamma(z)}\left(h_c(z,\lambda) - \overline{h^*}(z)\right), \qquad (98)$$

where γ is defined by (56). As an example, $h_c(z,\lambda)$ for given $\bar{h}(z)$, $\overline{h^*}(z)$, z_B and a constant $h_c(z_B,\lambda) = h_M$ is shown by the broken lines in Fig. 6 and $\gamma(z)$ is shown in Fig. 7. In the figures the pressure coordinate is used. For $\lambda = 0$, $\eta(z,0) \equiv 1$ and the second term in the bracket of (97) vanishes. Then $h_c(z,0) = h_M$ for all z. As λ increases, $h_c(z,\lambda)$ is more reapidly diluted by $\bar{h}(z)$. From (90), the detrainment level is given by the intersections of the broken lines with the curve $\overline{h^*}(z)$. The curve $p_D(\lambda)$, the pressure at the detrainment level thus obtained, ~~, which~~ is shown in Fig. 8. The figure shows that smaller clouds (larger λ) have lower detrainment levels than larger clouds (smaller λ), because smaller clouds, which have a larger entrainment rate, lose positive buoyancy more quickly than larger clouds.

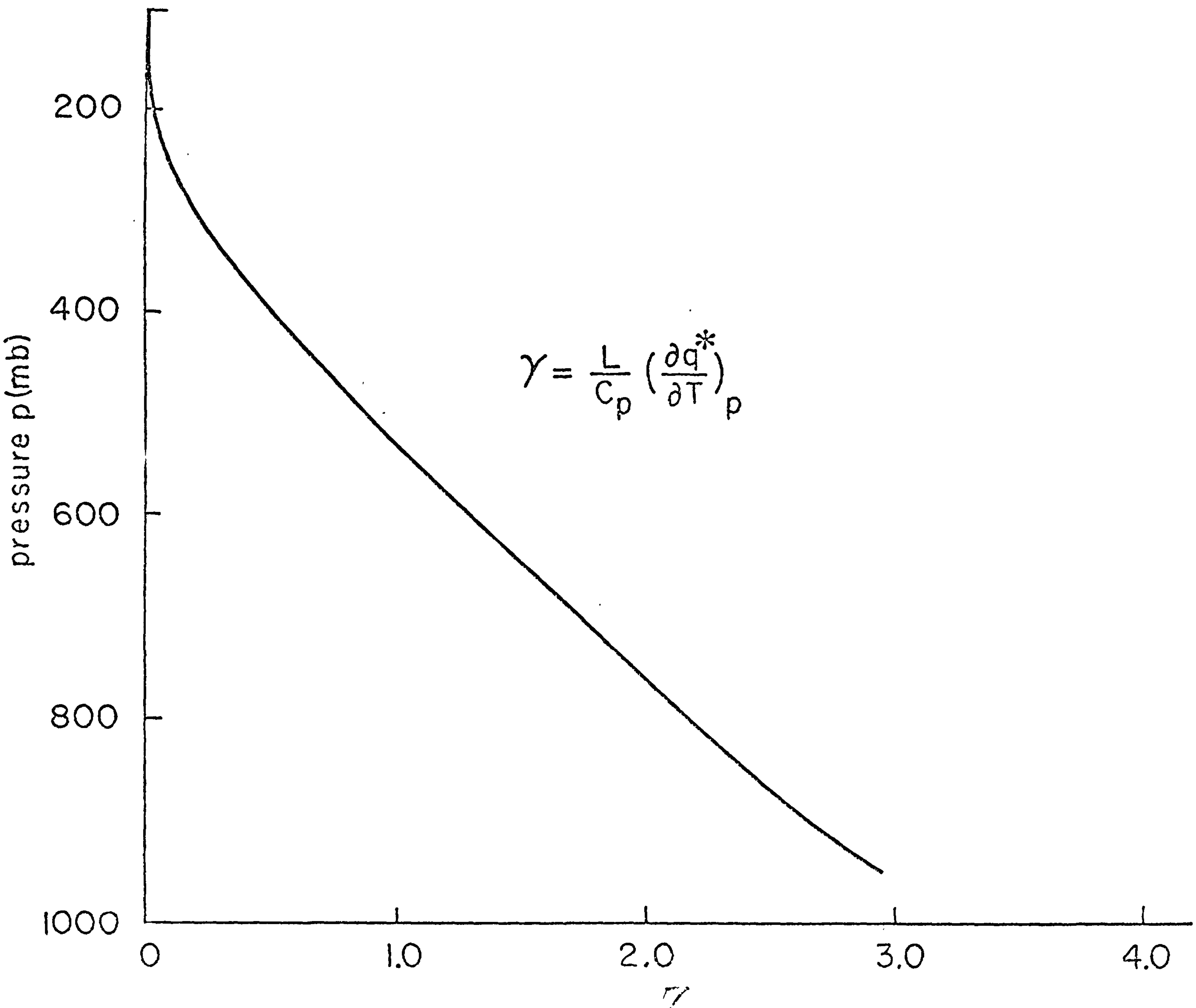
To find the mixing ratio of liquid water at the vanishing buoyancy level, $\hat{\ell}(z)$, which appears in (74), we integrate (85) with respect to height, from a given 'initial' condition $q_c(z_B,\lambda)$. In addition, a parameterization of the rainfall rate, $r(z,\lambda)$, is necessary. A very crude, but perhaps adequate parameterization is being used in the authors' application of this theory (Schubert and Arakawa, 1973); Arakawa and Chao, 1973).

The remaining problems are to find the base level variables z_B, $h_c(z_B,\lambda)$, $q_c(z_B,\lambda)$ and, most importantly, the mass flux distribution function, $m_B(\lambda)$. Up to this point our theory is not substantially different from

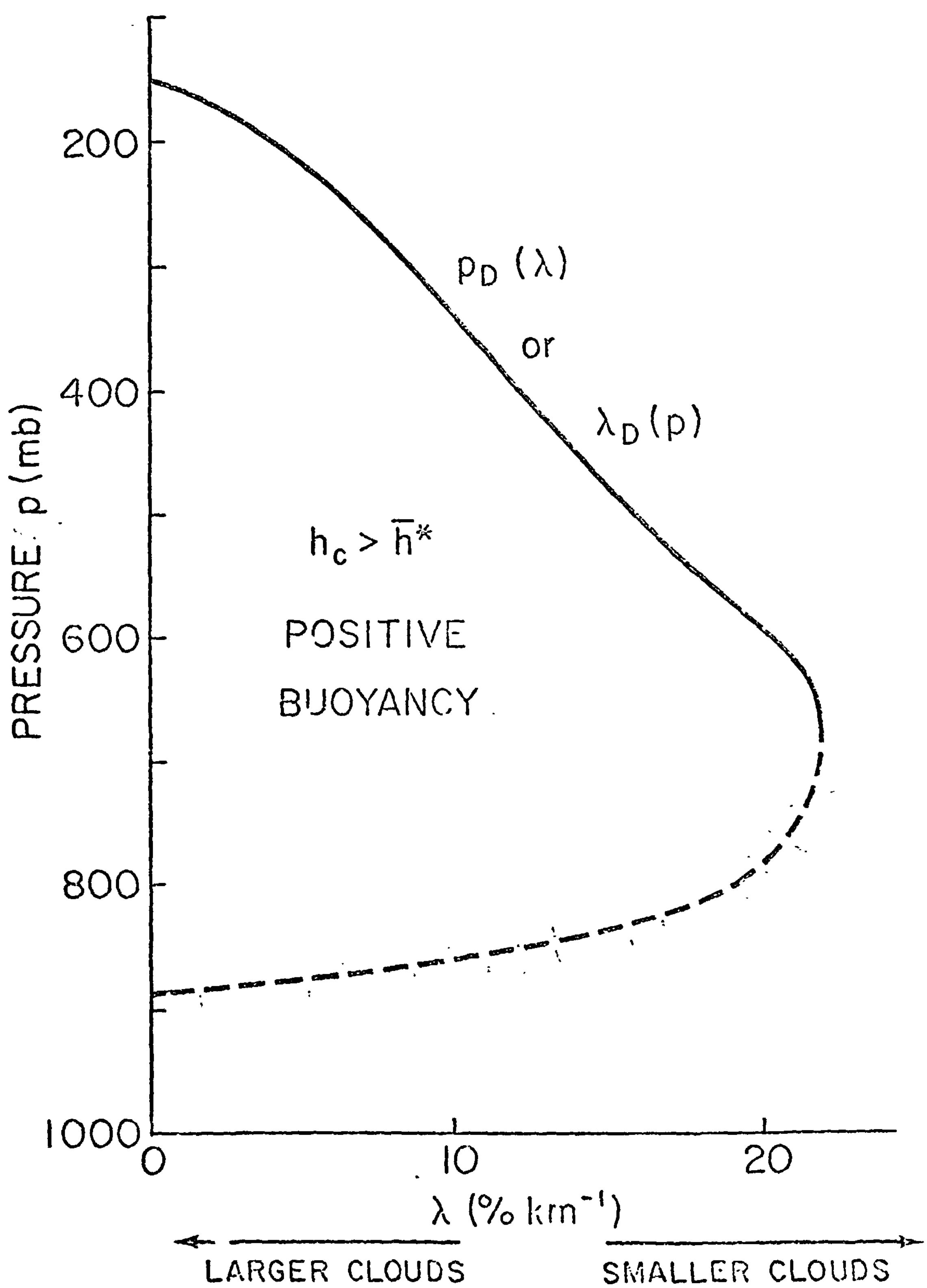


pressure p (mb)
200
400
600
800
1000
78
79
80
81
82
h̄, h̄* and h_c (cal/gm)
h̄
h̄*
λ
0 % km⁻¹
4
8
12
16
20
h_M

