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16. Abstract

A survey is presented of current research in the theory of gravitational radiation. The mathematical structure of gravitational radiation is stressed. Furthermore, the radiation problem is treated independently from other problems in gravitation. The development proceeds candidly through three points of view-scalar, rector, and tensor radiation theory - and the corresponding results are stated.

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GRAVITATIONAL RADIATION THEORY

BY

THOMAS L. WILSON

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IN PARTIAL FULFILLMENT OF THE
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ABSTRACT

GRAVITATIONAL RADIATION THEORY

BY
THOMAS L. WILSON

A SURVERY IS PRESENTED OF CURRENT RESEARCH IN THE THEORY OF GRAVITATIONAL RADIATION. IRONICALLY, THEORETICAL WORK ON GRAVITATIONAL RADIATION FIRST APPEARS IN THE ELECTROMAGNETIC (VECTOR) THEORIES OF GRAVITATION - WHERE IT HAS SINCE BEEN FORGOTTEN. THIS IS IN PART DUE TO AN OVER ENCHANTMENT WITH THE MORE GENERAL TENSOR THEORY OF RADIATION, ALTHOUGH SUCH ENCHANTMENT IS WARRANTED IN THE SENSE THAT IT MAY BE THE PROPER SOLUTION TO THE PROBLEMS IN RADIATION THEORY.

SPECIAL CARE IS TAKEN TO STAY AWAY FROM THE METAPHYSICS OF GRAVITATION THEORY AND TO STICK WITH THE MATHEMATICAL STRUCTURE OF GRAVITATIONAL RADIATION. FURTHERMORE, THE RADIATION PROBLEM IS TREATED IN A FASHION ENTIRELY DIVORCED FROM OTHER PROBLEMS IN GRAVITATION. THE DEVELOPMENT PROCEEDS CANDIDLY THROUGH THREE POINTS OF VIEW - SCALAR, VECTOR, AND TENSOR RADIATION THEORY - AND THE CORRESPONDING RESULTS ARE STATED.

GRAVITATIONAL RADIATION THEORY

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SCENARIO

(Enter Horatio, Brutus, & Cassius)

Horatio: Alas. My experiment must not work. I have been searching for gravitational radiation for months now and I have found nothing.

Brutus: If you see nothing, then there's nothing there. The stars must not emit gravitational radiation.

Horatio: Not necessarily. Perhaps we are looking for the wrong thing.

Cassius: The fault, dear Brutus, is not in the stars but in ourselves - that we are poor Masters of human thought. And not until we conquer ourselves shall we ever master the secrets of the Universe.

Brutus: Perhaps. Nevertheless, if there's nothing there then the linear tensor theory is wrong.

Horatio: That is incomprehensible. If the tensor theory is wrong we have no basis for an expanding Universe.

Cassius: Horatio, dear Horatio, here you are - standing upon a speck of dust trapped in a sunbeam - and you have the audacity to state that the Universe is expanding.

Horatio: Cassius, I know the Universe is expanding. I have measured the cosmological redshift. Indeed, that is why the heavens are black at night.

Cassius: Horatio, what you have discovered is a footprint in the sand. And behold - it is your own.

CHAPTER I

INTRODUCTION

The notion of gravitational radiation is an old one. It has as its beginnings the scalar theory of Newtonian mechanics, it becomes manifest and explicit in the electromagnetic vector theories of gravitation, and it is finally extended to its most general form in the tensor theories of relativity. The purpose here is to consider the nature of gravitational radiation. Thence, it is our hope to investigate and to understand the linear and the nonlinear characteristics of scalar, vector, and tensor radiation theory.

Early considerations of gravitational waves can be found in the work of Hooke (1671) on the physical cause of gravitation and of Laplace (1802) on Newtonian cosmology, but not until the advent of classical electromagnetism does gravitational radiation take a very definite form. This is the investigation of Maxwell (1865) into an electromagnetic theory of gravitation. Subsequently, theoreticians are found applying numerous features of the laws of classical electrodynamics to planetary orbits in an effort to account for several anomalies in celestial mechanics. The most popular problem to be found is the anomalous precession of the perihelion of the planet Mercury which is explained by mechanisms (all of them due to Newton) such as solar oblateness and variations in the inverse-square law. However, the concepts of electrodynamics (not forseen by Newton) are found to introduce suggestions of a velocity-dependent force fashioned by Höllzmüller (1870) after the electron theories of Weber (1846) and Riemann (1861). Adopting the precessional behaviour of Mercury, for example, as an experimental basis for the determination of the velocity of propagation of a retarded gravitational potential, one can arrive at the value of Gerber (1895): $c = 305{,}500$ km/sec, within 2% of the speed of light.

The lasting dividend of these 19th Century investigations is the gravitational wave. The shortcomings of Newtonian mechanics thus result in the electromagnetic (vector) theory of gravitation, definitively stated by Heaviside (1893) - not to mention the work of Lorentz, Poincaré, Lodge, Ishiwara (1914), and H.A. Wilson (1921). Vector gravitational waves are treated explicitly by Gans (1905, 1912) and Abraham (1911, 1912, 1913), who are the first to identify the annoying features of negative Poynting vectors, negative energy densities, and particle instabilities under gravitational radiation reaction. But the vector theories of gravitation are found to be inadequate by some and consequently this is one of the justifications for the more general theory of gravitation, the tensor theory due to Einstein (1915, 1916, 1918). Einstein's treatment of mass quadrupole radiation is the first for tensor gravitational waves. As closer study demonstrates, however, there is nothing intrinsic in the tensor theory which excludes the possibility of negative energy states discussed by Maxwell in 1865.

The nonlinear nature of Einstein's tensor theory of gravitational radiation is its most difficult, provocative, and yet rewarding feature. But most of the success in its application has been due to the many analogies with electromagnetism and the numerous linearization schemes, simply because the exact solutions of the nonlinear field equations in this theory are formidable, difficult to interpret, and beset with mathematical singularities which cannot be associated with radiative sources.

There is even controversy in the tensor theory of gravitational radiation as to its very existence, due to its covariance and often intractable nonlinearities. In the vector theory, however, there is really no such question; rather it is the physical interpretation of the difference between electromagnetic and vector gravitational waves and their resonant interactions (it is a unified field theory). Regardless, the concept of gravitational radiation now offers an exciting new vista in theoretical physics and may even become a definitive test of the wealth of existing theories of electromagnetism and gravitation.

CHAPTER II

ELECTROMAGNETIC THEORIES OF GRAVITATIONAL RADIATION

The Scalar Theory Of Gravitational Radiation

The concept of a gravitational wave follows from a wave eq[illegible] Recall from the Newtonian theory of gravitation that a force $\underset{\sim}{F}$ ex[illegible]ed upon [illegible] point mass m at a distance $\underset{\sim}{r}$ from another mass M is the equation of motion

$$\underset{\sim}{F} = -\frac{GMm}{r^3}\underset{\sim}{r} \tag{2-1}$$

where G is the Newtonian constant of gravitation. This relation in turn defines a field intensity $\underset{\sim}{g}(\underset{\sim}{x})$ which is a force per unit mass,

$$\underset{\sim}{g} = -\frac{GM}{r^3}\underset{\sim}{r} \tag{2-2}$$

For a multip[illegible]e distribution of matter $\rho(\underset{\sim}{x}')$ it follows that the field intensity $\underset{\sim}{g}$ is derivable from a scalar potential ϕ as

$$\underset{\sim}{g} = -\underset{\sim}{\nabla}\phi \tag{2-3}$$

where

$$\phi = \int_{\infty}^{r} \frac{GM}{r^2}\,dr = -G\int \frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,d^3\underset{\sim}{x}' = -\frac{GM}{r}. \tag{2-4}$$

Taking the divergence of Eq (2-3) one obtains the differential form of Gauss' law and a zero curl

$$\nabla\cdot\underset{\sim}{g} = -4\pi G\rho \tag{2-5a}$$

$$\nabla\times\underset{\sim}{g} = 0 \tag{2-5b}$$

Hence, the field is irrotational. Eqs (2-5) and (2-3) together constitute Poisson's Equation

$$\nabla^2\phi = -4\pi G\rho \qquad (2\text{-}6)$$

for this static and stationary case of a scalar Newtonian potential ϕ. However, the theory is as yet nonradiative. It provides only for the inductive transfer of energy between Newtonian masses, such as by the mechanism of tidal friction (Appendix G & S).

To obtain a radiative scalar theory one borrows from his understanding of the vector theory - which has a scalar component - and argues that a nonstationary scalar potential due to an oscillatory multipole distribution $\rho = \rho_0 \exp(i\omega t)$ extends Poisson's Equation to the wave equation

$$\Box\phi = -\kappa\rho \qquad (2\text{-}7)$$

where $\Box = \nabla^2 - c^{-2}\partial_t^2$, where $\kappa = 4\pi G$, and where c is the velocity of propagation of the scalar gravitational wave ϕ. The solution of this wave equation is essentially (2-4),

$$\phi = -G\int \frac{[\rho]}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, d^3\underset{\sim}{x}' \qquad (2\text{-}8a)$$

except that the brackets qualify the retarded solution of Lorenz (1867), the advanced solution of Ritz (1908), or standing waves (both).

A multipole expansion of ϕ in (2-8) states that

$$\phi = -\frac{GM}{r} - \frac{G\underset{\sim}{p}\cdot\underset{\sim}{r}}{cr^2} - \cdots \qquad (2\text{-}8b)$$

The Newtonian (Coulomb) scalar potential, furthermore, has a spherical harmonic decomposition (equivalent to 2-8b) into the 2^L-th multipoles Q_M^L which derives from an expansion of the denominator in (2-8a)

$$\phi^{(L)} = \frac{1}{r^{L+1}} \sum_{M=-L}^{L} \sqrt{\frac{4\pi}{2L+1}}\, Q_M^{(L)} Y_L^M(\theta,\phi) \qquad (2\text{-}8c)$$

where

$$Q_M^{(L)} = \sum_\alpha m'_\alpha r'^L_\alpha \sqrt{\frac{4\pi}{2L+1}}\, Y_L^M(\theta'_\alpha,\phi'_\alpha) \qquad (2\text{-}8d)$$

and where the time-dependence has been suppressed. (This spherical harmonic decomposition is given for the sake of comparison with the tensor decomposition in Chapter 3).

Arguing that the dipole $\underset{\sim}{p}$ contribution cannot radiate and conserve momentum, the first radiative contribution

$$P \sim \frac{c}{G}\int (\nabla\phi)^2 d^3\underset{\sim}{x}' \tag{2-9}$$

is due to the quadrupole (L=2)

$$P \sim \frac{G}{c^5}\dddot{Q}_{\alpha\beta}^{\;2} \tag{2-10}$$

The Vector Theory Of Gravitational Radiation

The vector theory of radiation derives from Maxwell's theory of electromagnetism. One can argue against establishing an electromagnetic theory of gravitation on the basis that the Newtonian field equations (2-5) are irrotational. They have no vector potential. But from the rotation of the planets in the solar system one can present a reasonable argument for the existence of a gravitational vector potential.* Indeed, the tensor theory of gravitation does just this (Appendix E, Eq E-10 & E-18).

Maxwell's Equations will be written as follows:

$$\nabla\cdot\underset{\sim}{E} = \frac{1}{\epsilon}\rho \tag{2-11a}$$

$$\nabla\times\underset{\sim}{E} = -\partial\underset{\sim}{B}/\partial t \tag{2-11b}$$

$$\nabla\cdot\underset{\sim}{B} = 0 \tag{2-11c}$$

$$\nabla\times\underset{\sim}{B} = \mu\underset{\sim}{J} + \mu\epsilon\,\partial\underset{\sim}{E}/\partial t \tag{2-11d}$$

where μ is the permeability, ϵ is the permittivity, $\underset{\sim}{E}$ is the electric field intensity, $\underset{\sim}{B}$ is the magnetic induction, $\underset{\sim}{J}$ is the current, and ρ is the charge density.

Comparison of the Newtonian field equations (2-5) with the Maxwell field equations (2-11) shows that the gravitational analogue of $\underset{\sim}{E}$ is $\underset{\sim}{g}$. Similarly,

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*This also derives from the notion of the Coriolis force as well as the remark of Wilson (1923) that there is a relation between the rotation of mass and the origin of magnetic fields (see Appendix F).

an entire development of a gravitational vector theory follows that of Maxwell. The resultant field equations (using lower case symbols for gravitation) are

$$\nabla\cdot\underset{\sim}{g} = -\frac{4\pi G}{\epsilon}\rho \tag{2-12a}$$

$$\nabla\times\underset{\sim}{g} = -\partial\underset{\sim}{b}/\partial t \tag{2-12b}$$

$$\nabla\cdot\underset{\sim}{b} = 0 \tag{2-12c}$$

$$\nabla\times\underset{\sim}{b} = -4\pi G\mu\underset{\sim}{j} + \mu\epsilon\,\partial\underset{\sim}{g}/\partial t \tag{2-12d}$$

where μ and ϵ are the gravitational permeability and permittivity, and where

$$\underset{\sim}{g} = -\nabla\phi - \partial\underset{\sim}{A}/\partial t \tag{2-13a}$$

$$\underset{\sim}{b} = \nabla\times\underset{\sim}{A} \tag{2-13b}$$

are derivable from scalar and vector potentials. The axial vector $\underset{\sim}{b}$ can be associated with the Coriolis force law, for example.

The constitutive relations are

$$\underset{\sim}{d} = \underset{=}{\epsilon}\,\underset{\sim}{g} \tag{2-14a}$$

$$\underset{\sim}{b} = \underset{=}{\mu}\,\underset{\sim}{h} \tag{2-14b}$$

These are important in the nonlinear theory of electromagnetic gravitation (in the fashion of nonlinear optics and Maxwell-Dirac Spinor electrodynamics). The Lorentz equations of motion are

$$\rho\dot{\underset{\sim}{v}} = \begin{cases} \rho[\underset{\sim}{g} + \underset{\sim}{v}\times\underset{\sim}{b}] \\ \rho[-\nabla\phi - \frac{\partial A}{\partial t} + \underset{\sim}{v}\times\nabla\times\underset{\sim}{A}] \\ \rho\underset{\sim}{g} + \underset{\sim}{j}\times\underset{\sim}{b} \end{cases} \tag{2-15}$$

There is likewise a gravitational Ohm's Law (See Appendix F)

$$\dot{\underset{\sim}{j}} = \underset{=}{\sigma}\,\underset{\sim}{g} = \rho\underset{\sim}{v} \tag{2-16}$$

which defines a gravitational conductivity $\underset{=}{\sigma}$, providing diffusion equations for gravitational vector waves as well as gravitational hydrodynamics. In what follows, $\underset{=}{\sigma}$ and its associated diffusion are neglected. (See Appendix F)

The continuity condition is

$$\nabla\cdot\underset{\sim}{j} + \frac{\partial\rho}{\partial t} = 0 \qquad (2\text{-}17)$$

while the gravitational Poynting vector is simply

$$\underset{\sim}{S} = \underset{\sim}{g}\times\underset{\sim}{h} \quad . \qquad (2\text{-}18)$$

It is straightforward to determine the wave equations for $\underset{\sim}{g}$, $\underset{\sim}{b}$, $\underset{\sim}{A}$, and ϕ. They are all of type (2-25). Their solutions are ϕ in Eq (2-8) as well as $\underset{\sim}{A}$ given by

$$\phi = -\frac{G}{\epsilon}\int\frac{[\rho]}{|\underset{\sim}{x}-\underset{\sim}{x}'|}d^3\underset{\sim}{x}' \qquad (2\text{-}19a)$$

$$\underset{\sim}{A} = -\mu G\int\frac{[\underset{\sim}{j}]}{|\underset{\sim}{x}-\underset{\sim}{x}'|}d^3\underset{\sim}{x}' \qquad (2\text{-}19b)$$

where $\mu\epsilon = c^{-2}$. Alternatively

$$\phi = -G\int\frac{[\rho]}{|\underset{\sim}{x}-\underset{\sim}{x}'|}d^3\underset{\sim}{x}' \qquad (2\text{-}19c)$$

$$\underset{\sim}{A} = -\frac{G}{c^2}\int\frac{[\underset{\sim}{j}]}{|\underset{\sim}{x}-\underset{\sim}{x}'|}d^3\underset{\sim}{x}' \quad . \qquad (2\text{-}19d)$$

The quadrupole radiation is (e = ellipticity of the ellipsoid)

$$P = \frac{G}{180c^5}\dddot{Q}_{\alpha\beta}^{\,2} = \frac{G}{380\epsilon c^5}I^2e^2\Omega^6 \quad . \qquad (2\text{-}19e)$$

In 4-vector notation the gravitational field equations are the Maxwell ones deriveable from the field tensor

$$f_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \qquad (2\text{-}20a)$$

$$f_{ij} = c^{-2}\epsilon_{ijk}b_k \qquad f_{0i} = -c^{-2}g_i \quad . \qquad (2\text{-}20b)$$

These field equations are (letting $\kappa = 4\pi G/\epsilon$)

$$\begin{aligned} \eta^{\alpha\beta}f_{\mu\alpha,\beta} &= -\kappa j_\mu \\ f_{\beta\gamma,\alpha} + f_{\gamma\alpha,\beta} + f_{\alpha\beta,\gamma} &= 0 \quad . \end{aligned} \qquad (2\text{-}21)$$

The gravitational stress-energy tensor is

$$t^{\mu\nu} = -\frac{1}{\kappa}\left(f^{\mu\alpha}f_\alpha{}^\nu + \frac{1}{4}\eta^{\mu\nu}f^{\alpha\beta}f_{\alpha\beta}\right) \qquad (2\text{-}22a)$$

where the field energy density c^2t^{00} is

$$c^2 t^{00} = -\frac{1}{2\varkappa}\left(\underset{\sim}{g}^2 + c^2\underset{\sim}{b}^2\right), \qquad (2\text{-}23a)$$

and where the Poynting vector is

$$S_i = c^2 t^{i0} \quad . \qquad (2\text{-}23b)$$

Explicitly in terms of the fields, the stress-energy tensor is

$$t^{\mu\nu} = -\frac{1}{2\varkappa}\left(\underset{\sim}{g}^2 + c^2\underset{\sim}{b}^2\right)\delta^{\mu\nu} - \frac{1}{\varkappa}\left(g^\mu g^\nu + c^2 b^\mu b^\nu\right). \qquad (2\text{-}22b)$$

Subject to the Lorentz gauge condition

$$A^\mu{}_{,\mu} = \nabla\cdot\underset{\sim}{A} + \mu\epsilon\frac{\partial\phi}{\partial t} = 0 \qquad (2\text{-}24)$$

the 4-potential $A^\mu = (\phi/c^2, \underset{\sim}{A})$ has the wave equation

$$\Box A^\mu = -\varkappa j^\mu \qquad (2\text{-}25)$$

where $j^\mu = \rho v^\mu$ obeys the continuity condition

$$j^\mu{}_{,\mu} = 0 \qquad (2\text{-}17)$$

in Eq (2-17). Note that (2-25) contains the scalar case (2-7) from Newtonian mechanics.

This, then, is the electromagnetic theory of gravitation. It is a vector theory whose equations of motion are (2-15), whose field equations are (2-12), and whose wave equation is the vector one (2-25) subject to the gauge condition (2-24). The theory is readily quantized, consisting of Spin-1 gravitons. If it be desired the development can be extended to Proca's Equations, providing for a massive graviton and a cut-off (Laplace-1846) in the gravitational field. Furthermore, if the development is assumed to satisfy the restricted theory of relativity, the entire theory is Lorentz covariant.

Such a theory does not account for the apparent nature of gravitation that all matter attracts; indeed, it maintains that in the physical world there could well exist states of negative matter (negative mass) or antimatter which are repulsive. Such speculation, in fact, has been pursued by Föppl (1896,

1898) and Schuster (1898). However, the principle embarassment in such a Maxwellian vector theory of gravitational radiation is that Maxwell himself abandons it (in 1865). He does so because the energy density of the gravitational field in Eq (2-23) is negative-definite, if the force between masses is to remain attractive. Maxwell accordingly invokes a universal background $\mathcal{E}_0$ in order to maintain a positive-definite energy $\mathcal{E}$,

$$\mathcal{E} = \mathcal{E}_0 - \frac{1}{2\kappa}\left[\underset{\sim}{g}^2 + c^2\underset{\sim}{b}^2\right]. \qquad (2\text{-}26)$$

The error in Maxwell's judgement is that he has divorced the gravitational theory from the electromagnetic one. This is to say that the total stress-energy tensor is composed of the electromagnetic one $T'^{\mu\nu}$ and the gravitational one $t^{\mu\nu}$. The total stress-energy tensor must be required to remain positive-definite

$$\left(T'^{\mu\nu} + t^{\mu\nu}\right) > 0, \qquad (2\text{-}27a)$$

which (along with the continuity equation) couples the gravitational vector fields with the electromagnetic ones - or else the force between masses must be allowed to become repulsive upon the appearance of negative-energy densities. More explicitly, Eq (2-27a) gives the coupling between electromagnetic and gravitational fields as

$$\frac{1}{2\kappa}\left[(\underset{\sim}{E}^2 + c^2\underset{\sim}{B}^2) - (\underset{\sim}{g}^2 + c^2\underset{\sim}{b}^2)\right]\delta^{\mu\nu} + \frac{1}{\kappa}\left[(E^\mu E^\nu + c^2 B^\mu B^\nu) - (g^\mu g^\nu + c^2 b^\mu b^\nu)\right] \geq 0. \qquad (2\text{-}27b)$$

This coupling is also supplemented by the continuity condit . (2-17) for both fields,

$$\left(J^\mu + j^\mu\right)_{,\mu} = 0 \qquad (2\text{-}27c)$$

or

$$\left(\nabla\cdot\underset{\sim}{J} + \frac{\partial\rho_e}{\partial t}\right) + \left(\nabla\cdot\underset{\sim}{j} + \frac{\partial\rho_g}{\partial t}\right) = 0. \qquad (2\text{-}27d)$$

In the absence of charge (ρ_e=0), for example, an electromagnetic current can manifest itself in (2-27d) due to the flow or rotation of mass - constituting a basis for the origin of magnetic fields. Further queries into the nature of the electromagnetic theory of gravitation are given in Appendix F.

The Tensor Theories Of Gravitational Radiation

Recalling now the scalar and vector wave equations, these can be readily generalized and one further step can be taken by introducing a tensor wave equation

$$\Box A^{\mu\nu} = -\varkappa T'^{\mu\nu} \tag{2-28}$$

for gravitational waves. In this equation $T'^{\mu\nu}$ represents the stress-energy tensor for all matter except those effects due to the gravitational fields themselves. Wave Eq (2-28) therefore represents a linear tensor wave theory. However, in order to make the energy of the gravitational field positive-definite, the auxiliary condition

$$A^{\mu\nu}{}_{,\nu} = 0 \tag{2-29}$$

must be imposed. But if condition (2-29) is imposed then it must be true from (2-28) that

$$T'^{\mu\nu}{}_{,\nu} = 0 \tag{2-30}$$

which is not true. The total stress-energy tensor must be divergence-free:

$$\left(T'^{\mu\nu} + t^{\mu\nu}\right)_{,\nu} = 0\,, \tag{2-31}$$

where $t^{\mu\nu}$ is the contribution from the gravitational fields themselves. Hence, the tensor wave equation must satisfy

$$\Box A^{\mu\nu} = -\varkappa\left(T'^{\mu\nu} + t^{\mu\nu}\right) \tag{2-32}$$

subject to the divergence condition (2-31). If we assume that $t^{\mu\nu}$ is at least quadratic in the gravitational field variables, then the tensor wave equation (2-32) becomes nonlinear. If ever quantized, it should contain Spin-2 gravitons.

In summary, then, the scalar, vector, and tensor theories of gravitational radiation are respectively

$$\Box\phi = -\varkappa\rho \tag{2-7}$$

$$\Box A^{\mu} = -\varkappa\left(J^{\mu}+j^{\mu}\right) \tag{2-25}$$

$$\Box A^{\mu\nu} = -\varkappa\left(T'^{\mu\nu}+t^{\mu\nu}\right). \tag{2-32}$$

They all constitute gravitational radiation, subject only to appropriate boundary conditions of incoming and outgoing radiation at infinity. These are field equations, and not necessarily equations of motion. It is the latter which address the meaning of radiation reaction and the energy content of such waves.

By far the most sophisticated theory is the tensor one, which we are now prepared to discuss.

CHAPTER III

GENERAL RELATIVISTIC THEORIES OF GRAVITATIONAL RADIATION

Introduction

In the development of General Relativity, Einstein's theory is confronted with the task of accounting for a number of accepted gravitational phenomena, known to most any student of natural philosophy. Amongst these are the deflection of light in a gravitational field as predicted by Newton in Book IIIof Opticks (1704) and calculated by Soldner (1801), the anomalous precession of the perihelion of Mercury which preoccupies more than 100 years of physics, as well as the inadequacies stated by Abraham and others of the restricted theory of relativity for an explanation of gravitational behaviour.

Gravitational radiation is another such example. By 1905 this is being explicitly addressed in the work of Gans (1905, 1912) and several years later by Abraham (1911, 1912, 1913), as we have mentioned. The theoretical problem at this time has developed to the point of demonstrating the existence of a gravitational instability due to gravitational radiation reaction (the radiating source gains energy - a problem which still plagues the theory).

When Einstein does proceed with his own theory of gravitation, it is to borrow an idea from Harry Bateman* and to "geometrize" gravitation, that is, to put the theory in the form of a relation between geometry and the sources rather than between the field and the sources. The resultant theory of gravitation is described by Einstein's tensor field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \qquad (3\text{-}1)$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature, and

$$T_{\mu\nu} = T'_{\mu\nu} + t_{\mu\nu} \qquad (3\text{-}2)$$

*The equation $ds^2 = g_{\mu\nu} dX^\mu dX^\nu$ might properly be called "Bateman's Equation."

is the total stress-energy tensor consisting of all nongravitational effects $T'_{\mu\nu}$ as well as the stress-energy pseudo-tensor components $t_{\mu\nu}$ due to the gravitational fields themselves. The left-hand side of the field equations (3-1), referred to as the Einstein tensor $G_{\mu\nu}$, is subject to the Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$ which implies that the right-hand side must also be divergence free

$$T^{\mu\nu}{}_{;\nu} = \left(T'^{\mu\nu} + t^{\mu\nu}\right)_{;\nu} = 0 \,. \qquad (3\text{-}3)$$

The right-hand side of the field equations is made nonlinear by the presence of $t_{\mu\nu}$ in (3-1) and (3-2) since the latter is at least quadratic in the field quantities. Furthermore, the equivalence of mass and energy due to Poincaré (1904)

$$E = Mc^2 \qquad (3\text{-}4)$$

is assumed throughout, although this postulate has never been demonstrated to have an experimental basis for gravitational phenomena. In effect, this postulate depletes the mass monopole source of gravitational radiation, in contrast to the invariance of classical electromagnetic charge under electrodynamic radiation where (3-4) has a firmer basis (Cockroft & Walton, 1932). The dilatation of mass

$$M = \gamma m = \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right)m \qquad (3\text{-}5)$$

is also postulated, which presents complications in the theory of gravitational radiation reaction. It is the origin of the "slow motion" approximation which will be discussed shortly when the linearized theory is addressed.

Note that Einstein's cosmological term $\Lambda g_{\mu\nu}/2$ is not included in the field equations (3-1), simply because its presence revokes all of the existing results in the general relativistic theory of gravitational radiation. It will not be discussed further.

Approximation Methods & Coordinate Conditions

Although there do exist exact solutions of Einstein's nonlinear field equations (3-1) (Chapter 4), the physical interpretation and meaning of such results is difficult to determine. As a consequence, most of the theoretical

work on gravitational radiation has been performed with approximation methods which are valuable because they provide some understanding of the nature of the problem - although they are actually inadmissible because they destroy the nonlinearity of the equations.

These approximations consist of linearization and perturbation techniques which make various assumptions about the nature of the metric tensor $g_{\mu\nu}$. For example, the "weak-field" approximation assumes

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3-6}$$

where $h_{\mu\nu}$ constitutes a weak ($h \ll g$) perturbation of the background geometry $\eta_{\mu\nu}$, such as the Minkowski or the Schwarzschild backgrounds. Evaluation of the Ricci tensor $R_{\mu\nu}$ to first order in $h_{\mu\nu}$ then transforms the field equations (3-1) in vacuum ($T_{\mu\nu} = 0$) into a system of linear differential equations. Another method, known as the "k-approximation" or "fast motion" method,

$$g_{\mu\nu} = \sum_{n=0}^{\infty} k^n \overset{(n)}{g}_{\mu\nu} = \overset{(0)}{g}_{\mu\nu} + k \overset{(1)}{g}_{\mu\nu} + k^2 \overset{(2)}{g}_{\mu\nu} + \cdots \tag{3-7}$$

assumes that the metric tensor $g_{\mu\nu}$ can be represented as a power series expansion in the parameter k, which is proportional to the gravitation constant G. Next, there is a "slow motion" approximation which expands $g_{\mu\nu}$ in inverse powers of the speed of light c:

$$\left.\begin{aligned} g_{00} &= -1 + \overset{(2)}{g}_{00} + \overset{(4)}{g}_{00} + \cdots \\ g_{0k} &= \overset{(3)}{g}_{0k} + \overset{(5)}{g}_{0k} + \cdots \\ g_{ik} &= \delta_{ik} + \overset{(2)}{g}_{ik} + \overset{(4)}{g}_{ik} + \cdots \end{aligned}\right\} \tag{3-8}$$

This technique is the basis of the so-called "EIH approximation" which has played a significant role in the development of the equations of motion in General Relativity and the determination of what are referred to as "post-Newtonian" results (the precession of the apse, the deflection of light, and the gravitational red-shift). The numbers in parentheses in (3-8) reflect the "order" of the approximation - namely the power of c^{-n}, such as c^{-2}, c^{-3}. One last technique that must be mentioned is the "double series" approach which consists of an expansion of the metric tensor $g_{\mu\nu}$ in powers of a mass parameter m as well as powers of c^{-n} like in (3-8):

$$\begin{aligned} g_{\mu\nu} = \sum_{p=0}^{\infty}\sum_{s=0}^{\infty} m^p c^{-s} \overset{(ps)}{g}_{\mu\nu} &= \overset{(00)}{g}_{\mu\nu} + \frac{1}{c}\overset{(01)}{g}_{\mu\nu} + \frac{1}{c^2}\overset{(02)}{g}_{\mu\nu} + \cdots \\ &\quad + m\overset{(10)}{g}_{\mu\nu} + m^2\overset{(20)}{g}_{\mu\nu} + \cdots + \frac{m}{c}\overset{(11)}{g}_{\mu\nu} + \frac{m}{c^2}\overset{(12)}{g}_{\mu\nu} + \cdots \end{aligned} \tag{3-9}$$

where now the sum of the numbers or exponents in parentheses represents the "order" of the approximation.

Coordinate conditions constitute the next significant aspect of linearized General Relativity. Just as the Lorentz gauge condition (2-24)

$$A^{\mu}{}_{,\mu} = 0 \tag{2-24}$$

is invoked in classical electrodynamics in order to obtain the vector wave equation (2-25)

$$\Box A^{\mu} = -\varkappa j^{\mu} \tag{2-25}$$

so do similar gauge conditions exist in general relativity. Einstein (1918) defines the auxiliary function

$$\gamma^{\mu\nu} = \frac{1}{2}\left[h^{\mu\nu} - \frac{1}{2}\delta^{\mu\nu}h\right] \tag{3-10}$$

which is subject to the Einstein coordinate condition (Appendix B)

$$\gamma^{\mu\nu}{}_{,\nu} = 0 \tag{3-11}$$

resulting in a tensor wave equation of the form (2-32) from the field equations (3-1)

$$\Box \gamma^{\mu\nu} = -\varkappa T^{\mu\nu}. \tag{3-12}$$

Such gauge conditions and coordinate conditions have led to much controversy in General Relativity, primarily because they destroy the general covariance of the theory. Fictitious gravitational waves also manifest themselves which must be removed by coordinate transformations. Furthermore, gravitational waves can be created in the linearized theory or they can be annihilated, simply by a coordinate condition. Upon this basis, some authors question the very existence of gravitational radiation.

Two more features of the linearized theory are significant: (a) Boundary conditions, and (b) Radiation reaction. Boundary conditions such as the Sommerfeld radiation condition (outgoing radiation at infinity) are important because they license the theoretician to obtain whatsoever experiment implores

- radiation, or no radiation. Radiation reaction is important because many authors employ the geodesic postulate of Einstein whereby gravitational radiation does not result in radiation reaction, in the linear theory. To be sure, such solutions are inconsistent with the general nonlinear theory.

It should be evident that there is a wealth of theoretical work, both controversial and thorough, which exists in the tensor theory of gravitation. It is now best to begin with Einstein's original derivation of gravitational radiation in the linearized approximation.

Einstein's Original Derivation Of Gravitational Radiation

The geometrical nature of the field equations (3-1) is made more transparent if the latter are contracted with $g^{\mu\nu}$

$$R = \kappa T \quad , \tag{3-13}$$

where $\kappa = 8\pi G/c^4$ and $T = T^{\mu}{}_{\mu}$ is the trace of the stress-energy tensor (3-2). Substitution of this relation (3-13) for the scalar curvature R in the original equations (3-1) gives

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right). \tag{3-14}$$

This form of Einstein's field equations states more explicitly the correspondence between the structure of geometry in the Ricci tensor $R_{\mu\nu}$ and the structure of the sources of matter implicit in the stress-energy tensor $T_{\mu\nu}$.

Einstein (1918) makes a "weak-field" approximation and thereby linearizes the Ricci tensor in (3-14), as shown in Appendix A. Employing the Einstein coordinate condition (Appendix B) in Eq (B-8), the first-order Ricci tensor (A-6) becomes (B-9) and the field equations (3-14) above simplify to

$$R^{(1)}_{\mu\nu} = \tfrac{1}{2} \Box h_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\right). \tag{3-15}$$

In other words, the wave equatior and its coordinate condition (3-11) for the tensor theory are, to first order in h,

$$\Box h_{\mu\nu} = -2\kappa\left(T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T\right) \qquad (3\text{-}16)$$

$$\gamma^{\mu\nu}{}_{,\nu} = 0 \quad . \qquad (3\text{-}11)$$

The wave equation (3-16) is still nonlinear due to the nonlinearity of $T_{\mu\nu}$, as discussed in (3-2). Its solution is represented by

$$h_{\mu\nu} = -\frac{\kappa}{2\pi}\int \frac{[T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T]}{r'}\, d^3\underset{\sim}{X}' \qquad (3\text{-}17)$$

where $r' = |\underset{\sim}{X} - \underset{\sim}{X}'|$ and where the brackets represent retarded, advanced, or standing wave solutions. To these solutions can be added those of the homogeneous wave equation

$$\Box h_{\mu\nu} = 0 \qquad (3\text{-}18)$$

which are plane waves. The stress-energy tensor is now linearized by making a "slow motion" approximation such that $M \approx m$ in Eq (3-5) while $T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T$ is represented by $\tau_{\mu\nu}$. For distances r' much greater than the extent of the sources, in the radiation zone, $r' \approx r$ whereby (3-17) becomes

$$h_{\mu\nu} = -\frac{\kappa}{2\pi r}\int [\tau_{\mu\nu}]\, d^3\underset{\sim}{X}' \quad . \qquad (3\text{-}19)$$

Imposing the boundary condition that $\tau_{\mu\nu} = 0$ at $r = \infty$ and using Gauss' theorem one can demonstrate that

$$\int \tau_{\mu\nu}\, d^3\underset{\sim}{X}' = \frac{1}{2}\frac{\partial^2}{\partial X_0^2}\int \tau_{00}\, X^\mu X^\nu\, d^3\underset{\sim}{X}' \, . \qquad (3\text{-}20)$$

Since $\tau_{00} \approx mc^2$ in a slow-motion approximation and $X_0 = ct$, then (3-19) and (3-20) give

$$h_{\mu\nu} = \frac{\kappa}{4\pi r}\frac{\partial^2}{\partial t^2}\int m X^\mu X^\nu\, d^3\underset{\sim}{X}' \, . \qquad (3\text{-}21)$$

It is advantageous to use the mass quadrupole tensor (the mass monopole and dipole make no radiative contribution)

$$Q_{\mu\nu} \equiv \int m\left(3X^\mu X^\nu - \delta_{\mu\nu}X_\gamma^2\right) d^3\underset{\sim}{X}' \qquad (3\text{-}22)$$

and express $h_{\mu\nu}$ in (3-21) as follows (recalling that $\kappa = 8\pi G/c^4$):

$$h_{\mu\nu} = \frac{2G}{3c^4 r}\left\{\ddot{Q}_{\mu\nu} - 2\delta_{\mu\nu}\int m\dot{X}_\gamma X_\gamma d^3\underset{\sim}{X}'\right\}. \qquad (3\text{-}23)$$

This completes the determination of the tensor wave $h_{\mu\nu}$ in the linearized tensor theory.

By means of a coordinate transformation one may go to the principal axes of the quadrupole and choose an axis, say X^1, to be an axis of symmetry such that $Q_{1\nu} = 0$. All longitudinal-longitudinal (h_{11}, h_{10}, h_{00}) and longitudinal-transverse (h_{12}, h_{13}, h_{20}, h_{30}) perturbations vanish under such a gauge, leaving only a transverse-transverse set of perturbations (h_{22}, h_{33}, h_{23}) propagating at velocity c in the X^1 direction:

$$h_{23} = \frac{2G}{3c^4 r}\ddot{Q}_{23} \qquad (3\text{-}24a)$$

$$h_{22} = \frac{2G}{3c^4 r}\left\{\ddot{Q}_{22} - 2\int m\dot{X}_\gamma X_\gamma d^3\underset{\sim}{X}'\right\}$$

$$h_{33} = \frac{2G}{3c^4 r}\left\{\ddot{Q}_{33} - 2\int m\dot{X}_\gamma X_\gamma d^3\underset{\sim}{X}'\right\}$$

$$(h_{22} - h_{33}) = \frac{2G}{3c^4 r}\left\{\ddot{Q}_{22} - \ddot{Q}_{33}\right\}. \qquad (3\text{-}24b)$$

One now wishes to determine the radiation spectrum. The gravitational stress-energy is given by the pseudo-tensor $t^{\mu\nu}$. For the perturbations h_{22}, h_{33}, h_{23} propagating along X^1, the energy flux $c\tau^{10}$ of the stress-energy pseudotensor (Appendix C, Eq (C-11)) simplifies to

$$c\tau^{10} = -\frac{c^4}{32\pi G}\left\{\frac{\partial h_{22}}{\partial X^1}\frac{\partial h_{22}}{\partial t} + \frac{\partial h_{33}}{\partial X^1}\frac{\partial h_{33}}{\partial t} + 2\frac{\partial h_{23}}{\partial X^1}\frac{\partial h_{23}}{\partial t}\right\}, \qquad (3\text{-}25)$$

which becomes for retarded solutions $X^1 = ct$

$$c\tau^{10} = \frac{c^3}{16\pi G}\left\{\dot{h}_{23}^{\,2} + \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2\right\}. \qquad (3\text{-}26)$$

Substitution of (3-24) into (3-26), noting that $\kappa = 8\pi G/c^4$, gives one

$$c\tau^{10} = \frac{G}{36\pi c^{3}r^2}\left\{\dddot{Q}_{23}^{\,2} + \left(\frac{\dddot{Q}_{22} - \dddot{Q}_{33}}{2}\right)^2\right\}. \qquad (3\text{-}27)$$

Generalizing this result to an arbitrary direction of propagation n^i, the radiated power per unit solid angle $d\Omega'$ is (this particular representation is due to Landau & Lifshitz, 1962, but their work offers nothing that is not contained in Einstein's paper)

$$\frac{dP}{d\Omega'} = \frac{G}{36\pi c^5}\left\{\frac{1}{4}\left(\dddot{Q}_{ij}n^i n^j\right)^2 + \frac{1}{2}\dddot{Q}_{ij}^2 - \dddot{Q}_{ij}\dddot{Q}_{ik}n^j n^k\right\}. \qquad (3\text{-}28)$$

The total power radiated is now determined by averaging this energy flux over all solid angles. The resultant energy loss due to linearized gravitational radiation is thus

$$-\frac{d\mathcal{E}}{dt} = \frac{G}{45c^5}\dddot{Q}_{ij}\dddot{Q}^{ij} \quad . \qquad (3\text{-}29)$$

The mass quadrupole and the moments of inertia are related as $Q_{ij}=I\delta_{ij}-3I_{ij}$. Hence for a spinning rod $\langle\dddot{Q}_{ij}\rangle \sim I\Omega^3$, whereby

$$-\frac{d\mathcal{E}}{dt} = \frac{32}{5}\frac{G}{c^5}I^2\Omega^6 \quad , \qquad (3\text{-}30)$$

twice the value obtained by Einstein (1918), as corrected by Eddington (1922). [Eq (3-30) is correct only for an ellipticity of unity; see Eq (5-1).]

For the particular case of a rod spinning about the $\hat{z}$-axis ($\Omega=\dot{\phi}$), Einstein's results are demonstrated by Park (1955) to give the radiation pattern shown in Figure 1. Transforming Eddington's (1922) Eq (18) into spherical coordinates, the power radiated per unit solid angle in the direction $\hat{n}$ is

$$\frac{d}{d\Omega'}P(\theta,\phi) = \frac{4G}{\pi c^5}I^2\Omega^6\left[\frac{1}{4}\sin^2\theta\sin^2 2\phi + \cos^2\theta\right]. \qquad (3\text{-}31)$$

Integrating this over the azimuth ϕ, Park obtains the radiation pattern in Figure 1:

$$\frac{d}{\sin\theta\, d\theta}P(\theta) = \frac{4G}{\pi c^5}I^2\Omega^6\left[\cos^2\theta + \frac{1}{8}\sin^4\theta\right]. \qquad (3\text{-}32)$$

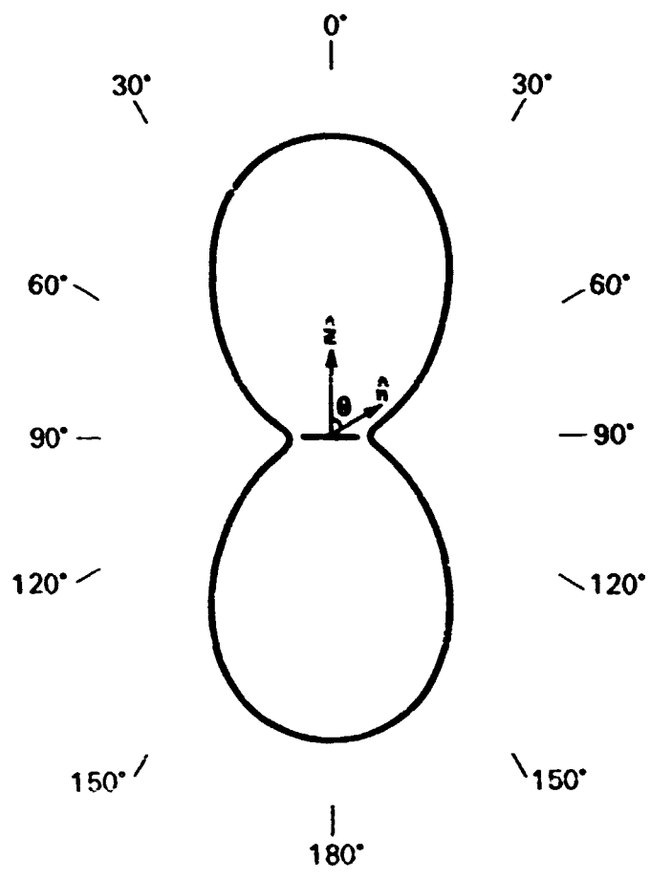

FIGURE 1

Averaged over Ω, the remaining part of the solid angle, (3-32) yields (3-30). Figure 1 also applies to the circular Kepler problem, as we shall see in Chapter 5.

The angular momentum $M^{\mu\nu}$ of the radiation* is likewise determinable from the gravitational pseudo-tensor $t^{\mu\nu}$ in (3-25), using Eq (D-4) from Appendix D - although Einstein neglects this aspect of the problem:

$$M^{\mu\nu} = \frac{1}{c}\int\left\{X^{\mu}t^{\nu\lambda} - X^{\nu}t^{\mu\lambda}\right\}(-g)\,dS_{\lambda} \quad . \qquad \text{(D-4')}$$

Einstein's original treatment of the linearized theory also considers the plane wave solutions of

$$\Box h_{\mu\nu} = 0 \quad . \qquad \text{(3-18)}$$

Defining appropriately the Einstein polarization tensor $\epsilon_{\mu\nu}$ (Einstein calls it $\alpha_{\mu\nu}$; Weyl and Eddington call it $a_{\mu\nu}$) the general plane wave solution of the homogeneous equation (3-18) is

$$h_{\mu\nu}(x) = \epsilon_{\mu\nu}\, e^{ikx} + \epsilon^{*}_{\mu\nu}\, e^{-ikx} \qquad \text{(3-33)}$$

which satisfies (3-18) provided

$$k^2 = k_{\mu}k^{\mu} = 0 \qquad \text{(3-34)}$$

and (3-11) if

$$k_{\mu}\epsilon^{\mu}{}_{\nu} = \frac{1}{2}k_{\nu}\epsilon^{\mu}{}_{\mu} \quad . \qquad \text{(3-35)}$$

Under a gauge condition (B-1) where $X'^{\mu} = X^{\mu} + \xi^{\mu}$ the polarization tensor $\epsilon_{\mu\nu}$ transforms as

$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + k_{\mu}\xi_{\nu} + k_{\nu}\xi_{\mu} \quad . \qquad \text{(3-36)}$$

For an Einstein-Eddington-Weyl plane-wave (3-33) propagating in the X^1 direction with $k^1=k^0=k$ while $k^2=k^3=0$

*See Hansen (1972).

$$\epsilon'_{22} = \epsilon_{22} \qquad \epsilon'_{23} = \epsilon_{23}$$
$$\epsilon'_{21} = \epsilon_{21} + k\xi_2 \qquad \epsilon'_{31} = \epsilon_{31} + k\xi_3 \qquad (3\text{-}37)$$
$$\epsilon'_{11} = \epsilon_{11} + 2k\xi_1 \qquad \epsilon'_{00} = \epsilon_{00} - 2k\xi_0 \ .$$

Letting $\xi_0 = \epsilon_{00}/2k$, $\xi_2 = -\epsilon_{21}/k$, $\xi_1 = -\epsilon_{11}/2k$, and $\xi_3 = -\epsilon_{31}/k$, all components of (3-37) vanish except ϵ_{22} and ϵ_{23}. Those that remain (ϵ_{22} and ϵ_{23}) represent the physical components of the gravitational plane waves. Those that vanish are "fictitious" because they can be produced by mere coordinate transformations. The tensor plane wave (3-33) has helicities 0, $\pm$ 1, $\pm$ 2. But only helicity $\pm$ 2 cannot be transformed away with a coordinate transformation. Linearized gravitational waves are therefore considered to have Spin-2 (although they do possess "fictitious" Spin-1 components).

Einstein's work on the theory of linearized gravitational radiation is characteristically definitive in a number of respects. He obtains a wave equation from the linear field equations and therefrom obtains retarded solutions in strict analogy with the radiative solutions of classical electrodynamics. There is no gravitational dipole radiation because he invokes Newton's principle of equivalence* wherein gravitational and inertial mass are the same, and whereby a system of masses necessarily has the same gravitational-to-inertial mass ratio and the dipole radiation vanishes under conservation of linear momentum - just as it does in electrodynamics for a system of charges with identical charge-to-mass ratios. The velocity of propagation of the remaining quadrupole radiation is the same as that for electromagnetic phenomena, upon the basis of correspondence between the general theory and the restricted theory of relativity and Newtonian mechanics (However, see Appendix G).

Einstein's development is not definitive, however, in the sense that he fails to address the real nature of his theory, its nonlinearities. Although there does appear to exist radiation from a mechanically driven Jacobi ellipsoid possessing a time-varying mass quadrupole, there is nothing yet in the theory that demonstrates radiation exists for an isolated, gravitationally bound system such as the two-body Kepler problem. Neither has it been established that the decrease in energy of the radiating mass quadrupole (3-30) is equal to the energy carried away by the gravitational radiation, a problem intimately related to the question of radiation reaction and the equations

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*In order to write the equation of motion $m_I\ddot{\underline{r}} = -GMm_g\hat{r}/r^2$, Newton had to assume $m_I = m_g$. This form of the principle, then, is due to Newton.

of motion. These more subtle aspects of gravitational radiation theory remain untouched, and are left to plague the proponents of Einstein's theory.

The EIH Approximation

If one were considering possibilities for tensor field equations, these would necessarily involve the total stress-energy tensor $T_{\mu\nu}$ on the basis of a generalization of classical field theory. Furthermore, the conservation conditions $T^{\mu\nu}{}_{,\nu} = 0$ from classical electrodynamics would necessarily generalize to the covariant relation

$$T^{\mu\nu}{}_{;\nu} = 0 \quad . \tag{3-38}$$

Hence, if one takes the field equations in the form

$$G^{\mu\nu} = -\kappa T^{\mu\nu} \tag{3-39}$$

then the covariant divergence of this form (3-39) guarantees by virtue of (3-38) that

$$G^{\mu\nu}{}_{;\nu} = 0 \quad . \tag{3-40}$$

Because (3-40) is simply the contracted form of the Bianchi identities, then the reasonable choice for $G_{\mu\nu}$ is the left-hand side of (3-1), namely

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R \quad . \tag{3-41}$$

This argument, partially due to Hilbert (1915), constitutes the teleological basis of Einstein's nonlinear field equations (3-1).

The surprising result of this argument is that the equations of motion follow from (3-38), although this seems not to have been apparent at the time. Einstein (1916) originally introduces the assumption of geodesic motion, but Einstein & Grommer (1927) later succeed in demonstrating that the motion of a singularity in an external field must be along the geodesics of the external field. Finally, Einstein, Infeld, & Hoffmann (EIH) formulate an approximation method by which both the gravitational field and the equations of motion for its sources can be calculated from the field equations.

This EIH approximation method is essentially a quasi stationary weak-field

approximation which has resulted in the so-called "post-Newtonian" results of General Relativity (see Appendix E). It is best understood by drawing an analogy due to Infeld (1938) and Trautman (1958) with the scalar wave theory. If we consider the "near zone" (in the region of the sources) of the scalar wave equation

$$\Box \phi = -\kappa \rho \tag{2-9}$$

and further consider that ϕ may be expanded in powers of a small parameter $\epsilon = c^{-1}$,

$$\phi = \sum_{n}^{\infty} \epsilon^n \phi_n = \epsilon^0 \phi_0 + \epsilon^1 \phi_1 + \cdots \quad , \tag{3-42}$$

then operation upon ϕ with the d'Alembertian $\Box$ results in a series of simultaneous differential equations determined by setting coefficients of the same powers of ϵ equal. Because of the quasi-stationary condition that $\nabla^2\phi \gg \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi$, we obtain upon equating coefficients:

$$\begin{array}{lll}
\epsilon^0: & \nabla^2\phi_0 & = -\kappa\rho \\
\epsilon^1: & \nabla^2\phi_1 & = 0 \\
\epsilon^2: & \nabla^2\phi_2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi_0 & = 0 \\
\epsilon^3: & \nabla^2\phi_3 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi_1 & = 0 \\
\epsilon^4: & \nabla^2\phi_4 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi_2 & = 0 \\
\vdots & \vdots & \vdots
\end{array} \tag{3-43}$$

These near zone equations (3-43) exhibit the salient feature of the EIH approximation method. The wave equation associated with any particular power of ϵ^n is determined in the near zone by the solutions of lower order approximations, in this case ϵ^{n-2}. This behaviour is represented by the arrows to the left of the equations which demonstrate how the approximations couple together. There is likewise a set of such equations for the homogeneous case in the wave zone where $\rho = 0$. But the interesting characteristic of the radiation zone is that it couples back into the near zone, acting on the sources as radiation reaction.

