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THE EARLY HISTORY OF THE LUNAR INCLINATION

DAVID PARRY RUBINCAM

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ABSTRACT

The effect of tidal friction on the inclination of the lunar orbit to the earth's equator for earth-moon distances of less than 10 earth radii is examined. The results obtained bear on a conclusion drawn by Gerstenkorn and others which has been raised as a fatal objection to the fission hypothesis of lunar origin, namely, that the present nonzero inclination of the moon's orbit to the ecliptic implies a steep inclination of the moon's orbit to the earth's equatorial plane in the early history of the earth-moon system. This conclusion is shown to be valid only for particular rheological models of the earth. In the case of a viscous earth, the results indicate that the problem of wrenching the moon out of an equatorial orbit into an inclined orbit to account for the present tilt of the lunar orbit to the ecliptic must be faced in the accretion theory of the moon's origin and possibly the capture theory, as well as in the fission theory. In this respect all three theories are on the same footing. A solution to the inclination problem is presented.

The treatment of tidal friction adopted here employs the approach of George Darwin and pursues his suggested solution to the inclination problem in great detail. The earth is assumed to behave like a highly viscous fluid in response to tides raised in it by the moon. The moon is assumed to be tideless and in a circular orbit about the earth. The equations of tidal friction are integrated numerically to give the inclination of the lunar orbit as a function of earth-moon

distance. It is found that if the radius of the lunar orbit is greater than 3.83 earth radii, then the inclination of the moon's orbit to the earth's equator will increase if the moon is perturbed from an equatorial orbit, provided the earth's viscosity is greater than 10^{16} poises. The present inclination of the lunar orbit to the ecliptic can be explained if the moon's orbit is perturbed about 3° out of the equatorial plane at 3.83 earth radii, provided that the earth's viscosity is not less than 10^{18} poises. It is also found that if the viscosity is large (greater than 10^{16} poises), then, under certain conditions, the radius of the moon's orbit may actually decrease temporarily, and then increase; and further, that an upper limit can be placed on the inclination of the lunar orbit to the earth's equator when the moon is 3.83 earth radii distant from the earth, regardless of the moon's prior history.

PREFACE

Readers unfamiliar with tidal friction should find Chapter I, Section A and Appendix A of some value. A list of important quantities for this work is given in Table 4. A list of corrections of misprints in Peter Goldreich's important paper "History of the Lunar Orbit" is given in Appendix F. Page numbers of the reference "Darwin (1880)" refer to Darwin's paper as it appears in Scientific Papers by George Howard Darwin, Volume II, 1908.

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INTRODUCTION

Men have speculated about the tides for centuries. An ancient Chinese scholar suggested that the earth lived, that the ocean was its blood, and that the tides were the beating of the earth's pulse (Darwin 1962, pg. 76). An Arabian scholar explained the rising of the tide as being caused by the heating of the ocean by sunlight and moonlight (Darwin 1962, pp. 77-79). Others also suggested that the tides were somehow caused by the sun and moon (Darwin 1962, pp. 79-85), but it remained for Isaac Newton to advance the correct explanation for the cause of the tides (Newton 1966). Newton realized that the lunar tides were caused by a combination of gravitational pull and centrifugal effects which would make the water in the oceans collect on the sides of the earth directly under and directly away from the moon, thus giving the earth a bulge. A similar argument holds for the solar tides.

Newton's theory of the tides was carried forward, notably by Bernoulli, Laplace, Darwin, and Kelvin (Darwin 1962, pp. 86-88) to explain the rise and fall of the oceans on the earth. Their efforts culminated in the work of Doodson and Proudman (Doodson 1958).

George Howard Darwin, son of the famous Charles Darwin, considered not only the problem of the tides raised on the earth by the moon, but also a more subtle problem: the action of the tides on the motion of the moon (Darwin 1880). He included in his investigations not only ocean tides, but also tides raised in the bulk of the earth as well; these latter tides are called earth or body tides. Darwin recognized that friction attending the tides, whether they are raised in the oceans or in the earth, would have profound effects on the moon's orbit. In fact, tidal friction dominates the secular change in the moon's orbital elements.

Darwin assumed the earth to be a homogeneous, incompressible viscous fluid in which tides were raised by the moon; the moon itself was assumed to be a point-mass. He expanded the tidal disturbing function in a Fourier series and integrated the equations for the secular change of the moon's orbital elements backwards in time in an attempt to uncover the past history of the moon. Darwin found that the moon orbited very close to the earth at some time in the distant past. He speculated that the earth and moon were once a single primitive body, and that resonance vibrations set up in the body by the sun caused the body to fission, thus throwing the moon into orbit about the earth. Tidal friction then caused the moon to move away from the earth to its present distance.

Jeffreys (1930) found that dissipation in the primitive body would be so great that the vibrations would be damped out, making it impossible for the moon to be torn out by the action of the sun. The fission theory of the origin of the moon fell out of favor. It has been reproposed in more recent times by Cameron (1963), Wise (1963), and O'Keefe (1969).

Modern interest in tidal friction was rekindled by Gerstenkorn (Alfven 1963), who invoked tidal friction in a new hypothesis of lunar origin: capture of the moon. Gerstenkorn's analysis lead him to propose that the moon was once an independent planet in an orbit which carried the moon close to the earth. The tidal interaction between the moon and earth captured the moon in a retrograde orbit, which subsequently flipped over into the prograde orbit we see today. MacDonald (1964) supported a many-moon hypothesis of lunar origin to overcome a time-scale difficulty in tidal evolution. Singer (1968) investigated the problem of prograde capture, while the analysis of Goldreich (1966) lead him to favor accretion of the moon from a swarm of particles in orbit about the earth.

The problems associated with the intimately connected questions of the moon's origin and its orbital evolution are seen to be of surpassing interest today. We will investigate here one of the problems of the early history of the moon: the inclination of the lunar orbit.

CHAPTER I

TIDAL FRICTION AND THE INCLINATION PROBLEM

A. Qualitative Aspects of Tidal Friction

Some of the qualitative aspects of tidal friction will now be examined; for fuller discussions see Goldreich (1972); MacDonald (1964); and Jeffreys (1962). We will begin by dealing with a simplified picture of the earth-moon system. The earth and moon are assumed to be the only two bodies in existence, with the moon orbiting the earth in a circular orbit lying in the plane of the earth's equator. In addition, the moon is assumed to be perfectly spherical, and the earth to be without atmosphere or oceans, so that we are concerned only with body tides in the earth.

Figure 1(a) shows the case where the earth exhibits no internal friction. In this case if the earth behaved like a solid it would be perfectly elastic; if the earth behaved like a liquid, it would have no viscosity. The tidal forces acting on the earth cause it to bulge along the line joining the centers of the earth and moon. The part of the bulge nearest the moon is raised by the pull of the moon's gravity, which is greatest on the side closest to the moon. The part of the bulge opposite the moon may be thought of as being thrown out by the centrifugal force associated with the motion of the two bodies about their common center of mass. In this case, there would be no evolution of the moon's orbit. The moon would still revolve about the earth in a circular orbit, with only a slight change in the earth's gravitational force from its value for an undistorted earth.

The situation changes, however, when friction is present in the earth; this case is shown in Figure 1(b). In the simplest picture, the action of tidal friction makes the axis of the bulge swing away from the line joining the centers of the earth and moon and reduces the size of the bulge. From the viewpoint of inertial space (a frame fixed with respect to the distant galaxies), the bulge may be thought of as being carried around by the earth's rotation. Note that in this case, where the angular velocity of the earth is greater than the angular velocity of the moon (relative to inertial space), the bulge leads the moon. The behavior of the bulge may also be understood from the viewpoint of an observer standing on the earth's equator. The observer would see the moon rise in the east and set in the west because the angular velocity of the moon relative to the observer is clockwise. In the frictionless case, a high tide would occur when the moon reaches the observers zenith. If friction is present, however, the high tide does not occur until after the moon has passed the zenith, since friction causes a delay. Hence to the observer standing on the earth, the tidal bulge lags behind the moon. If two lines are now drawn, one along the axis of the bulge and one joining the centers of the earth and moon, we get exactly the case shown in Figure 1(b). The tidal lag angle is the angle between the two lines.

If the angular velocity of the moon were greater than the angular velocity of the earth, as in the case of Phobos orbiting Mars, or if the moon revolved in a sense opposite to that of the earth, as in the case of Triton orbiting Neptune, then the bulge would lag behind the moon (as viewed from inertial space).

We return to the simple system shown in Figure 1(b). The moon's gravity pulls on the nearer part of the bulge with greater force than it pulls on the farther part of the bulge, producing a net torque on the earth. This torque acts in a sense opposite to the earth's rotation; hence the earth slows down. By reaction, the bulge will exert a torque on the moon, causing the moon to "speed up" and move away from the earth. Thus the moon was closer to the earth in earlier times. The total angular momentum of the system is conserved in this process, but the total mechanical energy decreases as friction dissipates the energy into heat.

Unfortunately, things are not as simple in reality as shown in Figure 1(b). For one thing, the moon's orbit does not lie in the earth's equatorial plane; nor is it circular but elliptical, which means the distance between the earth and moon is continually changing. Also, body tides are present in the moon, complicating the tidal interaction (MacDonald 1964). Further, the sun also raises tides on the earth; lunar and solar gravity act on both the lunar and solar bulges. The presence of the sun also creates a three body problem. The earth and moon are not spherical even in the absence of tidal forces; the earth is flattened by rotation, for example. Also, the actual shape of the tidal bulge is not necessarily as simple as shown in Figure 1(b). The shape depends upon the model chosen for the earth's properties. In general, the tidal forces distort the earth into a figure resembling a triaxial ellipsoid.

The present-day earth has ocean tides and atmospheric tides as well as body tides. The varying depths of the oceans, the flow of tidal currents, and the irregular shape of coast lines make the ocean tides quite complex. The ocean tides may be responsible for most of the dissipation of energy (see below). The atmospheric tide is an observed semidiurnal variation in atmospheric pressure caused by solar heating and not solar or lunar gravitational forces. This tide lags behind the sun as viewed from inertial space, so that the gravitational torque on the atmospheric tide tends to speed up the earth. This torque may be comparable in magnitude to the solar ocean torque, tending to cancel it (Jeffreys 1962).

Observational evidence for tidal friction comes from a variety of sources. Body tides are observed with sensitive gravimeters (Tomaschek 1957) from which the lag angle may be deduced (MacDonald 1964). The tidal bulge can be observed by its perturbing effects on the orbits of earth satellites (Newton 1968). Celestial observations, both modern and ancient (Newton 1969), reveal the secular acceleration of the moon and the deceleration of the earth's spin. Agreement between the different methods is rough, but they indicate the following (Goldreich 1972): the present-day lag angle in the simple picture of Figure 1(b) is between 2° and 3°, with energy being dissipated at the rate of $\sim 2.6 \times 10^{19}$ ergs/sec. The moon is moving away from the earth at the rate of 3 cm per year, with the earth's daily rotation period slowing down by 2×10^{-5} seconds per year. The work of Miller (1966) suggests that two thirds of the energy dissipation takes place in the shallow seas, but the figure is uncertain and the actual seat of most of the dissipation (whether in the oceans or in the earth) is unknown.

Remarkable evidence for tidal friction in the distant past exists in the form of daily growth bands found in fossil coral and shellfish. The work of Wells (1963) on fossil coral suggests that the year was about 400 days long 380 million years ago, which is consistent with the current rate of slowdown of the earth's rotation. A constant rate of slowdown over this time period has been called into question by the fossil evidence found by Pannella et al. (1968), however.

It should be mentioned that tidal friction is important not only in planet-satellite systems but also in sun-planet and binary star systems as well.

B. The Inclination Problem

We will investigate possible early histories of the inclination of the lunar orbit. The problem of the inclination in the early stages of the moon's history has been cogently summarized by O'Keefe (1972).

Goldreich (1966) investigated the history of the lunar orbit using the assumptions of a circular orbit and weak tidal friction; this latter assumption entails either low viscosity or imperfect elasticity in the earth. Goldreich found that if the moon were 10 earth radii distance from the earth the inclination of the moon's orbit to the earth's equator would have been about 10°; at closer distances the inclination would have been even higher.

This result appears to rule out the fission theory of the moon's origin, since if the rapidly rotating primitive body fissioned, the moon would be thrown into an equatorial orbit around the earth, in contradiction to Goldreich's findings.

Darwin (1880), making assumptions similar to Goldreich's, came to much the same conclusion; but Darwin originated the fission theory. How could Darwin believe in a contradiction?

The crux of the matter is the assumptions that are made in modeling the properties of the earth. Goldreich and other modern investigators (MacDonald 1964; Gerstenkorn (Alfven 1963)) considered the effects of weak tidal friction, as did Darwin; but Darwin went on and examined the effects of high viscosity: strong tidal friction (Darwin 1880).

Darwin found that if the moon were perturbed slightly out of an equatorial orbit, then, under certain conditions, the tidal forces acting on the moon would cause the inclination to grow. An earth which had a high viscosity in its early history might then solve the inclination problem. Darwin seized upon this as the answer and did not investigate the matter further.

We will follow Darwin's treatment in assuming a highly viscous earth in the early stages of the earth-moon system and examine the inclination in more detail; specifically, under what conditions the inclination will increase for small initial perturbations.

C. Proper Planes and the Inclination Problem

Let us now examine the inclination problem from the aspect of the proper planes of the moon and earth. These planes were discovered by Laplace (1966) and were used by Darwin (1880) in his treatment of tidal friction.

Laplace found that the plane of a satellite's orbit about an oblate planet tended to maintain a constant inclination to a plane which he called the proper plane of the satellite. We will follow Darwin and call this inclination J.

The proper plane lies intermediate between the plane of the planet's orbit about the sun (the ecliptic plane for the earth) and the invariable plane (the plane perpendicular to the total angular momentum vector of the planet-satellite system). The angle between the ecliptic and proper planes Darwin called $J_{/}$ If a satellite orbits far from a planet, the sun's influence dominates the inclination and the proper plane is nearly parallel to the ecliptic ($J_{/} \approx 0$), so that the satellite has very nearly a constant inclination to the ecliptic. If a satellite orbits close to a planet, the oblateness of the planet dominates the inclination and the proper plane is nearly parallel to the invariable plane. In this case the satellite tends to maintain a constant inclination to the equatorial plane of the planet (as shown later).

These two limiting cases are referred to by Goldreich (1966), who found that the transition between the two came at a distance which he called the critical distance. This is the distance where the torque exerted on the satellite by the planet's bulge is equal in magnitude to the torque exerted on the satellite by the sun. The critical distance is about 10 earth radii for the earth-moon system.

At the present time the moon is about 60 earth radii distance from the earth, so that the orbital plane of the moon keeps a nearly constant tilt to the ecliptic. The angle $J_{/}$ between the proper plane and the ecliptic is about 8"

(Darwin 1880) and the angle J between the proper plane and the moon's orbital plane is about 5°9'.

The earth also has a proper plane; the earth's equatorial plane tends to maintain a constant inclination to its proper plane. The angle between the two is called $I_{/}$ by Darwin, with the angle between the earth's proper plane and the ecliptic being I. At present $I_{/}$ is about 9″ (Darwin 1880) and I is 23°27'.

While the orbital plane of a satellite tends to maintain a constant angle J to its proper plane, the orbital plane precesses in space, so that the vector normal to the orbital plane sweeps out a cone around the vector normal to the proper plane. This is diagrammed in Figure 2(a). Likewise, the earth's axis sweeps out a cone around the vector normal to its proper plane. Both precessions have the same speed and direction.

At small distances between the planet and satellite (i.e. when solar influence can be neglected), the poles of the two proper planes merge with the pole of the invariable plane, so that the orbital plane of the satellite and the equatorial plane of the planet maintain a constant tilt to the invariable plane and each other, as shown in Figure 2(b). Hence in this case the orbital plane of the satellite has a constant inclination to the equatorial plane of the planet as mentioned earlier.

If the moon somehow formed or arrived in the earth's equatorial plane at a distance of less than 10 earth radii, then the orbital plane, equatorial plane, and invariable plane would all coincide and the inclination J of the moon's orbit to its proper plane would be essentially zero. Subsequently, if tidal friction did nothing to affect J as it pushed the moon steadily away from the earth, then at the present time the moon would have essentially an ecliptic orbit (J and $J_{/} \approx 0$). However, the present value of J is about 5°9'. Thus, if it is assumed that the moon did form close to the earth in the equatorial plane, then it must be explained why the moon's orbit now has a five degree tilt to the ecliptic and not a virtually zero tilt.

To put it another way, if the present five degree inclination is extrapolated back into the past, the plane of the moon's orbit would be steeply inclined to the equatorial plane of the earth when the moon was close to the earth. Therefore, those theories of the origin of the moon which postulate the moon's formation in the earth's equatorial plane must explain this discrepancy.

The effect of tidal friction in Goldreich's (1966) formulation on J, the inclination of the moon's orbit to its proper plane, can be extracted from his Figure 7, which is reproduced here as Figure 3.

Goldreich's figure shows the inclination of the moon's orbital plane to the ecliptic as a function of earth-moon distance. The inclination $J_{/}$ of the moon's proper plane to the ecliptic is so small at the present distance of 60 earth radii that it is imperceptible in the figure, so that the moon's orbital plane appears to keep a constant five degree inclination to the ecliptic for this distance. The normals to the ecliptic, proper plane, and the cone swept out by the normal to the lunar orbit for this case are shown in Figure 4(a). Note that the normal to the ecliptic is inside the cone.

At about 30 earth radii the angle $J_{/}$ becomes large enough to be noticeable in Goldreich's figure, so that the variation in angle between the ecliptic and orbital planes is clearly visible and the curve branches, showing the maximum and minimum inclination. This situation is diagrammed in Figure 4(b). Note that the normal to the ecliptic still lies inside the cone. Reference to this figure should make clear that the inclination J of the lunar orbit to its proper plane can be found from Goldreich's figure by adding the maximum and minimum inclinations and dividing by 2. The inclination $J_{/}$ of the proper plane to the ecliptic is then found by subtracting the minimum inclination from J.

At about 17.5 earth radii in Goldreich's figure the normal to the ecliptic lies in the surface of the cone, so that the minimum inclination is zero and the lower branch of the curve touches the horizontal axis. This is shown in Figure 4(c).

Between 17.5 and 3 earth radii the normal to the ecliptic falls outside the cone, as shown in Figure 4(d). The inclination $J_{/}$ in this region is then found by drawing a curve equidistant between two branches in Goldreich's figure and measuring from the horizontal axis to that curve. The angle J is half the difference between the two branches.

Figure 20 shows J as a function of earth-moon distance as extracted from Goldreich's figure by the above process (dashed curve). Note that the inclination of the moon's orbit to the proper plane <u>increases</u> as the distance decreases below 13 earth radii. Darwin's small viscosity model gives a remarkably similar result (dotted line; see Chapter IV). This indicates that small tidal lag angles cannot be invoked to drive the moon out of the earth's equatorial plane; if it could, then J would decrease as distance decreases for small distances, instead of increasing.

CHAPTER II

DARWIN'S APPROACH TO TIDAL FRICTION

We will now briefly outline George Darwin's approach to the problem of tidal friction (Darwin 1879, 1880). Although Darwin treats the case of a planet attended by two tide-raising satellites (such as the earth attended by the moon and sun, where the latter may be treated as a satellite of the earth), we will restrict the discussion in this chapter to the moon and earth as an isolated system; i.e. the presence of the sun will be neglected.

The following assumptions are made by Darwin: the earth is a homogeneous, viscous, incompressible sphere. Body tides are raised in the earth by the moon. The moon is taken to be a point-mass without rotational angular momentum. The tide-raising potential generated by the moon is a second degree spherical harmonic. The tidal disturbing potential generated by the earth is expressed as a sum of second degree spherical harmonics. The effects of inertia are neglected when solving for the response of the earth to the tide-raising force.

We will further restrict the discussion to a circular orbit for the moon about the earth.

Before plunging into a discussion of Darwin's treatment we will discuss the effects of the earth's rotational bulge on the motion of the moon.

Goldreich (1966) has shown in an elegant manner that the rotational flattening of the earth produces no secular change in the magnitude of the orbital angular momentum of the earth-moon system or in the earth's rotational angular momentum; we denote these two quantities by L_M and L_E, respectively.

The orbital angular momentum L_M of the system about the center of mass of the system is

$$L_M = M\Omega d_1^2 + m\Omega d_2^2$$

where

M = mass of the earth

m = mass of the moon

Ω = angular velocity of the earth and moon about the center of mass

d_1 = distance of the earth from the center of mass

d_2 = distance of the moon from the center of mass

Now by Kepler's third law

$$\Omega = \frac{\sqrt{G(M+m)}}{r^{3/2}}$$

where

G = universal gravitational constant

r = earth-moon distance

Also

$$d_1 = \left(\frac{m}{M+m}\right) r$$

$$d_2 = \left(\frac{M}{M+m}\right) r$$

We may now write

$$L_M = \sqrt{\frac{G}{M+m}}\, M m\, r^{1/2}$$

L_E is easily seen to be

$$L_E = Cn$$

where

n = rotational angular velocity of the earth

C = polar moment of inertia of the earth.

We may show the constancy of other important quantities by using Goldreich's result of the constancy of L_M and L_E.

Clearly there is no secular change in the earth-moon distance r if there is no secular change in L_M. Likewise, there is no secular change in the rotational angular velocity of the earth n if there is no secular change in L_E. Therefore r and n are constant in the secular sense for the case of the rotational bulge.

We refer now to Figure 5, which shows the angular momentum triangle for the earth-moon system. $\vec{L}_T$ is the total angular momentum of the system and is constant in both magnitude and direction because the system is isolated. It is clear from the diagram that if L_M, L_E, and L_T are unchanging, then j, the angle between the plane of the lunar orbit and the invariable plane, and i, the angle between the plane of the earth's equator and the invariable plane, are constant. Thus the rotational bulge of the earth produces no secular change in r, n, j, or i.

The flattening of the earth does make the lunar orbit precess in space. It is clear from Figure 5 that the earth must precess at the same rate and in the same sense as the lunar orbit. By conservation of angular momentum $\vec{L}_T = \vec{L}_M + \vec{L}_E$, so that the three vectors must lie in the same plane.

It should be clear, then, that if the longitude of the moon's node N is measured along the invariable plane from the descending node of the intersection of the earth's equatorial plane and the invariable plane, N must be zero.

In this investigation we are chiefly concerned with secular changes in r, n, j, and i; hence further consideration of the effects of the rotational flattening on the moon's motion will be dispensed with.

It should be mentioned that the sun also causes no secular change in r, n, j and i to the order of approximation carried out by Goldreich.

Let us now investigate the response of the earth to the tide-raising force and the effect of the earth's response on the moon (see Appendix A for a derivation of the tide-raising potential and the tidal disturbing function).

The tide-raising potential at some point (x^*, y^*, z^*) in the earth is given by Equation (A-6) of Appendix A as

$$V_t = \frac{3}{2} \frac{Gm}{r} \left(\frac{r^*}{r}\right)^2 \left[\cos^2 \Theta - \frac{1}{3}\right]$$

and is the first equation of §4 of Darwin (1880) with some notational changes. r^* is the distance from the center of the earth to (x^*, y^*, z^*) and the angle Θ (which Darwin calls PM) is shown in Figure 21.

If the earth were a frictionless fluid the tide-raising force would raise a tide on the earth, with the height of the tide σ_t being given by (Darwin 1879, Equation 13):

$$\sigma_t = \frac{15}{4} \frac{Gm}{g} \frac{a^2}{r^3} \left(\cos^2 \Theta - \frac{1}{3}\right)$$

where a is the radius of the earth and g is the gravitational acceleration at the earth's surface. The earth would clearly bulge in this case as shown in Figure 1(a). Note that the height of the tide is inversely proportional to the cube of the earth-moon distance.

Darwin chose axes fixed in the earth and expanded $\left(\cos^2 \Theta - \frac{1}{3}\right)$ in terms of the direction cosines of both (x^*, y^*, z^*) and the position of the moon (x, y, z). Letting ξ, η and ζ be the direction cosines of (x^*, y^*, z^*) and M_1, M_2, and M_3 the direction cosines of the moon, then $\cos \Theta = \xi M_1 + \eta M_2 + \zeta M_3$ and we may write

$$\sigma_t = \frac{15}{4} \frac{G m}{g} \frac{a^2}{r^3} \left\{ 2 \xi \eta M_1 M_2 + 2 \frac{\xi^2 - \eta^2}{2} \frac{M_1^2 - M_2^2}{2} + 2 \eta \zeta M_2 M_3 \right.$$
$$\left. + 2 \xi \zeta M_1 M_3 + \frac{3}{2} \frac{\xi^2 + \eta^2 - 2 \zeta^2}{3} \frac{M_1^2 + M_2^2 - 2 M_3^2}{3} \right\} \tag{II-1}$$

after some algebraic rearrangements.

M_1, M_2, and M_3 depend upon i and j, the respective angles of the earth's equator and the plane of the moon's orbit to a fixed plane, which, in the two body problem, we take to be the invariable plane; n, the rotational angular velocity of the earth; Ω, the angular velocity of the moon in its orbit; N, the longitude of the node of the moon's orbit measured from the descending intersection of the earth's equatorial plane and the fixed plane along the fixed plane; and t, the time. For example, M_1 is given by Equation (20) of Darwin (1880) as

$$M_1 = P^2 p^2 \cos (\chi - \ell - N) + P^2 q^2 \cos (\chi + \ell - N) + Q^2 p^2 \cos (\chi + \ell + N)$$
$$+ Q^2 q^2 \cos (\chi - \ell + N) + 2 P Q p q \left[\cos (\chi + \ell) - \cos (\chi - \ell)\right]$$

Here $P = \cos \left(\frac{1}{2} i\right)$, $Q = \sin \left(\frac{1}{2} i\right)$, $p = \cos \left(\frac{1}{2} j\right)$, $q = \sin \left(\frac{1}{2} j\right)$, $\chi = nt + \chi_0$ with χ_0 a constant, and $\ell = \Omega t + \epsilon$, where ϵ is the longitude of epoch. M_2 and M_3 may be written in a similar fashion. Notice that M_1 is expressed as a sum of terms periodic in time, whose periods depend on linear combinations of n and Ω; the same is true of M_2 and M_3.

