CORRECTION OF LASER RANGE TRACKING DATA FOR ATMOSPHERIC REFRACTION AT ELEVATIONS ABOVE 10 DEGREES

J. W. MARINI
C. W. MURRAY, JR.

NOVEMBER 1973

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND
CORRECTION OF LASER RANGE TRACKING DATA FOR ATMOSPHERIC
REFRACTION AT ELEVATIONS ABOVE 10 DEGREES

J. W. Marini
C. W. Murray, Jr.

November 1973

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland
CORRECTION OF LASER RANGE TRACKING DATA FOR ATMOSPHERIC REFRACTION AT ELEVATIONS ABOVE 10 DEGREES

J. W. Marini
C. W. Murray, Jr.

ABSTRACT

A formula for correcting laser measurements of satellite range for the effect of atmospheric refraction is given. The corrections apply above 10° elevation to satellites whose heights exceed 70 km. The meteorological measurements required are the temperature, pressure, and relative humidity of the air at the laser site at the time of satellite pass.

The accuracy of the formula was tested by comparison with corrections obtained by ray-tracing radiosonde profiles. The standard deviation of the difference between the refractive retardation given by the formula and that calculated by ray-tracing was less than about 0.04% of the retardation or about 0.5 cm at 10° elevation, decreasing to 0.04 cm near zenith.
# CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>REFRACTIVITY AT OPTICAL FREQUENCIES</td>
<td>1</td>
</tr>
<tr>
<td>GEOMETRY AND NOTATION</td>
<td>2</td>
</tr>
<tr>
<td>EXPANSION FORMULA</td>
<td>2</td>
</tr>
<tr>
<td>EVALUATION OF INTEGRALS</td>
<td>5</td>
</tr>
<tr>
<td>CORRECTION FORMULA</td>
<td>5</td>
</tr>
<tr>
<td>TEST OF ACCURACY</td>
<td>6</td>
</tr>
<tr>
<td>SUMMARY</td>
<td>7</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>18</td>
</tr>
</tbody>
</table>

## Appendices

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A1-1</td>
</tr>
<tr>
<td>2</td>
<td>A2-1</td>
</tr>
<tr>
<td>3</td>
<td>A3-1</td>
</tr>
<tr>
<td>4</td>
<td>A4-1</td>
</tr>
</tbody>
</table>

## ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

PRECEDING PAGE BLANK NOT FILMED
ILLUSTRATIONS (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Tropospheric Range Error at About 80 Degrees Elevation for Laser Frequency of 0.6943 Microns</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>Tropospheric Range Error at About 40 Degrees Elevation for Laser Frequency of 0.6943 Microns</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>Tropospheric Range Error at About 40 Degrees Elevation for Laser Frequency of 0.6943 Microns</td>
<td>11</td>
</tr>
<tr>
<td>6</td>
<td>Tropospheric Range Error at About 20 Degrees Elevation for Laser Frequency of 0.6943 Microns</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>Tropospheric Range Error at About 20 Degrees Elevation for Laser Frequency of 0.6943 Microns</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>Tropospheric Range Error at About 15 Degrees Elevation for Laser Frequency of 0.6943 Microns</td>
<td>14</td>
</tr>
<tr>
<td>9</td>
<td>Tropospheric Range Error at About 15 Degrees Elevation for Laser Frequency of 0.6943 Microns</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>Tropospheric Range Error at About 10 Degrees Elevation for Laser Frequency of 0.6943 Degrees</td>
<td>16</td>
</tr>
<tr>
<td>11</td>
<td>Tropospheric Range Error at About 10 Degrees Elevation for Laser Frequency of 0.6943 Microns</td>
<td>17</td>
</tr>
</tbody>
</table>
INTRODUCTION

The correction of tracking data for atmospheric refraction has been exhaustively studied, and many correction formulas have been published [1-6]. For certain earth and ocean physics applications, however, position accuracies of better than a few centimeters are desirable [7], and these accuracies are much greater than required for most previous applications. Out of the work cited, only the approach given by Marini [3], and the expansion and integral evaluations of Saastamoinen [5, 6] provide the desired accuracy at lower elevation angles (10° - 20°). In this report Saastamoinen's integral evaluations are incorporated into Marini's continued fraction form to provide relatively simple algorithms for correcting laser range-data using surface meteorological measurements.

REFRACTIVITY AT OPTICAL FREQUENCIES

There are a number of formulas [8-11] for the refractive index n of air and for the corresponding refractivity

$$N = 10^6 (n - 1)$$

all of which have sufficient accuracy for use here. The formula employed is [12]

$$N = \left( \frac{287.604 + 1.6288}{\lambda^2} + \frac{0.0136}{\lambda^4} \right) \left( \frac{P}{1013.25} \right) \left( \frac{1}{1 + 0.003661 t} \right)$$

$$-0.055 \left( \frac{760}{1013.25} \right) \left( \frac{e}{1 + 0.00366 t} \right)$$

where

$$\lambda = \text{wavelength of radiation in microns}$$
$$P = \text{atmospheric pressure in millibars}$$
$$e = \text{partial water vapor pressure in millibars}$$
$$t = \text{temperature in degrees Celsius}$$

Because air is dispersive at optical frequencies, the group refractivity $N_g$ is also required

$$N_g = \frac{d}{df}(IN) = N - \lambda \frac{dN}{d\lambda}$$
where \( f \) is the frequency. The expression for the group refractivity can be written as

\[
N_g = 80.343 f(\lambda) \frac{P}{T} - 11.3 \frac{e}{T}
\]  

(4)

where

\( P = \) Total air pressure in millibars
\( e = \) Partial pressure of water vapor (mb)
\( T = \) Temperature (°K)

and

\[
f(\lambda) \equiv 0.9650 + \frac{0.0164}{\lambda^2} + \frac{0.000228}{\lambda^4}
\]