Recalling tnat in General Relativity the scalar wave functions ϕ are replaced by the metric tensor components $g_{\mu\nu}$, the above analogy must be reconsidered

from the point of view of Einstein's field equations (3-1). As in the scalar case (3-42), Einstein's field equations for empty space may also be expanded in a power series

$$\hat{G}_{\mu\nu} \equiv \sqrt{-g}\, G_{\mu\nu} = \sum_{n=2}^{\infty} \epsilon^{n} \underset{(n)}{G}_{\mu\nu} = 0 \quad . \qquad (3\text{-}44)$$

However, in the case of (3-44) the coefficients of ϵ cannot in general be set equal to zero, as was done in (3-43). A word of caution about the literature is likewise in order due to a certain amount of confusion in notation. Most of the original EIH development is in terms of $\lambda = c^{-1}$, rather than ϵ. The treatments are equivalent, except that $\lambda^m = \epsilon^{n+2}$, for the following reason (see also Appendix E). From the geodesic equation of motion, radiation reaction on the particle in motion is described by

$$\underset{(\lambda)}{\ddot{X}^{\mu}} = -\underset{(\epsilon)}{\Gamma^{\mu}_{\alpha\beta}}\, \dot{X}^{\alpha}\dot{X}^{\beta} \quad . \qquad (3\text{-}45)$$

The equations of motion are of order λ, but the metric components $g_{\mu\nu}$ in the Christoffel symbol of (3-45) are of order ϵ. The two time derivatives ($\partial_0 = c^{-1}\partial_t = \lambda\partial_t$) raise the order of ϵ by two. To make matters even a little more miserable, some authors also get an extra factor of c^{-1} because they express energy as $P^0 = c^{-1}E = \lambda E$ (they get radiation reaction in λ^{10} instead of λ^9).

The EIH method is summed up by expressing $g_{\mu\nu}$ as in Figure 2. This is a representation of the expansion of $g_{\mu\nu}$ in (3-8) as powers of $\epsilon = c^{-1}$, where the numbers in parenthesis reflect the order of ϵ. It illustrates the "post-Newtonian" effects in General Relativity and their origin in the metric tensor as depicted by the EIH method. There are two types of contributions, radiative and non-radiative. By examining, for example, the quadrupole or "post$^{2.5}$-Newtonian" terms connected by the line indicated, one can see that g_{ik} and g_{00} are odd-powers of ϵ while g_{0k} is even in ϵ. For non-radiative contributions, g_{00} and g_{ik} are even in ϵ while g_{0k} is odd. By virtue of conservation of linear momentum and the equivalence of inertial and gravitational mass the dipole radiation (represented by the dotted line) of seventh-order in λ can be made to disappear. It also can be annihilated by a coordinate transformation, as demonstrated by Infeld and Scheidegger (1951, 1953, 1955, 1960). Boundary conditions likewise determine the behaviour of Figure 2. Standing waves, as an example, manifest themselves if all of the radiative lines cancel and vanish from the diagram.

EIH APPROXIMATION

$$g_{oo} = -1 + \underset{2}{g}_{oo} + \underset{4}{g}_{oo} + \underset{5}{g}_{oo} + \underset{6}{g}_{oo} + \underset{7}{g}_{oo}$$

$$g_{ok} = + \underset{3}{g}_{ok} + \underset{4}{g}_{ok} + \underset{5}{g}_{ok} + \underset{6}{g}_{ok} + \cdots$$

$$g_{ik} = \delta_{ik} + \underset{2}{g}_{ik} + \underset{3}{g}_{ik} + \underset{4}{g}_{ik} + \underset{5}{g}_{ik} + \cdots$$

Dipole

Newtonian Forces | Post$^{1.0}$ Forces | Post$^{2.0}$ Forces | Post$^{2.5}$ Forces: Quadrupole Radiation Reaction

FIGURE 2

POST-NEWTONIAN RELATIVITY

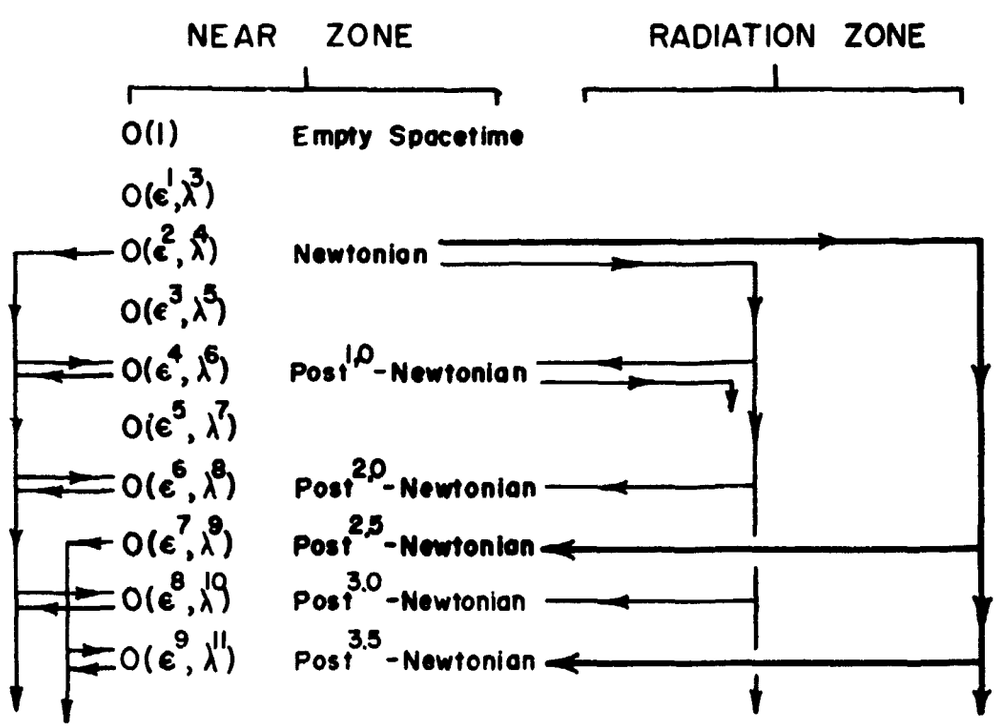

FIGURE 3

Infeld (1938) establishes the radiation terms in Figure 2, demonstrating that they do not contribute to the equations of motion (radiation reaction) up to the seventh-order in λ. Hu (1947) carries the approximation on to 9th-order, being the first to study explicitly the problem of radiation reaction in the two-body problem. Infeld & Scheidegger (1951) investigate the elimination of radiation reaction lines by coordinate transformations, but Goldberg (1955) finds solutions which cannot be eliminated in this fashion. By 1958, the definitive paper of Trautman (1958) firmly establishes the EIH formalism depicted in Figure 2 as a linear perturbation method with which to approximate the nonlinear field equations of Einstein in the weak-field, slow-motion case.

The final step in understanding the EIH method is one of re-establishing Figure 2 in a manner that reflects the coupling discussed in Eq (3-43). This is done in the schematic representation of Figure 3 (partially due to Thorne, 1969), which reflects the work of the Infeld school through 1960. As with Eq (3-43), the arrows in Figure 3 represent how the various levels of the approximation couple together. In the near zone there is the same scalar coupling of the even orders in ϵ, except that an additional scalar coupling in the odd orders manifests itself at the $post^{2.5}$ level. Likewise, the solutions of the homogeneous wave equations (far from the sources) couple in the radiation zone. And with proper matching - the method of EIH or asymptotic matching, to be discussed later in this chapter - at the boundary between the near and radiation zones, the solutions are consistent.

However, the most significant aspect of the EIH method, and any other linearized approximation of the nonlinear field equations for that matter, is the radiation reaction which occurs in the 9th-order (λ^9) or $post^{2.5}$-Newtonian approximation of Figures 2 & 3. If the boundary condition at infinity is the Sommerfeld condition (outgoing radiation) then the energy of the radiative sources must be depleted by an amount equivalent to that carried off in the gravitational radiation, or soaked up by some nonlinear mechanism in the near zone. This behaviour constitutes radiation reaction.

A demonstration of this result is first conducted by Trautman (1958) and Peres (1959). Peres succeeds in carrying the EIH method to the "$post^{2.5}$-Newtonian" approximation (7th-order in ϵ), doing so at a critical time when the integrity of Einstein's linearized quadrupole formula (3-29) and the very existence of gravitational radiation has been a controversial subject. This

same result has been obtained before as we have seen, by Hu (1947) who works the radiation reaction problem out to 9th-order in $\lambda(\lambda^9)$, at the suggestion of Pauli at Princeton. Both Hu (1947) and Peres (1959) consider the application of their results to the two-body Kepler problem and ascertain that the system gains energy* under gravitational radiation, corroborating the results of Gans (1905) and Maxwell (1865) - although Hu does not point this out. That the problem is a boundary-value problem, however, is maintained by Peres (1960), whereupon he obtains a decrease in energy as depicted in (3-29) for the Sommerfeld radiation condition. For the Kepler problem, Einstein's quadrupole radiation formula (3-30) reduces to

$$-\frac{d\mathcal{E}}{dt} = \frac{32}{5}\frac{G^4}{c^5}\frac{(m_1m_2)^2(m_1+m_2)}{r^5}, \qquad (3\text{-}44)$$

which is the result obtained by Peres (1960) - and by Hu (1947), except for the sign. The source mass likewise decreases by this amount.

Radiation reaction in the EIH mass quadrupole and two-body problem is subsequently pursued by several authors, such as Ryteń (1963), Demianski & Infeld (1963), and Infeld & Michalska-Trautman (1966, 1969). The EIH method, furthermore, is employed by Chandrasekhar, et al (1965-1970) and Thorne, et al (1967-1970) in certain astrophysical applications where the earlier work is extended to the hydrodynamics of perfect fluids. The significance of all of these results is that the EIH approximation method, while accounting for some of the nonlinearities of the theory, corroborates Einstein's linearized quadrupole radiation.

The derivation of Ryteń (1963) of the mass quadrupole radiation in the EIH approximation provides, for the interested reader, a detailed example of how the method is pursued to the 9th-order where quadrupole radiation reaction appears. Her final radiation formula,

$$-\frac{d\mathcal{E}}{dt} = \frac{G}{5c^5}\left\{\dddot{Q}_{ij}\dddot{Q}^{ij} + \frac{d}{dt}\left[\frac{1}{30}Q_{ij}\frac{d^5}{dt^5}Q^{ij} + \frac{17}{18}\dot{Q}_{ij}\frac{d^4}{dt^4}Q^{ij} - \frac{1}{3}\ddot{Q}_{ij}\dddot{Q}^{ij}\right]\right\}, \qquad (3\text{-}45)$$

is identical with the linearized approximation (3-29).

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*It is of interest to note that Eddington (1922, P.251) encounters this problem also.

The "Fast Motion" Approximation

The EIH approximation is both a "weak-field" and a "slow-motion" approximation, particularly suited to the study of such behaviour as planetary motion and post-Newtonian General Relativity. But such a technique is limited in its applications. Consequently, another method known as the "fast motion" or the "k-approximation" method has established itself. It does not make a "slow-motion" assumption.

This procedure consists of a scheme of successive approximations, like EIH, but with a power series expansion of the metric tensor $g_{\mu\nu}$ in terms of a small parameter k

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{m=1}^{\infty} k^m g_{\underset{(m)}{\mu\nu}} = \eta_{\mu\nu} + k g_{\underset{(1)}{\mu\nu}} + k^2 g_{\underset{(2)}{\mu\nu}} + \cdots \qquad (3\text{-}7)$$

where k is proportional to the gravitation constant G: $k = GM/R$, M is the mean particle mass, and R is the mean interparticle separation. It is not quasi-stationary; that is, $\partial_0\phi \approx \nabla\phi$. The energy-momentum tensor is likewise expanded in a power series in k.

Einstein (1916, 1918) is the first to use it, assuming that the field is sufficiently weak that the nonlinear terms can be neglected. This is the so-called "1st-approximation" in k, which is again addressed by Bertotti (1956) and deWitt & Ging (1960). Bertotti & Plebanski (1960) then establish a generalized Green's function method for nonlinear field theory and develop the equations of motion up to the "2nd-approximation" in k. Their work, however, contains infinite self-action terms - although these are removed with a renormalization process by Kuhnel (1964). Bock (1957, 1959) and Bonnor (1958) also discuss solutions of the "2nd-approximation."

More recently, the "fast motion" approximation is utilized by Havas (1957), Havas & Goldberg (1962), and Havas & Smith (1965) to address the question of the existence of gravitational radiation from a freely gravitating system, such as a Kepler problem. Havas, Goldberg, & Smith confine themselves to the "1st-approximation," investigating its particular contribution to radiation damping. They obtain antidamping and a gain of energy for the radiating two-body problem, as does Hu with the EIH method. However, their work is criticized by Peters (1970) as improperly neglecting the stresses in the system.*

*See also Lind (1972) and Peters (1972).

The radiation reaction associated with the "post$^{2.5}$-Newtonian" EIH effects appears in the "3rd-approximation" which none of these authors has addressed. Furthermore, Infeld (1961) has criticized the "k-approximation" method as a "step backwards," partly because it does not give the EIH results, which are well established, in the limit of low velocity. Nevertheless, the "fast motion" approximation will always have theoretical appeal because it has many realistic astrophysical applications.

The "Double Series" Approximation

In connection with the "fast motion" approximation just discussed, there is another method which is very similar - due to Bonnor (1959) and employed by Carmeli (1964, 1965). Using the "double series" expansion of the metric tensor introduced by Bonnor (1959)

$$g_{\mu\nu} = \eta_{\mu\nu} + M\overset{(10)}{g}_{\mu\nu} + m\overset{(01)}{g}_{\mu\nu} + M^2\overset{(20)}{g}_{\mu\nu} + Mm\overset{(11)}{g}_{\mu\nu} + \cdots \qquad (3\text{-}9)$$

where M and m are the inertial masses of a two particle system, Carmeli also imposes the "slow-motion" assumption in order to expand the "k-approximation" terms of the metric $\overset{(pq)}{g}_{\mu\nu}$ as a power series in c^{-1}

$$\overset{(pq)}{g}_{\mu\nu} = \underset{(0)}{\overset{(pq)}{g}}_{\mu\nu} + \underset{(1)}{\overset{(pq)}{g}}_{\mu\nu} + \underset{(2)}{\overset{(pq)}{g}}_{\mu\nu} + \cdots$$

just as in the EIH approximation. The result is effectively a synthesis of both methods with a proper correspondence between the two, thus addressing the criticism of Infeld (1961). Carmeli obtains the equations of motion in the "3rd-approximation" which are to the 9th-order in c^{-1} (λ^9), where the radiation reaction manifests itself. Although he does not solve the resultant equations, he does present an argument maintaining that the radiating two-body problem is damped, losing energy, under an outgoing radiation condition.

The Method Of Matched Asymptotic Expansions

A variation of the EIH approximation has been presented by Burke & Thorne (1969, 1970, 1971) based upon the methods of singular perturbation theory for nonlinear differential equations. Their technique establishes no new results in gravitational radiation theory, but it does succeed in reproducing the results of the EIH method for gravitational radiation from the two-body problem found by Peres (1960) and Ryteń (1963). The work of Burke & Thorne

borrows extensively from the EIH method, and the paper of Infeld (1938).

The formalism is the method of matched asymptotic expansions, an understanding of which follows from the continued analogy with radiation in electromagnetism originally due to Infeld (1938) and Trautman (1958). It begins with exactly the same EIH representation of the metric tensor $g_{\mu\nu}$ as in (3-8) or Figure 2,

$$\begin{aligned} g_{00} &= -1 + \tfrac{1}{2}\epsilon^2\psi + \epsilon^4\left(\tfrac{1}{2}d + \tfrac{1}{2}H^c{}_c\right) + \cdots \\ g_{0a} &= -\epsilon^3 V_a + \cdots \\ g_{ab} &= \delta_{ab} + \tfrac{1}{2}\epsilon^2\psi\delta_{ab} + \epsilon^4\left(\tfrac{1}{2}d\delta_{ab} + H_{ab} - \tfrac{1}{2}H^c{}_c\delta_{ab}\right) + \cdots \end{aligned} \tag{3-46}$$

except that the various components have been expressed explicitly as certain functions: the g_{00} are scalars, the g_{0a} are vectors, and the g_{ab} are tensors. (These choices follow by analogy with the components of the energy-momentum tensor where T^{00} is the scalar energy densiry, T^{0a} is the vector momentum flux, and T^{ab} is the Maxwell stress tensor.) Plugging (3-46) into the Einstein tensor $G_{\mu\nu}$ in (3-41) gives the following EIH result that

$$\begin{aligned} G_{00} &= -\tfrac{1}{2}\epsilon^2\nabla^2\psi + \cdots \\ G_{0a} &= \tfrac{1}{2}\epsilon^3\left[\nabla^2 V_a - \left(\underset{\sim}{\nabla}\cdot\underset{\sim}{V} + \partial_0\psi\right)_{,a}\right] + \cdots \\ G_{ab} &= \epsilon\Big\{-\tfrac{1}{2}\nabla^2 H_{ab} + \tfrac{1}{8}\psi_{,a}\psi_{,b} + \tfrac{1}{4}\psi\psi_{,ab} + \tfrac{1}{2}\left(H^c{}_{a;c} + V_{a,0}\right)_{;b} \\ &\quad + \tfrac{1}{2}\left(H^c{}_{b;c} + V_{b,0}\right)_{;a} - g_{ab}\left(\tfrac{3}{16}\nabla\psi\cdot\nabla\psi + \tfrac{1}{4}\psi\nabla^2\psi\right) \\ &\quad - \tfrac{1}{2}g_{ab}\left[\nabla\cdot\left(\nabla\cdot\underset{\sim}{H} + \partial_0\underset{\sim}{V}\right) + \partial_0\left(\nabla\cdot\underset{\sim}{V} + \partial_0\psi\right)\right]\Big\} + \cdots \end{aligned} \tag{3-47}$$

where $\underset{\sim}{H}$ is a 3-dyadic.

Burke (1971) then imposes the auxiliary conditons

$$\begin{aligned} \underset{\sim}{\nabla}\cdot\underset{\sim}{V} + \partial_0\psi &= 0 \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{H} + \partial_0\underset{\sim}{V} &= 0 \end{aligned} \tag{3-48}$$

which succeed in simplifying the Einstein tensor considerably. The Einstein field equations (3-1) or (3-39)

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \tag{3-39}$$

now reduce to

$$\begin{aligned} \nabla^2\psi &= 2\kappa\rho \\ \nabla^2\underset{\sim}{V} &= 2\kappa\underset{\sim}{J} \\ \nabla^2\underset{\sim}{H} &= 2\kappa\left[\underset{\sim}{S} + {}^G\underset{\sim}{S}\right], \end{aligned} \tag{3-49}$$

where

$$\begin{aligned} \rho &\equiv T_{00} \\ J_a &\equiv -T_{0a} \\ S_{ab} &\equiv T_{ab} \\ {}^{G}S_{ab} &\equiv \left(-\frac{\pi}{8}\right)\left[\frac{1}{8}\psi_{,a}\psi_{,b}+\frac{1}{4}\psi\psi_{,ab}-g_{ab}\left(\frac{3}{16}\nabla\psi\cdot\nabla\psi+\frac{1}{4}\psi\nabla^2\psi\right)\right], \end{aligned} \tag{3-50}$$

defining ρ as the energy density, J_a as the momentum flux, S_{ab} as the matter stress, and ${}^{G}S_{ab}$ as the gravitational stress.

Figure 2 for the EIH approximation becomes in this method of matched asymptotic expansions Figure 4. Due to the auxiliary conditions (3-48) and the EIH field equations (3-49), the sources must satisfy

$$\begin{aligned} \nabla\cdot\underset{\sim}{J} + \partial_0\rho &= 0 \\ \nabla\cdot\left(\underset{=}{S}+{}^{G}\underset{=}{S}\right) + \partial_0\underset{\sim}{J} &= 0 \end{aligned} \tag{3-51}$$

as conservation laws.

The gravitational force is determined by

$$-\nabla\cdot{}^{G}\underset{=}{S} = +\frac{1}{4}\rho\nabla\psi \tag{3-52}$$

where $\psi=4\phi$, with ϕ representing the Newtonian potential.

In effect, gravitationally bound systems such as the two-body problem are found to radiate quadrupole radiation while creating additional resistive fields which couple back from the radiation zone into the near or induction zone as pictured in Figure 3, causing radiation damping and a loss of energy.

The method of matched asymptotic expansions and singular perturbation theory in effect criticizes the EIH method as being strictly valid only in the near or inner zone, due to the fact that "slow-motion" expansions are not necessarily valid at distances far from the cources in nonlinear theories. A separate outer zone expansion is required there where the stresses are the nonlinear ones due to the gravitational waves themselves. The inner and outer expansions must then be properly matched. Although this technique has only succeeded in duplicating the results of the EIH method at the present time, its authors have argued that it simplifies calculations while providing a consistent and systematic frame work for the EIH theory. It could prove to be a significant step in the direction of a solution of some of the more formidable problems that remain unscathed by existing techniques.

$$g_{oo} = -1 + \epsilon^2 \frac{\psi}{2} + \epsilon^4 \frac{d}{2} - \epsilon^6 \frac{e}{2} + \epsilon^7 \frac{f}{2}$$

$$g_{ao} = -\epsilon^3 V_a + \epsilon^5 O_a + \epsilon^6 X_a$$

$$g_{ab} = \delta_{ab} \quad \epsilon^2 \frac{\psi}{2} \delta_{ab} + \epsilon^4 H_{ab} + \epsilon^5 X_{ab}$$

Radiation

Newtonian Forces | Post$^{1.0}$ Forces | Post$^{2.0}$ Forces | Radiation Reaction

FIGURE 4

The Regge-Wheeler-Zerilli Formalism

Another interesting perturbation technique for solving Einstein's nonlinear field equations appears in the very important paper by Regge & Wheeler (1957). They address the question of the stability of the Schwarzschild (1916) metric by considering a nonspherical perturbation of the Schwarzschild background $\eta_{\mu\nu}$, which is defined by

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
$$e^\nu = e^{-\lambda} = (1-2m/r) \quad , \quad m \equiv GM/c^2 \quad . \tag{3-53}$$

Based upon a weak-field approximation as before in (3-6)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \tag{3-6}$$

the perturbations $h_{\mu\nu}$ are decomposed into a generalized nonspherical form involving tensor spherical harmonics and comprising a separable solution of the type

$$h_{\mu\nu}(r,\theta,\phi,t) = h(r) Y^M_{L\,\mu\nu}(\theta,\phi) e^{-ikt} \quad . \tag{3-54}$$

The Einstein field equations are solved to first order in h, and it is determined that the time-dependence exp(-ikt) cannot diverge because imaginary frequencies k are disallowed, thereby demonstrating the stability - or so maintain Regge & Wheeler - of the Schwarzschild solution. Their technique is contested by Peres & Rosen (1959) as unsatisfactory because it does not take into account properly the nonlinear effects of gravitational radiation. Indeed, Peres & Rosen maintain that small oscillations cannot be stable for any gravitational field assumed to be asymptotically flat at infinity - but this criticism has been either ignored or forgotten. The work of Regge & Wheeler is pursued further by Manasse (1963), Brill & Hartle (1964), Doroshkevich, et al (1965), Peters (1966), Vishveshwara (1968, 1970), Vishveshwara & Edelstein (1970), and Zerilli (1969, 1970).

Regge & Wheeler note that Einstein's field equations (3-1) for the exterior Schwarzschild background $\eta_{\mu\nu}$ are simply

$$R_{\mu\nu}(\eta) = 0 \quad , \tag{3-55a}$$

and they argue that a perturbation of the metric as in (3-6) likewise results in a perturbation of the field equations which still equal zero

$$R_{\mu\nu}(\eta+h) = R_{\mu\nu}(\eta) + \delta R_{\mu\nu}(h) = 0 . \qquad (3\text{-}55b)$$

By virtue of (3-54), then (3-55) reduces to

$$R^{(1)}_{\mu\nu} = \delta R_{\mu\nu}(h) = 0 , \qquad (3\text{-}56a)$$

which is given in Appendix A, Eq (A-6). This perturbation can likewise be represented by a relation due to Eisenhart (1926)

$$R^{(1)}_{\mu\nu} = \delta R_{\mu\nu} = -\delta\Gamma^{\beta}_{\mu\nu;\beta} + \delta\Gamma^{\beta}_{\mu\beta;\nu} \qquad (3\text{-}56b)$$

noting that $\delta\Gamma^{\beta}_{\mu\nu}$ is a tensor although $\Gamma^{\beta}_{\mu\nu}$ is not. Eq (3-56) is the covariant generalization in curved space of the Schrödinger equation for a massless Spin-2 particle in flat space (Regge, 1957).

The most general form of $h_{\mu\nu}$ consists of an odd-parity $(-1)^{L+1}$ case

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & -h_0(t,r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & h_0(t,r)\sin\theta\frac{\partial}{\partial\theta} \\ 0 & 0 & -h_1(t,r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & h_1(t,r)\sin\theta\frac{\partial}{\partial\theta} \\ 0 & 0 & h_2(t,r)\left[\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right] & * \\ * & * & \frac{1}{2}h_2(t,r)\left[\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} + \cos\theta\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial^2}{\partial\theta^2}\right] & -h_2(t,r)\left[\sin\theta\frac{\partial^2}{\partial\theta\partial\phi} - \cos\theta\frac{\partial}{\partial\phi}\right] \end{bmatrix} Y_L^M(\theta,\phi) \qquad (3\text{-}57a)$$

as well as an even-parity $(-1)^L$ case

$$h_{\mu\nu} = \begin{bmatrix} \varepsilon^{\nu} H_0(t,r) & H_1(t,r) & h_0(t,r)\frac{\partial}{\partial\theta} & h_0(t,r)\frac{\partial}{\partial\phi} \\ H_1(t,r) & \varepsilon^{\lambda} H_2(t,r) & h_1(t,r)\frac{\partial}{\partial\theta} & h_1(t,r)\frac{\partial}{\partial\phi} \\ * & * & r\left[K(t,r) + G(t,r)\frac{\partial^2}{\partial\theta^2}\right] & * \\ * & * & r^2 G(t,r)\left[\frac{\partial^2}{\partial\theta\partial\phi} - \cos\theta\frac{\partial}{\partial\phi}\right] & r^2\left[K(t,r)\sin^2\theta + G(t,r)\left(\frac{\partial^2}{\partial\phi^2} + \cos\theta\sin\theta\frac{\partial}{\partial\theta}\right)\right] \end{bmatrix} Y_L^M(\theta,\phi) \qquad (3\text{-}57b)$$

where the asterisk $*$ means $h_{ij}=h_{ji}$. By performing a gauge transformation - known as the Regge-Wheeler gauge (Appendix I) - such as is discussed in

Appendix (B-5), the perturbations $h_{\mu\nu}$ of (3-57) can be reduced to the following canonical form:

Odd-Parity Perturbations

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix} e^{-ikt} \left[\sin\theta \frac{\partial}{\partial\theta}\right] P_L(\cos\theta) \tag{3-58a}$$

Even-Parity Perturbations

$$h_{\mu\nu} = \begin{bmatrix} H_0 e^{\nu} & H_1 & 0 & 0 \\ * & H_2 e^{\lambda} & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 K \sin^2\theta \end{bmatrix} e^{-ikt} P_L(\cos\theta) \tag{3-58b}$$

where aximuthal symmetry has been assumed (M=0). This particular form of $h_{\mu\nu}$ is what is meant by the "Regge-Wheeler gauge." It consists of two odd-parity (lower case) radial functions h_0 and h_1 as well as four even-parity (upper case) radial functions H_0, H_1, H_2, and K.

The Regge-Wheeler canonical perturbations of the Schwarzschild metric, then, substituting (3-58) into (3-6) and (3-53), amount to

Odd-Parity Schwarzschild Metric

$$\begin{aligned} ds^2 = {} & \left[e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \\ & + 2\left(h_0(r)dt + h_1(r)dr\right) d\phi\, e^{-ikt} \left(\sin\theta \frac{\partial}{\partial\theta}\right) P_L(\cos\theta) \end{aligned} \tag{3-59a}$$

Even-Parity Schwarzschild Metric

$$\begin{aligned} ds^2 = {} & \left(1 + H_0 P_L e^{-ikt}\right) e^{\nu} dt^2 + \left(1 + H_2 P_L e^{-ikt}\right) e^{\lambda} dr^2 \\ & + r^2\left(1 + K P_L e^{-ikt}\right)(d\theta^2 + \sin^2\theta\, d\phi^2) + 2 H_1 P_L e^{-ikt} dt\, dr. \end{aligned} \tag{3-59b}$$

The problem remaining is to determine the odd-parity and even-parity radial functions. These are obtained by solving the differential equations (Appendix H) which result from the substitution of $h_{\mu\nu}$ in (3-58) back into the perturbed Ricci tensor and field equations (3-56). For the odd-parity case Regge & Wheeler obtain the following "Schrödinger" type equation

$$\frac{d^2Q}{dr^{*2}} + \left(k^2 - V_{eff}\right)Q = 0 \qquad \text{(H-6)}$$

where $Q = e^{\nu}h_1/r$ and r^* is given by

$$r^* = r + 2m \ln\left|\frac{r}{2m} - 1\right| . \qquad \text{(H-5)}$$

The effective potential V_{eff} is

$$V_{eff} = \left(1 - \frac{2m}{r}\right)\left[\frac{L(L+1)}{r^2} - \frac{6m}{r^3}\right] . \qquad \text{(H-8)}$$

For the odd-parity quadrupole oscillations (L=2) the effective potential V_{eff} in (H-8) can be shown to be that in Figure 5.

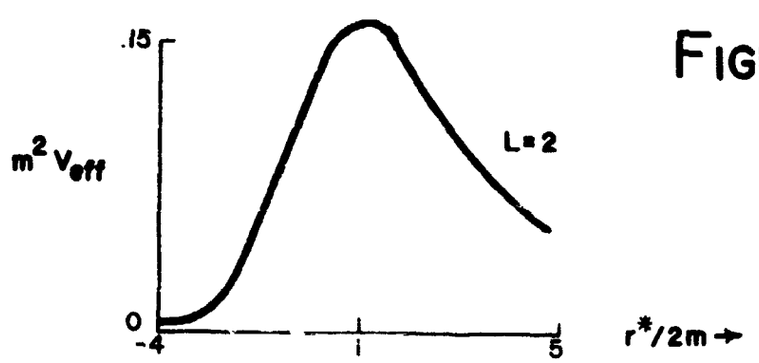


FIGURE 5

Vishveshwara (1970) treats the scattering of gravitational radiation from such a Schrödinger-type effective potential.

The even-parity radial equations (Appendix, H-10) prove to be more difficult and are not reduced to a single second-order differential equation until the work of Zerilli (1970b), which is presented in detail in Appendix J. He obtains a homogeneous "Schrödinger" equation (J-7) for the radial function $\hat{K}_{LM}(r^*)$ from which $h_{\mu\nu}$ in (3-58b) can be determined:

$$\frac{d^2\hat{K}_{LM}}{dr^{*2}} + \left[k^2 - V_L(r)\right]\hat{K}_{LM} = 0 \qquad \text{(J-7)}$$

where r^* is given by (H-5) and where

$$V_L(r) = \left(1-\frac{2m}{r}\right)\left[\frac{2\lambda^2(\lambda+1)r^3+6\lambda^2 mr^2+18\lambda m^2 r+18m^3}{r^3(\lambda r+3m)^2}\right] \qquad \text{(J-8)}$$

$$\lambda = \frac{1}{2}(L-1)(L+2) \quad . \qquad \text{(J-5)}$$

$V_L(r)$ in (J-8) is the even-parity dual of (H-8). Eqs (J-7) and (J-8) are sometimes referred to as Zerilli's Equations.

Tensor Harmonic Decomposition With Sources

The original treatment due to Regge & Wheeler (1957) is the first decomposition into tensor harmonics of perturbations on a spherically symmetric background. Tensor harmonic decompositions are useful because they provide for a separation of variables in the differential equations, and for this reason are further analyzed by Mathews (1962), Peters & Mathews (1963), Thorne, et al (1967-1970), and Zerilli (1970a,c). Such decompositions are also important in generalizing the linear perturbation technique used on the Schwarzschild background to include sources.

Recall that Regge & Wheeler do not account for sources. Theirs is a canonical nonspherical perturbation $h_{\mu\nu}$ of the Ricci tensor such that $R^{(1)}_{\mu\nu} = 0$, as in (3-56). They thereby find Green's functions from which solutions of inhomogeneous equations with sources can be constructed. On the other hand, in order to establish realistic perturbations with astrophysical applications one can account for sources with a non-zero stress-energy tensor $T_{\mu\nu}$. For example, this treatment of the problem is reasonable in the study of the perturbation due to a small particle falling into the Schwarzschild geometry from infinity.

Such an analysis is formulated by Peters (1966) who considers first-order linear perturbations of the Einstein tensor $\delta G_{\mu\nu}$ rather then $\delta R_{\mu\nu}$, and accounts for sources of small mass m ($m \ll M_s$) by means of a variation in the stress-energy tensor $T_{\mu\nu}$. It is a weak-field approximation (3-6) and must be classified as a slow-motion one also, to the extent that the dilatation of mass (3-5) retains the condition m(particle)$\ll M_s$(Schwarzschild). Peters' linear perturbation of the Einstein equations (3-1) results in the following field equations for a background $\eta_{\mu\nu}$ of constant curvature (Ricci flat with $R_{\mu\nu}=0$):

$$\left[h_{\mu\nu;\alpha}{}^{;\alpha} - (f_{\mu;\nu}+f_{\nu;\mu}) + 2R^{\rho}{}_{\mu}{}^{\alpha}{}_{\nu}h_{\rho\alpha} + h^{\alpha}{}_{\alpha;\mu;\nu}\right] + g_{\mu\nu}\left(f_{\alpha}{}^{;\alpha} - h^{\alpha}{}_{\alpha;\lambda}{}^{;\lambda}\right) = -2\kappa\,\delta T_{\mu\nu}. \qquad \text{(3-60)}$$

In (3-60), f_μ represents an arbitrary gauge vector introduced in order to reduce these field equations to a canonical form, once the unperturbed background metric $\eta_{\mu\nu}$ is chosen. Furthermore, in order that (3-60) be consistent, $\delta T_{\mu\nu}$ must obey a conservation law. Taking the variation of (3-38) shows that the representation of the perturbing source $\delta T_{\mu\nu}$ must satisfy

$$(\delta T_{\mu\nu})^{;\nu} = f_\alpha T_\mu{}^\alpha + h^{\alpha\beta} T_{\mu\alpha;\beta} + \tfrac{1}{2} h^{\alpha\beta}{}_{;\mu} T_{\alpha\beta} = 0 . \qquad (3\text{-}61)$$

Peters represents the part of $\delta T_{\mu\nu}$ due to the perturbing particle as

$$\delta T_{\mu\nu}{}^{(m)} = g_{\mu\alpha} g_{\nu\beta} \delta T^{\alpha\beta\,(m)} \qquad (3\text{-}62a)$$

where

$$\delta T^{\alpha\beta(m)} = m \int ds\, \delta^4(x - z(s)) \frac{dz^\alpha}{ds} \frac{dz^\beta}{ds} . \qquad (3\text{-}62b)$$

$z^\mu(s)$ is the space-time position of the particle in the metric. Peters finishes his treatment with a Green's function method for curved space and specializes his results to the Schwarzschild background.

Peters' field equations (3-60) and conservation law (3-61) <u>assume</u>* the stability of the background metric $\eta_{\mu\nu}$; this work is not a stability analysis as is that of Regge & Wheeler. Furthermore, the perturbation is of first-order and thereby neglects radiation reaction. The perturbing particle is assumed to follow a geodesic in the unperturbed background $\eta_{\mu\nu}$ for the computation of $\delta T_{\mu\nu}$ in (3-62) - an assumption which is not correct. The perturbing particle follows a geodesic in the perturbed metric $g_{\mu\nu}$ and not a geodesic in $\eta_{\mu\nu}$, according to Einstein's theory. (See Figure 20)

Nevertheless, Peters' representation of Schwarzschild perturbations is a very useful approximation, as has been demonstrated by Zerilli (1970a,c), who decomposes Peters' field equations (3-60) into tensor harmonics using the Regge-Wheeler gauge.

Zerilli (1970c) develops an orthonormal set of tensor harmonics (Appendix K) for application to gravitational radiation theory, based upon the earlier work of Regge & Wheeler (1957) and Mathews (1962). Because Peters' field equations (3-60) are of the form

$$Q[h_{\mu\nu}] = -2\varkappa \delta T_{\mu\nu} , \qquad (3\text{-}63)$$

---

* Peters and Zerilli assume $h_{\mu\nu}$ is small if $\delta T_{\mu\nu}$ is small.

where Q is a rotationally invariant operator, Zerilli expands both sides of (3-63) in tensor harmonics in order to separate the angular variables θ,ϕ from the radial equations. Choosing the Regge-Wheeler gauge (Appendix I) the $h_{\mu\nu}$ decompose as

$$\underline{\underline{h}}^{(m)}_{LM} = \frac{i}{r}\sqrt{2L(L+1)}\left\{ i h_{0_{LM}} \underline{\underline{c}}^{(o)}_{LM} + h_{1_{LM}} \underline{\underline{c}}_{LM} \right\} \tag{3-64}$$

$$\underline{\underline{h}}^{(e)}_{LM} = \left(1-\frac{2m}{r}\right) H_{0_{LM}} \underline{\underline{a}}^{(o)}_{LM} - i\sqrt{2}\, H_{1_{LM}} \underline{\underline{a}}^{(1)}_{LM} + \left(1-\frac{2m}{r}\right)^{-1} H_{2_{LM}} \underline{\underline{a}}_{LM} + \sqrt{2} K_{LM} \underline{\underline{g}}_{LM}$$

where tensor subscripts "$\mu\nu$" have been replaced by the double stroke in order to simplify tensor notation. The superscript (e) corresponds to the electric or even-parity case while (m) stands for magnetic or odd-parity. Referring to the tensor harmonics in Appendix K, one can verify that (3-64) is essentially the same as the Regge-Wheeler gauge in (3-58).

Likewise the stress-energy tensor for the perturbing particle $\delta T_{\mu\nu}$ can be expanded into the tensor harmonics of Appendix K:

$$\delta \underline{\underline{T}} = \sum_{L,M} \Big\{ A^{(o)}_{LM} \underline{\underline{a}}^{(o)}_{LM} + A^{(1)}_{LM} \underline{\underline{a}}^{(1)}_{LM} + A_{LM} \underline{\underline{a}}_{LM} + B^{(o)}_{LM} \underline{\underline{b}}^{(o)}_{LM} + B_{LM} \underline{\underline{b}}_{LM} + Q^{(o)}_{LM} \underline{\underline{c}}^{(o)}_{LM} + Q_{LM} \underline{\underline{c}}_{LM} + G_{LM} \underline{\underline{g}}_{LM} + D_{LM} \underline{\underline{d}}_{LM} + F_{LM} \underline{\underline{h}}_{LM} \Big\}. \tag{3-65}$$

The respective coefficients $A_{LM},\ldots,F_{LM}$ are defined in Appendix L and represent the amplitudes of the various tensor harmonic components of the stress-energy tensor which "drive" the perturbation of the metric.

Substitution of the tensor decompositions (3-64) and (3-65) back into Peters' field equations (3-63) or (3-60) results in one equation for the magnetic (odd or L+1) parity and one for the electric (even or L) parity case with tensor harmonics on both sides. The coefficients of the tensor harmonics on each side of these equations muct then be set equal, which gives the radial equations for the Peters-Zerilli analysis in the Regge-Wheeler gauge with sources (Appendix O). The sources, in turn, must satisfy the conservation conditions specified in (3-61) and Appendix M.

Zerilli next takes the Fourier transform of the radial equations (Appendix P) and then reduces them (P-1 and P-2) to second-order "Schrodinger" equations with sources, in the same fashion as is performed with the source-free Regge-Wheeler problem discussed earlier. Introducing two auxiliary radial functions $R^{(m)}_{LM}$ and $R^{(e)}_{LM}$ he obtains Eqs (3-66) & (3-67):

Magnetic-Parity "Schrödinger" Equation With Sources

$$\frac{d^2 R_{LM}^{(m)}}{dr^{*2}} + \left[\omega^2 - V_L^{(m)}(r)\right] R_{LM}^{(m)} = S_{LM}^{(m)} \tag{3-66a}$$

where

$$V_L^{(m)}(r) = \left(1 - \frac{2m}{r}\right)\left[\frac{L(L+1)}{r^2} - \frac{6m}{r^3}\right] \tag{3-66b}$$

$$S_{LM}^{(m)} = -\mathcal{H} i \left[\tfrac{1}{2} L(L+1)(L-1)(L+2)\right]^{-1/2} \left\{ \frac{d}{dr}\left[r(r-2m) D_{LM}\right] + 2(r-2m) D_{LM} + (r-2m)\sqrt{(L-1)(L+2)}\, Q_{LM} \right\}. \tag{3-66c}$$

Electric-Parity "Schrödinger" Equation With Sources

$$\frac{d^2 R_{LM}^{(e)}}{dr^{*2}} + \left[\omega^2 - V_L^{(e)}(r)\right] R_{LM}^{(e)} = S_{LM}^{(e)} \tag{3-67a}$$

where

$$V_L^{(e)}(r) = \left(1 - \frac{2m}{r}\right)\left[\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 m r^2 + 18\lambda m^2 r + 18 m^3}{r^3(\lambda r + 3m)^2}\right] \tag{3-67b}$$

$$S_{LM}^{(e)} = -i\left(1 - \frac{2m}{r}\right)\frac{d}{dr}\left[\frac{(r-2m)^2}{r(\lambda r + 3m)}\left(\frac{i r^2}{r-2m}\tilde{C}_{1_{LM}} + \tilde{C}_{2_{LM}}\right)\right] + i\frac{(r-2m)^2}{r(\lambda r+3m)^2}\left[\left(\frac{\lambda(\lambda+1)r^2 + 3\lambda m r + 6m^2}{r^2}\right)\tilde{C}_{2_{LM}} + i\left(\frac{\lambda r^2 - 3\lambda m r - 3m^2}{r-2m}\right)\tilde{C}_{1_{LM}}\right] \tag{3-67c}$$

$$\lambda = \tfrac{1}{2}(L-1)(L+2) \quad .$$

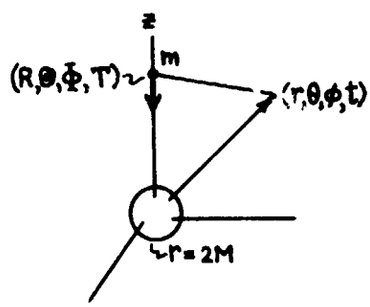
The original radial functions h_0, h_1, H_0, H_1, H_2, and K can now be derived from R_{LM} in (3-66) and (3-67). These relations as well as those for the auxiliary source coefficients C_1, C_2, and B are all given in Appendix Q. Equations (3-66) and (3-67) are also referred to as "Zerilli's Equations."

Upon evaluation of the coefficients of Appendix L for a Schwarzschild geodesic, Eq (L-8), one finds that for a particle falling radially (say along the z-axis) into the Schwarzschild geometry, all magnetic (odd or L+1) parity contributions and all $M \neq 0$ electric perturbations vanish. For this case, the source function $S^{(e)}_{LM=0}$ in (3-67) reduces to

$$S^{(e)}_L = \frac{4M}{\lambda r+3M}\sqrt{L+\tfrac{1}{2}}\left(1-\frac{2M}{r}\right)\left[\sqrt{\frac{r}{2M}} - \frac{i\,2\lambda}{\omega(\lambda r+3M)}\right]e^{i\omega T(r)} \tag{3-68a}$$

$$= \left[S^1_L + i S^2_L\right]e^{i\omega T(r)} \tag{3-68b}$$

where



$$S^1_L = -2m\sqrt{L+\tfrac{1}{2}}\left(\frac{r-2M}{\lambda r+3M}\right)\left[\frac{dT}{dr}\frac{\sqrt{r}}{r} + \frac{2}{r-2M}\sqrt{\frac{r}{r}}\right] \tag{3-68c}$$

$$S^2_L = -2m\left(\frac{r-2M}{\lambda r+3M}\right)\left[\frac{\sqrt{2(2L+1)}}{r}\,\frac{2\lambda}{\omega(\lambda r+3m)}\right] \tag{3-68d}$$

$$T(r) = -\frac{4}{3}\left(\frac{r}{3M}\right)^{3/2} - 4\sqrt{\frac{r}{2M}} + 2\ln\left|\frac{\sqrt{r/2M}+1}{\sqrt{r/2M}-1}\right|. \tag{3-68e}$$

These equations are of current astrophysical interest, in that several groups are evaluating them for higher L by numerical integration techniques, as we shall see in Chapter 5.

Zerilli solves his equations in $h_{\mu\nu}$ for L=0 and L=1 in the above case of a particle in orbit about a Schwarzschild geodesic. The magnetic monopole (L=0) is identically zero. The electric monopole (L=0) case correspond to a mass perturbation which simply adds the particle mass m to the Schwarzschild mass M (γ_0 is a constant), depending upon the position r of the observer:

$$e^{\nu(r,t)} = \begin{cases} 1-\dfrac{2M}{r} & , r<R(t) \\ 1-\dfrac{2(M+m\gamma_0)}{r} & , r>R(t) \end{cases} . \tag{3-69a}$$

The magnetic dipole (L=1) contribution is the angular momentum of the equatorial orbit assumed, which is

$$\left.\begin{aligned} L_z &= ma \\ a &= R^2\sin^2\Theta\,\dot{\Phi} \end{aligned}\right\} , \tag{3-69b}$$

while the electric dipole (L=1) represents a coordinate transformation of $g_{\mu\nu}$ by $h_{\mu\nu}$, as we know it should - but not contributing dipole radiation due to conservation of momentum. For L>1, analytic solutions have not been obtained.

The Bondi News Function & Multipole Expansions

An entirely different approximation technique is the method of multipole expansions. Multipole approximations have historically provided a great deal of insight into the study of radiation theory, certainly in the classical vector theory of electromagnetic and gravitational radiation, as discussed earlier. Likewise, multipole approximations have found their place in the tensor theory of gravitational radiation.

In order to illustrate the multipole approximation in General Relativity, it is best to consider the work of Bondi (1960, 1962, 1965) who has been expecially concerned about the meaning and the physical existence of gravitational radiation from gravitationally bound and otherwise isolated systems. On the basis of causality arguments, Bondi (1962) considers only retarded solutions and he places great emphasis upon the requirement that gravitational radiation must result in a loss of energy and hence mass of the source in order to be consistent with the spirit of General Relativity. This means that for gravitational radiation from a mass quadrupole, there must be a secular change in the mass monopole. Thus the monopole must be coupled to the radiative multipoles, such as the quadrupole, of the multipole approximation.

Bondi's development (Appendix R) treats the stationary-radiative-stationary transition of an outgoing gravitational "sandwich" wave (Figure 6) as it passes through some retarded hypersurface "u=t-r/c=constant" in an asymptotically flat region of space:

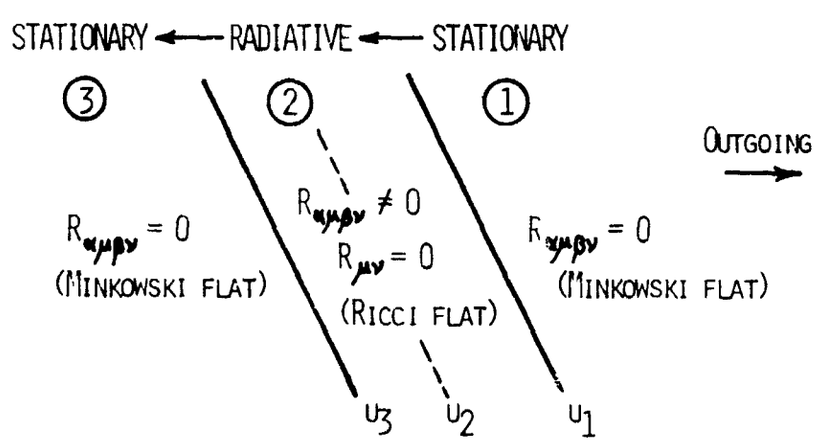
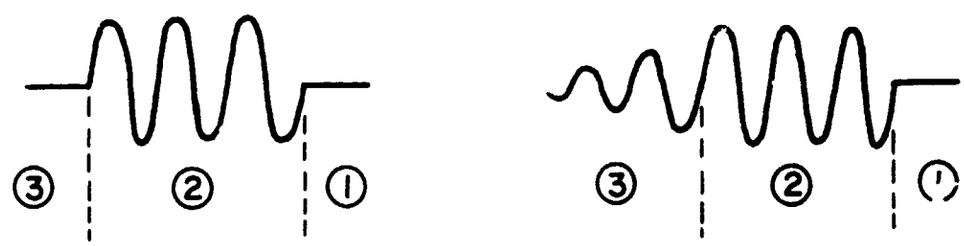


FIGURE 6

In Figure 6, regions 1 and 3 are stationary and Minkowski flat with no curvature. Region 2, however, is radiative and empty, behaving in much the same fashion as the Regge-Wheeler treatment presented earlier. Such a transition is assumed in order for the Bondi formalism to be consistent with Huygens' principle. It is tantamount to the statement that gravitational waves in a stationary-radiative-stationary transition have no "tails":

FIGURE 7



(A) Bondi's (Huygens) Assumption (B) Gravitational "Tails"

The existence of "tails" (Figure 7b) means that the background geometry continues to oscillate as a consequence of the "shock" perturbations induced by the passage of the gravitational wavefront. It has since been demonstrated, however, that Bondi's assumption of no "tails" (Figure 7a) is indeed invalid. Papapetrou (1969) and Hallidy & Janis (1970) have investigated the problem of the existence of a final stationary state (Region 3 of Figures 6 & 7a) and they have concluded that stationary-radiative-stationary transitions cannot occur for axially symmetric (Bondi) radiation of a finite multipole expansion. Nevertheless, it is informative to consider Bondi's representation because Sachs (1962) has demonstrated that the condition of axial symmetry can be relaxed, and the "tails" eliminated.

The critical and controversial question of outgoing radiation at infinity is addressed by Bondi by choosing the axially symmetric (independent of ϕ) metric

<u>Bondi's Axially Symmetric Metric</u>

$$ds^2 = A du^2 + 2B du dr - r^2\left[e^{2\gamma}(d\theta - F du)^2 + e^{-2\gamma}\sin^2\theta d\phi^2\right] \qquad (3\text{-}70)$$

which is extremely well suited to the problem of the Sommerfeld radiation condition. The Bondi coordinates are $X^\mu = (u=t-r, r, \theta, \phi)$. Asymptotically the Bondi metric reduces to its Minkowski form

$$ds_0^2 = du^2 + 2du dr - r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right] .$$

For sufficiently large values of r, Bondi (Appendix R) assumes for the arbitrary functions in his metric (3-70) or (R-1) the following:

$$\left.\begin{aligned} \gamma &= \frac{c}{r} + O\left(\frac{1}{r^2}\right) \\ F &= \frac{f}{r} + O\left(\frac{1}{r^2}\right) \\ B &= \left[1+\frac{b}{r}\right] + O\left(\frac{1}{r^2}\right) \\ A &= \left[1-\frac{2M}{r}\right] + O\left(\frac{1}{r^2}\right) \end{aligned}\right\} \quad . \tag{3-71}$$

He shows that f=b=0. Then he derives from the field equations $R_{\mu\nu} = 0$ in Region 2 of Figure 6 a differential equation for the time rate of change of the function $M(u,\theta)$ in (3-71)

$$\frac{\partial M}{\partial u} = -\left(\frac{\partial c}{\partial u}\right)^2 + \frac{1}{2}\frac{\partial}{\partial u}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(c\sin^2\theta\right) \quad . \tag{3-72 (R-12a)}$$

Based upon correspondence arguments with the static case, the "mass aspect" $M(u,\theta)$ is shown to be related to the mass of the source $m(u)$ by

$$m(u) = \frac{1}{2}\int_0^{\pi} M(u,\theta)\sin\theta\, d\theta \quad . \tag{3-73}$$

Differentiation of $m(u,\theta)$ in (3-73) with respect to u and substitution of M_0 from (3-72) gives Bondi's result that

$$\frac{dm}{du} = -\frac{1}{2}\int_0^{\pi}\left(\frac{\partial c}{\partial u}\right)^2 \sin\theta\, d\theta \quad . \tag{3-74}$$

Bondi calls the function $c=c(u,\theta)$ a "news function" because from it alone can be determined the entire behaviour of his metric. In particular, from relation (3-74) one can see that knowledge of $c(u,\theta)$ on some hypersurface u=constant would determine the secular change of the mass monopole source. Furthermore, if the news function is not zero, then from (3-74) there must be a monotonic <u>decrease</u> of the mass $m(u)$ of the source. Bondi's method is effectively a method of asymptotic linearization, and is illustrated in Figure 8.