The combinations of M_1, M_2, and M_3 that appear in Equation (II-1) ($M_1 M_2$, $\frac{M_1^2 - M_2^2}{2}$, etc.) can also be written as sums of simple harmonics whose angular speeds are linear combinations of n and Ω. (Table 1 gives the total number of angular speeds which arise.) For instance, after considerable work Darwin shows that $M_1 M_2$ may be written (Darwin 1880, Equation 25):

$$M_1 M_2 = \frac{\sqrt{-1}}{4} \Big\{ \pi^4 e^{2\sqrt{-1}(\chi - \theta)} + 2\pi^2 \underline{\kappa}^2 e^{2\sqrt{-1}\chi} + \underline{\kappa}^4 e^{2\sqrt{-1}(\chi + \theta)} \tag{II-2}$$

$$- \underline{\pi}^4 e^{-2\sqrt{-1}(\chi - \theta)} - 2\underline{\pi}^2 \kappa^2 e^{-2\sqrt{-1}\chi} - \kappa^4 e^{-2\sqrt{-1}(\chi + \theta)} \Big\}$$

Here $\pi = Pp - Qq\, e^{+\sqrt{-1}N}$; $\kappa = Qp + Pq\, e^{\sqrt{-1}N}$; $\underline{\pi} = Pp - Qq\, e^{-\sqrt{-1}N}$; $\underline{\kappa} = Qp + Pq\, e^{-\sqrt{-1}N}$; and $\theta = \ell + N$. Darwin put the sines and cosines in exponential form for convenience in later work.

Equation (II-1) could now be written

$$\sigma_t = \frac{15}{4} \frac{Gm}{g} \frac{a^2}{r^3} \Big\{ 2\xi\eta \Big[\frac{\sqrt{-1}}{4} \Big(\pi^4 e^{2\sqrt{-1}(\chi - \theta)} + 2\pi^2 \underline{\kappa}^2 e^{2\sqrt{-1}\chi} + \ldots \Big\}$$

by substituting in it the complicated expressions for $M_1 M_2$, etc.

The equation above gives the displacement of the earth's surface when the earth is composed of a frictionless fluid. What we now wish to find is the expression for σ_t when friction is present inside the earth.

It is assumed that the effects of friction are such that each simple time harmonic that appears in the expression for σ_t is multiplied by a factor to reduce the amplitude of the harmonic, and its phase is altered by a certain lag angle. For example, $M_1\ M_2$ now becomes in the presence of friction (Darwin 1880, Equation 33)

$$\frac{\sqrt{-1}}{4}\Big\{F_1\pi^4 e^{\sqrt{-1}\,[2(\chi-\theta)-2f_1]} + F\,2\pi^2\underline{\kappa}^2 e^{\sqrt{-1}\,[2\chi-2f]}$$

$$+ F_2\underline{\kappa}^4 e^{\sqrt{-1}\,[2(\chi+\theta)-2f_2]} - F_1\underline{\pi}^4 e^{-\sqrt{-1}\,[2(\chi-\theta)-2f_1]}$$

$$- F\,2\underline{\pi}^2\kappa^2 e^{\sqrt{-1}\,[-2\chi+2f]} - F_2\kappa^4 e^{-\sqrt{-1}\,[2(\chi+\theta)-2f_2]}\Big\}$$

F_1, F, and F_2 are amplitude factors and $2f_1$, $2f$, and $2f_2$ are the respective phase angles. (Table 1 gives the amplitude factors and phase angles for all the speeds.)

Darwin calls the above expression $\mathfrak{X}\mathfrak{Y}$. $M_1^2 - M_2^2$ becomes $\mathfrak{X}^2 - \mathfrak{Y}^2$, etc., so that now in place of (II-1), we have

$$\sigma_t = \frac{15}{4}\,\frac{Gm}{g}\,\frac{a^2}{r^3}\Big\{2\xi\eta\,\mathfrak{X}\mathfrak{Y} + 2\,\frac{\xi^2-\eta^2}{2}\,\frac{\mathfrak{X}^2-\mathfrak{Y}^2}{2} + 2\eta\zeta\,\mathfrak{Y}\mathfrak{Z}$$

$$+ 2\xi\zeta\,\mathfrak{X}\mathfrak{Z} + \frac{3}{2}\,\frac{\xi^2+\eta^2-2\zeta^2}{3}\,\frac{\mathfrak{X}^2+\mathfrak{Y}^2-2\mathfrak{Z}^2}{3}\Big\} \qquad \text{(II-3)}$$

as the equation for the earth's surface in the presence of friction (Darwin 1880, Equation 30).

The exact values of the amplitude factors and phase angles depend upon the model chosen for the earth's properties. The model we are interested in is the case where the earth is a fluid exhibiting Newtonian viscosity. Darwin (1879) found the amplitude factors and phase angles for this case, which are given by the following relations:

$$\tan\,[\text{lag angle}] \;=\; [\text{angular speed}] \times \frac{19\,\nu}{2\,g\,a\,\rho}$$

$$[\text{amplitude factor}] \;=\; \cos\,[\text{lag angle}]$$

where ρ is the density of the earth and ν is its viscosity.

An example of the above relations is

$$\tan\,2\,f_1 \;=\; 2\,(n - \Omega)\,\frac{19\,\nu}{2\,g\,a\,\rho}$$

$$F_1 \;=\; \cos\,2\,f_1$$

for the angular speed $2\,(n - \Omega)$.

Now that σ_t has been found, our next step is to find the tidal disturbing function R_t acting on the moon. It has been derived in Appendix A and is given by Equation (A-15):

$$R_t\,(r', \alpha', \beta') \;=\; \frac{4}{5}\,\pi\,G\left(\frac{M + m}{M}\right)\rho\,a\left(\frac{a}{r'}\right)^3 \sigma_t\,(\alpha', \beta') \qquad \text{(II-4)}$$

where r' (which was called Δ in Appendix A) is the earth-moon distance, α' and β' are the longitude and colatitude of the moon in the earth-fixed frame, and ρ is the earth's density. Primes are placed on the variables for reasons discussed below and in Appendix A. Note that σ_t implicitly depends on r^{-3} (see Equation II-3), so that the disturbing function is proportional to the inverse sixth power of the earth-moon distance.

We now wish to know how the disturbing function changes the orbital elements of the moon. Here Darwin uses the Lagrange equations for the time derivatives of the osculating orbital elements. These equations are derived e.g. in Brouwer and Clemence (1961). Darwin uses four of the six equations in his 1880 paper, which we reproduce in his notation (Darwin 1880, Equations 1-4):

$$\frac{dc}{dt} = \frac{2\Omega c^2}{G(M+m)} \frac{\partial R}{\partial \epsilon}$$

$$\frac{de}{dt} = \frac{\Omega c}{G(M+m)} \left[\frac{1-e^2}{e} \frac{\partial R}{\partial \epsilon} - \frac{\sqrt{1-e^2}}{e} \left(\frac{\partial R}{\partial \epsilon} + \frac{\partial R}{\partial \pi}\right)\right]$$

$$-\frac{dj}{dt} = \frac{\Omega c}{G(M+m)} \frac{1}{\sqrt{1-e^2}} \left[\frac{1}{\sin j} \frac{\partial R}{\partial N} + \tan \frac{1}{2} j \left(\frac{\partial R}{\partial \epsilon} + \frac{\partial R}{\partial \pi}\right)\right]$$

$$\sin j \frac{dN}{dt} = \frac{\Omega c}{G(M+m)} \frac{1}{\sqrt{1-e^2}} \frac{\partial R}{\partial j}$$

The only quantities not defined by us thus far are c, the semi-major axis of the orbit; e, the eccentricity; π, the longitude of perigee (not to be confused with π used elsewhere in Darwin); and R, any disturbing function in general. Since we are interested in the effects of the tides on the moon, we set R_t equal to R.

Darwin alters the form of the equations to make them more convenient to use for his purposes. For example, in the case of a circular orbit the first and third equations become (Darwin 1880, Equations 11 and 13):

$$\frac{1}{k} \frac{d\xi}{dt} = \frac{\partial W}{\partial \epsilon}$$

$$-\frac{\xi}{k} \frac{dj}{dt} = \frac{1}{\sin j} \frac{\partial W}{\partial N} + \tan \frac{1}{2} j \frac{\partial W}{\partial \epsilon}$$

where now $\xi = \left(\frac{c}{c_0}\right)^{1/2}$, with c_0 being some reference distance, and c is now the radius of the orbit; k is $\frac{C}{GMm} \Omega_0 c_0$, with Ω_0 being the angular velocity of the moon at the reference distance c_0; and W is $\frac{Mm}{M+m} \frac{1}{C}$ R, with $C = \frac{2}{5} Ma^2 = \frac{8}{15} \pi \rho a^5$.

The next step is to express W in terms of ξ, i, j, N, and ϵ and evaluate the derivatives of W to find the time rate of change of the moon's parameters. Before we proceed, however, we must heed the warning given in Appendix A not to confuse the moon's parameters as they enter in the role of tide-raising body and in the role of tidally disturbed body, even though here the bodies are one and the same (the moon). Let us therefore follow Darwin and place primes on the parameters of the disturbed body (which we have already done in Equation II-4). Our expression for W becomes (Darwin 1880, Equation 31):

$$W = \frac{\tau\tau'}{\mathfrak{g}}\left[2X'Y'\mathfrak{X}\mathfrak{Y} + 2\,\frac{X'^2 - Y'^2}{2}\,\frac{\mathfrak{X}^2 - \mathfrak{Y}^2}{2} + 2Y'Z'\mathfrak{Y}\mathfrak{Z} + 2X'Y'\mathfrak{X}\mathfrak{Z} + \frac{3}{2}\,\frac{X'^2 + Y'^2 - 2Z'^2}{3}\,\frac{\mathfrak{X}^2 + \mathfrak{Y}^2 - 2\mathfrak{Z}^2}{3}\right]$$

where X', Y', and Z' are the direction cosines of the moon in its role as disturbed body, $\tau = \tau_0/\xi^6$ with $\tau_0 = \frac{3}{2}\frac{Gm}{c_0^3}$ (with a similar expression for τ'), and $\mathfrak{g} = \frac{2}{5}\frac{g}{a}$.

X' Y' is the same as Equation (II-2) save primed variables replacing the unprimed variables; and similarly for $\frac{X'^2 - Y'^2}{2}$, Y' Z', etc. Primes must also be placed on the parameters in the variational equations:

$$\frac{1}{k'}\frac{d\xi'}{dt} = \frac{\partial W}{\partial \epsilon'}$$

$$-\frac{\xi'}{k'}\frac{dj'}{dt} = \frac{1}{\sin j'}\frac{\partial W}{\partial N'} + \tan\frac{1}{2}j'\,\frac{\partial W}{\partial \epsilon'}$$

since they refer to the motion of the disturbed body. After differentiation the primes may be dropped without fear of confusion. In fact, primes are not needed on χ, ξ, Ω, j, since these quantities are not differentiated in the above equations; nor are primes needed on k and τ_0; hence primes on them may be dropped before differentiation.

Terms in W which remain periodic after differentiation and after the primes have been dropped may be deleted at once from W, since we are interested in only secular changes in the orbital parameters.

To illustrate the procedure the term

$$2\,X'Y'\mathfrak{X}\,\mathfrak{Y} + 2\,\frac{X'^2 - Y'^2}{2}\;\frac{\mathfrak{X}^2 - \mathfrak{Y}^2}{2}$$

appearing in W is (Darwin 1880, Equation 37)

$$\frac{1}{4}\left\{F_1\,\pi^4\,\underline{\pi}'^4\,e^{\sqrt{-1}\,[2(\theta'-\theta)-2f_1]} + 4F\,\pi^2\,\underline{\kappa}^2\,\underline{\pi}'^2\,\kappa'^2\,e^{-2f\sqrt{-1}}\right.$$

$$\left. F_2\,\underline{\kappa}^4\,\kappa'^4\,e^{-\sqrt{-1}\,[2(\theta'-\theta)+2f_2]}\right\} + \frac{1}{4}\left\{F_1\,\underline{\pi}^4\,\pi'^4\,e^{-\sqrt{-1}\,[2(\theta'-\theta)-2f_1]}\right.$$

$$\left. + 4F\,\underline{\pi}^2\,\kappa^2\,\pi'^2\,\underline{\kappa}'^2\,e^{\sqrt{-1}\,2f} + F_2\,\kappa^4\,\underline{\kappa}'^4\,e^{\sqrt{-1}\,[2(\theta'-\theta)+2f_2]}\right\}$$

Periodic terms have been deleted. If χ and χ' had been included in the above expression $2(\chi - \chi')$ would appear in the exponentials of the first term in curly brackets and $-2(\chi - \chi')$ in the exponentials of the second curly bracketed term; but since primes are unnecessary on χ, these terms disappear. Also, $\theta = \Omega t + \epsilon$ and $\theta' = \Omega' t + \epsilon'$; but $\Omega = \Omega'$, so that $\theta' - \theta$ becomes $\epsilon' - \epsilon$.

We will now apply one of the variational equations to the terms in W in which the lag angle $2f_1$ appears:

$$W_{2f_1} = \frac{\tau^2}{g}\,\frac{1}{4}\,F_1\left[\pi^4\,\underline{\pi}'^4\,e^{\sqrt{-1}\,[2(\epsilon'-\epsilon)-2f_1]} + \underline{\pi}^4\,\pi'^4\,e^{-\sqrt{-1}\,[2(\epsilon'-\epsilon)-2f_1]}\right]$$

To find this term's contribution to $\frac{1}{k}\frac{d\xi}{dt}$ we must find $\frac{\partial W_{2f_1}}{\partial \epsilon'}$. Now ϵ' does not appear in π^4, π'^4, $\underline{\pi}^4$, or $\underline{\pi}'^4$, so that the derivative operates only on the exponential terms. We obtain after differentiation

$$\frac{\partial W_{2f_1}}{\partial \epsilon'} = -\frac{\tau^2}{g}\frac{1}{2\sqrt{-1}} F_1 \left[\pi^4 \underline{\pi}'^4 e^{\sqrt{-1}\,[2(\epsilon'-\epsilon)-2f_1]} - \underline{\pi}^4 \pi'^4 e^{-\sqrt{-1}\,[2(\epsilon'-\epsilon)-2f_1]}\right]$$

Dropping the primes, we have

$$\frac{\partial W_{2f_1}}{\partial \epsilon'} = \frac{\tau^2}{g} F_1 \sin 2f_1 \, \pi^4 \underline{\pi}^4$$

where

$$\pi = Pp - Qq\, e^{\sqrt{-1}\,N}, \quad \underline{\pi} = Pp - Qq\, e^{-\sqrt{-1}\,N},$$

and

$$\sin 2f_1 = \frac{e^{2\sqrt{-1}\,f_1} - e^{-2\sqrt{-1}\,f_1}}{2\sqrt{-1}}.$$

N is equal to zero when the earth and moon are the only two bodies in existence [as discussed earlier], so that $\pi = \underline{\pi} = \cos\left[\frac{1}{2}(i+j)\right]$. Also, in the case of a viscous earth $F_1 = \cos 2f_1$, making the contribution of the W_{2f_1} term to $\frac{1}{k}\frac{d\xi}{dt}$

$$\frac{1}{2}\frac{\tau^2}{g}\pi^8 \sin 4f_1$$

The other terms in W may be evaluated in similar fashion to finally give

$$\frac{1}{k}\frac{d\xi}{dt} = \frac{1}{2}\frac{\tau^2}{g}\Big[\pi^8 \sin 4 f_1 - \kappa^8 \sin 4 f_2 + 4\pi^6 \kappa^2 \sin 2 g_1 - 4\pi^2 \kappa^6 \sin 2 g_2 - 6\pi^4 \kappa^4 \sin 4 h\Big] \tag{II-5}$$

as the secular rate of change of ξ, where $\kappa = \sin\left[\frac{1}{2}(i + j)\right]$.

Similarly

$$\frac{-\xi}{k}\frac{dj}{dt} = \frac{\tau^2}{g}\Big[\frac{1}{2}\pi^7 \kappa \sin 4 f_1 + \pi^3 \kappa^3 \sin 4 f + \frac{1}{2}\pi \kappa^7 \sin 4 f_2 + \frac{3}{2}\pi^3 \kappa^3 (\pi^2 - \kappa^2) \sin 4 h - \frac{1}{2}\pi^5 \kappa [\pi^2 - 3\kappa^2] \sin 2 g_1 + \frac{1}{2}\pi \kappa (\pi^2 - \kappa^2)^2 \sin 2 g + \frac{1}{2}\pi \kappa^5 (3\pi^2 - \kappa^2) \sin 2 g_2\Big] \tag{II-6}$$

gives the secular rate of change of j.

Equations (II-5) and II-6) are respectively Equations (73) and (71) of Darwin (1880).

The secular rates of change of the earth's angular velocity n and the inclination i of the earth's equator to the invariable plane can be derived from (II-5) and (II-6) by application of the law of conservation of angular momentum:

$$\frac{dn}{dt} = \frac{-\tau^2}{g}\Big[\frac{1}{2}\pi^8 \sin 4 f_1 + 2\pi^4 \kappa^4 \sin 4 f + \frac{1}{2}\kappa^8 \sin 4 f_2 + \pi^6 \kappa^2 \sin 2 g_1 + \pi^2 \kappa^2 (\pi^2 - \kappa^2)^2 \sin 2 g + \pi^2 \kappa^6 \sin 2 g_2\Big] \tag{II-7}$$

$$n\,\frac{di}{dt} = \frac{\tau^2}{g}\left[\frac{1}{2}\,\pi^7\kappa\,\sin 4f_1 - \pi^3\kappa^3(\pi^2-\kappa^2)\,\sin 4f - \frac{1}{2}\,\pi\kappa^7\,\sin 4f_2\right.$$

$$+\frac{1}{2}\,\pi^5\kappa\,(\pi^2+3\kappa^2)\,\sin 2g_1 - \frac{1}{2}\,\pi\kappa\,(\pi^2-\kappa^2)^3\,\sin 2g$$

$$\left.-\frac{1}{2}\,\pi\kappa^5\,(3\pi^2+\kappa^2)\,\sin 2g_2 - \frac{3}{2}\,\pi^3\kappa^3\,\sin 4h\right] \qquad \text{(II-8)}$$

These two equations are the last two equations of § 11 of Darwin (1880). Equations (II-5) through (II-8) are central to our discussion.

CHAPTER III

TIDAL FRICTION AND LARGE VISCOSITY

A. Inclination of the Lunar Orbit to the Earth's Equator

In this chapter we discuss the secular motion of the moon for small inclinations of the lunar orbit to the earth's equatorial plane for earth-moon distances less than 10 earth radii. Equations (II-5) - (II-8) are not valid beyond 10 earth radii because solar influence would have to be considered (see Chapter I, Section C). The effects of the sun beyond 10 earth radii are considered later in Chapter IV.

The viscosity of the earth will be assumed constant throughout this discussion. Variable viscosity is considered in Section D of this chapter.

Let us first write Equations (II-5) - (II-8) in slightly altered form.

From Chapter II the moon's orbital angular momentum is

$$L_M = \sqrt{\frac{G}{M+m}}\, M m\, c^{1/2}$$

The earth's rotational angular momentum is

$$L_E = C n$$

The constant k introduced on page 21 is

$$k = \frac{C}{GMm} \Omega_0 c_0$$

We wish to express our results with reference to a particular earth-moon distance c_0; hence we can rewrite the first equation as

$$L_M = \sqrt{\frac{G}{M+m}}\, M m\, c_0^{1/2} \left(\frac{c}{c_0}\right)^{1/2} = b\,\xi$$

where

$$b = \sqrt{\frac{G}{M+m}}\, M m\, c_0^{1/2}.$$

Since

$$k = \frac{C}{G M m}\, \Omega_0\, c_0$$

$$bk = \frac{C}{\sqrt{G(M+m)}}\, \Omega_0\, c_0^{3/2}.$$

But by Kepler's third law

$$\Omega_0^2\, c_0^3 = G(M+m)$$

Hence

$$bk = C$$

Equations (II-5) - (II-8) may now be written as

$$\frac{dL_M}{dt} = \frac{1}{2}\,\frac{\tau^2}{g}\, C \left[\pi^8 \sin 4f_1 - \kappa^8 \sin 4f_2 + 4\pi^6 \kappa^2 \sin 2g_1 - 4\pi^2 \kappa^6 \sin 2g_2 - 6\pi^4 \kappa^4 \sin 4h\right]$$

(III-1)

$$\frac{dj}{dt} = -\frac{\tau^2}{g} C \frac{1}{L_M} \left[\frac{1}{2} \pi^7 \kappa \sin 4 f_1 + \pi^3 \kappa^3 \sin 4 f + \frac{1}{2} \pi \kappa^7 \sin 4 f_2 \right.$$

$$+ \frac{3}{2} \pi^3 \kappa^3 (\pi^2 - \kappa^2) \sin 4h - \frac{1}{2} \pi^5 \kappa [\pi^2 - 3\kappa^2] \sin 2 g_1$$

$$\left. + \frac{1}{2} \pi \kappa (\pi^2 - \kappa^2)^2 \sin 2g + \frac{1}{2} \pi \kappa^5 (3\pi^2 - \kappa^2) \sin 2 g_2 \right] \qquad \text{(III-2)}$$

$$\frac{dL_E}{dt} = -\frac{\tau^2}{g} C \left[\frac{1}{2} \pi^8 \sin 4 f_1 + 2\pi^4 \kappa^4 \sin 4 f + \frac{1}{2} \kappa^8 \sin 4 f_2 \right. \qquad \text{(III-3)}$$

$$\left. + \pi^6 \kappa^2 \sin 2 g_1 + \pi^2 \kappa^2 (\pi^2 - \kappa^2)^2 \sin 2g + \pi^2 \kappa^6 \sin 2 g_2 \right]$$

$$\frac{di}{dt} = \frac{\tau^2}{g} \frac{C}{L_E} \left[\frac{1}{2} \pi^7 \kappa \sin 4 f_1 - \pi^7 \kappa^3 (\pi^2 - \kappa^2) \sin 4 f - \frac{1}{2} \pi \kappa^7 \sin 4 f_2 \right.$$

$$+ \frac{1}{2} \pi^5 \kappa (\pi^2 + 3\kappa^2) \sin 2 g_1 - \frac{1}{2} \pi \kappa (\pi^2 - \kappa^2)^3 \sin 2g$$

$$\left. - \frac{1}{2} \pi \kappa^5 (3\pi^2 + \kappa^2) \sin 2 g_2 - \frac{3}{2} \pi^3 \kappa^3 \sin 4h \right] \qquad \text{(III-4)}$$

Let ψ be the inclination of the lunar orbit to the earth's equatorial plane; then

$$\psi = i + j$$

$$\pi = \cos\left[\frac{1}{2}(i + j)\right] = \cos\frac{\psi}{2}$$

$$\kappa = \sin\left[\frac{1}{2}(i + j)\right] = \sin\frac{\psi}{2}$$

For small ψ (on the order of a few degrees or less) $\pi \cong 1$ and $\kappa \ll 1$. Assuming small ψ and keeping terms through κ^2, Equations (III-1) - (III-4) become

$$\frac{dL_M}{dt} \cong \frac{1}{2}\frac{\tau_0^2\,C}{g}\,\frac{1}{\xi^{12}}\left[\pi^8 \sin 4f_1 + 4\pi^6\kappa^2 \sin 2g_1\right] \qquad \text{(III-5)}$$

$$\frac{dj}{dt} \cong \frac{\tau_0^2}{g}\,C\,\frac{1}{L_M}\,\frac{1}{\xi^{12}}\left[\frac{1}{2}\pi^7\kappa \sin 2g_1 - \frac{1}{2}\pi^7\kappa \sin 4f_1 - \frac{1}{2}\pi^5\kappa \sin 2g\right] \qquad \text{(III-6)}$$

$$\frac{dL_E}{dt} \cong -\frac{\tau_0^2}{g}\,C\,\frac{1}{\xi^{12}}\left[\frac{1}{2}\pi^8 \sin 4f_1 + \pi^6\kappa^2 \sin 2g_1 + \pi^6\kappa^2 \sin 2g\right] \qquad \text{(III-7)}$$

$$\frac{di}{dt} \cong \frac{\tau_0^2\,C}{g}\,\frac{1}{L_E}\,\frac{1}{\xi^{12}}\left[\frac{1}{2}\pi^7\kappa \sin 2g_1 + \frac{1}{2}\pi^7\kappa \sin 4f_1 - \frac{1}{2}\pi^7\kappa \sin 2g\right] \qquad \text{(III-8)}$$

where we have explicitly written τ_0/ξ^6 for τ.