(5)

which, at the 0.6943 micron wavelength of the ruby laser becomes

\[
f(0.6943) = 1.0000
\]

(6)

GEOMETRY AND NOTATION

The geometry of the satellite-tracking station configuration is shown in Figure 1. Spherical symmetry is assumed, i.e., the refractivity is taken to be a function of height only. The height \( h \) is measured from the tracking station upward. The subscript "0" designates quantities evaluated at the tracking station, the subscript "1", quantities evaluated at the satellite. The ray or phase path between tracking station and satellite is shown as a curved line. \( \lambda \) true range \( R \) is the distance along the straight line connecting the tracking station and the satellite, and the true elevation angle \( E \) is the angle between this line and the horizontal at the station. The nominal earth radius used is \( r_e = 6378 \text{ km} \), and \( H \) is the height of the tracking station above sea level. The latitude of the tracking station is \( \phi \) degrees above the equator.

EXPANSION FORMULA

The apparent range \( R_e \) between the ground station and the satellite as measured by a pulsed system is given by the integral of the group index of refraction along the phase path [13, 14]
Figure 1. Geometry
\[ R_c = \int_{r_0}^{r_1} \frac{n_g}{\sin \theta} \, dr \]  

(7)

where the angle \( \theta \) is given by Snell's law for a spherically stratified medium

\[ nr \cos \theta = n_0 r_0 \cos \theta_0 \]  

(8)

The correction sought is the difference between the measured and the true value of the range

\[ \Delta R = R_c - R \]  

(9)

The expansion of \( \Delta R \) in inverse powers of \( \sin \theta_0 \), following Marini [3] gives

\[
\Delta R = 10^{-6} \int hN_g \, dh \cdot \frac{1}{\sin \theta_0} \\
- \left[ 10^{-6} \int \frac{hN_g}{r_0} \, dh - 10^{-12} \int N_0 \, dh \right] \\
+ 10^{-12} \int (NN_g - \frac{1}{2} N^2) \, dh \cdot \frac{1}{\sin^3 \theta_0} \\
+ \ldots \]

(10)

where the range of integration is from the tracking station \((h = 0)\) upward to above the atmosphere \((h = \infty)\). The terms containing the satellite range \( R \) that appear in reference [3] can be neglected, as shown in Appendix 1, because (10) is to be applied only where \( E > 10^\circ \) and \( h_i > 70 \text{ km} \).

The expansion (10) is not the most useful one for many orbit determination programs because the correction is expressed as a function of arrival angle \( \theta_0 \), which may not even be measured, rather than as a function of elevation angle \( E \), which is computed. To convert (10) to the desired form, the first term of the expansion of the angular correction is used

\[ \theta_0^\circ - E = 10^{-6} N_0 \cot E \]  

(11)

substituting (11) into (10), and making suitable approximations
\[
\Delta R = \left[ 10^{-6} \int N_\theta \, dh \right] \frac{1}{\sin \hat{E}} \bigg] + 0.6 \left[ \frac{10^{-6}}{r_0} \int hN_\theta \, dh + 10^{-12} \int (N N_\theta \cdot \frac{1}{2} N^2) \, dh \right] \frac{1}{\sin \hat{E}}; \quad (12)
\]

Equation (12) above is the expansion that provides the basis for the correction formula that is the subject of this report.

**EVALUATION OF INTEGRALS**

The evaluation of the integrals, appearing in (12), as functions of the pressure, temperature, and relative humidity of the surface air at the tracking station, has been treated by Saastamoinen [6]. For completeness, and because they differ in detail, our evaluations are given in Appendix 2. The results are

\[
10^{-6} \int N_\theta \, dh = \frac{f(\lambda)}{f(\phi,H)} \left[ 0.002357 P_0 + 0.000141 e_0 \right] \quad (13)
\]

\[
\frac{10^{-6}}{r_0} \int hN_\theta \, dh = f(\lambda) (1.084 \times 10^{-8}) P_0 T_0 K \quad (14)
\]

\[
10^{-12} \int (N N_\theta \cdot \frac{1}{2} N^2) \, dh = f(\lambda) (4.734 \times 10^{-8}) \frac{P_0^2}{T_0} \cdot \frac{2}{3 \cdot 1/K} \quad (15)
\]

where

\[
f(\phi,H) = 1 - 0.0026 \cos 2\phi - 0.00031H \quad (16)
\]

and

\[
K = 1.163 - 0.00968 \cos 2\phi - 0.00104 T_0 + 0.00001435 P_0 \quad (17)
\]

**CORRECTION FORMULA**

The formula for calculating the range error \( \Delta R \) from the satellite elevation \( \hat{E} \) is obtained by approximating (12) by a continued fraction form.
\[
\Delta R = \frac{f(\lambda)}{f(\varphi, H)} \cdot \frac{A + B}{\sin \lambda + \frac{B/(A + B)}{\sin \lambda + 0.01}}
\]

where

\[
A = 0.002357 P_0 + 0.000141 e_0
\]

(19)

\[
B = (1.084 \times 10^{-8}) P_0 T_0 K + (4.734 \times 10^{-8}) \frac{P_0^2 T_0}{T_0} (3 - 1/K)
\]

(20)

\[
K = 1.163 - 0.00968 \cos 2\varphi - 0.00104 T_0 + 0.00001435 P_0
\]

(21)