In a paper due to Penrose & Newman (1965) a set of ten geometrical quantities, defined for asymptotically flat space-times, are shown to be conserved during the emission of gravitational radiation. One of these quantities is particularly relevant to the Bondi representation and states that a

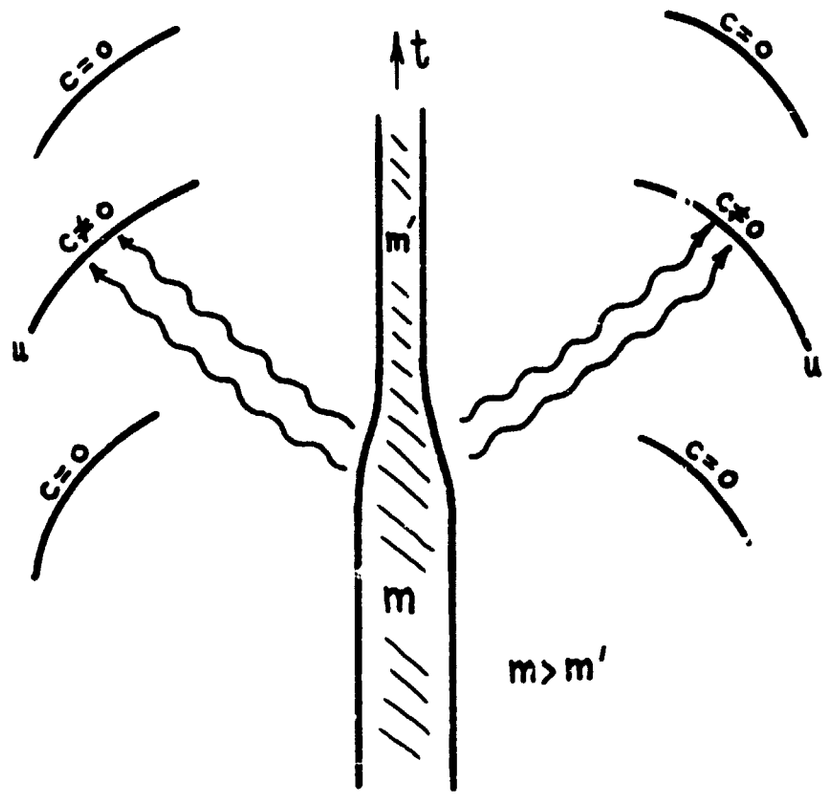


FIGURE 8

transition between two stationary states (Figures 6 & 7a) is allowed if

$$D^2 - mQ = \text{Conserved} \quad , \tag{3-75}$$

where D is the dipole, m is the mass, and Q is the quadrupole moment of the source. (3-75) is not sufficient to insure that the final state is stationary, however.

Bondi's work is generalized by Sachs (1962) who relaxes the condition of spatial symmetry on the fields, allowing for two polarizations instead of Bondi's one. Sach's generalization is based upon a complex news function c, from which he determines the rate of loss of mass due to gravitational radiation (Figure 8) as

$$\dot{m} = -\langle \dot{c}\dot{c}^* \rangle \quad , \tag{3-76}$$

which is identical to Bondi's (3-74), except that carets are used to represent the integral or average over the sphere at infinity. Morgan & Peres (1963) identify the Sachs news function with the mass quadrupole tensor Q_{ij} and then from (3-76) derive the relation

$$\dot{m} = -\frac{G}{5c^5}\dddot{Q}_{ij}\dddot{Q}^{ij} \quad , \tag{3-77}$$

in close agreement with the results of Einstein's linearized quadrupole radiation (3-29) and of the EIH approximation (3-41). Sach's paper (1962) has gained more significance with the work of Papapetrou (1969) and Hallidy & Janis (1970) mentioned earlier because the problem of gravitational "tails" in the Bondi representation does not exist if the condition of axial symmetry is removed.

Early criticism of Bondi's approximation method is also presented by Bonnor (1963) and Bonnor & Rotenberg (1966), who make use only of Bondi's metric (3-70) & (R-1). Bonnor (1963) argues that only part of the news function $c(u,\theta)$, the linear part, is known to the observer in Bondi's approximation method. The nonlinear and critical part is indeterminable. If you plug only the linear part of $c(u,\theta)$ for a quadrupole oscillator, as an example, into Bondi's formalism (R-7) the oscillator goes unstable. Bonnor & Rotenberg (1966) eliminate this shortcoming of the Bondi method by using instead Bonnor's (1959) "double series" approximation (3-9). Upon passing to nonlinear approximations, they demonstrate that under forced oscillations

the source loses mass due to quadrupole-quadrupole interaction and experiences radiation recoil under quadrupole-octupole interaction. They also point out that the axial symmetry of the Bondi method produces "tails." Bonnor & Rotenberg conclude, furthermore, with the statement that the existence of radiation from a freely gravitating system is still an outstanding problem (1965).

The error in Bondi's original approximation and the basis of the origin of the "tails" is a naive one. Using the multipole approximation method in the linear vector theory of electromagnetism, one can determine the solutions for a stationary-radiative-stationary transition (Figure 6) by satisfying a finite number of linear conditions. Bondi errs, however, in extending this result to a nonlinear tensor theory of radiation. In the general relativistic tensor problem, Bondi needs an infinite number of nonlinear conditions to make his approximation work for the axially symmetric case.

Nevertheless, the Bondi formalism and its association with the notion of gravitational news or data on a radiative hypersurface has inspired a great deal of research into methods of asymptotic linearization - in particular, the tetrad formalisms of the next chapter.

The High Frequency Approximation

Relevant to any discussion of approximations in General Relativity is the high-frequency approximation of Isaacson (1968) who arrives at an effective stress tensor by analogy with the geometrical optics limit of electromagnetism. In the high-frequency limit the gravitational fields uncouple from their sources and attain an existence all their own. The Isaacson stress-tensor is gauge-invariant, second-rank, symmetric, and (most important) not a pseudotensor (like the Landau-Lifshitz pseudotensor of Appendix C). Treatments using the Isaacson approximation include Price & Thorne (1969, II), Ipser (1971), and Thorne (1968).

The Isaacson effective stress tensor linearizes the radiation zone, placing the burden of the nonlinearities upon the region of the sources - as does the Bondi formalism.

CHAPTER IV

EXACT SOLUTIONS & ASYMPTOTIC APPROXIMATION

Looking for exact solution's of Einstein's nonlinear field equations is very much like taking a peak inside Pandora's box. In fact, one might simply make this statement and leave the whole thing at that. Nevertheless, any treatise on gravitational radiation theory must assess the physical meaning of radiative solutions, in particular those which have been found.

Indeed, it is precisely the physical interpretation of the exact solutions which makes progress in the nonlinear theory of radiation extremely awkward. All of the radiative solutions either contain naked singularities, or for those singularities which can be identified with a source, there always appear singular 2-surfaces which transport energy and/or momentum from infinity into the source, where it can then be radiated outward again. Consequently, the most profound theoretical question about nonlinear gravitational radiation - that of energy transfer - remains unanswered. And until it is answered, the very existence of nonlinear tensor radiation will remain in doubt.

Exact Plane Wave Solutions

Plane wave solutions of Einstein's empty space field equations $G_{\mu\nu} = 0$ or $R_{\mu\nu} = 0$ are investigated by Brinkmann (1925) with the metric

$$ds^2 = du\left\{2dV + (Y^2 - X^2)h(u)du\right\} - \left\{dX^2 + dY^2\right\} \qquad (4\text{-}1)$$

although he does not characterize the solutions as radiation. They are more formally addressed by Rosen (1937) whose metric is derivable from (4-1)

$$ds^2 = 2L\,du\,dv - \left(F^2 dx^2 + G dy^2\right) \qquad (4\text{-}2a)$$

or as stated by Bondi (1957)

$$ds^2 = e^{2\alpha}(d\tau^2 - d\xi^2) - u^2\left\{e^{2\beta}d\eta^2 + e^{-2\beta}d\zeta^2\right\}, \qquad (4\text{-}2b)$$

where $u = \tau - \xi$, $\beta = \beta(u)$, $\alpha = \alpha(u)$, and $\alpha_u = \partial_u \alpha = u\beta_u^2$. Rosen concludes, however that the solutions of (4-2) cannot exist because they possess physical singularities. That this conclusion is too severe is pointed out by Bondi (1957) and Bondi, Pirani, & Robinson (1959) who maintain that Rosen does not distinguish adequately between coordinate singularities and physical singularities. Rosen's metric is empty, and it is flat ($R_{\alpha\beta\gamma\delta}=0$) if

$$u\beta_{uu} + 2\beta_u = u^2\beta_u^3 \quad . \qquad (4\text{-}3)$$

As pointed out by Bondi (1957), the coordinate transformation

$$u = \tau - \xi = ct - x \qquad e^{2\alpha}(\tau + \xi) = ct + x - (y^2+z^2)/u$$
$$y = u\eta e^{\beta} \qquad z = u\zeta e^{-\beta} \qquad (4\text{-}4)$$

is nonsingular for $u>0$ and (4-2b) becomes

$$ds^2 = c^2dt^2 - (dx^2+dy^2+dz^2) - \beta_u^2(c^2t^2-x^2)(cdt-dx)^2 \qquad (4\text{-}5)$$
$$+2\beta_u\left[(ydy-zdz)(cdt-dx) - (y^2-z^2)(cdt-dx)^2/(ct-x)\right]$$

which represents a non-flat region between two flat ones provided $\beta_u \neq 0$ in the last term. This is a "plane-wave"* zone of finite extent whose amplitude is determined by $\beta=\beta(u)$. A more general plane-wave, with a variable plane of polarization is given by Bondi (1957) as

$$ds^2 = e^{2\alpha}(c^2dt^2 - d\xi^2) - (ct-\xi)^2\left[\cosh 2\beta\,(d\eta^2+d\zeta^2) + \right.$$
$$\left.\sinh 2\beta\cos 2\theta\,(d\eta^2-d\zeta^2) - 2\sinh 2\beta\sin 2\theta\, d\eta\, d\zeta\right], \qquad (4\text{-}6)$$

where $\theta=\theta(u)$. The empty-space condition for (4-5) is

$$2\alpha_u = (ct-\xi)\left[\beta_u^2 + \theta_u^2\sinh^2 2\beta\right] . \qquad (4\text{-}7)$$

A discussion of singularities is presented by Bonnor (1957), while Kundt (1961) investigates the general nature of "plane-fronted" gravitational waves.

Recent work by Szekeres (1970, 1972) and Penrose & Kahn (1971) has made significant progress in the study of exact solutions for colliding plane

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*They are not really "plane" because the departure from flatness $R_{\alpha\beta\gamma\delta}$ depends upon y and z, for propagation along x.

waves (essentially a sort of "graviton-graviton" interaction). The nonlinearities are taken fully into account, which is of the utmost importance, because the very notion of linear superposition (Maxwell's theory) is at stake. Indeed, as we have seen in Figure 8, the Bondi-Sachs asymptotic representation essentially linearizes the field equations in the radiation zone by virtue of the fact that there is superposition of the news functions ($c = c_1 + c_2$) - which places the burden of the nonlinearity upon the near zone in the region of the sources. Szekeres (1970, 1965) maintains that superposition is simply invalid for colliding "sandwich waves," in conflict with the Bondi-Sachs formalism. Penrose & Kahn employ impulsive δ-function plane-waves, arriving at similar results as Szekeres. In either case singularities exist, which make the physical interpretation difficult to assess; perhaps the singularities disappear for more realistic, curved wave-fronts.

Exact Cylindrical Wave Solutions

Exact cylindrical waves for $R_{\mu\nu} = 0$ are presented by Einstein & Rosen (1937) and Rosen (1937, 1954, 1956, 1958), based upon the static axially-symmetric metric of Weyl (1918) and Levi-Civita (1919),

$$ds^2 = e^{2\psi} c^2 dt^2 - e^{-2\psi} \rho^2 d\phi^2 - e^{2\gamma - 2\psi}(d\rho^2 + dz^2) \qquad (4\text{-}8a)$$

which reduces to the Laplace equation (in culindrical polar coordinates)

$$\nabla^2 \psi = \psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz} = 0 \qquad (4\text{-}8b)$$

$$\gamma_z = 2\rho\psi_\rho\psi_z \qquad \gamma_\rho = \rho(\psi_\rho^2 - \psi_z^2) \quad . \qquad (4\text{-}8c)$$

Rosen's cylindrical metric is obtained by merely interchanging the roles of z and t in (4-8), whereby

$$ds^2 = e^{2\gamma - 2\psi}(c^2 dt^2 - d\rho^2) - e^{-2\psi}\rho^2 d\phi^2 - e^{2\psi} dz^2 \quad . \qquad (4\text{-}9a)$$

Substituting in $R_{\mu\nu}=0$, the Rosen metric (4-9a) gives

$$\psi_{\rho\rho} - \frac{1}{\rho}\psi_\rho - \frac{1}{c^2}\psi_{tt} = 0 \qquad (4\text{-}9b)$$

$$\gamma_\rho = \rho\left[\psi_\rho^2 + \frac{1}{c^2}\psi_t^2\right] \qquad (4\text{-}9c)$$

$$\gamma_t = \frac{2}{c}\rho\psi_\rho\psi_t \quad . \qquad (4\text{-}9d)$$

For the Sommerfeld radiation condition, the solutions of (4-9) for $k = \omega\rho/c$ are (Rosen (1954))

$$\left.\begin{aligned}\psi &= A J_0(k)\cos\omega t + N_0(k)\sin\omega t \\ \gamma &= \tfrac{1}{2}A^2 k\Big\{J_0(k)J_{0,k}(k) + N_0(k)N_{0,k}(k) + k\big[J_0^2(k) + J_{0,k}^2(k) \\ &\quad + N_0^2(k) + N_{0,k}^2(k)\big] + \big[J_0(k)J_{0,k}(k) - N_0(k)N_{0,k}(k)\big]\cos\omega t \\ &\quad + \big[J_0(k)N_{0,k}(k) + N_0(k)J_{0,k}(k)\big]\sin 2\omega t\Big\} - \tfrac{3}{4}A^2\omega t .\end{aligned}\right\} \quad (4\text{-}10)$$

The last term in (4-10) is a secular variation of the metric which can be interpreted as a loss of energy due to radiation. Rosen (1956) objects to this interpretation, claiming that there is no transport of energy and momentum because the pseudotensor vanishes. Weber & Wheeler (1957) demonstrate that it is nonzero. Marder (1958, 1961, 1969, 1972) studies exact cylindrical waves also.

Although the singularity along the axis of symmetry can be identified with a source such as a thing rod, which makes it more tenable, this rod is infinite in length. Thus the nice cylindrical solutions which radiate energy outwards require an infinite source. Furthermore, there is reason to believe that energy is transported back around and down the axis of symmetry along the source singularity with zero net transfer of energy.

Exact Spherical Wave Solutions

Birkhoff (1927) proves that any general spherically symmetric solution of the empty space field equations can always be transformed into the static Schwarzschild metric. By virtue of Birkhoff's theorem, no spherically symmetric gravitational radiation can exist, an important result for astrophysical theory. Spherical gravitational waves in the linearized theory have been investigated by Boardman & Bergmann (1959) and Bonnor (1959). As stated by Weber & Wheeler (1957), however, spherical gravitational waves can never be truly spherically symmetric, just as is the case with electromagnetic ones (the fixed point theorem of topology).

Robinson & Trautman (1960, 1961), nevertheless, have discovered a class of exact solutions which correspond to a form of expanding radiation which they classify as spherical gravitational waves.

Other Solutions

Additional solutions can be found in the work of Peres (1959) and Takeno (1956, 1957, 1958). Takeno treats the problem of plane, cylindrical, and spherical waves in the nonsymmetrical unified field theory.

Invariant Formulation Of Gravitational Radiation Theory

As we have seen in some detail, coordinate or gauge transformations can be used to create and destroy apparent gravitational radiation. But that radiation which cannot be transformed away is considered physical. The Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ likewise cannot be transformed away by means of a change of coordinates. As a consequence, theoreticians - enchanted by the prospect of relating the Riemann curvature tensor to the existence of physical radiation - have begun to classify its radiative characteristics.

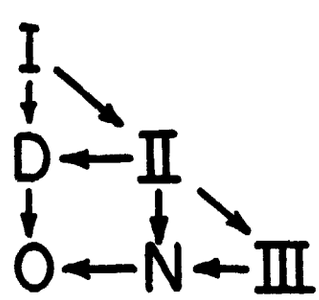
Recalling that the transversality of $\underset{\sim}{E}\cdot\underset{\sim}{H} = 0$ and $|\underset{\sim}{E}| = |\underset{\sim}{H}|$ in the radiation zone of electromagnetism can be expressed in terms of the null vectors k^α (defined by $k^\alpha k_\alpha = 0$) as

$$k^\alpha F_{\alpha\beta} = 0 \qquad\qquad k^\alpha F^*_{\alpha\beta} = 0 \quad ,$$

it is argued that the Riemann tensor possesses similar symmetry properties. Based upon a theorem due to Debever (1958) and Sachs (1961), a similar set of null vectors k^α satisfy the following equations (see Bonnor (1963) for a discussion):

Petrov Metric Type	Equations For Null Vectors	No. Distinct Rays
I	$k_{[\alpha}R_{\beta]ij[\gamma}R_{\delta]}k^i k^j = 0$	4
II	$R_{\alpha\beta\gamma[\delta}k_{\varepsilon]}k^\alpha k^\gamma = 0$	3
D	$R_{\alpha\beta\gamma[\delta}k_{\varepsilon]}k^\alpha k^\gamma = 0$ $R_{\alpha\beta\gamma[\delta}m_{\varepsilon]}m^\alpha m^\gamma = 0$	2
III	$R_{\alpha\beta\gamma[\delta}k^\gamma k_{\varepsilon]} = 0$	2
N	$R_{\alpha\beta\gamma\delta}k^\delta = 0$	1

where the Petrov (1954) types (Appendix U) have been applied by Pirani (1958) as a means of classifying the radiative properties of the Riemann curvature tensor. This can be stated in terms of a Penrose (1960) diagram:



Penrose Diagram

which represents the substructure of the Petrov metrics. There are three spaces (three vertical columns) which may or may not be composed of a subspace. The left-hand column (reading down) is composed of "Petrov-type I nondegenerate," "Petrov-type I degenerate" (D), and flat-space (O). The middle column is "Petrov-type II," and "Petrov-type II null" (N). All the metrics are "algebraically special" except type I (the most general type).

The Riemann tensor has the form

$$R_{\alpha\beta\gamma\delta} = \frac{N_{\alpha\beta\gamma\delta}}{r} + \frac{III_{\alpha\beta\gamma\delta}}{r^2} + \frac{II_{\alpha\beta\gamma\delta}}{r^3} + \frac{I_{\alpha\beta\gamma\delta}}{r^4} + \frac{I'_{\alpha\beta\gamma\delta}}{r^5} + \cdots \quad (4\text{-}11a)$$

or dropping the subscripts

$$R = \frac{N}{r} + \frac{III}{r^2} + \frac{II}{r^3} + \cdots \quad (4\text{-}11b)$$

Hence, Petrov-type N is radiative (it goes as r^{-1}). The same formalism also applies to the Weyl tensor. In terms of the Bondi news function, the Petrov types in Eq (4-11b) are (Trautman (1962))

$$N \sim \frac{\partial^2 c}{\partial u^2}, \quad III \sim \frac{\partial^2}{\partial u \partial \theta} c \sin^2\theta, \quad D \sim 2M + \frac{\partial c^2}{\partial \theta}. \quad (4\text{-}11c)$$

Appendix U contains details on all of the exact solutions discussed earlier and their associated Petrov-Pirani classification.

The Petrov classification of fields using the Riemann tensor, then, is introduced by Pirani (1957) into gravitational radiation theory with the hope of constructing an invariant basis for establishing the existence of gravitational radiation. Referring to Appendix U, one can see that this endeavor is not entirely successful.

Invariant Formulation & Asymptotic Approximations

The embodiment of the Pirani invariant formulation of gravitational radiation using the Petrov classification can also be designated as a tetrad calculus. Witten(1959) and Penrose (1960) likewise develop a two-component spinor formalism. The tetrad and spinor calculus are then combined together by Newman & Penrose (1962) in a formalism with emphasis upon an expansion of the Weyl tensor rather than an expansion of the metric tensor in inverse

powers of r, due to Bondi et al (1962). Hopefully, the Newman-Penrose method will offer a reconciliation of the invariant formulation with the asymptotic approximation of Bondi.

Exact Solution For A Gravitationally Bound System

An exact radiative solution from a system representing a bound self-gravitating state is now available, and is of great importance in that it has physically meaningful singularities associated only with the material sources and constitutes the first exact solution of any kind for a system in motion. It is the work of Bonnor & Swaminarayan (1964), Swaminarayan (1966), and Bičák (1968, 1971). It is an analogue of the classical Born (1909) problem of the uniform acceleration of two charges (see also Rohrlich (1965), 118).

The solution is "Petrov-type I, nondegenerate." It comes from the metric (Bičák, 1971)

$$ds^2 = -e^{\lambda} d\rho^2 - \rho^2 e^{-\mu} d\phi^2 + (z^2-t^2)^{-1}\Big\{(z^2 e^{\mu} - t^2 e^{\lambda})dt^2 - (z^2 e^{\lambda} - t^2 e^{\mu})dz^2 + 2zt(e^{\lambda}-e^{\mu})\,dz\,dt\Big\} \tag{4-12}$$

where

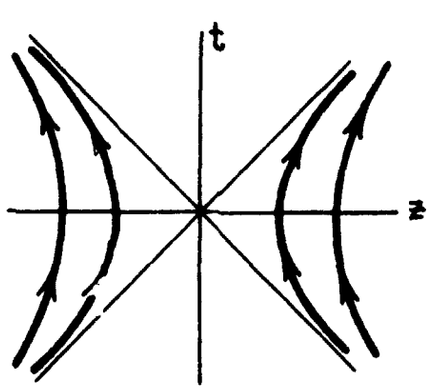
$$\mu = -\frac{2a_1}{R_1} - \frac{2a_2}{R_2} + \frac{2a_1}{h_1} + \frac{2a_2}{h_2} + \ln k$$

$$\lambda = \frac{a_1 a_2}{(h_1-h_2)^2} f - \rho^2(z^2-t^2)\left[\frac{a_1^2}{R_1^4} + \frac{a_2^2}{R_2^4}\right] + \frac{2a_1 R}{h_1 R_1} + \frac{2a_2 R}{h_2 R_2} + \ln k$$

$$R_i = \sqrt{(R-h_i)^2 + 2\rho^2 h_i}\,, \quad (i=1,2) \qquad R = \tfrac{1}{2}(\rho^2 - z^2 - t^2)$$

$$f = \frac{4}{R_1 R_2}\Big\{\rho^2(z^2-t^2) + (R-\rho^2-h_1)(R-\rho^2-h_2) - R_1 R_2\Big\}$$

and where a_1, a_2, $h_1>0$, $h_2>0$, and $k>0$ are arbitrary constants. The solution describes two pairs of mass points which are represented by the world lines



$$\left.\begin{array}{l} \rho = 0 \\ z_i = \pm\sqrt{t^2+2h_i} \end{array}\right\} \tag{4-13}$$

and are thus uniformaly accelerated by $\pm\sqrt{(2h_i)}$ in a Minkowski background. There are four possible cases for particle mass:

$$a_1 = \frac{(h_1-h_2)^2}{2h_2} \qquad a_2 = -\frac{(h_1-h_2)^2}{2h_1} \qquad k = 1 \tag{4-14a}$$

$$a_1 = \frac{(h_1-h_2)^2}{2h_1} \qquad a_2 > 0 \qquad k = 1 \tag{4-14b}$$

$$a_1 \text{ arbitrary} \qquad a_2 = -\frac{(h_1-h_2)^2}{2h_1} \qquad k = \exp\left[\frac{(h_1-h_2)^2}{h_1 h_2} - \frac{2a_1}{h_1}\right] \tag{4-14c}$$

$$\left.\begin{array}{ll} a_1 > 0 & a_2 = 0 \\ a_1 = 0 & a_2 > 0 \end{array}\right\} \tag{4-14d}$$

Eq (4-14a) corresponds to two mass dipoles (positive and negative mass); Eq (4-14b) corresponds to four masses of positive mass with stresses; Eq (4-14c) corresponds to a pair with negative mass moving under stress, while the other two particles move freely; and Eq (4-14d) represents only two particles with the same positive mass, connected by stress.

Transforming to spherical coordinates, the line element (4-12) becomes very involved, but Bičák gets for the Bondi news function

$$C = \frac{1}{2}\left[\frac{1 - e^{\beta} - \alpha_u \sin^2\theta}{e^{\beta}\sin^2\theta}\right] \tag{4-15a}$$

where u = t - r and

$$\alpha = -\frac{2a_1}{U_1} - \frac{2a_2}{U_2} \qquad (\alpha_u = \partial_u \alpha) \tag{4-15b}$$

$$U_i = \sqrt{u^2 + 2h_i \sin^2\theta} \tag{4-15c}$$

$$\beta = \frac{4a_1a_2}{(h_1-h_2)^2}\left\{\frac{u^2+(h_1+h_2)\sin^2\theta}{U_1U_2} - 1\right\} + \sin^4\theta\left\{\frac{a_1^2}{U_1^4} + \frac{a_2^2}{U_2^4}\right\} - \left\{\frac{2a_1u}{h_1U_1} + \frac{2a_2u}{h_2U_2}\right\}. \tag{4-15d}$$

The Bondi news function (4-15) begins with an asymptotic expansion containing r^{-1}, irrespective of the choice of a_i, h_i, k (that is, for all cases 4-14). This is to say, the asymptotic expansion of the Riemann tensor tetrad is radiative according to Bondi's method. Bičák (1971) demonstrates that the radiation pattern, furthermore, for the Born solution of electrodynamics is

$$(dP/d\Omega)_{Born} = \frac{e^6}{4\pi}\,\frac{9\cos^4\theta\,\sin^2\theta}{(2h)^6}\,\frac{1}{r^2} \tag{4-16a}$$

while that for the Bonnor-Swaminarayan solution of gravitational radiation is

$$(dP/d\Omega)_{Grav} = (dP/d\Omega)_{Born}\tan^2\theta\ . \tag{4-16b}$$

Bičák, in addition, points out that the particle masses in the Bonnor-Swaminarayan solution are the same as those occurring in Bondi's (1957) paper on negative mass in general relativity. There is strong evidence that the particles in the Bonnor-Swaminarayan solution are gravitational monopoles.

CHAPTER V

GRAVITATIONAL RADIATION IN ASTROPHYSICS

Let us turn now to astrophysics. A wealth of astrophysical applications exists for the linearized solutions of the tensor theory of gravitational radiation - even though the exact, radiative solutions which are known have been difficult to interpret. Such applications must be astrophysical in nature simply because the order of magnitude of the radiation is so small ($G/c^5 = 2.7\times10^{-60}$). For quadrupole radiation alone,

$$-\frac{d\mathcal{E}}{dt} = 6.1\times10^{-62}\,\dddot{Q}_{\alpha\beta}^{\,2}\ \text{erg/sec}\,. \qquad (3\text{-}29)$$

Until exact solutions which are physically meaningful are established, it is nevertheless informative to adopt the linearized theory, to forget any controversy which may exist in regard to radiation from the Kepler problem, and to consider the observable phenomena which might result.

Gravitational Radiation From A Spinning Jacobi Ellipsoid

Consider a spinning Jacobi ellipsoid (triaxial inertias I_{11}, I_{22}, I_{33} and semi-axes a, b, c) whose body axes are chosen to coincide with the principal axes and whose angular velocity Ω is along the X^3-axis. The Einstein-Eddington quadrupole radiation formula (3-30) becomes in this case

$$-\frac{d\mathcal{E}}{dt} = \frac{32}{5}\frac{G}{c^5}I^2e^2\Omega^6 \qquad (5\text{-}1a)$$

$$-\frac{d\mathcal{E}}{dt} = \frac{32}{5}\frac{G}{c^5}(I_{11}-I_{22})^2\Omega^6 \qquad (5\text{-}1b)$$

where $I = I_{11}+I_{22}$ and $e = (I_{11}-I_{22})/I = (a-b)/\sqrt{(ab)}$ is the equatorial ellipticity. For an ellipticity of unity ($e=1$, $I_{22}=0$) this formula reduces to the original quadrupole radiation (3-30) for a spinning rod. On the other hand, if $I_{11}=I_{22}$ the Jacobi ellipsoid degenerates into the Maclaurin spheroid and

ceases to generate gravitational radiation because it has zero ellipticity (e=0 in 5-1).

The radiation reaction torque acting on the ellipsoid due to gravitational radiation would appear to contribute a loss in angular momentum L by an amount

$$\frac{dL}{dt} = \frac{1}{\Omega}\frac{d\mathcal{E}}{dt} = -\frac{32}{5}\frac{G}{c^5}\left(I_{11}-I_{22}\right)^2\Omega^5 . \qquad (5\text{-}2a)$$

For the triaxial ellipsoid, this is (Chandrasekhar, 1970)

$$\frac{d}{dt}\left[(a^2+c^2)\Omega\right] = -\frac{32}{5}\frac{GM}{c^5}(a^2-b^2)^2\Omega^5 . \qquad (5\text{-}2b)$$

Naively treated, Eq (5-2) should result in a spinning down of the ellipsoid. Consequently, the exponent n of Ω^n in $-d\mathcal{E}/dt$ is known as the "slowing exponent" and has been used to indicate the presence of multipole radiation mechanisms,

$$P = \left[\frac{2}{3c^3}\ddot{\underset{\sim}{D}}^2 + \frac{2}{3c^3}\ddot{\underset{\sim}{M}}^2 + \frac{1}{180c^5}\dddot{Q}_{\alpha\beta}^2\right]_{EM} + \left[\frac{G}{45c^5}\dddot{Q}_{\alpha\beta}^2\right]_{GRAV} \qquad (5\text{-}2c)$$

The Crab pulsar has been observed to have (e.g. Ruffini & Wheeler, 1969)

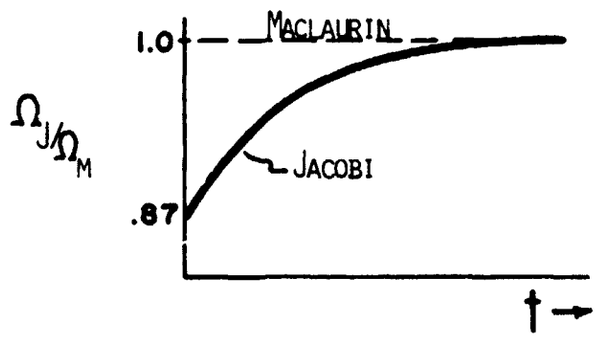
$$n = 5.76 \pm 0.65 \quad \text{(Optical)}$$
$$n = 3.6 \pm 0.6 \quad \text{(Radio)}$$

which would imply the existence of a quadrupole radiation mechanism (n=6) such as (5-1). Assuming a mass of 0.786 $M_\odot$, a mean radius of 9.75km, and eccentricity of 8×10^{-4}, and a period of 33 msec, then a "neutron" star spinning as a triaxial ellipsoid in (5-1) can account for the observed rate of change of the Crab pulsar's period of 4×10^{-13}sec/sec. The power radiated would be

$$-d\mathcal{E}/dt = 2 \times 10^{38} \text{ erg/sec.}$$

However, energy dissipation and angular velocity cannot be related in this fashion. Chandrasekhar (1969, 1970) has pursued the evolution of the Jacobi ellipsoid under gravitational radiation in the context of the "post$^{2.5}$-Newtonian" (EIH) approximation. He has arrived at the surprising result that a triaxial ellipsoid <u>increases</u> in angular velocity as it loses angular momentum and radiates energy (5-1). It asymptotically approaches the Maclaurin spheroid, whereupon it ceases to radiate, at the point of bifurcation (see Figure 9). The Maclaurin spheroid is then dynamically unstable under

gravitational radiation reaction, exhibiting the possibility of fragmentation.



$$(1 - B/A) = K_1 E^{-T/\tau}$$

$$(\Omega_M - \Omega_J) = K_2 E^{-2T/\tau}$$

FIGURE 9

The dissipation of energy, then, under gravitational radiation can be derived from potential and internal energy, and need not be at the expense of rotational kinetic energy. Chandrasekhar's results have significance for gravitational collapse and the formation of "black holes"* in astrophysics, because asymmetries during collapse should be radiated away as gravitational radiation if his approximations are applicable. The collapsing object should evolve into a spheroidal (Maclaurin), nonradiating "black hole" - which is secularly and dynamically unstable. It could fragment and even bifurcate under this EIH analysis, in contrast to the results of Penrose (1972) and Hawking (1971).

Figure 9 also explains the "glitches" or sharp drops in the rotational periods of pulsars. Starquakes briefly create a Jacobi ellipsoid which quickly becomes Maclaurin.

Gravitational Radiation From The Kepler Problem

Another system of astrophysical interest is the two-body Kepler problem which has already been discussed at some length in the EIH approximation. It is particularly interesting because it is a case of a self-gravitating system not undergoing forced oscillations but yet experiencing a time-varying quadrupole moment.

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*"Black hole" is the acronym attributed to the geometry in (3-53) when the radius of the mass M is less than the Schwarzschild radius $2GM/c^2$.

Recalling for a circular orbit that $\Omega^2=G(m_1+m_2)/r^3$ and that the ellipticity is unity (e=1), then (Landau & Lifshitz, 1962) Eq (5-1) becomes

$$-\frac{d\mathcal{E}}{dt} = \frac{32}{5}\frac{G^4}{c^5}\frac{(m_1m_2)^2(m_1+m_2)}{r^5} \quad . \qquad (5\text{-}3)$$

This formula should represent the grav ional mass quadrupole radiation from a stellar binary, provided such radiation exists and boundary conditions allow for it. Noting that the total energy of a Keplerian conic is

$$\mathcal{E} = -\frac{Gm_1m_2}{2r} \quad \Rightarrow \quad \dot{r} = \left(\frac{2r^3}{Gm_1m_2}\right)\frac{d\mathcal{E}}{dt} \quad , \qquad (5\text{-}4)$$

then substitution of (5-3) into $\dot{r}$ of (5-4) gives

$$\dot{r} = -\frac{64}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1+m_2)}{r^3} \quad , \qquad (5\text{-}5)$$

which represents the rate of decay of the binary - but only approximately because the circular orbit assumption must be treated adiabatically as the conic starts to collapse. Eq (5-5) is often used by astrophysicists when considering the lifetimes or relaxation times of binary systems undergoing gravitational radiation. Such arguments, for example, are used to dispel binaries as possible pulsar mechanisms because they collapse too rapidly. This interpretation of gravitational radiation reaction (binary collapse) effectively ignores or considers invalid the results of Hu* (1947), Peres (1959), Havas & Goldberg (1962), and Havas & Smith (1965) who all find a dilatation of the binary system under gravitational radiation - which is consistent with the behaviour of observed pulsar periods. Thinking of the binary as an ellipsoid, however, it should increase in angular velocity (collapse) under Chandrasekhar's interpretation of the EIH method.

The first adequate treatment of the general Kepler problem subjected to gravitational radiation is due to Peters & Mathews (1963) and Peters (1964), who study the case of eccentric orbits using Mathews' (1962) tensor harmonics.

---

*Hu also argues that such a behaviour of gravitational radiation reaction should account for the cosmological expansion of the Universe.

Instead of Eq (5-3), they obtain

$$-\frac{d\mathcal{E}}{dt} = \frac{32}{5}\frac{G^4}{c^5}\frac{(m_1 m_2)^2(m_1+m_2)}{a^5} f(\epsilon) \qquad (5\text{-}6a)$$

where

$$f(\epsilon) = \frac{1 + \left(\frac{73}{24}\right)\epsilon^2 + \left(\frac{37}{96}\right)\epsilon^4}{(1-\epsilon^2)^{7/2}} \quad , \qquad (5\text{-}6b)$$

while "ϵ" is the eccentricity and "a" represents the semi-major axis or apse. If we let

$$P_o = \frac{1}{\pi}\frac{G^4}{c^5}\frac{(m_1 m_2)^2(m_1+m_2)}{a^5} \qquad (5\text{-}6c)$$

then (5-6a) is simply

$$-\frac{d\mathcal{E}}{dt} = \frac{32\pi}{5} P_o f(\epsilon) \quad . \qquad (5\text{-}6d)$$

But recalling our circular orbit assumption, where the enhancement factor $f(\epsilon=0)=1$, P_o in (5-6c) is actually

$$P_o = \frac{1}{\pi}\frac{G}{c^5} I^2 \Omega^6 \qquad (5\text{-}7a)$$

which is equivalent to (5-1) (and 3-30 for e=1):

$$-\frac{d\mathcal{E}}{dt} = \frac{32\pi}{5} P_o e^2 \, . \qquad (5\text{-}7b)$$

In terms of (5-7a), the Einstein-Eddington-Park power formulae (3-30) and (3-31) that we obtained earlier can now be abbreviated as

$$\frac{d}{d\Omega} P(\theta,\phi) = P_o\left[4\cos^2\theta + \sin^4\theta \sin^2 2\phi\right] \qquad (5\text{-}8a)$$

$$\frac{d}{\sin\theta\, d\theta} P(\theta) = 4P_o\left[\cos^2\theta + \frac{1}{8}\sin^4\theta\right] = \frac{1}{2}P_o\left[1 + 6\cos^2\theta + \cos^4\theta\right] . \qquad (5\text{-}8b)$$

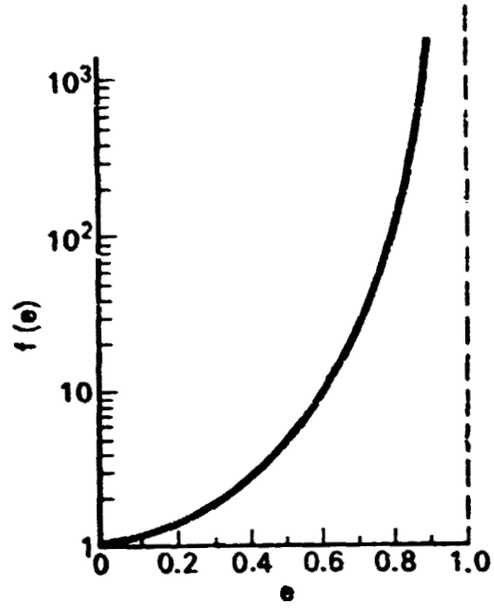


FIGURE 10a

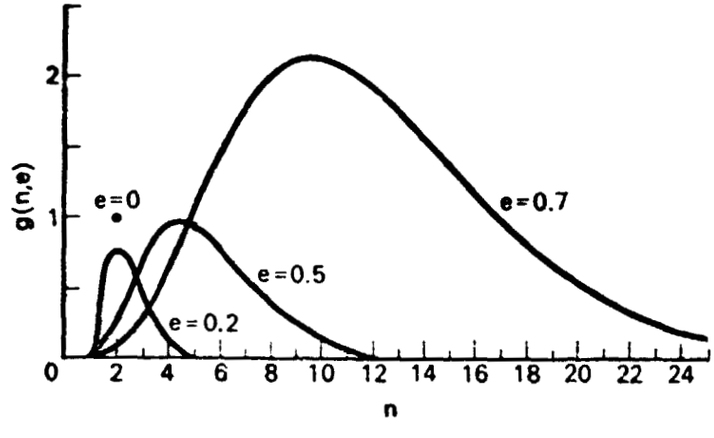


FIGURE 10b

The results of Peters & Mathews (1963) reduce identically to these angular power distributions in the case of gravitational radiation from a circular orbit. The circular Kepler orbit, then, like the spinning rod of Einstein, is represented by Figure 1.

For the more realistic case of an eccentric orbit, the enhancement factor $f(\varepsilon)$ becomes important, as shown in Figure 10a. The power radiated in the nth harmonic, Figure 10b, is

$$P(n) = \frac{32\pi}{5} P_0 \, g(n,\varepsilon) \qquad (5\text{-}9a)$$

where $g(n,\varepsilon)$ is derived from the Bessel functions $J_n(n\varepsilon)$ as (5-9b)

$$g(n,\varepsilon) = \frac{n^4}{32}\left\{\left[J_{n-2} - 2\varepsilon J_{n-1} + \frac{2}{n}J_n + 2\varepsilon J_{n+1} - J_{n+2}\right]^2 + (1-\varepsilon^2)\left[J_{n-2} - 2J_n + J_{n+2}\right]^2 + \frac{4}{3n^2}J_n^2\right\}.$$

The enhancement factor $f(\varepsilon)$ is merely the sum of all of the harmonic contributions in (5-9b):

$$f(\varepsilon) = \sum_{n=1}^{\infty} g(n,\varepsilon) \quad . \qquad (5\text{-}9c)$$

Figure 10 illustrates the important result that most of the power radiated is emitted at the higher harmonics for greater eccentricities.

Before leaving the question of gravitational radiation from the freely gravitating Kepler problem for two or more bodies, the role of tidal friction must be assessed. The treatments presented so far have been adequate only for point masses (actually, they are even marginal for point masses - if we recall Appendix G). Consequently, the energy dissipation represented by the radiation - dE/dt in (5-3) necessarily exhibits itself as a loss in orbital angular momentum. However, for macroscopic bodies tidal phenomena manifest themselves, providing an energy dissipation mechanism deriving from spin-orbit interaction. The resultant tidal friction has the opposite effect on the binary system as does gravitational radiation reaction (Appendix S). The mechanism of tidal friction, furthermore, cannot be differentiated from variations in the velocity of propagation of gravitational radiation, as we have already stated (Appendices G & S).

Gravitational radiation and tidal friction remain in equilibrium until the spin angular momentum of the primary is extinguished. Acting as a primary

in a binary system, a "black hole" can play host to another companion which will radiate away its angular momentum as gravitational radiation, and the system will not collapse. Local nonlinear effects can vary the velocity of propagation and cause the same result.

Gravitational Bremsstrahlung & Synchrotron Radiation

A natural extension of the bound ($\varepsilon<1$, $\mathcal{E}<0$) two-body Kepler problem just treated is the hyperbolic scattering of a small mass m as it passes in the vicinity of a large mass M such as a "black hole, at an impact parameter b (Figure 11a). On the basis of an electromagnetic analogy, one expects from the linearized tensor theory of gravitation that the small mass m should emit a "burst" or "splash" of multipole radiation as it experiences a transverse acceleration due to M.

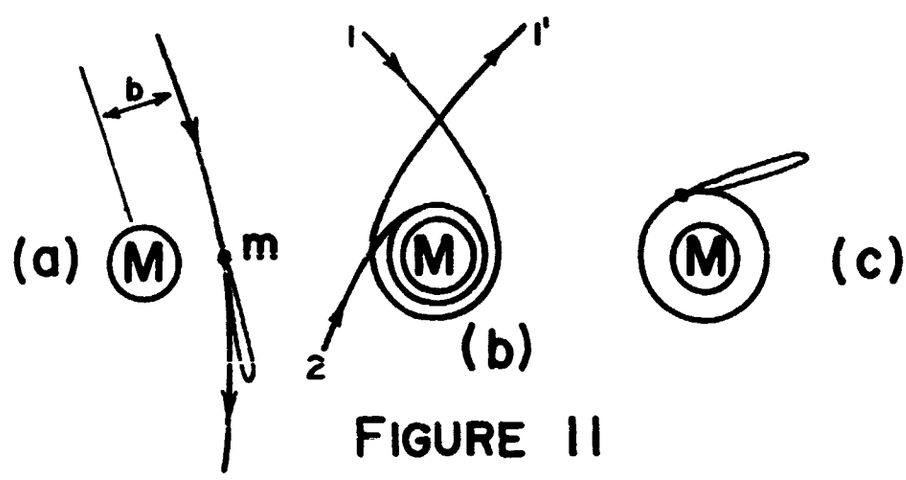


FIGURE 11

Such an analysis is presented by Peters (1970) as a generalization of the non-relativistic bound orbits treated by Peters & Mathews (1963). Peters removes the "slow-motion" assumption employed in the derivation of the quadrupole radiation (3-29), (5-1), and (5-6), and he considers the relativistic case of high velocity - assuming that the impact parameter is large enough ($b>>M$), which keeps the scattering looking like the classical one of Figure 11a rather than the general relativistic behaviour at small impact parameters ($3M\leq b\leq 6M$) such as depicted by Figure 11b. (The orbital theory of Figure 11b is due to Forsyth (1920), Morton (1921), and Darwin (1959) who qualify the conditions for capture, stability, and scatter.*)

*See also de Felice (1968) for a gravitational Störmer-type representation of general relativistic orbit theory. Also read Ruffini & Wheeler (1969, 1971).

Peters develops his own representation of relativistic radiation at high velocities, different from the "fast-motion" approximation which has already been discussed. In fact, his paper (1970) presents an excellent critique of the "fast-motion" method used by Smith & Havas (1965), who treat the Kepler problem by improperly neglecting the stresses in the system.

In the "slow-motion weak-field" limit, Peters shows that the energy radiated in Figure 11a is ($\beta \ll 1$ or $v \ll c$)

$$\Delta E = \frac{37\pi}{15}\frac{G^3}{c^4}M^2 m^2 \frac{\beta}{b^3} \quad , \; v \ll c \qquad (5\text{-}10)$$

which can be derived from the Einstein quadrupole radiation formula (3-29) by calculating the quadrupole tensor Q_{ij} of the mass m from its Schwarzschild geodesic in Figure 11a. The same result is obtained by Ruffini & Wheeler (1969), being discussed further by Ruffini (1973). However, for the relativistic case Peters' approximation gives

$$P \sim \frac{G^3 M^2 m^2}{b^4 c^3 (1-\beta^2)^2} \qquad (5\text{-}11a)$$

$$\Delta E \sim \frac{G^3 M^2 m^2}{b^3 c^4 (1-\beta^2)^{3/2}} \qquad (5\text{-}11b)$$

for the radiated power P and energy ΔE, where the precise coefficients of Eqs (5-11) must be determined by numerical integration. The results are given in Figure 12, evaluated in the equatorial plane ($\theta = \pi/2$ on the angular distribution of the bound conic in Figure 1) of the hyperbolic orbit. The forward beaming or bremsstrahlung is manifest in Peters' analysis.

In light of Peters' relativistic gravitational bremsstrahlung, the next reasonable question to consider is that of the highly relativistic (high velocity) bound conics in close circular orbits about a massive object M, such as a Schwarzschild "black hole." The tighter the circular orbit, the higher its angular velocity (neglecting the Roche limit) and consequently the greater its chances of behaving like a rotating "searchlight," radiating out Peters' bremsstrahlung (Figure 11c). Such a mechanism constitutes gravitational synchrotron radiation.

Circular Darwin (1959) orbits are allowed for radii $r>3M$, although they are unstable in the region $3M<r<4M$. Consequently, Misner et al (1972) have

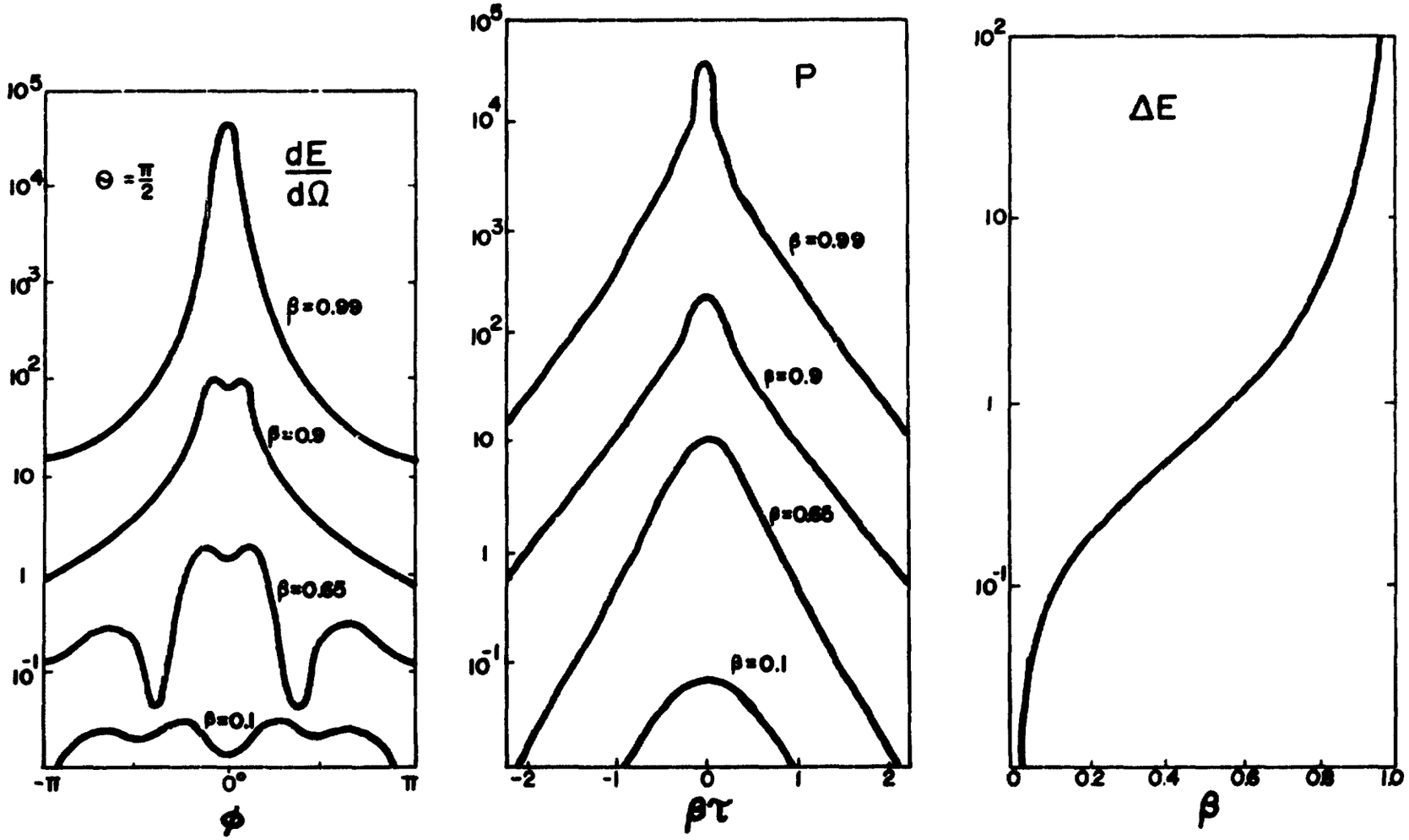

$\Theta = \frac{\pi}{2}$
$\frac{dE}{d\Omega}$
β=0.99
β=0.9
β=0.65
β=0.1
10^5
10^4
10^3
10^2
10
1
10^{-1}
-π
0°
π
φ
P
β=0.99
β=0.9
β=0.65
β=0.1
-2
-1
0
1
2
βτ
ΔE
10^2
10
1
10^{-1}
0
0.2
0.4
0.6
0.8
1.0
β

FIGURE 12

made the suggestion that gravitational synchrotron radiation should be emitted by such orbits, although they are unstable and rapidly plunge into the "black hole" under radiative perturbations. Nevertheless, Misner and associates present an analysis of scalar* gravitational synchrotron radiation based upon the Regge-Wheeler-Zerilli formalism, using standard JWKB methods. They obtain the result that particles in high relativistic orbits which are coupled to a scalar gravitational field radiate strongly into narrow synchrotron beams at high harmonics of the orbital frequency.