Using the approximations

$$\kappa = \sin\frac{\psi}{2} \approx \frac{\psi}{2}$$

$$\pi = \cos\frac{\psi}{2} \approx 1 - \frac{\psi^2}{8}$$

We write Equations (III-5) - (III-8) as

$$\frac{dL_M}{dt} \cong \frac{1}{2}\frac{\tau_0^2\,C}{g}\,\frac{1}{\xi^{12}}\left[(1-\psi^2)\sin 4f_1 + \psi^2 \sin 2g_1\right] \qquad \text{(III-9)}$$

$$\frac{dj}{dt} \cong \frac{1}{4}\frac{\tau_0^2}{g}\,C\,\frac{1}{L_M}\,\frac{1}{\xi^{12}}\left[\sin 2g_1 - \sin 4f_1 - \sin 2g\right]\psi \qquad \text{(III-10)}$$

$$\frac{dL_E}{dt} \cong \frac{-\tau_0^2}{g} C \frac{1}{\xi^{12}} \left[\frac{1}{2}(1-\psi^2)\sin 4f_1 + \frac{\psi^2}{4}\sin 2g_1 + \frac{\psi^2}{4}\sin 2g\right] \quad \text{(III-11)}$$

$$\frac{di}{dt} \cong \frac{1}{4}\frac{\tau_0^2}{g} C \frac{1}{L_E}\frac{1}{\xi^{12}}[\sin 2g_1 + \sin 4f_1 - \sin 2g]\,\psi \quad \text{(III-12)}$$

neglecting powers of ψ higher than 2.

If Equations (III-10) and (III-12) are added together we obtain the rate of change of ψ in time:

$$\frac{d\psi}{dt} \cong \frac{1}{4}\frac{\tau_0^2}{g}\frac{C}{\xi^{12}}\left[\left(\frac{1}{L_M}+\frac{1}{L_E}\right)\sin 2g_1 - \left(\frac{1}{L_M}-\frac{1}{L_E}\right)\sin 4f_1 - \left(\frac{1}{L_M}+\frac{1}{L_E}\right)\sin 2g\right]\cdot\psi \quad \text{(III-13)}$$

Note that

$$\frac{d\psi}{dt} \propto \psi$$

so that for $\psi = 0$, $\frac{d\psi}{dt} = 0$.

If the moon orbits the earth exactly in the equatorial plane, then the inclination will remain zero.

If the moon is slightly perturbed out of the equatorial plane so that $\psi > 0$, then the moon will move toward or away from the equatorial plane depending upon whether

$$\left(\frac{1}{L_M}+\frac{1}{L_E}\right)\sin 2g_1 \gtrless \left(\frac{1}{L_M}-\frac{1}{L_E}\right)\sin 4f_1 - \left(\frac{1}{L_M}+\frac{1}{L_E}\right)\sin 2g \quad \text{(III-14)}$$

is a negative or positive quantity, i.e.

(a) $\frac{d\psi}{dt} < 0$ if (III-14) is negative

(b) $\frac{d\psi}{dt} > 0$ if (III-14) is positive

For case (a) an equatorial orbit would be stable since small perturbations in ψ would drive the moon back toward the equatorial plane. In case (b) an equatorial orbit would be unstable, because small perturbations in ψ would cause the inclination to grow at a rate proportional to ψ. It is this second case in which we are mainly interested; we therefore want to examine (III-14) in detail to learn whether the tides can drive the moon away from the equatorial plane.

We start with the coefficients of the sines of the lag angles.

$\left(\frac{1}{L_M} + \frac{1}{L_E}\right)$ is always positive

$\left(\frac{1}{L_M} - \frac{1}{L_E}\right)$ is positive for $c < 21$ earth radii

Thus both these terms are positive in the region of interest. We next turn our attention to the lag angle terms.

Equations (III-9) - (III-13) indicate that the tides which govern the evolution of the earth-moon system for small inclinations are the tides with speeds $n - 2\Omega$, $2(n - \Omega)$, and n, with the lag angles being g_1, $2f$, and g, respectively. These tides are called O, M_2, and K_1 in Darwin (1883).

To learn something of the nature of these tides we refer to Figures 6 and 7.

Figure 6 shows Ω and n as a function of earth-moon distance for $\psi = 0$. Here use is made of the equations

$$\Omega = \frac{\sqrt{G(M + m)}}{c^{3/2}}$$

which is Kepler's third law, and

$$L_T^2 = L_E^2 + L_M^2 + 2 L_E L_M \cos \psi$$

which is derived from the conservation of angular momentum. From this latter equation we obtain (remembering $\psi = 0$)

$$n = \frac{L_E}{C} = \frac{L_T - L_M}{C} = \frac{L_T - \sqrt{\frac{G}{M + m}} M m c^{1/2}}{C}$$

Figure 7 shows the angular speeds of the tides as a function of distance. The region to the left of the dashed line is inside the Roche limit (2.89 earth radii) where the moon would be torn apart by the tidal stress if the moon lacked cohesiveness; thus distances greatly inside the Roche limit are not physically realistic. Note that both n and $2(n - \Omega)$ are positive for distances greater than the Roche limit, while $n - 2\Omega$ changes sign at 3.83 earth radii (dotted line).

The distance where $n - 2\Omega = 0$ makes a convenient reference distance (at this distance the earth's rotation period is about 5.25 hours and the moon's orbital period about 10.5 hours). We henceforward take c_0 as the earth-moon distance where $n = 2\Omega$:

$$c_0 = 3.83 \cdot a$$

where a is the present radius of the earth (6.37×10^8 cm). All quantities with zeros as subscripts refer to their values at this distance.

We now examine the sines of the lag angles. Using the identity

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

and the assumption of viscosity we have

$$\sin 4 f_1 = \frac{4 (n - \Omega) \zeta}{1 + 4 (n - \Omega)^2 \zeta^2} \tag{III-15}$$

$$\sin 2 g = \frac{2 n \zeta}{1 + n^2 \zeta^2} \tag{III-16}$$

$$\sin 2 g_1 = \frac{2 (n - 2\Omega) \zeta}{1 + (n - 2\Omega)^2 \zeta^2} \tag{III-17}$$

where

$$\zeta = \frac{19 \nu}{2 g a \rho}$$

and use has been made of the tangent formula for the lag angles (see page 20).

The signs of (III-15) - (III-17) are of the same signs as the respective speeds; thus $\sin 4 f_1$ and $\sin 2 g$ are positive while $\sin 2 g_1$ is negative for $n < 2\Omega$ and positive for $n > 2\Omega$.

From the above considerations we can assert that

$$\frac{d\psi}{dt} \leq 0 \quad \text{for} \quad c < c_0$$

for all values of viscosity since each of the three terms in (III-14) is negative. Hence an equatorial orbit is stable at least up to 3.83 earth radii distance.

The sign of (III-14) for $c > c_0$ depends upon the viscosity of the earth. To prove this, we examine this expression in the limit of low viscosity and then in the limit of high viscosity.

In the limit of low viscosity

$$(n - \Omega)\,\zeta \ll 1$$

where from Figure 7

$$(n - \Omega) \approx 10^{-4} \text{ sec.}$$

This implies $\nu \ll 10^{15}$ poises.

In this case Equations (III-15) - (III-17) can be written as

$$\sin 4 f_1 = \frac{4(n - \Omega)\,\zeta}{1 + 4(n - \Omega)^2 \zeta^2} \approx 4(n - \Omega)\,\zeta$$

$$\sin 2g = \frac{2n\,\zeta}{1 + n^2 \zeta^2} \approx 2n\,\zeta$$

$$\sin 2g_1 = \frac{2(n - 2\Omega)\,\zeta}{1 + (n - 2\Omega)^2 \zeta^2} \approx 2(n - 2\Omega)\,\zeta$$

It is convenient at this point to introduce Darwin's notation (Darwin 1880)

$$\lambda = \frac{\Omega}{n}.$$

λ decreases monotonically as the earth-moon distance increases (still remembering $\psi \cong 0$) and obviously has value 0.5 at c_0; see Figure 8.

Using this notation we have

$$\sin 4 f_1 \approx n \zeta \, 4 (1 - \lambda)$$

$$\sin 2 g \approx n \zeta \cdot 2$$

$$\sin 2 g_1 \quad n \zeta \, 2 (1 - 2\lambda)$$

and (III-14) becomes

$$n \zeta \left[2 \left(\frac{1}{L_M} + \frac{1}{L_E} \right) (1 - 2\lambda) - 4 \left(\frac{1}{L_M} - \frac{1}{L_E} \right) (1 - \lambda) - 2 \left(\frac{1}{L_M} + \frac{1}{L_E} \right) \right]$$

This expression (see Figure 9(a)) was found to be negative by numerically computing the expression in square brackets for various distances. Thus, for small viscosities $\frac{d\psi}{dt} < 0$ everywhere outside the Roche limit and an equatorial orbit is stable.

In the limit of large viscosity for which

$$(n - \Omega) \, \zeta \gg 1$$

which implies

$$\nu \gg 10^{15} \text{ poises}$$

Equations (III-15) - (III-17) become

$$\sin 4 f_1 = \frac{4 (n - \Omega) \zeta}{1 + 4 (n - \Omega)^2 \zeta^2} \approx \frac{1}{(n - \Omega) \zeta} = \frac{1}{\zeta n} \left(\frac{1}{1 - \lambda} \right)$$

$$\sin 2g = \frac{2n\zeta}{1 + n^2\zeta^2} \approx \frac{2}{n\zeta} = \frac{1}{n\zeta}\cdot 2$$

$$\sin 2g_1 = \frac{2(n - 2\Omega)\zeta}{1 + (n - 2\Omega)^2\zeta^2} \approx \frac{2}{(n - 2\Omega)\zeta} = \frac{1}{n\zeta}\,\frac{2}{1 - 2\lambda}$$

This last expression holds only in the regions away from c_0; see Figure 9(b). earlier $\sin 2g_1 = 0$ at $c_0 = c$ while the expression above approaches infinity as c approaches c_0. The behavior of $\sin 2g_1$ near c_0 will be examined later.

For large viscosity (III-14) becomes

$$\frac{1}{\zeta n}\left[\left(\frac{1}{L_M} + \frac{1}{L_E}\right)\frac{2}{1 - 2\lambda} - \left(\frac{1}{L_M} - \frac{1}{L_E}\right)\frac{1}{1 - \lambda} - \left(\frac{1}{L_M} + \frac{1}{L_E}\right)2\right]$$

The expression in square brackets is positive for $c > c_0$; see Figure 9(b).

We next inquire about the behavior of $\sin 2g_1$ near c_0 where $n \cong 2\Omega$. We make use of

$$\Omega = \frac{\sqrt{G(M + m)}}{c^{3/2}}\,; \qquad n \cong \frac{L_T - \dfrac{b}{c_0^{1/2}}\,c^{1/2}}{C}\,.$$

Let

$$c = c - c_0 + c_0 = c_0 + x$$

where

$$x = c - c_0\,.$$

x measures deviations in distance from c_0. The expressions for Ω and n become

$$\Omega = \frac{\sqrt{G(M+m)}}{c_0^{3/2}\left(1+\frac{x}{c_0}\right)^{3/2}} = \frac{\Omega_0}{\left(1+\frac{x}{c_0}\right)^{3/2}}$$

$$n = \frac{L_T - \frac{b}{c_0^{1/2}}\, c_0^{1/2}\left(1+\frac{x}{c_0}\right)^{1/2}}{C} = \frac{L_T - b\left(1+\frac{x}{c_0}\right)^{1/2}}{C}$$

If $\frac{x}{c_0} << 1$, then by Taylor series

$$\Omega \cong \Omega_0 - \frac{3}{2}\,\frac{\Omega_0}{c_0}\,x$$

$$n \cong \frac{L_T - b - \frac{b}{2\,c_0}\,x}{C} = n_0 - \frac{b}{2\,c_0\,C}\,x$$

keeping only first order terms in x.

So we have

$$n - 2\Omega \cong n_0 - \frac{b}{2\,c_0\,C}\,x - 2\Omega_0 + \frac{3\,\Omega_0}{c_0}\,x$$

$$= n_0 - 2\,\Omega_0 + \left[\frac{3\,\Omega_0}{c_0} - \frac{b}{2\,c_0\,C}\right]x$$

$$= 0 + \frac{x}{\zeta\,\epsilon}$$

where

$$\epsilon = \left[\zeta\left(3\,\frac{\Omega_0}{c_0} - \frac{b}{2\,c_0\,C}\right)\right]^{-1}$$

Equation (III-17) becomes

$$\sin 2g_1 \cong \frac{2\left(\frac{x}{\epsilon}\right)}{1+\left(\frac{x}{\epsilon}\right)^2}$$

Note that this expression is antisymmetric in x.

If $\epsilon << c_0$ ($\nu >> 10^{15}$ poises), then $\sin 2g_1$ has the features shown in Figure 10. $\text{Sin } 2g_1$ ranges from -1 at $x = -\epsilon$ to 0 at $x = 0$ to +1 at $x = +\epsilon$. The peaks become sharper as the viscosity increases (and ϵ decreases).

Both $\sin 4f_1$ and $\sin 2g$ are only slowly varying for $\nu >> 10^{15}$ poises and are virtually constant between $x = -\epsilon$ and $x = +\epsilon$. Further, both $\sin 4f_1 << 1$ and $\sin 2g << 1$ for $\nu >> 10^{15}$ poises. Expression (III-14) is then seen to be zero for c slightly greater than c_0 and the zero approaches c_0 as the viscosity increases; we may then speak of (III-14) as being zero for $c = c_0$ and positive for $c > c_0$ with negligible error for large viscosities ($>>10^{15}$ poises).

We conclude that $\frac{d\psi}{dt}$ becomes strongly negative for $c < c_0$ and strongly positive for $c > c_0$. Thus for viscosities $>> 10^{15}$ poises an equatorial orbit is unstable for $c > c_0$ and the moon will be driven away from the equatorial plane if perturbed.

The transition of (III-14) from negative to positive for $c > c_0$ was found to occur at a viscosity between 10^{15} and 10^{16} poises by numerical computation.

We summarize the major results of this section.

(i) $\frac{d\psi}{dt} \leq 0$ for $c < c_0$ for all viscosities and an equatorial orbit is stable.

(ii) $\frac{d\psi}{dt} \leq 0$ for $c > c_0$ for viscosities less than about 10^{16} poises and an equatorial orbit is stable.

(iii) $\frac{d\psi}{dt} \geq 0$ for $c > c_0$ for viscosities greater than 10^{16} poises and an equatorial orbit is unstable.

These results hold for small values of ψ (the inclination of the lunar orbit to the earth's equator).

B. Variation in the Earth-Moon Distance

Let us turn our attention to Equation (III-9) and write it as

$$\frac{d\xi}{dt} \cong \frac{1}{2}\,\frac{\tau_0^2 C}{g\,b}\,\frac{1}{\xi^{12}}\left[(1-\psi^2)\sin 4f_1 + \psi^2 \sin 2g_1\right]$$

$$\xi = \left(\frac{c}{c_0}\right)^{1/2} \tag{III-19}$$

to show explicitly that we are discussing essentially the variation of the earth-moon distance in time and not the orbital angular momentum.

In the limit of extremely small angles the ψ^2 terms can be neglected and (III-19) becomes

$$\frac{d\xi}{dt} = \frac{1}{2}\,\frac{\tau_0^2 C}{g\,b}\,\frac{1}{\xi^{12}} \sin 4f_1 \tag{III-20}$$

so that only one tide governs the variation in distance. If $\nu << 10^{15}$ poises, then by the approximations of the previous section $\frac{d\xi}{dt} \propto \nu$; and if $\nu >> 10^{15}$ poises, then $\frac{d\xi}{dt} \propto \nu^{-1}$.

The right side of Equation (III-20) is positive because $\sin 4f_1$ is positive; therefore the moon is driven away from the earth. Also, $\frac{d\xi}{dt}$ is greatest when $\sin 4f_1 = 1$, which occurs at a viscosity of about 10^{15} poises.

Equation (III-20) can be integrated to give

$$\frac{1}{13}\left(\xi_2^{13} - \xi_1^{13}\right) = \frac{1}{2}\frac{\tau_0^2 C}{g\,b}\int_{t_1}^{t_2} \sin 4 f_1\, dt$$

If $\sin 4f_1$ is only slowly varying, we have approximately

$$\frac{1}{13}\left(\xi_2^{13} - \xi_1^{13}\right) \cong \frac{1}{2}\frac{\tau_0^2 C}{g\,b} \sin 4 f_1 (t_2 - t_1)$$

for the dependence of distance on time.

Returning to Equation (III-20), if the equation is divided into Equation (III-10) and into (III-12) we obtain equations which eliminate the time:

$$\frac{dj}{d\xi} = \frac{1}{2}\frac{b}{L_M}\left[\frac{\sin 2 g_1}{\sin 4 f_1} - 1 - \frac{\sin 2 g}{\sin 4 f_1}\right]\psi$$

$$\frac{di}{d\xi} = \frac{1}{2}\frac{b}{L_E}\left[\frac{\sin 2 g_1}{\sin 4 f_1} + 1 - \frac{\sin 2 g}{\sin 4 f_1}\right]\psi$$

If each side of these two equations is multiplied by kn, then

$$kn\,\frac{dj}{d\xi} = \frac{1}{2}\frac{L_E}{L_M}\left[\frac{\sin 2 g_1}{\sin 4 f_1} - 1 - \frac{\sin 2 g}{\sin 4 f_1}\right]\psi$$

$$kn\,\frac{di}{d\xi} = \frac{1}{2}\left[\frac{\sin 2 g_1}{\sin 4 f_1} + 1 - \frac{\sin 2 g}{\sin 4 f_1}\right]\psi$$

where we have used $kb = C$ and $L_E = Cn$.

Now because $\psi = i + j$ is very small, examination of Figure 5 shows that

$$L_E\, i \approx L_M\, j$$

giving

$$\psi = i + j \cong \left(1 + \frac{L_M}{L_E}\right) j \cong \left(1 + \frac{L_E}{L_M}\right) i$$

Substituting, we have

$$kn \frac{dj}{d\xi} = \frac{1}{2}\left(1 + \frac{L_E}{L_M}\right)\left[\frac{\sin 2 g_1}{\sin 4 f_1} - 1 - \frac{\sin 2 g}{\sin 4 f_1}\right] j$$

$$kn \frac{di}{d\xi} = \frac{1}{2}\left(1 + \frac{L_E}{L_M}\right)\left[\frac{\sin 2 g_1}{\sin 4 f_1} + 1 - \frac{\sin 2 g}{\sin 4 f_1}\right] i$$

or finally

$$kn \frac{d \log j}{d\xi} = -\frac{1}{2}\left(1 + \frac{L_E}{L_M}\right)\left[1 - \frac{\sin 2 g_1}{\sin 4 f_1} + \frac{\sin 2 g}{\sin 4 f_1}\right]$$

$$kn \frac{d \log i}{d\xi} = \frac{1}{2}\left(1 + \frac{L_E}{L_M}\right)\left[1 + \frac{\sin 2 g_1}{\sin 4 f_1} - \frac{\sin 2 g}{\sin 4 f_1}\right]$$

Using our previous approximations for $\sin 2g_1$, $\sin 4f_1$, and $\sin 2g$ in the limit of low viscosity we obtain

$$kn \frac{d \log j}{d\xi} = -\frac{1}{2}\left(1 + \frac{L_E}{L_M}\right)\left[\frac{1}{1 - \lambda}\right]$$

$$kn \frac{d \log i}{d\xi} = \frac{1}{2}\left(1 + \frac{L_E}{L_M}\right)\left[\frac{1 - 2\lambda}{1 - \lambda}\right]$$

These equations are found in §19 of Darwin's 1880 paper (Darwin 1880 pg. 312).

In the limit of large viscosity

$$kn \frac{d \log j}{d\xi} = -\frac{1}{2}\left(1 + \frac{L_E}{L_M}\right)\left[1 - \frac{4\lambda(1-\lambda)}{1-2\lambda}\right] \qquad \text{(III-21)}$$

$$kn \frac{d \log i}{d\xi} = \frac{1}{2}\left(1 + \frac{L_E}{L_M}\right)\left[1 + \frac{4\lambda(1-\lambda)}{1-2\lambda}\right] \qquad \text{(III-22)}$$

These last equations are found in § 20 of the 1880 paper (Darwin 1880, pg. 317).

Equations (III-21) and (III-22) will be discussed when analyzing Darwin's theory of the moon's origin.

We now return to Equation (III-19) and write it as

$$\frac{d\xi}{dt} = \frac{1}{2}\,\frac{\tau_0^2 C}{g\, b\, \xi^{12}}\left[\sin 4f_1 - \psi^2 \sin 4f_1 + \psi^2 \sin 2g_1\right].$$

It is not generally true that

$$|\psi^2 \sin 2g_1| \ll |\sin 4f_1|$$

for small ψ for viscosities greater than 10^{16} poises because $|\sin 2g_1|$ may be on the order of 1.

An example will illustrate this. Take

$$\nu = 10^{18} \text{ poises}$$

$$\sin 2g_1 = 1$$

$$\psi = 3^\circ = 0.0052 \text{ radians}$$

$$n - \Omega = 1.66 \times 10^{-4} \text{ sec}^{-1}$$

Then

$$\sin 4f_1 \approx \frac{1}{(n-\Omega)\,\zeta} = 0.0022$$

$$\psi^2 \sin 2\,g_1 \;=\; 0.0027$$

$$\psi^2 \sin 4\,f_1 \;=\; 0.000006$$

Obviously in this case

$$\psi^2 \sin 2\,g_1 \;>\; \sin 4\,f_1$$

The $\psi^2 \sin 4f_1$ term is generally quite small and may be neglected giving

$$\frac{d\xi}{dt} \cong \frac{1}{2}\,\frac{\tau_0^2\,C}{g\,b\,\xi^{12}}\left[\sin 4\,f_1 + \psi^2 \sin 2\,g_1\right]$$

The point of this example is that even for small ψ (on the order of a degree) neglect of the $\psi^2 \sin 2g_1$ term may lead to serious error. In fact, this term may have profound effects on the lunar orbit. We demonstrate this by examining some possible histories of the lunar orbit.

Figures 11, 12, and 13, show ψ; $\sin 4f_1$, $\psi^2 \sin 2g$, and $\sin 2g_1$; and $\frac{d\xi}{dt}$ respectively as functions of x for a viscosity of 10^{20} poises. (All computations for Figures 11-14 were carried out with the computer program described in the next section.) The initial conditions are chosen to be $\psi \approx 0.4°$ at $x = -0.4 \times 10^{-3}$ earth radii; it is labelled A in Figure 13.

Since

$$\sin 4\,f_1 + \psi^2 \sin 2\,g_1 \;>\; 0$$

for the chosen starting condition, $\frac{d\xi}{dt} > 0$ at A and the radius of the moon's orbit increases, so that the moon moves away from the earth (to the right in the figures). As the moon moves toward point B, its outward rate of motion becomes slower and slower as $\psi^2 \sin 2g_1$ becomes more and more negative. Past point B

$\frac{d\xi}{dt}$ increases and reaches a maximum at c. From c onward the rate of motion decreases. Note that ψ decreases for $x < 0$ and increases for $x > 0$.

Now if the radius of the moon's orbit is initially less than $c_0 - \epsilon$ and the radius of the orbit is expanding, it must be that

$$\frac{d\xi}{dt} \geq 0$$

at all points along its outward journey if the moon is to reach the outer regions past c_0. In other words,

$$\sin 4 f_1 + \psi^2 \sin 2 g_1 \geq 0$$

Now at $x = -\epsilon$, $\sin 2 g_1 = -1$ and the above condition becomes

$$\sin 4 f_1 - \psi^2 \geq 0$$

or

$$\psi \leq \sqrt{\sin 4 f_1}$$

at $x = -\epsilon$.

Hence for initial conditions for which the radius of the lunar orbit is less than $c_0 - \epsilon$ and expanding, the above restriction must hold: the inclination ψ must decrease below a certain value at $x = -\epsilon$ for the moon to gain the outer regions. Thus, if such an orbit has a large inclination, $\frac{d\xi}{dt}$ becomes small as the moon approaches $x = -\epsilon$ and the moon "waits" near $x = -\epsilon$ until ψ has decreased enough to allow the moon past $x = -\epsilon$ ($\frac{d\psi}{dt}$ is negative for $x < 0$) and into the outer regions.

The net effect is that the distance $x = -\epsilon$ acts as a "barrier" and will not let the moon through until ψ has dropped below a critical angle, which we label ψ_c:

$$\psi_c = \sqrt{\sin 4 f_1} \approx \sqrt{\frac{1}{(n - \Omega)\,\zeta}}$$

Critical angles for various viscosities are given in Table 2.

Information regarding the history of the inclination for orbits which have $\psi > \psi_c$ for $x < -\epsilon$ is lost at $x = -\epsilon$, since ψ must be less than or equal to ψ_c in all cases to get past the barrier.

Figure 14 gives an example of ψ initially so large that

$$\sin 4 f_1 + \psi^2 \sin 2 g_1 < 0$$

so that now

$$\frac{d\xi}{dt} < 0$$

and the moon moves <u>toward</u> the earth. It continues to do so until

$$\sin 4 f_1 + \psi^2 \sin 2 g_1 = 0$$

at point D in the figure. Here $\frac{d\xi}{dt}$ changes sign and the moon moves away from the earth. Thereafter the moon's possible motion is as described before. In this particular case ψ is quite large as the moon approaches the barrier, so that the moon must wait until ψ drops below 2.7° (the critical angle for 10^{18} poises) before moving into the region past $x = 0.0$.

The effect of the barrier and the moon's orbit shrinking and then expanding can occur only if the radius of the lunar orbit is less than c_0. If the moon formed or somehow arrived at a distance greater than c_0, then there is no restriction on the inclination and the moon moves continually outwards, since $\sin 2g_1 > 0$ for $c > c_0$.

We summarize the major results of this section.