Here

\[
\Delta R = \text{Range correction (meters)}
\]

\[
E = \text{True elevation of satellite}
\]

\[
P_0 = \text{Atmospheric pressure at the laser site (millibars)}
\]

\[
T_0 = \text{Atmospheric temperature at the laser site (degrees Kelvin)}
\]

\[
e_0 = \text{Water vapor pressure at the laser site (millibars)}
\]

\[
f(\lambda) = 1 \text{ for a ruby laser, and is given by (5) otherwise}
\]

\[
f(\varphi, H) = 1 \text{ for a laser site at 45° latitude and at sea level, and is given by (16) for sites at different latitudes } \varphi \text{ and elevations H (in km)}
\]

The water vapor pressure \(e_0\) may be calculated from a relative humidity measurement \(R_h\) (%)

\[
e_0 = \frac{R_h}{100} \times 6.11 \times 10^{-1} \frac{7.5 (T_0 - 273.15)}{237.3 + (T_0 - 273.15)}
\]

(22)

In (18) the quantity 0.01 is an empirical constant that serves to compensate for the neglect of higher order terms. The divisor \(f(\varphi, H)\) can be factored out of the series (12) and consequently the fraction (18) because the error thereby incurred in the second term of (12) is negligible. The use of the sum \(A + B\) where it appears in (18) instead of using \(A\) alone is an optional adjustment used to reduce at elevations near 90° a small bias that occurs in the expansion (12) because of approximations made in its derivation.

TEST OF ACCURACY

To test the accuracy of formula (18), which is based on surface measurements, range corrections obtained using the formula were compared with corrections...
obtained by ray-tracing radiosonde refractivity profiles. The ray-trace corrections are considered to have state-of-the-art accuracy, so that the differences between these corrections and those calculated from the simpler formulas represent the penalty paid for simplicity in calculation and measurement.

The data used in Figures 2–11 was obtained from the National Climatic Center at Asheville, North Carolina. It consists of radiosonde observations taken near Dulles Airport, Virginia, during the year 1967.

Using the procedure described in Appendices 3 and 4, 634 refractivity profiles were calculated up to a height of 1000 kilometers from the radiosonde observations. The calculated profiles were ray-traced [16] at arrival angles of 10°, 15°, 20°, 40°, and 80°, and the tropospheric errors in range and elevation angle were obtained. The histograms of these errors are shown in Figures 2, 4, 6, 8, and 10. The correction formula (18) was applied using only surface data and the known elevation angle to obtain approximate tropospheric corrections. The differences between these algorithm corrections and the ray-trace corrections were calculated. The histograms of these differences is shown in Figures 3, 5, 7, 9, and 11. The maximum bias of the error remaining after correction was -0.1 cm, and the maximum standard deviation was 0.49 cm at 10°, decreasing to 0.04 cm at 80°.

In addition formula (18) was compared with range corrections obtained by ray tracing (at arrival angles of 10°, 15°, 20°, 40°, and 80°) radiosonde refractivity profiles calculated at Jananarive (85 profiles), Fairbanks, Alaska (200 profiles), Athens, Georgia (200 profiles), Greensboro, North Carolina (200 profiles), and Nashville, Tennessee (135 profiles). The maximum standard deviation of the error in the algorithm at 10° was 1 centimeter and the maximum at 80° was 0.06 centimeters. The maximum mean error of the algorithm at 10° was 0.16 cm and the maximum at 80° was 0.07 cm.

CONCLUSIONS

An equation that corrects laser range data for atmospheric refraction using surface meteorological measurements has been derived, and a comparison made between the corrections calculated using this equation (equation 18) and the corrections calculated by ray-tracing through a radiosonde profile. The comparison (Figures 2–11) indicates that the differences between the corrections calculated by the two methods are negligible for practical applications. Hence accurate refraction correction of laser range data can be made without the requirement for radiosonde measurements or lengthy ray-tracing algorithms.
Figure 2. Tropospheric Range Error at About 80 Degrees Elevation for Laser Frequency of 0.6943 Microns

MEAN = 2.417 meters
STANDARD DEVIATION = 1.62 cm

NUMBER OF OBSERVATIONS
Figure 3. Tropospheric Range Error at About 80 Degrees Elevation for Laser Frequency of 0.6943 Microns
Figure 4. Tropospheric Range Error at About 40 Degrees Elevation for Laser Frequency of 0.6943 Microns
Figure 5. Tropospheric Range Error at About 40 Degrees Elevation for Laser Frequency of 0.6943 Microns
Figure 6. Tropospheric Range Error at About 20 Degrees Elevation for Laser Frequency of 0.6943 Microns

Mean = 6.910 meters
Standard Deviation = 4.67 cm
Figure 7. Tropospheric Range Error at About 20 Degrees Elevation for Laser Frequency of 0.6943 Microns

MEAN = -0.05 cm
STANDARD DEVIATION = 0.12 cm

ERROR AFTER CORRECTION USING SURFACE DATA (cm)
Figure 8. Tropospheric Range Error at About 15 Degrees Elevation for Laser Frequency of 0.6943 Microns

MEAN = 9.079 meters
STANDARD DEVIATION = 6.18 cm
Figure 9. Tropospheric Range Error at About 15 Degrees Elevation for Laser Frequency of 0.6943 Microns
Figure 11. Tropospheric Range Error at About 10 Degrees Elevation for Laser Frequency of 0.6943 Microns
It should be pointed out that only the relative accuracy of the two procedures has been tested, and that errors caused by factors common to both methods are not in evidence. For example, equation (4) for the group refractive index is used both in (18) and in the ray-trace equations, and any error in its magnitude would reflect equally in the corrections. Similarly, the hydrostatic equation used in equation (2-1) and hence (18) is also implicit in the ray-tracing method because the heights that appear in radiosonde profiles are not measured quantities but rather are calculated from the measured pressures, temperatures, and relative humidities using the hydrostatic equation. Also, both methods assume horizontal homogeneity. Saastamoinen [6] has estimated the standard error from such sources to be less than 1 or 2 centimeters at 10° elevation.