Because of the importance of the synchrotron mechanism in the experimental verification (the energy of the source can be much less) of gravitational radiation, Misner's work on the scalar theory has prompted Davis, Ruffini, Tiomno, & Zerilli (1972) and Breuer, et al (1973) to investigate the general problem of scalar (Spin-0), vector (Spin-1), and tensor (Spin-2) synchrotron radiation from $r = (3+\delta)M$ circular, unstable orbits. If $\omega = m\omega_0$ is the frequency of the radiation and ω_0 represents the frequency of the orbit $\omega_0 = \sqrt{(M/r_0^3)}$, then the power emitted is given in all three cases (scalar, vector, tensor) by

$$P(\omega) = \sum_{l,m} \frac{\omega^2}{2\pi}\left[\left|R^{lm}_{(M)}\right|^2 + \left|R^{lm}_{(E)}\right|^2\right] \qquad (5\text{-}12)$$

in the Regge-Wheeler formalism. The functions $R^{lm}_{(M)}$ and $R^{lm}_{(E)}$ are of magnetic (odd) and electric (even) parity respectively, being defined explicitly in Appendix R. They are computed in the asymptotic region ($r \to +\infty$) from a Green's function technique using the solutions $u(r_*)$ and $v(r_*)$ of the "Schrodinger" equations (3-66) and (3-67) without sources:

$$\left.\begin{aligned} \frac{d^2u}{dr_*^2} + (\omega^2 - V_{eff})u &= 0 \\ \frac{d^2v}{dr_*^2} + (\omega^2 - V_{eff})v &= 0 \end{aligned}\right\} . \qquad (5\text{-}13)$$

u must be outgoing at $r_* = +\infty$ and v must be ingoing at $r_* = -\infty$ (into the "black hole" at r=2M). The structure of V_{eff} depends upon the field (scalar, vector, tensor).

The results of Davis, et al (1972) are given in Figure 13, which represents the power radiated at a circular orbit r=3.05M (δ=.05). The scalar spectrum corroborates that of Misner et al (1972), determined by the JWKB method, but the vector and tensor power spectra do not at all behave as do conventional

*Scalar radiation occurs in the Brans-Dicke (1961) theory. See also Morganstern & Chiu (1967).

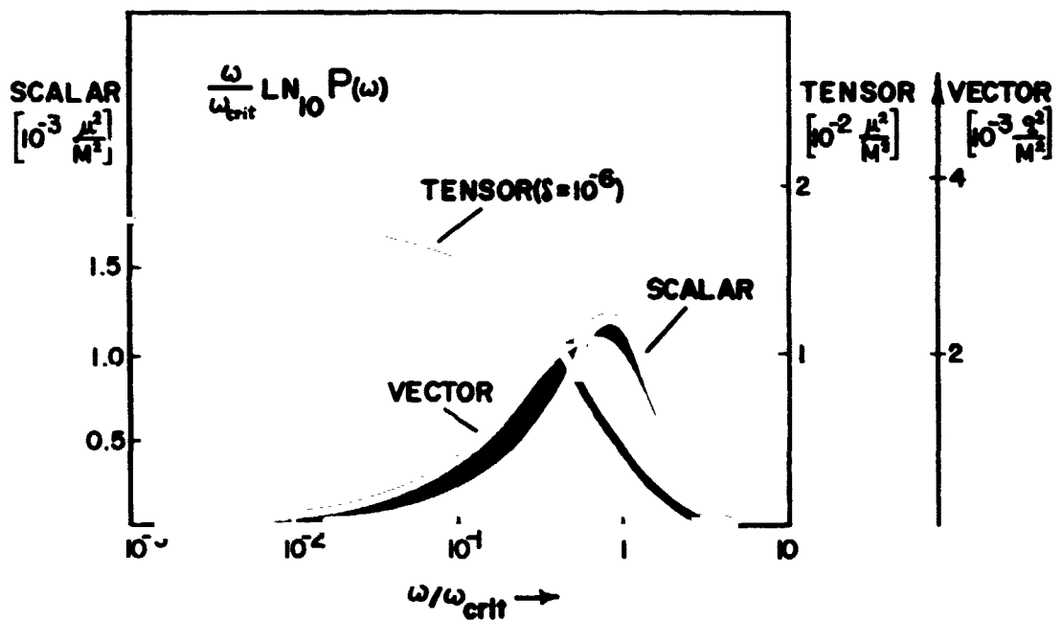

SCALAR
$[10^{-3}\ \frac{\mu^2}{M^2}]$
$\frac{\omega}{\omega_{crit}} LN_{10} P(\omega)$
TENSOR
$[10^{-2}\ \frac{\mu^2}{M^2}]$
VECTOR
$[10^{-3}\ \frac{\mu^2}{M^2}]$
TENSOR($\delta = 10^{-6}$)
SCALAR
VECTOR
1.5
1.0
0.5
2
1
4
2
10^{-2}
10^{-1}
1
10
ω/ω_{crit} →

FIGURE 13

synchrotron mechanisms. Synchrotron radiation is enhanced at high multipoles, which accounts for the beaming effect. From Figure 13 the vector and tensor contributions are significant at lower multipoles with an associated radiation at wider beam angles. As a consequence, vector (electromagnetic or gravitational) and gravitational tensor radiation do not concentrate energy very effectively into the orbital plane of the source.

The critical frequency ω_{crit} is defined as

$$\omega_{crit} = \omega_0 m_{crit} = \frac{4M}{\pi(r_0-3M)} = \frac{4}{\pi\delta} \quad . \qquad (5\text{-}14)$$

The power spectrum $P(\omega)$ in (5-12) is found to vary as

$$P(\omega) \sim \omega^{1-s} \exp\left[-2\,\omega/\omega_{crit}\right] \qquad (5\text{-}15)$$

for all three spins: s=0 (scalar), s=1 (vector), and s=2 (tensor).

Particle Falling Into A Schwarzschild Black Hole

Also of wide astrophysical interest has been Zerilli's (1969, 1970) study of the radiative behaviour of a particle falling radially into a Schwarzschild "black hole," deriving from his spectral decomposition of the Regge-Wheeler and Peters formalisms into tensor harmonics (Chapter 3). This particular problem is very much like Peters' relativistic gravitational bremsstrahlung of Figure 11a, except that Zerilli is treating a geodesic which is captured (impact parameter $b<3M$) - more like trajectory 2 of Figure 11b. In particular, Zerilli treats a radial, unperturbed Schwarzschild geodesic (Eq 3-68, along the z-axis) to characterize his particle (which neglects and avoids the critical issue of radiation reaction). This is the case of zero impact parameter (b=0 in Figure 11a).

Zerilli's equations (the electric, even parity ones in 3-67) have been numerically integrated by several authors. Davis & Ruffini (1971) and Davis, Ruffini, Press, & Price (1971) investigate the asymptotic behaviour of the outgoing burst of gravitational radiation due to a particle falling from rest at infinity - as suggested by Zerilli. The results are given in Figure 14 for the effective potential, the energy flux, and the tidal stresses of the Riemann curvature tensor (these are of interest to experimentalists). Forward

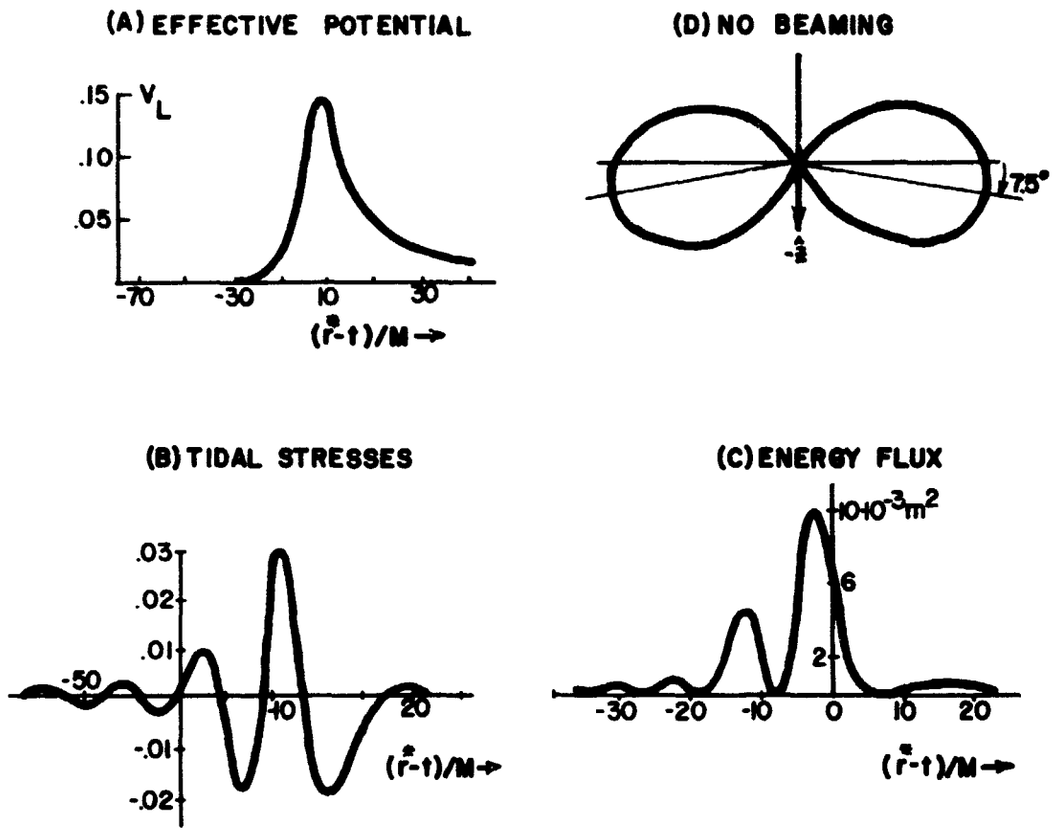

(A) EFFECTIVE POTENTIAL
.15
.10
.05
V_L
-70
-30
10
30
$(\tilde{r}-t)/M \rightarrow$
(D) NO BEAMING
7.5°
$-\hat{z}$
(B) TIDAL STRESSES
.03
.02
.01
-.01
-.02
-50
10
20
$(\tilde{r}-t)/M \rightarrow$
(C) ENERGY FLUX
$10 \times 10^{-3} m^2$
6
2
-30
-20
-10
0
10
20
$(\tilde{r}-t)/M \rightarrow$

FIGURE 14

bremsstrahlung is not manifest (Figure 14d) as it is in Peters' analysis of larger (b>>M) impact parameters (Figure 12).* Davis, et al find an outgoing radiated energy of

$$E_{out} \sim 0.0104\left(\frac{m}{M}\right)mc^2 \qquad (5\text{-}16a)$$

where mc^2 is the particle's rest energy and M is the mass of the "black hole." The energy radiated into the "black hole" proves to have contributions at all higher multipoles L (hence divergent), but assuming a cutoff $L_{max}=\pi/2\ (M/m)$ the total ingoing radiation is finite:

$$E_{in} \sim \left(\frac{\pi}{8}\right)m \qquad (5\text{-}16b)$$

which is independent of the mass of the "black hole" (?). $(E_{in}+E_{out})$ in (5-16) is still a small percentage of the particle's rest energy mc^2, however which is an important result due to the Schwarzschild geodesic assumption. This is to say, neglecting radiation reaction (by assuming the radiated energy is small enough for a sufficiently small mass m) results in a small loss of energy - if you can rationalize away the ingoing divergence. The spectrum of gravitational radiation which Davis, et al obtain is given in Figure 15a for the quadrupole (L=2) contribution.

Ruffini (1973), however, has addressed the more practical case of Zerilli's problem wherein the particle is assumed to possess a nonzero kinetic energy at infinity. Solving Zerilli's Equation (3-67) for a Fourier-transformed source $S^{(e)}_{LM}$ given by

$$S_L = -\frac{8\pi(r-2M)}{r}\,\frac{\lambda r+3M}{\omega}\,\frac{d}{dr}\left[\frac{r(r-2M)}{(\lambda r+3M)^2}\,\frac{A^{(1)}_L(\omega,r)}{\sqrt{2}}\right]+\frac{r(r-2M)}{\lambda r+3M}A_L(\omega,r) \qquad (5\text{-}17a)$$

where

$$A_L(\omega,r)=\left(\frac{m}{2\pi}\right)\left[L+\frac{1}{2}\left(\gamma^2-1+\frac{2m}{r}\right)\right]^{1/2}\frac{e^{i\omega T(r)}}{(r-2m)^2} \qquad (5\text{-}17b)$$

$$A^{(1)}_L(\omega,r)=-i\left(\frac{m}{2\pi}\right)\gamma\sqrt{2L+1}\,\frac{e^{i\omega T(r)}}{r(r-2m)} \qquad (5\text{-}17c)$$

$$\gamma = \left(1-\frac{2m}{r}\right)dt/d\tau \quad , \qquad (5\text{-}17d)$$

Ruffini demonstrates that the spectrum does not vanish at low frequencies (as it does in Figure 15a for γ=1). His results (Figure 15b-d) illustrate,

*This result appears contradictory if one allows Peters' scattering problem of Figure 11a to evolve into trajectory 2 of Figure 11b (capture, r<3M) and finally lets b=0. But for b=0 there is no curvature of the trajectory.

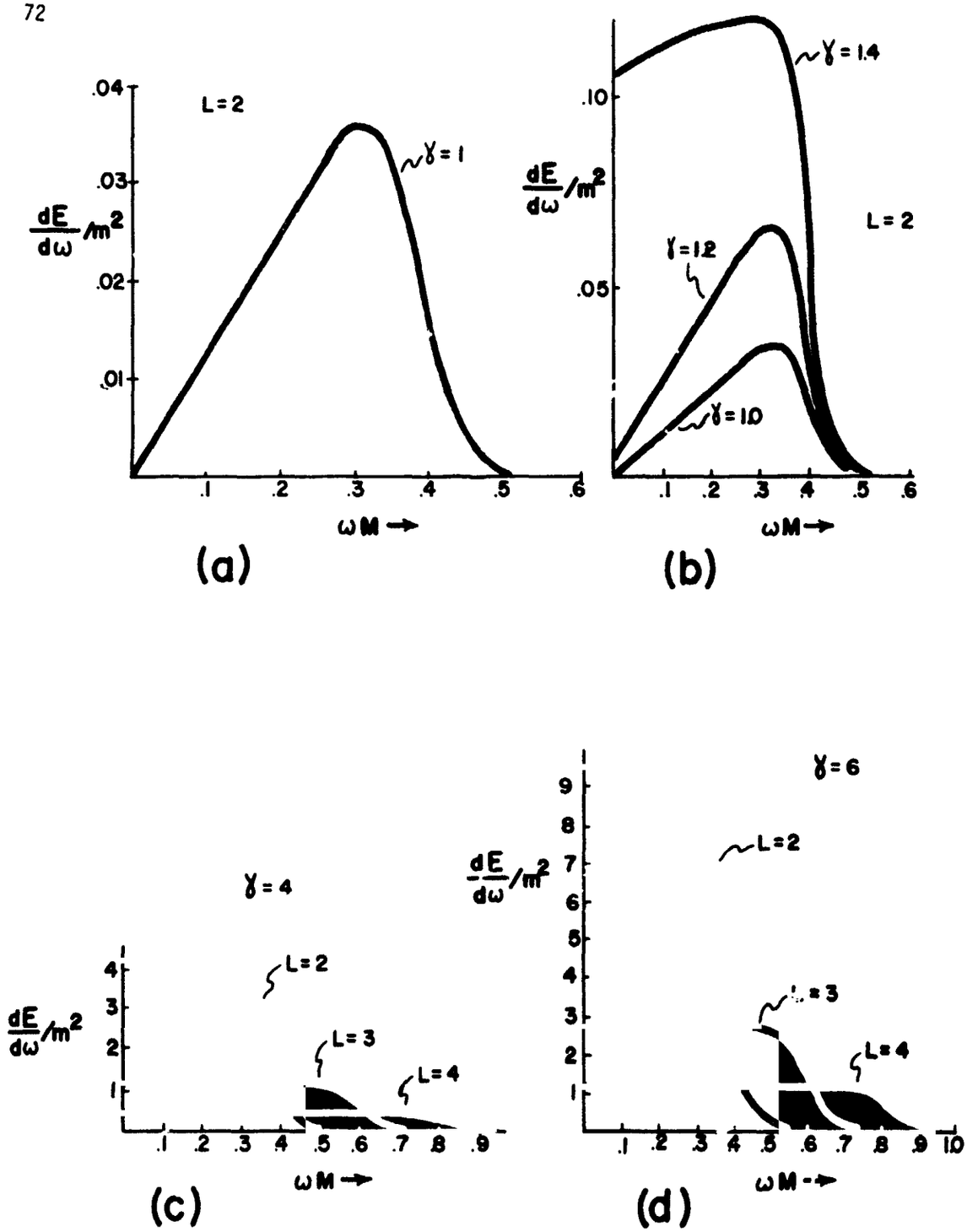


FIGURE 15

in fact, that large amounts of energy are emitted there. In Figure 15b, his comparison of quadrupole (L=2) contributions for varying γ demonstrates the dramatic effect of assuming that the particle has a velocity at infinity ($\gamma>1$). Furthermore, as shown in Figure 15c-d, the gravitational tensor radiation emitted outward is flat at the lower harmonics for quadrupole (L=2) and higher multipoles. This further substantiates the earlier discussion (Figure 13) of the inefficiency of tensor radiation as a synchrotron mechanism. There is no bremsstrahlung in the sense of Figure 12.

Umklapp & Gravitational Dipole Radiation

The multipole expansions considered earlier contain dipole contributions which are usually argued away, based upon the Newtonian principle of equivalence that inertial and gravitational mass are the same. This means that if the gravitational-to-inertial mass ratio is identical for all components of a massive system, gravitational dipole radiation vanishes* in the center of momentum system - due to conservation of linear momentum. Although neither the Newtonian nor the Einstein theory of gravitation assumes that mass must be positive-definite, introducing the concept of negative mass does not necessarily change this argument against gravitational dipole radiation (the sign of the gravitational constant G is also important). Nor is the question made simple if we recall Bondi's multipole representation wherein the mass monopole, dipole, and quadrupole are all non-zero and coupled together, as seen in the Newman-Penrose conserved quantity D^2-mQ in Eq (3-75), or in Eq (R-15).

To certain observers momentum simply need not be conserved, and in such frames gravitational dipole radiation may serve a meaningful purpose. The quickest example of such a process is found in solid state physics where thermal resistance cannot be accounted for by the normal convention of momentum conservation. The notion of "umklapp," due to Peierls (1929, 1955) must be invoked.

Conventional conservation of energy [illegible] momentum in the linear quantum theory are represented by (assume [illegible] $\underset{\sim}{k}$-vectors)

$$\hbar\omega_1 + \hbar\omega_2 = \hbar\omega_3 \qquad \text{(5-18a)}$$

$$\hbar\underset{\sim}{k}_1 + \hbar\underset{\sim}{k}_2 = \hbar\underset{\sim}{k}_3 \qquad \text{(5-18b)}$$

*This does not rule out mass dipole pulsations, however (Campolattaro & Thorne, 1970).

If you now consider a crystal, and you feed momentum into one side of it in the laboratory, then according to the normal process of (5-18) all of that momentum has to come out the other side. There is nothing in the theory which accounts for the thermalization of phonons; the crystal lattice, hence, has infinite thermal conductivity or zero thermal resistance. Eq (5-18b) cannot explain the known behaviour of crystals for an observer in the laboratory center-of-momentum frame.

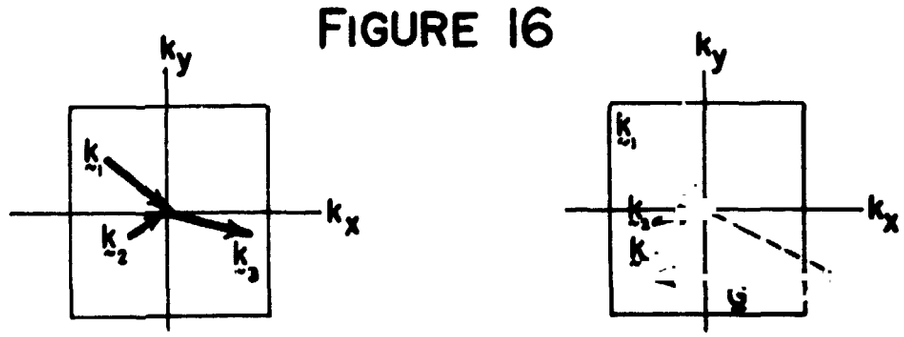
Admittedly, this equation for momentum conservation is a linear one, implying the need for a nonlinear mechanism to account for the thermalization of phonon populations.* But Peierls presents a linear mechanism instead of (5-18b), which works, and it is known as "umklapp":

$$\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3 + \underset{\sim}{G} \quad , \qquad \text{(5-18c)}$$

where $\underset{\sim}{G}$ is a reciprocal lattice vector. One way of visualizing umklapp is to recall moments in old Western movies when the wagon wheels appear to spin backwards. This is an oversimplification, however, because thermalization and entropy are real.

For a periodic lattice, the wave-vectors k_3 and (k_3+G) are indistinguishable. Normal momentum conservation (5-18b) and umklapp (5-18c) are represented in Figure 16 for the 1st-Brillouin zone in two-dimensional k-space.

FIGURE 16



$\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3$ $\qquad$ $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3 + \underset{\sim}{G}$

(a) Normal $\qquad$ (b) Umklapp

Whensoever the normal process of momentum conservation (5-18b) exceeds the

*The Maxwell-Dirac theory (second quantization) is nonlinear, for example. There, Eqs (5-18a,b) are inadequate due to photon-photon interaction.

boundary of the 1st-Brillouin zone, umklapp ensues and brings it back into the zone, as in Figure 16b. Umklapp destroys momentum and actually changes the direction of the "Poynting vector" or energy flow of Eq (5-8a). The onset of umklapp also constitutes the onset of an increase in system entropy.

The notion of umklapp can also serve as a linearization of some of the nonlinear effects of nonlinear radiation theory - and hence, its applicability to gravitational radiation. The nonlinear wave-wave interaction of colliding plane waves can be treated in such a fashion (Figure 16b). The nonlinear scattering of gravitational radiation can be analyzed this way. And, in fact, scattering of gravitational radiation from the background geometry itself (such as the Schwarzschild background) can occur with an umklapp mechanism. Lastly, it can be used to analyze gravitational "tails."

Umklapp does not necessarily have to be gravitational dipole radiation; it simply provides for a nonlinear source of momentum in a linear fashion. Hence, it has a meaningful place in the astrophysical theory of thermalization due to gravitational interaction. If a deity feeds nonlinear gravitational plane waves into one side of a crystal (periodic structure) Universe, no one expects them all to come out the other side. There is also an obvious implication of a lattice structure in geometrization itself, which can be characterized by gravitational umklapp. The perturbations $h_{\mu\nu}$ due to one gravitational wave give the background geometry a periodic structure, which behaves like a grating or moving mirror and scatters other gravitational waves (Brillouin scattering, 1922). Thus, there is a Bragg condition for gravitational waves.

Gravitational umklapp, however, does readily treat dipole radiation mechanisms. Two particular problems come to mind: Motion of the center of mass in two-body systems (which is known to exist in certain theories), and theories which dispel the Newtonian principle of equivalence ($m_I \neq m_g$).

Acceleration of the center of mass of the two-body problem is considered to exist in general relativity by Levi-Civita (1937), but this is demonstrated to be in error by Robertson (1938) and Robertson & Noonan (1968).* However, many other theories of gravitation do predict such an effect. Defining K as

$$K = \frac{\pi}{c^2} \frac{G^{3/2} m_1 m_2 (m_1 - m_2)}{[m a (1-e^2)]^{3/2}} , \qquad (5\text{-}19)$$

then from Whitrow & Morduch (1965) the parameterized theories compare as

*See also Schild (1962).

Gravitation Theory	Acceleration Of Center Of Mass
General Relativity	0
Nordström (1912)	-K
Scalar Potential (Variable c)	3K or 5K
Vector Potential - Parameterized (p)	pK
Kustaanheimo (1957)	0
Birkhoff (1943)	K
Whitehead (1922)	K
Whitehead - Parameterized (b)	(1-4b)K
Poincaré (1904, 1906)	0

All of the theories which exhibit an acceleration of the center-of-mass in the two-body problem also emit gravitational dipole radiation as a consequence of radiation reaction and conservation of momentum. The background metric can be thought of a recoiling under gravitational umklapp.

A breakdown in the equivalence principle (mass is a tensor, not a scalar: $m_{\alpha\beta} \neq m\delta_{\alpha\beta}$) also results in gravitational dipole radiation, in addition to producing mass anisotropies. Recent work by Nordtvedt (1968, 1969), Dicke (1969), Will (1971), Thorne & Will (1971), and Will & Nordtvedt (1972) studies mass anisotropies in the motions of massive, self-gravitating systems. This "Nordtvedt effect" results in gravitational dipole radiation and gravitational umklapp, if the effect exists.* In particular, Brans-Dicke-Nordtvedt "b[illegible]oles" should emit dipole radiation during collisions.

Backscatter (Parker, 1972), like gravitational "tails," and gravitational shock waves are also admirably treated by gravitational umklapp. Umklapp can also occur in the periodic lattice of a neutron star. This is pertinent not only for gravitational radiation but also for the propagation of optical and acoustical (sound) phonon branches there.

Gravitational dipole radiation and umklapp are discussed further in Appendix D, based upon angular momentum conservation laws in general relativity.

Gravitational Zitterbewegung

Standing waves are important in physics, for example in characterizing the electrical properties of insulators in solid state physics (the electron velocities form standing waves). Standing waves are also important in the theory of waveguides. Furthermore, they manifest themselves in the theory of gravitation.

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*See also Hawking (1972).

The standing wave solutions of the scalar, vector, and tensor wave equations in radiation theory give rise to zitterbewegung. The assumption of both incoming and outgoing radiation at infinity for boundary conditions results in advanced (Ritz, 1908) and retarded (Lorenz, 1867) solutions which together comprise standing waves with no outgoing radiation at all. The classical hydrogen atom, for example, is stable for standing wave solutions; quantum mechanics is not necessary on this basis alone.

Standing waves likewise occur in gravitational radiation theory. A small mass in the presence of such standing waves will go into a jittering oscillation between the nodes of the standing waves and resonate at twice (quadrupole) the frequency of the radiative source.

Gravitational standing waves appear in the analysis of cylindrical waves by Marder (1958, II) as well as the work of Thorne (1969, III) on nonradial pulsations of neutron stars using the Regge-Wheeler formalism. Thorne (1968) also presents a rigorous treatment of gravitational radiation damping in terms of a standing wave analysis which concentrates attention on the poles of the S-matrix (the complex frequencies $\omega=\omega+i/\tau$ at which the incoming radiation vanishes). For all resonances in the quadrupole standing-wave normal modes of various neutron star models, the poles all lie in the upper half of the complex frequency plane - which corresponds to damping (and not anti-damping) of the exponential $\exp(i\omega t)$. If a particle is in the presence of such an astrophysical object, zitterbewegung ensues and the smaller particle oscillates in resonance until it radiates itself away.

Scattering & Absorption Of Gravitational Radiation

A generalization of the vector (electromagnetic) scattering problem to tensor plane-waves results in an optical theorem and scattering cross-section for gravitational radiation. If we consider the scattering of $h_{\mu\nu}$ in (3-33) from a tensor scattering center $f_{\mu\nu}$ (an antenna, for example)

$$h_{\mu\nu}(\mathbf{r},t) \sim \left[\epsilon_{\mu\nu} e^{i\mathbf{k}\cdot\mathbf{r}} + f_{\mu\nu}\frac{e^{i\omega r}}{r}\right] e^{-i\omega t} , \qquad (5\text{-}20)$$

a Gegenbauer expansion of the incident plane-wave part gives in the usual fashion (using the Einstein polarization tensor $\epsilon_{\mu\nu}$)

$$\sigma_{tot} = \frac{4\pi\, \mathrm{Im}\left\{\epsilon^{\lambda\nu *} f_{\lambda\nu} - \frac{1}{2}\epsilon^{\lambda *}_{\lambda} f^{\nu}{}_{\nu}\right\}}{\omega\left(\epsilon^{\lambda\nu *}\epsilon_{\lambda\nu} - \frac{1}{2}|\epsilon^{\lambda}_{\lambda}|^2\right)} , \qquad (5\text{-}21)$$

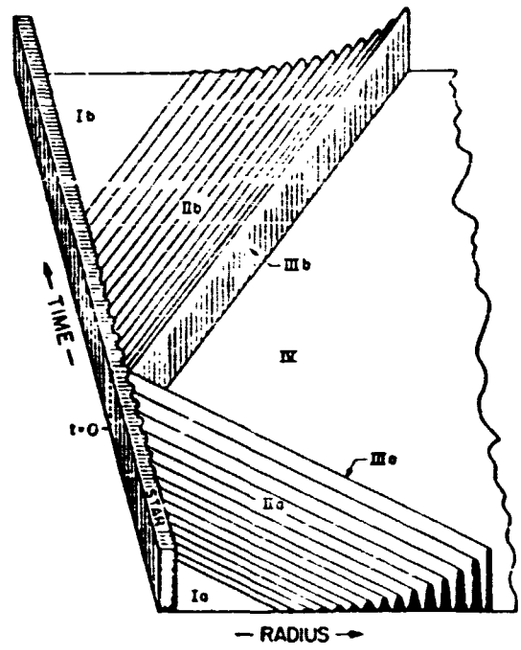

FIGURE 17

which is the optical theorem for tensor scattering. For an antenna with an axis of circular symmetry, the antenna cross-section reduces to (see Weinberg, 1972, or Ruffini & Wheeler, 1969)

$$\sigma_{tot} = \frac{15\pi\eta c^2}{4\omega^2}\left[\frac{\Gamma^2/4}{(\omega-\omega_0)^2+\Gamma^2/4}\right]\sin^4\theta \,, \qquad (5\text{-}22)$$

where ω_0 is the natural frequency of free oscillation of the antenna, Γ is the total decay rate of free oscillation, Γ_{grav} is the decay rate of free oscillation due to the re-emission of gravitational radiation, and $\eta=\Gamma_{grav}/\Gamma$. This cross-section is maximum when the antenna is perpendicular to the gravitational wave ($\theta=\pi/2$).

The significance of an optical theorem (also known as the Ewald-Oseen Extinction Theorem or the Bohr-Peierls-Placzek relation) for linearized tensor radiation is that it represents a unitarity of the scattering-matrix. It also constitutes Huyghen's principle, which we know is not valid if there exist gravitational tails - from our discussion of the Bondi news function. An idea of the behaviour of gravitational tensor scattering is represented by Figure 17 from Price & Thorne (1969), for a neutron star.

It is interesting to note that the scattered spectrum of Price & Thorne is identical to (5-8) for the Kepler problem and the spinning rod of Figure 1, provided L=2 and M=2 for even-parity, quasi-normal pulsations. Vishveshwara (1970) also considers such scattering from the effective potential of Figure 5, maintaining that the scattered radiation contains the signature of the Schwarzschild "black hole." The scattering of scalar gravitons from rotating Kerr "black holes" is discussed by Misner (1972) and Press & Teukolsky (1972) with the interesting conclusion that the scalar wave is amplified as it scatters off the hole.

In the vector theory of gravitation, of course, there are the direct analogues of electromagnetic scattering phenomena such as Mie scattering of gravitational (Spin-1) waves.

Detection & Experimental Verification Of Gravitational Radiation

If you recall from Newtonian mechanics that for any nonuniform gravitational there exist gravitational gradients, you realize that you can readily distinguish between gravitational and nongravitational acceleration effects. If you are in free-fall in an inverse-square field (in an elevator, if you wish), a measurement with inertial sensors of the gravitational gradient components in

your elevator frame will not only distinguish such accleration from non-gravitational ones, but it will give you the value of the field as well. Einstein's principle of equivalence, important for correspondence with the Minkowski frames of restricted relativity, is valid only in a uniform gravitational field* ("fictitious" forces arising from the choice of coordinates and hence $g_{\mu\nu}$ are not questioned here). But such uniform fields are not known to exist in astrophysics; hence his principle is a tautology.

The Riemann curvature tensor $R_{\alpha\mu\beta\nu}$ is essentially a glorified version of the same thing, the Newtonian gravitational gradient or tidal component in classical mechanics. Instead of taking the gradient of the scalar field intensity with multipoles (quadrupoles, etc.) in Eq (2-3), tidal stresses can be determined from the $R^{i}{}_{0i0}$ components of the Riemann tensor (Eq A-2 of Appendix A) in the tensor theory of gravitation. These tidal forces or Riemann stresses lie at the basis of the construction of gravitational radiation detectors. To first order in the linear perturbations $h_{\mu\nu}$ of $g_{\mu\nu}$, Appendix A determines that

$$R^{(1)}_{\alpha\mu\beta\nu} = \frac{1}{2}\left[h_{\mu\nu,\alpha\beta} + h_{\alpha\beta,\mu\nu} - h_{\alpha\nu,\mu\beta} - h_{\mu\beta,\alpha\nu}\right] . \qquad (5\text{-}23)$$

From Einstein's quadrupole radiation in (3-26) and (3-27), which lead to the linearized quadrupole power formulae (3-29) and (5-1), the non-zero Newtonian tide producing components due to a gravitational plane wave propagating in the X^1-direction are caused by h_{23} and $(h_{22}-h_{33})$ in (3-26) or (3-37). From Eq (5-23) above, using (3-24),

$$\begin{aligned} R^{2}{}_{020} &= -\frac{1}{2}\frac{\partial^2}{\partial t^2} h_{22} = \frac{G}{6c^4}\left(\ddddot{Q}_{22} - \ddddot{Q}_{33}\right)/r \\ &= -R^{3}{}_{030} \\ R^{2}{}_{030} &= R^{3}{}_{020} = -\frac{1}{2}\frac{\partial^2}{\partial t^2} h_{23} = -\frac{G}{3c^4}\ddddot{Q}_{23}/r \ . \end{aligned} \qquad (5\text{-}24)$$

Note that $R^{i}{}_{0i0}$ has zero trace $(R^{1}{}_{010}+R^{2}{}_{020}+R^{3}{}_{030}=0)$ which means that the quadrupole detector Q_{ij} responding to the Einstein-Eddington-Weyl plane-wave (in Chapter 3) maintains a constant volume under the Newtonian tide components of stress. These components of transverse stress simply deform the object by exciting its quadrupole moment Q_{ij}. A cylinder (Figure 18a), for example, has two transverse modes of polarization, separated by 45^0 due to quadrupole symmetry. A disk (Figure 18b) also exhibits a radial stress which is indicative of a scalar component of radiation. No detectors have been designed for measuring stresses due to electromagnetic (vector) theories

*Einstein (1911) did not understand this when he first established the principle.

of gravitation, nor is there consideration of dipole mechanisms.

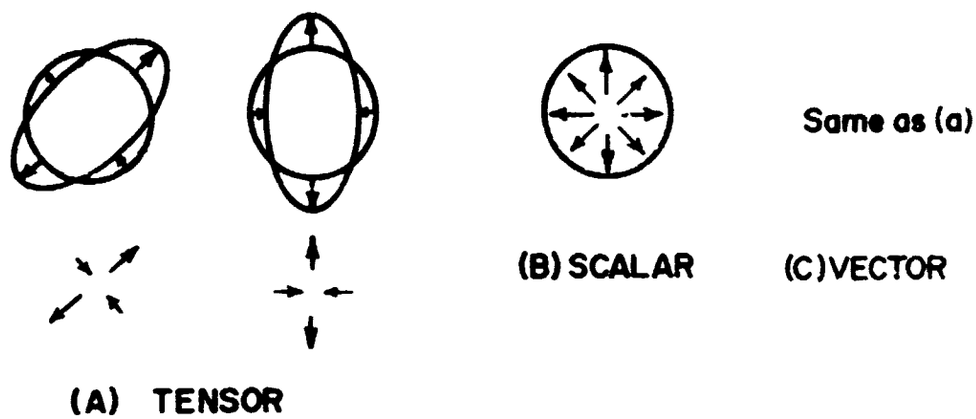

FIGURE 18

Experimental work on gravitational radiation has been primarily fostered by Weber, et al (1960-1972), who measure responses of a gravitational quadrupole antenna to supposed radiation sources. Although they have found coincidences between their two cylindrical antennas, these measurements have not been duplicated. The unusually large flux (10^4 ergs/cm^2-sec) given by Weber's measurements (Figure 19) indicates unphysically short lifetimes of astrophysical objects such as the Galaxy, and consequently a great deal of theoretical work on tensor synchrotron mechanisms is a direct result - as we have seen. One should also note in Figure 19 that the coincidences occur not only when the galactic center is in view but also 12 hours later when it is occulted by the earth. Others (Beron & Hofstadter, 1969) present arguments that high-energy cosmic rays can excite phonon oscillations of quadrupole symmetry in gravitational antennas (or telescopes) and be mistaken for gravitationally induced oscillations (which are of the order of a nuclear radius!).* Some work on the generation of gravitational radiation in the laboratory is available (Weber, 1960; and Forward, 1966), although the quadrupole power factor of 10^{-61} erg/sec in Eq (3-29) is more than enough to discourage the most ardent experimentalist. A good survey of experimental work is presented by Logan (1973).

Attempts other than synchrotron mechanisms to explain the high fluxes measured by Weber include the amplification of gravitons upon scattering from rotating Kerr "black holes" (Misner, 1972; and Press & Teukolsky, 1972)

*This suggestion is rebuffed by Ezrow, et al (1970), however.

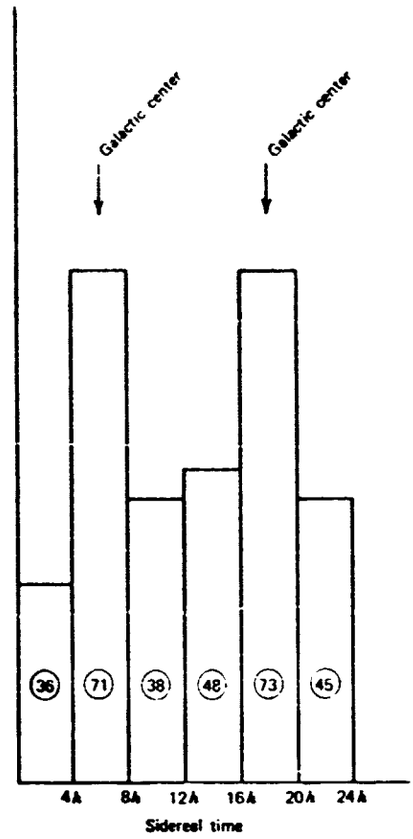
Galactic center
Galactic center
36
71
38
48
73
45
4h
8h
12h
16h
20h
24h
Sidereal time

FIGURE 19

as well as the collision of black holes which cannot bifurcate (Hawking, 1972). Hawking's mechanism is reminiscent of using a power saw to cut saplings.

But rather than build detectors, some observers have been using natural gravitational antennas - for hundreds of years. Hooke (1671) considers the attraction of two spheres, that is, gravitational attraction itself, as a "gravitational antenna" indicating the existence of gravitational waves in order to explain the mechanism of gravitation. Laplace uses the planetary orbits of the solar system as a means of determining perturbations caused by finite velocities of propagation of fields, and hence of waves. And now experiments are being performed using the quadrupole modes of the earth, the moon, the earth-moon system, and the Mariner 6 spacecraft as gravitational antennas. Dyson (1969), in the process of considering the seismic response of the earth to gravitational radiation, makes the suggestion that coincidences might be autocorrelated between seismic data and astrophysical pulsars. This suggestion gives negative results from some authors (Wiggins & Press, 1969), but an interesting flury of experimental work is being carried out by Dror Sadeh (1972), who claims to have correlated terrestrial and lunar seismic activity with pulsar CP 1133 - much to the consternation of seismologists such as Mast et al (1972).

In order to make this discussion of the tidal stresses complete, it is worthwhile to consider the Riemann stress components in the Regge-Wheeler-Zerilli formalism For a wave propagating along the X^1-axis as before, the asymptotic tide-producing, transverse components are

$$R^3{}_{030}(r^*,t) = -R^2{}_{020} = \sum_{L,M} \ddot{R}_L(r^*,t) W_{LM}(\theta,\phi)/2r \qquad (5\text{-}25a)$$

where

$$W_{LM}(\theta,\phi) = \left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_L^M(\theta,\phi) \qquad (5\text{-}25b)$$

which appears in the tensor harmonics of Appendix K. The physical interpretation of these stresses is found by referring to Figure 14b. Tidal accelerations are also discussed by Finley (1971). Tidal deformation is treated by Manasse (1963).

Astrophysical Miscellania

There are numerous sources of gravitational radiation in astrophysics which have been examined with the theoretical results of Einstein's nonlinear tensor theory of gravitation. The nonradial, asymmetric pulsations of stars, in particular neutron stars, have been studied quite extensively by Thorne et al (1967-1970), using the familiar Regge-Wheeler formalism but applied to general relativistic hydrodynamics. The odd-parity perturbations describe differential rotation, while the even-parity perturbations describe the pulsations. Thorne's work is a thorough treatment of the damping of stellar oscillations under gravitational radiation and a study of the associated relaxation times. However, it is limited in that Ruderman (1968, 1969) has presented models of neutron stars with crystalline mantels which support shear stresses. Ipser (1971) extends these investigations to slowly rotating neutron stars.

Pulsating stars prove to have another significance in the astrophysical theory of gravitation, as pointed out by Morganstern & Chiu (1967). In the Brans-Dicke (1961) theory a transverse scalar (Spin-0) component of the radiative Riemann tensor is predicted, whose source is a monopole oscillation (see Robinson & Winicour, 1969). For a radially symmetric pulsation, scalar waves can be radiated away but not tensor waves. Such scalar radiation, furthermore, damps out the radial pulsation of the neutron star in a matter of seconds. The existence of the scalar field can hence be ruled out if neutron stars are found to pulsate.

Gravitational collapse with asymmetries is another aspect of gravitational radiation theory which has astrophysical importance. Symmetric collapse, of course, does not radiate tensor radiation due to Birkhoff's theorem. Asymmetric collapse is treated by de la Cruz, Chase, & Israel (1970) who determine that ingoing radiation (down the Schwarzschild "black hole") causes external asymmetries to decay quickly with a brief relaxation time on the order of $2GM/c^3$. They conclude that "black holes" cannot serve as tensor synchrotron mechanisms, a result consistent with Chandrasekhar's triaxial Jacobi ellipsoid of Figure 9 wherein asymmetries are quickly radiated away during collapse and the "black hole" becomes a Maclaurin spheroid. The radiation of such asymmetries is also discussed by Israel (1971), Penrose & Floyd (1971), and Penrose (1972).

Excited "black holes"* are discussed by Fackerell (1971), Peres (1971), and Goebel (1972). They are also relevant in the context of "black hole" collisions, wherein the gravitational radiation is entirely tensor radiation according to the analysis of Hawking (1972). This is simply because Brans-Dicke "black holes" have no scalar field. Hence, no Brans-Dicke scalar radiation can be emitted during the collision of two "black holes."

In the many and varied efforts to study the behaviour of nonlinear tensor gravitational radiation, low frequency and high frequency approximations have been made. Low frequency work, of course, is what leads Ruffini (1973) to the important conclusions of Figure 15. Likewise, low frequency analysis is the basis of Bergmann's (1971) investigation and of the Mariner spacecraft tracking studies of Anderson (1971) (see also Gibbons, 1971) in an effort to observe scintillations due to gravitational waves. High frequency approximations, on the other hand, are discussed by Isaacson (1968) as well as Halpern (1971), Isaacson & Winicour (1972), and Kafka (1970). The significance of Isaacson's work has already been stated in the discussion of his resultant stress-tensor. Kafka assumes the possibility that gravitational radiation might be of primordial origin. Primordial gravitational radiation is further pursued at very long wavelength by Rees (1971) and critiqued by Jackson (1972). This work all emphasizes the cosmological significance of gravitational radiation.

Misner's (1968) mixmaster universe is another interesting embodiment of gravitational radiation in relativistic astrophysics. It essentially treats the Universe as a gravitational waveguide, much like the work of Campbell & Morgan (1971) which employs a gravitational Debye potential formalism and the work of Marder (1958, II) on cylindrical standing waves. Lastly, Lapedes & Jacobs (1972) maintain that tachyons should produce gravitational Cerenkov radiation, although tachyonic systems are not very popular among astrophysicists.

Meditations Upon A Few Thoughts From Classical Physics

There are a number of topics in the classical astrophysics of gravitational radiation which must not go unnoticed. They are controversial. And yet we know that controversial thought is traditionally wherein lie many of the great advancements in human science. Consequently, it is a greater heresy

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* They can be excited by a particle falling into the geometry, as in the Zerilli problem.

to ignore such thoughts than to weigh their merit - although many of them should not be dignified by a great deal of detail.

The concept of mass is fundamental to any theory of gravitation and hence to any theory of gravitational radiation. But is mass positive-definite, or does Nature provide for states of negative mass (Föppl, 1898) and negative matter or anti-matter (Schuster, 1898)? We know, for example, that there exist negative energy instabilities in nonlinear plasma theory (Sagdeev & Galeev, 1969). Can the Dirac theory of negative energy states be re-interpreted, perhaps, in terms of negative mass* (using the negative mass shell of $p^2=m^2$), and if so are there such things as negative energy photons and gravitons (e.g. Misner, 1972) associated with jumping the energy gap? Such conjectures have been made in explaining the behaviour of quasi-stellar sources. Indeed, in view of the negative mass solutions manifesting themselves in the only known exact radiative solution of Einstein's nonlinear theory, we are forced to address this problem. It is no longer a metaphysical whim of Arthur Schuster.

It is likewise believed that mass and energy are equivalent in relativity theory, as stated by Eq (3-4). This assumption results in an epistemological contradiction with the related postulate of classical electrodynamics that charge is invariant. Gravitational radiation is produced by mass, and it transports mass (according to Einstein's theory), while electromagnetic radiation is produced by charge - but does not transport charge. This irreconcilable difference is just cause for rejecting any grounds for an electromagnetic analogy in the linearized tensor theory of Einstein. It is an epistemological flaw of the first magnitude and deteriorates any foundation for a unified field theory based upon geometrization. The linear Maxwell theory and the nonlinear Einstein theory of energy transport cannot be reconciled.

Also, we must note that negative energy densities characterize all gravitation theory. There is nothing intrinsic in the scalar, vector, or tensor theory that denies this. Time-symmetric arguments (Brill, 1960) for positive-definite energy are tautological in the sense that they assume a reversal of the Poynting vector ($g_{io} \cdot -g_{io}$) under time-reversal (reversal of incoming and outgoing radiation conditions). They assume what they attempt to demonstrate.

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*Velocity is anti-parallel to momentum for negative mass. The helicity $\sigma p/p$ of anti-particles is directly accounted for if anti-particles possess negative mass. However, see Morrison & Gold (1957) and Schiff (1958) on this point.

Furthermore, mass is assumed to dilatate as stated by Eq (3-5), in the Einstein theory of relativity. The structure of gravitational radiation theory must not collapse just because this single postulate proves to be untenable if there are future developments in electrodynamics. Likewise, there is a "second postulate of general relativity." In other words, convention has it that the velocity of a photon is independent of the velocity of its source.* If the nonlinear tensor theory of gravitational radiation is allowed to be linearized and characterized with all of the other attributes of electromagnetic radiation, then must not gravitons likewise be independent of the velocity of their source? In contrast, there is a ballistic theory of gravitational radiation. The Ewald-Oseen Extinction Theorem (the Optical Theorem) renders the second postulate indeterminable by experimental method for both photons and linearized gravitons.

Is a gravitational wave stable? The contribution of the Regge-Wheeler (1957) study of the stability of the Schwarzschild metric is the implication that no physical theory can be considered seriously until it has been demonstrated to be stable. Whence, if the integrity of $\eta_{\mu\nu}$ is questioned then why not question the stability of the gravitational waves $h_{\mu\nu}$? The regenerative nature of self-interaction in the nonlinear theory of general relativity is important in such considerations.

Is the graviton stable? This aspect of the question of stability is taken from the quantized linear point of view rather than the general nonlinear wave and resonance instability discussed above. It becomes important if experimentalists fail to detect such radiation or if there exists a "cosmological redshift" of the gravitational radiation power spectrum (which there should be in order to avoid an Olbers-Chesaux-Halley paradox). Decay mechanisms for gravitons, with and without rest-mass, can be readily improvised. The dispersion relation for vector (Spin-1) gravitons includes an effective mass term from the Proca theory or from the propagation of gravitons through matter using the Weyl tensor (in direct analogy with "plasmons," which are photons of effective mass propagating through an electron gas). The vector (Spin-1) massive graviton is readily extended to the tensor (Spin-2) theory by adding Proca terms to the tensor wave equation.

Lastly, we mention that there is work in the literature on composite field theories, such as the neutrino theory of light. A candid discussion of these is presented within the context of gravitational radiation theory in Appendix V.

*This postulate has marginal experimental basis (Wilson, 1972).

Nonlinear Effects In Astrophysics

Several aspects of the nonlinear nature of the tensor theory of gravitational radiation are significant and must be kept in mind as one interprets the astrophysical properties of the Universe. The first characteristic of Einstein's geometrized theory of gravitation which asserts itself is the remarkable feature that it has no radiation reaction problem, as does the linear, classical theory of Maxwell and Lorentz. This is originally due to Einstein's geodesic postulate, but actually it is a consequence of the fact that the EIH equations of motion follow from the nonlinear field equations because of (3-38). Referring to Figure 20, an empty Schwarzschild geometry is traversed by a geodesic on the background $\eta_{\mu\nu}$.

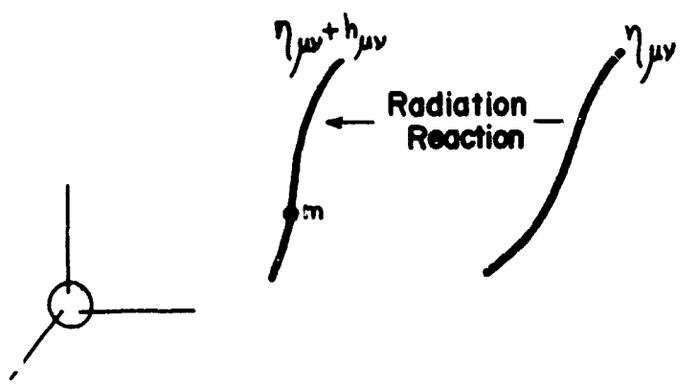

FIGURE 20

However, upon inserting a particle of small mass m into the Schwarzschild geometry, the particle does not follow the geodesic for $\eta_{\mu\nu}$ but rather that for $\eta_{\mu\nu} + h_{\mu\nu}$. The perturbations $h_{\mu\nu}$ are those imposed upon the structure of the background by m and are determined by the very existence of m in the geometry. If you can ever solve Einstein's nonlinear field equations you get as a direct dividend the new geodesic. Radiation reaction is automatically accounted for by the nonlinear nature of the theory. Linearizations of Einstein's theory, however, such as the Regge-Wheeler-Peters-Zerilli formalism are confronted with the problem of radiation reaction because they neglect it. Zerilli's treatment of the particle falling in, if you recall, assumes the particle follows $\eta_{\mu\nu}$ - which is not the trajectory it actually takes. Only for vanishingly small mass ($m \to 0$) is the linearized treatment tenable.