(i) $$\frac{d\xi}{dt} \cong \frac{1}{2} \frac{\tau_0^2 C}{g\, b\, \xi^{12}} [\sin 4 f_1]$$

for equatorial orbits

(ii) $$\frac{d\xi}{dt} \cong \frac{1}{2} \frac{\tau_0^2 C}{g\, b\, \xi^{12}} [\sin 4 f_1 + \psi^2 \sin 2 g_1]$$

for viscosities greater than 10^{16} poises and ψ on the order of a degree

(iii) $$\psi \leq \sqrt{\sin 4 f_1} \quad \text{at} \quad c = c_0 - \epsilon$$

for an expanding orbit

(iv) if $\sin 4f_1 + \psi^2 \sin 2g_1 \leq 0$ (for $c < c_0$) an orbit will contract, and then expand.

C. Computational Results

The integration of Equations (III-5) - (III-8) was carried out numerically with a FORTRAN computer program using double precision variables.* The program is given in Appendix C. It was run on an IBM 360/91 computer at the Goddard Space Flight Center, as well as on a Univac 1108 at the University of Maryland.

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*This program was also used to obtain results similar to those of Gerstenkorn (Alfven 1963).

The program has the capability of integrating the full Equations (III-1) - (III-4), but the contribution of the neglected terms was found to be insignificant in the computations discussed below.

The viscosity ν, time interval Δt, and the initial values of t, ψ, and ξ are read into the program. From the initial data i, j, n, and Ω are computed, as well as L_M and L_E.

The program then iterates equations (III-5) - (III-8) by computing the changes in i, j, L_E, and L_M according to the simple formula

$$\Delta X = \frac{dX}{dt} \Delta t$$

where X is i, j, L_E, or L_M. The new values become

$$X_{New} = X_{Old} + \Delta X$$

$\frac{dX}{dt}$ is then recomputed from the new values and the process is repeated. At each step the new values of ξ, n, ψ, Ω, i, and j are printed, as well as Δi, Δj, and $\Delta \psi$.

After a certain chosen number of steps NQ Δt is adjusted so that the step change in ξ $\Delta \xi$ is constant for the remainder of the run. The reason for switching from constant Δt to constant $\Delta \xi$ is to insure that the time intervals at the beginning of the run are small enough so that the peaks in $\frac{d\psi}{dt}$ near $\xi = 1.0$ are not missed (most of the runs start near $\xi = 1.0$). Later, as ξ increases and the change in ξ and the change in angles i, j, and ψ in time become small, constant $\Delta \xi$ is used to keep the run from becoming extremely long.

If at any step $|\Delta i/i|$ or $|\Delta j/j|$ exceeds some chosen fraction called CRIT in the program the time interval for that step is halved and the step is repeated until both $|\Delta i/i|$ and $|\Delta j/j|$ are less than CRIT. The purpose of introducing CRIT is to avoid large changes in angle at any one step which would lead to cumulative errors after many steps.

When ξ exceeds some chosen value XIMAX, or the total number of steps exceeds NLAST, the run is terminated. At the end of each run the total angular momentum is computed from the values of the last step of the run and is compared to the initial angular momentum. This serves as a check on how well the approximations used in writing (III-5) - (III-8) conserve angular momentum.

After a run is completed its accuracy can be checked by halving the time interval of each step, doubling the number of steps, and repeating the run.

The program also has the capability of integrating backward in time as well as forward.

The program was run for various viscosities for which the moon is perturbed from an equatorial orbit near $c = c_0$. The relevant data for these runs is summarized in Table 3. All runs stopped when the moon reached 10 earth radii distance from the earth; beyond 10 earth radii solar influence must be taken into account. No viscosities above 10^{21} poises were considered because of the unrealistically long time scales involved.

Figure 15 shows ψ as a function of earth-moon distance for an initial perturbation of 3° for viscosities of 10^{15}, 10^{16}, and 10^{17} poises at $c = c_0$. Note that ψ decreases as a function of distance for 10^{15} poises, but increases for 10^{16} and 10^{17} poises. This behavior is to be expected from the discussion given in Section A of this chapter.

The program was run next for $\psi = \psi_c$ at $c = c_0 - \epsilon$ for 10^{18}, 10^{19}, 10^{20} and 10^{21} poises (Figure 16). This is the largest possible value ψ can have near

$c = c_0$ without suffering some further perturbation from an equatorial orbit. Smaller initial angles at $c = c_0 - \epsilon$ invariably gave smaller final angles at 10 earth radii.

Finally, Figure 17 shows ψ as a function of distance for viscosities of 10^{18} and 10^{21} poises for initial perturbations of 1°, 2°, and 3°. Figures 18 and 19 give i and j respectively for the given initial perturbations. Curves for viscosities between 10^{18} and 10^{21} poises fall between the respective curves given in the figures.

Note that only when the initial perturbation is about 3° does ψ reach near 10° at 10 earth radii as required in Goldreich's model; or equivalently, does j reach 6° at 10 earth radii.

D. Variable Viscosity

It was next assumed that the viscosity was not constant, but that the viscosity ν was a function of absolute temperature T and that the earth was cooling down from an initially molten state. The purpose in doing this was to see whether the earth could cool off enough to be solid and have a high viscosity by the time the moon moved from the Roche limit (2.89 earth radii) to c_0 (3.83 earth radii). If so, the mechanism for driving the moon out of the earth's equatorial plane may have been operative.

The dependence of ν on T was assumed to have the form

$$\nu = \nu_0 \, e^{E^*/kT} \qquad \text{(III-23)}$$

where

ν_0 = a constant

E^* = activation energy per atom

k = the Boltzmann constant.

A theoretical derivation of this equation is given by Glasstone, Laidler, and Eyring (1941). We have ignored the dependence of ν_0 on T and have assumed it to be a constant. Experimental data shows that this equation holds fairly well for silicate melts (Clark 1966), with

$$\nu_0 \approx 10^{-4} \text{ poises}$$

$$E^* \approx 2\text{-}5 \text{ eV/atom}$$

Data on molten rocks are uncertain; the activation energy E* has approximately the range given above, but ν_0 may vary by orders of magnitude.

A cooling law for the earth was required to give the temperature T as a function of time t. The law adopted here is derived in Appendix B. From Equation (B-5) of Appendix B we take the form of the cooling law as

$$T(t) = \frac{T_0}{\left[1 + ZST_0^3(t - t_0)\right]^{1/3}}$$

where

T_0 = temperature at time t_0

$$Z = \frac{\text{surface temperature}}{\text{average temperature of the earth}}$$

$$S = \frac{12\pi a^2 \sigma}{M C_p}$$

where

a = radius of the earth = 6.37×10^8 cm

σ = Stefan-Boltzmann constant = 5.72×10^{-5} erg cm^{-2} sec^{-1} deg^{-4}

M = mass of the earth = 5.98×10^{27} g

C_p was taken to be 1.0×10^7 erg/g-deg, giving

$$S = 1.46 \times 10^{-20} \frac{\mathrm{deg}^{-3}}{\mathrm{sec}}$$

The computer program given in Appendix D was used to give the moon's distance, and the earth's temperature and viscosity, as a function of time. The program differs from the program of Appendix C only in that the viscosity is allowed to vary in time rather than remain constant. The moon was initially at the Roche limit and in the equatorial plane of the earth, with the earth at a temperature T_0. Various values of ν_0, E^*, Z, and T_0 were used to see if the earth could cool down near the melting point of rocks (about 1500°K) by the time the moon reached c_0.

The results may be briefly summarized. For $E^* \gtrsim 5$ eV and $\nu_0 > 10^{-4}$ poises the temperature of the earth at c_0 did not drop below about 1500° for initial temperatures between 2000°K and 3000°K with $Z \approx \frac{1}{3}$ to $\frac{1}{2}$. For $E^* \approx$ 4.3 eV and ν_0 in the neighborhood of 10^{-4} poises, the temperature of the earth could fall below 1250° for the same ranges of initial temperature and values of Z. Apparently large values of E^* and ν_0, which increase the viscosity at any given temperature, hasten the moon past c_0 before the temperature has a chance to fall very low.

Due to the wide variation in results and ignorance of the interval condition of the earth, it appears that we can make no definite statement as to whether the earth could cool down sufficiently from a molten state to have a large viscosity when the moon reaches c_0.

CHAPTER IV

SOLAR INFLUENCE

The history of the lunar orbit for distances greater than 10 earth radii, where solar influence must be considered, will now be investigated. Our discussion will be restricted to the behavior of J, the angle between the plane of the lunar orbit and its proper plane. The angle j, the angle between the plane of the lunar orbit and the invariable plane, is essentially J for distances less than 10 earth radii (see Chapter I, Section C). Hence at 10 earth radii we will join our previous solutions for j as a function of distance to those we obtain for J as a function of distance.

Darwin obtains the rate of change of J with respect to ξ, with solar tides included, in Section III of his 1880 paper. It was found by assuming the inclinations of the earth's equator and the lunar orbit to the ecliptic are small and applying the variations of parameters technique in solving differential equations. After a quite lengthy analysis he obtains (Darwin 1880, eq. 250, pg. 297):

$$\frac{d \log J}{d\xi} = \frac{1}{kn}\left[\frac{1}{(\kappa_1 - \kappa_2)^2}\left\{-(\kappa_1 + \alpha)(a' - \beta') - a'b\,\frac{\kappa_1 + \alpha}{\kappa_2 + \alpha} - b'a\right\}\right. \tag{IV-1}$$

$$\left. + \frac{1}{(\kappa_1 - \kappa_2)}\left\{\Gamma(\kappa_2 + \alpha) + \Delta(\kappa_1 + \alpha) + bG - aD\right\}\right]$$

where

$$\alpha = m + \frac{\tau'}{\tau}\,\frac{1}{2\lambda e} \qquad a = m \qquad \beta = 1 + \frac{\tau'}{\tau} \qquad b = 1$$

$$\alpha' = m\left\{\frac{\tau'}{\tau}\,\frac{3}{2\lambda e} - \left[2\left\{1+\left(\frac{\tau'}{\tau}\right)^2\right\} + 7m\right]\right\}$$

$$a' = -m\left\{2\left[1+\left(\frac{\tau'}{\tau}\right)^2\right] + 7m\right\}$$

$$\beta' = -\left\{1 + \frac{\tau'}{\tau} + \left(\frac{\tau'}{\tau}\right)^2 + \left(\frac{\tau'}{\tau}\right)^3 + 6m\right\}$$

$$b' = -\left\{1 + \left(\frac{\tau'}{\tau}\right)^2 + 6m\right\}$$

$$\Gamma = \frac{1}{2}m\,\frac{\sin 4f_1 - \sin 2g_1 + \sin 2g}{\sin 4f_1}$$

$$\Delta = \frac{(\sin 4f_1 + \sin 2g_1 - \sin 2g) - 2\left(\frac{\tau'}{\tau}\right)\sin 2g + \left(\frac{\tau'}{\tau}\right)^2 \sin 4f}{2\sin 4f_1}$$

$$bG - aD = \frac{1}{2}m\,\frac{2\left(1+\frac{\tau'}{\tau}\right)\sin 2g - 2\sin 2g_1}{\sin 4f_1}$$

$$m = \frac{kn}{\xi} \qquad \lambda = \frac{\Omega}{n} \qquad e = \frac{1}{2}\,\frac{n^2}{g}$$

$$\kappa_1 + \kappa_2 = -\alpha - \beta \qquad \kappa_1 - \kappa_2 = -\sqrt{(\alpha-\beta)^2 + 4ab}$$

$$\tau = \frac{\tau_0}{\xi^6} \qquad \tau' = \frac{3}{2}\,\frac{GM_\odot}{c_\odot^3}$$

$M_{\odot}$ = mass of the sun

$c_{\odot}$ = earth-sun distance

The change of ξ with time is given by (Darwin 1880, eq. 227, pg. 293):

$$\frac{1}{k}\,\frac{d\xi}{dt} = \frac{1}{2}\,\frac{\tau^2}{g}\,\sin 4\,f_1 \qquad \text{(IV-2)}$$

This is just the same equation as (III-20), where the only tide-raising body was the moon. The two are the same because solar tides and the direct gravitational force of the sun on the moon produce no secular change in the moon's distance.

The computer program of Appendix E integrates Equation (IV-1) from 10 to 60 earth radii for any desired viscosity. It assumes a constant step size in ξ.

The angular velocity of the earth n is computed by assuming the total angular momentum of the earth-moon system is conserved and that the moon remains in the equatorial plane of the earth. (The neglect of the frictional effects of solar tides and inclination leads to only small corrections in the final results.) These assumptions make the right side of the equation independent of angle, so that the solution of the equation has the form

$$J_2 = e^{\int_{\xi_1}^{\xi_2} F(\nu,\xi)\,d\xi} \cdot J_1$$

where $F(\nu, \xi)$ is the right side of the equation and J_2 is J at ξ_2, and J_1 is J at the initial distance ξ_1. A graph of J versus earth-moon distance c has the same shape for a given viscosity regardless of the initial value of J; larger or small initial values of J merely shift the curve up or down.

The program was first run for the present-day values J = 5°9' and ξ = 3.96 (60 earth radii) for a viscosity of 10^{12} poises to obtain the small viscosity limit.

The result is shown in Figure 20 (dotted line). Note the close agreement with Goldreich's curve (dashed line), where the lag angles are assumed to be equal. The program was run again for a viscosity of 10^{10} poises. It showed negligible difference in results from 10^{12} poises, so that the dashed line indeed represents the small viscosity limit.

The program was run next for a viscosity of 10^{18} poises to extend the curve for j shown in Figure 19 for an initial perturbation in ψ of 3°. The resulting curve is the upper solid line shown in Figure 20. The program was run again for a viscosity of 10^{21} poises; it produced little change in the shape of the curve from 10 to 60 earth radii; hence the curve shows the large viscosity limit in that region. Note that the character of the large viscosity curve is quite different from that of the small viscosity curve.

If the earth behaved as though it had a large viscosity from the time the moon was at 3.83 earth radii to the present time, then an initial perturbation in ψ of about 2.5° at 3.83 earth radii would be required to give the present value of J of 5°9'. This is shown as the lower solid curve in Figure 20. However, viscosities greater than about 10^{17} poises give time scales of the orbital evolution greater than the age of the solar system.

What is more likely is that the earth behaved like a liquid with high viscosity in its early history and then like an anelastic solid or liquid with low viscosity later on, which is what is observed today; so that the inclination J in Figure 20 started on the upper solid curve at 3.83 earth radii and switched over to the dotted line, possibly somewhere in the region where the two curves merge beyond 15 earth radii. Darwin (1880, § 32, pg. 363) discusses the possibility of this kind of behavior.

CHAPTER V

DISCUSSION

A. Critique of Assumptions Made

We shall now examine the important assumptions made in obtaining our results for strong tidal friction.

One important assumption we have made is that the orbit of the moon remains circular throughout its history, i.e. the eccentricity e of the lunar orbit is zero. The work of Darwin (1880, Section V), Singer (1968), and MacDonald (1964) shows that weak tidal friction decreases the eccentricity as we look back into the past until the moon reaches about 3 earth radii from the earth, where the eccentricity undergoes rapid changes. Since the present value of the eccentricity is 0.055, this would imply that neglect of the eccentricity when the moon was at the reference distance of 3.83 earth radii would lead to negligible error in considering weak tidal friction.

However, use of Darwin's treatment of the eccentricity for viscosities greater than 10^{17} poises indicates that e increases with time until the moon reaches about 16 earth radii distance from the earth; at larger distances the eccentricity rapidly decreases. This indicates that the eccentricity could have been large for earth-moon distances of less than 16 earth radii. However, a nearly circular orbit for the moon over its whole history is by no means excluded. The earth could have behaved as though it had a large viscosity when the moon was less than 16 earth radii from it; beyond 16 earth radii the earth could have behaved as though it had a small viscosity. If this were the case, then if the moon were in a nearly circular orbit at 3.83 earth radii, the eccentricity would slowly grow to its present value as the moon moved outward to its present distance.

Another assumption which we have made is that harmonics higher than the second degree may be neglected in the tidal disturbing function (Equation II-4). To show that this approximation is a good one, we note that the second degree harmonics in the disturbing function are multiplied by $\left(\frac{a}{r}\right)^6$, where a is the radius of the earth and r is the earth-moon distance; this may be seen from Equations (II-3) and (II-4). If third degree harmonics were included in the disturbing function, then they would be multiplied by $\left(\frac{a}{r}\right)^8$; likewise, fourth degree harmonics would be multiplied by $\left(\frac{a}{r}\right)^{10}$, etc. Hence the third degree terms are reduced by a factor of $\left(\frac{a}{r}\right)^2$ from the second degree terms. For $r = 3.83a$, $\left(\frac{a}{r}\right)^2 = 0.068$. Also, the contribution of the third degree tides to the rate of change in time of the inclination is small compared to that of the second degree tides (see the discussion in the last paragraph of this section). Thus the restriction to the second degree terms in the disturbing function leads to only a small error.

We have further assumed that the moment of inertia of the earth C had its present-day value of 8.11×10^{40} g-cm^2 = 0.33 Ma^2. This implies that the earth's core had already formed. Darwin assumed that the earth was homogeneous ($C = 0.4$ Ma^2), as well as incompressible, etc., for reasons of tractability in solving for the response of the earth to the tidal force. Changing the moment of inertia to its value for a homogeneous earth would lead to only slight corrections in our results.

Heating of the earth by the dissipation associated with the friction does not appear to be significant. The energy deposited in the earth as the moon moves from the Roche limit at 2.89 earth radii to 3.83 earth radii amounts to 1.43 ×

10^{36} ergs. Assuming a specific heat of 10^7 ergs/g-deg, the average change in temperature of each gram of matter in the earth is only 24°K.

Our most crucial assumption was that the earth behaved like a highly viscous liquid ($\nu >> 10^{15}$ poises). Whether the earth could have behaved in this manner when the moon was at 3.83 earth radii depended upon the rheological properties of the earth at that time; these properties are unknown.

O'Keefe (1972) points out that since the tidal potential varies like the inverse cube of distance (Equation A-6, Appendix A), the tidal forces acting on the earth were 4000 times greater when the moon was at 3.83 earth radii than they are today, so that the material in the earth may have been near the elastic limit. In such circumstances the earth may have behaved like a highly viscous liquid.

At the present time the mantle of the earth responds to the tidal forces like an anelastic solid, with the tidal lag angles being small (MacDonald 1964). However, the mantle responds to deformations of the earth's surface caused by ice loads as though it had a viscosity of about 10^{21} poises (Gutenberg 1959, Chapter 9), requiring thousands of years to rebound after the removal of the loads. (This may be explainable in terms of diffusion creep; see Kaula 1968, pgs. 101-104). Now the period of the O tide with speed $n - 2\Omega$ is given by $2\pi/(n - 2\Omega)$, so that the period ranges from infinity to about 5 hours as the moon moves through 3.83 earth radii distance. Hence if the earth has a characteristic response time between these two extremes, then it should be excited by the O tidal force as the moon passes through 3.83 earth radii. If the dissipation were great, then the lag angle of the tide would be large. Hence it is by no means clear that the earth would not respond as we have assumed, even with the present internal conditions in the earth, where the characteristic response time is thousands of years.

Our last important assumption was that the moon may have been perturbed out of an equatorial orbit by 2.5 to 3° at 3.83 earth radii distance from the earth, thus explaining the present inclination of the lunar orbit to the ecliptic. Whether the moon could suffer such a perturbation is not clear; conditions at that time may have been chaotic enough to produce it. However, several sources of the perturbation may be ruled out. The first obvious source of perturbation is a collision of a large meteoritic object with the moon. If such a collision occurred, then large amounts of meteoritic nickel might be expected to spatter over the moon's surface*. Large amounts of nickel are not observed in lunar samples. The third degree harmonic in the earth's figure will not give rise to long period perturbations in the inclination if the moon's orbit is circular. Further, it may be shown from Equation 38 of Kaula (1964) that the tides associated with the third degree harmonics in the tidal disturbing function give $\frac{d\psi}{dt} \propto \psi$, just as in the second degree harmonics, but that these terms are much less important than those discussed here. Also, the disturbance in the inclination caused by the precession of the earth's axis and the moon's orbit may be shown to be quite small ($\ll 1°$). The question of the source of the perturbation remains open.

B. Relation of the Results to Theories of the Moon's Origin

We will now examine how our results relate to the theories of the origin of the moon. The three principal theories, namely fission, accretion, and capture are reviewed by Kaula (1971).

Darwin (1880) proposed that a primitive body rotating with a period close to its natural oscillation period was disrupted into the earth and moon by resonance oscillations induced in it by the sun. (That this was not at all likely was shown

*I am indebted to Dr. O'Keefe on this point.

by Jeffreys 1930.) The moon would necessarily be thrown out in the equatorial plane of the earth. Darwin was forced to assume that the primitive earth had a very high viscosity to solve the inclination problem. He derived Equations (III-21) and (III-22) which give rate of change of inclination with distance in the limit of infinite viscosity. After commenting on the absurdity implied by the equations that the rate of change of angle was infinite when the earth rotated twice as fast as the moon revolved, he assumed that the viscosity merely had to be very large to increase the inclination of the moon's orbit to the equatorial plane from an infinitesimal disturbance to an appreciable angle. Darwin took this as the solution to the inclination problem and let the matter rest.

Our detailed analysis (Chapter III) shows that the initial perturbation in the inclination of the moon's orbit to the earth's equator must be about 2.5-3.0° to explain the present inclination, with the viscosity of the earth being greater than 10^{17} poises.

O'Keefe (1969) in his version of the fission theory suggested that the primitive body had greater mass and twice the angular momentum of the present earth-moon system. As the primitive body spun up, its figure progressed along the sequence of the well-known Jacobi ellipsoids and pear-shaped figures (Jeans 1961) until it fissioned into the earth and moon. The system then lost mass and angular momentum through intense heating. While taking over Darwin's results, O'Keefe further suggested that even if the earth were molten after the moon and earth separated, the moon's orbital evolution would not begin until the earth cooled off appreciably, so that the moon would not arrive at 3.83 earth radii until the earth's viscosity was quite high.

In Chapter III, Section D, we investigated the orbital evolution of the moon as the earth cooled off for a number of different activation energies and coefficients in Equation (III-23). In view of the wide variety of results obtained in the temperature of the earth when the moon arrives at 3.83 earth radii, it appears that the self-regulating mechanism proposed by O'Keefe does not exist.

The accretion theory states that the moon formed from a ring of particles in orbit about the earth. The particles collided with each other and stuck together, ultimately building up into the moon.

The ring of particles would be expected to lie in the proper plane. The orbits of particles inclined to the proper plane would precess, thus lowering the chances of collision; at least all the orbits would intersect the proper plane, favoring accretion there. If the moon accreted from the ring much beyond 3.83 earth radii, then the moon would tend to remain near the proper plane, so that its present inclination to the ecliptic could not be explained. However, if the moon formed in the proper plane within 3.83 earth radii (essentially in the equatorial plane), then the mechanism proposed here for driving the moon out of the earth's equatorial plane could have come into play.

At any rate, regardless of how the moon arrived at 3.83 earth radii in the equatorial plane of the earth, whether by fission or accretion, if the viscosity of the earth was greater than 10^{17} poises, and the moon suffered a 2.5–3.0° perturbation in inclination at 3.83 earth radii, then the present inclination to the ecliptic could be explained.

The capture theory has a simple answer to the inclination problem: the moon was captured in a highly inclined orbit to begin with and tidal friction has acted to decrease the inclination to its present value (Gerstenkorn 1969, MacDonald 1964). Of course, these theories begin with the present inclination of the moon's orbit to the ecliptic and solve the equations of tidal friction backward in time to discover the moon's inclination at capture.

These theories once again assume weak tidal friction with small lag angles. However, if the viscosity of the earth is greater than 10^{16} poises, and the moon arrives at a distance less than 3.83 earth radii in an inclined orbit, then the inclination must drop below the critical angle ψ_c before the orbit can expand past 3.83 earth radii (Chapter III, Section B). Thus for large viscosities (greater than 10^{17} poises) the orbit becomes nearly equatorial and we are faced with the same problem as before.

C. Summary of the Important Results

Assuming that the earth behaves like a viscous liquid in responding to the tidal force, and that the moon is in a circular orbit about the earth, with the inclination of the lunar orbit to the earth's equator $\lesssim 20°$, then:*

(a) If the moon is less than 3.83 earth radii distance from the earth and the inclination of the orbit to the earth's equator is steep, then the orbit may contract and then expand, provided the viscosity of the earth is greater than 10^{17} poises. The orbit will expand monotonically if the viscosity is less than 10^{16} poises regardless of the inclination.

(b) The inclination of the lunar orbit to the earth's equator will decrease or remain zero if the moon is closer to the earth than 3.83 earth radii, regardless of the viscosity.

(c) The inclination of the lunar orbit to the earth's equator must be less than 2.7° when the moon is at 3.83 earth radii if the earth's viscosity is 10^{18} poises; at higher viscosities the inclination must be even lower.

(d) The lunar orbit will expand monotonically if the moon is at a distance greater than 3.83 earth radii from the earth, regardless of the viscosity.

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*$\lesssim 20°$ so that Equations (III-9) through (III-12) hold.

(e) The inclination of the lunar orbit to the earth's equator will increase, or decrease (or remain zero) for earth-moon distances greater than 3.83 earth radii depending upon whether the viscosity of the earth is greater than, or less than 10^{16} poises.

(f) If the viscosity of the earth is greater than or equal to 10^{18} poises, <u>and</u> the plane of the lunar orbit is perturbed about 2.5 to 3 degrees out of the equatorial plane of the earth when the moon is just beyond 3.83 earth radii, then the present five degree inclination of the moon to the ecliptic may be explained.

APPENDIX A

DERIVATION OF THE TIDE-RAISING POTENTIAL AND TIDAL DISTURBING FUNCTION

A. Derivation of the Tide-Raising Potential

Consider the top diagram in Figure 21. The center of mass of the earth is located at point O; the center of mass of the moon is at Q; and the center of mass of the earth-moon system is located at point P. The earth and moon have masses M and m respectively. The vector $\vec{h}$ is directed from P to O.