REFERENCES


APPENDIX 1

NEGLECT OF SATELLITE RANGE
APPENDIX 1

NEGLECT OF SATELLITE RANGE

The correction

\[ \Delta R = \int_{r_0}^{r_1} \frac{n_g}{\sin \theta} \, dr - R \]  

(1-1)

can be written as

\[ \Delta R = 10^{-6} \int_{r_0}^{r_1} \frac{N_g}{\sin \theta} \, dr + \left[ \int_{r_0}^{r_1} \frac{1}{\sin \theta} \, dr - R \right] \]  

(1-2)

The expansion of the first term in (1-2), using suitable approximations [3], gives

\[ 10^{-6} \int_{r_0}^{r_1} \frac{N_g}{\sin \theta} \, dr = \frac{10^{-6}}{\sin \theta_0} \int_{0}^{\infty} N_g \, dh \]

\[ - \frac{1}{\sin^2 \theta_0} \left[ \frac{10^{-6}}{r_0} \int h N_g \, dh \right] 

-10^{-12} \int N_g (N_0 - N) \, dh \]  

(1-3)

The expansion of the bracketed second term in (1-2), which represents the difference \( \Delta R_g \) between the geometrical lengths of the phase and the straight-line paths between the satellite and the tracking station, can be obtained by expanding equation (A5) of reference [3] in inverse powers of \( \sin \theta_0 \) giving

\[ \Delta R_g = \frac{1}{\sin^3 \theta_0} \cdot \frac{1}{2} 10^{-12} \int N^2 \, dh \]  

(1-4)

\[ - \frac{1}{2} 10^{-12} (f N \, dh)^2 \cdot \frac{\cos^2 \theta_0}{R} \cdot \frac{1}{\sin \theta_0^4} \]
The relative error incurred in neglecting the last term in (1-4) is estimated by dividing it by the (dominant) first term in (1-3), ignoring the small difference between the magnitudes of \( N \) and \( N_8 \)

\[
\text{relative error} = \frac{1}{2} \frac{10^{-6} f N dh}{R} \frac{\cos^2 \theta_0}{\sin^3 \theta_0}
\]  

(1-5)

The satellite height \( h_1 \) is roughly approximated by approximated by \( R \sin \theta_0 \), and the zenith integral is about 2 meters:

\[
\text{relative error} \approx \frac{1}{h_1 \tan^2 \theta_0}
\]  

(1-6)

where \( h_1 \) is the satellite height in meters. Taking \( h_1 \geq 70 \text{ km} \) and \( \theta_0 \geq 10^\circ \), the error calculated from (1-6) is less than 0.05% which can be neglected.
APPENDIX 2

EVALUATION OF INTEGRALS

From the perfect-gas law, the law of partial pressures, and the hydrostatic equation

\[ \frac{dP}{dh} = \frac{Mg(P-e)}{RT} + \frac{Mwge}{RT} \]

\[ = \frac{MgP}{RT} + \frac{0.378 Mge}{RT} \]  \hspace{1cm} (2-1)

where [15]

- \( M = 28.966 \) = Molecular weight of dry air
- \( Mw = 18.016 \) = Molecular weight of water vapor
- \( R = 8314.36 \) Joules (°K)^{-1} (Kg - Mole)^{-1}
  = Universal gas constant
- \( g \) = acceleration of gravity (m/s)
- \( h \) = height (m)

Combining (2-1) and (4)

\[ \int N_e \, dh = -80.343 f(\lambda) \frac{R}{M} \int \frac{1}{g} \, dP \]

\[ + [30.5 f(\lambda) - 11.3] \int \frac{e}{T} \, dh \]  \hspace{1cm} (2-2)

The first integral on the right side of (2-2) above can be evaluated using the approximation [15].

\[ g = 9.806 \left[ 1 - 0.0026 \cos2\varphi - 0.00031 (H + h) \right] \] \hspace{1cm} (2-3)

\[ \frac{1}{g} = \frac{1}{9.806} \left[ 1 + 0.0026 \cos2\varphi + 0.00031 (H + h) \right] \] \hspace{1cm} (2-4)

A2-1
from which, integrating the last term by parts,

\[- \int \frac{1}{g} \, dP = P_0 \frac{1}{9.806} \left[ 1 + 0.0026 \cos 2\varphi + 0.00031 \left( H + \frac{1}{P_0} \int P \, dh \right) \right] = \frac{P_0}{\bar{g}} \]

(2-5)

where \( \bar{g} \) is the value of \( g \) at the height

\[ \bar{h} = \frac{1}{P_0} \int P \, dh \]

(2-6)

above the tracking station or \( H + \bar{h} \) above sea level. Saastamoinen uses a gravitational constant evaluated at*

\[ H + \bar{h} = 7.3 + 0.9 \, \text{H km} \]

(2-7)

From (2-7) and (2-3)

\[ \bar{g} = 9.784 \left( 1 - 0.0026 \cos 2\varphi - 0.00028 \, H \right) \]

(2-8)

\[ \equiv 9.784 \, f(\varphi, H) \]

where \( H \) is the station elevation in kilometers. Saastamoinen has also evaluated the integral

\[ \int \frac{e}{T} \, dh = \frac{R}{4M_g} \epsilon_0 \]

(2-9)

where the \( \bar{g} \) appearing in (2-9) is set equal to \( \bar{g} \) in (2-8) as a convenient approximation. The expression for the zenith integral becomes

\[ \int N \, dh = 80.343 \, f(\lambda) \frac{R}{M_g} P_0 \]

\[ + [30.5 \, f(\lambda) - 11.3] \frac{R}{4M_g} \epsilon_0 \]

(2-10)

*An equivalent result can be obtained by numerically estimating \( \bar{h} \) using (2-17) with \( T_0 \) set equal to \( T_e + 36.7 \) where \( T_e \) is the sea level temperature.
10^{-6} \int N \, dh = \frac{f(\lambda)}{I(\phi, \Pi)} \left[ 0.002357 P_0 + \frac{(30.5 - 11.3/f(\lambda))}{19.2} 0.000141 c_0 \right] \quad (2-11)

Neglecting small errors in the second term of (2-11), equation (13) results.