5



pressure p(mb)
200
400
600
800
1000
0
1.0
2.0
3.0
4.0
γ
$\gamma = \frac{L}{C_p}\left(\frac{\partial q^*}{\partial T}\right)_p$

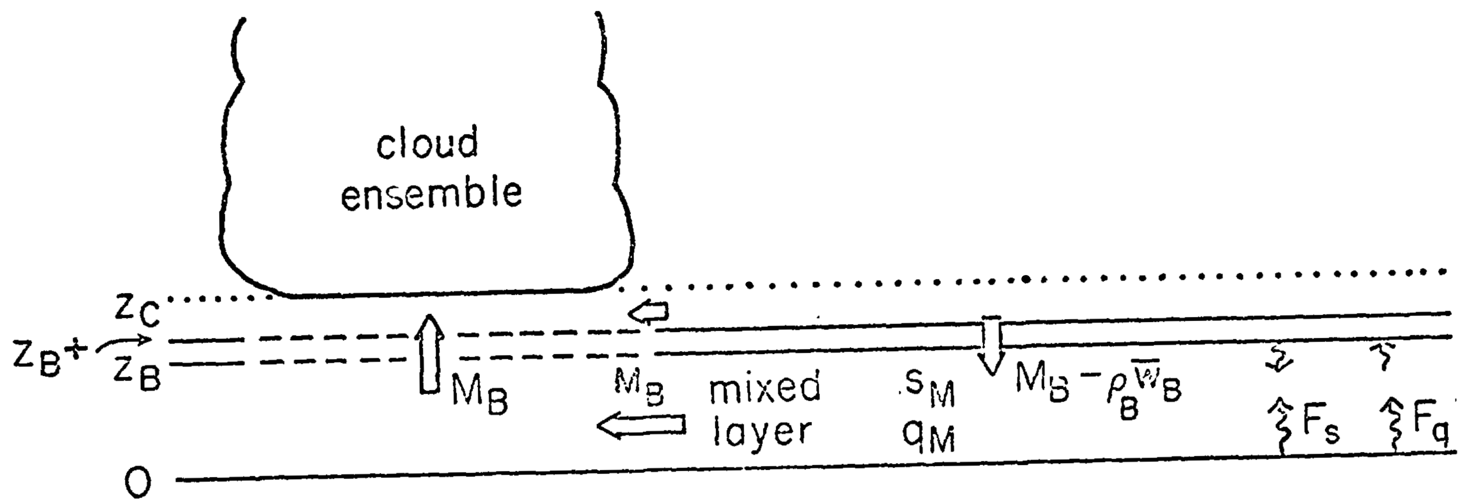

PRESSURE p (mb)
200
400
600
800
1000
0
10
20
λ (% km⁻¹)
LARGER CLOUDS
SMALLER CLOUDS
$p_D(\lambda)$
or
$\lambda_D(p)$
$h_c > \overline{h}^*$
POSITIVE
BUOYANCY

that of Ooyama (1971), as far as the basic logic is concerned. He concluded that the problem of parameterization of cumulus convection reduces to finding what he calls, "the dispatcher function", the rate of generation of buoyant bubbles as a function of the initial state of the bubbles. However, Ooyama left the determination of this dispatcher function an open question and, therefore, his parameterizatic.. was not complete.

5. Budgets of static energy and moisture for the mixed layer

In this section we present a model of the subcloud mixed layer, which interacts with the cumulus ensemble. Observations over the Carribbean sea by Bunker et al. (1949) and by Malkus (1958) show that between the ocean surface and the cloud base level there typically exists a mixed layer in which the potential temperature, θ, and the mixing ratio of water vapor, q, and therefore s and h are approximately constant with height. The top of the mixed layer is somewhat (about 200 m) lower than the cloud base level. Except for the region right below the clouds, there typically exists a very thin transition layer, immediately above the mixed layer, in which θ and, therefore, s rapidly increase and q rapidl: decreases with height.

We denote the height of the mixed layer by z_B (see Fig. 9). z_B is assumed to be lower than the cloud base, z_c, which is approximately the level of lifting condensation. Therefore, we consider only non-saturated mixed layers. We model the transition layer as a discontinuity in s and q at z_B.



cloud
ensemble
z_C
z_B+
z_B
M_B
M_B
mixed
layer
s_M
q_M
$M_B - \rho_B \overline{w}_B$
F_s
F_q
0

In this respect, our approach is similar to those given by Ball (1960), Lilly (1968), Deardorff (1972), and Betts (1972). We define

$$\Delta s \equiv \bar{s}(z_B+) - s_M, \tag{99}$$

$$\Delta q \equiv \bar{q}(z_B+) - q_M, \tag{100}$$

$$\Delta h \equiv \bar{h}(z_B+) - h_M, \tag{101}$$

where $\bar{s}(z_B+)$, $\bar{q}(z_B+)$ and $\bar{h}(z_B+)$ are values of $\bar{s}$, $\bar{q}$, and $\bar{h}$ evaluated just above the discontinuity at z_B and s_M, q_M and h_M are the mixed layer values of s, q and h, respectively. Δh is given in terms of Δs and Δq by

$$\Delta h = \Delta s + L \Delta q. \tag{102}$$

In typical cumulus situations $L\Delta q$ dominates over Δs, making Δh negative. Δh^* is given by

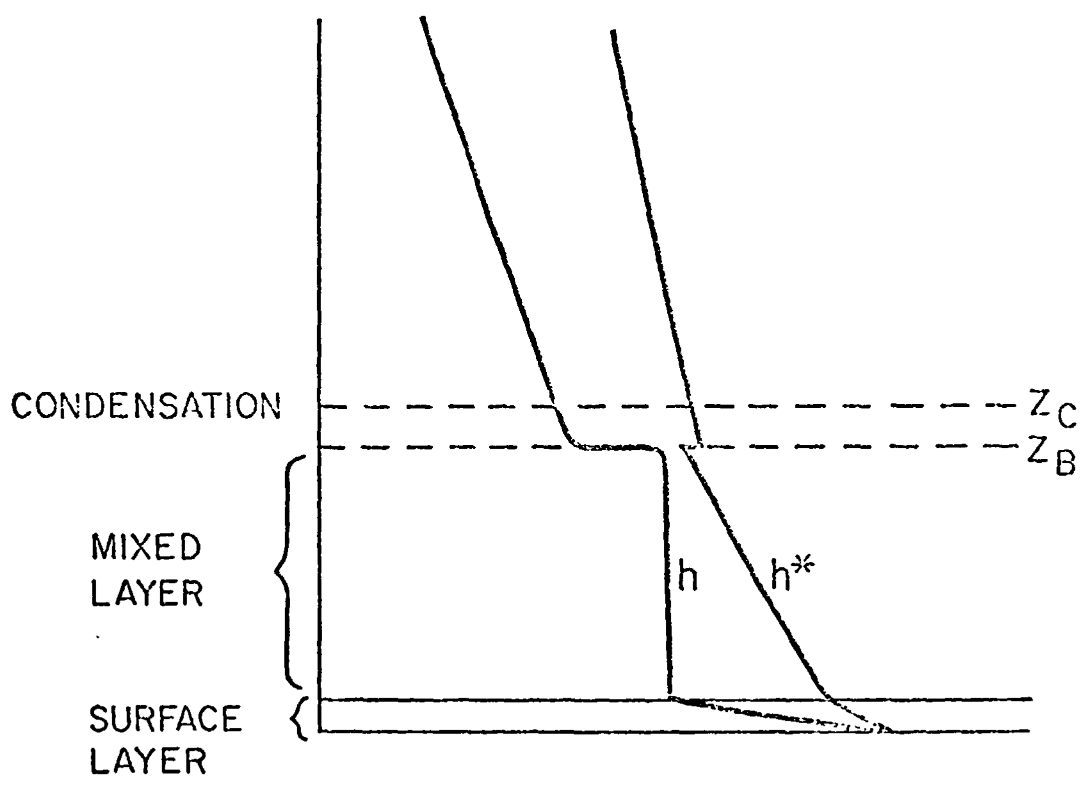
$$\Delta h^* = \frac{1}{1+\gamma(z_B)} \Delta s, \tag{102)'}$$

where γ is defined by (56). Typical vertical profiles of h and h^* near the mixed layer are schematically shown in Fig. (10). Because of the sign of Δh, the cumulus cloud updraft prefers to originate from the mixed layer. In this paper, we shall only treat that type of situation.