The center of mass of the system is taken to be at rest in inertial space, with the earth and moon orbiting about P in circular orbits. The angular velocity of either the earth or moon about P is $\vec{\Omega}$.

The x^* y^* z^* coordinate system has its origin at O and is rigidly attached to the earth. The earth rotates about the Z^* axis with angular velocity $\vec{n}$ with respect to inertial space. The vector $\vec{r}^* = (x^*, y^*, z^*)$ is the position vector of some unit mass element in the earth in the starred system.

The bottom diagram in Figure 21 shows that the moon is located at $\vec{r} = (x, y, z)$ in the starred system, and the angle between $\vec{r}^*$ and $\vec{r}$ is Θ. $\vec{s} = \vec{r} - \vec{r}^*$ is the vector directed from the mass element to the moon. We take $|\vec{s}| = s$, $|\vec{r}^*| = r^*$ and $|\vec{r}| = r$, so that the earth and moon are separated by a distance r.

We wish to know the forces acting on the unit mass located at position $\vec{r}^*$. Let us denote the total force on the unit mass as $\vec{f}_T$. Let us further write $\vec{f}_T$ as the sum of two forces, one being the gravitational pull of the moon $\vec{f}_m$, and the other being the sum of all other forces $\vec{f}$ (such as the earth's gravity, viscous forces, etc.). Hence we have

$$\vec{f}_T = \vec{f} + \vec{f}_m \tag{A-1}$$

$\vec{f}_T$ represents the total force on the unit mass as seen from an inertial frame.

We now wish to find the forces acting on the unit mass in the frame in which we have reduced the earth to rest, i.e. the forces as seen in the starred frame. Since the starred frame is non-inertial, fictitious forces will be introduced.

Following Symon, Mechanics, Chapter 7, we write

$$\vec{f}_T{}^* = \vec{f}_T - \vec{n} \times (\vec{n} \times \vec{r}^*) - 2\vec{n} \times \vec{v}^* - \frac{d\vec{n}}{dt} \times \vec{r}^* - \frac{d^2\vec{h}}{dt^2} \tag{A-2}$$

$\vec{f}_T{}^*$ is the total force acting on the unit mass as seen from the starred system. The second term on the right-hand side of Equation (A-2) is the centrifugal force caused by the rotation of the earth on its axis. The third term is the Coriolis force, with $\vec{v}^*$ being the velocity of the unit mass in the starred system. The fourth term arises from any variations in $\vec{n}$. The last term arises from the earth's motion about the center of mass of the system.

The third and fourth terms on the right-hand side of (A-2) will be assumed to be negligible, as they would be if the earth were changing its rotation rate only slowly and velocities relative to the earth were small. Equation (A-2) becomes in that case

$$\vec{f}_T{}^* = \vec{f} + \vec{f}_m - \vec{n} \times (\vec{n} \times \vec{r}^*) - \frac{d^2\vec{h}}{dt^2} \tag{A-3}$$

where we have explicitly written $\vec{f} + \vec{f}_m$ for $\vec{f}_T$.

Now

$$\frac{d\vec{h}}{dt} = \vec{\Omega} \times \vec{h}$$

and

$$\frac{d^2\vec{h}}{dt^2} = \vec{\Omega} \times (\vec{\Omega} \times \vec{h}).$$

But $\vec{\Omega} \times (\vec{\Omega} \times \vec{h}) = \Omega^2 \frac{h}{r} \vec{r}$, where $h = |\vec{h}| = \frac{m}{M + m} r$. By Kepler's third law $\Omega^2 = \frac{G(M + m)}{r^3}$ so that we may write

$$\frac{d^2\vec{h}}{dt^2} = \frac{G(M + m)}{r^3} \frac{m}{M + m} r \frac{\vec{r}}{r} = \frac{Gm}{r^3} \vec{r}$$

We could have written this directly by recognizing that $\frac{d^2\vec{h}}{dt}$ is just the acceleration of the earth's center of mass due to the gravitational pull of the moon.

Equation (A-3) becomes

$$\vec{f}_T^* = \vec{f} + \vec{f}_m - \vec{n} \times (\vec{n} \times \vec{r}^*) - \frac{Gm}{r^3} \vec{r} \qquad \text{(A-4)}$$

Let us now examine the $\vec{f}_m$ term in Equation (A-4). $\vec{f}_m$ is the gravitational force of the moon on the unit mass located at $\vec{r}^*$. We can write $\vec{f}_m$ as the gradient of a potential:

$$\vec{f}_m = \vec{\nabla}^* V(x^*, y^*, z^*, x, y, z). \qquad \text{(A-5)}$$

Here

$$\vec{\nabla}^* = \frac{\partial}{\partial x^*} \vec{i}^* + \frac{\partial}{\partial y^*} \vec{j}^* + \frac{\partial}{\partial z^*} \vec{k}^*$$

denotes the gradient operator operating in the starred system; $\vec{i}^*$, $\vec{j}^*$, and $\vec{k}^*$, are unit vectors along the x^*, y^*, and z^* axes, respectively. V is the gravitational potential of the moon at $\vec{r}^*$. Note that V is a function of both the coordinates of the unit mass and the coordinates of the moon, but that $\vec{\nabla}^*$ acts only on the starred coordinates.

Taking the moon to be a point mass gives

$$V = \frac{G\,m}{s}$$

We now proceed to expand V in spherical harmonics about the center of the earth in the usual manner (Kaula 1968, Eq. 2.1.22):

Since

$$\vec{s} = \vec{r} - \vec{r}^*, \quad s = (\vec{s} \cdot \vec{s})^{1/2} = (\vec{r} \cdot \vec{r} + \vec{r}^* \cdot \vec{r}^* - 2\,\vec{r} \cdot \vec{r}^*)^{1/2}$$

$$= (r^2 + r^{*2} - 2\,r\,r^* \cos\Theta)^{1/2} = r\left(1 + \left(\frac{r^*}{r}\right)^2 - 2\left(\frac{r^*}{r}\right)\cos\Theta\right)^{1/2} .$$

We can thus write

$$V = \frac{G\,m}{r}\left(1 + \left(\frac{r^*}{r}\right)^2 - 2\left(\frac{r^*}{r}\right)\cos\Theta\right)^{-1/2}$$

Now

$$\left(1 + \left(\frac{r^*}{r}\right)^2 - 2\left(\frac{r^*}{r}\right)\cos\Theta\right)^{-1/2}$$

is of the form $(1 + q)^n$ where

$$q = \left(\frac{r^*}{r}\right)^2 - 2\left(\frac{r^*}{r}\right)\cos\Theta$$

and

$$n = -\frac{1}{2}.$$

Expanding $(1 + q)^n$ as a binominal series gives

$$(1 + q)^n = 1 + n\,q + \frac{n\,(n-1)}{2\,!}\,q^2 + \ldots$$

so that our expression for V becomes

$$V = \frac{G\,m}{r}\left\{1 + \left(\frac{r^*}{r}\right)\cos\Theta + \frac{3}{2}\left(\frac{r^*}{r}\right)^2\left[\cos^2\Theta - \frac{1}{3}\right] + \ldots\right\}$$

where we have been careful to gather together terms in powers of $\frac{r^*}{r}$. This is nothing more than the familiar expression

$$V = \frac{G\,m}{r}\left\{\sum_{m=0}^{\infty} P_m\,(\cos\Theta)\left(\frac{r^*}{r}\right)^m\right\}$$

where P_m $(\cos\Theta)$ is the Legendre polynomial of order m.

If $\frac{r^*}{r} \ll 1$ we can ignore powers of $\frac{r^*}{r}$ higher than 2 so that we can write

$$V = \frac{G\,m}{r}\left\{1 + \left(\frac{r^*}{r}\right)\cos\Theta + \frac{3}{2}\left(\frac{r^*}{r}\right)^2\left[\cos^2\Theta - \frac{1}{3}\right]\right\}$$

Let us set

$$V_t = \frac{G\,m}{r}\cdot\frac{3}{2}\left(\frac{r^*}{r}\right)^2\left[\cos^2\Theta - \frac{1}{3}\right] \qquad \text{(A-6)}$$

which we will call the tide-raising potential. Note that V_t is a second degree spherical solid harmonic function and $\left[\cos^2\Theta - \frac{1}{3}\right]$ is a second degree surface harmonic.

We finally have

$$V = \frac{Gm}{r} + \frac{Gm}{r}\left(\frac{r^*}{r}\right)\cos\Theta + V_t$$

If we again write $\vec{f}_m$ as the gradient of V we have

$$\vec{f}_m = \vec{\nabla}^*\left(\frac{Gm}{r}\right) + Gm\,\vec{\nabla}^*\left(\frac{r^*}{r^2}\cos\Theta\right) + \vec{\nabla}^* V_t$$

The first term on the right-hand side of the above equation is zero because $r^2 = x^2 + y^2 + z^2$ and nowhere contains x^*, y^*, or z^*. The function that the operator $\vec{\nabla}^*$ acts on in the second term can be written

$$\frac{r^*}{r^2}\cos\Theta = \frac{r^* r\cos\Theta}{r^3} = \frac{x^*x + y^*y + z^*z}{r^3}$$

so that

$$\vec{\nabla}^*\left(\frac{x^*x + y^*y + z^*z}{r^3}\right) = \frac{\vec{r}}{r^3}$$

We are left with

$$\vec{f}_m = Gm\frac{\vec{r}}{r^3} + \vec{\nabla}^* V_t \qquad \text{(A-7)}$$

Substituting the above equation in (A-4) gives

$$\vec{f}_T^* = \vec{f} + Gm\frac{\vec{r}}{r^3} + \vec{\nabla}^* V_t - \vec{n}\times(\vec{n}\times\vec{r}^*) - Gm\frac{\vec{r}}{r^3}$$

or finally

$$\vec{f}_T^* = \vec{f} + \vec{\nabla}^* V_t - \vec{n}\times(\vec{n}\times\vec{r}^*) \qquad \text{(A-8)}$$

Note that the first term on the right side of Equation (A-7) cancels the term associated with the motion of the earth about the center of mass of the system in (A-4), leaving the $\vec{\nabla}^* V_t$ term as the only term in (A-8) generated by the moon's gravity. We were therefore justified in calling V_t the tide-raising potential.

We note in passing that the centrifugal force may also be written as the gradient of a potential:

$$-\vec{n} \times (\vec{n} \times \vec{r}^*) = \vec{\nabla}^* V_c\ (x^*, y^*, z^*)$$

where

$$V_c = \frac{1}{2} n^2 (x^{*2} + y^{*2})$$

with $n = |\vec{n}|$.

In this case Equation (A-8) becomes

$$\vec{f}_T^* = \vec{f} + \vec{\nabla}^* V_t + \vec{\nabla}^* V_c \qquad \text{(A-9)}$$

B. Derivation of the Disturbing Function

In this section we show how deviations from sphericity of the earth give rise to a disturbing function.

Let us again take the starred coordinate system to be fixed in the earth with its origin at the center of mass of the earth, and let $\vec{\delta} = (\tilde{x}, \tilde{y}, \tilde{z})$ be the position vector of some mass element in the earth and $\vec{\Delta} = (x', y', z')$ be the position vector of some point E exterior to the earth (see Figure 22). Let $\delta = |\vec{\delta}| = (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{1/2}$ and $\Delta = |\vec{\Delta}| = (x'^2 + y'^2 + z'^2)^{1/2}$. $\vec{\Gamma} = \vec{\Delta} - \vec{\delta}$ is the vector directed from the mass element to the exterior point E and has length $\Gamma = |\vec{\Gamma}|$.

The gravitational potential at (x', y', z') is

$$U(x', y', z') = \int \frac{G\rho}{\Gamma} \, d\tilde{V} \qquad \text{(A-10)}$$

where ρ is the density and $d\tilde{V}$ the volume of the mass element, with the volume integral evaluated over the volume of the earth.

The force on a unit mass at (x', y', z') would be given by $\vec{\nabla}' U$, where

$$\vec{\nabla}' = \frac{\partial}{\partial x'} \vec{i}^* + \frac{\partial}{\partial y'} \vec{j}^* + \frac{\partial}{\partial z'} \vec{k}^* .$$

Note that the gradient operator acts only on the primed coordinates.

We can expand Γ in a binomial series just as we did previously for s; Equation (A-10) then becomes

$$U(x', y', z') = \int \frac{G\rho}{\Delta} \left\{ 1 + \left(\frac{\delta}{\Delta}\right) \cos \Psi + \frac{3}{2} \left(\frac{\delta}{\Delta}\right)^2 \left[\cos^2 \Psi - \frac{1}{3}\right] + \dots \right\} d\tilde{V}$$

where Ψ is the angle between $\vec{\delta}$ and $\vec{\Delta}$.

If we neglect powers of $\left(\frac{\delta}{\Delta}\right)$ higher than 2 we have

$$U(x', y', z') = \int \frac{G\rho}{\Delta} \, d\tilde{V} + \int \frac{G\rho}{\Delta} \left(\frac{\delta}{\Delta}\right) \cos \Psi \, d\tilde{V}$$

$$+ \int \frac{G\rho}{\Delta} \frac{3}{2} \left(\frac{\delta}{\Delta}\right)^2 \left[\cos^2 \Psi - \frac{1}{3}\right] d\tilde{V} \qquad \text{(A-11)}$$

The first term in Equation (A-11) is easily evaluated:

$$\int \frac{G\rho}{\Delta} \, d\tilde{V} = \frac{G}{\Delta} \int \rho \, d\tilde{V} = \frac{GM}{\Delta}$$

This term gives the inverse square force. The second term of (A-11) is zero by virtue of having taken the origin of the coordinate system at the center of mass of the earth. The third term is the disturbing function R_I:

$$R_I\,(x', y', z') \;=\; \frac{3}{2}\frac{G}{\Delta^3} \int \rho\, \delta^2 \left[\cos^2 \Psi - \frac{1}{3}\right] d\tilde{V} \qquad \text{(A-12)}$$

with the subscript I reminding us that $\vec{\nabla}' R_I$ gives the force per unit mass in inertial space.

Let us now write Equation (A-12) in spherical polar coordinates, with $\tilde{\alpha}$ and $\tilde{\beta}$ being the longitude and colatitude respectively of the mass element, and α' and β' being the corresponding longitude and colatitude of the exterior point; then

$$R_I\,(\Delta, \alpha', \beta') \;=\; \frac{3}{2}\frac{G}{\Delta^3} \int_0^{r_s} \int_0^{\pi} \int_0^{2\pi} \rho\, \delta^2 \left[\cos^2 \Psi - \frac{1}{3}\right] \delta^2 \sin \tilde{\beta}\; d\tilde{\alpha}\; d\tilde{\beta}\; d\delta$$

where

$$\cos \Psi \;=\; \cos \alpha' \sin \beta' \cos \tilde{\alpha} \sin \tilde{\beta} + \sin \alpha' \sin \beta' \sin \tilde{\alpha} \sin \tilde{\beta} + \cos \beta' \cos \tilde{\beta}$$

and

$$d\tilde{V} \;=\; \sin \tilde{\beta}\; \delta^2\; d\tilde{\alpha}\; d\tilde{\beta}\; d\delta$$

and r_s is the distance from the earth's center to the surface of the earth.

Let us now write

$$r_s\,(\tilde{\alpha}, \tilde{\beta}) \;=\; a + \sigma\,(\tilde{\alpha}, \tilde{\beta})$$

where a is the mean radius of the earth and $\sigma\,(\tilde{\alpha}, \tilde{\beta})$ the "surface inequality" which accounts for deviations of the earth's surface from sphericity, regardless of how those deviations arise.

We introduce several assumptions at this point: first, that ρ is a function of radial distance δ only and not a function of $\tilde{\alpha}$ and $\tilde{\beta}$; second, that $\sigma \ll a$; and third, that $\sigma(\tilde{\alpha}, \tilde{\beta})$ may be written as a sum of second degree surface spherical harmonics $Y_m^\ell(\tilde{\alpha}, \tilde{\beta})$ ($\ell = 2$). Of course any surface displacement in general may be expressed as a sum of surface spherical harmonics of all degrees. We retain only the second degree harmonics since these are the most important.

The second assumption allows us to write $R_I(\Delta, \alpha', \beta')$ as the sum of two terms, with the first term containing the volume integral evaluated over the spherical earth and the second term taking care of the "surface inequality":

$$R_I(\Delta, \alpha', \beta') = \frac{3}{2}\frac{G}{\Delta^3}\int_0^a \int_0^\pi \int_0^{2\pi} \rho\,\delta^2 \left[\cos^2\Psi - \frac{1}{3}\right] \sin\tilde{\beta}\,\delta^2\, d\tilde{\alpha}\, d\tilde{\beta}\, d\delta$$

$$+ \frac{3}{2}\frac{G}{\Delta^3}\int_0^\pi \int_0^{2\pi} \sigma(\tilde{\alpha}, \tilde{\beta})\,\rho_s\, a^2 \left[\cos^2\Psi - \frac{1}{3}\right] \sin\tilde{\beta}\, a^2\, d\tilde{\alpha}\, d\tilde{\beta}$$

where ρ_s is the density at the earth's surface.

The first term vanishes by the first assumption because the integral over the angular part is zero. We are left with

$$R_I(\Delta, \alpha', \beta') = \frac{3}{2}\frac{G\rho_s}{\Delta^3}\, a^4 \int_0^\pi \int_0^{2\pi} \sigma(\tilde{\alpha}, \tilde{\beta}) \left[\cos^2\Psi - \frac{1}{3}\right] \sin\tilde{\beta}\, d\tilde{\alpha}\, d\tilde{\beta}$$

Now by the third assumption $\sigma(\tilde{\alpha}, \tilde{\beta})$ is a sum of second degree surface harmonics; $\cos^2\Psi - \frac{1}{3}$ is also a sum of second degree harmonics. We then recognize that the integral in the above equation is a sum of inner products of spherical harmonics. By use of the orthogonality of the $Y_m^\ell(\tilde{\alpha}, \tilde{\beta})$'s and keeping track of normalization constants we may finally write

$$R_I(\Delta, \alpha', \beta') = \frac{4}{5}\pi G \rho_s a \left(\frac{a}{\Delta}\right)^3 \sigma(\alpha', \beta') \tag{A-13}$$

(See Kaula 1968, pgs. 65-67, for a general expansion in surface harmonics.)

The disturbing function R_I at longitude α' and colatitude β' is seen to be proportional to the displacement of the surface at that same longitude and latitude; hence a body in the vicinity of the earth is acted upon by a disturbing function which is proportional to the height of the displacement of the earth's surface where the position vector from the center of the earth to the body pierces the earth's surface.

If harmonics of degree n where $n > 2$ had been included in our expression for the surface displacement, then they would appear in our expression for the disturbing function correspondingly multiplied by $\left(\frac{a}{\Delta}\right)^{n+1}$ For distances far from the earth $\left(\frac{a}{\Delta}\right) \ll 1$, so that these higher order terms are less important than the second degree terms.

C. Conversion of R_I to R

Equation (A-13) gives the disturbing function as seen inertial space. Generally we want the disturbing function acting on the moon referred to the (accelerated) earth. We show how to find it below.

Let $\vec{r}_1$ be the position vector of the earth in some inertial coordinate system. Likewise let $\vec{r}_2$ be the position vector of the moon in this same coordinate system. (We still assume that the earth and moon are the only two bodies in existence.) Let $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$ be the position of the moon as seen from the earth.

If V is the potential of the earth, then by Newton's second law

$$M\ddot{\vec{r}}_1 = -m\vec{\nabla}V$$

$$m\ddot{\vec{r}}_2 = m\vec{\nabla}V$$

and

$$\ddot{\vec{r}}_{12} = \frac{M+m}{M}\vec{\nabla}V = \vec{\nabla}\left(\frac{M+m}{M}V\right)$$

$\ddot{\vec{r}}_{12}$ is the acceleration of the moon as seen from the earth, and $\frac{M+m}{M}V$ represents the potential of the earth as seen from the moon. It is then clear that we must write

$$R(\Delta, \alpha', \beta') = \frac{M+m}{M} R_I$$

$$= \frac{4}{5}\pi G\left(\frac{M+m}{M}\right)\rho_s\, a\left(\frac{a}{\Delta}\right)^3 \sigma(\alpha', \beta') \qquad \text{(A-14)}$$

as the disturbing function acting on the moon as seen from the earth.

D. The Tidal Disturbing Function

The forces acting in Equation (A-9) displace mass on the earth; the displaced mass acts gravitationally on the moon and affects its motion.

Assume that the earth responds separately to the centrifugal and the tide-raising force so that we may write

$$\sigma(\tilde{\alpha}, \tilde{\beta}) = \sigma_c(\tilde{\alpha}, \tilde{\beta}) + \sigma_t(\tilde{\alpha}, \tilde{\beta})$$

where σ_c is the displacement of the earth caused by the centrifugal force and σ_t is the displacement of the surface caused by the tide-raising force. Equation (A-14) then becomes

$$R(\Delta, \alpha', \beta') = R_c(\Delta, \alpha', \beta') + R_t(\Delta, \alpha', \beta')$$

where again the subscripts "c" and "t" mean "centrifugal" and "tidal" respectively. We will call

$$R_t(\Delta, \alpha', \beta') = \frac{4}{5}\pi G\left(\frac{M+m}{M}\right)\rho_s\, a\left(\frac{a}{\Delta}\right)^3 \sigma_t(\alpha', \beta') \tag{A-15}$$

the tidal disturbing function.

At this point we must take great pains to make clear the distinction between the tide-raising and tidally disturbed body. The two are not necessarily the same and must in any case be kept mathematically distinct to avoid incorrect derivations. We explain this below.

Suppose we wished to find the action of the tides raised on Mars by Phobos on Mars' other moon Deimos. In the above discussion Phobos (the tide-raiser) is at point $\vec{r} = (x, y, z)$, and Deimos (the tidally disturbed body) is at $\vec{\Delta} = (x', y', z')$. The force per unit mass on Deimos caused by Phobos' tides is

$$\vec{\nabla}' R_t = \frac{\partial R_t}{\partial x'}\vec{i}^* + \frac{\partial R_t}{\partial y'}\vec{j}^* + \frac{\partial R_t}{\partial z'}\vec{k}^* .$$

Even though R_t depends on both x, y, z and x', y', z' the gradient operator acts only on the x', y', z' coordinates. So much should be clear.

Now suppose we have the case we are interested in, namely the action of the lunar tides raised on the earth on the moon itself. Here the tide-raiser is also the tidally disturbed body, and the positions (x', y', z') and (x, y, z) are the same. We cannot drop the primes appearing in R_t and apply the gradient

$$\frac{\partial}{\partial x}\vec{i}* + \frac{\partial}{\partial y}\vec{j}* + \frac{\partial}{\partial z}\vec{k}*$$

to find the force per unit mass of the moon however, since the gradient operator can act only on the disturbed body's coordinates to retain the proper meaning of force per unit mass; thus the distinction between the tide-raiser and tidally disturbed body must be kept, even though they may be one and the same object.

Darwin keeps the distinction clear in his 1880 paper by introducing the interesting artifice of giving the earth two satellites; the tide-raiser he calls Diana and the tidally disturbed body is the moon. When considering the action of lunar tides on the moon, Diana and the moon are, of course, the same object.

APPENDIX B

COOLING OF A PLANET BY RADIATIVE LOSS

Let us suppose that a planet in empty space is cooling off by radiating heat into space from its surface according to the Stefan-Boltzmann law. Solar heating is neglected, and the planet is not surrounded by an atmosphere.

Let us assume that the temperature distribution inside the planet has the form

$$T(r, t) = T_s(t)\, F(r) \tag{B-1}$$

where

t = time

r = radial distance

$T(r, t)$ = temperature at distance r and time t

$T_s(t)$ = surface temperature at time t

$F(r)$ = some function of radial distance

Note that the temperature distribution has spherical symmetry.

The planet radiates like a black body so that the amount of energy dQ given off in a time dt is

$$dQ = -4\pi R^2 \sigma T_s^4(t)\, dt \tag{B-2}$$

Here

R = radius of planet

σ = Stefan-Boltzmann constant

As the planet cools off, each element of mass dm inside the earth gives up an amount of heat

$$dQ_{dm} = C_p \, dT(r, t) \, dm$$

in time dt, where C_p is the specific heat at constant pressure.

The total amount of heat given off by the planet in time dt is then

$$dQ = \int_{\substack{\text{Mass of} \\ \text{the planet}}} C_p \, dT(r, t) \, dm$$

This must be equal to Equation (B-2), so

$$\int C_p \, dT(r, t) \, dm = -4\pi R^2 \sigma T_s^4(t) \, dt \tag{B-3}$$

Now from Equation (B-1)

$$dT(r, t) = dT_s(t) \, F(r)$$

gives the change in temperature with time, so that Equation (B-3) becomes

$$\int C_p \, dT_s(t) \, F(r) \, dm = dT_s(t) \int C_p \, F(r) \, dm$$

$$= -4\pi R^2 \sigma T_s^4(t) \, dt$$

Let

$$\int C_p \, F(r) \, dm = I .$$

Note that if we assume ρ has spherical symmetry, we may write $dm = \rho(r) \, 4\pi r^2 \, dr$ to show dm as an explicit function of r.