SECOND INTEGRAL

In equation (12), the magnitude of the coefficient of $1/\sin E$ is about 2.4 meters, while the coefficient of $1/\sin^3 E$, is only about $\frac{1}{3}$ centimeters. At $E = 10^\circ$, the magnitude of the first term is about 12 meters, while the second is about half a meter. Consequently the second term need not be as accurately evaluated as the first, and it is sufficient to use the approximation

$$
\frac{10^{-6}}{r_0} \int h N \, dh = \frac{10^{-6}}{r_e} \int \frac{80.343 f(\lambda) P}{T} \, dh
$$

(2-12)

where $r_0$ is a nominal earth radius (6378 km) and the air is assumed to be dry. It is also sufficient to treat $g$ as a constant throughout.

From (2-1), and integrating by parts

$$
\int \frac{P}{T} \, dh = \frac{R}{Mg} \int P \, dh
$$

(2-13)

The pressure $P$ in (2-13) is obtained by integrating (2-1)

$$
P = P_0 \exp \left[ \frac{Mg}{R} \int \frac{1}{T} \, dh \right]
$$

(2-14)

The temperature $T$ is assumed to have a linear slope

$$
T = T_0 + \beta h
$$

(2-15)

and the integration in (2-14) is carried out giving

$$
P = P_0 \left( \frac{T}{T_0} \right)^{-Mg/R\beta}
$$

(2-16)
The integration in (2-13) may now be performed

\[
\int P \, dh = P_0 \cdot \frac{R T_0}{Mg} \cdot \frac{1}{1 - \frac{R \beta}{Mg}}
\]  

(2-17)

From (2-12), (2-19), and (2-17)

\[
\frac{10^{-6}}{r_0} \int h N_g \, dh = f(\lambda) \frac{10^{-6} (80.343)R^2}{r_0 M^2 g^2} P_0 T_0 K
\]

\[
= f(\lambda) (1.084 \times 10^{-8}) P_0 T_0 K
\]

(2-18)

where \( g \) has been set equal to 9.784 and the factor

\[
K \equiv \frac{1}{1 - \frac{R \beta}{Mg}}
\]

(2-19)

is equal to unity in an isothermal atmosphere (\( \beta = 0 \)) and is equal to about 0.8 in an atmosphere in which the temperature lapse rate is a constant 6°/km (\( \beta = -6°/km \)).

Rather than use the theoretical value for \( K \) given by (2-19), which is based on a constant lapse rate, the value of \( K \) used in the corrections equations is taken to be an empirical constant which was determined by solving (2-18) for \( k \) and calculating its value by numerically integrating through the atmospheres of the U.S. Standard Atmosphere Supplements, 1966. Using linear regression on the values so obtained, the formula

\[
K = 1.163 - 0.00968 \cos \varphi - 0.00104 T_0 + 0.00001435 P_0
\]

(2-20)

resulted. Here \( \varphi \) is the latitude of the tracking station.

THIRD INTEGRAL

The contribution from the third integral in (12) is only marginally significant, and the term can be approximated by

\[
\frac{1}{2} 10^{-12} \int N_g N \, dh = \frac{1}{2} 10^{-12} (80.343)^2 f(\lambda) \int \frac{p^2}{T^2} \, dh
\]

(2-21)
Assuming a constant temperature gradient, and using (2-16)

\[
10^{12} \int \left( N N_g - \frac{1}{2} N^2 \right) dh = \frac{10^{12}}{4} \left( 80.343 \right)^2 f(\lambda) \frac{R P_0^2}{M_g T_0} \cdot \frac{1}{1 + \frac{R P}{2mg}}
\]

(2-22)

The last factor in (2-22) can be expressed in terms of K using (2-19), giving (15).
APPENDIX 3

PROGRAM FOR CALCULATING REFRactivity
profiles FROM Radiosonde data
APPENDIX 3

PROGRAM FOR CALCULATING REFRACTIVITY PROFILES FROM RADIOSONDE DATA*

RADIOSONDE DATA

Radiosonde observations are measurements of pressure, temperature, and humidity taken from the surface up to the point where the balloon that carries the sensors bursts [1]. The values of temperature, pressure and relative humidity measured at certain standard and significant levels during each balloon ascent from numerous weather stations is available from the National Climatic Center. This data can be used to construct continuous refractivity profiles from the surface up to the point of highest measurement. Above the latter point, the refractivity profile can be extended by assuming a suitable temperature profile.