In the mixed layer

$$\rho \frac{\partial s_M}{\partial t} = -\rho \bar{\underset{\sim}{V}} \cdot \nabla s_M - \frac{\partial F_s}{\partial z} + Q_R, \tag{103}$$

$$\rho \frac{\partial q_M}{\partial t} = -\rho \bar{\underset{\sim}{V}} \cdot \nabla q_M - \frac{\partial F_q}{\partial z}, \tag{104}$$


CONDENSATION
Z_C
Z_B
MIXED LAYER
h
h*
SURFACE LAYER

where F_s and F_q are the vertical turbulent eddy fluxes of s and q. Integration of (103) and (104) with respect to z from zero to z_B gives

$$\rho_M \frac{\partial s_M}{\partial t} = -(\rho \tilde{\bar{V}})_M \cdot \nabla s_M + \frac{1}{z_B}\left[(F_s)_0 - (F_s)_B\right] + (Q_R)_M, \tag{105}$$

$$\rho_M \frac{\partial q_M}{\partial t} = -(\rho \tilde{\bar{V}})_M \cdot \nabla q_M + \frac{1}{z_B}\left[(F_q)_0 - (F_q)_B\right], \tag{106}$$

where

$$\rho_M \equiv \frac{1}{z_B}\int_0^{z_B} \rho \, dz, \tag{107}$$

$$(\rho \tilde{\bar{V}})_M \equiv \frac{1}{z_B}\int_0^{z_B} \rho \tilde{\bar{V}} \, dz, \tag{108}$$

$$(Q_R)_M \equiv \frac{1}{z_B}\int_0^{z_B} Q_R \, dz. \tag{109}$$

$(F_s)_0$ and $(F_q)_0$ are the fluxes of s and q at the surface, while $(F_s)_B$ and $(F_q)_B$ are the fluxes of s and q just beneath z_B (see Fig. 9). The turbulent eddy fluxes jump to zero across z_B since the turbulence is confined below this level.

To derive equations for the time change of z_B we consider the heat and moisture budgets in the infinitely thin transition layer shown in Fig. 9. M_B is given by

$$M_B \equiv M_c(z_B) = \int_0^{\lambda_{max}} m_B(\lambda) \, d\lambda, \tag{110}$$

which is the total vertical mass flux toward the cloud base at level z_B and $M_B - \rho_B \bar{w}_B$ is the subsidence between the clouds at the top of the mixed layer. The mass flux into the mixed layer, the thickness of which may be changing with time, is given by $\rho_B (Dz_B/Dt - \bar{w}_B) + M_B$.

The downward fluxes of s and q through the top of the transition layer, at $z = z_B +$, are

$$(s_M + \Delta s)\left[\rho_B\left(\frac{Dz_B}{Dt} - \bar{w}_B\right) + M_B\right], \tag{111}$$

$$(q_M + \Delta q)\left[\rho_B\left(\frac{Dz_B}{Dt} - \bar{w}_B\right) + M_B\right]. \tag{112}$$

The downward fluxes of s and q through the bottom of the transition layer, at z_B, are

$$s_M\left[\rho_B\left(\frac{Dz_B}{Dt} - \bar{w}_B\right) + M_B\right] - (F_s)_B, \tag{113}$$

$$q_M\left[\rho_B\left(\frac{Dz_B}{Dt} - \bar{w}_B\right) + M_B\right] - (F_q)_B. \tag{114}$$

Here we have defined

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{\mathbf{v}}_B \cdot \nabla. \tag{115}$$

The continuity of heat and moisture fluxes across z_B yields

$$\rho_B \frac{Dz_B}{Dt} = -(M_B - \rho_B \bar{w}_B) - \frac{1}{\Delta s}(F_s)_B, \tag{116}$$

$$\rho_B \frac{Dz_B}{Dt} = -(M_B - \rho_B \bar{w}_B) - \frac{1}{\Delta q}(F_q)_B. \tag{117}$$

This effect is of critical importance when the upper part of the mixed layer is saturated and has a layer of stratus cloud (see Lilly (1968)), but not with our unsaturated mixed layer. Consistency of (116) and (117) requires

$$(F_q)_B = \frac{\Delta q}{\Delta s}(F_s)_B \, . \tag{118}$$

A relation between the fluxes of virtual s at the bottom and top of the mixed layer can be derived from the turbulent energy balance as given by Lilly (1968). We let

$$(F_{sv})_B = -k(F_{sv})_0 \, . \tag{119}$$

The flux of virtual s is given in terms of the fluxes of s and q by

$$F_{sv} = F_s + c_p T \delta F_q \tag{120}$$

where T is a reference temperature, which is assumed to be constant, and δ is equal to 0.61. Lilly (1968) discussed the extreme cases $k = 0$ and $k = 1$, which he called the minimum and [illegible]ntrainment cases. The minimum entrainment case corresponds to total friction dissipation of turbulent kinetic energy, while the maximum entrainment case corresponds to zero frictional dissipation of turbulent kinetic energy. Deardorff et al. (1969) suggested $k \doteq 0.10$, from laboratory experiments on non-steady penetrative convection in a water tank. Betts (1972) suggested $k \doteq 0.25$.

(118) and (119) can be combined to give

$$(F_s)_B = -\frac{k\Delta s}{\Delta s_v}(F_{sv})_0 \, , \tag{121}$$

$$(F_q)_B = -\frac{k\Delta q}{\Delta s_v}(F_{sv})_0 \, , \tag{122}$$

where

$$\Delta s_v = \Delta s + c_p T \delta \Delta q \, . \tag{123}$$

Positive Δs_v is required for dry convective stability. (121) and (122) show that with an upward surface flux of virtual s, there is a downward flux of s and an upward flux of q at z_B. This is shown schematically in Fig. 8.

Eqs. (105), (106), (116), (121), and (122) combine to give

$$\rho_M \frac{\partial s_M}{\partial t} = -(\rho \bar{\mathbf{V}})_M \cdot \nabla s_M + \frac{1}{z_B}\left[(F_s)_0 + k\frac{\Delta s}{\Delta s_v}(F_{sv})_0\right] + (Q_R)_M, \tag{124}$$

$$\rho_M \frac{\partial q_M}{\partial t} = -(\rho \bar{\mathbf{V}})_M \cdot \nabla q_M + \frac{1}{z_B}\left[(F_q)_0 + k\frac{\Delta q}{\Delta s_v}(F_{sv})_0\right], \tag{125}$$

$$\rho_B \frac{D z_B}{Dt} = -(M_B - \hat{\rho}_B \bar{w}_B) + \frac{k}{\Delta s_v}(F_{sv})_0. \tag{126}$$

Eqs. (124) and (125) give

$$\rho_M \frac{\partial h_M}{\partial t} = -(\rho \bar{\mathbf{V}})_M \cdot \nabla h_M + \frac{1}{z_B}\left[(F_h)_0 + k\frac{\Delta h}{\Delta s_v}(F_{sv})_0\right] + (Q_R)_M, \tag{127}$$

where $(F_h)_0$ is the surface flux of h.