The above equation can be written

$$\frac{dT_s(t)}{T_s^4(t)} = -\frac{4\pi R^2 \sigma}{I} dt$$

This may be integrated to give

$$-\frac{1}{3}\left[T_s^{-3}(t) - T_s^{-3}(t_0)\right] = -\int_{t_0}^{t} \frac{4\pi R^2\sigma}{I} dt$$

$$= -\frac{4\pi R^2\sigma}{I}(t - t_0)$$

which may be written

$$T_s(t) = \frac{T_s(t_0)}{\left[1 + \frac{12\pi R^2\sigma}{I} T_s^3(t_0)(t - t_0)\right]^{1/3}} \qquad \text{(B-4)}$$

This expression gives the surface temperature as a function of time. If F (r) is known, the temperature at any point r at time t is given by

$$T(r, t) = \frac{T_s(t_0) \cdot F(r)}{\left[1 + \frac{12\pi R^2\sigma T_s^3(t_0)}{I}(t - t_0)\right]^{1/3}} \qquad \text{(B-5)}$$

Now suppose that C_p is a constant so that we may write

$$I = \int C_p F(r)\, dm = C_p \int F(r)\, dm$$

$$= \frac{C_p M}{T_s(t)} \frac{\int T_s(t) F(r)\, dm}{M}$$

$$= \frac{C_p M}{T_s(t)} \frac{\int T(r, t)\, dm}{M} = C_p M \frac{\langle T(t) \rangle}{T_s(t)}$$

where M is the mass of the planet and

$$\langle T(t) \rangle = \frac{\int T(r, t)\, dm}{M}$$

is the average temperature inside the planet weighted by mass.

Set

$$Z = \frac{T_s(t)}{\langle T(t) \rangle} = \frac{\text{surface temperature}}{\text{average temperature}}$$

Z varies from ~ 1 to probably $\sim \frac{1}{5}$ for any plausible temperature distribution.

Also set

$$S = \frac{12\pi R^2 \sigma}{M C_p}$$

S has a characteristic value for each planet.

Equation (B-4) becomes using this notation

$$T_s(t) = \frac{T_s(t_0)}{\left[1 + Z S\, T_0^3(t_0)\,(t - t_0)\right]^{1/3}} \tag{B-6}$$

To give an example, for the earth

$$R = 6.37 \times 10^8 \text{ cm}$$

$$M = 5.98 \times 10^{27} \text{ g}$$

$$C_p \simeq 1 \times 10^7 \text{ erg/g-deg}$$

$$S = 1.47 \times 10^{-20} \frac{\text{deg}^{-3}}{\text{sec}}$$

For $Z = \frac{1}{3}$ and $T_s(t_0) = 3000°\text{K}$ T_s falls to half its value in about 1700 years.

To demonstrate that a temperature distribution of the form

$$T(r, t) = T_s(t)\, F(r) \tag{B-1}$$

is not unrealistic, we note that the adiabatic temperature gradient is given by

$$\frac{dT}{dr} = -\frac{\alpha T g}{C_p} \tag{B-7}$$

where

α = coefficient of compressibility

g = gravitational acceleration

The above equation may be rewritten as

$$\frac{dT}{T} = -\frac{\alpha g}{C_p}\, dr$$

If the right side depends solely on r, we have

$$\log\left(\frac{T}{T_s}\right) = -\int_R^r \frac{\alpha g}{C_p}\, dr$$

or

$$T = T_s\, e^{-\int \frac{\alpha g}{C_p} dr} = T_s\, F(r)$$

where

T_s = surface temperature

and

$$F(r) = e^{-\int \frac{\alpha g}{C_p} dr}$$

If the planet cools off in such a manner as to maintain the adiabatic temperature gradient at all times, T_s becomes a function of time and

$$T = T_s(t)\, F(r)$$

This is exactly the form which we assumed the temperature distribution had.

APPENDIX C

COMPUTER PROGRAM FOR CONSTANT VISCOSITY

The computer program given in this appendix is discussed in Chapter III, Section C, and in the comments listed in the program itself.

```
CC******************
CC******************
C*****THIS IS TIDE1
C*****     THIS PROGRAM INTEGRATES DARWINS (1880) EQUATIONS TO GIVE THE
C      EVOLUTION OF THE MOONS (CIRCULAR) ORBIT FOR A CONSTANT VISCOSITY
C      OF THE EARTH.
C          DARWIN (1880) IS
C      ON THE SECULAR CHANGES IN THE ELEMENTS OF THE ORBIT OF A SATELLITE
C      REVOLVING ABOUT A TIDALLY DISTORTED PLANET
C                IN
C      SCIENTIFIC PAPERS BY SIR GEORGE HOWARD DARWIN, VOL. 2, PP 208-382
C      CAMBRIDGE UNIVERSITY PRESS, 1908.
C          THE EQUATIONS ARE ON PGS 240,241, AND 242 (EQUATIONS 71, 73,
C      AND THE TWO PRECEEDING EQUATION 75.)
C          THE PAPER CAN ALSO BE FOUND IN
C      PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY, VOL. 171, 1880,
C      PP 713 - 891.
C          THE PROGRAM ALLOWS TWO APPROACHES - TO USE ALL THE TERMS IN
C      DARWINS EQUATIONS, OR KEEP ONLY TERMS UP TO AND INCLUDING SECOND
C      ORDER IN K=SIN((I+J)/2). THE LATTER WE CALL THE SECOND ORDER
C      APPROXIMATION.
C          THE EQUATIONS CAN BE INTEGRATED FORWARD OR BACKWARD IN TIME,
C      DEPENDING UPON WHETHER MTIME=+1 OR -1.
C          THE PROGRAM STARTS BY INTEGRATING WITH CONSTANT TIME INTERVALS
C      IT DOES THIS FOR NO STEPS. AFTER NO STEPS IT SWITCHES OVER TO
C      CONSTANT LN INTERVALS (CHOSEN IN THE PROGRAM TO GIVE DXI= TO ABOUT
C      0.001.) ALL THIS ASSUMES THAT THE PER CENT CHANGE IN I OR J IS
C      LESS THAN CRIT. IF THE FRACTIONAL CHANGE IS GREATER THAN CRIT, THE
C      INTERVAL IS HALVED UNTIL THE FRACTIONAL CHANGE IS LESS THAN CRIT.
C          CRIT WAS INTRODUCED TO PREVENT LARGE CHANGES IN ANGLE TO AVOID
C      CUMULATIVE ERROR. THE STRATAGEM OF SWITCHING FROM CONSTANT DELT TO
C      CONSTANT DLN IS TO KEEP THE ITERATIONS FROM TAKING FOREVER, SINCE
C      AT LARGE XI DXI IS QUITE SMALL FOR CONSTANT DELT.
C
C.....SOME OF DARWINS NOTATION.
C.....C-ZERO =REFERENCE DISTANCE
C.....OMEGA-ZERO=OMEGA AT C-ZERO
C.....SMALL K=C*(OMEGA-ZERO)*(C-ZERO)/(BIG G)*(BIG M)*(SMALL M)
C.....TAU-ZERO=(3/2)*(BIG G)*(SMALL M)/(C-ZERO)**3
C.....GOTHIC SMALL G=(2/5)*(SMALL G)/(SMALL A)
C.....BIG G=UNIVERSAL GRAVITATIONAL CONSTANT
C.....SMALL A=RADIUS OF THE EARTH
C.....BIG M=MASS OF THE EARTH
C.....SMALL M=MASS OF THE MOON
C.....SMALL G=GRAVITATIONAL CONSTANT AT THE EARTHS SURFACE
C.....W IS THE DENSITY OF THE EARTH
C.....WE HAVE SUBSTITUTED BIG G FOR DARWINS MU ABOVE.
C
C*****THIS SECTION DEFINES THE MOST IMPORTANT QUANTITIES.
C.....XI IS SQRT(EARTH-MOON DISTANCE/REFERENCE DISTANCE).
C.....DZERO IS THE REFERENCE DISTANCE, HERE IN UNITS OF EARTH RADII,
C      AND WHERE N=2*OMEGA.
C.....DS=SQRT(DZERO).  C-ZERO=SMALL A*DS.
C.....N IS THE ROTATIONAL ANGULAR VELOCITY OF THE EARTH IN 10**-4 /SEC
C      (I.E. MULTIPLY THE VALUE GIVEN IN THE PROGRAM BY 10**-4 TO GET THE
C      VALUE IN CGS UNITS.)
```

```
C.....OMEGA IS THE ORBITAL ANGULAR VELOCITY OF THE MOON IN UNITS OF
C      10**-4 /SEC.
C.....PSI IS THE ANGLE BETWEEN THE EARTHS EQUATORIAL PLANE AND THE
C      PLANE OF THE MOONS ORBIT. PSI=I + J.
C.....I IS THE ANGLE BETWEEN THE EARTHS EQUATORIAL PLANE AND THE
C      INVARIABLE PLANE.
C.....J IS THE ANGLE BETWEEN THE MOONS ORBITAL PLANE AND THE INVARIABLE
C      PLANE.
C.....PSI, I, AND J ARE IN RADIANS. XPSI,XII, AND XJ ARE THE SAME ANGLES
C      IN DEGREES.
C.....DPSI, DI, AND DJ ARE THE CHANGES IN THE RESPECTIVE ANGLES.
C.....VIS IS THE EARTHS VISCOSITY IN UNITS OF 10**16 CGS.
C.....LM IS THE ORBITAL ANGULAR MOMENTUM OF THE SYSTEM IN UNITS OF
C      10**40 CGS.
C.....LE IS THE ROTATIONAL ANGULAR MOMENTUM OF THE EARTH IN UNITS OF
C      10**40 CGS. LE=C*N.
C.....DLM AND DLE ARE THE RESPECTIVE CHANGES IN LM AND LE.
C.....LT IS THE TOTAL ANGULAR MOMENTUM OF THE SYSTEM IN UNITS OF
C      10**40 CGS.
C.....C IS THE MOMENT OF INERTIA OF THE EARTH IN UNITS OF 10**44 CGS.
C      C=SMALL K*B.
C.....T IS THE TIME IN UNITS OF 10**9 SEC. DELT IS THE CHANGE IN T.
C*****END SECTION.
C
C.....A1 IS (SMALL K)*(TAU-ZERO)**2/(GOTHIC SMALL G) (IN DARWINS
C      NOTATION) TIMES B (AS DEFINED HERE) IN UNITS OF 10**31 CGS.
C.....B IS SQRT((BIG G*SMALL A)/(BIG M+SMALL M))*(BIG M)*(SMALL M)*DS IN
C      UNITS OF 10**40 CGS. LM=B*XI.
C.....A2 IS 19/(2*(SMALL G)*(SMALL A)*(SMALL W))IN UNITS OF 10**-12 CGS.
C.....A3 IS SQRT((BIG G)*(BIG M+SMALL M)/(SMALL A)**3)/DS**3 IN UNITS OF
C      10**-4 SEC. OMEGA=A3/(XI**3).
C
C*****THIS SECTION EXPLAINS THE INITIAL INPUT DATA.
C.....NRUN IS THE NUMBER OF DATA CARDS TO BE READ. ALL INITIAL DATA FOR
C      A SINGLE RUN IS CONTAINED ON A SINGLE CARD.
C.....CRIT IS THE MAXIMUM CHANGE PERMITTED IN THE ABSOLUTE VALUES OF
C.....MTIME=+1 FOR A BATCH OF RUNS INTEGRATED FORWARD IN TIME, AND
C      MTIME=-1 FOR INTEGRATION BACKWARD IN TIME.
C      DI/I AND DJ/J IN A SINGLE STEP.
C.....ANGLE = INITIAL VALUE FOR PSI=I + J  IN RADIANS.
C.....XIF = INITIAL VALUE OF XI.
C.....VISF = VISCOSITY OF THE EARTH IN UNITS OF 10**16 CGS.
C.....DELTIF = STEP SIZE IN TIME IN UNITS OF 10**9 SEC.
C      DELTIF SHOULD ALWAYS BE POSITIVE.
C.....XIMAX = THE MAXIMUM VALUE OF XI TO WHICH THE PROGRAM INTEGRATES.
C      THE RUN STOPS WHEN XI = XIMAX.
C.....TSTART = INITIAL VALUE OF TIME IN UNITS OF 10**9 SEC.
C.....NP = THE NUMBER OF ITERATIONS DESIRED USING CONSTANT STEP SIZE IN
C      TIME.
C.....NF=1, NL=1 MEANS THE PROGRAM USES THE SECOND ORDER APPROXIMATION.
C.....NF=2, NL=2 MEANS THE PROGRAM USES ALL THE TERMS IN DARWINS
C      EQUATIONS.
C.....NF=1, NL=2 MEANS THE PROGRAM RUNS BOTH THE SECOND ORDER
C      APPROXIMATION AND DARWINS FULL EQUATIONS ON THE INITIAL DATA.
C.....NC1=1 GIVES A CHECK ON THE DATA RUN FOR THE SECOND ORDER
C      APPROXIMATION BY HALVING THE STEP SIZES AND DOUBLING THE NUMBER OF
C      STEPS AND REPEATING THE RUN. IF THE CHECK IS NOT DESIRED, NC1=0.
C.....NC2 PERFORMS THE SAME FUNCTION AS NC1 FOR THE FULL EQUATIONS.
```

```
              C.....NLAST IS THE TOTAL NUMBER OF STEPS PERMITTED IN ANY ONE RUN. THE
              C        RUN TERMINATES IF NLAST IS EXCEEDED.
              C*****END SECTION.
              C
              C
ISN 0002               DOUBLE PRECISION DZERO,C,LT,A1,A2,A3,A4,B,ANGLE,XI,VIS,LM,CC,
                      1 LE,N,OMEGA,P,K,TF1,SF1,TG,SG,TG1,SG1,DLE,DLM,    LTC,T,DELT,DS,
                      2 XIMAX
ISN 0003               DOUBLE PRECISION AY,DI,DJ,DELT1
ISN 0004               DOUBLE PRECISION TF,SF,TF2,SF2,TG2,SG2,TH,SH
ISN 0005               DOUBLE PRECISION PSI,DPSI
ISN 0006               DOUBLE PRECISION XIF,VISF,DELT1F,TSTART
ISN 0007               DOUBLE PRECISION T1,T2,T3,T4,T5,T6,T7,T8,T9,T10,T11,T12,T13,T14,
                      1 T15,T16,T17,T18,T19,T20,T21,T22,T23,T24,T25
ISN 0008               DOUBLE PRECISION SS,SJ,J,I
ISN 0009               DOUBLE PRECISION L,R1,R2,R3,DII,DJJ
ISN 0010               DOUBLE PRECISION U1
ISN 0011               DZERO=3.83387305D0
ISN 0012               C=8.11D0
ISN 0013               LT=34.2D0
ISN 0014               DS=DSQRT(DZERO)
ISN 0015               A1=1.313157D4*C/DZERO**6
ISN 0016               A2=2.756D0
ISN 0017               B=3.681701D0*DS
ISN 0018               A3=12.451850D0/(DZERO*DS)
ISN 0019               U1=0.0D0
ISN 0020               READ (5,1) NRUN,CRIT,MTIME
ISN 0021        1      FORMAT (I5,F10.2,I5)
ISN 0022               DO 100 NR=1,NRUN
              C*****THIS SECTION READS IN THE INITIAL DATA. ANGLE IS IN RADIANS.
ISN 0023               READ (5,2) ANGLE,XIF,VISF,DELT1F,XIMAX,TSTART,NP,NF,NL,NC1,NC2,
                      1 NLAST
ISN 0024        2      FORMAT (D9.2,D11.2,4D10.2,I5,4I2,I7)
              C*****END SECTION.
ISN 0025               DO 100 NA=NF,NL
ISN 0026               IF (NA .EQ. 1) ND=NC1
ISN 0028               IF (NA .EQ. 2) ND=NC2
ISN 0030               NC= 1 + ND
ISN 0031               DO 100 NB=1,NC
ISN 0032               NCHECK=-1
ISN 0033       40      XI=XIF
ISN 0034               NQ=NP
ISN 0035               IF (NB .EQ. 2) NQ=2*NP
ISN 0037               VIS=VISF
ISN 0038               DELT1=DELT1F
ISN 0039               T=TSTART
ISN 0040               A4=VIS*A2
              C*****THIS SECTION WRITES OUT THE INPUT DATA.
ISN 0041               WRITE (6,6) VIS
ISN 0042        6      FORMAT (1H1,///,10X,10HVISCOSITY=,D10.2,10H 10**16CGS,///)
ISN 0043               WRITE (6,18) DELT1F,XIMAX,NQ,NF,NL,NC1,NC2,NLAST
ISN 0044       18      FORMAT (5X,6HDELT1=,D10.3,1X,9H10**9 SEC,5X,6HXIMAX=,D10.3,5X,
                      1 3HNQ=,I5,5X,3HNF=,I1,5X,3HNL=,I1,5X,4HNC1=,I1,5X,4HNC2=,I1,5X,
                      2 6HNLAST=,I5,///)
ISN 0045               WRITE (6,52) CRIT,MTIME
ISN 0046       52      FORMAT (5X,5HCRIT=,F10.4,5X,6HMTIME=,I3,///)
              C*****END SECTION.
              C*****THIS SECTION WRITES THE HEADING.
```

```
                 C.....ALL QUANTITIES PRINTED OUT ARE IN UNITS GIVEN IN THE PROGRAM.
                 C     NOTE THAT UNDER HEADINGS PSI, I, AND J XPSI, XII, AND XJ ARE
                 C     PRINTED, THUS GIVING ALL ANGLES IN DEGREES. ALSO, THE FIRST DI AND
                 C     DJ LISTED (READING FROM LEFT TO RIGHT) IS FOR THE INFINITE
                 C     VISCOSITY LIMIT. THE INFINITE VISCOSITY CHANGE IN ANGLE IS TO BE
                 C     COMPARED TO THE ACTUAL CHANGE IN ANGLE (SECOND DI, DJ HEADING.)
ISN 0047               WRITE (6,7)
ISN 0048          7    FORMAT (6X,4HTIME,9X,2HXI,9X,1HN,8X,5H PSI ,3X,5HOMEGA,7X,2HDI,10X
                      1,2HDJ,8X,3H  I,8X,3H  J,7X,2HDI,9X,2HDJ,9X,4HDPSI,///)
                 C*****END SECTION.
                 C*****THIS SECTION COMPUTES THE INITIAL VALUES OF LE, N, OMEGA, I, AND J
ISN 0049               NT=0
ISN 0050               LM=B*XI
ISN 0051               CC=DCOS(ANGLE)
ISN 0052               PSI=ANGLE
ISN 0053               XPSI=180.0*ANGLE/3.14159
ISN 0054         15    DELT=DELT1
ISN 0055         17    CONTINUE
ISN 0056         22    LE=-LM*CC + DSQRT((LM*CC)**2 + LT**2 - LM**2)
ISN 0057               XI=LM/B
ISN 0058               N=LE/C
ISN 0059               OMEGA=A3/(XI**3)
ISN 0060               SS=DSIN(ANGLE)
ISN 0061               SJ=SS/DSQRT(SS**2 + (CC + LM/LE)**2)
ISN 0062               J=DARSIN(SJ)
ISN 0063               I=ANGLE-J
ISN 0064               XII=180.0*I/3.14159
ISN 0065               XJ=180.0*J/3.14159
                 C*****END SECTION.
                 C.....UI IS USED ONLY HERE TO FILL IN ZEROS.
ISN 0066               WRITE (6,2) T,XI,N,XPSI,OMEGA,UI,UI,XII,XJ
                 CC*******************
                 CC*******************
ISN 0067          5    AY=0.500*PSI
ISN 0068               IF (NA .EQ. 1) GO TO 26
                 C*****THIS SECTION COMPUTES ALL THE TERMS IN DARWINS EQUATIONS.
ISN 0070               K=DSIN(AY)
ISN 0071               P=DCOS(AY)
                 C.....COMPUTE THE TANGENTS, SINES OF THE LAG ANGLES.
ISN 0072               TF=2.0*N*A4
ISN 0073               SF=2.0*TF/(1.0+TF**2)
ISN 0074               TF2=2.0*(N+OMEGA)*A4
ISN 0075               SF2=2.0*TF2/(1.0+TF2**2)
ISN 0076               TG2=(N+2.0*OMEGA)*A4
ISN 0077               SG2=2.0*TG2/(1.0+TG2**2)
ISN 0078               TH=2.0*OMEGA*A4
ISN 0079               SH=2.0*TH/(1.0+TH**2)
ISN 0080               TF1=2.0*(N-OMEGA)*A4
ISN 0081               SF1=2.0*TF1/(1.0+TF1**2)
ISN 0082               TG=N*A4
ISN 0083               SG=2.0*TG/(1.0+TG**2)
ISN 0084               TG1=(N-2.0*OMEGA)*A4
ISN 0085               SG1=2.0*TG1/(1.0+TG1**2)
                 C.....COMPUTE THE TERMS.
ISN 0086               T1=0.500*P**8*SF1
ISN 0087               T2=2.0*P**4*K**4*SF
ISN 0088               T3=0.500*K**8*SF2
ISN 0089               T4=P**6*K**2*SG1
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```
ISN 0090             T5=P**2*K**2*((P**2-K**2)**2)*SG
ISN 0091             T6=P**2*K**6*SG2
ISN 0092             T7=P**8*SF1
ISN 0093             T8=K**8*SF2
ISN 0094             T9=4.0*P**6*K**2*SG1
ISN 0095             T10=4.0*P**2*K**6*SG2
ISN 0096             T11=6.0*P**4*K**4*SH
ISN 0097             T12=0.5D0*P**7*K*SF1
ISN 0098             T13=P**3*K**3*SF
ISN 0099             T14=0.5D0*P*K**7*SF2
ISN 0100             T15=1.5D0*P**3*K**3*(P**2-K**2)*SH
ISN 0101             T16=0.5D0*P**5*K*(P**2-3.0*K**2)*SG1
ISN 0102             T17=0.5D0*P*K*((P**2-K**2)**2)*SG
ISN 0103             T18=0.5D0*P*K**5*(3.0*P**2-K**2)*SG2
ISN 0104             T19=0.5D0*P**7*K*SF1
ISN 0105             T20=P**3*K**3*(P**2-K**2)*SF
ISN 0106             T21=0.5D0*P*K**7*SF2
ISN 0107             T22=0.5D0*P**5*K*(P**2+3.0*K**2)*SG1
ISN 0108             T23=0.5D0*P*K*((P**2-K**2)**3)*SG
ISN 0109             T24=0.5D0*P*K**5*(3.0*P**2+K**2)*SG2
ISN 0110             T25=1.5D0*P**3*K**3*SH
           C*****END SECTION.
           C*****THIS SECTION COMPUTES THE CHANGES IN LE, LM, J, AND I FOR DARWINS
           C     FULL EQUATIONS.
ISN 0111    37       DLE=-A1*(T1+T2+T3+T4+T5+T6)*DELT/(XI**12)
ISN 0112             DLM=0.5D0*A1*(T7-T8+T9-T10-T11)*DELT/(XI**12)
ISN 0113             DJ=-A1*(T12+T13+T14+T15-T16+T17+T18)*DELT/(LM*XI**12)
ISN 0114             DI=A1*(T19-T20-T21+T22-T23-T24-T25)*DELT/(LE*XI**12)
           C*****END SECTION.
ISN 0115             GO TO 27
           C*****THIS SECTION COMPUTES THE TERMS IN THE SECOND ORDER APPROXIMATION.
ISN 0116    26       K=DSIN(AY)
ISN 0117             P=DCOS(AY)
           C.....COMPUTE THE TANGENTS, SINES OF THE LAG ANGLES.
ISN 0118             TF1=2.0*(N-OMEGA)*A4
ISN 0119             SF1=2.0*TF1/(1.0 + TF1**2)
ISN 0120             TG=N*A4
ISN 0121             SG=2.0*TG/(1.0 + TG**2)
ISN 0122             TG1=(N-2.0*OMEGA)*A4
ISN 0123             SG1=2.0*TG1/(1.0 + TG1**2)
ISN 0124             TH=2.0*OMEGA*A4
ISN 0125             SH=2.0*TH/(1.0 + TH**2)
ISN 0126             TF=2.0*N*A4
ISN 0127             SF=2.0*TF/(1.0 + TF**2)
           C.....COMPUTE THE TERMS.
ISN 0128             T1=0.5D0*P**8*SF1
ISN 0129             T2=P**6*K**2*SG1
ISN 0130             T3=P**6*K**2*SG
ISN 0131             T4=P**8*SF1
ISN 0132             T5=4.0*P**6*K**2*SG1
ISN 0133             T6=0.5D0*P**7*K*SF1
ISN 0134             T7=P**3*K**3*SF
ISN 0135             T8=1.5D0*P**5*K**3*SH
ISN 0136             T9=0.5D0*P**7*K*SG1
ISN 0137             T10=1.5D0*P**5*K**3*SG1
ISN 0138             T11=0.5D0*P**5*K*SG
ISN 0139             T12=P**3*K**3*SG
ISN 0140             T13=0.5D0*P**7*K*SF1
```