GEOPOTENTIAL ALTITUDE

The equations used to calculate the refractivity profiles employ the geopotential altitude \( H \) [2, p. 217], which is given by

\[
H = \frac{1}{G} \int_0^Z g \, dZ
\]

where \( Z \) is the geometric altitude, and the lower limit of integration is from sea level \( (Z = 0) \). \( H \) is in geopotential meters when \( G \) equals 9.8 m/sec². The local acceleration of gravity is calculated from the latitude \( \phi \) by [2, p. 488]

\[
g_0 = 9.780356 \left(1 + 0.0052835 \sin^2 \phi - 5.9 \times 10^{-6} \sin^2 2\phi\right)\]

and [2, p. 217]

\[
g = \frac{g_0 r_0^2}{(r_0 + Z)^2} \text{ (m/sec}^2)\]

Here \( r_0 \) is an effective earth radius given by [2, p. 218]

\[
r_0 = \frac{2g_0}{\frac{3.085462\times 10^{-6} + 2.27 \times 10^{-9} \cos 2\phi - 2 \times 10^{-12} \cos 4\phi}{}}\text{ (m)}
\]

*This appendix is self-contained. It has separate references, and the notation used differs from that in the rest of the report.
From (1) and (3) the conversion between geopotential and geometric altitude is given by [2, p. 218]

\[ H = \frac{g_0}{G} \left( \frac{r_0 Z}{r_0 + Z} \right) \]  

(5)

and

\[ Z = \frac{r_0 H}{\frac{g_0 r_0}{G} - H} \]  

(6)

VIRTUAL TEMPERATURE

The calculations also make use of the virtual temperature \( T_v \) [3] which is related to the ordinary temperature \( T (°K) \) by

\[ T_v = \frac{T}{1 - 0.379 \frac{e}{P}} \]  

(7)

where \( e \) is the partial pressure of the water vapor in the air, and is given by [4, p. 343]

\[ e = \left( \frac{R_h}{100} \right) (6.11) 10^{7.5 (T - 273.15)} \left( \frac{T - 273.15}{10^2 237.3 + (T - 273.15)} \right) \text{ (mbar)} \]  

(8)

\( R_h \) being the relative humidity in percent.*

CALCULATION OF GEOPOTENTIAL ALTITUDES

The first step in the calculation of refractivity profiles from the radiosonde measurements of pressure, temperature, and relative humidity is to establish a

---

*If the dewpoint temperature \( T_d (°K) \) is given instead of the relative humidity, \( e \) can be calculated from (8) by setting \( R_h = 100 \) and \( T = T_d \).
table of pressure, temperature, and virtual temperature versus geopotential altitude. The virtual temperatures at the given points are calculated from the measured values of $P$, $T$, and $R_h$ using (8) and (7).

To calculate the geopotential altitudes, it is necessary to assume hydrostatic equilibrium [3]

$$dP = -\rho g \, dZ$$

(9)

The density $\rho$ is given with sufficient accuracy by [3]

$$\rho = \frac{MP}{RT_v}$$

(10)

The apparent molecular weight of dry air is taken to be [2, p. 289]

$$M = 28.966$$

(11)

and the universal gas constant [2, p. 289]

$$R = 8314.36 \text{ Joules \,(°K)^{-1} \,(Kg\cdot\text{mole})^{-1}}$$

(12)

Using the assumption that the virtual temperature is a linear function of geopotential height between any two adjacent measured points $H_1$ and $H_2$, (9) may be integrated with the use of (1) and (10) to give

$$\frac{P_2}{P_1} = \left(\frac{T_{v1}}{T_{v2}}\right)^{GM(H_2 - H_1)}$$

(13)

which may be written as

$$H_2 = H_1 + \left(\frac{RT_{v1}}{GM}\right) \frac{x \ln \left(\frac{P_2}{P_1}\right)}{1 \ln (1 + x)}$$

$$= H_1 + \left(\frac{RT_{v1}}{GM}\right) \ln \left(\frac{P_2}{P_1}\right) \left(1 - \frac{x}{2} + \frac{x^2}{3} \ldots\right)^{-1}$$

(14)

A3-3
where

\[ x = \frac{\left( T_{v2} - T_{v1} \right)}{T_{v1}} \]  \hspace{1cm} (15)

Equation (14) can be used stepwise starting at the known geopotential elevation of the radiosonde station to compute the geopotential altitudes.* In this way the required table of pressure, temperature, and virtual temperature versus height is established.

**CALCULATION OF REFRACTIVITY PROFILES**

The radio refractivity \( N \) is given by the formula\(^\dagger \) [5, p. 7]

\[ N = 77.6 \frac{P}{T} + 3.73 \times 10^5 \frac{e}{T^2} \]  \hspace{1cm} (16)

with \( P \) and \( e \) expressed in millibars and \( T \) in degrees Kelvin.

To calculate \( N \) at a given height, i.e., to obtain a point of a refractivity profile, it is necessary to know the values of \( P \), \( T \) and \( e \) at that height. These are obtained as follows:

The height is converted to a geopotential altitude by adding it to the geometric station elevation to obtain the geometric altitude \( Z \), and applying (5). Using the geopotential altitude so calculated, the temperature and the virtual temperature at the given height are obtained from the table of \( P \), \( T \), and \( T_v \) vs. \( H \) by linear interpolation. The pressure at the given height is calculated using (13) with \( P_2 \), \( T_{v2} \) and \( H_2 \) replaced by the values associated with the given height. Finally the vapor pressure \( e \) is calculated from (7). Substitution into (16) then gives the required refractivity.

---

*The geopotential altitudes are computed at the radio-sonde stations and are included in the data stored at the National Climatic Center. The altitudes are recomputed both as a check of the self-consistency of the data and also to generate geopotential altitudes consistent with the values of the fundamental constants (\( R \) and \( M \), for example) adopted.

\^At optical frequencies (2) and (4) of the main text are used.

\[ N = 77.6 \frac{P}{T} \left( 1 + \frac{7.52 \times 10^{-3}}{\lambda^2} \right) \]

where the wavelength \( \lambda \) is in microns.
A listing of the FORTRANH program with a sample profile calculated from meteorological data taken at Dulles airport on 1 January 1967 is shown in Appendix 4.

Also shown are the surface measurements of temperature, pressure and relative humidity, the tropospheric range error obtained from ray-trace (RANGE ERROR), the tropospheric elevation angle error, the tropospheric range error approximation (RANGE ERROR APPROX) obtained from using equation (18) of the main text, and the difference between the ray-trace and the approximation (RANGE DIFF) for arrival angles of 10°, 15°, 20°, 40° and 80°.