The regenerative feature of Einstein's nonlinear field equations is another crucial aspect of their properties. The fields act as their own sources (Gupta 1957; Thirring, 1959, 1961).

Having emphasized the nonlinear nature of the tensor radiation theory, we must finally ask ourselves what it means to treat the theory in a linear fashion, as with energy and momentum relations like Eqs (5-18a,b) from quantum theory. Surely we can write for energy

$$E = h(2\nu) \quad , \tag{5-26}$$

but what does that mean? From the work of Pauli & Fierz (1939) and Gupta (1954, 1962) tensor radiation is Spin-2, in the linearized theory of gravitation. However, this does not mean that Spin-2 "gravitons" exist in Einstein's nonlinear theory, for it has never been quantized. A casual look at the behaviour of colliding plane waves (Szekeres, 1970, 1972; and Penrose & Kahn, 1971) and radiation scattering (Torrence & Janis, 1967) illustrates the naiveté one must have in order to consider energy and momentum in the normal linear fashion (superposition)

$$\omega_1 + \omega_2 = \omega_3 \tag{5-18a}$$

$$\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3 \quad . \tag{5-18b}$$

The self-interaction (See e.g. Torrence & Couch, 1970) of nonlinear radiation and the behaviour of gravitational "tails" (DeWitt & Brehme, 1960; Couch et al, 1968; and Hallidy & Janis, 1970) also are formidable problems - although significant progress has been made since the advent of Bondi's asymptotic approximation (Bondi et al, 1962) and of the Newman-Penrose tetrad formalism (Newman & Penrose, 1962).

We can invoke gravitational umklapp (5-18c) but we must ask ourselves questions about gravitational dispersion of the frequency ν in Eq (5-26). Gravitational radiation must traverse interstellar distances through a "medium" of nonlinear wave-wave interactions. It would be foolish to suppose that the frequency detected by a terrestrial observer ν' would be the theoretical one for quadrupole radiation (3-29), namely: $\nu'=2\nu$. For this reason optical and radio (see Charman et al, 1970) correlations with gravitational radiation detection coincidences are of major significance because they can help ascertain the dispersive nature of intergalactic space to gravitational radiation.

The failure to find any electromagnetic and gravitational coincidence* so far is a basis (assuming the gravitational data are real) for maintaining that there is a high degree of shift in frequency - meaning that the frequency of the source ν (and hence its angular momentum) can be much less than that measured here on earth. Synchrotron mechanisms are then unnecessary in order to rationalize Weber's data. One over-estimates the energy of the source because he misinterprets the energy density of the gravitational wave - due to the linearization of the theory.

Another way of presenting the problem of nonlinear radiation is to turn again to Eq (5-26). What is Planck's constant** "h" for gravitational phenomena (which have never been quantized in the tensor theory)? Is it constant? Simply be increasing the ratio h_{grav}/h_{em} one can easily account for apparently high quadrupole fluxes.

And what about the uncertainty principle? Is there an uncertainty principle for gravitational phenomena?

$$\Delta X \Delta P \geqslant \hbar \qquad (5\text{-}27)$$

Again, what is h? How do we know that we can even measure gravitational radiation?

Indeed, one must not abuse relationships for the Maxwell theory and the quantum theory which have not yet been established for gravitation theory. The closest thing yet to gravitational quantization is the tensor harmonic decomposition of the linearized field equations.

With this we end our query on the nature of gravitational radiation theory.

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*Electromagnetic and gravitational coincidences are further complicated by the assumption that their velocities of propagation are equal - which may not be true.

**To demonstrate the epistemological inconsistency in gravitational and quantum physics, one can use the identical Bohr quantization condition employed to quantize the orbital angular momentum of the Coulomb Kepler problem, and derive h for the gravitational Kepler problem (the solar system). On the basis of the orbit of Mercury (this is really the second "Bohr" orbit if Bode's law is relevant), $\hbar = mva_g \approx 10^{39}$ joule-sec, rather than 10^{-34}. In making this analogy, one sees another distinctive feature of the gravitational quantization problem: charge is quantized and so is its field theory - while mass is not, neither is its gravitational field, at the present time.

CHAPTER VI

CONCLUSIONS

Gravitational radiation theory has been discussed from the point of view of its scalar, vector, and tensor counterparts. This discussion has not ventured to ascertain the existence of such radiation in gravitation theory but rather to take a candid look and to explore its meaning in the context of contemporary physics.

We have found that there is a theoretical basis for such radiation in the vector and tensor theories of gravitation, but in doing so we have been forced to address all of the fundamental questions about radiation theory. These include the notions of radiation reaction, the behaviour of advanced and retarded solutions, the choice of boundary conditions, the transport of energy, the meaning of energy, the nonlinear nature of radiation - and even the meaning of a wave. We have made analogies with classical radiation theory which are sound in the linear electromagnetic (vector) theory of gravitation, but they become unsuitable when extended to the nonlinear tensor theory of general relativity. There they have as yet no experimental basis. Nonlinear electromagnetism, on the other hand, does have some foundation in experimental fact, such as Maxwell-Dirac spinor electrodynamics and nonlinear optics (both employ nonlinear constitutive relations 2-14). This in turn provides reason for analogy with a nonlinear vector theory of gravitational radiation. Nevertheless, we know and understand very little about nonlinear gravitational waves.

To be sure, superposition must be sacrificed in the nonlinear theory. The linearization, furthermore, of the nonlinear field equations inherently destroys any possibility of accounting for the interaction of the gravitational field with itself.

The exact radiative solutions of the tensor theory of gravitation likewise do not yet render an adequate understanding of the problem. The only exact solution for moving sources in general relativity allows for negative mass, which like the concept of negative energy densities, may be very metaphysical

in nature. Indeed, enchantment with nonlinear tensor theory teaches us more about the tyranny of mathematics than it reveals about the physical nature of gravitational radiation.

However, by means of linearized tensor radiation theory, we have studied the behaviour of mass quadrupole oscillators such as the Jacobi ellipsoid and the two-body problem. As paradigms in astrophysics, they have taken a useful place in the life and death of neutron stars, binaries, and "black holes."

By means of linearization, we have also gained insight into characteristic features of the tensor theory which can eventually lead to a decision as to which of the scalar, vector, and tensor theories is most consistent with experimental data.

Perhaps there has been undue interest here in the vector theory of gravitational radiation, but it is intimately consistent with classical physics, quantum mechanics, and the quantum theory of radiation - which the nonlinear tensor theory is not. There is [redacted] a prodiguous amount of comparison of physical phenomena with electromagnetism -- and there always will be. Accordingly, an understanding of the real nature of the vector theory is of fundamental significance.

Nevertheless, this query on the nature of gravitational radiation has led us to understand better the characteristics of linear and nonlinear radiation. With a little bit of luck, we may even be on our way towards a unified picture of the nature of the physical world.

APPENDIX A: LINEARIZATION OF RIEMANNIAN GEOMETRY

In an arbitrary geometry one is concerned with its affine connections. If the geometry is Riemannian, these affine connections are represented by the Christoffel symbols

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} = \frac{1}{2} g^{\alpha\beta}\left[g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta} \right]. \qquad \text{(A-1)}$$

The Riemann tensor for this geometry is defined as

$$R^{\alpha}_{\mu\beta\nu} \equiv \Gamma^{\alpha}_{\mu\beta,\nu} - \Gamma^{\alpha}_{\mu\nu,\beta} + \Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\mu\beta} - \Gamma^{\alpha}_{\epsilon\beta}\Gamma^{\epsilon}_{\mu\nu} . \qquad \text{(A-2)}$$

The geometry is "flat" if $R_{\alpha\mu\beta\nu} = 0$. When contracted ($\alpha=\beta$), $R^{\alpha}{}_{\mu\alpha\nu}$ gives the Ricci tensor $R_{\mu\nu}$

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\epsilon\nu}\Gamma^{\epsilon}_{\mu\alpha} - \Gamma^{\alpha}_{\epsilon\alpha}\Gamma^{\epsilon}_{\mu\nu}. \qquad \text{(A-3)}$$

For a "weak-field" approximation of the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{(A-4)}$$

then to first order in $h=h^{\lambda}{}_{\lambda}=\eta^{\sigma\lambda}h_{\sigma\lambda}$, the two right-hand terms in (A-2) and (A-3) are negligible, giving for (A-2)

$$R^{(1)}_{\alpha\mu\beta\nu} = \frac{1}{2}\left[h_{\alpha\beta,\nu\mu} + h_{\nu\mu,\alpha\beta} - h_{\mu\beta,\nu\alpha} - h_{\nu\alpha,\mu\beta} \right] \qquad \text{(A-5)}$$

and instead of (A-3),

$$R^{(1)}_{\mu\nu} = \overset{(1)}{R}{}^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} . \qquad \text{(A-6)}$$

Evaluating (A-6) by taking the appropriate partial derivatives of (A-1) gives

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\Box h_{\mu\nu} - h^{\lambda}{}_{\mu,\lambda\nu} - h^{\lambda}{}_{\nu,\lambda\mu} + h_{,\mu\nu} \right] \qquad \text{(A-7)}$$

to first order in h. This may be re-written as

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\Box h_{\mu\nu} - \left(h^{\lambda}_{\mu} - \tfrac{1}{2}\eta^{\lambda}_{\mu} h\right)_{,\lambda\nu} - \left(h^{\lambda}_{\nu} - \tfrac{1}{2}\eta^{\lambda}_{\nu} h\right)_{,\lambda\mu} \right]. \qquad \text{(A-8)}$$

Similarly

$$\begin{aligned} R^{(2)}_{\mu\nu} = & -\frac{1}{2} h^{\lambda\kappa}\left[h_{\lambda\kappa,\nu\mu} - h_{\mu\kappa,\nu\lambda} - h_{\lambda\nu,\kappa\mu} + h_{\mu\nu,\kappa\lambda} \right] \\ & + \frac{1}{4}\left[2h^{\kappa}_{\sigma,\kappa} - h^{\kappa}_{\kappa,\sigma} \right]\left[h^{\sigma}_{\mu,\nu} + h^{\sigma}_{\nu,\mu} - h_{\mu\nu,}{}^{\sigma} \right] \\ & - \frac{1}{4}\left[h_{\sigma\nu,\lambda} + h_{\sigma\lambda,\nu} - h_{\lambda\nu,\sigma} \right]\left[h^{\sigma}{}_{\mu}{}^{,\lambda} + h^{\sigma\lambda}{}_{,\mu} - h^{\lambda}{}_{\mu}{}^{,\sigma} \right] . \end{aligned} \qquad \text{(A-9)}$$

to second order in h.

APPENDIX B: COORDINATE CONDITIONS & GAUGE INVARIANCE

Gauge Invariance

If one considers an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \tag{B-1}$$

the metric tensor is changed as follows

$$g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\rho}} g^{\lambda\rho} \quad . \tag{B-2}$$

Because, furthermore, $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$ (raising indices) then

$$h'^{\mu\nu} = h^{\mu\nu} - \xi^{\mu}{}_{,\lambda}\eta^{\lambda\nu} - \xi^{\nu}{}_{,\rho}\eta^{\rho\mu} . \tag{B-3}$$

Hence, if $h_{\mu\nu}$ is a solution of the wave equation (A-6)

$$R^{(1)}_{\mu\nu} = 0 \tag{B-4}$$

then so is

$$h'_{\mu\nu} = h_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}) . \tag{B-5}$$

This demonstrates the gauge invariance of the field equation (B-4). If the perturbation of $h_{\mu\nu}$ in (B-5) is null

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0 \tag{B-6}$$

one has what are referred to as "Killing's Equations." A gauge transformation, then, is comprised of inhomogeneous Killing vectors.

Einstein Coordinate Condition

Einstein (1916, 1918), in his original derivation of the gravitational radiation associated with linearized General Relativity, chooses to define a tensor wave function

$$\gamma^{\mu\nu} = \frac{1}{2}\left[h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h\right] \tag{B-7}$$

subject to the following coordinate condition

$$\gamma^{\mu\nu}{}_{,\nu} = 0 \quad . \tag{B-8}$$

It appears to be a common misconception that relation (B-8) is due to de Donder (1921), and the latter is often associated with his name. It should be more properly credited to Einstein.

The virtue of the Einstein coordinate condition (B-8) is realized if one recalls the first-order linearized Ricci tensor in Appendix A. With (B-7) and (B-8), Eq (A-6) simplifies directly to

$$R^{(1)}_{\mu\nu} = \tfrac{1}{2}\Box h_{\mu\nu} \quad . \tag{B-9}$$

If $\gamma^{\mu\nu}$ does not satisfy (B-8), then one can always find a $\gamma'^{\mu\nu}$ and hence $h'^{\mu\nu}$ which do, by performing a gauge transformation (B-1) where $\Box\xi^{\mu}=\gamma^{\mu\nu}{}_{\nu}$.

Harmonic Coordinates

Another coordinate condition much like (B-8) due to Einstein is the "harmonic coordinate" condition employed by de Donder (1921) and Lanczos (1922), and later exploited by Fock (1939, 1959). It is defined as

$$\Gamma^{\alpha} \equiv g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = 0 \quad , \tag{B-10}$$

which is not generally covariant and which thereby destroys the general covariance of the theory. It is "harmonic" because the coordinates themselves satisfy $\Box X^{\mu}=0$ due to (B-10). Defining a metric tensor desity $\hat{g}^{\mu\nu}=\sqrt{(-g)}g^{\mu\nu}$, there is the condition

$$\hat{g}^{\mu\nu}{}_{,\nu} = 0 \quad , \tag{B-11}$$

equivalent to (B-10). The difference between (B-11) and (B-8) is a very academic one.

APPENDIX C: ENERGY MOMENTUM PSEUDOTENSOR

The energy-momentum pseudotensor is demonstrated by Landau & Lifshitz (1962) to be

$$
\begin{aligned}
t^{\mu\nu} = \frac{c^4}{16\pi G}\Big\{ & \left(2\Gamma^{\alpha}_{\gamma\delta}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\gamma\beta}\Gamma^{\beta}_{\delta\alpha} - \Gamma^{\alpha}_{\gamma\alpha}\Gamma^{\beta}_{\delta\beta}\right)\left(g^{\mu\gamma}g^{\nu\delta} - g^{\mu\nu}g^{\gamma\delta}\right) \\
& + g^{\mu\gamma}g^{\delta\alpha}\left(\Gamma^{\nu}_{\gamma\beta}\Gamma^{\beta}_{\delta\alpha} + \Gamma^{\nu}_{\delta\alpha}\Gamma^{\beta}_{\gamma\beta} - \Gamma^{\nu}_{\alpha\beta}\Gamma^{\beta}_{\gamma\delta} - \Gamma^{\nu}_{\gamma\delta}\Gamma^{\beta}_{\alpha\beta}\right) \\
& + g^{\nu\gamma}g^{\delta\alpha}\left(\Gamma^{\mu}_{\gamma\beta}\Gamma^{\beta}_{\delta\alpha} + \Gamma^{\mu}_{\delta\alpha}\Gamma^{\beta}_{\gamma\beta} - \Gamma^{\mu}_{\alpha\beta}\Gamma^{\beta}_{\gamma\delta} - \Gamma^{\mu}_{\gamma\delta}\Gamma^{\beta}_{\alpha\beta}\right) \\
& + g^{\gamma\delta}g^{\alpha\beta}\left(\Gamma^{\mu}_{\gamma\alpha}\Gamma^{\nu}_{\delta\beta} - \Gamma^{\mu}_{\gamma\delta}\Gamma^{\nu}_{\alpha\beta}\right)\Big\} .
\end{aligned}
\tag{C-1}
$$

Defining a metric density $\hat{g}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}$, this is also

$$
\begin{aligned}
(-g)t^{\mu\nu} = \frac{c^4}{16\pi G}\Big\{ & \hat{g}^{\mu\nu}{}_{,\gamma}\hat{g}^{\gamma\delta}{}_{,\delta} - \hat{g}^{\mu\gamma}{}_{,\gamma}\hat{g}^{\nu\delta}{}_{,\delta} + \tfrac{1}{2}g^{\mu\nu}g_{\gamma\delta}\hat{g}^{\gamma\alpha}{}_{,\beta}\hat{g}^{\beta\delta}{}_{,\alpha} \\
& - \left(g^{\mu\gamma}g_{\delta\alpha}\hat{g}^{\nu\alpha}{}_{,\beta}\hat{g}^{\delta\beta}{}_{,\gamma} + g^{\nu\gamma}g_{\delta\alpha}\hat{g}^{\mu\alpha}{}_{,\beta}\hat{g}^{\delta\beta}{}_{,\gamma}\right) \\
& + g_{\gamma\delta}g^{\alpha\beta}g^{\mu\gamma}{}_{,\alpha}\hat{g}^{\nu\delta}{}_{,\beta} + \tfrac{1}{8}\left(2g^{\mu\gamma}g^{\delta\nu} - g^{\mu\nu}g^{\gamma\delta}\right)\left(2g_{\alpha\beta}g_{\epsilon\eta} - g_{\beta\epsilon}g_{\alpha\eta}\right)\hat{g}^{\alpha\eta}{}_{,\gamma}\hat{g}^{\beta\epsilon}{}_{,\delta}\Big\} .
\end{aligned}
\tag{C-2}
$$

In the weak-field linear perturbation approximation, however, the pseudotensor $t_{\mu\nu}$ can be found more directly than by Eqs (C-1) or (C-2). The procedure is mentioned, for example, by Weber (1962). Multiplying the wave Eq (3-16)

$$\Box \gamma_{\alpha\beta} = -2\kappa T_{\alpha\beta} \tag{3-16}$$

by $h^{\cdot\beta}{}_{,\mu}$ one arrives at

$$h^{\alpha\beta}{}_{,\mu}\Box\gamma_{\alpha\beta} = -2\kappa h^{\alpha\beta}{}_{,\mu}T_{\alpha\beta} . \tag{C-3}$$

The covariant divergence in (3-3) of the stress energy tensor is

$$T^{\mu\nu}{}_{;\nu} = \left(T'^{\mu\nu} + t^{\mu\nu}\right)_{;\nu} = \frac{1}{\sqrt{-g}}\left(T^{\mu\nu}\sqrt{-g}\right)_{,\nu} - \frac{1}{2} g_{\alpha\beta,}{}^{\mu} T^{\alpha\beta} = 0. \tag{C-4}$$

Furthermore, the stress energy tensor density $\hat{T}^{\mu\nu}$ and the stress energy pseudotensor density $\hat{t}^{\mu\nu}$ are related by

$$\hat{T}^{\mu\nu}{}_{,\nu} + \hat{t}^{\mu\nu}{}_{,\nu} = 0 \ . \tag{C-5}$$

From (C-5) and (C-4), then

$$\hat{t}^{\mu\nu}{}_{,\nu} = -\frac{1}{2} g_{\alpha\beta,}{}^{\mu} \hat{T}^{\alpha\beta} \approx -\frac{1}{2} h_{\alpha\beta}{}^{,\mu} T^{\alpha\beta}$$

$$h^{\alpha\beta}{}_{,\mu} T_{\alpha\beta} = -2\hat{t}_{\mu\nu}{}^{,\nu} . \tag{C-6}$$

Hence, (C-3) can be re-stated, with (C-6), as

$$h^{\alpha\beta}{}_{,\mu} \Box \gamma_{\alpha\beta} = +4\kappa \hat{t}_{\mu\nu}{}^{,\nu} . \tag{C-7}$$

Expanding the left-hand side of (C-7), one obtains (C-8)

$$h^{\alpha\beta}{}_{,\mu}\left[h_{\alpha\beta,\nu}{}^{,\nu} - \frac{1}{2}\eta_{\alpha\beta} h_{,\nu}{}^{,\nu}\right] = \left[h^{\alpha\beta}{}_{,\mu} h_{\alpha\beta}{}^{,\nu} - \frac{1}{2}\eta_{\mu}{}^{\nu} h_{\alpha\beta,\epsilon} h^{\alpha\beta,\epsilon} - \frac{1}{2} h_{,\mu} h^{,\nu} + \frac{1}{4} h_{,\epsilon} h^{,\epsilon} \eta_{\mu}{}^{\nu}\right]_{,\nu}$$

Direct comparison of (C-7) and (C-8) - recalling Eq (B-7) for $\gamma^{\mu\nu}$ - gives the first-order stress energy pseudotensor density $\hat{t}^{\mu\nu}=\sqrt{-g}\ t^{\mu\nu}$: (C-9)

$$\hat{t}_{\mu}{}^{\nu} = \frac{1}{4\kappa}\left[2h^{\alpha\beta}{}_{,\mu} h_{\alpha\beta}{}^{,\nu} - h_{,\mu} h^{,\nu} + \eta_{\mu}{}^{\nu}\left(\tfrac{1}{2} h_{,\epsilon} h^{,\epsilon} - h_{\alpha\beta,\epsilon} h^{\alpha\beta,\epsilon}\right)\right].$$

For the particular case of a perturbation $h_{\mu\nu}$ propagating in the X^1 direction (discussed in Chapter 3, Eq 3-25), and noting that $\sqrt{-g}=1$ for a flat background $\eta_{\mu\nu}$, the (1,0) component of $\hat{t}_{\mu}{}^{\nu}$ in (C-9) is

$$t_1{}^0 = \frac{1}{4\pi\sqrt{-g}}\left[2h^{\alpha\beta}{}_{,1} h_{\alpha\beta}{}^{,0} - \cancel{h_{,1} h^{,0}} + \cancel{\eta_{1}{}^{0}(\cdots)}\right]. \tag{C-10}$$

Whence, the gravitational energy flux (Poynting vector) is determined by

$$t^{10} = -\frac{c^4}{32G}\left[\frac{\partial h_{22}}{\partial x^1}\frac{\partial h_{22}}{\partial(ct)} + \frac{\partial h_{33}}{\partial x^1}\frac{\partial h_{33}}{\partial(ct)} + 2\frac{\partial h_{23}}{\partial x^1}\frac{\partial h_{23}}{\partial(ct)}\right] . \tag{C-11}$$

The first-order pseudotensor $t_{\mu}{}^{\nu}$ in (C-9) provides the basis for studying radiation spectra and energy radiated in this approximation.

APPENDIX D: CONSERVATION OF MOMENTUM & THE EQUIVALENCE OF GRAVITATIONAL & INERTIAL MASS

Angular Momentum & The Pseudotensor

Having addressed the Landau-Lifshitz pseudotensor in Appendix C, Eq (C-1) and (C-2), we need to assess its relation to conservation laws. The conservation law

$$\frac{\partial}{\partial x^\nu}(-g)\left(T'^{\mu\nu}+t^{\mu\nu}\right)=0 \qquad \text{(D-1)}$$

is satisfied by the 4-momentum P^μ

$$P^\mu=\frac{1}{c}\int(-g)\left(T'^{\mu\nu}+t^{\mu\nu}\right)dS_\nu \qquad \text{(D-2)}$$

which can be written in the form of a 3-space integral

$$P^\mu=\frac{1}{c}\int(-g)\left(T'^{\mu 0}+t^{\mu 0}\right)dV\ . \qquad \text{(D-3)}$$

Because the quantity in the integrand of (D-2) is symmetric in the indices μ,ν we obtain a conservation law of angular momentum

$$M^{\mu\nu}=\int\left(x^\mu dP^\nu-x^\nu dP^\mu\right)=\frac{1}{c}\int\left\{x^\mu\left(T'^{\nu\lambda}+t^{\nu\lambda}\right)-x^\nu\left(T'^{\mu\lambda}+t^{\mu\lambda}\right)\right\}(-g)\,dS_\lambda, \qquad \text{(D-4)}$$

for a closed gravitating system in General Relativity. Because of this conservation condition, one can define a center of inertia system moving under uniform motion. The center of inertia is defined by M^{0i}= const in (D-4), whereby its coordinates are determined by

$$R^\alpha=\frac{\int x^\alpha\left(T'^{00}+t^{00}\right)(-g)\,dV}{\int\left(T'^{00}+t^{00}\right)(-g)\,dV}\ . \qquad \text{(D-5)}$$

By performing gauge transformations such as (D-5) one can create or annihilate the pseudotensor $t^{\mu\nu}$, which characterizes the energy and angular momentum of gravitational waves - the argument of Scheidegger & Infeld. However, $t^{\mu\nu}$ does not vanish globally under such transformations.

Equivalence Of Gravitational & Inertial Mass

Landau & Lifshitz consider the purely gravitational case ($T'^{\mu\nu}=0$) and define

$$\eta^{\mu\nu\lambda} = \frac{1}{2\kappa}\frac{\partial}{\partial x^\rho}\left[\hat{g}^{\mu\nu}\hat{g}^{\lambda\rho} - \hat{g}^{\nu\lambda}\hat{g}^{\mu\rho}\right] \qquad \text{(D-6)}$$

whereby P^μ in (D-2) is an ordinary 3-space integral (i=1,2,3)

$$P^\mu = \frac{1}{c}\oint \eta^{\mu 0 i}\, dV_i \ . \qquad \text{(D-7)}$$

Demanding that the metric $g_{\mu\nu}$ must be asymptotically flat (Minkowskian) at infinity, they evaluate (D-6) for the metric (where M is the total mass of the system; $\hat{n}$ is a unit vector along $\underset{\sim}{r}$)

$$g_{00} = 1 - \frac{2GM}{c^2 r} \qquad g_{ij} = -\delta_{ij} - \frac{2GM}{c^2}\frac{n_i n_j}{r} \qquad g_{0i} = 0 \qquad \text{(D-8a)}$$

which gives

$$h^{i0j} = 0 \qquad h^{00i} = \frac{Mc^2}{4\pi}\frac{n^i}{r^2} \qquad \text{(D-8b)}$$

and (D-7) becomes

$$\underset{\sim}{P} = 0 \qquad\qquad P^0 = Mc \ . \qquad \text{(D-9)}$$

In the center-of-inertia asymptotically-flat frame (D-8a), 4-momentum is determinable and it is conserved. This is a statement of the Newtonian equivalence of gravitational and inertial mass discussed in the text. There is no outgoing or ingoing gravitational dipole radiation at infinity because

$$\mathbb{P}_{D1} \sim \dot{\underset{\sim}{P}}^2 = 0 \ , \qquad \text{(D-10)}$$

from (D-9).

Failure Of Conservation Laws

One cannot always formulate a principle of conservation of angular momentum, whereupon the notion of a center of inertia loses its mathematical basis. Such is the case for certain interactions, or if the integrand in (D-2) is not symmetric in the indices μ,ν - which is precisely what happens when one includes the Electromagnetic stress-energy tensor ($T'^{\mu\nu}\neq 0$). Landau & Lifshitz determine (p. 311) that (D-2) becomes

$$P_\mu = \frac{1}{c}\int\left\{ T'^{\nu}_{\mu} + \frac{1}{2\kappa}\left[\hat{G}\delta^{\nu}_{\mu} + \Gamma^{\nu}_{\lambda\delta}\hat{g}^{\lambda\delta}{}_{,\mu} - \Gamma^{\lambda}_{\delta\lambda}\hat{g}^{\delta\nu}{}_{,\mu}\right]\right\} dS_\nu , \tag{D-11}$$

where

$$\hat{G} = \sqrt{-g}\, g^{\mu\nu}\left(\Gamma^{\delta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\delta} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\delta}_{\lambda\delta}\right). \tag{D-11a}$$

This result is relevant to an inequivalence of gravitational and inertial mass. Eq (D-11) is an expression for gravitational umklapp.

APPENDIX E: POST-NEWTONIAN GENERAL RELATIVITY

We are concerned with the equations of motion (3-45)

$$\underset{(\lambda)}{\ddot{X}^{\mu}} = -\underset{(\epsilon)}{\Gamma^{\mu}_{\alpha\beta}}\dot{X}^{\alpha}\dot{X}^{\beta} \qquad (3\text{-}45)$$

in the EIH approximation of Figures 2 & 3 to some order $\epsilon^n = c^{-n}$. The equation of motion (3-45) represents a geodesic defined by the affine connections $\Gamma^{\mu}_{\alpha\beta}$ which can be expanded as follows:

$$\underset{(\lambda^{\mu}=\epsilon^{n+2})}{\ddot{X}^{i}} = -\underset{(\epsilon^{n})}{\Gamma^{i}_{00}} - 2\underset{(\epsilon^{n-1})}{\Gamma^{i}_{0\beta}}\dot{X}^{\beta} - \underset{(\epsilon^{n-2})}{\Gamma^{i}_{\alpha\beta}}\dot{X}^{\alpha}\dot{X}^{\beta} + \left[\underset{(\epsilon^{n-1})}{\Gamma^{0}_{00}} + 2\underset{(\epsilon^{n-2})}{\Gamma^{0}_{0\beta}}\dot{X}^{\beta} + \underset{(\epsilon^{n-3})}{\Gamma^{0}_{\alpha\beta}}\dot{X}^{\alpha}\dot{X}^{\beta}\right]\dot{X}^{i}. \qquad (E\text{-}1)$$

The powers of ϵ on the right-hand side of (E-1) reflect the order to which the Christoffel symbol must be expanded in $g_{\mu\nu}$ in Eq (A-1). Note that each time-differentiation (the dots) lowers the order of c by c^{-1}.

<u>Newtonian Approximation $O(\epsilon^2, \lambda^4)$</u>

This is the left-hand line of Figure 2 in the text. Analyzing (E-1), to order $O(\epsilon^2,\lambda^4)$ there are the following contributions:

$$\epsilon^2: \ \Gamma^{\mu}_{00}$$

$$\epsilon^1: \ \Gamma^{\mu}_{0\beta}, \Gamma^{0}_{00} \ .$$

Thus, Eq (E-1) is

$$\ddot{X}^{i} = -\Gamma^{i}_{00} + \cancelto{\simeq 0}{\left[\Gamma^{0}_{00} - 2\Gamma^{i}_{0j}\right]}\dot{X}^{j}. \qquad (2)$$

From (A-1) for a stationary field to 1st-order in $h_{\mu\nu}$

$$\Gamma^{\mu}_{00} = -\frac{1}{2}g^{\mu\nu}g_{00,\nu} \simeq -\frac{1}{2}\eta^{\mu\nu}h_{00,\nu}.$$

Hence

$$\underset{\sim}{\ddot{X}} = \frac{1}{2}\nabla h_{00} = -\frac{1}{c^2}\nabla\phi .$$

Correspondence with Newtonian mechanics gives, then,

$$h_{00} = -\frac{2\phi}{c^2}$$

whereby from (A-4)

$$g_{00} = -(1 + 2\phi/c^2). \qquad (E\text{-}3)$$

Post$^{0.5}$-Newtonian Approximation $O(\varepsilon^3,\lambda^5)$

As depicted in Figure 2, there is no contribution at order $O(\varepsilon^3,\lambda^5)$. This is due to the conservation of energy and momentum.

Post$^{1.0}$-Newtonian Approximation $O(\varepsilon^4,\lambda^6)$

The post$^{1.0}$-Newtonian forces manifest themselves as the second line of Figure 2. They are comprised of the metric contributions $\underset{4}{g}_{oo}$, $\underset{3}{g}_{ok}$, and $\underset{2}{g}_{ik}$. From the equation of motion (E-1), we require

$$\epsilon^4:\ \Gamma^{\mu}_{oo} \qquad\qquad \epsilon^2:\ \Gamma^{\mu}_{\alpha\beta},\ \Gamma^{o}_{o\beta}$$

$$\epsilon^3:\ \Gamma^{\mu}_{o\beta},\ \Gamma^{o}_{oo} \qquad\qquad \epsilon^1:\ \Gamma^{o}_{\alpha\beta}$$

Calculating the Christoffel symbols $\Gamma^{\mu}_{\alpha\beta}$ to these orders of c^{-n} using (A-1):

$$\begin{aligned}
\epsilon^4:&\ \left\{ \overset{(4)}{\Gamma}{}^{i}_{oo} = -\tfrac{1}{2}\overset{(4)}{g}_{oo,i} + \overset{(3)}{g}_{io,o} + \tfrac{1}{2}\overset{(2)}{g}_{ij}\overset{(2)}{g}_{oo,j} \right. \\
\epsilon^3:&\ \left\{ \begin{aligned} \overset{(3)}{\Gamma}{}^{i}_{oj} &= \tfrac{1}{2}\left[\overset{(3)}{g}_{io,j} + \overset{(2)}{g}_{ij,o} - \overset{(3)}{g}_{jo,i}\right] \\ \overset{(3)}{\Gamma}{}^{o}_{oo} &= -\tfrac{1}{2}\overset{(3)}{g}_{oo,o} \end{aligned} \right. \\
\epsilon^2:&\ \left\{ \begin{aligned} \overset{(2)}{\Gamma}{}^{i}_{oo} &= -\tfrac{1}{2}\overset{(2)}{g}_{oo,i} \\ \overset{(2)}{\Gamma}{}^{o}_{oi} &= -\tfrac{1}{2}\overset{(2)}{g}_{oo,i} \end{aligned} \right. \\
\epsilon^1:&\ \left\{ \overset{(1)}{\Gamma}{}^{o}_{ij} = 0 \right.
\end{aligned} \qquad \text{(E-4)}$$

For the Ricci tensor (A-3) to this order of ε^4, one has

$$\begin{aligned}
\overset{(4)}{R}_{oo} &= \overset{(3)}{\Gamma}{}^{i}_{oi,o} - \overset{(4)}{\Gamma}{}^{i}_{oo,i} + \overset{(2)}{\Gamma}{}^{o}_{oi}\overset{(2)}{\Gamma}{}^{i}_{oo} - \overset{(2)}{\Gamma}{}^{i}_{oo}\overset{(2)}{\Gamma}{}^{j}_{ij} \\
\overset{(3)}{R}_{io} &= \overset{(2)}{\Gamma}{}^{j}_{ij,o} - \overset{(2)}{\Gamma}{}^{j}_{oi,j} \\
\overset{(3)}{R}_{oo} &= -\overset{(2)}{\Gamma}{}^{i}_{oo,i} \\
\overset{(2)}{R}_{ij} &= \overset{(2)}{\Gamma}{}^{o}_{io,j} + \overset{(2)}{\Gamma}{}^{k}_{ik,j} - \overset{(2)}{\Gamma}{}^{k}_{ij,k}
\end{aligned} \qquad \text{(E-5)}$$

Plugging (E-4) into (E-5) we get the following Ricci equations:

$$\begin{aligned}
\overset{(4)}{R}_{oo} &= \tfrac{1}{2}\overset{(2)}{g}_{ii,oo} - \overset{(3)}{g}_{io,io} + \tfrac{1}{2}\nabla^2\overset{(4)}{g}_{oo} - \tfrac{1}{2}\overset{(2)}{g}_{ij}\overset{(2)}{g}_{oo,ij} - \tfrac{1}{2}\overset{(2)}{g}_{ij,j}\overset{(2)}{g}_{oo,i} + \tfrac{1}{4}\overset{(2)}{g}_{oo,i}\overset{(2)}{g}_{oo,i} + \tfrac{1}{4}\overset{(2)}{g}_{oo,i}\overset{(2)}{g}_{ij,i} \\
\overset{(3)}{R}_{io} &= \tfrac{1}{2}\overset{(2)}{g}_{jj,io} - \tfrac{1}{2}\overset{(3)}{g}_{jo,ij} - \tfrac{1}{2}\overset{(2)}{g}_{ij,jo} + \tfrac{1}{2}\nabla^2\overset{(3)}{g}_{io} \\
\overset{(2)}{R}_{oo} &= \tfrac{1}{2}\nabla^2\overset{(2)}{g}_{oo} \\
\overset{(2)}{R}_{ij} &= -\tfrac{1}{2}\overset{(2)}{g}_{oo,ij} + \tfrac{1}{2}\overset{(2)}{g}_{kk,ij} - \tfrac{1}{2}\overset{(2)}{g}_{ik,kj} - \tfrac{1}{2}\overset{(2)}{g}_{kj,ki} + \tfrac{1}{2}\nabla^2\overset{(2)}{g}_{ij} \quad .
\end{aligned} \qquad \text{(E-6)}$$

Transforming (E-6) to harmonic coordinates (B-10), the Ricci equations simplify to

$$
\begin{aligned}
\overset{(4)}{R}_{00} &= \tfrac{1}{2}\nabla^2 \overset{(4)}{g}_{00} - \tfrac{1}{2}\overset{(2)}{g}_{00,00} + \tfrac{1}{2}\nabla^2 \overset{(2)}{g}_{00} - \tfrac{1}{2}\overset{(2)}{g}_{ij}\overset{(2)}{g}_{00,ij} + \tfrac{1}{2}\left(\nabla \overset{(2)}{g}_{00}\right)^2 \\
\overset{(3)}{R}_{0i} &= \tfrac{1}{2}\nabla^2 \overset{(3)}{g}_{i0} \\
\overset{(2)}{R}_{00} &= \tfrac{1}{2}\nabla^2 \overset{(2)}{g}_{00} \\
\overset{(0)}{R}_{ij} &= \tfrac{1}{2}\nabla^2 \overset{(2)}{g}_{ij}
\end{aligned}
\qquad \text{(E-7)}
$$

Einstein's field equations now follow from (3-14), using (E-7). Taking caution with the c^{-2} factors in the stress-energy tensor $T^{\mu\nu}$, substitution of (E-7) into (3-14) gives to order $O(\varepsilon^4, \lambda^6)$:

$$
\nabla^2 \overset{(4)}{g}_{00} = \overset{(2)}{g}_{00,00} + \overset{(2)}{g}_{ij}\overset{(2)}{g}_{00,ij} - \overset{(2)}{g}_{00,i}\overset{(2)}{g}_{00,i} - \frac{8\pi G}{c^2}\left[\overset{(2)}{T}{}^{00} - \frac{2}{c^2}\overset{(0)}{g}_{00}\overset{(0)}{T}{}^{00} + \overset{(2)}{T}{}^{ii}\right] \qquad \text{(E-8a)}
$$

$$
\nabla^2 \overset{(3)}{g}_{i0} = \frac{16\pi G}{c^2}\overset{(1)}{T}{}^{i0} \qquad \text{(E-8b)}
$$

$$
\nabla^2 \overset{(2)}{g}_{00} = -\frac{8\pi G}{c^2}\overset{(0)}{T}{}^{00} \qquad \text{(E-8c)}
$$

$$
\nabla^2 \overset{(0)}{g}_{ij} = -\frac{8\pi G}{c^2}\delta_{ij}\overset{(0)}{T}{}^{00} \qquad \text{(E-8d)}
$$

POST$^{1.0}$-NEWTONIAN FIELD EQUATIONS

From Eq (E-8c) we arrive at the Newtonian result in (E-3)

$$
\overset{(2)}{g}_{00} = -2\frac{\phi}{c^2} \quad . \qquad \text{(E-3')}
$$

The Newtonian potential ϕ is that defined by Poisson's Eq (2-6)

$$
\nabla^2\phi = 4\pi G\overset{(0)}{T}{}^{00}
$$

$g_{\mu\nu}$ must be asymptotically flat. Hence, in order that $\overset{(2)}{g}_{00}$ vanish at infinity, the scalar potential ϕ is identical to (2-8)

$$
\boxed{\phi = -G\int \frac{[T^{00}]}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d^3\underset{\sim}{x}'} \qquad \text{(E-9)}
$$

The solution of (E-8b) is the vector potential $A_i = \overset{3}{g}_{io}$:

$$\boxed{A_i = -\frac{4G}{c^2}\int \frac{\left[\overset{(1)}{T}{}^{io}\right]}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, d^3\underset{\sim}{x}'} \qquad \text{(E-10)}$$

which is identical to that in the electromagnetic theory of gravitation discussed in Chapter 2.

In order that $\overset{2}{g}_{ij}$ vanish at infinity the solution of (E-8d) is

$$\overset{(2)}{g}_{ij} = -2\,\delta_{ij}\frac{\phi}{c^2} \qquad \text{(E-11)}$$

The remaining field equation (E-8a) is solved using (E-3') and (2-6') whereby

$$\nabla^2\left[\overset{(4)}{g}_{oo} + 2\left(\frac{\phi}{c^2}\right)^2\right] = -\frac{2}{c^2}\left[\phi_{,oo} + \frac{4\pi G}{c^2}\left(\overset{(2)}{T}{}^{oo} + \overset{(2)}{T}{}^{ii}\right)\right] . \qquad \text{(E-12)}$$

If we assume (Weinberg, 1972, following Møller, 1962)

$$\overset{(4)}{g}_{oo} = -2\left(\frac{\phi}{c^2}\right)^2 - 2\psi \quad , \qquad \text{(E-13)}$$

then (E-11) becomes

$$\nabla^2\psi = \frac{1}{c^2}\left[\phi_{,oo} + 4\pi G\left(\overset{(2)}{T}{}^{oo} + \overset{(2)}{T}{}^{ii}\right)\right] \qquad \text{(E-14)}$$

which has the scalar solution

$$\boxed{\psi = -\frac{G}{c^2}\int \frac{\left[\frac{1}{4\pi G}\phi_{,oo} + \overset{(2)}{T}{}^{oo} + \overset{(2)}{T}{}^{ii}\right]}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, d^3\underset{\sim}{x}'} \qquad \text{(E-15)}$$

The harmonic condition used earlier imposes the following relation between the scalar potential ϕ and the vector potential A:

$$\frac{4}{c^3}\frac{\partial\phi}{\partial t} + \nabla\cdot\underset{\sim}{A} = 0 \quad . \qquad \text{(E-16)}$$

Using (E-3'), (E-11), and (E-13) in conjunction with the solutions ϕ, $\underset{\sim}{A}$, and ψ, the Christoffel connections (E-4) become

$$
\begin{aligned}
\overset{(4)}{\Gamma}{}^{i}_{00} &= \left[2\left(\frac{\phi}{c^2}\right)^2+\psi\right]_{,i} + A_{i,0} \\
\overset{(3)}{\Gamma}{}^{i}_{0j} &= \frac{1}{2}\left[A_{i,j}-A_{j,i}\right] - \delta_{ij}\left(\frac{\phi}{c^2}\right)_{,0} \\
\overset{(3)}{\Gamma}{}^{0}_{00} &= \frac{1}{c^2}\phi_{,0} \\
\overset{(3)}{\Gamma}{}^{i}_{00} &= \frac{1}{c^2}\phi_{,i} \qquad\qquad \text{(E-17)} \\
\overset{(3)}{\Gamma}{}^{0}_{0i} &= \frac{1}{c^2}\phi_{,i} \quad .
\end{aligned}
$$

Substituting (E-17) back into E-1) we obtain the "post$^{1.0}$-Newtonian" equations of motion: (E-18)

$$
\boxed{
\begin{aligned}
c^2\ddot{\underset{\sim}{X}} = &-\underset{\sim}{\nabla}\left[\phi+2\left(\frac{\phi}{c}\right)^2+c^2\psi\right]-\frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t}+\underset{\sim}{v}\times(\nabla\times\underset{\sim}{A}) \\
&+\frac{3}{c}\underset{\sim}{v}\frac{\partial\phi}{\partial t}+4\underset{\sim}{v}(\underset{\sim}{v}\cdot\underset{\sim}{\nabla})\phi-v^2\underset{\sim}{\nabla}\phi
\end{aligned}
}
$$

POST$^{1.0}$-NEWTONIAN EQUATION OF MOTION

Note carefully that these are simply scalar and vector potentials. Furthermore, note that Eq (E-18) contains the Lorentz equation of motion (2-15) from the electromagnetic theory of gravitation in Chapter 2.

APPENDIX F:

SOME FEATURES OF THE ELECTROMAGNETIC THEORY OF GRAVITATION

Having derived the equations of motion for the "post$^{1.0}$-Newtonian" approximation of the tensor (Einstein) theory (Appendix, Eq E-18), we observe the remarkable similarities which exist between it and the vector (electromagnetic) theory of gravitation - primarily the use of a vector potential. This sort of correspondence also manifests itself in the matched asymptotic expansion method which decomposes the tensor field equations into scalar, vector, and Maxwell stress-tensor components in Eq (3-49).

The Classical Basis For The Vector Potential In Gravitation Theory

A simple hydrodynamic argument constitutes the teleological basis for invoking a vector potential into the Newtonian field equations (2-5) of Chapter 2. These equations for ϕ are irrotational. Hydrodynamically, a particle orbiting in an irrotational fluid acquires no intrinsic spin. If it does, the fluid is rotational, possessing a curl. Now extending this argument to the solar system, one is struck by the experimental fact that all of the planets possess an intrinsic angular momentum ($\underset{\sim}{S}$) in addition to their orbital angular momentum ($\underset{\sim}{L}$).

Of course, we know that a ponderable, elastic body (with Love numbers) which is some form of ellipsoid (formed cosmogonically) will acquire an intrinsic spin $\underset{\sim}{S}$ which is commensurable with its orbital angular momentum $\underset{\sim}{L}$ in the scalar potential $\phi \sim r^{-1}$ of Newtonian theory. This is due to the gravitational gradient, inductive energy dissipation, and the spin-orbit interaction mechanism of, say, tidal friction (Appendices G & S). In other words, just because planets spin one does not necessarily need a vector potential to supplement Eq (2-5).

Nevertheless, the gravitational hydrodynamics of the solar system does establish an experimental basis for invoking a vector potential, and thereby achieve an electromagnetic theory of gravitation, as in Eq (2-12). The vector theory of gravitation in Eq (2-12) becomes even more interesting when one considers the origin of magnetic fields and the nature of the Coriolis force.

Rotation Of Mass & The Origin Of Magnetic Fields

H.A. Wilson (1921, 1923) points out that there may exist a correlation between the rotation of mass and the origin of magnetic fields - basing his query upon the experimental fact that the ratio of the magnetic moment to the angular momentum for the sun is the same as it is for the earth. Wilson also investigates a similar line of reasoning in the electromagnetic theory of gravitation by supposing the "relativity" of gravitational fields. In the same fashion that a uniformly moving electric charge exhibits an apparent magnetic field, then perhaps a moving neutral mass possesses an apparent magnetic field. Wilson then performs an experiment, but measures no such effect.

Blackett (1947) pursues Wilson's ideas and advocates the formula

$$\underset{\sim}{P} = -\beta \frac{G^{1/2}}{c} \underset{\sim}{U} \qquad \text{(F-1)}$$

as the linear relation between the angular momentum $\underset{\sim}{U}$ and the magnetic moment $\underset{\sim}{P}$ (where β is a dimensionless constatn of the order of unity). Fuchs (1948) generalizes (F-1) as

$$P = \tfrac{1}{2}\mu\sqrt{\epsilon G}\, U = \mu c \mu_B U \,, \qquad \text{(F-2)}$$

where μ is the gravitational permeability, ϵ is the permittivity, and μ_B is the Bohr magneton.

The Lorentz Force & The Coriolis Term

Wilson's conjecture is too naive - as his experimental results determine. That is, he considers the analog of the Lorentz force stated in Eq (2-15)

$$\underset{\sim}{f} = \rho\left[\underset{\sim}{g} + \underset{\sim}{v} \times \underset{\sim}{b}\right] \qquad \text{(2-15)}$$

and surmises that the axial vector $\underset{\sim}{b}$ for the "gravitational induction" field is a magnetic field $\underset{\sim}{B}$ - just as in Maxwell's relations (2-11).

However, from classical mechanics we already have a force term exactly like the Larmor-Lorentz term in Eq (2-15), namely the Coriolis force

$$f_{\text{Coriolis}} = +2\rho\,\underset{\sim}{\omega} \times \underset{\sim}{v} = -\rho\,\underset{\sim}{v} \times (2\underset{\sim}{\omega}) \,. \qquad \text{(F-3)}$$

Whence, we can make the identity

$$\underset{\sim}{b} = -2\underset{\sim}{\omega} \,, \qquad \text{(F-4)}$$

whereby the Lorentz force (2-15) becomes

$$\underset{\sim}{f} = \rho\left[\underset{\sim}{g} + \underset{\sim}{v} \times (2\underset{\sim}{\omega})\right]. \qquad \text{(F-5)}$$

Result (F-5) consists of the Newtonian (Coulomb) force and the Coriolis force.

A re-interpretation of Wilson's conjecture, then, is to state that there is a relation between mass in uniform motion with spin and the Coriolis force. Likewise, in the same fashion that one has motionally induced electric fields

$$\underset{\sim}{E} = -\underset{\sim}{v} \times \underset{\sim}{B} \qquad \text{(F-6a)}$$

for equilibrium states ($\underset{\sim}{E}$=0 or high conductivity), one also achieves motionally induced gravitational fields

$$\underset{\sim}{g} = -\underset{\sim}{v} \times (2\underset{\sim}{\omega}) \quad . \qquad \text{(F-6b)}$$

Eq (F-6b) is the electrodynamic way of stating that "fictitious forces" behave like gravitational ones and are both characterized by $g_{\mu\nu}$ in $\Gamma^{\alpha}_{\mu\nu}$ (Eq A-1) of the equation of motion

$$\ddot{X}^{\alpha} + \Gamma^{\alpha}_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} = 0 \quad . \qquad \text{(3-45)}$$

It should also be transparent that the electromagnetic theory of gravitation is Machian* in nature, attributing inertia to the inductive effect of distant cosmological matter.

The gravitational vector potential $\underset{\sim}{A}$ in Eq (2-19) arises in the presence of a hydrodynamic mass current

$$\dot{\underset{\sim}{j}} = \rho \underset{\sim}{v} \qquad \text{(2-16)}$$

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*Epistemologically speaking, however, the Machian school of thought finds its principal experimental basis in electrodynamics. Hence electrodynamics and the electromagnetic theory of gravitation stand on their own. Mach's work, in fact, is a philosophical derivative of Maxwell's electromagnetism - particularly in light of Maxwell's work on gravitation, available to Mach. The vector theory of gravitation does not have to answer to Mach.

in Eq (2-16). If $\underset{\sim}{v}$ is rotational, that is, if

$$\left.\begin{aligned} \underset{\sim}{v} &= \underset{\sim}{\omega} \times \underset{\sim}{r} \\ 2\underset{\sim}{\omega} &= \underset{\sim}{\nabla} \times \underset{\sim}{v} \end{aligned}\right\} \qquad \text{(F-7)}$$

we have directly from (2-19), if $\underset{\sim}{v}$ can be moved through the integral sign,

$$\underset{\sim}{A} = -\frac{G}{c}\int \frac{\rho \underset{\sim}{v}}{r'} dv' = -\underset{\sim}{v}\frac{G}{c^2}\int \frac{\rho}{r'} dv' = -\underset{\sim}{v}\,\phi/c \,. \qquad \begin{matrix}\text{(2-19)}\\ \text{(F-8)}\end{matrix}$$

$$\underset{\sim}{b} = \nabla \times \underset{\sim}{A} = \frac{\underset{\sim}{v}}{c} \times \nabla\phi - \frac{\phi}{c}\nabla \times \underset{\sim}{v} \,. \qquad \text{(F-9)}$$

In order to establish a basis for the origin of magnetic fields in rotating mass, one should turn to the continuity equation (2-27d). For neutral matter (ρ_e=0)

$$\nabla \cdot \underset{\sim}{J} = -\nabla \cdot \underset{\sim}{j} - \frac{\partial \rho_J}{\partial t} \,. \qquad \text{(2-27d')}$$

One can then argue that electromagnetic currents $\underset{\sim}{J}$ arise in the presence of mass currents $\underset{\sim}{j}$. This sort of reasoning can be used to justify Blackett's relation (F-1).

The Coriolis Vector Potential & Gravitational Radiation

If we now temporarily adopt Eq (F-9) as a definition of the gravitational vector potential $\underset{\sim}{A}$, from which derives the Coriolis force, we can quickly surmise the behaviour of the field equations.