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ISN 0141            T14=P**5*K**3*SF
ISN 0142            T15=0.5D0*P**7*K*SG1
ISN 0143            T16=1.5D0*P**5*K**3*SG1
ISN 0144            T17=0.5D0*P**7*K*SG
ISN 0145            T18=1.5D0*P**5*K**3*SG
ISN 0146            T19=1.5D0*P**3*K**3*SH
              C*****END SECTION.
              C*****THIS SECTION COMPUTES THE CHANGES IN LE, LM, J, AND I FOR THE
              C     SECOND ORDER APPROXIMATION.
ISN 0147       36   DLE=-A1*(T1+T2+T3)*DELT/(XI**12)
ISN 0148            DLM=A1*0.5D0*(T4+T5)*DELT/(XI**12)
ISN 0149            DJ=A1*0.5D0*(P**7*K*SG1-P**7*K*SF1-P**5*K*SG)*DELT/(LM*XI**12)
ISN 0150            DI=A1*0.5D0*(P**7*K*SG1+P**7*K*SF1-P**7*K*SG)*DELT/(LE*XI**12)
              C*****END SECTION.
ISN 0151       27   CONTINUE
              C*****THIS SECTION INSURES THAT DI/I OR DJ/J NEVER EXCEEDS CRIT.
ISN 0152            IF (I - 0.0) 19,19,38
ISN 0153       38   IF (J - 0.0) 19,19,39
ISN 0154       39   RI=DI/I
ISN 0155        .   IF (RI - 0.0) 47,46,46
ISN 0156       47   RI=-RI
ISN 0157       46   CONTINUE
ISN 0158            RJ=DJ/J
ISN 0159            IF (RJ - 0.0) 49,48,48
ISN 0160       49   RJ=-RJ
ISN 0161       48   CONTINUE
ISN 0162            IF (RI - CRIT) 34,34,35
ISN 0163       34   IF (RJ - CRIT) 19,19,35
ISN 0164       35   DELT=DELT/2.0
ISN 0166            IF (NA .EQ. 1) GO TO 36
ISN 0167            IF (NA .EQ. 2) GO TO 37
ISN 0169       19   CONTINUE
              C*****END SECTION.
              C*****THIS SECTION HALVES THE STEP SIZE WHEN NC1 OR NC2 EQUALS 1. NCHECK
              C     KEEPS TRACK OF WHETHER THE STEP SIZE IS HALVED FOR THE CHECK.
ISN 0170            IF (NB .EQ. 2) GO TO 41
ISN 0172            GO TO 42
ISN 0173       41   NCHECK=-NCHECK
ISN 0174            IF (NCHECK .EQ. 1) GO TO 43
ISN 0176            GO TO 42
ISN 0177       43   DELT=DELT/2.0
ISN 0178            IF (NA .EQ. 1) GO TO 36
ISN 0180            IF (NA .EQ. 2) GO TO 37
ISN 0182       42   CONTINUE
              C*****END SECTION.
              C*****THIS SECTION INCREMENTS THE IMPORTANT QUANTITIES.
ISN 0183            DPSI=DI + DJ
ISN 0184            I=I + CI
ISN 0185            J=J + CJ
ISN 0186            XII=180.0*I/3.14159
ISN 0187            XJ=180.0*J/3.14159
ISN 0188            PSI=PSI + DI + DJ
ISN 0189            XPSI=180.0*PSI/3.14159
ISN 0190            LE=LE+DLE
ISN 0191            LM=LM+DLM
ISN 0192            N=LE/C
ISN 0193            OMEGA=A3/(XI**3)
ISN 0194            XI=LM/B
```

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ISN 0195              T=T+DELT
             C.....NT=NUMBER OF ITERATIONS DONE SO FAR IN A RUN.
ISN 0196              NT=NT+1
             C*****END SECTION.
             C*****THIS SECTION COMPUTES THE CHANGES IN I AND J IN THE LIMIT OF
             C        INFINITE VISCOSITY (DARWIN 1880 PAGE 317.)
             C.....COMPUTATION IS NOT BEGUN UNTIL XI=1.0001 TO AVOID DIVIDING BY 0.
ISN 0197              IF (XI - 1.0001D0) 31,32,32
ISN 0198        31    DII=0.0D0
ISN 0199              DJJ=0.0D0
ISN 0200              GO TO 33
ISN 0201        32    L=C*OMEGA/(LT-LM)
ISN 0202              R1=4.0*L*(1.0-L)/(1.0-2.0*L)
ISN 0203              R2=0.5D0*(1.0/(LT-LM) + 1.0/LM)*(1.0 + R1)
ISN 0204              R3=-0.5D0*(1.0/(LT-LM) + 1.0/LM)*(1.0-R1)
ISN 0205              DII=DLM*R2*I
ISN 0206              DJJ=DLM*R3*J
ISN 0207        33    CONTINUE
             C*****END SECTION.
             C*****THIS SECTION PRINTS OUT THE NEW VALUES FOR THE IMPORTANT
             C        QUANTITIES.
ISN 0208              WRITE (6,3) T,XI,N,XPSI,OMEGA,DII,DJJ,XII,XJ,DI,DJ,DPSI
ISN 0209         3    FORMAT (1X,D10.4,1X,D13.8,1X,D10.5,1X,F7.3,2(1X,D10.5),1X,D10.4,
                    1 2(1X, F9.3),3(1X,D10.4))
             C*****END SECTION.
             C*****THIS SECTION DECIDES WHETHER CONSTANT DELT OR CONSTANT DLM SHOULD
             C        BE USED.
             C.....STATEMENT 12 GIVES CONSTANT STEP SIZE IN TIME.
             C.....STATEMENTS 13 AND 50 GIVE CONSTANT STEP SIZE IN LM.
ISN 0210              IF (NT - NQ)12,12,13
ISN 0211        13    DELT=DABS(0.0072089D0*XI**12/(A1*0.5D0*(T4+T6)))
ISN 0212              IF (NA .EQ. 2) GO TO 50
ISN 0214              GO TO 51
ISN 0215        50    DELT=DABS(0.0072089D0*XI**12/(A1*0.5D0*(T7-T8+T9-T10-T11)))
ISN 0216        51    CONTINUE
ISN 0217              GO TO 14
ISN 0218        12    DELT=DELT1
ISN 0219        14    CONTINUE
             C*****END SECTION.
             C.....SHOULD INTEGRATION BE FORWARD OR BACKWARD IN TIME?
ISN 0220              IF (MTIME .EQ. -1) DELT=-DELT
             C.....IS NLAST EXCEEDED?
ISN 0222              IF (NT-NLAST) 4,4,99
             C.....IS XIMAX EXCEEDED?
ISN 0223         4    IF (XI-XIMAX) 5,5,99
             C*****THIS SECTION COMPUTES THE TOTAL ANGULAR MOMENTUM AT THE END OF THE
             C        RUN. IT SERVES AS A CHECK ON HOW WELL THE ITERATION SCHEME WORKS.
ISN 0224        99    CC=DCOS(PSI)
ISN 0225              LTC=DSQRT(LE**2+LM**2+2.0*LE*LM*CC)
ISN 0226              WRITE (6,11) LT,LTC
ISN 0227        11    FORMAT (///10X,18HINITIAL ANG. MOM.=,D14.8,10X,16HFINAL ANG. MOM.=
                    1,D14.8)
ISN 0228       100    CONTINUE
ISN 0229              STOP
ISN 0230              END
```

APPENDIX D

COMPUTER PROGRAM FOR VARIABLE VISCOSITY

The computer program given in this appendix is discussed in Chapter III, Section D, and in the comments listed in the program itself.

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CC*******************
CC*******************
C*****THIS IS TIDE2
C*****   THIS PROGRAM INTEGRATES DARWINS (1880) EQUATIONS TO GIVE THE
C       EVOLUTION OF THE MOONS (CIRCULAR) ORBIT FOR A VARIABLE VISCOSITY
C       OF THE EARTH.
C            DARWIN (1880) IS
C       ON THE SECULAR CHANGES IN THE ELEMENTS OF THE ORBIT OF A SATELLITE
C       REVOLVING ABOUT A TIDALLY DISTORTED PLANET
C                   IN
C       SCIENTIFIC PAPERS BY SIR GEORGE HOWARD DARWIN, VOL. 2, PP 208-382
C       CAMBRIDGE UNIVERSITY PRESS, 1908.
C            THE EQUATIONS ARE ON PGS 240,241, AND 262 (EQUATIONS 71, 73,
C       AND THE TWO PRECEEDING EQUATION 75.)
C            THE PAPER CAN ALSO BE FOUND IN
C       PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY, VOL. 171, 1880,
C       PP 713 - 891.
C            THE PROGRAM ALLOWS TWO APPROACHES - TO USE ALL THE TERMS IN
C       DARWINS EQUATIONS, OR KEEP ONLY TERMS UP TO AND INCLUDING SECOND
C       ORDER IN K=SIN((I+J)/2). THE LATTER WE CALL THE SECOND ORDER
C       APPROXIMATION.
C            THE EQUATIONS CAN BE INTEGRATED FORWARD OR BACKWARD IN TIME,
C       DEPENDING UPON WHETHER MTIME=+1 OR -1.
C            THE PROGRAM STARTS BY INTEGRATING WITH CONSTANT TIME INTERVALS
C       IT DOES THIS FOR NQ STEPS. AFTER NQ STEPS IT SWITCHES OVER TO
C       CONSTANT LM INTERVALS (CHOSEN IN THE PROGRAM TO GIVE DXI= TO ABOUT
C       0.001.) ALL THIS ASSUMES THAT THE PER CENT CHANGE IN I OR J IS
C       LESS THAN CRIT. IF THE FRACTIONAL CHANGE IS GREATER THAN CRIT, THE
C       INTERVAL IS HALVED UNTIL THE FRACTIONAL CHANGE IS LESS THAN CRIT.
C            CRIT WAS INTRODUCED TO PREVENT LARGE CHANGES IN ANGLE TO AVOID
C       CUMULATIVE ERROR. THE STRATAGEM OF SWITCHING FROM CONSTANT DELT TO
C       CONSTANT DLM IS TO KEEP THE ITERATIONS FROM TAKING FOREVER, SINCE
C       AT LARGE XI DXI IS QUITE SMALL FOR CONSTANT DELT.
C
C.....SOME OF DARWINS NOTATION.
C.....C-ZERO =REFERENCE DISTANCE
C.....OMEGA-ZERO=OMEGA AT C-ZERO
C.....SMALL K=C*(OMEGA-ZERO)*(C-ZERO)/(BIG G)*(BIG M)*(SMALL M)
C.....TAU-ZERO=(3/2)*(BIG G)*(SMALL M)/(C-ZERO)**3
C.....GOTHIC SMALL G=(2/5)*(SMALL G)/(SMALL A)
C.....BIG G=UNIVERSAL GRAVITATIONAL CONSTANT
C.....SMALL A=RADIUS OF THE EARTH
C.....BIG M=MASS OF THE EARTH
C.....SMALL M=MASS OF THE MOON
C.....SMALL G=GRAVITATIONAL CONSTANT AT THE EARTHS SURFACE
C.....W IS THE DENSITY OF THE EARTH
C.....WE HAVE SUBSTITUTED BIG G FOR DARWINS MU ABOVE.
C
C*****THIS SECTION DEFINES THE MOST IMPORTANT QUANTITIES.
C.....XI IS SQRT(EARTH-MOON DISTANCE/REFERENCE DISTANCE).
C.....DZERO IS THE REFERENCE DISTANCE, HERE IN UNITS OF EARTH RADII,
C       AND WHERE N=2*OMEGA.
C.....DS=SQRT(DZERO).  C-ZERO=SMALL A*DS.
C.....N IS THE ROTATIONAL ANGULAR VELOCITY OF THE EARTH IN 10**-4 /SEC
C       (I.E. MULTIPLY THE VALUE GIVEN IN THE PROGRAM BY 10**-4 TO GET THE
C       VALUE IN CGS UNITS.)
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C.....OMEGA IS THE ORBITAL ANGULAR VELOCITY OF THE MOON IN UNITS OF
C     10**-4 /SEC.
C.....PSI IS THE ANGLE BETWEEN THE EARTHS EQUATORIAL PLANE AND THE
C     PLANE OF THE MOONS ORBIT. PSI=I + J.
C.....I IS THE ANGLE BETWEEN THE EARTHS EQUATORIAL PLANE AND THE
C     INVARIABLE PLANE.
C.....J IS THE ANGLE BETWEEN THE MOONS ORBITAL PLANE AND THE INVARIABLE
C     PLANE.
C.....PSI, I, AND J ARE IN RADIANS. XPSI,XII, AND XJ ARE THE SAME ANGLES
C     IN DEGREES.
C.....DPSI, DI, AND DJ ARE THE CHANGES IN THE RESPECTIVE ANGLES.
C.....VIS IS THE EARTHS VISCOSITY IN UNITS OF 10**16 CGS.
C.....LM IS THE ORBITAL ANGULAR MOMENTUM OF THE SYSTEM IN UNITS OF
C     10**40 CGS.
C.....LE IS THE ROTATIONAL ANGULAR MOMENTUM OF THE EARTH IN UNITS OF
C     10**40 CGS. LE=C*N.
C.....DLM AND DLE ARE THE RESPECTIVE CHANGES IN LM AND LE.
C.....LT IS THE TOTAL ANGULAR MOMENTUM OF THE SYSTEM IN UNITS OF
C     10**40 CGS.
C.....C IS THE MOMENT OF INERTIA OF THE EARTH IN UNITS OF 10**44 CGS.
C     C=SMALL K*B.
C.....T IS THE TIME IN UNITS OF 10**9 SEC. DELT IS THE CHANGE IN T.
C*****END SECTION.
C
C.....A1 IS (SMALL K)*(TAU-ZERO)**2/(GOTHIC SMALL G) (IN DARWINS
C     NOTATION) TIMES B (AS DEFINED HERE) IN UNITS OF 10**31 CGS.
C.....B IS SQRT((BIG G*SMALL A)/(BIG M+SMALL M))*(BIG M)*(SMALL M)*DS IN
C     UNITS OF 10**40 CGS. LM=B*XI.
C.....A2 IS 19/(2*(SMALL G)*(SMALL A)*(SMALL W))IN UNITS OF 10**-12 CGS.
C.....A3 IS SQRT((BIG G)*(BIG M+SMALL M)/(SMALL A)**3)/DS**3 IN UNITS OF
C     10**-4 SEC. OMEGA=A3/(XI**3).
C
C*****THE TIME VARIATION OF THE VISCOSITY IS GIVEN BY
C           VIS=VISZ*EXP(BB/TEMP)
C.....VISZ=COEFFICIENT OF VISCOSITY IN UNITS OF 10**16 CGS.
C.....BB=ACTIVATION TEMPERATURE IN DEGREES KELVIN.
C.....TEMP IS THE TEMPERATURE OF THE EARTH IN DEGREES KELVIN.
C.....THE TIME VARIATION OF TEMPERATURE OF THE EARTH IS GIVEN BY
C     TEMP=TEMPZ/((1.0+BETA*(TEMPZ**3)*(T-TSTART))**0.333)
C.....TEMPZ=TEMPERATURE OF THE EARTH AT TIME TSTART, GIVEN IN DEGREES K.
C.....BETA=3*(AREA OF EARTH)*(STEFAN-BOLTZ. CONST)/(BIG M*SPECIFIC HEAT)
C           TIMES
C     SURFACE TEMP./AVG. TEMP. OF EARTH, ROUGHLY FROM 1/2 TO MAYBE 1/5.
C     FOR SPF. HEAT=10**7 ERG/GRAM*DEG BETA=1.47*10**-11/DEG**3*10**9SEC
C.....TLOWER IS THE LOWEST TEMPERATURE PERMITTED. IF TEMP DROPS BELOW
C     TLOWER IN THE EQUATION FOR TEMP, THEN TLOWER IS USED IN THE
C     VISCOSITY EQUATION. TLOWER IN DEGREES KELVIN.
C
C*****THIS SECTION EXPLAINS THE INITIAL INPUT DATA.
C.....NRUN IS THE NUMBER OF RUNS TO BE MADE. ALL INITIAL DATA FOR A RUN
C     IS READ FROM TWO CONSECUTIVE CARDS.
C.....CRIT IS THE MAXIMUM CHANGE PERMITTED IN THE ABSOLUTE VALUES OF
C     DI/I AND DJ/J IN A SINGLE STEP.
C.....MTIME=+1 FOR A BATCH OF RUNS INTEGRATED FORWARD IN TIME, AND
C     MTIME=-1 FOR INTEGRATION BACKWARD IN TIME.
C.....ANGLE = INITIAL VALUE FOR PSI=I + J  IN RADIANS.
C.....XIF = INITIAL VALUE OF XI.
C.....VISF = VISCOSITY OF THE EARTH IN UNITS OF 10**16 CGS.
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                 C     VISF IS NOT USED IN THIS PROGRAM, SO ANY VALUE MAY BE USED.
                 C.....DELT1F = STEP SIZE IN TIME IN UNITS OF 10**9 SEC.
                 C     DELT1F SHOULD ALWAYS BE POSITIVE.
                 C.....XIMAX = THE MAXIMUM VALUE OF XI TO WHICH THE PROGRAM INTEGRATES.
                 C     THE RUN STOPS WHEN XI = XIMAX.
                 C.....TSTART = INITIAL VALUE OF TIME IN UNITS OF 10**9 SEC.
                 C.....NP = THE NUMBER OF ITERATIONS DESIRED USING CONSTANT STEP SIZE IN
                 C     TIME.
                 C.....NF=1, NL=1 MEANS THE PROGRAM USES THE SECOND ORDER APPROXIMATION.
                 C.....NF=2, NL=2 MEANS THE PROGRAM USES ALL THE TERMS IN DARWINS
                 C     EQUATIONS.
                 C.....NF=1, NL=2 MEANS THE PROGRAM RUNS BOTH THE SECOND ORDER
                 C     APPROXIMATION AND DARWINS FULL EQUATIONS ON THE INITIAL DATA.
                 C.....NC1=1 GIVES A CHECK ON THE DATA RUN FOR THE SECOND ORDER
                 C     APPROXIMATION BY HALVING THE STEP SIZES AND DOUBLING THE NUMBER OF
                 C     STEPS AND REPEATING THE RUN. IF THE CHECK IS NOT DESIRED, NC1=0.
                 C.....NC2 PERFORMS THE SAME FUNCTION AS NC1 FOR THE FULL EQUATIONS.
                 C.....NLAST IS THE TOTAL NUMBER OF STEPS PERMITTED IN ANY ONE RUN. THE
                 C     RUN TERMINATES IF NLAST IS EXCEEDED.
                 C*****END SECTION.
ISN 0002               DOUBLE PRECISION DZERO,C,LT,A1,A2,A3,A4,B,ANGLE,XI,VIS,LM,CC,
                     1 LE,N,OMEGA,P,K,TF1,SF1,TG,SG,TG1,SG1,DLE,DLM,    LTC,T,DELT,DS,
                     2 XIMAX
ISN 0003               DOUBLE PRECISION AY,DI,DJ,DELT1
ISN 0004               DOUBLE PRECISION TF,SF,TF2,SF2,TG2,SG2,TH,SH
ISN 0005               DOUBLE PRECISION PSI,DPSI
ISN 0006               DOUBLE PRECISION XIF,VISF,DELT1F,TSTART
ISN 0007               DOUBLE PRECISION T1,T2,T3,T4,T5,T6,T7,T8,T9,T10,T11,T12,T13,T14,
                     1 T15,T16,T17,T18,T19,T20,T21,T22,T23,T24,T25
ISN 0008               DOUBLE PRECISION SS,SJ,J,I
ISN 0009               DOUBLE PRECISION L,R1,R2,R3,DII,DJJ
ISN 0010               DOUBLE PRECISION TEMP,TEMPZ,BETA,TLOWER,BB,E1,EE,VISZ
ISN 0011               DZERO=3.83387305D0
ISN 0012               C=8.11D0
ISN 0013               LT=34.2D0
ISN 0014               DS=DSQRT(DZERO)
ISN 0015               A1=1.313157D4*C/DZERO**6
ISN 0016               A2=2.756D0
ISN 0017               B=3.681701D0*DS
ISN 0018               A3=12.49185D0/(DZERO*DS)
ISN 0019               U1=0.0D0
ISN 0020               READ (5,1) NRUN,CRIT,MTIME
ISN 0021         1     FORMAT (I5,F10.2,I5)
ISN 0022               DO 100 NR=1,NRUN
                 C*****THIS SECTION READS IN THE INITIAL DATA. ANGLE IS IN RADIANS.
ISN 0023               READ (5,2) ANGLE,XIF,VISF,DELT1F,XIMAX,TSTART,NP,NF,NL,NC1,NC2,
                     1 NLAST
ISN 0024         2     FORMAT (D9.2,D11.2,4D10.2,I5,4I2,I7)
ISN 0025               READ (5,52) BB,VISZ,TEMPZ,BETA,TLOWER
ISN 0026         52    FORMAT (5D10.4)
                 C*****END SECTION.
ISN 0027               DO 100 NA=NF,NL
ISN 0028               IF (NA .EQ. 1) ND=NC1
ISN 0030               IF (NA .EQ. 2) ND=NC2
ISN 0032               NC= 1 + ND
ISN 0033               DO 100 NB=1,NC
ISN 0034               NCHECK=-1
ISN 0035         40    XI=XIF
```

```
ISN 0036            NQ=NP
ISN 0037            IF (NB .EQ. 2) NQ=2*NP
ISN 0039            VIS=VISF
ISN 0040            DELT1=DELT1F
ISN 0041            T=TSTART
          C*****THIS SECTION COMPUTES THE INITIAL VISCOSITY.
ISN 0042            TEMP=TEMPZ/((1.0+BETA*(TEMPZ**3)*(T-TSTART))**0.333)
ISN 0043            IF (TEMP - TLOWER) 50,50,51
ISN 0044       50   TEMP=TLOWER
ISN 0045       51   E1=BB/TEMP
ISN 0046            EE=DEXP(E1)
ISN 0047            VIS=VISZ*EE
          C*****END SECTION.
ISN 0048            A4=VIS*A2
          C*****THIS SECTION WRITES OUT THE INPUT DATA.
ISN 0049            WRITE (6,6)
ISN 0050        6   FORMAT (1H1)
ISN 0051            WRITE (6,53) BB,VISZ,BETA
ISN 0052       53   FORMAT(5X,3HBB=,D10.4,1X,7HDEGREES,5X,5HVISZ=,D10.4,1X,10H10**16 C
                   1GS,5X,5HBETA=,D10.4,1X,10H/10**9 SEC,/)
ISN 0053            WRITE (6,54) TEMPZ,TLOWER
ISN 0054       54   FORMAT (5X,6HTEMPZ=,D10.4,1X,7HDEGREES,5X,7HTLOWER=,D10.4,1X,7HDEG
                   1REES,//)
ISN 0055            WRITE (6,18) DELT1F,XIMAX,NQ,NF,NL,NC1,NC2,NLAST
ISN 0056       18   FORMAT (5X,6HDELT1=,D10.3,1X,9H10**9 SEC,5X,6HXIMAX=,D10.3,5X,
                   1 3HNQ=,I5,5X,3HNF=,I1,5X,3HNL=,I1,5X,4HNC1=,I1,5X,4HNC2=,I1,5X,
                   2 6HNLAST=,I5,///)
ISN 0057            WRITE (6,57) CRIT,MTIME
ISN 0058       57   FORMAT (5X,5HCRIT=,F10.4,5X,6HMTIME=,I3,///)
          C*****END SECTION.
          C*****THIS SECTION WRITES THE HEADING.
          C.....ALL QUANTITIES PRINTED OUT ARE IN UNITS GIVEN IN THE PROGRAM.
          C     NOTE THAT UNDER HEADINGS PSI, I, AND J XPSI, XII, AND XJ ARE
          C     PRINTED, THUS GIVING ALL ANGLES IN DEGREES.
ISN 0059            WRITE (6,7)
ISN 0060        7   FORMAT (6X,4HTIME,9X,2HXI,9X,1HN,8X,5H PSI ,3X,5HOMEGA,7X,3HVIS,9X
                   1,4HTEMP,5X,3H  I,8X,3H  J,8X,2HDI,9X,2HDJ,9X,4HDPSI,///)
          C*****END SECTION.
          C*****THIS SECTION COMPUTES THE INITIAL VALUES OF LE, N, OMEGA, I, AND J
ISN 0061            NT=0
ISN 0062            LM=B*XI
ISN 0063            CC=DCOS(ANGLE)
ISN 0064            PSI=ANGLE
ISN 0065            XPSI=180.0*ANGLE/3.14159
ISN 0066       15   DELT=DELT1
ISN 0067       17   CONTINUE
ISN 0068       22   LE=-LM*CC + DSQRT((LM*CC)**2 + LT**2 - LM**2)
ISN 0069            XI=LM/B
ISN 0070            N=LE/C
ISN 0071            OMEGA=A3/(XI**3)
ISN 0072            SS=DSIN(ANGLE)
ISN 0073            SJ=SS/DSQRT(SS**2 + (CC + LM/LE)**2)
ISN 0074            J=DARSIN(SJ)
ISN 0075            I=ANGLE-J
ISN 0076            XII=180.0*I/3.14159
ISN 0077            XJ=180.0*J/3.14159
          C*****END SECTION.
ISN 0078            WRITE (6,3) T,XI,N,XPSI,OMEGA,VIS,TEMP,XII,XJ
```