REFERENCES FOR APPENDIX 3


APPENDIX 4

PROGRAM LISTING AND EXAMPLE CALCULATION
102 1 = 1 + 1

102 1 = 1 + 1

102 1 = 1 + 1

102 1 = 1 + 1
PROCEDURE GENERATE REFERENCE GEOMETRIC HEIGHTS IN MRT.45.

C

ISN 0160
- 
I = 1

ISN 0161
- IF( (HGRMF(1)/7000.000) > 170.500) THEN 613, 912

ISN 0162
- 912 HGRMF(1) = HGRMF(1) + 170.500

ISN 0163
- 913 HGRMF(1) = HGRMF(1) + 0.005 * GPHD(HGRMF(1)/20000.000)

ISN 0164
- 914 IF( HGRMF(1) > 1000.000) THEN 040, 910

ISN 0165
- I = 1

ISN 0166
- GO TO 911

ISN 0167
- 911 N = I + 1

ISN 0168
- HGRMF(N) = 1000.000

ISN 0169
- CONTINUE

ISN 0170
- FORMAT(2X,F15.6)

ISN 0171
- C CONVERT STATION HEIGHT FROM GROUND LEVEL METERS TO GEOMETRIC HEIGHTS

ISN 0172
- TUF = TPH(S)

ISN 0173
- BUT = T(100*PSG) = HSPR(B)

ISN 0174
- CADD + TOP/GOT

ISN 0175
- C ADJUST STATION HEIGHT IN METERS (GEOMETRIC) TO REFERENCE GEOMETRIC HEIGHTS

ISN 0176
- DD(760) = 1.0

ISN 0177
- 760 GNPFP(I) = HGRMF(I) +CONST

ISN 0178
- C CONVERT GNPFP(I) - IN GEOMETRICAL METERS TO GEOPOTENTIAL HEIGHTS

ISN 0179
- AUVFR SA LEVEL GMPRF(I)

ISN 0180
- GMPRF(I) = GMR(1)

ISN 0181
- D0 761 = 2.0

ISN 0182
- GO TO 760

ISN 0183
- 760 GMR = GMR + GNPFP(I)

ISN 0184
- GMR = GMR + GMRFP(I)

ISN 0185
- GMR = GMR + GMRFP(I)

ISN 0186
- GMR = GMR + GMRFP(I)

ISN 0187
- C CADD + TOP/GOT

ISN 0188
- BUT = GMR + HGRMF(I)

ISN 0189
- C ASSUME LINEARITY IN TEMPERATURE TEMPK(I) AND TMPK(i) WITH READ-

ISN 0190
- LINE VALUES OR HEIGHT -IN GEOPOTENTIAL METERS - COMPILE SLOPE.

ISN 0191
- SLK(I) AND SLRK(I) IN DEGREES KELVIN PER GEOPOTENTIAL METER

ISN 0192
- NTH = N - 1

ISN 0193
- D0 762 = 1.0

ISN 0194
- GO TO 761

ISN 0195
- 761 TMRK(I) = TEMPK(I)

ISN 0196
- TOP = TEMPK(I) = TEMPK(I)

ISN 0197
- BUT = HGRMF(I) = HGRMF(I)

ISN 0198
- SLK(I) = TOP/GOT

ISN 0199
- SLRK(I) = TOP/GOT

ISN 0200
- C LINEARLY INTERPOLATE BETWEEN READ-IOR VALUES OF TEMPK(I) AND

ISN 0201
- TEMPK(i) TO OBTAIN VALUES OF TEMPERATURE AT THE FIXED LEVELS.

ISN 0202
- C

ISN 0203
- NTH = N - 1

ISN 0204
- D0 710 = 1.0

ISN 0205
- GO TO 711

ISN 0206
- 711 FGRPFP(1) + LEHGRMN(I) + ANGRHFP(I + 1) + GTVHGRM(N) ) NT = 1

ISN 0207
- D0 712 = 1.0

ISN 0208
- C

ISN 0209
- D0 722 = 1.0

ISN 0210
- D0 722 = 1.0

ISN 0211
- 722 FGRPFP(I) + LEHGRMN(I) + ANGRHFP(I + 1) + GTVHGRM(N) ) NT = 4

ISN 0212
- D0 722 = 1.0

ISN 0213
- 722 FGRPFP(I) + LEHGRMN(I) + ANGRHFP(I + 1) + GTVHGRM(N) ) NT = 7

ISN 0214
- D0 722 = 1.0

ISN 0215
- C GET TEMPERATURE ABOVE LAST READ-IN TEMPERATURE TEMPK(N) EQUAL

ISN 0216
- C TO THE LAST READ-IN TEMPERATURE.