First of all, the energy density in (2-23a) is

$$\mathcal{E} = -\frac{2c^2}{\mathcal{K}}\omega^2 \,. \qquad \text{(F-10)}$$

The "gravitational induction" pressure (magnetic pressure) is

$$p_g = \frac{(2\omega)^2}{8\pi} = \frac{\omega^2}{2\pi} \qquad \text{(F-11a)}$$

$$\left(p_e = \frac{B^2}{8\pi}\right)$$

while (from plasma theory) the pressure of matter p is related to it by

$$p + \frac{\omega^2}{2\pi} = \text{Const.} \qquad \text{(F-11b)}$$

$$\left(p + \frac{B^2}{8\pi} = \text{Const.}\right)$$

The cyclotron frequency (eB/m) becomes

$$\omega_c = \frac{mb}{m} = b = 2\omega, \qquad \text{(F-12)}$$

while the Larmor frequency is

$$\omega_L = \frac{1}{2}\omega_c = \frac{1}{2}b = \omega. \qquad \text{(F-13)}$$

The spin $\underset{\sim}{\omega}$ of the vector gravitational field is precisely the Larmor frequency of electromagnetism.

The $\underset{\sim}{E}\times\underset{\sim}{B}$ drift velocity perpendicular to $\underset{\sim}{\omega}$ or $\underset{\sim}{b}$ is

$$\underset{\sim}{v}_{d_\perp} = 2\,\frac{\underset{\sim}{g}\times\underset{\sim}{\omega}}{\omega^2}, \qquad \text{(F-14)}$$

while there is an adiabatic invariant (Coriolis mirror) given by

$$\frac{\sin\alpha}{b} = \frac{\sin\alpha}{2\omega} = \frac{1}{2\omega_L} = \text{Const.} \qquad \text{(F-15)}$$

Likewise, a "plasma oscillation" follows in the usual fashion from the Maxwell equations (2-12) and the continuity equation (2-17),

$$\omega_p = \sqrt{\frac{nm^2}{\epsilon_0 m}} = \sqrt{nm/\epsilon_0} = \sqrt{4\pi c^2 nm} = 2c\sqrt{\pi nm}. \qquad \text{(F-16)}$$

$$\left(\omega_p = \sqrt{\frac{ne^2}{\epsilon_0 m}}\right)$$

The magnetic moment is precisely the mechanical angular momentum $\underset{\sim}{L}$,

$$\underset{\sim}{L} = \frac{1}{2}\int \rho_g (\underset{\sim}{\xi}\times\underset{\sim}{v})\,dV \qquad \text{(F-17)}$$

$$\left(\underset{\sim}{\mu} = \frac{1}{2}\int \rho_e (\underset{\sim}{\xi}\times\underset{\sim}{v})\,dV\right)$$

The torque acting upon a spinning mass of angular momentum $\underset{\sim}{L}$ in the Coriolis field $2\underset{\sim}{\omega}$ is

$$T = \underset{\sim}{L}\times\underset{\sim}{b} \qquad \text{(F-18)}$$

$$(T = \underset{\sim}{\mu}\times\underset{\sim}{B})$$

The Poynting vector is $\underset{\sim}{S} = \underset{\sim}{g} \times \underset{\sim}{h}$ in Eq (2-18), or

$$\underset{\sim}{S} = \frac{1}{\mu} \underset{\sim}{g} \times \underset{\sim}{b} = \frac{2}{\mu} \underset{\sim}{g} \times \underset{\sim}{\omega} \qquad \text{(F-19)}$$

where the polarization of $\underset{\sim}{S}$ is determined by $\underset{\sim}{g}$.

Finally, it should be pointed out that the field intensity $\underset{\sim}{g}$ is derived from (2-13a)

$$\underset{\sim}{g} = -\nabla\Phi - \frac{\partial \underset{\sim}{A}}{\partial t} \quad . \qquad \text{(2-13a)}$$

Φ is really a centrifugal potential, containing both the Newtonian ϕ as well as centrifugal effects due to rotation of the source of the field. $\partial_t \underset{\sim}{A}$ then includes the time-dependence of the Coriolis term.

Arbitrary Vector Potential

There is nothing implicit in the field equations (2-12) that demands a strict identification of the gravitational induction $\underset{\sim}{b}$ with the Coriolis effect $2\underset{\sim}{\omega}$, although it should be consistent with it. One can consider a ring current $\underset{\sim}{j}$ of mass, using the vector potentials from electromagnetism, and determine the gravitational induction field $\underset{\sim}{b}$ in spherical coordinates as a solution of (2-19) (Smythe, p. 294-295),

$$\underset{\sim}{A} = -\frac{2\pi G}{c^2}\Phi j \sum_{n=1} \frac{\sin\alpha}{n(n+1)} \left(\frac{r}{a}\right)^n P_n^1(\cos\alpha) P_n^1(\cos\theta) \quad , \qquad \text{(F-20)}$$

where the ring is at some colatitude α, and a is the radius of the mass. For r>a,

$$b_r = +\frac{2\pi G}{c^2}\frac{j}{a}\sum_{n\ \text{odd}}^{\infty} (-1)^{\frac{n+1}{2}} N \left(\frac{a}{r}\right)^{n+2}$$

$$b_\theta = -\frac{2\pi G}{c^2}\frac{j}{a}\sum_{n\ \text{odd}}^{\infty} (-1)^{\frac{n+1}{2}} N \left(\frac{n}{n+1}\right)\left(\frac{a}{r}\right)^{n+2} \qquad \text{(F-21)}$$

$$N \equiv \frac{1\cdot 3\cdot 5 \cdots n\cdot(n+1)}{2\cdot 4\cdot 6\cdots(n-1)(n+1)} \quad .$$

Then integrating the current loop over the volume of the spherical mass, which is simply the moment of inertia of the mass, one gets approximately

$$b_r = -\frac{GI\omega\sin\theta}{2c^2r^3}$$
$$b_\theta = -\frac{GI\omega\cos\theta}{c^2r^3} \quad . \qquad \text{(F-22)}$$

This is the vector gravitational induction field about a mass spinning at angular velocity $\underset{\sim}{\omega}$.

The orbital angular momentum $\underset{\sim}{L}$ of another particle orbiting about this spinning mass interacts with $\underset{\sim}{b}$, creating a spin-orbit and a spin-spin interaction. First, there is a precession of the Kepler conic by an amount

$$\Omega_L = \langle b_r \rangle$$

for a polar orbit. Averaging over the orbit (where I is the Keplerian moment of inertia $I=Mr^2/2$)

$$\Omega_L = -\frac{2GI}{c^2r^3}\omega_L = -\frac{GM}{c^2r}\omega_L \quad . \qquad \text{(F-23)}$$

There is likewise a spin-spin interaction, similar to the gyroscopic effect predicted by Schiff (Phys Rev Lett 4, 215, 1960) in post-Newtonian General Relativity (Appendix E). If the second particle (satellite) possesses an intrinsic spin angular momentum $\underset{\sim}{s}=I'\underset{\sim}{\omega}_s$, it experiences a torque due to $\underset{\sim}{b}$ of

$$\underset{\sim}{T} = \frac{1}{2}\,\underset{\sim}{s}\times\underset{\sim}{b} \qquad \text{(F-24)}$$

which results in a precession of the spin axis of

$$\Omega_s = \frac{T}{s} = \frac{1}{2}\,\frac{|\underset{\sim}{s}\times\underset{\sim}{b}|}{s} \quad .$$

In an equatorial orbit

$$\Omega_s = \frac{1}{2}\langle b_\theta \rangle = -\frac{GI'}{c^2r^3}\omega_s \, . \qquad \text{(F-25)}$$

Compiling (F-23) and (F-25)

$$\Omega = -\frac{GM}{c^2r}\omega_L - \frac{GI'}{c^2r^3}\omega_s \qquad \text{(F-26)}$$

which is Schiff's result, except that the first term must be multiplied by -3/2.

Generalized Ohm's Law

The classical relation for Ohm's Law (2-16) generalizes as follows:

$$\frac{\partial \underset{\sim}{j}}{\partial t} + \underset{\sim}{j} \times \underset{\sim}{b} = \rho\left[\underset{\sim}{g} + \underset{\sim}{v} \times \underset{\sim}{b}\right] - \rho \nabla \cdot \psi \ , \qquad \text{(F-27)}$$

where ψ represents a pressure of matter. For the Coriolis interpretation, this is

$$\frac{\partial \underset{\sim}{j}}{\partial t} + \underset{\sim}{j} \times (2\omega) = \rho\left[\underset{\sim}{g} + \underset{\sim}{v} \times (2\omega)\right] - \rho \nabla \cdot \psi \ . \qquad \text{(F-28)}$$

MacCullagh's Equations & Gravitational Radiation

It is pertinent to remark that MacCullagh's (1838) equations can be used to derive Maxwell's Equations for electromagnetism. Likewise, they can be used in the vector theory of gravitation. MacCullagh's theory, in effect, is a hydrodynamic theory of electromagnetism, and in this context, of gravitation. He assumes that velocity $\underset{\sim}{v}$ is given by our (F-7)

$$\nabla \times \underset{\sim}{v} = 2\underset{\sim}{\omega} \qquad \text{(F-7)}$$

where $\underset{\sim}{\omega} = \dot{\underset{\sim}{\phi}}$. His equation of motion is then assumed to be

$$\rho \frac{\partial \underset{\sim}{v}}{\partial t} = -\frac{k}{2} \nabla \times \underset{\sim}{\phi} \ . \qquad \text{(F-29)}$$

(F-7) and (F-29) are subjected to the zero divergence conditions (incompressibility or transversality)

$$\nabla \cdot \underset{\sim}{v} = 0$$
$$\nabla \cdot \underset{\sim}{\phi} = 0 \ . \qquad \text{(F-30)}$$

From (F-7), (F-29), and (F-30) one can derive the source-free Maxwell equations (with displacement current in 1838!). If you recall the hydrodynamics of current

$$\underset{\sim}{j} = \rho \underset{\sim}{v} = \sigma \underset{\sim}{E} \ , \qquad \text{(2-16)}$$

then we have the implicit use of MacCullagh's ideas if we simply state from (2-16) that

$$\underset{\sim}{V} = \frac{\sigma}{\rho} \underset{\sim}{E} \quad . \tag{F-31}$$

For arbitrary constants of proportionality α, β,

$$\begin{aligned} &\text{Case (a)} \quad \underset{\sim}{V} = \pm\alpha \underset{\sim}{E} \quad \underset{\sim}{\phi} = \mp\beta \underset{\sim}{H} \\ &\text{Case (b)} \quad \underset{\sim}{V} = \pm\alpha \underset{\sim}{H} \quad \underset{\sim}{\phi} = \pm\beta E \end{aligned} \tag{F-32}$$

we obtain Maxwell's relations (2-11) for $\rho=0$ and $\underset{\sim}{j}=0$. We can identify

$$\begin{aligned} &\text{Case (a)} \quad \epsilon_0 = \tfrac{1}{2\pi} 2\alpha/\beta \quad \mu_0 = 2\beta/\alpha \\ &\text{Case (b)} \quad \epsilon_0 = 2\beta/\alpha \quad \mu_0 = \tfrac{1}{2\pi} 2\alpha/\beta \, . \end{aligned} \tag{F-33}$$

Sources can be readily included. The purpose in going into this discussion of MacCullagh's theory is to demonstrate that all of the "magnetohydrodynamic" analogies in the vector theory of gravitational radiation are not magnetohydrodynamic at all. Magnetohydrodynamics was established by MacCullagh over a century ago. Again, the electromagnetic theory of gravitation goes back to fundamentals.

A Generalization Of Bateman's Equations

Starting with Maxwell's Equations (2-11), Bateman (1915) makes a simplification which in effect is reminiscent of the original MacCullagh theory. Bateman assumes a complex electromagnetic field

$$\underset{\sim}{M} = \underset{\sim}{E} + i \underset{\sim}{H} \tag{F-34}$$

whereby Maxwell's Equations reduce to and derive from

$$\begin{aligned} \nabla \cdot \underset{\sim}{M} &= \rho \\ \nabla \times \underset{\sim}{M} &= -i \frac{\partial \underset{\sim}{M}}{\partial t} \quad . \end{aligned} \tag{F-35}$$

These are simply a complex formulation of MacCullagh's Equations (F-29) and (F-30), with sources. (E.g. let $\underset{\sim}{M} = \underset{\sim}{v} + i\underset{\sim}{\phi}$).

The neat thing about Bateman's Equations is that they provide a simple way of synthesizing vector electromagnetism and vector gravitation, as a complex unified field theory. Let

$$\left.\begin{aligned} \underset{\sim}{M} &= \hat{\underset{\sim}{E}} + \hat{\underset{\sim}{H}} \\ \hat{\underset{\sim}{E}} &= \underset{\sim}{E} + i\underset{\sim}{g} \\ \hat{\underset{\sim}{H}} &= \underset{\sim}{H} + i\underset{\sim}{h} \\ \rho &= \rho_e - i\rho_m \end{aligned}\right\} . \qquad \text{(F-36)}$$

Substitution of the complex fields (F-36) into the Bateman Eqs (F-35) gives the vector Maxwell Equations (2-11) for electromagnetism and (2-12) for gravitation. The Lorentz force (2-15) has a complex component,

$$\underset{\sim}{F} = \rho_e(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{H}) + \rho_m(\underset{\sim}{g} + \underset{\sim}{v}\times\underset{\sim}{h}) - i\left[\rho_m(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{H}) - \rho_e(\underset{\sim}{g} + \underset{\sim}{v}\times\underset{\sim}{h})\right] . \qquad \text{(F-37)}$$

Or

$$\underset{\sim}{F} = \underset{\sim}{F}_e + \underset{\sim}{F}_g - i\left[\rho_e \underset{\sim}{F}_e - \frac{\rho_e}{\rho_m}\underset{\sim}{F}_g\right]_{\text{Inter.}} \qquad \text{(F-38)}$$

One is interested only in the real part of F in (F-38).

The interesting feature of this approach is the implication that charge and mass are the real components of the same thing - a single complex quantity. That is, the Newtonian and Coulomb forces are unified into a single complex operation. The same is true for the Larmor-Lorentz term.

The field intensities are determined by

$$\begin{aligned} \hat{E}^2 &= E^2 + g^2 \\ \hat{H}^2 &= H^2 + h^2 \end{aligned} . \qquad \text{(F-39)}$$

There is no negative energy density.

$$\mathcal{E} \sim \hat{E}^2 + \hat{H}^2 . \qquad \text{(F-40)}$$

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APPENDIX G: TIDAL FRICTION & THE VELOCITY OF PROPAGATION

Eddington (1920) discusses an important physical application of the retarded Lienard-Wiechert potentials when he considers the question of the existence of a couple or torque acting on the two-body Kepler problem. Referring to Figure G-1, Eddington points out that the acceleration $\underset{\sim}{a}$ acting on the present position of either component must be in the direction of the other binary component

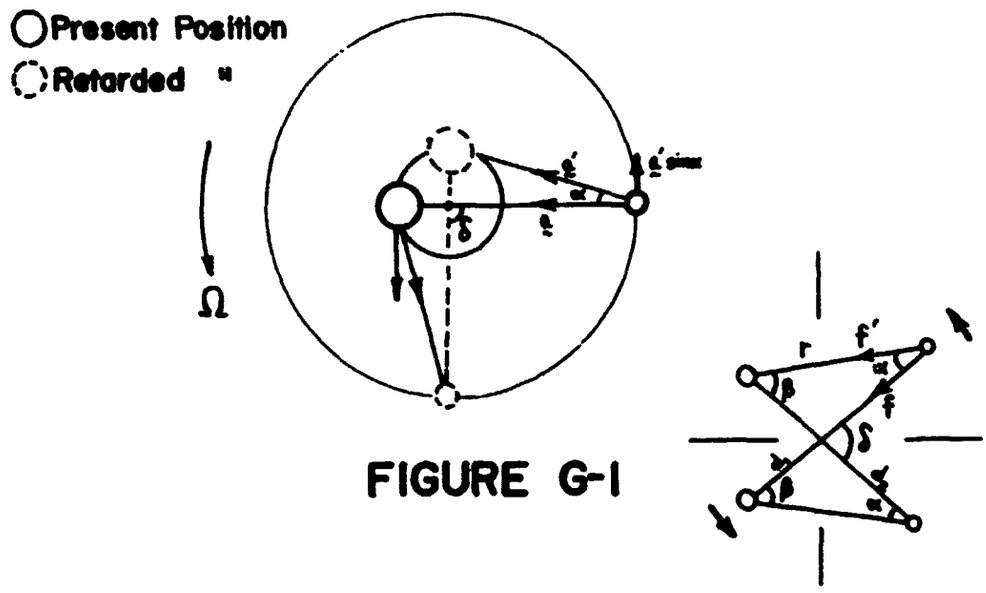

FIGURE G-1

or a resultant torque T acts upon the system. Assuming a phase lag δ between the retarded and present positions, this osculating torque is

$$T = \frac{G m_1 m_2}{r^3}(d_1^2 + d_2^2)\sin\delta = \frac{G m_1 m_2}{r^3}\left[\frac{m_1^2 + m_2^2}{(m_1+m_2)^2}\right] d^2 \sin\delta, \tag{G-1}$$

where

$$r^2 = d_1^2 + d_2^2 + 2 d_1 d_2 \cos\delta \ ,$$

$$d_1 = \frac{m_1}{m_1+m_2} d \ , \qquad d = d_1 + d_2 \ , \qquad d_2 = \frac{m_2}{m_1+m_2} d.$$

Note that this is an inductive energy dissipation mechanism (it goes as r^{-1}), characteristic of scalar (Newtonian) gravitation theory. According to the Lienard-Wiechert scalar potential

$$\phi = -G\int \frac{d\rho}{r - \underset{\sim}{r}\cdot\underset{\sim}{u}/c} \qquad \text{(G-2)}$$

$$\left(\underset{\sim}{A} = -\frac{G}{c^2}\int \frac{\underset{\sim}{u}\, d\rho}{r - \underset{\sim}{r}\cdot\underset{\sim}{u}/c} \right)$$

the force per unit mass or field intensity $\underset{\sim}{g} = -\nabla\phi$ in Eq (2-3) (assuming zero vector potential $\underset{\sim}{A}$) acting on either component in Figure G-1 is

$$\underset{\sim}{g}_i = \frac{GM_j}{r_0^2}\hat{r}_0 \qquad (i,j=1,2) \qquad \text{(G-3)}$$

where $M=\gamma m$ for a circular orbit and where $\hat{r}_0$ is a unit vector directed toward the present position and not the retarded one $\hat{r}$. Hence,

$$\delta = 0 \qquad \text{(G-4)}$$

and there is no torque.

However, this equilibrium condition (no couple or torque) derives from the fact that the velocity of propagation of gravitational information c is just right. If we perturb the velocity of propagation

$$c' = c + \epsilon \qquad \text{(G-5)}$$

the gravitational action is no longer in the "present" direction $\hat{r}_0$, $\delta \neq 0$, and a torque manifests itself. The conic then dilatates or contracts until once again it is in equilibrium, whereby the phase angle disappears and the torque vanishes. These results apply to point masses.

The first matter that needs to be dispelled is the misconception of Laplace (1829) and others that variations in the velocity of propagation ϵ necessarily manifest themselves as anomalous behaviour of the Kepler conic. This is not true, because the retardation effect above cannot be distinguished from the mechanism of tidal friction (Appendix S) in the case of real, ponderable bodies. The second matter of importance is the assumption of a constant velocity of propagation c in the derivation of the Lienard-Wiechert potential (G-2). In the presence of an index of refraction n ($n = 1 + 2m/r$ for light in a Schwarzschild background) whereby $c' = c/n$, the two-body problem expands or contracts until the torque disappears.

Hence, the effective inductive friction in the two-body problem due to retardation behaves a great deal like tidal friction, except that it occurs for point masses as well as elastic bodies with Love numbers. In turn, both of these regulate the behaviour of gravitational radiation damping (or anti-damping). Furthermore, a nonlinear field theory can affect the velocity of propagation c', producing retardation, and in turn neutralizing radiation reaction mechanisms. By means of this orbit-orbit interaction a nonlinear mechanism can vary the velocity of propagation in the vicinity of a "black hole" binary; there can be gravitational radiation (assuming it exists for the Kepler problem); and there does not have to be collapse.

APPENDIX H: DIFFERENTIAL EQUATIONS FOR PERTURBATIONS OF THE SCHWARZSCHILD METRIC

The differential equations representing the nonspherical perturbations of the Schwarzschild metric

$$\delta R_{\mu\nu} = -\delta\Gamma^{\beta}_{\mu\nu;\beta} + \delta\Gamma^{\beta}_{\mu\beta;\nu} \qquad (3\text{-}56b)$$

$$\delta R_{\mu\nu} = 0 = R^{(1)}_{\mu\nu} \qquad (3\text{-}56a)$$

are first derived by Regge & Wheeler (1957) with a number of errors which are later corrected by Vishveshwara & Edelstein (1970). They are derived using spherical harmonics and are comprised of odd-parity and even-parity solutions.

Odd-Parity Equations

In the case of odd (magnetic) parity $(-1)^{L+1}$, Eqs (3-56b) take the following form in the canonical Regge-Wheeler gauge (3-58):

$$\delta R_{23} = -\frac{1}{2}\left\{ik e^{-\nu} h_0 + e^{-\lambda}\left[\frac{1}{2}(\nu_r - \lambda_r)h_1 + \frac{dh_1}{dr}\right]\right\}\left(\cos\theta\frac{d}{d\theta} - \sin\theta\frac{d^2}{d\theta^2}\right)P_L e^{-ikt} \qquad (H\text{-}1a)$$

$$\delta R_{13} = \frac{1}{2}\left\{ik e^{-\nu}\left(\frac{dh_0}{dr} - 2\frac{h_0}{r}\right) + \left[\frac{L(L+1)}{r^2} - e^{-\nu}k^2 + \frac{e^{-\lambda}}{r}\left(\lambda_r - \nu_r - \frac{2}{r}\right)\right]h_1\right\}\sin\theta\frac{dP_L}{d\theta}e^{-ikt} \qquad (H\text{-}1b)$$

$$\delta R_{03} = \left\{\frac{1}{2}e^{-\lambda}\frac{d^2h_0}{dr^2} + ik\left[\frac{1}{2}e^{-\lambda}\frac{dh_1}{dr} + \frac{e^{-\lambda}}{r}h_1 - \frac{e^{-\lambda}}{4}(\nu_r + \lambda_r)h_1\right] - \frac{e^{-\lambda}}{4}(\nu_r + \lambda_r)\frac{dh_0}{dr} + \left(\frac{e^{-\lambda}}{r}\nu_r - \frac{L(L+1)}{2r^2}\right)h_0\right\}\sin\theta\frac{dP_L}{d\theta}e^{-ikt}, \qquad (H\text{-}1c)$$

where the subscripts (such as ν_r) represent derivatives ($\nu_r = d\nu/dr$). Eqs (H-1) are then set equal to zero in order to determine (3-56a). For L=1, all of the angular factors in (H-1) vanish. Also for L=1, $\delta R_{23}=0$. For higher L, Eq (3-56a) implies the terms in brackets vanish, providing three radial equations. The second-order radial equation resulting from (H-1c) is actually derivable from the other two, provided

$$(\nu - \lambda)_{rr} + \frac{1}{2}(\nu_r - \lambda_r)^2 - \frac{1}{2}\left[\nu_r^2 - \lambda_r^2\right] + \frac{1}{r}(\nu_r - 3\lambda_r) = 0, \qquad (H\text{-}2)$$

which indeed is the case for the Schwarzschild metric where

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} \,. \qquad \text{(H-3)}$$

From these other two radial equations (H-1a) and (H-1b), h_0 can be eliminated, resulting in a single second-order radial equation

$$\frac{d^2 h_1}{dr^2} + \left[\frac{3}{2}(\nu_r - \lambda_r) - \frac{2}{r}\right]\frac{dh_1}{dr} + \left[\frac{1}{2}(\nu - \lambda)_{rr} + \frac{1}{4}(\nu_r - \lambda_r)^2 - e^{\lambda}\frac{L(L+1)}{r^2} + k^2 e^{\lambda - \nu} + \frac{2}{r^2}\right] h_1 = 0 . \qquad \text{(H-4)}$$

If one defines

$$Q \equiv e^{\frac{1}{2}(\nu - \lambda)}\frac{h_1}{r} = \left(1 - \frac{2m}{r}\right)\frac{h_1}{r} \qquad \text{(H-5a)}$$

$$r^* = r + 2m \ln\left|\frac{r}{2m} - 1\right| \qquad \text{(H-5b)}$$

then from (H-4) one obtains the odd-parity "Schrödinger" equation first given by Regge & Wheeler

$$\frac{d^2 Q}{dr^{*2}} + \left(k^2 - V_{eff}\right) Q = 0 \qquad \text{(H-6)}$$

where V_{eff} is an effective potential given by

$$V_{eff} = \frac{e^{\nu} L(L+1)}{r^2} - \frac{3}{r}\frac{d}{dr^*}\left(e^{\frac{1}{2}(\nu - \lambda)}\right) . \qquad \text{(H-7)}$$

In the case of the Schwarzschild background

$$V_{eff} = \left[\frac{L(L+1)}{r^2} - \frac{6m}{r^3}\right]\left(1 - \frac{2m}{r}\right) . \qquad \text{(H-8)}$$

<u>Even-Parity Equations</u>

Continuing in the Regge-Wheeler gauge, and using upper-case letters to represent the even (electric) parity $(-1)^L$ radial functions in order to distinguish them from the lower-case odd-parity functions, the first-order perturbations of the Ricci tensor in (3-56b) are

$$\delta R_{01} = \left\{ik\left[\frac{dK}{dr} + \left(\frac{1}{r} - \frac{\nu_r}{2}\right)K - \frac{H_2}{r}\right] + e^{-\lambda}\left[\frac{e^{\lambda} L(L+1)}{2r^2} - \frac{\nu_{rr}}{2} - \frac{\nu_r^2}{4} + \frac{\nu_r \lambda_r}{4} - \frac{\nu_r}{4}\right] H_1\right\} P_L e^{-ikt} \qquad \text{(H-9a)}$$

$$\delta R_{12} = \left[\frac{1}{2} ik e^{-\nu} H_1 + \frac{1}{2}\frac{dH_0}{dr} - \frac{1}{2}\frac{dK}{dr} + \left(\frac{\nu_r}{4} - \frac{1}{2r}\right)H_0 + \left(\frac{\nu_r}{4} + \frac{1}{2r}\right)H_2\right]\frac{dP_L}{d\theta} e^{-ikt} \qquad \text{(H-9b)}$$

$$\delta R_{02} = \left[\frac{1}{2} ik(H_2 + K) + \frac{1}{2} e^{-\lambda}\frac{dH_1}{dr} + \frac{1}{4}(\nu_r - \lambda_r) e^{-\lambda} H_1\right]\frac{dP_L}{d\theta} e^{-ikt} \qquad \text{(H-9c)}$$

$$\delta R_{00} = \Big\{ \tfrac{1}{2}k^2(H_2+2K) + ik\Omega^{-\lambda}\Big[\Big(\tfrac{\lambda_r}{2}-\tfrac{2}{r}\Big)H_1 - \tfrac{dH_1}{dr}\Big] - \tfrac{1}{2}\Omega^{\nu-\lambda}\tfrac{d^2H_2}{dr^2} + \Omega^{\nu-\lambda}\Big(\tfrac{\lambda_r}{4}-\tfrac{\nu_r}{2}-\tfrac{1}{r}\Big)\tfrac{dH_2}{dr}$$
$$-\tfrac{1}{4}\Omega^{\nu-\lambda}\nu_r\tfrac{dH_2}{dr} + \tfrac{1}{2}\Omega^{\nu-\lambda}\nu_r\tfrac{dK}{dr} + \Omega^{\nu-\lambda}\Big[\tfrac{\nu_r\lambda_r}{4}-\tfrac{\nu_{rr}}{2}-\tfrac{\nu_r^2}{4}-\tfrac{\nu_r}{r}+\tfrac{1}{2r^2}\Omega^{\lambda}L(L+1)\Big]H_0$$
$$+\Omega^{\nu-\lambda}\Big[\tfrac{\nu_r\lambda_r}{4}-\tfrac{\nu_{rr}}{2}-\tfrac{\nu_r^2}{4}-\tfrac{\nu_r}{r}\Big]H_2\Big\}P_L\Omega^{-ikt} \qquad \text{(H-9d)}$$

$$\delta R_{11} = \Big[ik\Omega^{-\nu}\Big(\tfrac{dH_2}{dr}-\tfrac{\lambda_r}{2}H_1\Big) - \tfrac{1}{2}k^2\Omega^{\lambda-\nu}H_2 + \tfrac{1}{2}\tfrac{d^2H_2}{dr^2} - \tfrac{d^2K}{dr^2} + \Big(\tfrac{\nu_r}{2}-\tfrac{\lambda_r}{4}\Big)\tfrac{dH_2}{dr}$$
$$+\Big(\tfrac{\nu_r}{4}+\tfrac{1}{r}\Big)\tfrac{dH_2}{dr} + \Big(\tfrac{\lambda_r}{2}-\tfrac{2}{r}\Big)\tfrac{dK}{dr} + \tfrac{L(L+1)}{2r^2}\Omega^{\lambda}H_2\Big]P_L\Omega^{-ikt} \qquad \text{(H-9e)}$$

$$\delta R_{22} = \Big\{-\tfrac{1}{2}k^2\Omega^{-\nu}r^2K + ikr\Omega^{-(\lambda+\nu)}H_1 - \tfrac{1}{2}\Omega^{-\lambda}r^2\tfrac{d^2K}{dr^2} + \tfrac{1}{2}r\Omega^{-\lambda}\tfrac{dH_2}{dr} + \tfrac{1}{2}r\Omega^{-\lambda}\tfrac{dH_2}{dr}$$
$$+\Big(\tfrac{1}{4}r^2\Omega^{-\lambda}\lambda_r - \tfrac{1}{4}r^2\Omega^{-\lambda}\nu_r - 2r\Omega^{-\lambda}\Big)\tfrac{dK}{dr} + \Big[\tfrac{1}{2}r\Omega^{-\lambda}(\nu_r-\lambda_r)+\Omega^{-\lambda}\Big]H_2$$
$$+\Big[\tfrac{1}{2}r\Omega^{-\lambda}(\lambda_r-\nu_r)-\Omega^{-\lambda}+\tfrac{1}{2}L(L+1)\Big]K\Big\}P_L\Omega^{-ikt} + \tfrac{1}{2}(H_0-H_2)\tfrac{d^2P_L}{d\theta^2}\Omega^{-ikt}. \qquad \text{(H-9f)}$$

Derivatives of the Legendre polynomials vanish for L=0. For L=1, the two angular factors in δR_{22} of (H-9f) are not independent. For L>1, substitution of (H-9) into (3-56a) gives the following even-parity radial equations:

$$H \equiv H_0 = H_2 \qquad \text{(H-10a)}$$

$$\frac{dK}{dr} + \frac{1}{r}(K-H) - \frac{m}{r^2}\Big(1-\frac{2m}{r}\Big)^{-1}K - i\frac{L(L+1)}{2kr^2}H_1 = 0 \qquad \text{(H-10b)}$$

$$\frac{d}{dr}\Big[\Big(1-\frac{2m}{r}\Big)H_1\Big] + ik(H+K) = 0 \qquad \text{(H-10c)}$$

$$ikH_1 + \Big(1-\frac{2m}{r}\Big)\Big(\frac{dH}{dr}-\frac{dK}{dr}\Big) + \frac{2m}{r^2}H = 0 \qquad \text{(H-10d)}$$

$$\Big(1-\frac{2m}{r}\Big)^2\frac{d^2H}{dr^2} + \frac{2}{r}\Big(1-\frac{2m}{r}\Big)\frac{dH}{dr} - k^2H - L(L+1)\Big(1-\frac{2m}{r}\Big)\frac{H}{r^2} + 2ikm\frac{H_1}{r} + 2ik\Big(1-\frac{2m}{r}\Big)\frac{1}{r^2}\frac{d}{dr}(r^2H_1) \qquad \text{(e)}$$
$$-2k^2K - 2\Big(1-\frac{2m}{r}\Big)\frac{m}{r^2}\frac{dK}{dr} = 0$$

$$\frac{d}{dr}\Big\{\Big(1-\frac{2m}{r}\Big)\Big[\frac{d}{dr}(r^2K)-2rH\Big]\Big\} - L(L+1)K + k^2r^2\Big(1-\frac{2m}{r}\Big)^{-1}K - 2ikrH_1 = 0 \qquad \text{(H-10g)}$$

$$2ik\Big(1-\frac{2m}{r}\Big)\frac{dH_1}{dr} + 2ikm\frac{H_1}{r^2} - k^2H + \Big(1-\frac{2m}{r}\Big)^2\Big[\frac{1}{r^2}\frac{d}{dr}\Big(r^2\frac{dH}{dr}\Big) - \frac{2}{r^2}\frac{d}{dr}\Big(r^2\frac{dK}{dr}\Big)\Big] \qquad \text{(H-10f)}$$
$$+\Big(1-\frac{2m}{r}\Big)\Big[\frac{4m}{r^2}\frac{dH}{dr} - \frac{2m}{r^2}\frac{dK}{dr} + L(L+1)\frac{H}{r^2}\Big] = 0.$$

Eqs (10e,f,g), of second order, can all be derived from (10b,c,d), of first order, provided

$$F(r) \equiv -\left[\frac{6m}{r} + (L-1)(L+2)\right] H + \left[(L-1)(L+2) + \frac{2m}{r} - 2\left(1-\frac{2m}{r}\right)^{-1}\left(\frac{m^2}{r^2} + r^2k^2\right)\right] K + \left[2ikr - \frac{iL(L+1)}{k}\frac{m}{r^2}\right] H_1 = 0. \tag{H-11}$$

This relation (H-11), in conjunction with the first-order equations (H-10b,c,d), gives one second-order differential equation:

$$\frac{d^2\bar{S}}{dx^{*2}} + \left[(L-1)(L+2) + \frac{3}{x}\right]\frac{4}{x}\left(1-\frac{3}{2x}\right)\frac{1}{D(x)}\frac{d\bar{S}}{dx^*} + \left\{\bar{k}^2 - \frac{1}{D(x)}\left(1-\frac{3}{2x}\right)\left[8\bar{k}^2 - 2L(L+1)\frac{1}{x^3} - \frac{4(L-1)(L+2)}{x^2}\left(1-\frac{1}{x}\right) - \frac{12}{x^3}\left(1-\frac{1}{x}\right)\right] - \frac{2}{x^2}\left(1-\frac{3}{2x}\right)\left(1-\frac{1}{x}\right) - \frac{L(L+1)}{x^2}\left(1-\frac{1}{x}\right)\right\}\bar{S} = 0 \tag{H-12}$$

where

$$\bar{S} \equiv \left(1-\frac{2m}{r}\right)\frac{H_1}{r} \tag{H-13a}$$

$$x = \frac{r}{2m} \qquad x^* = r^*/2m \qquad \bar{k} = 2mk \tag{H-13b}$$

$$D(x) = \frac{4}{x} + 2(L-1)(L+2) - 2\left(\frac{1}{4x^2} + \bar{k}^2x^2\right)\left(\frac{x}{x-1}\right). \tag{H-13c}$$

The second-order even-parity differential equation for $S=H_1/r$ is then

$$\frac{d^2S}{dx^{*2}} + \left\{\frac{2}{x} + \frac{1}{D(x)}\left[(L-1)(L+2) + \frac{3}{x}\right]\frac{4}{x}\left(1-\frac{3}{2x}\right)\right\}\frac{dS}{dx^*} + \left\{\bar{k}^2 - \frac{1}{D(x)}\left(1-\frac{3}{2x}\right)\left[8\bar{k}^2 - \frac{2L(L+1)}{x^3} - \frac{4(L-1)(L+2)}{x^2} - \frac{12}{x^3}\right] - \frac{2}{x^2}\left(1-\frac{3}{2x}\right) - \frac{L(L+1)}{x^2}\left(1-\frac{1}{x}\right)\right\}S = 0 \tag{H-14}$$

Further details are provided by Edelstein & Vishveshwara (1970). It is safer to follow their development as it does not contain the errors made in the original Regge-Wheeler (1957) paper.

APPENDIX I: REGGE-WHEELER GAUGE

Recall from Appendix B that in Einstein's field equations for the first-order perturbations, $h_{\mu\nu}$ remain invariant under the gauge transformation

$$h'_{\mu\nu} = h_{\mu\nu} - \left(\xi_{\mu;\nu} + \xi_{\nu;\mu}\right). \tag{B-5}$$

Odd-Parity Gauge

Upon consideration of $h_{\mu\nu}$ in (3-57a), Regge & Wheeler let

$$\begin{aligned} \xi^0 &= \xi^1 = 0 \\ \xi^i &= \Lambda(t,r)\epsilon^{ij}\frac{\partial}{\partial x^j}Y_L^M \qquad (i,j=2,3) \end{aligned} \tag{I-1}$$

whereby (B-5) transforms (3-57a) into (3-58a). Λ is arbitrary.

Even-Parity Gauge

Similarly, if for arbitrary M_0, M_1, and M one selects

$$\begin{aligned} \xi_0 &= M_0(t,r)Y_L^M(\theta,\phi) \\ \xi_1 &= M_1(t,r)Y_L^M(\theta,\phi) \\ \xi_2 &= M(t,r)\frac{\partial}{\partial\theta}Y_L^M(\theta,\phi) \\ \xi_3 &= M(t,r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Y_L^M(\theta,\phi), \end{aligned} \tag{I-2}$$

then (B-5) transforms (3-57b) into (3-58b).

APPENDIX J: EVEN-PARITY "SCHRÖDINGER" EQUATION

Zerilli's (1970b) derivation of the second-order "Schrödinger" equation for the even-parity Regge-Wheeler perturbation problem is as follows. He substitutes $kR=H_1$ in the first-order even-parity equations (H-10b,c,d) and the algebraic relation (H-11), and then uses (H-11) to reduce (H-10b,c,d) to

$$\left.\begin{aligned} \frac{dK}{dr} &= \left[\alpha_0(r)+\alpha_2(r)k^2\right]K + \left[\beta_0(r)+\beta_2(r)k^2\right]R \\ \frac{dR}{dr} &= \left[\gamma_0(r)+\gamma_2(r)k^2\right]K + \left[\delta_0(r)+\delta_2(r)k^2\right]R \end{aligned}\right\} \qquad \text{(J-1)}$$

where α,β,γ, and δ are functions of r,L, and M - but not $k=\omega$. Next, the following transformations are assumed:

$$\left.\begin{aligned} K &= f(r)\hat{K} + g(r)\hat{R} \\ R &= \ell(r)\hat{K} + j(r)\hat{R} \end{aligned}\right\} \qquad \text{(J-2)}$$

$$dr = n(r)\,dr^* = (1-2m/r)\,dr^* \ . \qquad \text{(J-3)}$$

n(r) in (J-3) is Eddington's (1920) refractive index. Zerilli lets

$$\left.\begin{aligned} f(r) &= \frac{[\lambda(\lambda+1)r^2+3\lambda mr+6m^2]}{r^2(\lambda r+3m)} \\ g(r) &= 1 \\ \ell(r) &= i\,\frac{[-\lambda r^2+3\lambda mr+3m^2]}{(r-2m)(\lambda r+3m)} \\ j(r) &= \frac{-ir^2}{(r-2m)} \end{aligned}\right\} \qquad \text{(J-4)}$$

where

$$\lambda = \tfrac{1}{2}(L-1)(L+2)\ , \qquad \text{(J-5)}$$

noting that throughout the L and M subscripts on all radial functions have been suppressed. By virtue of (J-4), (J-5), and the integral of (J-3) which is (H-5b), then

$$\frac{d\hat{K}}{dr^*} = \hat{R} \quad ; \quad \frac{d\hat{R}}{dr^*} = \left[V(r^*)-k^2\right]\hat{K} \ . \qquad \text{(J-6)}$$

These may be re-written in second-order as

$$\frac{d^2\hat{K}_{LM}}{dr^{*2}} + \left[k^2 - V_L(r)\right]\hat{K}_{LM} = 0 \qquad \text{(J-7)}$$

$$V_L(r) = \left(1-\frac{2m}{r}\right)\left[\frac{2\lambda^2(\lambda+1)r^3+6\lambda^2mr^2+18\lambda m^2r+18m^3}{r^3(\lambda r+3m)^2}\right], \qquad \text{(J-8)}$$

where angular momentum L and azimuth M have been re-instated.

APPENDIX K: TENSOR SPHERICAL HARMONICS

The following are the orthonormal set of tensor harmonics in 4-space as developed by Zerilli (1970a,c):

$$\underset{=}{a}^{(0)}_{LM} = \begin{bmatrix} Y_{LM} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \underset{=}{a}^{(1)}_{LM} = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & Y_{LM} & 0 & 0 \\ Y_{LM} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \underset{=}{a}_{LM} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{LM} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{=}{b}^{(0)}_{LM} = i r [2L(L+1)]^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & \partial_\theta Y_{LM} & \partial_\phi Y_{LM} \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix}, \quad \underset{=}{b}_{LM} = r [2L(L+1)]^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_\theta Y_{LM} & \partial_\phi Y_{LM} \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{bmatrix}$$

$$\underset{=}{c}^{(0)}_{LM} = r [2L(L+1)]^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & \frac{1}{\sin\theta}\partial_\phi Y_{LM} & -\sin\theta\,\partial_\theta Y_{LM} \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix}, \quad \underset{=}{c}_{LM} = i r [2L(L+1)]^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sin\theta}\partial_\phi Y_{LM} & -\sin\theta\,\partial_\theta Y_{LM} \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{bmatrix}$$

$$\underset{=}{d}_{LM} = -i r^2 [2L(L+1)(L-1)(L+2)]^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sin\theta}X_{LM} & \sin\theta\, W_{LM} \\ 0 & 0 & * & \sin\theta\, X_{LM} \end{bmatrix}, \quad \underset{=}{g}_{LM} = \frac{r^2}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2\theta \end{bmatrix} Y_{LM}$$

$$\underset{=}{f}_{LM} = r^2 [2L(L+1)(L-1)(L+2)]^{-\frac{1}{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_{LM} & X_{LM} \\ 0 & 0 & * & \sin\theta\, W_{LM} \end{bmatrix},$$

where

$$X_{LM} \equiv 2\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\theta} - \cot\theta\right) Y_{LM} \qquad W_{LM} \equiv \left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{LM} .$$

$\underset{=}{a}$, $\underset{=}{b}$, and $\underset{=}{c}$ are orthogonal to the 2-sphere (θ,ϕ). $\underset{=}{d}$, $\underset{=}{e}$, and $\underset{=}{h}$ are tangent. $\underset{=}{f} \sim (\underset{=}{e} + \underset{=}{h})$ and $\underset{=}{g} \sim (\underset{=}{e} - \underset{=}{h})$ are, however, orthogonal. Thus $\{\underset{=}{a},\underset{=}{b},\underset{=}{c},\underset{=}{d},\underset{=}{g},\underset{=}{f}\}$ is an orthonormal set. Upon generalizing to a 4-dimensional pseudo-Euclidean space, we obtain 4 more tensor harmonics: $\underset{=}{a}^{(0)}$, $\underset{=}{a}^{(1)}$, $\underset{=}{b}^{(0)}$, $\underset{=}{c}^{(0)}$. Regge & Wheeler use $\underset{=}{e}$ in place of $\underset{=}{f}$. Hence, theirs is not an orthonormal set.

APPENDIX L: COEFFICIENTS OF SOURCE DECOMPOSITION

Any symmetric covariant tensor can be expanded in terms of the tensor harmonics in Appendix K:

$$\delta \underline{\underline{T}} = \sum \Big\{ A^{(0)}_{LM} \underline{\underline{a}}^{(0)}_{LM} + A^{(1)}_{LM} \underline{\underline{a}}^{(1)}_{LM} + A_{LM} \underline{\underline{a}}_{LM} + B^{(0)}_{LM} \underline{\underline{b}}^{(0)}_{LM} + B_{LM} \underline{\underline{b}}_{LM} + Q^{(0)}_{LM} \underline{\underline{c}}^{(0)}_{LM} + Q_{LM} \underline{\underline{c}}_{LM} + G_{LM} \underline{\underline{g}}_{LM} + D_{LM} \underline{\underline{d}}_{LM} + F_{LM} \underline{\underline{h}}_{LM} \Big\}. \tag{L-1}$$

The coefficients $A_{LM}, \ldots, F_{LM}$ are determined by the inner products

$$(\underline{\underline{T}}, \underline{\underline{S}}) \equiv \iint \underline{\underline{T}}^{*} : \underline{\underline{S}} \, d\Omega \tag{L-2}$$

where

$$\underline{\underline{T}} : \underline{\underline{S}} = \eta^{\mu\lambda} \eta^{\nu\kappa} T_{\mu\nu} S_{\lambda\kappa} \tag{L-3}$$

and $\eta_{\mu\nu}$ is the background metric. Accordingly,

$$\begin{aligned}
A^{(0)}_{LM} &= (\underline{\underline{a}}^{(0)}_{LM}, \underline{\underline{T}}) \\
A^{(1)}_{LM} &= -2(\underline{\underline{a}}^{(1)}_{LM}, \underline{\underline{T}}) \\
A_{LM} &= (\underline{\underline{a}}_{LM}, \underline{\underline{T}}) \\
B^{(0)}_{LM} &= -\frac{2r^2}{L(L+1)} (\underline{\underline{b}}^{(0)}_{LM}, \underline{\underline{T}}) \\
B_{LM} &= \frac{2r^2}{L(L+1)} (\underline{\underline{b}}_{LM}, \underline{\underline{T}}) \\
Q^{(0)}_{LM} &= \frac{-2}{L(L+1)} (\underline{\underline{c}}^{(0)}_{LM}, \underline{\underline{T}}) \\
Q_{LM} &= \frac{2}{L(L+1)} (\underline{\underline{c}}_{LM}, \underline{\underline{T}}) \\
G_{LM} &= \tfrac{1}{2} r^4 (\underline{\underline{g}}_{LM}, \underline{\underline{T}}) \\
D_{LM} &= \frac{2r^2}{L(L+1)(L-1)(L+2)} (\underline{\underline{d}}_{LM}, \underline{\underline{T}}) \\
F_{LM} &= \frac{r^4}{L(L+1)(L-1)(L+2)} (\underline{\underline{h}}_{LM}, \underline{\underline{T}})
\end{aligned} \tag{L-4}$$

Now, the coefficient of $\underset{=}{a}_{LM}$ is $A_{LM} = (\underset{=}{a}_{LM}, \underset{=}{T})$. So from (L-2) through (L-4)

$$A_{LM}(r,t) = \iint \eta^{\mu\rho}\eta^{\nu\sigma}\left(\underset{=}{a}^{*}_{LM}\right)_{\mu\nu} T_{\rho\sigma}\, d\Omega . \tag{L-5}$$

Thus, from Appendix K

$$A_{LM}(r,t) = \iint Y^{*}_{LM}(\Omega)\, T_{11}\, d\Omega \quad . \tag{L-6}$$

But from Appendix N, if the Schwarzschild geodesic postulate is assumed,

$$T_{11} = m_o \frac{dT}{ds} \frac{r^2}{(r-2m)^2}\left(\frac{dR}{dt}\right)^2 \frac{\delta(r-R(t))}{r^2}\, \delta^{(2)}(\Omega-\Omega(t)) . \tag{L-7}$$

Hence, letting $\gamma = dT/dS^0$ (and similarly for the other coefficients):

$$\begin{aligned}
A_{LM}(r,t) &= m_o\gamma\left(\frac{dR}{dt}\right)^2 \frac{\delta(r-R(t))}{(r-2m)^2} Y^{*}_{LM}(\Omega(t)) \\
A^{(0)}_{LM}(r,t) &= m_o\gamma\frac{(r-2m)^2}{r^4}\,\delta(r-R(t))\, Y^{*}_{LM}(\Omega(t)) \\
A^{(1)}_{LM}(r,t) &= -2m_o\gamma\frac{dR}{dt}\,\frac{\delta(r-R(t))}{r^2}\, Y^{*}_{LM}(\Omega(t)) \\
B^{(0)}_{LM}(r,t) &= -\frac{2m_o\gamma}{L(L+1)}\left(\frac{r-2m}{r}\right)\delta(r-R(t))\left[\frac{\partial Y^{*}_{LM}}{\partial\Theta}\frac{d\Theta}{dt} + \frac{\partial Y^{*}_{LM}}{\partial\Phi}\frac{d\Phi}{dt}\right] \\
B_{LM}(r,t) &= \frac{2m_o\gamma}{L(L+1)}\left(\frac{r}{r-2m}\right)\delta(r-R(t))\frac{dR}{dt}\left[\frac{\partial Y^{*}_{LM}}{\partial\Theta}\frac{d\Theta}{dt} + \frac{\partial Y_{LM}}{\partial\Phi}\frac{d\Phi}{dt}\right] \\
Q^{(0)}_{LM}(r,t) &= \frac{-2m_o\gamma}{L(L+1)}\left(\frac{r-2m}{r^2}\right)\delta(r-R(t))\, i\left[\frac{1}{\sin\Theta}\frac{\partial Y^{*}_{LM}}{\partial\Phi}\frac{d\Theta}{dt} - \sin\Theta\frac{\partial Y^{*}_{LM}}{\partial\Theta}\frac{d\Phi}{dt}\right] \\
Q_{LM}(r,t) &= \frac{2m_o\gamma}{L(L+1)}\,\frac{\delta(r-R(t))}{r-2m}\frac{dR}{dt}\, i\left[\frac{1}{\sin\Theta}\frac{\partial Y^{*}_{LM}}{\partial\Phi}\frac{d\Theta}{dt} - \sin\Theta\frac{\partial Y^{*}_{LM}}{\partial\Theta}\frac{d\Phi}{dt}\right] \\
D_{LM} &= -\frac{2m_o\gamma i r\,\delta(r-R(t))}{L(L+1)(L-1)(L+2)}\left\{\left[\left(\frac{d\Theta}{dt}\right)^2 - \sin^2\Theta\left(\frac{d\Phi}{dt}\right)^2\right]\frac{1}{\sin\Theta}\frac{\partial}{\partial\Phi}\left(\frac{\partial}{\partial\Theta} - \cot\Theta\right)Y^{*}_{LM}\right. \\
&\qquad \left. - \sin\Theta\left(\frac{\partial^2 Y^{*}_{LM}}{\partial\Theta^2} - \cot\Theta\frac{\partial Y^{*}_{LM}}{\partial\Theta} - \frac{1}{\sin^2\Theta}\frac{\partial^2 Y^{*}_{LM}}{\partial\Phi^2}\right)\frac{d\Phi}{dt}\frac{d\Theta}{dt}\right\} \\
F_{LM} &= \frac{m_o\gamma r^2\,\delta(r-R(t))}{2L(L+1)(L-1)(L+2)}\left\{4i\frac{\partial}{\partial\Phi}\left(\frac{\partial Y^{*}_{LM}}{\partial\Theta} - \cot\Theta\, Y^{*}_{LM}\right)\frac{d\Theta}{dt}\frac{d\Phi}{dt}\right. \\
&\qquad \left. + \left(\frac{\partial^2 Y^{*}_{LM}}{\partial\Theta^2} - \cot\Theta\frac{\partial Y^{*}_{LM}}{\partial\Theta} - \frac{1}{\sin\Theta}\frac{\partial^2 Y^{*}_{LM}}{\partial\Phi^2}\right)\left[\left(\frac{d\Theta}{dt}\right)^2 - \sin^2\Theta\left(\frac{d\Phi}{dt}\right)^2\right]\right\} \\
G_{LM} &= \frac{m_o\gamma r^2}{2}\,\delta(r-R(t))\left[\left(\frac{d\Theta}{dt}\right)^2 + \sin^2\Theta\left(\frac{d\Phi}{dt}\right)^2\right] Y^{*}_{LM}
\end{aligned} \tag{L-8}$$

APPENDIX M: DIVERGENCE CONDITION FOR SOURCES

The divergence of the source stress-energy tensor $\delta T_{\mu\nu}$ in Eq (3-61) must be zero. Since $(\delta T_{\mu\nu})^{;\nu}$ is a vector, expanding it in tensor harmonics gives

$$\delta T_{\mu\nu}{}^{;\nu} = \sum_{L,M} \left\{ \alpha_{LM} \hat{\underset{t}{e}} Y_{LM} + \beta_{LM} \hat{\underset{r}{e}} Y_{LM} + \gamma_{LM} \underset{\sim}{\nabla} Y_{LM} + \delta_{LM} \underset{\sim}{L} Y_{LM} \right\} . \tag{M-1}$$

Because of (3-61)

$$\delta T_{\mu\nu}{}^{;\nu} = 0$$

then $\forall\, L,M$

$$\alpha_{LM} = \beta_{LM} = \gamma_{LM} = \delta_{LM} = 0 . \tag{M-2}$$

A simple calculation gives the following divergence conditions:

$$\alpha_{LM} = -\frac{1}{2}\left(1-\frac{2m}{r}\right)\frac{\partial A^{(1)}_{LM}}{\partial r} - \left(\frac{r-m}{r^2}\right)A^{(1)}_{LM} + \left(\frac{r}{r-2m}\right)\frac{\partial A^{(0)}_{LM}}{\partial t} + \frac{1}{2}\frac{L(L+1)}{r^2}B^{(0)}_{LM} = 0 \tag{M-3}$$

$$\beta_{LM} = -\left(\frac{r-2m}{r}\right)\frac{\partial A_{LM}}{\partial r} - \left(\frac{2r-m}{r^2}\right)A_{LM} + \frac{1}{2}\left(\frac{r}{r-2m}\right)\frac{\partial A^{(1)}_{LM}}{\partial t} - \frac{m}{(r-2m)^2}A^{(0)}_{LM} + \frac{1}{2}\frac{1}{r^2}L(L+1)B_{LM} + \frac{3}{r^3}G_{LM} = 0$$

$$\gamma_{LM} = \frac{1}{2}\left(\frac{r}{r-2m}\right)\frac{\partial B^{(0)}_{LM}}{\partial t} - \frac{1}{2}\left(\frac{r-2m}{r}\right)\frac{\partial B_{LM}}{\partial r} - \left(\frac{r-m}{r^2}\right)B_{LM} - \frac{1}{r^2}G_{LM} + \frac{1}{r^3}(L-1)(L+2)F_{LM} = 0$$

$$\delta_{LM} = \frac{1}{2}\left(\frac{r}{r-2m}\right)Q^{(0)}_{LM} - \frac{1}{2}\left(\frac{r-2m}{r^2}\right)\frac{\partial}{\partial r}(rQ_{LM}) - \left(\frac{r-m}{r^2}\right)Q_{LM} - \frac{1}{2r^3}(L-1)(L+2)D_{LM} = 0 .$$

DIVERGENCE CONDITIONS

Upon integration of these conditions (Appendix C of Zerilli's thesis) one obtains the equations of motion for a point particle falling along a geodesic of the Schwarzschild geometry $\eta_{\mu\nu}$ - demonstrating consistency with the geodesic postulate.