In (124) through (127), Δs_v appears as a denominator. When Δs_v is small, the direct estimate of Δs_v may be inaccurate and cause a difficulty. Following Deardorff et al. (1969) and Betts (1972), we can derive the following alternative approximate expression for Δs_v, which may be used when Δs_v is small (see Appendix I):

$$\Delta s_v \doteq \frac{k}{1+k}\, z_B \left(\frac{\partial \bar{s}_v}{\partial z}\right)_{z=z_B+} \quad \text{for small } \Delta s_v. \tag{128}$$

Eq. (126) reflects the fact that what really determines the mass inflow into the mixed layer is the entrainment due to turbulent eddies, which depends on the turbulent eddy flux of virtual s at the surface. Without the entrainment, the top of the mixed layer is simply pushed down by the subsidence $M_B - \rho_B \bar{w}_B$.

Without cumulus clouds ($M_B = 0$), the depth of the mixed layer increases with time, when $\rho_B w_B + k(F_{SV})_0/\Delta s_v > 0$. With cumulus clouds, however, the cumulus induced subsidence between the clouds counteracts the deepening. When the cumulus ensemble is very active, the subsidence may even make the mixed layer shallower. However, the mixed layer cannot be made too shallow, because the shallower the mixed layer becomes, the less is the fraction of air which enters the cloud from the mixed layer and the greater is the fraction of air which enters the cloud from the environment above the mixed layer; and that environmental air has not been reached by the turbulent upward transport of moisture. A shallow mixed layer is therefore not favorable for maintaining an intensely active cumulus ensemble. In this sense, the variable z_B is a part of the mechanism which controls the total mass flux into the clouds from the mixed layer. Betts (1972) obtained M_B from (126) and (128), for a given $\bar{w}_B$ and $(F_{SV})_0$, assuming $z_B = z_C$. We do not assume $z_B = z_C$, however, but we let z_B vary in time, in order to let the thickness of the mixed layer be one of the controls on the intensity of the cumulus convection. A more quantitative formulation of this mechanism will be given later in this paper.

In this section, we have shown that z_B, s_M, q_M and, therefore, h_M can be determined prognostically. However, whether we can represent $s_c(z_B,\lambda)$, $q_c(z_B,\lambda)$ and $h_c(z_B,\lambda)$ by the characteristic values

s_M, q_M and h_M in the mixed layer, is a difficult question. It depends on whether the cumulus clouds have their roots in the thermodynamical variables within the mixed layer. It has been reported that the roots are not observed for trade wind cumuli (Bunker et al., 1949; see also Riehl, 1954). The evaporation from falling precipitation would give $s(z_B,\lambda)$ lower than s_M and $q(z_B,\lambda)$ higher than q_M, but would leave h_M unmodified. $\hat{\ell}(z)$ depends on $q(z_B,\lambda)$, but even more on how we parameterize $r(z,\lambda)$. Apparently, some perturbations of the mixed layer are necessary for triggering the onset of a cloud within an otherwise uniform environment. But even there, the perturbation could be on z_B, rather than on s_M or q_M (see Malkus,). We postulate that the primary role of the mixed layer is to supply its moisture and static energy to the cumulus clouds, rather than triggering the onset of the clouds, and we let

$$s_c(z_B,\lambda) = s_M, \tag{129}$$

$$q_c(z_B,\lambda) = q_M, \tag{130}$$

$$h_c(z_B,\lambda) = h_M. \tag{131}$$

However, the existence of more or less organized updrafts at level z_B, toward the cloud base, is required for the mixed layer to supply its moisture to the clouds.

6. The cloud work function

Our final problem is to find the mass flux distribution function, $m_B(\lambda)$. What we have done up to this point is relatively straightforward. Of course, our models of cumulus clouds, cloud environment, subcloud mixed layer, and of their interactions are highly idealized. But even so, the real conceptual difficulty in parameterizing cumulus convection starts from this point. We must answer the question: how do the large-scale processes control the spectral distribution of clouds, in terms of the mass flux distribution function, $m_B(\lambda)$, assuming that they do so at all? This is the essence of the parameterization problem.

In the special case when the mass flux distribution function has a sharp maximum around a certain λ , the entrainment relationship (94) will give the predominant size of the clouds. In this particular case, therefore, finding $m_B(\lambda)$ will also solve the problem of the cumulus size. However, as Simpson stated (1971), "Although cumulus size appears to be at least roughly proportional to the horizontal convergence in the synoptic regime, what really determines the scale of convection remains one of the critical unsolved problems in meteorology".

The solution of $m_B(\lambda)$ may be even more difficult, because we must determine the spectral distribution, and not only the predominant size. However, in a parameterization theory, it is necessary to find only the most probable statistical properties of the cumulus ensemble, under given ~~place~~ large-scale conditions, and not the properties of an individual cloud at a given place and time. Also, what we must obtain is the mass flux distribution, and not necessarily the population distribution in λ-space. These two are generally not equivalent.

One might take the view that the mass flux distribution function is determined entirely by subcloud layer processes. Such a point of view is supported by one-dimensional cumulus cloud models, in which the horizontal velocity components and induced pressure gradients are neglected. With the exception of precipitation effects, such one-dimensional cloud models do not allow dynamical interaction between the upper and lower parts of the cloud. An initial cloud base condition determines the solution only along a characteristic line in the z-t plane. While such a model can predict the height and some other properties of the cloud top reasonably well, the prediction of the properties of the cloud air which follows the cloud top is doubtful, unless proper cloud base conditions are given as a continuous time sequence of 'initial' conditions. Such 'initial' conditions and, therefore, the time integrated mass flux at cloud base can be specified independently from the dynamics of the cloud, because there is no way for the dynamical processes above cloud base to control them, as long as an updraft is produced near the cloud base. Therefore, only local subcloud layer processes remain as a possible mechanism for determining the cloud base conditions.

The situation is different in models which have more than one dimension. An impulse is still needed, if the initial condition is otherwise unform horizontally, but only for the first cloud to get started. After the initial time, the cloud base conditions for the first cloud and for all subsequent clouds cannot be specified but are determined as a part of the solution of the entire system of equations, which includes the dynamics of both the cloud and subcloud layers. Small turbulent perturbations in the mixed layer below the clear environment are not likely to trigger new clouds, if the top of the mixed layer is sufficiently far below the condensation level. Thus, the formation of new clouds between existing clouds usually requires stronger impulses, possibly stimulated by the downdrafts associated with neighboring clouds. Otherwise, a new cloud (or a new active part of a cloud) is likely to form in the wake of a preceding cloud (or a preceding active part of a cloud), because the solenoidal field associated with the preceding cloud (or the preceding active part of the cloud) produces a circulation in a vertical plane.

It was pointed out in section 4 that the sub-ensemble mass flux is the population times the mass flux of a single cloud, averaged in time over its entire life. At level z_B, we have

$$\mathcal{M}_B(\lambda)\,d\lambda = \frac{n(\lambda)\,d\lambda}{\tau(\lambda)}\, m_B(\lambda). \qquad (132)$$

Here $n(\lambda)d\lambda$ is the population of the sub-ensemble. $\tau(\lambda)$ is the lifetime and $m_B(\lambda)$ is the vertical mass flux, at level z_B, of <u>a single cloud</u> integrated over its entire life time. Because all clouds which exist at time t must have formed during the time interval $(t-\tau(\lambda), t)$, $n(\lambda)\,d\lambda/\tau(\lambda)$ is the rate of cloud formation. This corresponds to the "dispatcher function" of Ooyama (1971). $m_B(\lambda)$ is the total mass which passes level z_B through the entire life of a single cloud. If we represent each cloud by a spherical bubble, as Ooyama (1971) did, the total mass becomes the mass of the bubble, which depends on its radius only. Because the radius is related to the fractional rate of entrainment λ through an entrainment relation similar to (94), the total mass $m_B(\lambda)$ becomes a prescribed function of λ, which remains the same regardless of the large-scale conditions. Then the mass flux distribution function $\mathcal{M}_B(\lambda)$ for different large-scale conditions is controlled only through different "dispatcher functions". This agrees with Ooyama's conclusion.

However, if we do not assume sphericity, or any other prescribed geometry which relates the vertical dimension to the horizontal size, the functional form for $m_B(\lambda)$ is unknown. $m_B(\lambda)$ is a gross measure of the activity of a single cloud of type λ, and it is highly probable that large-scale conditions control the mass flux distribution function by giving different functional forms to $m_B(\lambda)$.