A4-3
A4-8
```
COMPILE OPTIONS - NAME: MAIN
FREE=104011,INCLUD=IOC11,MACROS=H

SUBROUTINE PATTERN
C

THAYE METHOD FOR TRACING
C

INPUT-DIMENSIONS TO PRINTS AND DENSITY PROFILE PLOTTER
C

C

C

C

C

C

C

DIMENSION R(151), A(151)
DIMENSION REFRA(1), REFRA(1)
DIMENSION 47,
DO 6 I = 1,N
4N-0010 R(I) = HGT(I) + 6378.5CD
DO 6 I = 2N
4N-0012 R(I) = (1,CD-6)*(REFRAT(1)-REFRA(1))
4N-0013 P = R(I)/R(I-1)
4N-0015 TIP = 1,000 - (P2+0,0) + (P+2/1,5)J
4N-0017 BUT = 1,000 - (0,02?JUG + (0+2/1,5)J)
4N-0018 0 A(I) = (R(TIP)/4WBD)
4N-0019 DO T I = 2N
4N-0020 AA = (R(I)/2,0)0(1,1)
4N-0021 BB = 2,000*USIN(THETA(1)+E,50)+2
4N-0022 CC = (R(I)-P(I))/P(I)
4N-0024 DD = (REFRA(1) - REFRA(1))/USIN(1)
4N-0025 EL = (1,CD-6)*CCOS(THETA(1))
4N-0026 S10 = AA*BB + CC - CDME1
4N-0027 SINA = I SORT(CINS1)
4N-0028 ARG = DARSIN(SINA)
4N-0029 7 THETA(I) = 1,000*ARG
4N-0029 TAU = 0.000
DO 9 I = 2N
4N-0300 AA = THETA(I) - THETA(I-1)
4N-0301 BB = A(I)/1,000 + A(I)
4N-0302 CC = AAAB
4N-0303 9 TAL = TAL + CC
4N-0304 PHI = 0.000
DO 11 I = 2N
4N-0305 AA = THETA(I) - THETA(I-1)
4N-0306 BU = 1,000/A(A(I) + 1,000)
4N-0308 CC = AAAB
4N-0309 11 PHI = PHI + CC
4N-0309 RR = 0.000
DO 14 I = 1, N
4N-0303 A(I) = 1,000 + (1,CD-6)*REF(A(I))
DO 13 I = 2N
4N-0303 AA = A(I)+1,0R1-1,050*USIN(THETA(I-1))
4N-0305 BU = OTAN(THETA(I)) - OTAN(THETA(I-1))
4N-0308 CC = 1,000 + A(I)
4N-0308 EN = (AAAB)/CC
4N-0309 13 HE = RF + EN
4N-0300 RANG = (R(I) - R(I-1))*2*24,000 + (1,000*R(SIN(3.14159265358)))
4N-0301 HAGE = DesORT(RANG)
4N-0302 ENCHR = RE = RANG
4N-0302 TOP = CCOS(TAU) = CSIN(TAU)*TAN(THETA(N))
4N-0304 TOT = TOP - (REFIN(N)/REFING(1))
4N-0305 HOT = (REING(N)/REING(1))*TAN(THETA(1))
4N-0306 UJT = HCT - CSIN(TAU)
4N-0307 HCT = HCT = CSIN(TAU)
4N-0307 LEP = RANG(TOFHCT)
4N-0308 RETURN
4N-0308 END
```
<table>
<thead>
<tr>
<th>MKH</th>
<th>TFM(K)</th>
<th>PRI(MB)</th>
<th>MV(MB)</th>
<th>RN(PCT)</th>
<th>REF#ET</th>
<th>FEFCRY</th>
<th>REFRAT</th>
<th>GR REFRAT</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.55</td>
<td>273.9</td>
<td>190.2</td>
<td>4.586</td>
<td>5.204</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.50</td>
<td>270.7</td>
<td>190.3</td>
<td>4.584</td>
<td>5.206</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.45</td>
<td>267.5</td>
<td>187.9</td>
<td>4.582</td>
<td>5.216</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.40</td>
<td>264.3</td>
<td>185.6</td>
<td>4.580</td>
<td>5.226</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.35</td>
<td>261.1</td>
<td>183.3</td>
<td>4.578</td>
<td>5.236</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.30</td>
<td>257.9</td>
<td>181.0</td>
<td>4.576</td>
<td>5.246</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.25</td>
<td>254.7</td>
<td>178.7</td>
<td>4.575</td>
<td>5.256</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.20</td>
<td>251.5</td>
<td>176.5</td>
<td>4.573</td>
<td>5.266</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.15</td>
<td>248.3</td>
<td>174.3</td>
<td>4.572</td>
<td>5.276</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.10</td>
<td>245.1</td>
<td>172.0</td>
<td>4.571</td>
<td>5.286</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td>0.05</td>
<td>241.9</td>
<td>169.7</td>
<td>4.570</td>
<td>5.296</td>
<td>0.233</td>
<td>0.277</td>
<td>211.5</td>
<td>270.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
STATION = 93734 YEAR = 67 MONTH = 1 DAY = 1 HOUR = 12
ELEVATION ANGLE = 0.4746382°, 0.00 RADIANS
RANGE = 0.277030160, 04 KILOMETERS
SUM OF REFRACTIVITY = -0.3566050000 -04
HEIGHT = 0.645722630, 01 KILOMETERS
PRESSURE = -0.4503066600, 04 MILLIBARS
TEMPERATURE = -0.4200000000 01 DEGREES CELCIUS
RELATIVE HUMIDITY = -0.7660000000, 02 PERCENT
ELEVATION ANGLE ERR = 0.7674443800, 02 RADIANS
RANGE ERROR APPROX = 0.7674443800, 02 METERS
RANGE DIFF = 0.7674443800, 02 KILOMETERS

STATION = 93734 YEAR = 67 MONTH = 1 DAY = 1 HOUR = 12
ELEVATION ANGLE = 0.4746382°, 0.00 RADIANS
RANGE = 0.277030160, 04 KILOMETERS
SUM OF REFRACTIVITY = -0.3566050000 -04
HEIGHT = 0.645722630, 01 KILOMETERS
PRESSURE = -0.4503066600, 04 MILLIBARS
TEMPERATURE = -0.4200000000 01 DEGREES CELCIUS
RELATIVE HUMIDITY = -0.7660000000, 02 PERCENT
ELEVATION ANGLE ERR = 0.7674443800, 02 RADIANS
RANGE ERROR APPROX = 0.7674443800, 02 METERS
RANGE DIFF = 0.7674443800, 02 KILOMETERS