APPENDIX N: TRAJECTORY OF PARTICLE & STRESS TENSOR

The stress-energy tensor due to Peters (1966) in Eq (3-62) is

$$T^{\mu\nu} = m_0 \int_{-\infty}^{\infty} \delta^{(4)}(x - z(s)) \frac{dz^\mu}{ds} \frac{dz^\nu}{ds} ds \qquad \text{(N-1)}$$

where $\delta^{(4)}$ is the invariant delta function defined by $\iiiint \delta^{(4)}(x)\sqrt{(-g)}d^4x = 1$. s is the affine parameter along the world line z(s) of the particle. In the case of the Schwarzschild metric

$$(-g) = e^{\nu} e^{-\nu} r^4 \sin^2\theta = r^4 \sin^2\theta$$

$$\therefore \quad \delta^{(4)}(x) = \delta(x^0)\delta(\underset{\sim}{x}) \ . \qquad \text{(N-2)}$$

Carrying out the integration (N-1), using (N-2):

$$T^{\mu\nu} = m_0 \frac{dT}{ds} \frac{dz^\mu}{dt} \frac{dz^\nu}{dt} \frac{\delta(r - R(t))}{r^2} \delta^{(2)}(\Omega - \Omega(t)) \qquad \text{(N-3)}$$

where $\Omega = (\Theta, \phi)$ and $\delta^{(2)}(\Omega) = \delta(\cos\Theta)\delta(\phi)$.

R(t), $\Theta(t)$, $\phi(t)$, and T(t) are the position of the perturbing particle in the Schwarzschild coordinates r, θ, ϕ, t.

This is the very critic[illegible]tion in the Peters-Zerilli analysis which adopts the Schwarzschild geodesics as the path of the radiating particle - in order to evaluate $T_{\mu\nu}$ as stated in (N-3). Radiation reaction is thus unaccounted for.

APPENDIX O: RADIAL EQUATIONS WITH SOURCES

The following radial equations are a consequence of the Zerilli tensor harmonic decompositions (3-64) and (3-65) of the Peters field equations (3-60):

Magnetic-Parity Harmonics

$$\frac{\partial^2 h_0}{\partial r^2} - \frac{\partial^2 h_1}{\partial r \partial t} - \frac{2}{r}\frac{\partial h_0}{\partial t} + \left[\frac{4m}{r^2} - \frac{L(L+1)}{r}\right]\frac{h_0}{r-2m} = -\chi r\left[\tfrac{1}{2}L(L+1)\right]^{-\frac{1}{2}} Q^{(m)}_{LM}$$

$$\frac{\partial^2 h_0}{\partial t^2} - \frac{\partial^2 h_1}{\partial r \partial t} + \frac{2}{r}\frac{\partial h_0}{\partial t} + (L-1)(L+2)(r-2m)\frac{h_1}{r^3} = \chi i(r-2m)\left[\tfrac{1}{2}L(L+1)\right]^{-\frac{1}{2}} Q_{LM}$$

$$\left(1-\frac{2m}{r}\right)\frac{\partial h_1}{\partial r} - \left(1-\frac{2m}{r}\right)^{-1}\frac{\partial h_0}{\partial t} + \frac{2m}{r^2}h_1 = \chi i r^2\left[\tfrac{1}{2}L(L+1)(L-1)(L+2)\right]^{-\frac{1}{2}} D_{LM} .$$

Electric-Parity Harmonics

$$\left(1-\frac{2m}{r}\right)^2\frac{\partial^2 K}{\partial r^2} + \left(1-\frac{2m}{r}\right)\left(3-\frac{5m}{r}\right)\frac{1}{r}\frac{\partial K}{\partial r} - \left(1-\frac{2m}{r}\right)^2\frac{1}{r}\frac{\partial H_2}{\partial r} - \left(1-\frac{2m}{r}\right)\frac{1}{r^2}(H_2-K) - \left(1-\frac{2m}{r}\right)\frac{1}{2r^2}L(L+1)(H_2+K) = \chi A^{(m)}_{LM}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial K}{\partial r} + \frac{1}{r}(K-H_2) - \frac{m}{r(r-2m)}K\right) - \frac{L(L+1)}{2r^2}H_1 = \frac{i\chi}{\sqrt{2}}A^{(1)}_{LM}$$

$$\left(1-\frac{2m}{r}\right)^{-2}\frac{\partial^2 K}{\partial t^2} - \left(\frac{1-\frac{m}{r}}{r-2m}\right)\frac{\partial K}{\partial r} - \left(\frac{2}{r-2m}\right)\frac{\partial H_1}{\partial t} + \frac{1}{r}\frac{\partial H_0}{\partial r} + \frac{1}{r(r-2m)}(H_2-K) + \frac{L(L+1)}{2r(r-2m)}(K-H_0) = \chi A_{LM}$$

$$\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)H_1\right] - \frac{\partial}{\partial t}(H_2+K) = -\chi i r\left[\tfrac{1}{2}L(L+1)\right]^{-\frac{1}{2}} B^{(m)}_{LM}$$

$$-\frac{\partial H_1}{\partial t} + \left(1-\frac{2m}{r}\right)\frac{\partial}{\partial r}(H_2-K) + \frac{2m}{r^2}H_0 + \left(\frac{1-\frac{m}{r}}{r}\right)(H_2-H_0) = -\chi(r-2m)\left[\tfrac{1}{2}L(L+1)\right]^{-\frac{1}{2}} B_{LM}$$

$$-\left(1-\frac{2m}{r}\right)^{-1}\frac{\partial^2 K}{\partial t^2} + \left(1-\frac{3m}{r}\right)\frac{\partial^2 K}{\partial r^2} + \left(1-\frac{m}{r}\right)\frac{2}{r}\frac{\partial K}{\partial r} - \left(1-\frac{2m}{r}\right)^{-1}\frac{\partial^2 H_2}{\partial t^2} + 2\frac{\partial^2 H_1}{\partial r \partial t} - \left(1-\frac{2m}{r}\right)\frac{\partial^2 H_2}{\partial r^2}$$
$$+ \frac{2}{r-2m}\left(1-\frac{m}{r}\right)\frac{\partial H_1}{\partial t} - \frac{1}{r}\left(1-\frac{m}{r}\right)\frac{\partial H_2}{\partial r} - \frac{1}{r}\left(1+\frac{m}{r}\right)\frac{\partial H_0}{\partial r} - \frac{1}{2r^2}L(L+1)(H_2-H_0) = -\chi\sqrt{2}\,G_{LM}$$

$$\frac{1}{2}(H_0-H_2) = -\chi r^2\left[\tfrac{1}{2}L(L+1)(L-1)(L+2)\right]^{-\frac{1}{2}} F_{LM} .$$

APPENDIX P: FOURIER TRANSFORMED FIELD EQUATIONS

We may write the Fourier transform of the field equations in Appendix O as:

Magnetic-Parity Equations

$$\omega^2 h_{1_{Lm}} - i\omega \frac{dh_{2_{Lm}}}{dr} + 2i\omega \frac{h_{2_{Lm}}}{r} - (r-2m)(L-1)(L+2)\frac{h_{2_{Lm}}}{r^2} = -i\mathcal{K}\left[\tfrac{1}{2}L(L+1)\right]^{\frac{1}{2}}(r-2m)Q_{Lm}^{(m)}(\omega,r)$$

$$(r-2m)\left(\frac{d^2h_{2_{Lm}}}{dr^2} + i\omega\frac{dh_{2_{Lm}}}{dr} + 2i\omega\frac{h_{2_{Lm}}}{r}\right) + \left[\frac{4m}{r^2} - \frac{L(L+1)}{r}\right]h_{2_{Lm}} = -\mathcal{K}\left[\tfrac{1}{2}L(L+1)\right]^{\frac{1}{2}} r^3 Q_{Lm}^{(m)}(\omega,r)$$

$$-(r-2m)\frac{dh_{2_{Lm}}}{dr} - \frac{i\omega r^2}{r-2m}h_{2_{Lm}} - \frac{2m}{r}h_{1_{Lm}} = \mathcal{K}\, i\left[\tfrac{1}{2}L(L+1)(L-1)(L+2)\right]^{\frac{1}{2}} r^3 D_{Lm}^{(m)}(\omega,r).$$

Electric-Parity Equations

$$\left(1-\frac{2m}{r}\right)^2\frac{d^2K}{dr^2} + \frac{1}{r}\left(1-\frac{2m}{r}\right)\left(3-\frac{5m}{r}\right)\frac{dK}{dr} - \frac{1}{r}\left(1-\frac{2m}{r}\right)^2\frac{dH_0}{dr} - \frac{1}{r^3}\left(1-\frac{2m}{r}\right)(H_2-K) - L(L+1)\frac{1}{2r^2}\left(1-\frac{2m}{r}\right)(H_2+K) = \mathcal{K}A_{Lm}^{(e)}(\omega,r)$$

$$-i\omega^2\left(1-\frac{2m}{r}\right)^{-2}K + L(L+1)\frac{1}{2r^2}\left(1-\frac{2m}{r}\right)^{-1}(K-H_0) + \frac{1}{r(r-2m)}(H_2-K) + \frac{2i\omega}{r}\left(1-\frac{2m}{r}\right)^{-1}H_1 - \frac{1}{r}\left(\frac{1-\frac{m}{r}}{1-\frac{2m}{r}}\right)\frac{dK}{dr} + \frac{1}{r}\frac{dH_0}{dr} = \mathcal{K}A_{Lm}^{(o)}(\omega,r)$$

$$-i\omega\left[\frac{dK}{dr} + \frac{1}{r}(K-H_2) - \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}K\right] - L(L+1)\left(\frac{1}{2r^2}\right)H_1 = \tfrac{1}{2}\mathcal{K}\, i\sqrt{2}\, A_{Lm}^{(i)}(\omega,r)$$

$$\frac{d}{dr}\left[\left(1-\frac{2m}{r}\right)H_1\right] + i\omega(H_2+K) = -\mathcal{K}\, i\left[\tfrac{1}{2}L(L+1)\right]^{\frac{1}{2}} r B_{Lm}^{(e)}(\omega,r)$$

$$i\omega H_1 + \frac{2m}{r^2}H_0 + \left(1-\frac{2m}{r}\right)\frac{d}{dr}(H_0-K) + \frac{1}{r}\left(1-\frac{m}{r}\right)(H_2-H_0) = -\mathcal{K}\left[\tfrac{1}{2}L(L+1)\right]^{\frac{1}{2}}(r-2m)B_{Lm}^{(o)}(\omega,r)$$

$$r^2\omega^2\left(1-\frac{2m}{r}\right)^{-1}(H_2+K) + r(r-2m)\frac{d^2}{dr^2}(K-H_0) - 2i\omega r^2\frac{dH_1}{dr} - 2i\omega r\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)^{-1}H_1$$
$$+ 2(r-2m)\frac{dK}{dr} - r\left(1-\frac{m}{r}\right)\frac{dH_2}{dr} - r\left(1+\frac{m}{r}\right)\frac{dH_0}{dr} - \tfrac{1}{2}L(L+1)(H_2-H_0) = -\mathcal{K}\sqrt{2}\, r^2 G_{Lm}(\omega,r)$$

$$\tfrac{1}{2}\left(H_{0_{Lm}} - H_{2_{Lm}}\right) = -2\mathcal{K}\left[2L(L+1)(L-1)(L+2)\right]^{\frac{1}{2}} r^2 F_{Lm}(\omega,r).$$

APPENDIX Q: THE AUXILIARY RADIAL FUNCTIONS

Having determined the auxiliary radial functions R_{LM} as the solutions of the "Schrödinger" equations (3-66) and (3-67), one can derive the original Regge-Wheeler radial functions and determine $h_{\mu\nu}$ with the following relations:

Magnetic (Odd) Parity Perturbations

$$h_{1_{LM}} = \left(\frac{r^2}{r-2m}\right) R^{(m)}_{LM} \tag{Q-1}$$

$$h_{0_{LM}} = \frac{i}{\omega}\frac{d}{dr}\left(rR^{(m)}_{LM}\right) - \frac{\mathcal{K} r(r-2m)}{\omega\left[\frac{1}{2}L(L+1)(L-1)(L+2)\right]^{\frac{1}{2}}} D^{(\omega,r)}_{LM} . \tag{Q-2}$$

Electric (Even) Parity Perturbations

$$K_{LM} = \left[\frac{\lambda(\lambda+1)r^2 + 3\lambda m r + 6m^2}{r^2(\lambda r + 3m)}\right] R^{(e)}_{LM} + \left(1 - \frac{2m}{r}\right)\frac{d}{dr} R^{(e)}_{LM} \tag{Q-3}$$

$$H_{1_{LM}} = -i\omega\left[\frac{\lambda r^2 - 3\lambda m r - 3m}{(r-2m)(\lambda r + 3m)}\right] R^{(e)}_{LM} - i\omega r \frac{d}{dr} R^{(e)}_{LM} \tag{Q-4}$$

$$H_{0_{LM}} = \left[\frac{\lambda r(r-2m) - \omega^2 r^4 + m(r-3m)}{(r-2m)(\lambda r + 3m)}\right] K_{LM} + \left[\frac{m(\lambda+1) - \omega^2 r^3}{i\omega r(\lambda r + 3m)}\right] H_{1_{LM}} - \tilde{B}_{LM} \tag{Q-5}$$

$$H_{2_{LM}} = H_{0_{LM}} + 2\mathcal{K} r^2 \left[\frac{1}{2}L(L+1)(L-1)(L+2)\right]^{-\frac{1}{2}} F_{LM} . \tag{Q-6}$$

Auxiliary Source Functions

$$\tilde{B}_{LM} = \frac{\mathcal{K} r^2(r-2m)}{\lambda r + 3m}\left\{A_{LM} + \left[\frac{1}{2}L(L+1)\right]^{-\frac{1}{2}} B_{LM}\right\} - \left(\frac{\frac{1}{2}\mathcal{K}\sqrt{2}}{\lambda r + 3m}\right)\frac{mr}{\omega} A^{(1)}_{LM} \tag{Q-7}$$

$$\tilde{C}_{1_{LM}} = -\frac{\mathcal{K}}{\sqrt{2}\,\omega} A^{(1)}_{LM} - \frac{1}{r}\tilde{B}_{LM} \tag{Q-8}$$

$$\tilde{C}_{2_{LM}} = \left[\frac{\mathcal{K} r^2}{i\omega}\frac{\left[\frac{1}{2}L(L+1)\right]^{-\frac{1}{2}}}{r-2m}\right] B^{(0)}_{LM} - \left(\frac{ir}{r-2m}\right)\tilde{B}_{LM} . \tag{Q-9}$$

APPENDIX R: THE BONDI NEWS FUNCTION

Bondi's representation of gravitational radiation begins with his investigations into negative mass in General Relativity (1957a) - negative mass originally being considered by Föppl (1896) and Schuster (1898) although being discussed again by Einstein & Infeld (1949) - and his rediscovery of certain plane-fronted waves (1957b, 1959). He publishes in 1960 the first results of his multipole approximation method, and then in 1962 in conjunction with van der Burg & Metzner, Bondi establishes a radiative solution for an isolated axisymmetric system whose mass decreases monotonically due to outgoing radiation.

Bondi's Metric & The "Main Equations"

Bondi's metric for axial symmetry is

$$ds^2 = \left(Vr^{-1}\Omega^{2\beta} - U^2 r^2 \Omega^{2\gamma}\right)du^2 + 2\Omega^{2\beta}\,du\,dr + 2Ur^2\Omega^{2\gamma}\,du\,d\theta - r^2\left(\Omega^{2\gamma}d\theta^2 + \Omega^{-2\gamma}\sin^2\theta\, d\phi^2\right) \tag{R-1}$$

for coordinates ($u=t-r$, r, θ, ϕ). That is to say,

$$g_{\mu\nu} = \begin{bmatrix} \left(\frac{V}{r}\Omega^{2\beta} - U^2 r^2 \Omega^{2\gamma}\right) & \Omega^{2\beta} & Ur^2\Omega^{2\gamma} & 0 \\ \Omega^{2\beta} & 0 & 0 & 0 \\ r^2\Omega^{2\gamma} & 0 & -r^2\Omega^{2\gamma} & 0 \\ 0 & 0 & 0 & -r^2\Omega^{-2\gamma}\sin^2\theta \end{bmatrix} \tag{R-2}$$

$$g^{\mu\nu} = \begin{bmatrix} 0 & \Omega^{-2\beta} & 0 & 0 \\ \Omega^{-2\beta} & -V\Omega^{-2\beta}/r & U\Omega^{-2\beta} & 0 \\ 0 & U\Omega^{-2\beta} & -\Omega^{-2\gamma}/r^2 & 0 \\ 0 & 0 & 0 & -\Omega^{2\gamma}/r^2\sin^2\theta \end{bmatrix}. \tag{R-3}$$

Bondi assumes throughout a stationary-radiative-stationary transition as depicted in Figure 6 where the radiative solution is represented by Einstein's field equations in vacuo,

$$R_{\mu\nu} = 0\,, \tag{3-54}$$

just as is assumed in the Regge-Wheeler formalism. Using arguments based upon a contraction of the Bianchi identities

$$g^{\alpha\epsilon}G_{\mu\alpha;\epsilon} = 0\,, \tag{R-4}$$

with $g^{\alpha\epsilon}$ in Eq (R-3), Bondi reduces the field equations $R_{\mu\nu} = 0$ to the "main equations" $R_{11} = R_{12} = R_{22} = R_{33} = 0$:

Bondi's "Main Equations"

$$0 = R_{11} = -4\left[\beta_1 - \tfrac{1}{2} r \gamma_1^2\right] r^{-1} \tag{R-5a}$$

$$0 = -2r^2 R_{12} = \left[r^4 e^{2(\gamma-\beta)} U_1\right]_1 - 2r^2\left[\beta_{12} - \gamma_{12} + 2\gamma_1\gamma_2 - 2\beta_2 r^{-1} - 2\gamma_1 \cot\theta\right] \tag{R-5b}$$

$$\begin{aligned} 0 = R_{22} e^{2(\beta-\gamma)} - r^2 R_3^3 e^{2\beta} &= 2V_1 + \tfrac{1}{2} r^4 e^{2(\gamma-\beta)} U_1^2 - r^2 U_{12} - 4rU_2 - r^2 U_1 \cot\theta \\ &\quad - 4rU\cot\theta + 2e^{2(\beta-\gamma)}\left[-1 - (3\gamma_2 - \beta_2)\cot\theta - \gamma_{22} + \beta_{22} + \beta_2^2 + 2\gamma_2(\gamma_2 - \beta_2)\right] \end{aligned} \tag{R-5c}$$

$$\begin{aligned} 0 = -R_3^3 e^{2\beta} r^2 &= 2r(r\gamma)_{01} + (1 - r\gamma_1)V_1 - (r\gamma_{11} + \gamma_1)V - r(1 - r\gamma_1)U_2 - r^2(\cot\theta - \gamma_2)U_2 \\ &\quad + r(2r\gamma_{12} + 2\gamma_2 + r\gamma_1\cot\theta - 3\cot\theta)U + e^{2(\beta-\gamma)}\left[-1 - (3\gamma_2 - 2\beta_2)\cot\theta - \gamma_{22} + 2\gamma_2(\gamma_2 - \beta_2)\right] \end{aligned} \tag{R-5d}$$

plus two supplementary conditions (Appendix 2 of Bondi, vd Burg, Metzner)

$$R_{00} = R_{02} = 0 \tag{R-6}$$

and the trivial equation $R_{01} = 0$. The various subscripts in Eqs (R-5) correspond to ordinary derivatives: $(0,1,2,3) \leftrightarrow (u,r,\theta,\phi)$. The Eqs (R-5a) and (R-5c) are referred to as the "hypersurface" equations because they involve only differentiation in the retarded hypersurface u=t-r=const, while Eq (R-5d) is known as the "standard" equation. Only the standard equation contains a derivative with respect to u.

Numerical Integration Scheme

Bondi's iterative solution of the "main equations" (R-5) consists of

$$\begin{array}{l} \rightarrow \text{Given } \gamma \quad \text{(for some hypersurface u)} \\ \qquad \downarrow \\ \qquad (\text{Eq } 5a) \rightarrow \beta \\ \qquad\qquad \downarrow \\ \qquad\qquad (\text{Eq } 5b) \rightarrow \left\{ \begin{array}{l} \left[r^4 e^{2(\gamma-\beta)}\right] U_1 - 6N(u,\theta) \\ U \quad + \quad L(u,\theta) \end{array} \right. \\ \qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad (\text{Eq } 5c) \rightarrow V \\ \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad (\text{Eq } 5d) \longrightarrow \gamma_0 + c_0(u,\theta)/r \quad . \\ \text{New } \gamma \leftarrow \text{New } u \leftarrow \gamma_0 \end{array} \tag{R-7}$$

The functions of integration

$$H(u,\theta),\ N(u,\theta),\ L(u,\theta),\ M(u,\theta),\ c(u,\theta) \qquad \text{(R-8)}$$

provide all the information or "news", as Bondi calls it, about the behaviour of the source. The functions (R-8) can be reduced to one single "news function."

Multipole Expansion & Sommerfeld Radiation Condition

Based upon causality arguments, Bondi assumes an expansion of γ in his metric (R-1) as

$$\gamma = \frac{f(t-r)}{r} + \frac{g(t-r)}{r^2} + \cdots \qquad \text{(R-9)}$$

in terms of retarded time $u = t - r$, which is equivalent to an outgoing (Sommerfeld) radiation condition.

Boundary conditions require that $L(u,\theta) = 0$, eliminating one of the functions in (R-8). Hence, the leading terms in the iterative scheme (R-7) are given by

$$\gamma = \frac{\overset{(1)}{c}(u,\theta)}{r} + \frac{c(u,\theta)}{r^2} + \cdots \qquad \text{(R-10a)}$$

$$\beta = H(u,\theta) - \tfrac{1}{4}c^2/r^2 + \cdots \qquad \text{(R-10b)}$$

$$U = 2H_2 e^{2H}/r + \cdots \qquad \text{(R-10c)}$$

$$V = r e^{2H}\left[1 + 2H_2\cot\theta + 4H_2^2 + 2H_{22}\right] + \cdots \qquad \text{(R-10d)}$$

Under coordinate or gauge transformations which do not change the Bondi metric (R-1), $H(u,\theta)$ can be eliminated. Furthermore, by virtue of the Sommerfeld radiation condition, Bondi argues that $g(t-r)$ in Eq(R-9) or $\overset{2}{c}(u,\theta)$ in (R-10a) must vanish. Thus, he arrives at the following canonical form of the "main equations," noting that $c = c(u,\theta)$ - and not the speed of light:

Bondi's Canonical "Main Equations"

$$\gamma = c(u,\theta)/r + \left[C(u,\theta) - c^3/6\right]/r^3 + \cdots \qquad \text{(R-11a)}$$

$$U = -(c_2 + 2c\cot\theta)/r^2 + \left[2N(u,\theta) + 3cc_2 + 4c^2\cot\theta\right]r^{-3} + \tfrac{1}{2}(3C_2 + 6C_1\cot\theta - 6cN - 8c^2c_2 - 8c^3\cot\theta)\, r^{-4} + \cdots$$

$$V = r - 2M(u,\theta) - \left[N_2 + N\cot\theta - c_2^2 - 4cc_2\cot\theta - \tfrac{1}{2}c^2(1 + 8\cot^2\theta)\right]r^{-1} - \tfrac{1}{2}\left[C_{22} + 3C_2\cot\theta - 2C_1 + 6N(c_2 + 2c\cot\theta) + 8c(c_2^2 + 3cc_2 + 2c^2\cot^2\theta)\right]r^{-2} + \cdots$$

$$4C_0 = 2c^2c_0 + 2cM + N\cot\theta - N_2 \qquad \text{(R-11d)}$$

The News Function $c(u,\theta)$

On the basis of the canonical form of the "main equations" (R-11), the supplementary conditions (R-6) reduce to

Supplementary Conditions

$$M_0 = -c_0^2 + \tfrac{1}{2}\left(c_{22} + 3c_2\cot\theta - 2c\right)_0 \qquad \text{(R-12a)}$$

$$-3N_0 = M_2 + 3cc_{02} + 4cc_0\cot\theta + c_0c_2 \ . \qquad \text{(R-12b)}$$

If M and N are given for one value of u (that is, on some hypersurface u = const), and $c = c(u,\theta)$ is given, the entire solution is determined. For this reason, Bondi calls $c(u,\theta)$ the "news function." Another appropriate name might be the "data function."

Correspondence of Bondi's equations with a static, non-radiating case reveals the physical nature and meaning of the functions of integration M, N, and C. An empty, axially symmetric metric such as Bondi's (R-1) can always be reduced to Weyl's form

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}\left[e^{2\sigma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right] \qquad \text{(R-13a)}$$

where

$$\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} = 0 \ . \qquad \text{(R-13b)}$$

Letting α represent the transformation parameter between (R-1) and (R-13), defined by

$$c = -\tfrac{1}{2}\left(\alpha_{22} - \alpha_2\cot\theta\right) \ , \qquad \text{(R-14)}$$

then one obtains (static case)

$$\begin{aligned} &\text{Bondi's Monopole} && M(u,\theta) = m \\ &\text{Bondi's Dipole} && N(u,\theta) = D\sin\theta - m\alpha_2 \\ &\text{Bondi's Quadrupole} && C(u,\theta) = \tfrac{1}{2}Q\sin^2\theta - \alpha_2 D\sin\theta + \tfrac{1}{2}m\alpha_2^2 \ , \end{aligned} \qquad \text{(R-15)}$$

where m, D, and Q are the mass monopole, dipole, and quadrupole respectively.

Depletion Of Mass Monopole & Radiation

$M(u,\Theta)$ in the "main equations" (R-11), based upon the correspondence to the static case (R-15), is called the "mass aspect." The mass of the system is defined as the mean value of $M(u,\Theta)$ over the sphere

$$m(u) = \frac{1}{2}\int_0^{\pi} M(u,\theta)\sin\theta\, d\theta \qquad \text{(R-16)}$$

Note that for the static case $M_0 = \partial M/\partial u = 0$ and (R-16) gives $m(u) = m$, as it should. Differentiating (R-16) with respect to u and using (R-12a),

$$m_0 = \frac{1}{2}\int_0^{\pi}\left[-c_0^2 + \frac{1}{2}\left(\cancelto{0}{c_{22} + 3c_2\cot\theta - 2c}\right)_0\right]\sin\theta\, d\theta$$

That is

$$m_0 = -\frac{1}{2}\int_0^{\pi} c_0^2 \sin\theta\, d\theta \qquad \text{(R-17a)}$$

or re-instating the derivative

$$\frac{\partial m}{\partial u} = -\frac{1}{2}\int_0^{\pi}\left(\frac{\partial c}{\partial u}\right)^2 \sin\theta\, d\theta \quad . \qquad \text{(R-17b)}$$

The mass m of the source or radiating system, then, is constant if and only if there is no "news":

$$\frac{\partial c}{\partial u} = 0 \quad .$$

If $c_0 \neq 0$ on some hypersurface u=const (Figure 8), then m_0 is negative and m decreases monotonically.

Outgoing radiation depletes the mass of the source, then, according to this formalism. The news function $c(u,\Theta)$ of Bondi is generalized to include another polarization $c(u,\Theta,\phi)$ by Sachs (1962) while cylindrical gravitational news is addressed by Stachel (1966).

APPENDIX S: GRAVITATIONAL RADIATION & TIDAL FRICTION

The inductive friction due to retardation and variations in the velocity of propagation in Figure G-1 of Appendix G must be contrasted with tidal friction, which constitutes spin-orbit interaction:

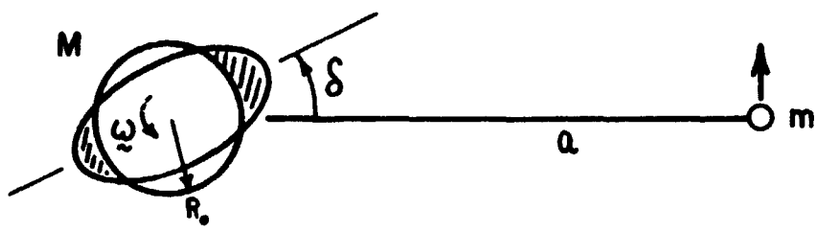

FIGURE S-1

It occurs in the earth-moon system and is due to imperfect elasticity with a resultant energy dissipation which characterizes real matter. Neglecting the eccentricity of the orbit, the rate of change of expansion of the conic $\dot{a}_T$ is approximately

$$\dot{a}_T \sim \left[3k_2\sqrt{\frac{G}{M}}\,R_\oplus^5 \sin 2|\delta|\right] a^{-11/2} \qquad \text{(S-1)}$$

where the Love number k_2 is defined in terms of the inertias C and A as

$$k_2 = \frac{3G(C-A)}{\omega^2 a^5} \quad . \qquad \text{(S-)}$$

Tidal friction (S-1) has the opposite effect as does gravitational radiation (that is, if gravitational radiation collapses the binary), which is given in Eq (5-5) as

$$\dot{a}_G = -\left[\frac{64}{5}\frac{G^3}{c^5} Mm(M+m)\right] a^{-3} . \qquad \text{(S-3)}$$

The critical value of the apse a_c where gravitational radiation and tidal friction remain in equilibrium is given by equating (S-1) and (S-3):

$$a_c = \frac{c^2}{G}\left[\frac{15}{64}\,\frac{k \sin 2|\delta|}{m(M+m)}\right]^{2/5} \frac{R_\oplus^2}{M^{3/5}} . \qquad \text{(S-4)}$$

For a binary system of two solar masses, for example, this is

$$a_c = \left[\left(\frac{15}{128}\right) k \sin 2|\delta|\right]^{2/5} \frac{(cR_o)^2}{GM_\odot^{7/5}} \qquad \text{(S-5)}$$

or

$$a_c = 2.25 \times 10^{-16} \left[k \sin 2|\delta|\right]^{2/5} R_o^2. \qquad \text{(S-6)}$$

An optimization of $k\sin 2|\delta|$ with $k_{max} = 3/2$ and $\delta = \pi/4$ gives

$$a_c = 128\, R_o^2 \qquad \text{(S-7)}$$

where R_o is the mean radius of the primary in Figure S-1.

A binary system collapses under gravitational radiation until it reaches a size of the order of (S-7) and then it attains a stable orbit, radiating away the angular momentum of the primary component until the latter's angular velocity ω in Figure S-1 is exhausted.

APPENDIX T: SCALAR, VECTOR, & TENSOR SYNCHROTRON MECHANISMS

The work of Davis, Ruffini, Tiomno, & Zerilli (1972) as well as Breuer, Ruffini, Tiomno, & Vishveshwara (1973) analyzes the power emitted from a particle in a circular orbit $r = (3 + \delta)M$ about a Schwarzschild "black hole." The treatment is general, so the vector case can readily be modified to treat vector theories of gravitation as discussed in Chapter 2. The even $(-1)^L$ or electric and the odd$(-1)^{L+1}$ or magnetic components for the power in Eq (5-12) are respectively:

Scalar

$$R^{\ell m}_{even} = \frac{4\pi s}{\sqrt{r_0(r_0-3M)}} Y^m_\ell(\tfrac{\pi}{2},0)\, G^{\ell m}_{even}$$

$$R^{\ell m}_{odd} : \text{None}$$

Vector

$$R^{\ell m}_{even} = \frac{4\pi q}{\sqrt{\ell(\ell+1)}} Y^m_\ell(\tfrac{\pi}{2},0) \frac{\partial}{\partial r_0} G^{\ell m}_{even}$$

$$R^{\ell m}_{odd} = -\frac{4\pi q}{\sqrt{\ell(\ell+1)}} \sqrt{\ell(\ell+1)-m(m+1)}\; Y^m_\ell(\tfrac{\pi}{2},0)\, G^{\ell m}_{odd}$$

Tensor

$$R^{\ell m}_{even} = 4\pi\mu u^0 Y^m_\ell(\tfrac{\pi}{2},0)\sqrt{\frac{(\ell-1)(\ell+2)}{\ell(\ell+1)}}\left\{\alpha(r_0)G^{\ell m}_{even} + \frac{1}{\lambda}\frac{\partial}{\partial r_0^*}\left[\beta(r_0)G^{\ell m}_{odd}\right]\right\}$$

$$R^{\ell m}_{odd} = 4\pi\mu u^0 Y^{m+1}_\ell(\tfrac{\pi}{2},0)\sqrt{\frac{\ell(\ell+1)-m(m+1)}{\ell(\ell-1)(\ell+1)(\ell+2)}}\frac{\partial}{\partial r_0^*}\left(r_0 G^{\ell m}_{odd}\right)$$

$$\lambda = \tfrac{1}{2}(\ell-1)(\ell+2) \qquad u^0 = (1-3M/r_0)^{-1/2} \approx \sqrt{3/\delta}$$

$$\alpha(r_0) = \frac{r-2M}{(r_0+3M/\lambda)^2}\left[1+\frac{1}{\lambda}+\frac{M}{\lambda r_0}-\frac{3M}{\lambda^2 r_0}+\frac{6M^2}{\lambda^2 r_0^2}\right]-\frac{2m^2-\ell(\ell+1)}{\ell(\ell+1)-2}r_0\omega_0^2 \qquad \beta(r_0) = \frac{1-2M/r_0}{1+3M/\lambda r_0}.$$

The Green's function terms are

$$G(r^*,r_0^*) = \tfrac{1}{4}i\omega_0^{-\frac{1}{2}}\ell^{-\frac{1}{2}}\frac{e^{-\pi\epsilon/8}}{\sqrt{\pi m\omega_0}}\,\ell^{-i\left[\frac{\pi}{8}+\frac{\epsilon}{4}\ln\frac{\ell}{4}\right]}\Gamma(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon)\,\ell^{i\omega r^*}$$

$$\frac{\partial G}{\partial r_0^*}(r^*,r_0^*) = \frac{1+i}{\sqrt{8\pi}}\sqrt{\ell\omega_0}\,\frac{e^{-\pi\epsilon/8}}{\sqrt{m\omega_0}}\,\ell^{-i\left[\frac{\pi}{8}+\frac{\epsilon}{4}\ln\frac{\ell}{4}\right]}\Gamma(\tfrac{3}{4}+\tfrac{1}{4}i\epsilon)\,\ell^{i\omega r^*}$$

where

$$\epsilon_{even} = 1+4p+|m|\delta \quad , 2p = \ell-|m|$$

$$\epsilon_{odd} = 3+4p+|m|\delta \quad , 2p = \ell-|m|-1 \quad .$$

APPENDIX U: THE PETROV-PIRANI CLASSIFICATION

The Petrov types in the Penrose diagram break down as follows:

(Most General)	I	Cylindrical Waves (Einstein&Rosen, Rosen, Marder, Bonnor, Weber & Wheeler) Axisymmetric field of Weyl (1918), Levi-Civita(1919) Empty space-times Isolated, bounded sources (oscillators)(Bonnor,1963) All real fields (Pirani, 1962) Field of an actual isolated system
	II	Robinson & Trautman (1960) Peres (1959)
	D	Schwarzschild Metric Robinson & Trautman (1960)
	III	Semi-far fields Robinson & Trautman (1960)
(Radiation)	N	Plane Waves (Brinkmann,1925; Rosen,1937;Takeno, Kundt) "Plane-Fronted" Waves (Robinson, 1956) Any radiating system of sources Far radiation zone fields Robinson & Trautman (1960)
(Flat)	O	Flat space-time

It should be apparent from these results of the Petrov-Pirani classification that the method is much too general.

APPENDIX V: GRAVITATIONAL RADIATION & COMPOSITE FIELD THEORY

One can consider the linearized graviton in the context of composite field theory. The composite nature of electromagnetic quanta is suggested by Jordan (1928) using statistical arguments. de Broglie (1932, 1934, 1936) suggests that a photon is composed of a neutrino and an anti-neutrino, establishing the basis of the neutrino theory of light. This is one form of a more general idea that elementary particles are composite particles built up out of a fundamental Spin-1/2 fermion, such as the neutrino. If a Spin-1/2 neutrino and a Spin-1/2 anti-neutrino couple together and form a Spin-1 boson, it is a photon (emission or creation); and if they mutually couple and cancel out they annihilate (absorption of a photon). In terms of hole theory, a neutrino jumps the energy gap of the neutrino sea, creating a neutrino-antineutrino pair (emission of a photon).

Jordan (1935) originally treats the de Broglie postulate in terms of a four-component neutrino theory, using a neutrino of momentum $\underset{\sim}{k}$ and an anti-neutrino of momentum $\underset{\sim}{p} - \underset{\sim}{k}$ (producing a Raman effect for neutrinos) However, this four-component work is brought to an abrupt end by Pryce (1938) when he demonstrates that it is not invariant under spatial rotation or arbitrary Lorentz transformation. Perkins (1965), however, overcomes this shortcoming in the Jordan-Pryce treatment and establishes an interesting four-component neutrino theory of the photon, formulated from two two-component Weyl equations. Perkins' four-component model is invariant under spatial rotation, although his resultant photon operators are not strictly Bose commutation relations, due to additional terms. However, Planck's radiation law still follows as it does from Bose statistics.

From the vector (Spin-1 boson) electromagnetic theory of gravitational radiation, then, there is a neutrino theory of the vector graviton - in direct analogy with the work of Jordan, de Broglie, Pryce, and Perkins. One can likewise construct a neutrino theory of the graviton comprised of four coupled neutrinos or two coupled photons, for the linear Spin-2 tensor theory of gravitational radiation. This implies, of course, that linear gravitons can decay into four neutrinos or two photons - e.g. upon interaction with matter or absorption. Wave resonances due to interaction between gravitons and photons both travelling at the same velocity of propagation (Gertsenshtein, 1962) are easily visualized from the aspect of composite field theory. Care must be taken, however, to account for the interaction of photons with charge and gravitons with mass.

One can speculate, furthermore, in a candid way with the bold conjecture that given a fundamental Spin-1/2 quantum (such as a Spin-1/2 boson) he could construct a unified field theory using composite field theory. If this quantum were a nonlinear Spin-1/2 graviton, for example, he could explain β-decay, μ-meson and π-meson decay in terms of gravitational radiation without need for the neutrino. Weber's high fluxes might then be due to μ-meson decay in the earth's atmosphere.

There is another reason for pursuing this line of reasoning. The creation and annihilation of photons is completely compatible with the Dirac treatment of his negative energy states, namely with positron-electron pairs. However, Dirac explicitly assumes that the infinite sea of negative energy exhibits no gravitational effect (if it does, we have Maxwell's postulate of an infinite sea of gravitational energy, in 1865). But if gravitational radiation exists and if we are to have a consistent quantum theory of radiation, then there must be a gravitational dual of Maxwell-Dirac spinor electrodynamics - for gravitons. Such a model may be afforded by neutrino theory, or by invoking (negative) mass conjugation rather than or as well as charge conjugation in the Dirac theory.

APPENDIX W: SPIN-2 & GRAVITATIONAL RADIATION

The sixteen (16) independent variables $h_{\mu\nu}$ in Einstein's field equations (3-1) and (3-6) must be coupled to a tensor source $T_{\mu\nu}$. Generally speaking, a tensor $T_{\mu\nu}$ consists of Spin-2, three Spin-1, and two Spin-0 admixtures:

$$[T_{\mu\nu}] \rightarrow \underset{\begin{pmatrix}\pm 2\\ \pm 1\\ 0\end{pmatrix}}{2} \oplus \underset{\begin{pmatrix}\pm 1\\ 0\end{pmatrix}}{1} \oplus \underset{\begin{pmatrix}\pm 1\\ 0\end{pmatrix}}{1} \oplus \underset{\begin{pmatrix}\pm 1\\ 0\end{pmatrix}}{1} \oplus \underset{(0)}{0} \oplus \underset{(0)}{0} \quad = 16 \qquad \text{(W-1)}$$

where the numbers in parentheses under Eq (W-1) represent the various spin components or helicities of each spin group. If $T_{\mu\nu}$ is symmetric,

$$T_{\mu\nu} = T_{\nu\mu} \qquad \text{(W-2)}$$

then (W-1) loses six (6) degrees of freedom or two of the vector Spin-1 admixtures and reduces to

$$[T_{\mu\nu}] \rightarrow \underset{\begin{pmatrix}\pm 2\\ \pm 1\\ 0\end{pmatrix}}{2} \oplus \underset{\begin{pmatrix}\pm 1\\ 0\end{pmatrix}}{1} \oplus \underset{(0)}{0} \oplus \underset{(0)}{0} \quad = 10 \quad . \qquad \text{(W-3)}$$

The spin parts are

$$[T_{\mu}{}^{\mu}] \rightarrow 0 \qquad \text{(W-4a)}$$

$$[T_{\mu\nu}{}^{,\nu}] \rightarrow 1 \oplus \odot \qquad \text{(W-4b)}$$

$$[T_{\mu\nu}{}^{,\mu\nu}] \rightarrow \odot \quad . \qquad \text{(W-4c)}$$

The dot is placed in the zero components of (W-4b) and (W-4c) to distinguish them as the same tensor Spin-0 contribution, distinct and different from the scalar Spin-0 in (W-4a) which is coupled to the trace of $T_{\mu\nu}$. Note that (W-1) through (W-4) are not equations but group-theoretical relations. It is (W-4a) which causes inconsistencies in massive radiation theory (Appendix X, Eq X-32), where $T_{\mu}{}^{\mu} \neq 0$ provides an excitation of the Spin-0 degree of freedom.

If energy is assumed conserved, then

$$T_{\mu\nu}{}^{,\nu} = 0 \quad . \qquad \text{(W-5)}$$

This means there is no source to which the Spin-1 and Spin-0 components in (W-4b) can couple. Energy conservation, then, causes these helicities to vanish, reducing the degrees of freedom by four (4) - eliminating namely: ± 1, 0, and 0. (W-5) also dictates that

$$T_{\mu\nu}{}^{,\mu\nu} = 0 \qquad \text{(W-6)}$$

is true. (See Fronsdal, 1958, on higher spins.)

Symmetry ($h_{\mu\nu}=h_{\nu\mu}$) and energy conservation have thus reduced the 16 unknowns to six (6). Invoking the Lorentz gauge condition

$$h_{\mu\nu}{}^{,\nu} = 0 \qquad \text{(W-7)}$$

on $h_{\mu\nu}$ eliminates four (4) more ("vector" gravitons) and reduces the degrees of freedom to two (2). These are the pure massless Spin-2 helicities, with all other spin admixtures removed. This result constitutes the so-called "transverse-traceless" gauge (Appendix Z).

In massive radiation theory for Spin-2 (Appendix X), on the other hand, the Lorentz (also called Hilbert-Lorentz) gauge condition does not produce four additional constraints, but rather it reduces the field equations in conjunction with energy conservation to provide a relation between the trace $n = h^{\mu}{}_{\mu}$ and the trace $T = T^{\mu}{}_{\mu}$ of the energy-momentum tensor. This gives only a fifth constraint, thereby reducing the six (6) independent variables to five (5), or 2S+1, where S is the spin. (See Eq X-28.)

In addition to the above group-theoretic representation of the gravitational Spin-2 problem in terms of reducible groups, we can recall the tensor spherical harmonics of Appendix K. Spin-0 is comprised of $\underset{\sim}{a}^{(0)}$, $\underset{\sim}{a}^{(1)}$, and $\underset{\sim}{a}$, Spin-1 of $\underset{\sim}{b}^{(0)}$, $\underset{\sim}{b}$, $\underset{\sim}{c}^{(0)}$, and $\underset{\sim}{c}$, and Spin-2 of $\underset{\sim}{d}$, $\underset{\sim}{g}$, and $\underset{\sim}{f}$. This representation, however, is often confusing because the respective harmonics consist of lower-spin admixtures and are reducible.

APPENDIX X: MASSIVE GRAVITATIONAL RADIATION

Massive radiation theory is an embodiment of the screening potential e^{-mr} due to Laplace (1846). It concerns itself with wave equations such as (2-7), (2-25), and (2-32) which include a finite rest-mass in the form of Klein-Gordon equations ($\hbar$=c=1):

$$\begin{aligned} (\Box - m_0^2)\,\phi &= -\varkappa\rho \\ (\Box - m_1^2)\,A^{\mu} &= -\varkappa J^{\mu} \\ (\Box - m_2^2)\,A^{\mu\nu} &= -\varkappa T^{\mu\nu} \quad . \end{aligned} \qquad \text{(X-1)}$$

The subscripts on the masses m_i (i = 0,1,2) correspond to Spin-0, Spin-1, and Spin-2 for the respective scalar, vector, and tensor wave equations.

Radiation theory with quanta of finite rest-mass has been well developed in electrodynamics, based upon the de Broglie postulate (1924) of a finite photon mass and its subsequent incorporation into Proca's vector meson theory (1936). The resultant massive electrodynamics is well-behaved since it reduces to Maxwell's theory as the photon mass goes to zero.

However, such is not the case with conventional treatments of massive gravitational radiation theory using Einstein's General Theory of Relativity. An assumption of a graviton rest-mass does not reduce to Einstein's theory in the massless limit, indicating either that (i) Einstein's theory is particularly unique, or that (ii) conventional representations of massive radiation theory are inadequate and bolder hypotheses may be justified. Einstein's theory is said to have no "neighbors," meaning it does not tolerate neighboring theories with a finite rest-mass.

Massive radiation theory is the contemporary expression of the cut-off potential of Laplace, later investigated by Neumann (1874, 1896), Seeliger (1895), and Yukawa (1935). That is, an assumption of a finite rest-mass m_0 for the scalar wave equation in (X-1) modifies the classical Newtonian potential with an exponential cut-off:

$$\phi = -\frac{\varkappa}{4\pi}\,\frac{e^{-m_0 r}}{r} = \phi_{m=0}\,e^{-m_0 r} \quad . \qquad \text{(X-2)}$$

In this fashion, the Newtonian or Coulombic field is screened and does not have an infinite cross-section, an artifice also employed in plasma theory.

Massive Electrodynamics

Massive electrodynamics is important because it is manifestly covariant, a feature which is not true for the massless Maxwell theory. It has been developed by many authors, among them Wentzel (1943), Kallen (1972), Jauch & Rohrlich (1955), Bjorken & Drell (1965), and Bogoliubov & Shirkov (1959).

de Broglie (1924) has demonstrated that if one assumes the photon to have a finite rest mass m_1, then the energy relation

$$E = h\nu = \frac{m_1 c^2}{\sqrt{1-\beta^2}} \qquad \text{(X-3)}$$

necessarily implies that the vacuum has a refractive index

$$n = \beta^{-1} = 1\Big/\sqrt{1-\frac{m_1^2 c^4}{h^2\nu^2}} \approx 1+\frac{1}{2}\left(\frac{m_1 c^2}{h}\right)^2 \nu^{-2} \quad , \qquad \text{(X-4)}$$

and behaves as a dispersive medium. This idea is important because it is the basis of Proca's extension of Maxwell's equations for massive quanta.

The massive Maxwell-Proca equations take the form

$$\begin{aligned} \nabla\cdot\underset{\sim}{E} &= 4\pi\rho - \mu^2\phi \\ \nabla\times\underset{\sim}{E} &= -\tfrac{1}{c}\,\partial\underset{\sim}{H}/\partial t \\ \nabla\cdot\underset{\sim}{H} &= 0 \\ \nabla\times\underset{\sim}{H} &= \tfrac{4\pi}{c}\underset{\sim}{J} + \tfrac{1}{c}\,\partial\underset{\sim}{E}/\partial t - \mu^2\underset{\sim}{A} \end{aligned} \qquad \text{(X-5)}$$

where $\mu = m_0 c/\hbar$. These equations represent a massive vector field coupled to a conserved current.

$$\begin{aligned} F_{\alpha\beta}{}^{,\beta} + \mu^2 A_\alpha &= \frac{4\pi}{c} J_\alpha \\ F_{\alpha\beta} &= \partial_\alpha A_\beta - \partial_\beta A_\alpha \quad . \end{aligned} \qquad \text{(X-6)}$$

Their wave equation is

$$(\Box - \mu^2) A_\alpha = \frac{4\pi}{c} J_\alpha \qquad \text{(X-7)}$$

with a Poynting vector

$$\underset{\sim}{S} = \frac{c}{4\pi}\left(\underset{\sim}{E}\times\underset{\sim}{B} + \mu^2 \underset{\sim}{A} A_0\right) \qquad \text{(X-8)}$$

and an energy density

$$\mathcal{E} = \frac{1}{8\pi}\left[\underset{\sim}{E}^2 + \underset{\sim}{H}^2 + \mu^2\left(\underset{\sim}{A}^2 + \phi^2\right)\right] . \qquad \text{(X-9)}$$

The addition of the massive quanta to the Maxwell theory increases the number of degrees of freedom in the field due to the fact that the Lorentz condition no longer reduces these degrees of freedom. Instead,

$$\mu^2\left(A^{\alpha}{}_{,\alpha}\right) = \frac{4\pi}{c} J^{\alpha}{}_{,\alpha} \qquad \text{(X-10)}$$

relates current conservation and the Lorentz condition. In addition to the two transverse polarizations of the photon, there are now a longitudinal photon and a scalar (time-like) photon due to the presence of a photon rest mass. (See Källén.)