The numerical simulation of a cloud by Ogura and Takahashi (1971), with a 'one and half' dimensional model, clearly shows that the time-integrated mass flux near the cloud base is highly sensitive to the reate of conversion of cloud droplets to rain drops (see Fig. 7 of their paper). When the conversion rate is sufficiently small, the cloud attains a steady state[1] (i.e. $m_B(\lambda)$ is infinitely large), while the cloud undergoes a life cycle ($m_B(\lambda)$ is

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[1] The time change of the environment is neglected in this model.

finite) with larger conversion rates. According to their interpretation, this difference is due to the different drag force by rain drops in the middle portion of the cloud where the buoyancy force is acting. This does not directly show that $m_B(\lambda)$ is sensitive to the large-scale conditions, but it does indicate that $m_B(\lambda)$ is sensitive to work done by forces in the middle portion of the cloud.

Recently, rapid progress towards the realistic simulation of cumulus clouds has been made with two-dimensional models, notably by Arnason et al. (1968, 1969), Murray et al. (1970, 1971), Orville et al. (1968, 1970), Takeda (1971) and Wilhelmson and Ogura (1972). But so far the studies have been limited to the formation and the decay of a single cloud. There has been no long-term numerical simulation of a cumulus cloud ensemble analogous to the numerical simulation of the general circulation of the atmosphere.

In this paper, we do not attempt to determine $h(\lambda)$, $\tau(\lambda)$ and $m_B(\lambda)$ separately, although that should be an eventual goal of statistical cumulus dynamics. But the preceding arguments suggest that we look into the generation of the kinetic energy of cumulus clouds.

The time change of the kinetic energy of each sub-ensemble can be written as

$$\frac{d\mathcal{K}(\lambda)}{dt} = A(\lambda)\, m_B(\lambda) - \mathcal{D}(\lambda), \tag{133}$$

where $\mathcal{K}(\lambda)\,d\lambda$ and $\mathcal{D}(\lambda)\,d\lambda$ are respectively the kinetic energy and the dissipation rate due to all the clouds with fractional entrainment rates between

λ and $\lambda + d\lambda$. The first term on the right of (133) is the rate of generation of kinetic energy by buoyancy forces. The work done by downward drag due to raindrops is included in the dissipation. It is important to note that (133) holds for three-dimensional clouds, as far as the clouds are driven mainly by buoyancy force. Mechanical interactions with the vertical shear of the horizontal velocity of the environment and with other types of clouds are neglected.

$A(\lambda)$ is the kinetic energy generation per unit $m_B(\lambda)\,d\lambda$ or the efficiency of kinetic energy generation for cloud type λ. It is given by

$$A(\lambda) = \int_{z_B}^{z_D(\lambda)} \frac{g}{c_p \bar{T}(z)} \eta(z,\lambda)\left(s_c(z,\lambda) - \bar{s}(z)\right) dz\,. \qquad (134)$$

We call $A(\lambda)$ the "cloud work function" for type λ. Eqs. (97), (98) and (131) give

$$\eta(z,\lambda)\left(s_c(z,\lambda)-\bar{s}(z)\right) = \frac{1}{1+\gamma(z)}\left[h_M + \lambda\int_{z_B}^{z}\eta(z',\lambda)\bar{h}(z')dz' - \eta(z,\lambda)\bar{h}^*(z)\right]$$

$$\text{for} \quad z_c(\lambda) \leqq z \leqq z_D(\lambda), \qquad (135)$$

and

$$\eta(z,\lambda)\left(s_c(z,\lambda)-\bar{s}(z)\right) = s_M + \lambda\int_{z_B}^{z}\eta(z',\lambda)\bar{s}(z')dz' - \eta(z,\lambda)\bar{s}(z)$$

$$\text{for} \quad z_B \leqq z \leqq z_c(\lambda). \qquad (136)$$

The condensation level is given by the consistency of (135) and (136) at $z = z_c(\lambda)$. An approximate expression for $z_c(\lambda)$ is given in Appendix II. From (134), (135) and (136), we can see that $A(\lambda)$ is a static property of the environment, including the subcloud mixed layer.

In the special case of $z_c(\lambda) = z_B^+$, substitution of (135) into (134) gives

$$A(\lambda) = \int_{z_{B^+}}^{z_D(\lambda)} \rho(z)\beta(z)\left[h_M + \lambda\int_{z_{B^+}}^{z}\eta(z',\lambda)\bar{h}(z')\,dz' - \eta(z,\lambda)\bar{h}^*(z)\right]dz, \quad (137)$$

and
$$\rho(z)\beta(z) \equiv \frac{g/c_p}{\bar{T}(z)(1+\gamma(z))}. \quad (138)$$

We have the identity

$$\eta(z,\lambda)\bar{h}^*(z) = \bar{h}^*(z_{B^+}) + \int_{z_{B^+}}^{z} \frac{\partial}{\partial z'}\left(\eta(z',\lambda)\bar{h}^*(z')\right)dz', \quad (139)$$

Using (139) and (95) in (137), we obtain

$$A(\lambda) = \int_{z_{B^+}}^{z_D(\lambda)} \rho(z)\beta(z)\left[h_M - \bar{h}^*(z_{B^+}) + \int_{z_{B^+}}^{z_D(\lambda)} \eta(z',\lambda)\left\{\lambda\left(\bar{h}(z') - \bar{h}^*(z')\right) - \frac{\partial \bar{h}^*(z')}{\partial z'}\right\}dz'\right]dz. \quad (140)$$

$\rho(z)\beta(z)\left(h_M - \bar{h}^*(z_{B^+})\right)$ is the buoyancy at cloud base. $-\partial\bar{h}^*/\partial z$ is positive when the lapse rate is steeper than the moist adiabatic lapse rate. When there is no buoyancy at cloud base and the environment is saturated $(\bar{h} = \bar{h}^*)$, (142) is simply a measure of the conditional instability. When the environment is not saturated, the contribution to $A(\lambda)$ by $\bar{h} - \bar{h}^*$ (i.e. $L(\bar{q} - \bar{q}^*)$) is always negative and opposes the positive contribution by the conditional instability. To have a positive $A(\lambda)$, which is necessary for kinetic energy generation, the environment must be not only conditionally unstable, but also moist enough to give a sufficiently small $|\bar{h} - \bar{h}^*|$. This effect becomes increasingly important as λ increases. Unentraining clouds, for which $\lambda = 0$, are of course not influenced by the environmental humidity above the cloud base.

$A(\lambda) > 0$ can therefore be considered as a generalized criterion for moist convective instability. Because of the entrainment, the criterion depends on cloud type. $A(\lambda) \leq 0$ for all λ gives a stable environment.

7. The quasi-equilibrium assumption

The purpose of cumulus parameterization is to relate the statistical properties of a cumulus cloud ensemble to the large-scale variables, and thereby to obtain a closed system of prognostic equations for the large-scale variables. However, there is no apriori reason to believe that this is always possible.

Suppose that initially there are no cumulus clouds but the vertical distributions of temperature and moisture are such that the cloud work function $A(\lambda)$ is positive for a certain range of λ. Cumulus clouds will then form and develop, and, sooner or later, will enter a non-linear regime. In this regime the environment will be modified by the cumulus clouds. Typically, the modification will be such that moisture is removed from the environment, especially from its lower part, and warming occurs in the environment, especially in its upper part. The cloud work function will then decrease in time and eventually approach zero for the entire range of λ

(i.e. a neutral state is reached). We call the time needed for this adjustment to a neutral state "the adjustment time scale", τ_{ADJ}. After this adjustment is over, there will be no generation of kinetic energy, and the clouds will dissipate. We call the time needed for dissipation "the dissipative time scales", $\tau_{DIS}(\lambda)$.

However, if there is a counteracting destabilization (a generation of the cloud work function) by large-scale processes, which we call "the large-scale forcing", the cumulus activity will be maintained. If the large-scale forcing is constant in time, it is probable that the cumulus ensemble will approach an equilibrium. The time needed to reach the equilibrium should be about the same as the adjustment time, at least when the large-scale forcing is weak.

When the large-scale forcing is changing in time, the cumulus ensemble will not reach an equilibrium. But when the time change of the forcing is sufficiently ~~slow,~~ slower than the adjustment time, we can expect the cumulus ensemble to approach a sequence of quasi-equilibria. In such a sequence of quasi-equilibria, the large-scale forcing and the cumulus ensemble vary in time in a coupled way, and, therefore, the time scale of the statistical properties of the ensemble is equal to the time scale of the large scale fields, τ_{LS}. We call this relationship "the quasi-equilibrium assumption". It is also an assumption on parameterizability, because, unless a cumulus ensemble is in quasi-equilibrium with the large-scale processes, there is no hope that we can uniquely relate the statistical properties of the ensemble to the large-scale variables.

The quasi-equilibrium assumption applied to the spectral distrib tion of the kinetic energy allows us to write

$$\frac{d\,K(\lambda)}{dt} \sim \frac{K(\lambda)}{\tau_{LS}} . \tag{141}$$

From the definition of the dissipative time scale, we have

$$D(\lambda) \sim \frac{K(\lambda)}{\tau_{DIS}(\lambda)} . \tag{142}$$

For $\tau_{LS} \gg \tau_{DIS}(\lambda)$ (typically, $\tau_{LS} \sim 10^5$ sec and $\tau_{DIS}(\lambda) \lesssim 10^3 sec$, we see that the time change of the kinetic energy is negligible in (133), and we have a balance between the kinetic energy generation and the dissipation. When cumulus clouds of type λ exist ($m_B(\lambda) > 0$) , we have

$$A(\lambda) = \frac{D(\lambda)}{m_B(\lambda)} . \tag{143}$$

The right hand side of (143) is the dissipation per unit $m_B(\lambda)\,d\lambda$ or the efficiency of dissipation for cloud type . We call this the "cloud dissipation function". Because the cloud dissipation function must be positive, the cloud work function must also be positive, for any cloud type which exists ($m_B(\lambda) > 0$).

8. Generation and destruction of the cloud work function

In this section, we consider the time derivative of the cloud work function. Because the cloud work function depends only on the vertical distributions of the static energy and the water vapor mixing ratio of the environment, including the subcloud mixed layer, the prognostic equations (74), (75), (124), (125) and (126), which respectively govern the time derivatives of $\bar{s}(z)$, $\bar{q}(z)$, s_M, q_M and z_B, are sufficient to give the time derivative of the cloud work function. Those prognostic equations involve terms of two types: "cloud terms" which depend on the mass flux distribution function either through the total vertical mass flux, $M_c(z)$ (or M_B) or through the total detrainment, $D(z)$; and "large-scale terms" such as large-scale advection, surface eddy fluxes and radiational heating terms, which do not depend on the mass flux distribution function. Then the time derivative of the cloud work function can also be expressed as a summation of cloud terms and large-scale terms. We may write

$$\frac{dA(\lambda)}{dt} = \left(\frac{dA(\lambda)}{dt}\right)_c + F(\lambda), \tag{144}$$

where the suffices c denote the cloud terms. $F(\lambda)$, which we call "large-scale forcing", represents large-scale terms. Positive $F(\lambda)$ means generation of the cloud work function (destabilization) for type λ by the large-scale processes.

The cloud terms $(dA/dt)_c$, linearly depend on $M_c(z)$ and $D(z)$. $M_c(z)$ is an integral transform of $m_B(\lambda)$ (see (77) and (80). $D(z)$ is also determined by $m_B(\lambda)$ (see (79) and (80)). Thus the whole spectrum of cloud types can participate in determining $M_c(z)$ and $D(z)$, and, therefore, in determining $(dA(\lambda)/dt)_c$. Then (144) may be rewritten as

$$\frac{dA(\lambda)}{dt} = \int_0^{\lambda_{max}} K(\lambda,\lambda') m_B(\lambda') d\lambda' + F(\lambda). \tag{145}$$

When the kernel $K(\lambda,\lambda')$ is negative, $K(\lambda,\lambda') m_B(\lambda') d\lambda'$ is the amount of destruction (stabilization) of the cloud work function for type λ through the modification of the environment by type λ'. However, the kernel may become positive for some combinations of λ and λ'.

The actual forms of $F(\lambda)$ and $K(\lambda,\lambda')$ are given below. A brief derivation is given in Appendix III. First we define

$$\rho(z)\alpha(z) \equiv \frac{g}{c_p \bar{T}(z)}, \qquad \rho(z)\beta(z) \equiv \frac{g}{(1+\gamma(z))\, c_p \bar{T}(z)}, \tag{146}$$

$$\rho(z)a(z,\lambda) \equiv \begin{cases} 0 & \text{for } z_c(\lambda) \leq z \leq z_D(\lambda) \\ \int_z^{z_c(\lambda)} \alpha(z')\rho(z')dz' & \text{for } z_B \leq z \leq z_c(\lambda) \end{cases} \tag{147}$$

$$\rho(z)\, b(z,\lambda) \equiv \begin{cases} \int_z^{z_D(\lambda)} \beta(z')\rho(z')dz' & \text{for } z_c(\lambda) \leq z \leq z_D(\lambda) \\ \int_{z_c(\lambda)}^{z_D(\lambda)} \beta(z')\rho(z')dz' & \text{for } z_B \leq z \leq z_c(\lambda) \end{cases} \tag{148}$$

The large-scale forcing is given by

$$F(\lambda) = F_c(\lambda) + F_M(\lambda), \tag{149}$$

where

$$F_c(\lambda) = \int_{z_B}^{z_D(\lambda)} \eta(z,\lambda) \left[-\alpha(z) \left(\frac{\partial \bar{s}}{\partial t} \right)_{LS} + \lambda \left\{ a(z,\lambda) \left(\frac{\partial \bar{s}}{\partial t} \right)_{LS} + b(z,\lambda) \left(\frac{\partial \bar{h}}{\partial t} \right)_{LS} \right\} \right] \rho(z)\, dz \tag{150}$$

and

$$F_M(\lambda) = \rho_B \left[a(z_B,\lambda) \frac{\partial s_M}{\partial t} + b(z_B,\lambda) \frac{\partial h_M}{\partial t} + \rho_B \left(\frac{\partial z_B}{\partial t} \right)_{LS} \left\{ \alpha(z_B) \Delta s - \lambda \left(\frac{A}{\rho_B} + a(z_B,\lambda) \Delta s + b(z_B,\lambda) \Delta h \right) \right\} \right]. \tag{151}$$

The suffix LS denotes large-scale terms as previously defined. We call $F_c(\lambda)$ the "cloud layer forcing" and $F_M(\lambda)$ the "mixed layer forcing".

The first term in the bracket of (150), which is the most dominant term, represents generation of the cloud work function due to the cooling of the environment by large-scale processes, typically by adiabatic cooling due to the large-scale upward motion. The term which depends on Δh seems to be at least one of the dominant terms in (151). When Δh is a large negative, the moist static energy in the mixed layer is large compared to that in the environment above the mixed layer. The effect of large-scale upward velocity at the top of the mixed layer on $\partial z_B / \partial t$ is positive, and, therefore, the cloud work function is generated. An examination of relative importance of the terms in the large-scale forcing (150) and (151) for actual observed situations will be presented in the author's subsequent paper (Schubert and Arakawa, 1973).

The kernel is given by

$$K(\lambda,\lambda') = K_V(\lambda,\lambda') + K_D(\lambda,\lambda') + K_M(\lambda), \tag{152}$$

where

$$K_V(\lambda,\lambda') = \int_{z_B}^{z_D(\lambda)} \eta(z,\lambda)\,\eta(z,\lambda')\left[-\alpha(z)\frac{\partial \bar{s}}{\partial z} + \lambda\left\{a(z)\frac{\partial \bar{s}}{\partial z} + b(z)\frac{\partial \bar{h}}{\partial z}\right\}\right] dz\,, \tag{153}$$

$$K_D(\lambda,\lambda') = \begin{cases} 0 & \text{for } \lambda' < \lambda, \\ L\,\eta(z_D',\lambda)\,\eta(z_D',\lambda')\left[\alpha(z_D')\,\hat{l}(z_D') + \lambda\, b(z_D',\lambda)\left(\bar{q}^*(z_D') - \bar{q}(z_D')\right)\right] & \text{for } \lambda' > \lambda, \end{cases} \tag{154}$$

$$K_M(\lambda) = -\alpha(z_B)\Delta s + \lambda\left\{\frac{A}{\rho_B} + a(z_B,\lambda)\Delta s + b\,(z_B,\lambda)\Delta h\right\}. \tag{155}$$