From all of the above equations it is apparent that the massless limit $\mu^2 = m_\gamma{}^2 = 0$ reduces to the Maxwell theory. However, massive radiation theory is not as simple as this because the most significant relation above is (X-9) for the energy density. The massive energy density must be positive semi-definite or everything will collapse into any negative energy states that happen to exist.*

For this reason, massive radiation theory is usually developed using an action principle. Provided the action I is expressed in a canonical "$p\dot{q}$ - H" form,

$$I = \int \mathcal{L}\, d^4x = \int (p\dot{q} - \mathcal{H})\, d^4x$$

then the Hamiltorian energy density $\mathcal{H}$ can be readily identified in the Lagrangian density $\mathcal{L}$ and hence examined for negative energy states.

---

*This problematic feature of Lagrangian field theory can be alleviated by a massive exclusion principle, discussed in the final section of this appendix.

The first-order Proca action coupled to a prescribed conserved source J^α

$$I=\int d^4x\left[\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}-\tfrac{1}{2}F^{\alpha\beta}(\partial_\alpha A_\beta-\partial_\beta A_\alpha)-\tfrac{1}{2}m^2A_\alpha A^\alpha+J^\alpha A_\alpha\right] \qquad (X\text{-}11)$$

has been decomposed by Deser (1972) into its transverse and longitudinal parts, demonstrating that the longitudinal mode constitutes a scalar field decoupled from the current. The energy density is

$$T_{00}=\left\{\tfrac{1}{2}\left[(\underset{\sim}{E}^T)^2+(\nabla\times\underset{\sim}{A}^T)^2+m^2(\underset{\sim}{A}^T)^2\right]+\tfrac{1}{2}(\underset{\sim}{E}^L)^2+\underset{\sim}{E}^T\cdot\underset{\sim}{E}^L+m^2\underset{\sim}{A}^T\cdot\underset{\sim}{A}^L\right\}$$
$$+\tfrac{1}{2}\left\{m^2(\underset{\sim}{A}^L)^2+\tfrac{1}{m^2}(\nabla\cdot\underset{\sim}{E}-J^0)^2\right\}. \qquad (X\text{-}12)$$

The "T" and "L" superscripts represent "transverse" and "longitudinal" respectively. The second term in braces is non-Maxwellian in that it remains coupled to gravitation in the m=0 limit, which may have implications about the frequency dependence of the gravitational deflection of light (See e.g. Woodward & Yourgrau, 1972).

Massive Gravitational Radiation

The current dilemma in the theory of massive gravitational radiation can be best characterized in terms of the source theory (Schwinger, 1970) associated with the Feynman amplitudes of quantum electrodynamics. One can represent the gravitational interaction as the exchange of a Feynman propagator $D_{\mu\nu,\alpha\beta}$ between one source $T^{\mu\nu}(X)$ and another $T^{\alpha\beta}(X')$:

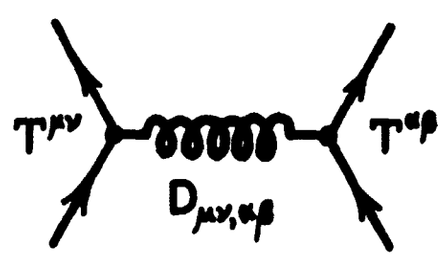

FIGURE X-1

This propagator is given in a momentum space representation by

$$D_{\mu\nu,\alpha\beta}(p) = \frac{\Pi_{\mu\nu,\alpha\beta}(p)}{p^2+m^2-i\epsilon} \qquad \text{(X-13)}$$

where $\Pi_{\mu\nu,\alpha\beta}$ is known as the spin projection operator. These are stated by Ogievetsky & Polubarinov (1965).

The resultant propagator for a <u>massive</u> graviton is (van Dam & Veltman, 1970, 1972):

$$D^{m\neq 0}_{\mu\nu,\alpha\beta} = \frac{\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha})-\frac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2+m^2-i\epsilon} \qquad \text{(X-14)}$$

which reduces in the massless limit (m=0) to

$$D^{m\to 0}_{\mu\nu,\alpha\beta} = \frac{\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha})-\frac{1}{3}\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2} \quad . \qquad \text{(X-15)}$$

In contrast, the graviton propagator for the massless case is

$$D^{m=0}_{\mu\nu,\alpha\beta} = \frac{\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha})-\frac{1}{2}\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2} \quad . \qquad \text{(X-16)}$$

Comparison of the second term in the numerator of (X-15) and (X-16) demonstrates that the two theories are incompatible. If we take the case where the delta functions are all unity, the numerators of (X-15) and (X-16) are respectively

$$\frac{1}{2}(2)-\frac{1}{3} = \frac{2}{3} \qquad \text{and} \qquad \frac{1}{2}(2)-\frac{1}{2} = \frac{1}{2} \quad .$$

This means that

$$D^{m=0}_{\text{massive}} = \frac{3}{4}\, D^{m=0}_{\text{massless}} \qquad \text{(X-17)}$$

whereby the deflection of light δ is changed as

$$\delta^{(m\to 0)}_{\text{massive}} = \frac{3}{4}\,\delta_{\text{massless}} \quad , \qquad \text{(X-18a)}$$

the massive radar time delays are likewise 3/4 of the massless theory, and the precession of the perihelion $\Delta\theta$ is

$$\Delta\theta^{(m\to 0)}_{\text{massive}} = \frac{2}{3}\Delta\theta_{\text{massless}} \quad . \qquad \text{(X-18b)}$$

The Jordan-Brans-Dicke (Brans & Dicke, 1962) scalar-tensor theory gives similar results when its dimensionless parameter ω=0.

$$\delta_{B\text{-}D} = \left(\frac{2\omega+3}{2\omega+4}\right)\delta_{\text{Massless Einstein}} \qquad \text{(X-19a)}$$

$$\Delta\theta_{B\text{-}D} = \left(\frac{3\omega+4}{3\omega+6}\right)\Delta\theta_{\text{Massless Einstein}} \quad . \qquad \text{(X-19b)}$$

$\omega=\infty$ in (X-19) gives the Einstein theory. The Eddington-Robertson metric

$$d\tau^2 = \left(1-2\alpha\frac{GM}{\rho}+2\beta\frac{G^2M^2}{\rho^2}+\cdots\right)dt^2 - \left(1+2\gamma\frac{GM}{\rho}+\cdots\right)\left(d\rho^2+\rho^2 d\Omega^2\right) \qquad \text{(X-20a)}$$

comparatively speaking is

$\alpha=\beta=\gamma=1$		Einstein (Massless)
$\alpha=\beta=1$	$\gamma=\frac{1}{2}$	Einstein (Massive, m=0)
$\alpha=\beta=1$	$\gamma=(\omega+1)/(\omega+2)$	Brans-Dicke

for these three representations. Drawing further upon the ω=0 analogy between the Brans-Dicke and the massive theory, the massive Eddington-Robertson metric in the massless limit (m=0) is:

$$d\tau^2 = \left(1-\frac{2GM}{\rho}+\frac{2G^2M^2}{\rho^2}+\cdots\right)dt^2 - \left(1+\frac{GM}{\rho}+\cdots\right)\left(d\rho^2+\rho^2 d\Omega^2\right) \qquad \text{(X-20b)}$$

This is not the Schwarzschild solution.

Massive Gravitational Action

Many of the problems associated with massive gravitation theory have been addressed by van Dam & Veltman (1970, 1972), Zeldovich (1970), Boulware & Deser (1972a, 197 b), Deser (1973), van Nieuwenhuizen (1973), Vainshtein (1972), Sexl (1967), Freund, et al (1969), and Ogievetsky & Polubarinov (1965). Massive two-tensor theories have been investigated by Isham, Salam, & Strathdee (1971), Aichelburg et al (1971), Aichelburg & Mansouri (1972), Aragone & Chela-Flores (1972), Aichelburg(1973), and Lawrence & Toton(1972).

The definitive work at present has proceeded with the Hilbert-Palatini formulation of the action principle (Arnowitt, Deser, & Misner, 1962) which represents the full Einstein action as

$$I = \int ({}^4R) d^4x = \int (\pi^{ij}\dot{g}_{ij} - NR^0 - N_i R^i) d^4x \quad , \qquad \text{(X-21)}$$

where Latin indices vary over 3-space and where π^{ij} is the conjugate momentum:

$$N = (-g_{00})^{-\frac{1}{2}} \qquad N_i = g_{0i} \qquad R^i = -2\pi^{ij}{}_{,j}$$
$$-R^0 = {}^3R + g^{-\frac{1}{2}}\left(\tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij}\right) \qquad \text{(X-21a)}$$

The linearized massless action (including the coupling to the source $\frac{1}{2}h_{\mu\nu}T^{\mu\nu}$) is

$$I_L(m=0) = \int d^4x \Big[\pi^{ij}\dot{g}_{ij} - \left(\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2\right) + 2N_i\pi^{ij}{}_{,j}$$
$$+ \tfrac{1}{2}h_{00}(\nabla^2 h_{ii} - h_{ij,ij}) + {}^3R + \tfrac{1}{2}h_{ij}T^{ij}$$
$$+ N_i T^{0i} + \tfrac{1}{2}h_{00}T^{00}\Big] . \qquad \text{(X-22)}$$

In this action (X-22), N_i and h_{00} constitute Lagrangian multipliers - upon whose variation are recovered the auxiliary conditions ($h=h_{ii}$)

$$\nabla^2 h - h_{ij,ij} = -T^{00} \qquad \text{(X-23a)}$$

$$\pi^{ij}{}_{,j} = -\tfrac{1}{2}T^{0i} \quad . \qquad \text{(X-23b)}$$

Alternatively, one can vary with respect to N and N_i in (X-21) and recover the constraint

$$R^\mu = 0 \quad . \qquad \text{(X-24)}$$

Hence, the field equations following from (X-21) or (X-22) consist of 16 unknowns which are reduced to 10 upon the assumption of symmetry ($h_{\mu\nu}=h_{\nu\mu}$). These 10 are reduced to 6 by energy conservation

$$T^{\mu\nu}{}_{;\nu} = 0 \qquad \text{(X-25)}$$

and further reduced to two (2) by the auxiliary conditions (X-23) or (X-24). Thence, the massless graviton has two (2) degrees of freedom (±2 helicity).

Now, the Fierz-Pauli (1939) massive action I_m (X-26)

$$I_m = -\frac{1}{4}m^2\int d^4x\left[h_{\mu\nu}h^{\mu\nu} - \left(h^{\alpha}_{\alpha}\right)^2\right] = -\frac{1}{4}m^2\int d^4x\left[h_{ij}^2 - h^2 + 2hh_{00} - 2N_i^2\right]$$

can be added to (X-22) to give a total massive action

$$I = I_L + I_m \quad . \qquad \text{(X-27)}$$

The immediate consequence is that variation of N_i and h_{00} in (X-27) no longer gives the massless auxiliary conditions (X-23) but rather

$$\nabla^2 h - h_{ij,ij} + T^{00} = m^2 h \qquad \text{(X-28a)}$$

$$2\pi^{ij}{}_{,j} + m^2 N_i = -T^{0i} \quad . \qquad \text{(X-28b)}$$

(X-28b), as a constraint, simply recovers N_i itself

$$N_i = m^{-2}\left(2\pi^{ij}{}_{,j} + T^{0i}\right) \qquad \text{(X-28c)}$$

and does not eliminate three (3) other variables. Hence only (X-28a) constrains the six (6) degrees of freedom subsequent to energy conservation in (X-25). (X-28a), then, reduces the degrees of freedom by one, to five (5). The massive theory has five (5) or 2S+1 helicities associated with a Spin-2 graviton of finite rest mass.

Orthogonal Decomposition Of Massive Action

Boulware & Deser (1972) have considered an orthogonal decomposition of h_{ij}

$$h_{ij} = h_{ij}^{TT} + h_{i,j}^{T} + h_{j,i}^{T} + \tfrac{1}{2}\left(\delta_{ij} - \nabla_i\nabla_j\nabla^{-2}\right)h^{T} + 2h_{,ij}^{L} \qquad \text{(X-29)}$$

which in turn decomposes I in (X-27) into a "transverse-traceless" (I_{TT}), a "vector" (I_V), and a "scalar" (I_S) part:

$$I = I_{TT} + I_V + I_S \quad . \qquad \text{(X-30)}$$

This is further decomposed by rescaling the action in order to eliminate problematic m^{-2} singularities and to get it in a canonical "$p\dot{q}$ - H" form:

$$I = I_{TT}^{(\pm 2)} + \left(I_C^{(1)} + I_V^{(0)}\right) + \left(I_C^{(0)} + I_S^{(0)}\right) \quad . \qquad \text{(X-31)}$$

I_C represents instantaneous Coulomb contributions, while the superscripts correspond to spin components or helicity. All contributions in (X-31) uncouple and vanish as the graviton mass $m \to 0$ except the first and last

$$I(m=0) = I_{TT}^{(\pm 2)} + I_S^{(0)} \quad . \qquad \text{(X-32)}$$

Hence, the massive theory retains a scalar component coupled to the trace $T^{\alpha}{}_{\alpha}$ of the energy-momentum tensor (See Appendix W). This massive theory, then, is a scalar-tensor theory. It is the Spin-0 or scalar contribution which creates the inconsistencies. The right-hand term in (X-32) is that which derives from the trace in (W-4a).

Speculations On The Problems Of Massive Limits

The literature has investigated ways around the incompatibilities which arise in massless limits, using such techniques as indefinite metrics, indefinite probability, broken symmetries, Goldstone bosons, and the cosmological term (Appendix Y). As remarked earlier, bolder hypotheses may be warranted due to the inadequacies of the conventional massive theory. Therefore, some additional approaches are stated here.

(A) One can suppress the Spin-0 helicity coupled to the trace $T^{\mu}{}_{\mu} \neq 0$ by employing both advanced and retarded Green's functions in the determination of the graviton propagator in Figure (X-1). This is tantamount to removing the Sommerfeld radiation condition for the Spin-0 exchange, and constitutes a rejection of the Feynman propagator.

(B) The Bel-Robinson tensor has zero trace in empty space. It is possible that a massive gravitation theory (following Einstein) could be developed

around it, guaranteeing that no source can couple to the problematic Spin-0 component. This tensor is fourth-order, but such a theory would be based upon its contracted, Spin-2 form.

(C) Recalling the gauge conditions discussed Appendix W for the Spin-2 problem, a new gauge prescription may provide possibilities. In Eq (W-5), the conservation of energy

$$T^{\mu\nu}{}_{,\nu} = 0 \qquad \text{(W-5)}$$

forces the Spin-0 helicity of $T^{\mu\nu}$ to be zero:

$$T^{\mu\nu}{}_{,\mu\nu} = 0 \quad . \qquad \text{(W-6)}$$

In other words, the Spin-0 contribution in the tensor source $T_{\mu\nu}$ is demanded to be zero. Instead, this tensor Spin-0 component could be used to cancel out the scalar Spin-0 component coupled to the trace $T=T^{\mu}{}_{\mu}$. That is, let

$$T = -T^{\mu\nu}{}_{,\mu\nu} \neq 0 \, . \qquad \text{(X-33)}$$

This gives a new gauge prescription

$$m^2\left(T^{\mu}{}_{\mu} + T^{\mu\nu}{}_{,\mu\nu}\right)^{,\alpha} = T^{\alpha\beta}{}_{,\beta} \qquad \text{(X-34)}$$

instead of the Hilbert-Lorentz gauge studied by Ogievetsky & Polubarinov (1965). In their notation, the spin projection operator for Spin-0

$$\Pi^{(0)}_{\mu\nu,\alpha\beta} = P^{(0)} = \frac{1}{3}\left\{(1+q)\delta_{\mu\alpha}\partial_\nu\partial_\beta + \frac{4}{\Box^2}\partial_\mu\partial_\alpha\partial_\nu\partial_\beta - \frac{1}{\Box}\left[\delta_{\mu\alpha}\partial_\nu\partial_\beta + \delta_{\nu\beta}\partial_\mu\partial_\alpha\right]\right\} \qquad \text{(X-34a)}$$

is broken up into two Spin-0 admixtures $P^{(0)}_{\text{scalar}}$ and $P^{(0)}_{\text{tensor}}$ using the Hilbert gauge (q is arbitrary)

$$m^2\left(h_{\mu\nu,\mu} + q\,h_{,\nu}\right) = 0 \qquad \text{(X-34b)}$$

such that

$$P^{(0)} = P_S^{(0)} + P_T^{(0)} = 1 \quad . \qquad \text{(X-34c)}$$

The suggestion here is to use (X-34) and suppress $P^{(0)}$ entirely:

$$P^{(0)} = P_S^{(0)} + P_T^{(0)} = 0 \qquad \text{(X-34d)}$$

whereby

$$P_S^{(0)} = - P_T^{(0)} \quad . \qquad \text{(X-34e)}$$

The price for this approach is that $T^{\mu\nu}$ is no longer conserved:*

$$T^{\mu\nu}{}_{,\nu} \neq 0 \quad .$$

(D) Lastly, one can speculate on the existence of negative mass (Wilson, 1972). In the Stückelberg-Feynman representation of Quantum Electrodynamics, the positron in a negative energy state is equivalent to a positive energy electron travelling backward in time. This view is based upon the fact that changing the sign of charge e and time s in the equation of motion (X-36) does not change the electromagnetic behaviour of the particle:

$$m\left(\frac{d^2X_\mu}{ds^2}\right) = e\left(\frac{dX_\nu}{ds}\right)F_{\mu\nu}$$
$$m\dot{v}^\mu = e\, v^\nu F_{\mu\nu} \quad . \qquad \text{(X-36)}$$

However, the equation of motion (X-36) is also not changed under a mass-time reversal. By attributing the characteristic of negative mass to anti-matter, then, a particle of negative mass in a negative energy state behaves like a particle of positive mass in a positive energy state, electromagnetically speaking.

Hence, by adopting an exclusion principle based upon intrinsic rest mass, only particles of negative rest mass can occupy the negative energy mass shell in Figure (X-2). Rest mass cannot change mass shells (if $\dot{m}_0=0$).

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*Rastall (1972) treats the case $T^\mu{}_{\nu;\mu} = \lambda R_{,\nu}$.

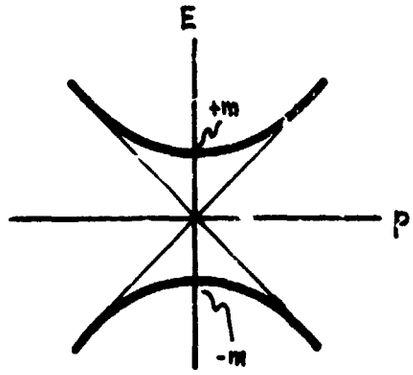

FIGURE X-2

As a consequence, a Lagrangian such as (X-26) and (X-27) can have a Hamiltonian energy density containing negative energy states and not undergo collapse of positive matter into those states. Instead of the Fierz-Pauli massive action (X-26), we can adopt the following one

$$I_m = \int d^4x(\mathcal{L})$$

where

$$\mathcal{L} = \frac{1}{4}m^2\left(h_{\mu\nu}^2 - h^2\right) - \frac{1}{24}\left(h_{,\mu}^2 + m^2 h\right) \qquad \text{(X-37)}$$

which gives the correct limit as $m \to 0$, namely Einstein's theory. Although the right-hand term in (X-37) is negative, no collapse of particles occupying positive energy states can occur into negative energy states, due to the exclusion principle invoked above. Problems associated with "ghosts" (as particles associated with these negative states are called) are eliminated.

A more rigorous treatment of the above conjecture must treat gravitational behaviour under mass-time inversion. This invariance may be surmised from the radiation reaction equations of Lind, et al (1972, Eq 11).

APPENDIX Y: THE COSMOLOGICAL TERM & GRAVITON REST MASS

The field equations (3-1) of General Relativity have numerous cosmological implications among them gravitational collapse and an expansion of the Universe. Concerned about the latter, Einstein (1917) has considered a more general form of the field equations which still satisfy the contracted Bianchi identities and energy-momentum conservation:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\varkappa T_{\mu\nu} \qquad \text{(Y-1a)}$$

or in terms of the Einstein tensor $G_{\mu\nu}$,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\varkappa T_{\mu\nu} \quad . \qquad \text{(Y-1b)}$$

The most general spherically symmetric solution (by virtue of Birkhoff's theorem) of (Y-1a) is the exterior Schwarzschild form with $\Lambda \neq 0$, namely

$$\begin{gathered} ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 d\Omega^2 \\ e^{\nu} = 1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2 = e^{-\lambda} \end{gathered} \qquad \text{(Y-1c)}$$

which for $m = GM/c^2 = 0$ is

$$ds^2 = -\left(1 - \frac{\Lambda}{3} r^2\right) dt^2 + \left(\frac{1}{1 - \frac{\Lambda}{3} r^2}\right) dr^2 + r^2 d\Omega^2 \quad . \qquad \text{(Y-1d)}$$

Because photons following geodesics in (Y-1d) do not travel at the speed of light c, $\Lambda \neq 0$ implies a photon rest mass. Furthermore, the orbital equation is

$$\frac{d^2 u}{d\phi^2} + u = \frac{m}{\ell^2} + 3mu^2 - \frac{\Lambda}{3} \frac{1}{\ell^2} u^{-3} \qquad \text{(Y-1e)}$$

and an additional precession of the periapsis appears as

$$\Delta\theta = 2\pi \left(\frac{1}{2} \frac{\Lambda \ell^6}{m^4}\right) = \Delta\pi \frac{\ell^6}{m^4} \quad . \qquad \text{(Y-1f)}$$

The cubic $e^{\nu} = 0$ now has two positive roots: $r = 2m$ and $r = \sqrt{(3/\Lambda)}$. For Λ not to effect Mercury's orbit by more than one arc sec per century,

$$\frac{\frac{\Lambda}{3}\frac{\ell^6}{m^4}}{\frac{3m^2}{\ell^2}} = \frac{\Lambda}{9}\frac{\ell^8}{m^6} = \frac{1^2}{43^2} \Rightarrow \Lambda \lesssim \frac{5m^6}{\ell^8} = 5\frac{m^2}{a^4(1-e)^4} \sim 5\times 10^{-17}\,(\mathrm{cm})^{-2}.$$

Teleologically speaking, the additional "cosmological term" governed by the "cosmological constant" Λ is meant to offset the expansion of the Universe. However, Hubble's (1929) interpretation of the cosmological red-shift as a Doppler effect has been invoked as a reason for rejecting* the term, maintaining that General Relativity predicts a cosmological expansion.

Nevertheless, a meaningful basis for retaining the cosmological term may be found in the problems associated with the theory of massive radiation (Appendix W). In such a context, the question before us is whether or not the cosmological constant Λ in (Y-1) is equivalent to or can be related to a graviton of non-zero rest mass. Some say yes (E.G. Tonnelet, 1965; Peak, 1972; Kurdgelaidze, 1965; and Freund et al, 1969) while others say no (Treder, 1968; Polievktov-Nikoladze, 1967). At the same time, some maintain that Λ cannot alleviate the inconsistencies of the massless limit in Appendix W (Boulware & Deser, 1972).

The wave equation for gravitational radiation on a non-flat background containing the cosmological term follows from the formalism of Peters (1966), Isaacson (1968), and Zerilli (1970) employed in Chapter 3. The variation of Eq (Y-1) for a stable** background $\eta_{\mu\nu}=g^{(0)}_{\mu\nu}$ is the following:

$$[h_{\mu\nu;\alpha}{}^{;\alpha} - h_{\mu\alpha;\nu}{}^{;\alpha} - h_{\nu\alpha;\mu}{}^{;\alpha} + h^{\alpha}{}_{;\mu;\nu}] + \eta_{\mu\nu}[h_{\alpha\lambda}{}^{;\alpha;\lambda} - h^{\alpha}{}_{;\lambda}{}^{;\lambda}]$$
$$+ h_{\mu\nu}(R-2\Lambda) - \eta_{\mu\nu} h_{\alpha\beta} R^{\alpha\beta} = -2\kappa\,\delta T_{\mu\nu}\,. \qquad \text{(Y-2)}$$

This equation can now be simplified by defining the function $\bar{h}_{\mu\nu}$ (this is the same as $\psi_{\mu\nu}$ in App. B, (B-7), and (3-10) except that the background is arbitrary)

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h \qquad \text{(Y-3)}$$

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*See, however, Eddington's (MTR,1922,p.154) interesting argument for $\Lambda\neq 0$. Λ provides a fundamental length. "An electron could never decide how large it ought to be unless there existed some length independent of itself for it to compare itself with."

**Stability of the background $\eta_{\mu\nu}$ must be taken for granted in order that $\delta T_{\mu\nu}$ can be assumed small.

and its divergence

$$f_\mu \equiv \bar{h}_{\mu\nu}{}^{;\nu} \quad . \qquad \text{(Y-4)}$$

Re-grouping terms and substituting (Y-3) and (Y-4) in (Y-2) we get a relation similar to (3-60)

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} - (f_{\mu;\nu} + f_{\nu;\mu}) + \eta_{\mu\nu} f_\alpha{}^{;\alpha} - 2\bar{h}_{\alpha\beta} R^\alpha{}_{\mu\nu}{}^\beta - \bar{h}_{\mu\alpha} R^\alpha{}_\nu$$
$$- \bar{h}_{\nu\alpha} R^\alpha{}_\mu + h_{\mu\nu}(R-2\Lambda) - \eta_{\mu\nu} h_{\alpha\beta} R^{\alpha\beta} = -2\kappa\delta T_{\mu\nu} \; . \qquad \text{(Y-5)}$$

Now we impose the Hilbert (Einstein-deDonder in Appendix B) gauge which sets (Y-4) to zero

$$f_\mu = \bar{h}_{\mu\nu}{}^{;\nu} = 0 \qquad \text{(Y-6)}$$

and suppresses the vector gravitons. ($f_\mu \neq 0$ can be retained for further simplification in some cases of $\eta_{\mu\nu}$, although problematic negative energy states may be associated with these degrees of freedom.) Wave Eq. (Y-5) hereby reduces to

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} - 2R^\alpha{}_{\mu\nu}{}^\beta \bar{h}_{\alpha\beta} - \bar{h}_{\mu\alpha} R^\alpha{}_\nu - \bar{h}_{\nu\alpha} R^\alpha{}_\mu - \eta_{\mu\nu} h_{\alpha\beta} R^{\alpha\beta}$$
$$+ h_{\mu\nu}(R-2\Lambda) = -2\kappa\delta T_{\mu\nu} \; . \qquad \text{(Y-7)}$$

In an empty ($T_{\mu\nu}=0$), Ricci-flat ($R_{\mu\nu}=0$) space with no cosmological constant ($R=4\Lambda=0$), (Y-7) reduces to

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} - 2R^\alpha{}_{\mu\nu}{}^\beta \bar{h}_{\alpha\beta} = -2\kappa\delta T_{\mu\nu} \; , \qquad \text{(Y-8)}$$

which is the starting point of the Regge-Wheeler-Zerilli formalism in Chapter 3.

However, since $\Lambda \neq 0$ we know that the field equations (Y-1a) demand that

$$4\Lambda - R = \kappa T \qquad \text{(Y-9)}$$

whereby (Y-1a) becomes

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = -\kappa\left[T_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} T\right]. \qquad \text{(Y-10)}$$

For an empty Universe ($T_{\mu\nu}=0$ and $T=0$), (Y-10) reduces to

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \qquad \text{(Y-11a)}$$

and (Y-9) to

$$R = 4\Lambda \quad . \qquad \text{(Y-11b)}$$

Substitution of (Y-11a,b) into (Y-7), using (Y-3), shows that the contributions due to $\Lambda \neq 0$ are now of second order in $h_{\mu\nu}$. Neglecting these terms (particularly if Λ is very, very small*), (Y-7) simplifies to

$$\boxed{\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} - 2R^{\alpha}{}_{\mu\nu}{}^{\beta}\bar{h}_{\alpha\beta} = -2\kappa\delta T_{\mu\nu}} \quad . \qquad \text{(Y-12)}$$

Note that one can arrive at (Y-12) to first order in $h_{\mu\nu}$ by using $g_{\mu\nu}$ as a raising and lowering operator rather than the background $\eta_{\mu\nu}$ - a result which leads Treder (1968) to the conclusion that Λ terms cancel out of the gravitational wave equation. Furthermore, note with caution that (Y-12) and (Y-8) are <u>not</u> the same wave equation.

Overtly, the cosmological terms have vanished from (Y-12), like (Y-8) where $\Lambda=0$; but the character of the Riemann tensor $R^{\alpha}{}_{\mu\nu}{}^{\beta}$ is significantly different in the two relations. For a space of constant curvature $K=1/R^2$, the cosmological constant $\Lambda \neq 0$ is still manifest. That is, the Riemann tensor

*Tolman (1934) places a lower limit $\Lambda \lesssim 10^{-57}$ cm$^{-2} \sim 10^{-21}$ (light years)$^{-2}$ upon Λ, which gives a ratio $(\Lambda r^2/3)/(2m/r) \sim 10^{-19}$ at Neptune's orbit.

$$R_{\gamma\mu\nu\delta} = +K\left(g_{\gamma\nu}g_{\mu\delta} - g_{\gamma\delta}g_{\mu\nu}\right) \tag{Y-13}$$

reverts to

$$R^{\alpha}{}_{\mu\nu}{}^{\beta} = +K\left(g^{\alpha}{}_{\nu}g_{\mu}{}^{\beta} - g^{\alpha\beta}g_{\mu\nu}\right) \tag{Y-14}$$

for use in (Y-12). This substitution (raising and lowering with $\eta_{\mu\nu}$) into (Y-12) now gives a K and a Λ term contribution

$$-2K\left[\left(\bar{h}_{\mu\nu} - \eta_{\mu\nu}\bar{h}\right) + \left(\bar{h}_{\alpha\mu}h^{\alpha}{}_{\nu} + \bar{h}_{\nu\beta}h^{\beta}{}_{\mu} - \bar{h}h_{\mu\nu} - \eta_{\mu\nu}h^{\alpha\beta}\bar{h}_{\alpha\beta}\right)\right]$$
$$+\Lambda\left[2\bar{h}_{\mu\alpha}h^{\alpha}{}_{\nu} + \eta_{\mu\nu}h^{2}_{\alpha\beta}\right] \tag{Y-15}$$

to second order in $h_{\mu\nu}$. Recalling that K is related to Λ by

$$K = \Lambda/3 \quad , \tag{Y-16}$$

(Y-12) is to first order

$$\boxed{\bar{h}_{\mu\nu;\alpha}{}^{;\alpha}} - \frac{2}{3}\Lambda\bar{h}_{\mu\nu} + \frac{2}{3}\Lambda\eta_{\mu\nu}\bar{h} = -2\kappa\delta T_{\mu\nu} \quad . \tag{Y-17}$$

Noting that

$$\bar{h} = h\left(1 - \tfrac{1}{2}\eta\right) \tag{Y-18}$$

then a traceless gauge $\bar{h} = 0$ means that

$$\bar{h} = 0 \quad \Rightarrow \quad h = 0 \quad \text{or} \quad \eta^{\alpha}{}_{\alpha} = 2. \tag{Y-19}$$

Whence, (Y-17) reduces ($n_\alpha{}^\alpha \neq 2$) to

$$\boxed{\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} - \frac{2}{3}\Lambda \bar{h}_{\mu\nu} = -2\kappa\delta T_{\mu\nu}} \quad . \tag{Y-20}$$

in a traceless-Hilbert gauge:

$$\boxed{\begin{aligned} \bar{h}_{\mu\nu}{}^{;\nu} &= 0 \\ \bar{h}_{\mu}{}^{\mu} &= 0 \end{aligned}} \quad . \qquad \begin{aligned} &(Y\text{-}6) \\ &(Y\text{-}19) \end{aligned}$$

At this point it is necessary to determine if the wave equation (Y-20) can be put in a Klein-Gordon form

$$(\Box - m^2)\bar{h}_{\mu\nu} = -2\kappa\delta T_{\mu\nu} \quad . \tag{Y-21}$$

To do so we must investigate $h_{\mu\nu;\alpha}{}^{;\alpha}$ in the curved background (Y-1d) where

$$\eta_{\mu\nu} = \begin{pmatrix} -e^{\nu} & & & \\ & e^{-\nu} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \qquad \eta^{\mu\nu} = \begin{pmatrix} -e^{-\nu} & & & \\ & e^{\nu} & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{Y-22}$$

and determine if the $\Lambda \neq 0$ term in (Y-20) gets cancelled out. To simplify the computations, assume $d\Omega = 0$ and note the $e^{-\nu} \to 1$ and $e^{\nu} \to 1$, as $r \to 0$. We wish to determine if $h_{\mu\nu;\alpha}{}^{;\alpha}$ goes to a flat-space d'Alembertian in a locally flat region of (Y-22). We can write

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} = g^{\alpha\beta}\bar{h}_{\mu\nu;\alpha;\beta} = g^{\alpha\beta}\left[\bar{h}_{\mu\nu,\alpha;\beta} - \left(\Gamma^{\epsilon}_{\alpha\mu}\bar{h}_{\epsilon\nu}\right)_{;\beta} - \left(\Gamma^{\epsilon}_{\alpha\nu}\bar{h}_{\mu\epsilon}\right)_{;\beta}\right] . \tag{Y-23}$$

Define

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} = \Box\bar{h}_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu} + C_{\mu\nu} \quad , \tag{Y-24}$$

where

$$\Box \bar{h}_{\mu\nu} = \bar{h}_{\mu\nu,\alpha}{}^{,\alpha} \tag{Y-24a}$$

$$A_{\mu\nu} = -\Gamma^{\epsilon}_{\beta\mu}\bar{h}_{\epsilon\nu}{}^{,\beta} - \Gamma^{\epsilon}_{\beta\nu}\bar{h}_{\mu\epsilon}{}^{,\beta} - \Gamma^{\epsilon}_{\beta\alpha}\bar{h}_{\mu\nu,\epsilon}\eta^{\alpha\beta} - \Gamma^{\epsilon}_{\alpha\mu}\bar{h}_{\epsilon\nu}{}^{,\alpha} - \Gamma^{\epsilon}_{\alpha\nu}\bar{h}_{\mu\epsilon}{}^{,\alpha} \tag{Y-24b}$$

$$B_{\mu\nu} = -\left(\Gamma^{\epsilon}_{\alpha\mu}\right)^{,\alpha}\bar{h}_{\epsilon\nu} - \left(\Gamma^{\epsilon}_{\alpha\nu}\right)^{,\alpha}\bar{h}_{\mu\epsilon} \tag{Y-24c}$$

$$\begin{aligned} C_{\mu\nu} = -\eta^{\alpha\beta}\Big[&\left(\Gamma^{\epsilon}_{\beta\delta}\Gamma^{\delta}_{\alpha\mu} - \Gamma^{\delta}_{\beta\alpha}\Gamma^{\epsilon}_{\delta\mu} - \Gamma^{\delta}_{\beta\mu}\Gamma^{\epsilon}_{\alpha\delta}\right)\bar{h}_{\epsilon\nu} - \Gamma^{\delta}_{\beta\epsilon}\Gamma^{\epsilon}_{\alpha\mu}\bar{h}_{\delta\nu} - \Gamma^{\delta}_{\beta\nu}\Gamma^{\epsilon}_{\alpha\mu}\bar{h}_{\epsilon\delta} \\ &+\left(\Gamma^{\epsilon}_{\beta\delta}\Gamma^{\delta}_{\alpha\nu} - \Gamma^{\delta}_{\beta\alpha}\Gamma^{\epsilon}_{\delta\nu} - \Gamma^{\delta}_{\beta\nu}\Gamma^{\epsilon}_{\alpha\delta}\right)\bar{h}_{\mu\epsilon} - \Gamma^{\delta}_{\beta\mu}\Gamma^{\epsilon}_{\alpha\nu}\bar{h}_{\delta\epsilon} - \Gamma^{\delta}_{\beta\epsilon}\Gamma^{\epsilon}_{\alpha\nu}\bar{h}_{\mu\delta}\Big]. \end{aligned} \tag{Y-24d}$$

$B_{\mu\nu}$ is the term of interest. $A_{\mu\nu}$ and $C_{\mu\nu}$ contain terms of second order, or terms that vanish in locally flat space ($r \ll 1$). Furthermore, only the first-order second derivatives in $B_{\mu\nu}$ remain as $r \to 0$. These terms are

$$B^{*}_{\alpha\mu\nu}{}^{\alpha} = -\tfrac{1}{2}\eta^{\epsilon\gamma}\left[\left(\eta_{\alpha\gamma,\mu}{}^{,\alpha} + \eta_{\mu\gamma,\alpha}{}^{,\alpha} - \eta_{\alpha\mu,\gamma}{}^{,\alpha}\right)\bar{h}_{\epsilon\nu} + \left(\eta_{\alpha\gamma,\nu}{}^{,\alpha} + \eta_{\nu\gamma,\alpha}{}^{,\alpha} - \eta_{\alpha\nu,\gamma}{}^{,\alpha}\right)\bar{h}_{\mu\epsilon}\right], \tag{Y-25a}$$

which we can define as

$$B^{*}_{\alpha\mu\nu}{}^{\alpha} = F_{\mu\nu} + G_{\mu\nu} + H_{\mu\nu} \quad , \tag{Y-25b}$$

where

$$F_{\mu\nu} = -\tfrac{1}{2}\eta^{\epsilon\gamma}\left[(\Box\eta_{\mu\gamma})\bar{h}_{\epsilon\nu} + (\Box\eta_{\nu\gamma})\bar{h}_{\mu\epsilon}\right] \tag{Y-25c}$$

$$G_{\mu\nu} = -\tfrac{1}{2}\eta^{\epsilon\gamma}\left[\eta_{\alpha\gamma,\mu}{}^{,\alpha}\bar{h}_{\epsilon\nu} + \eta_{\alpha\gamma,\nu}{}^{,\alpha}\bar{h}_{\mu\epsilon}\right] \tag{Y-25d}$$

$$H_{\mu\nu} = +\tfrac{1}{2}\eta^{\epsilon\gamma}\left[\eta_{\alpha\mu,\gamma}{}^{,\alpha}\bar{h}_{\epsilon\nu} + \eta_{\alpha\nu,\gamma}{}^{,\alpha}\bar{h}_{\mu\epsilon}\right]. \tag{Y-25e}$$

Next we note in this approximation that

$$\left.\begin{aligned} \Box &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \rightarrow \nabla^2 \\ \Box\eta_{00} &\rightarrow \nabla^2\eta_{00} = -2\Lambda/3 \\ \Box\eta_{11} &\rightarrow \nabla^2\eta_{11} \xrightarrow[r\downarrow 0]{} +2\Lambda/3 \end{aligned}\right\} \quad . \tag{Y-26}$$

The results in (Y-26) are not $\pm 2\Lambda$ because of the type of expansion in (Y-24).

We find that

$$F_{\mu\nu} = -\tfrac{1}{2}\eta^{00}\left[(\Box\eta_{\mu 0})\bar{h}_{\nu 0} + (\Box\eta_{\nu 0})\bar{h}_{\mu 0}\right] - \tfrac{1}{2}\eta^{11}\left[(\Box\eta_{\mu 1})\bar{h}_{1\nu} + (\Box\eta_{\nu 1})\bar{h}_{\mu 1}\right] , \qquad \text{(Y-27a)}$$

whereby (all other terms do not contribute)

$$F_{00} = -\eta^{00}\left[(\Box\eta_{00})\bar{h}_{00}\right] = -\tfrac{2}{3}\Lambda\bar{h}_{00} \qquad \text{(Y-27b)}$$

$$F_{11} = -\eta^{11}\left[(\Box\eta_{11})\bar{h}_{11}\right] = -\tfrac{2}{3}\Lambda\bar{h}_{11} \quad . \qquad \text{(Y-27c)}$$

Next,

$$G_{\mu\nu} = -\tfrac{1}{2}\eta^{11}\left[\eta_{11,\mu}{}^{,1}\bar{h}_{1\nu} + \eta_{11,\nu}{}^{,1}\bar{h}_{\mu 1}\right] , \qquad \text{(Y-28a)}$$

whereby (all other terms do not contribute)

$$G_{01} = -\tfrac{1}{3}\Lambda\bar{h}_{01} \qquad G_{10} = -\tfrac{1}{3}\Lambda\bar{h}_{10} \qquad G_{11} = -\tfrac{2}{3}\Lambda\bar{h}_{11} \quad . \qquad \text{(Y-28b)}$$

And lastly,

$$H_{\mu\nu} = \tfrac{1}{2}\eta^{11}\left[\eta_{\alpha\mu,1}{}^{,\alpha}\bar{h}_{1\nu} + \eta_{\alpha\nu,1}{}^{,\alpha}\bar{h}_{\mu 1}\right] , \qquad \text{(Y-29a)}$$

whereby

$$H_{00} = 0 \qquad H_{01} = \tfrac{1}{3}\Lambda\bar{h}_{01}$$
$$H_{11} = \tfrac{2}{3}\Lambda\bar{h}_{11} \qquad H_{10} = \tfrac{1}{3}\Lambda\bar{h}_{10} \quad . \qquad \text{(Y-29b)}$$

Substituting (Y-29b), (Y-28b), (Y-27b,c) back into (Y-25b), we find that $G_{\mu\nu} + H_{\mu\nu} = 0$. That is, $B^{*}_{\mu\nu} = F_{\mu\nu}$:

$$B^{*}_{00} = -\tfrac{2}{3}\Lambda\bar{h}_{00} \qquad B^{*}_{11} = -\tfrac{2}{3}\Lambda\bar{h}_{11} \quad . \qquad \text{(Y-30)}$$

Substitution of (Y-30) and $A_{\mu\nu} = C_{\mu\nu} = 0$ into (Y-24) determines that

$$(\Box - \tfrac{2}{3}\Lambda)\bar{h}_{00} = -2\kappa\delta T_{00}$$
$$(\Box - \tfrac{2}{3}\Lambda)\bar{h}_{11} = -2\kappa\delta T_{11} \quad . \qquad \text{(Y-31)}$$

The d'Alembertian in (Y-31) is not arbitrary because curvilinear (e.g. spherical) coordinates cannot be used. Curvilinear transformations have been lost in the expansion (Y-24). What is important is that the sign of $B_{\mu\nu}$ in (Y-30) and $\bar{h}_{\mu\nu}$ in (Y-20) is the same, with no cancellation.

The significant result is that there is not a constant term hidden in a decomposition of $\bar{h}_{\mu\nu;\alpha}{}^{;\alpha}$ which removes the potentially massive term $2\Lambda/3\ \bar{h}_{\mu\nu}$ in (Y-20). The cosmological term $\Lambda \neq 0$ does appear to constitute a massive contribution to the linearized gravitational wave equation (Y-20) in a traceless-Hilbert gauge.

APPENDIX Z: TRANSVERSE-TRACELESS GAUGE

Upon consideration of the nature of the spin admixtures inherent in tensor radiation theory (Appendices W & X), we know that the linearized massless theory of Einstein (1916, 1918) and Fierz & Pauli (1939) is a pure, transverse Spin-2 with no longitudinal Spin-0 scalar contribution. This scalar Spin-0 couples to the trace of $T_{\mu\nu}$, that is $T=T^{\mu}{}_{\mu}$, as in (W-4a). Hence, the massless graviton $h_{\mu\nu}$ and its energy momentum tensor $T_{\mu\nu}$ must be traceless in empty space.

Recalling the Regge-Wheeler perturbations of the Schwarzschild metric (3-57) or (3-58), however, it is not readily apparent that $h_{\mu\nu}$ is traceless. It is, on the other hand, true for the plane-wave discussed in (3-33) and (3-37). Of course, as in electrodynamics, such a transversality condition holds only in a source-free region of space-time.

The transverse-traceless (TT) gauge, then, is that coordinate condition (B-1) whereby the solution

$$h_{\mu\nu} = h^{TT}_{\mu\nu} \tag{Z-1}$$

has zero trace. That is,

$$h = h^{TT} = 0. \tag{Z-2}$$

Furthermore, it is transverse in the sense treated for the plane-wave of (3-24), (3-33), and (3-37).

Note that in the transverse-traceless gauge, $\gamma_{\mu\nu}$ of (3-10) and (B-7) and the Hilbert function (Y-3) are identical with $h_{\mu\nu}$, except for possible scale changes. The Hilbert-Einstein-deDonder coordinate condition (B-8) and the Hilbert gauge (Y-6) are likewise simplified.

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It is time that I should end these weak words,
and lay my poor garland
On the grave of this just and faithful knight of God.

(John Tyndall on Michael Faraday)

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REPLY TO
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NASA TECHNICAL MEMORANDUM X-58132

GRAVITATIONAL RADIATION THEORY

A Thesis Presented to the Faculty of the Graduate
School of Rice University in Partial Fulfillment
of the Requirements for the Degree of Master of
Arts in Physics

BY

Thomas L. Wilson

December 1973

LYNDON B. JOHNSON SPACE CENTER
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The attached pages 115 and 116 are a corrected version to replace the same numbered pages and to be attached to the above-named NASA Technical Memorandum.

Issue Date: 4/15/74

Including a current $\underset{\sim}{J}$, (F-35) is

$$\nabla\cdot\underset{\sim}{M}=\rho \qquad \nabla\times M=-i\left(\underset{\sim}{J}+\frac{\partial \underset{\sim}{M}}{\partial t}\right) \quad . \qquad \text{(F-35')}$$

Bateman's Equations, reminiscent of classical six-vectors and quaternions, provide a simple way of synthesizing vector electromagnetism and vector gravitation. As such they are the teleological basis of a complex unified field theory. Let

$$\hat{\underset{\sim}{A}}=\hat{\underset{\sim}{M}}+\hat{\underset{\sim}{G}} \qquad \begin{array}{l}\hat{\underset{\sim}{M}}=\underset{\sim}{E}+i\underset{\sim}{H}\\ \hat{\underset{\sim}{G}}=\underset{\sim}{g}+i\underset{\sim}{h}\end{array} \qquad \begin{array}{l}\rho=\rho_e-i\rho_m\\ \underset{\sim}{J}=\underset{\sim}{J}_e-i\underset{\sim}{J}_m\end{array} \quad . \qquad \text{(F-36)}$$

Substitution of the complex fields (F-36) into the Bateman Eqs (F-35') gives the vector Maxwell Equations (2-11) for electromagnetism and (2-12) for gravitation. The convention

$$\underset{\sim}{F}_e=\rho_e\left[\hat{\underset{\sim}{M}}+i\underset{\sim}{v}\times\hat{\underset{\sim}{M}}^*\right] \qquad \text{(F-37)}$$

$$\underset{\sim}{F}_m=\rho_m\left[\hat{\underset{\sim}{G}}+i\underset{\sim}{v}\times\hat{\underset{\sim}{G}}^*\right] \qquad \text{(F-38)}$$

provides the complex Lorentz force relations. One is interested only in the real parts of $\underset{\sim}{F}$ in (F-37 and F-38).

The interesting feature of this approach is the implication that charge and mass are the real components of the same thing - a single complex quantity. That is, the Newtonian and Coulomb forces are unified into a single complex operation. The same is true for the Larmor-Lorentz term.

The field intensities are determined by

$$\mathcal{E}=\hat{\underset{\sim}{A}}^2=\hat{\underset{\sim}{M}}^2+\hat{\underset{\sim}{G}}^2=(\underset{\sim}{E}^2+\underset{\sim}{H}^2)+(\underset{\sim}{g}^2+\underset{\sim}{h}^2) \quad . \qquad \text{(F-39)}$$

The above formulation is not derived from a Lagrangian, however.

A Quaternion Formulation Of Maxwell's Equations

Recalling the rule of multiplication of two quaternions $\underline{A}=a+\underset{\sim}{A}$ and $\underline{B}=b+\underset{\sim}{B}$,

$$\underline{A}\,\underline{B}=ab-\underset{\sim}{A}\cdot\underset{\sim}{B}+a\underset{\sim}{B}+b\underset{\sim}{A}+\underset{\sim}{A}\times\underset{\sim}{B} \qquad \text{(F-40)}$$

then Bateman's representation of Maxwell's Eqs is simply

$$*\underline{M}=\underline{J}_e \qquad \text{and} \qquad *\underline{G}=\underline{J}_m \qquad \text{(F-41)}$$

where

$$*\equiv\partial_t+i\underset{\sim}{\nabla}$$

$$\underline{M}=\underset{\sim}{E}+i\underset{\sim}{H} \qquad \underline{G}=\underset{\sim}{g}+i\underset{\sim}{h}$$

$$\underline{J}_e=-(i\rho_e+\underset{\sim}{J}_e) \qquad \underline{J}_m=(i\rho_m+\underset{\sim}{J}_m) \quad .$$

A Unified, Vector Field Theory

A derivation of the Maxwell Eqs. (2-11) follows from the Euler-Lagrange Eqs for a variation of the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - mc^2\sqrt{-\frac{dz^\mu}{d\lambda}\frac{dz_\mu}{d\lambda}} + J^\mu A_\mu \quad . \qquad \text{(F-42)}$$

To get the gravitational Maxwell Eqs. (2-12) one must change the sign of either the first or last term in (F-42). However, reversing the sign of $F_{\mu\nu}F^{\mu\nu}$ gives a negative energy density in the radiation field; hence, one must change the sign of the current coupling term $J^\mu A_\mu$. But in order to derive the Lorentz force (2-15) and get a Newtonian rather than a Coulombic interaction, the signs of both of the last two terms in (F-42) must change. This argument, however, then results in the wrong sign for the middle, kinetic term.

Nevertheless, if we adopt a complex vector potential

$$M_\mu = (A_\mu + i a_\mu)/\sqrt{2} \qquad \text{(F-43)}$$

where

$$A_\mu = (M_\mu + M^*_\mu)/\sqrt{2} \qquad\qquad a_\mu = (M_\mu - M^*_\mu)/\sqrt{2}$$

and where

$$G_{\mu\nu} = M_{\nu,\mu} - M_{\mu,\nu} \qquad \text{(F-44)}$$

then

$$\mathcal{L} = -\frac{1}{4}G^*_{\mu\nu}G^{\mu\nu} - mc^2\sqrt{-\frac{dz^\mu}{d\lambda}\frac{dz_\mu}{d\lambda}} + J^\mu M^*_\mu + J^*_\mu M^\mu \qquad \text{(F-45)}$$

results in the electromagnetic Maxwell Eqs (2-11)

$$G^{\mu\nu}{}_{,\nu} = J^\mu \qquad \text{(F-46)}$$

and the gravitational Maxwell Eqs (2-12)

$$G^{*}_{\mu\nu}{}^{,\nu} = J^*_\mu \qquad \text{(F-47)}$$

as well as the corresponding Lorentz force equations

$$F^\mu = G^{\mu\nu}J_\nu \qquad\qquad F^*_\mu = G^*_{\mu\nu}J^{*\nu} \quad . \qquad \text{(F-48)}$$

The current for M_μ

$$J^\mu = -i\left(M^*_\sigma M^{\sigma,\mu} - M^{*\sigma,\mu}M_\sigma\right) \qquad \text{(F-49)}$$

is conserved and does not transport charge or mass. The equations are linear as in the Maxwell theory. Interchanging M^*_μ and M_μ in (F-49) reverses the direction of the current, basic for the gravitational interaction. To make the theory nonlinear (transporting charge and mass) one simply adds the complex Klein-Gordon charge-mass terms (where $\rho = \rho_e + i\rho_m$) to (F-45)

$$\mathcal{L}_{nonlinear} = \frac{1}{2}\rho^2 M^*_\mu M^\mu = \frac{1}{2}(\rho_e^2 + \rho_m^2)M^*_\mu M^\mu \quad . \qquad \text{(F-50)}$$