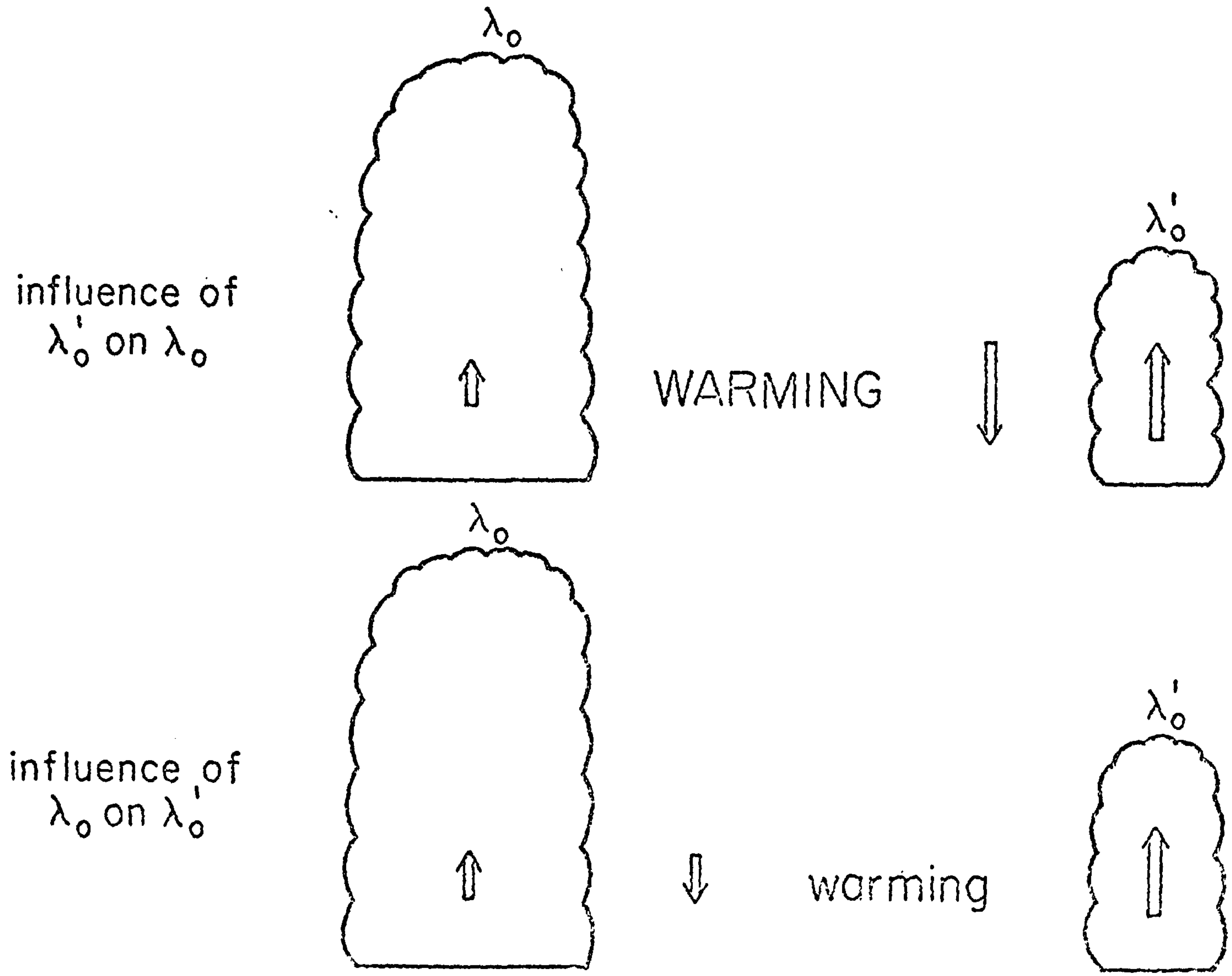
In (154), the symbol z_D' has been used for $z_D(\lambda')$. $K_V(\lambda,\lambda')$, $K_D(\lambda,\lambda')$ and $K_M(\lambda)$ originate respectively from $M_c(z)$ in (74) and (75), $D(z)$ in (74) and (75), and M_B in (126). We call $K_V(\lambda,\lambda')$ the "vertical mass flux kernel", $K_D(\lambda,\lambda')$ the "detrainment kernel '', and $K_M(\lambda)$ the "mixed layer kernel".

The most dominant term in (153) is the first term in the bracket. This term represents destruction of the cloud w rk function for type λ clouds through adiabatic warming of the environment due to the subsidence induced by type λ' clouds. Because $\eta(z,\lambda)$ vanishes above $z_D(\lambda)$, the upper limit of the integral can be replaced by $z_D(\lambda')$, if $z_D(\lambda') > z_D(\lambda)$. Then we can see that $K_V(\lambda,\lambda')$ is symmetric with respect to λ and λ', if other terms which follow the first term are neglected. This symmetry of $K_V(\lambda,\lambda')$ means that the amount of destruction of $A(\lambda)$ by type λ' clouds per unit $m_B(\lambda')d\lambda'$, (through the mechanism described above) is equal to the destruction of $A(\lambda')$ by type λ clouds per unit $m_B(\lambda)d\lambda$. This situation is illustrated in Fig. 11.

The detrainment kernel is always positive for $\lambda'>\lambda$. This means that shallower clouds generate the cloud work function of deeper clouds through cooling of the environment due to the evaporation of detrained liquid water and moistening of the environment due to the detrainment of more moist air from clouds. This situation is illustrated in Fig. 12.

The role of varying z_B as a control on the intensity of the cumulus convection was discussed in section 5. This role is represented by the last term in (155), which has a negative contribution to the mixed layer kernel because Δh is negative.

To Be Continued.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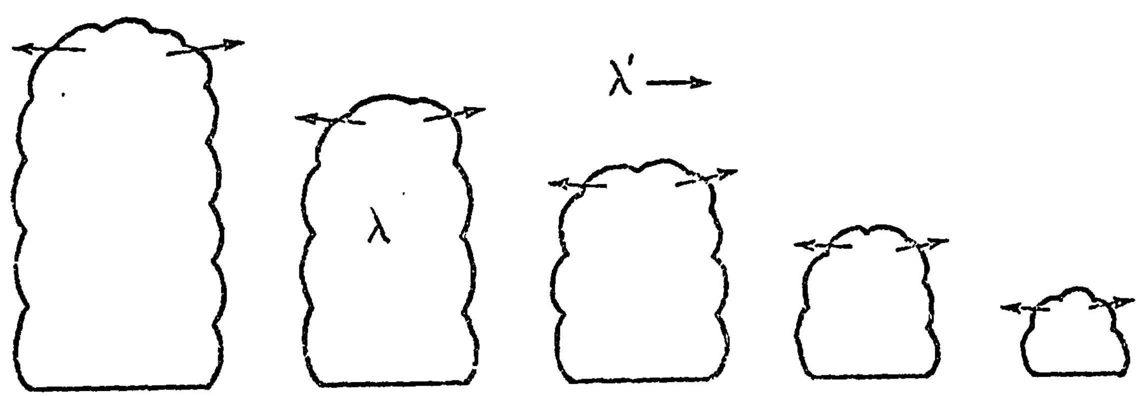


λ'
λ

Appendix III

Derivation of the Kernels and the Large Scale Forcing

Using the definition of $A(\lambda)$ given by (134) we can write

$$\frac{\partial A(\lambda)}{\partial t} = \int_{z_B}^{z_D(\lambda)} \rho(z)\alpha(z)\frac{\partial}{\partial t}\left\{\eta(z,\lambda)\left[s_c(z,\lambda)-\bar{s}(z)\right]\right\}dz - \rho_B \frac{\partial z_B}{\partial t}\alpha(z_B)\left[s_c(z_B,\lambda)-\bar{s}(z_B)\right] \quad \text{(III.1)}$$

where $\rho(z)\alpha(z)$ is defined by (146). A term involving the time change of $z_D(\lambda)$ does not appear since $z_D(\lambda)$ is a vanishing buoyancy level for cloud type λ. Using (129), (135), and (136), (III.1) can be written

$$\frac{\partial A(\lambda)}{\partial t} = \int_{z_B}^{z_c(\lambda)} \rho(z)\alpha(z)\frac{\partial}{\partial t}\left\{s_M + \lambda\int_{z_B}^{z}\eta(z',\lambda)\,\bar{s}(z')\,dz' - \eta(z,\lambda)\,\bar{s}(z)\right\}dz$$
$$+ \int_{z_c(\lambda)}^{z_D(\lambda)} \rho(z)\beta(z)\frac{\partial}{\partial t}\left\{h_M + \lambda\int_{z_B}^{z}\eta(z',\lambda)\,\bar{h}(z')\,dz' - \eta(z,\lambda)\,\bar{h}^*(z)\right\}dz$$
$$+ \rho_B\frac{\partial z_B}{\partial t}\alpha(z_B)\,\Delta S \quad \text{(III.2)}$$

where $\rho(z)\beta(z)$ is also defined by (146). After evaluation of the time derivatives in the integrands of (III.2), we can write

$$\frac{\partial A(\lambda)}{\partial t} = a(z_B,\lambda)\,\rho_B \frac{\partial s_M}{\partial t} + b(z_B,\lambda)\,\rho_B \frac{\partial h_M}{\partial t}$$

$$+ \int_{z_B}^{z_c(\lambda)} \rho(z)\alpha(z) \left\{ \lambda \int_{z_B}^{z} \eta(z',\lambda) \frac{\partial \bar{s}(z')}{\partial t} dz' - \eta(z,\lambda) \frac{\partial \bar{s}(z)}{\partial t} \right\} dz$$

$$+ \int_{z_c(\lambda)}^{z_D(\lambda)} \rho(z)\beta(z) \left\{ \lambda \int_{z_B}^{z} \eta(z',\lambda) \frac{\partial \bar{h}(z')}{\partial t} dz' - \eta(z,\lambda) \frac{\partial \bar{h}^*(z)}{\partial t} \right\} dz$$

$$+ \rho_B \frac{\partial z_B}{\partial t} \left\{ \alpha(z_B)\,\Delta s - \lambda \left[\frac{A(\lambda)}{\rho_B} + a(z_B,\lambda)\Delta s + b(z_B,\lambda)\Delta h \right] \right\}$$

where $\rho(z)a(z,\lambda)$ and $\rho(z)b(z,\lambda)$ are given by (147) and (148). The first double integral in (III.3) is over the hatched triangular region of figure while the second double integral is over the stipled area of the figure. The first double integral can be written as

$$\int_{z_B}^{z_c(\lambda)} \left\{ \int_{z'}^{z_c(\lambda)} \rho(z)\alpha(z)\,dz \right\} \lambda\,\eta(z',\lambda) \frac{\partial \bar{s}(z')}{\partial t} dz' = \int_{z_B}^{z_D(\lambda)} \eta(z,\lambda)\,\lambda\, a(z,\lambda)\,\rho(z) \frac{\partial \bar{s}(z)}{\partial t} dz \qquad \text{(III.4)}$$

while the second double integral can be written as

$$\int_{z_c(\lambda)}^{z_D(\lambda)} \left\{ \int_{z'}^{z_D(\lambda)} \rho(z)\beta(z)\,dz \right\} \lambda\,\eta(z',\lambda) \frac{\partial \bar{h}(z')}{\partial t} dz' + \int_{z_B} \left\{ \int_{z_c(\lambda)}^{z_D(\lambda)} \rho(z)\beta(z)\,dz \right\} \lambda\,\eta(z',\lambda) \frac{\partial \bar{h}(z')}{\partial t} dz'$$

$$= \int_{z_B}^{z_D(\lambda)} \eta(z,\lambda)\,\lambda\, b(z,\lambda)\,\rho(z) \frac{\partial \bar{h}(z)}{\partial t} dz \qquad \text{(III.5)}$$

Substitution of (III.4) and (III.5) into (III.3) yields

$$\frac{\partial A(\lambda)}{\partial t} = a(z_B,\lambda)\rho_B\frac{\partial s_M}{\partial t} + b(z_B,\lambda)\rho_B\frac{\partial h_M}{\partial t}$$

$$+ \int_{z_B}^{z_D(\lambda)} \eta(z,\lambda)\left\{\lambda\left[a(z,\lambda)\bar\rho(z)\frac{\partial \bar s(z)}{\partial t} + b(z,\lambda)\bar\rho(z)\frac{\partial \bar h(z)}{\partial t}\right] - \alpha(z)\bar\rho(z)\frac{\partial \bar s(z)}{\partial t}\right\}dz$$

$$+ \rho_B\frac{\partial z_B}{\partial t}\left\{\alpha(z_B)\Delta s - \lambda\left[\frac{A(\lambda)}{\rho_B} + a(z_B,\lambda)\Delta s + b(z_B,\lambda)\Delta h\right]\right\} \quad \text{(III.6)}.$$

Substitution of (74), (75), and (126) into (III.6) gives

$$\frac{\partial A(\lambda)}{\partial t} = \int_{z_B}^{z_D(\lambda)} \eta(z,\lambda)\left\{-\alpha(z)\frac{\partial \bar s(z)}{\partial z} + \lambda\left[a(z,\lambda)\frac{\partial \bar s(z)}{\partial z} + b(z,\lambda)\frac{\partial \bar h(z)}{\partial z}\right]\right\}M_c(z)dz$$

$$+ \int_{z_B}^{z_D(\lambda)} L\eta(z,\lambda)\left\{\alpha(z)\hat{\ell}(z) + \lambda\left[-a(z,\lambda)\hat{\ell}(z) + b(z,\lambda)\left(\bar q^*(z) - \bar q(z)\right)\right]\right\}D(z)dz$$

$$+ K_M(\lambda)M_B + F_c(\lambda) + F_M(\lambda) \qquad \text{(III.7)}$$

Using (77) and (79), and changing the detrainment term from an integral over z to an integral over λ', we obtain

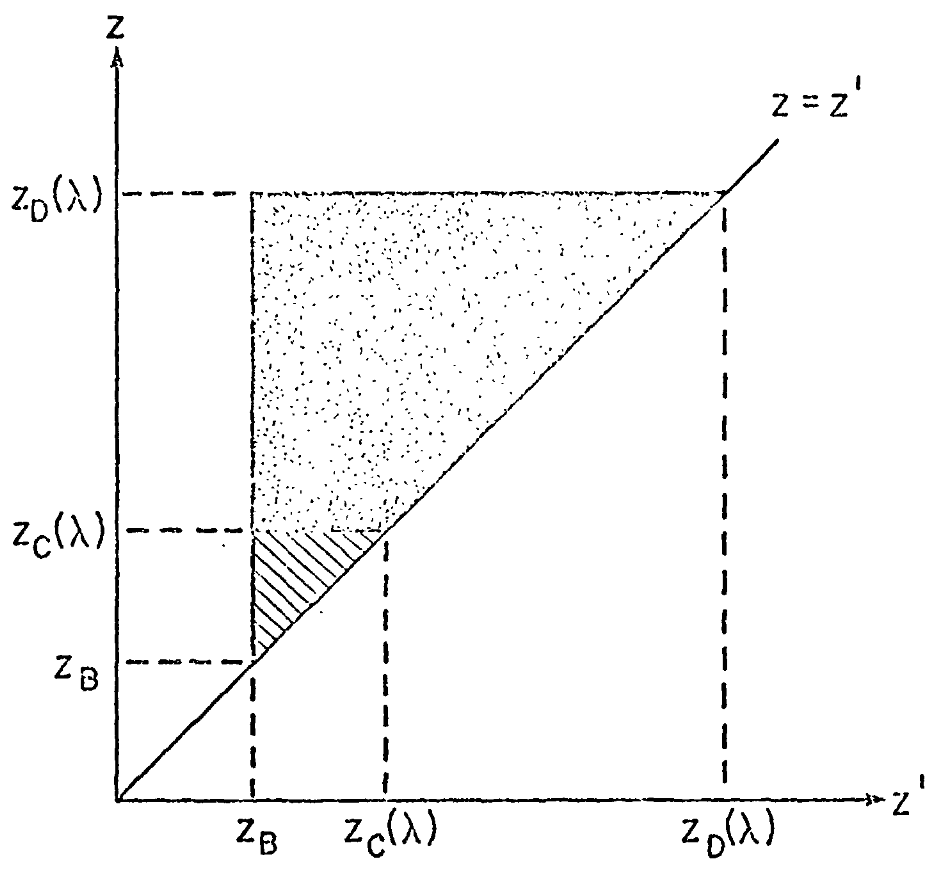
$$\frac{\partial A(\lambda)}{\partial t} = \int_0^{\lambda_{max}} K_v(\lambda,\lambda') m_B(\lambda') d\lambda' + \int_\lambda^{\lambda_{max}} K_D(\lambda,\lambda') m_B(\lambda') d\lambda'$$

$$+ \int_0^{\lambda_{max}} K_M(\lambda) m_B(\lambda') d\lambda' + F_c(\lambda) + F_M(\lambda) \tag{III.8}$$

The vertical mass flux kernel and the mixed layer kernel appear in terms of Fredholm type while the detrainment kernel appears in a term of Volterra type.

Neglecting the time change of $A(\lambda)$ and combining terms we obtain

$$\int_0^{\lambda_{max}} K(\lambda,\lambda') m_B(\lambda') d\lambda' + F(\lambda) = 0 \tag{III.9}$$



z
z = z'
$z_D(\lambda)$
$z_C(\lambda)$
z_B
z'
z_B
$z_C(\lambda)$
$z_D(\lambda)$