```
                  CC*******************
                  CC*******************
ISN 0079     5    AY=0.5D0*PSI
                  C*****THIS SECTION COMPUTES THE VISCOSITY.
ISN 0080          TEMP=TEMPZ/((1.0+BETA*(TEMPZ**3)*(T-TSTART))**0.333)
ISN 0081          IF (TEMP - TLOWER) 55,55,56
ISN 0082    55    TEMP=TLOWER
ISN 0083    56    E1=BB/TEMP
ISN 0084          EE=DEXP(E1)
ISN 0085          VIS=VISZ*EE
                  C*****END SECTION.
ISN 0086          A4=VIS*A2
ISN 0087          IF (NA .EQ. 1) GO TO 26
                  C*****THIS SECTION COMPUTES ALL THE TERMS IN DARWINS EQUATIONS.
ISN 0089          K=DSIN(AY)
ISN 0090          P=DCOS(AY)
                  C.....COMPUTE THE TANGENTS, SINES OF THE LAG ANGLES.
ISN 0091          TF=2.0*N*A4
ISN 0092          SF=2.0*TF/(1.0+TF**2)
ISN 0093          TF2=2.0*(N+OMEGA)*A4
ISN 0094          SF2=2.0*TF2/(1.0+TF2**2)
ISN 0095          TG2=(N+2.0*OMEGA)*A4
ISN 0096          SG2=2.0*TG2/(1.0+TG2**2)
ISN 0097          TH=2.0*OMEGA*A4
ISN 0098          SH=2.0*TH/(1.0+TH**2)
ISN 0099          TF1=2.0*(N-OMEGA)*A4
ISN 0100          SF1=2.0*TF1/(1.0+TF1**2)
ISN 0101          TG=N*A4
ISN 0102          SG=2.0*TG/(1.0+TG**2)
ISN 0103          TG1=(N-2.0*OMEGA)*A4
ISN 0104          SG1=2.0*TG1/(1.0+TG1**2)
                  C.....COMPUTE THE TERMS.
ISN 0105          T1=0.5D0*P**8*SF1
ISN 0106          T2=2.0*P**4*K**4*SF
ISN 0107          T3=0.5D0*K**8*SF2
ISN 0108          T4=P**6*K**2*SG1
ISN 0109          T5=P**2*K**2*((P**2-K**2)**2)*SG
ISN 0110          T6=P**2*K**6*SG2
ISN 0111          T7=P**8*SF1
ISN 0112          T8=K**8*SF2
ISN 0113          T9=4.0*P**6*K**2*SG1
ISN 0114          T10=4.0*P**2*K**6*SG2
ISN 0115          T11=6.0*P**4*K**4*SH
ISN 0116          T12=0.5D0*P**7*K*SF1
ISN 0117          T13=P**3*K**3*SF
ISN 0118          T14=0.5D0*P*K**7*SF2
ISN 0119          T15=1.5D0*P**3*K**3*(P**2-K**2)*SH
ISN 0120          T16=0.5D0*P**5*K*(P**2-3.0*K**2)*SG1
ISN 0121          T17=0.5D0*P*K*((P**2-K**2)**2)*SG
ISN 0122          T18=0.5D0*P*K**5*(3.0*P**2-K**2)*SG2
ISN 0123          T19=0.5D0*P**7*K*SF1
ISN 0124          T20=P**3*K**3*(P**2-K**2)*SF
ISN 0125          T21=0.5D0*P*K**7*SF2
ISN 0126          T22=0.5D0*P**5*K*(P**2+3.0*K**2)*SG1
ISN 0127          T23=0.5D0*P*K*((P**2-K**2)**3)*SG
ISN 0128          T24=0.5D0*P*K**5*(3.0*P**2+K**2)*SG2
ISN 0129          T25=1.5D0*P**3*K**3*SH
                  C*****END SECTION.
```

```
                  C*****THIS SECTION COMPUTES THE CHANGES IN LE, LM, J, AND I FOR DARWINS
                  C     FULL EQUATIONS.
ISN 0130      37        DLE=-A1*(T1+T2+T3+T4+T5+T6)*DELT/(XI**12)
ISN 0131                DLM=0.5D0*A1*(T7-T8+T9-T10-T11)*DELT/(XI**12)
ISN 0132                DJ=-A1*(T12+T13+T14+T15-T16+T17+T18)*DELT/(LM*XI**12)
ISN 0133                DI=A1*(T19-T20-T21+T22-T23-T24-T25)*DELT/(LE*XI**12)
                  C*****END SECTION.
ISN 0134                GO TO 27
                  C*****THIS SECTION COMPUTES THE TERMS IN THE SECOND ORDER APPROXIMATION.
ISN 0135      26        K=DSIN(AY)
ISN 0136                P=DCOS(AY)
                  C.....COMPUTE THE TANGENTS, SINES OF THE LAG ANGLES.
ISN 0137                TF1=2.0*(N-OMEGA)*AA
ISN 0138                SF1=2.0*TF1/(1.0 + TF1**2)
ISN 0139                TG=N*AA
ISN 0140                SG=2.0*TG/(1.0 + TG**2)
ISN 0141                TG1=(N-2.0*OMEGA)*AA
ISN 0142                SG1=2.0*TG1/(1.0 + TG1**2)
ISN 0143                TH=2.0*OMEGA*AA
ISN 0144                SH=2.0*TH/(1.0 + TH**2)
ISN 0145                TF=2.0*N*AA
ISN 0146                SF=2.0*TF/(1.0 + TF**2)
                  C.....COMPUTE THE TERMS.
ISN 0147                T1=0.5D0*P**8*SF1
ISN 0148                T2=P**6*K**2*SG1
ISN 0149                T3=P**6*K**2*SG
ISN 0150                T4=P**8*SF1
ISN 0151                T5=4.0*P**6*K**2*SG1
ISN 0152                T6=0.5D0*P**7*K*SF1
ISN 0153                T7=P**3*K**3*SF
ISN 0154                T8=1.5D0*P**5*K**3*SH
ISN 0155                T9=0.5D0*P**7*K*SG1
ISN 0156                T10=1.5D0*P**5*K**3*SG1
ISN 0157                T11=0.5D0*P**5*K*SG
ISN 0158                T12=P**3*K**3*SG
ISN 0159                T13=0.5D0*P**7*K*SF1
ISN 0160                T14=P**5*K**3*SF
ISN 0161                T15=0.5D0*P**7*K*SG1
ISN 0162                T16=1.5D0*P**5*K**3*SG1
ISN 0163                T17=0.5D0*P**7*K*SG
ISN 0164                T18=1.5D0*P**5*K**3*SG
ISN 0165                T19=1.5D0*P**3*K**3*SH
                  C*****END SECTION.
                  C*****THIS SECTION COMPUTES THE CHANGES IN LE, LM, J, AND I FOR THE
                  C     SECOND ORDER APPROXIMATION.
ISN 0166      36        DLE=-A1*(T1+T2+T3)*DELT/(XI**12)
ISN 0167                DLM=A1*0.5D0*(T4+T5)*DELT/(XI**12)
ISN 0168                DJ=A1*0.5D0*(P**7*K*SG1-P**7*K*SF1-P**5*K*SG)*DELT/(LM*XI**12)
ISN 0169                DI=A1*0.5D0*(P**7*K*SG1+P**7*K*SF1-P**7*K*SG)*DELT/(LE*XI**12)
ISN 0170      27        CONTINUE
                  C*****END SECTION.
                  C*****THIS SECTION INSURES THAT DI/I OR DJ/J NEVER EXCEEDS CRIT.
ISN 0171                IF (I - 0.0) 19,19,38
ISN 0172      38        IF (J - 0.0) 19,19,39
ISN 0173      39        RI=DI/I
ISN 0174                IF (RI - 0.0) 47,46,46
ISN 0175      47        RI=-RI
ISN 0176      46        CONTINUE
```

```
ISN 0177            RJ=DJ/J
ISN 0178            IF (RJ - 0.0) 49,48,48
ISN 0179      49    RJ=-RJ
ISN 0180      48    CONTINUE
ISN 0181            IF (RI - CRIT) 34,34,35
ISN 0182      34    IF (RJ - CRIT) 19,19,35
ISN 0183      35    DELT=DELT/2.0
ISN 0184            IF (NA .EQ. 1) GO TO 36
ISN 0186            IF (NA .EQ. 2) GO TO 37
ISN 0188      19    CONTINUE
              C*****END SECTION.
              C*****THIS SECTION HALVES THE STEP SIZE WHEN NC1 OR NC2 EQUALS 1. NCHECK
              C     KEEPS TRACK OF WHETHER THE STEP SIZE IS HALVED FOR THE CHECK.
ISN 0189            IF (NB .EQ. 2) GO TO 41
ISN 0191            GO TO 42
ISN 0192      41    NCHECK=-NCHECK
ISN 0193            IF (NCHECK .EQ. 1) GO TO 43
ISN 0195            GO TO 42
ISN 0196      43    DELT=DELT/2.0
ISN 0197            IF (NA .EQ. 1) GO TO 36
ISN 0199            IF (NA .EQ. 2) GO TO 37
ISN 0201      42    CONTINUE
              C*****END SECTION.
              C*****THIS SECTION INCREMENTS THE IMPORTANT QUANTITIES.
ISN 0202            DPSI=DI + DJ
ISN 0203            I=I + DI
ISN 0204            J=J + DJ
ISN 0205            XII=180.0*I/3.14159
ISN 0206            XJ=180.0*J/3.14159
ISN 0207            PSI=PSI + DI + DJ
ISN 0208            XPSI=180.0*PSI/3.14159
ISN 0209            LE=LE+DLE
ISN 0210            LM=LM+DLM
ISN 0211            N=LE/C
ISN 0212            OMEGA=A3/(XI**3)
ISN 0213            XI=LM/B
ISN 0214            T=T+DELT
              C*****END SECTION.
              C.....NT=NUMBER OF ITERATIONS DONE SO FAR IN A RUN.
ISN 0215            NT=NT+1
              C*****THIS SECTION COMPUTES THE CHANGES IN I AND J IN THE LIMIT OF
              C     INFINITE VISCOSITY (DARWIN 1880 PAGE 317.)
              C.....COMPUTATION IS NOT BEGUN UNTIL XI=1.0001 TO AVOID DIVIDING BY 0.
              C.....THIS SECTION IS USED ONLY IN THE CONSTANT VISCOSITY PROGRAM.
ISN 0216            GO TO 33
ISN 0217            IF (XI - 1.0001D0) 31,32,32
ISN 0218      31    DII=0.0D0
ISN 0219            DJJ=0.0D0
ISN 0220            GO TO 33
ISN 0221      32    L=C*OMEGA/(LT-LM)
ISN 0222            R1=4.0*L*(1.0-L)/(1.0-2.0*L)
ISN 0223            R2=0.5D0*(1.0/(LT-LM) + 1.0/LM)*(1.0 + R1)
ISN 0224            R3=-0.5D0*(1.0/(LT-LM) + 1.0/LM)*(1.0-R1)
ISN 0225            DII=DLM*R2*I
ISN 0226            DJJ=DLM*R3*J
ISN 0227      33    CONTINUE
              C*****END SECTION.
              C*****THIS SECTION PRINTS OUT THE NEW VALUES FOR THE IMPORTANT
```

```
              C     QUANTITIES.
ISN 0228            WRITE (6,3) T,XI,N,XPSI,OMEGA,VIS,TEMP,XII,XJ,DI,DJ,DPSI
ISN 0229       3    FORMAT (1X,D10.4,1X,D13.8,1X,D10.5,1X,F7.3,2(1X,D10.5),1X,D10.4,
                   1 2(1X, F9.3),3(1X,D10.4))
              C*****END SECTION.
              C*****THIS SECTION DECIDES WHETHER CONSTANT DELT OR CONSTANT DLM SHOULD
              C     BE USED.
              C.....STATEMENT 12 GIVES CONSTANT STEP SIZE IN TIME.
              C.....STATEMENTS 13 AND 60 GIVE CONSTANT STEP SIZE IN LM.
ISN 0230            IF (NT - NO)12,12,13
ISN 0231      13    DELT=DABS(0.007208900*XI**12/(A1*0.5D0*(T4+T5)))
ISN 0232            IF (NA .EQ. 2) GO TO 60
ISN 0234            GO TO 61
ISN 0235      60    DELT=DABS(0.007208900*XI**12/(A1*0.5D0*(T7-T8+T9-T10-T11)))
ISN 0236      61    CONTINUE
ISN 0237            GO TO 14
ISN 0238      12    DELT=DELT1
ISN 0239      14    CONTINUE
              C*****END SECTION.
              C.....SHOULD INTEGRATION BE FORWARD OR BACKWARD IN TIME?
ISN 0240            IF (MTIME .EQ. -1) DELT=-DELT
              C.....IS NLAST EXCEEDED?
ISN 0242            IF (NT-NLAST) 4,4,99
              C.....IS XIMAX EXCEEDED?
ISN 0243       4    IF (XI-XIMAX) 5,5,99
              C*****THIS SECTION COMPUTES THE TOTAL ANGULAR MOMENTUM AT THE END OF THE
              C     RUN. IT SERVES AS A CHECK ON HOW WELL THE ITERATION SCHEME WORKS.
ISN 0244      99    CC=DCOS(PSI)
ISN 0245            LTC=DSQRT(LE**2+LM**2+2.0*LE*LM*CC)
ISN 0246            WRITE (6,11) LT,LTC
ISN 0247      11    FORMAT (///10X,18HINITIAL ANG. MOM.=,D14.8,10X,16HFINAL ANG. MOM.=
                   1,D14.8)
ISN 0248     100    CONTINUE
ISN 0249            STOP
ISN 0250            END
```

APPENDIX E

COMPUTER PROGRAM FOR SOLAR INFLUENCE

The computer program given in this appendix is discussed in Chapter IV, and in the comments listed in the program itself.

```
C*****THIS IS TIDE4
C*****     THIS PROGRAM INTEGRATES THE FIRST OF DARWINS (1880) EQUATIONS
C      (250) TJ FIND THE ANGLE BETWEEN THE PLANE OF THE MOONS ORBIT AND
C      THE PROPER PLANE FOR ANY CHOSEN VISCOSITY OF THE EARTH.
C          J IS THE ANGLE BETWEEN THE PLANE OF THE LUNAR ORBIT AND THE
C      PROPER PLANE.
C          DARWIN (1880) IS
C      ON THE SECULAR CHANGES IN THE ELEMENTS OF THE ORBIT OF A SATELLITE
C      REVOLVING ABOUT A TIDALLY DISTORTED PLANET
C                IN
C      SCIENTIFIC PAPERS BY SIR GEORGE HOWARD DARWIN. VOL. 2. PP 208-382
C      CAMBRIDGE UNIVERSITY PRESS. 1908.
C          THE PAPER CAN ALSO BE FOUND IN
C      PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY. VOL. 171. 1880.
C      PP 713 - 891.
C
C.....THE PROGRAM INTEGRATES THE EQUATION FROM XI=3.95 TO XI=1.0 (60 TO
C      3.8 EARTH RADII).
C.....IF INTEGRATING FROM LARGE XI TO SMALL XI, THE INTEGRAL IS FOUND BY
C      SUBTRACTING THE NUMBER IN THE SUM COLUMN AT SMALL XI FROM THE
C      NUMBER IN THE SUM COLUMN AT LARGE XI. THE ANGLE J AT SMALL XI IS
C      THEN THE EXPONENT OF THE INTEGRAL TIMES J AT LARGE XI.
C
C*****THIS SECTION DEFINES THE MOST IMPORTANT QUANTITIES.
C.....XI IS SQRT(EARTH-MOON DISTANCE/REFERENCE DISTANCE).
C.....DXI IS THE CHANGE IN XI.
C.....DZERO IS THE REFERENCE DISTANCE, HERE IN UNITS OF EARTH RADII.
C      AND WHERE N=2*OMEGA.
C.....DS=SQRT(DZERO).  C-ZERO=SMALL A*DS.
C.....N IS THE ROTATIONAL ANGULAR VELOCITY OF THE EARTH IN 10**-4 /SEC
C      (I.E. MULTIPLY THE VALUE GIVEN IN THE PROGRAM BY 10**-4 TO GET THE
C      VALUE IN CGS UNITS.)
C.....OMEGA IS THE ORBITAL ANGULAR VELOCITY OF THE MOON IN UNITS OF
C      10**-4 /SEC.
C.....VIS IS THE EARTHS VISCOSITY IN UNITS OF 10**16 CGS.
C.....LT IS THE TOTAL ANGULAR MOMENTUM OF THE SYSTEM IN UNITS OF
C      10**40 CGS.
C.....C IS THE MOMENT OF INERTIA OF THE EARTH IN UNITS OF 10**44 CGS.
C      C=SMALL K*B.
C.....B IS SQRT((BIG G*SMALL A)/(BIG M+SMALL M))*(BIG M)*(SMALL M)*DS IN
C      UNITS OF 10**40 CGS. LM=B*XI.
C.....A2 IS 19/(2*(SMALL G)*(SMALL A)*(SMALL W))IN UNITS OF 10**-12 CGS.
C.....A3 IS SQRT((BIG G)*(BIG M+SMALL M)/(SMALL A)**3)/DS**3 IN UNITS OF
C      10**-4 SEC. OMEGA=A3/(XI**3).
C*****END SECTION.
C
C.....SOME OF DARWINS NOTATION.
C.....C-ZERO =REFERENCE DISTANCE
C.....OMEGA-ZERO=OMEGA AT C-ZERO
C.....SMALL K=C*(OMEGA-ZERO)*(C-ZERO)/(BIG G)*(BIG M)*(SMALL M)
C.....TAU-ZERO=(3/2)*(BIG G)*(SMALL M)/(C-ZERO)**3
C.....GOTHIC SMALL G=(2/5)*(SMALL G)/(SMALL A)
C.....BIG G=UNIVERSAL GRAVITATIONAL CONSTANT
C.....SMALL A=RADIUS OF THE EARTH
C.....BIG M=MASS OF THE EARTH
C.....SMALL M=MASS OF THE MOON
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```
              C.....SMALL G=GRAVITATIONAL CONSTANT AT THE EARTHS SURFACE
              C.....W IS THE DENSITY OF THE EARTH
              C.....WE HAVE SUBSTITUTED BIG G FOR DARWINS MU ABOVE.
              C
              C.....CORRESPONDENCE BETWEEN OUR NOTATION AND DARWINS.
              C      OUR XI IS DARWINS GREEK LETTER XI.
              C      TP - TAU PRIME
              C      TZERO - TAU ZERO FOR THE MOON
              C      GG - GOTHIC SMALL G
              C      LDA - GREEK LETTER LAMBDA
              C      E - GOTHIC SMALL E
              C      T - GREEK LETTER TAU
              C      M - GOTHIC SMALL M
              C      K1 - KAPPA SUB-1 , K2 - KAPPA SUB-2
              C
ISN 0002             DOUBLE PRECISION DXI,VIS,XI,DZERO,C,LT,DS,A2,TP,TZERO,GG,A3,B,A4,
                    1 S,OMEGA,N,LDA,E,T,M,Y,TG,SG,TG1,SG1,TF,SF,TF1,SF1,ALPHA,A,BETA,
                    2 BL,ALPHP,AP,BETAP,BP,GAM,DELTA,TERM,K1,K2,X1,X2,X3,Z1,X4,X5,X6,
                    3 Z2,DLOGJ
ISN 0003             DXI=0.0025D0
ISN 0004             DZERO=3.83387305D0
ISN 0005             C=8.11D0
ISN 0006             LT=34.2D0
ISN 0007             DS=DSQRT(DZERO)
ISN 0008             A2=2.756D0
ISN 0009             TP=5.946692D-14
ISN 0010             TZERO=0.28434D-7/(DZERO**3)
ISN 0011             GG=6.156862D-7
ISN 0012             A3=12.49185D0/(DZERO*DS)
ISN 0013             B=3.681701D0*DS
              C.....NVIS IS THE NUMBER OF VISCOSITIES TO BE READ.
ISN 0014             READ (5,4) NVIS
ISN 0015         4   FORMAT (I5)
ISN 0016             DO 3 J=1,NVIS
              C.....VIS IS THE CHOSEN VISCOSITY OF THE EARTH.
              C.....READ IN THE VISCOSITY.
ISN 0017             READ (5,5) VIS
ISN 0018         5   FORMAT (D10.5)
ISN 0019             WRITE (6,1)
ISN 0020         1   FORMAT (1H1)
ISN 0021             WRITE (6,6) VIS
ISN 0022         6   FORMAT (///,20X,10HVISCOSITY=,D10.5,1X,10H10**16 CGS,///)
ISN 0023             WRITE (6,7)
ISN 0024         7   FORMAT (///,15X,2HXI,11X,5HDLOGJ,12X,3HSUM,///)
ISN 0025             A4=VIS*A2
              C.....EVALUATE THE INTEGRAL.
ISN 0026             S=0.0
ISN 0027             XI=3.96D0
ISN 0028             DO 3 I=1,1188
ISN 0029             OMEGA=A3/(XI**3)
              C.....ANGULAR MOMENTUM OF THE EARTH = C*N.
              C.....ORBITAL ANGULAR MOMENTUM = B*XI.
              C.....N IS COMPUTED BY ASSUMING THE MOON STAYS IN THE EQUATORIAL PLANE.
ISN 0030             N=(LT-B*XI)/C
ISN 0031             LDA=OMEGA/N
ISN 0032             E=(0.5D0)*(N**2)*1.0D-8/(GG)
ISN 0033             T=TZERO/(XI**6)
ISN 0034             M=C*N/(B*XI)
```

```
ISN 0035              Y=TP/T
            C.....HERE THE TANGENTS AND SINES OF THE LAG ANGLES ARE COMPUTED.
ISN 0036              TG=N*A4
ISN 0037              SG=2.0*TG/(1.0 + TG**2)
ISN 0038              TG1=(N-2.0*OMEGA)*A4
ISN 0039              SG1=2.0*TG1/(1.0 + TG1**2)
ISN 0040              TF=2.0*N*A4
ISN 0041              SF=2.0*TF/(1.0 + TF**2)
ISN 0042              TF1=2.0*(N-OMEGA)*A4
ISN 0043              SF1=2.0*TF1/(1.0 + TF1**2)
            C.....HERE THE TERMS IN THE EQUATION ARE COMPUTED. THE NOTATION IS
            C     STRAIGHTFORWARD.
ISN 0044              ALPHA=M + Y*(1.0/(2.0*LDA*E))
ISN 0045              A=M
ISN 0046              BETA=1.0 + Y
ISN 0047              BL=1.0
ISN 0048              ALPHP=M*(Y*(3.0/(2.0*LDA*E))-(2.0*(1.0+Y**2) + 7.0*M))
ISN 0049              AP=-M*(2.0*(1.0+Y**2) + 7.0*M)
ISN 0050              BETAP=-(1.0 + Y + Y**2 + Y**3 + 6.0*M)
ISN 0051              BP=-(1.0 + Y**2 + 6.0*M)
ISN 0052              GAM=(0.500)*M*(SF1-SG1+SG)/SF1
ISN 0053              DELTA=(SF1+SG1-SG-2.0*Y*SG+(Y**2)*SF)/(2.0*SF1)
ISN 0054              TERM=(0.500*M)*(2.0*(1.0+Y)*SG-2.0*SG1)/SF1
ISN 0055              K1=(-ALPHA-BETA-DSQRT((ALPHA-BETA)**2+4.0*A*BL))/2.0
ISN 0056              K2=(-ALPHA-BETA+DSQRT((ALPHA-BETA)**2+4.0*A*BL))/2.0
ISN 0057              X1=-(K1+ALPHA)*(ALPHP-BETAP)
ISN 0058              X2=-AP*BL*(K1+ALPHA)/(K2+ALPHA)
ISN 0059              X3=-BP*A
ISN 0060              Z1=(X1+X2+X3)/((K1-K2)**2)
ISN 0061              X4=GAM*(K2+ALPHA)
ISN 0062              X5=DELTA*(K1+ALPHA)
ISN 0063              X6=TERM
ISN 0064              Z2=(X4+X5+X6)/(K1-K2)
            C.....DLOGJ IS THE CHANGE IN LOG J AT ANY ONE STEP.
ISN 0065              DLOGJ=(Z1+Z2)*B*DXI/(C*N)
            C.....SUM IS THE SUMMATION OF THE DLOGJ S (I.E. THE INTEGRAL.)
ISN 0066              S=S + DLOGJ
ISN 0067              WRITE (6,8) XI,DLOGJ,S
ISN 0068          8   FORMAT (10X,D12.5,3X,D12.5,3X,D12.5)
ISN 0069              XI=XI-DXI
ISN 0070          3   CONTINUE
ISN 0071              STOP
ISN 0072              END
```

APPENDIX F

ERRATA FOR GOLDREICH (1966)

The following are corrections of misprints in "History of the Lunar Orbit" by Peter Goldreich as the article appears in Reviews of Geophysics, vol. 4, pgs. 411-439, 1966. I do not claim to have caught all the misprints; some of the corrections may result from my own misunderstanding; but this list should be of use to readers of this classic paper.

pg. 416: Equation (9) should read:

$$\text{``} \cos S = \cos^2 \frac{I}{2} \cos(\Phi' - u) + \sin^2 \frac{I}{2} \cos(\Phi' + u) \text{''}$$

pg. 417: Equation (12) is derived from Equation (7) by using the approximation

$$\cos I \cong 1 - \beta^2/2.$$

If this approximation is not used, then the expression in braces in Equation (12) will read

$$\text{``} \frac{1}{4} - \frac{3}{8}\beta^2 + \left(\frac{3}{4} - \frac{3}{8}\beta^2\right) \cos 2\Phi' \cos 2u$$

$$+ \frac{3}{4} \cos I (\sin 2\Phi' \sin 2u) + \frac{3}{8}\beta^2 \cos 2\Phi' + \frac{3}{8}\beta^2 \cos 2u \text{''}$$

The derivation of Equation (13) is still permissible, since the terms containing ($\cos 2\Phi' \cos 2u$) and ($\sin 2\Phi' \sin 2u$) are periodic, so long as $\Phi' \neq u$.

pg. 417: Equation (14) and the line below it: How $M/\mathfrak{M}$ enters into the discussion is not apparent (to me).

pg. 419: Equations (19) should read:

$$\text{``}\ \frac{d\,(H\vec{a})}{dt} = \ldots\ \text{''} \quad \text{and} \quad \text{``}\ \frac{d\,(h\vec{b})}{dt} = \ldots\ \text{''}$$

Equations (22), (23), (43a), and (43b) likewise need parentheses around the whole quantity appearing in the differentiation operator.

pg. 419: Read " GM " for " μ " in the equation for K_1 in Equations (21).

pg. 420: Two lines above Equation (29):

" ... be derived by multiplying $2\,(\vec{a}\cdot\vec{c})\,\frac{K_1}{H}$ into equation 25, $2\,(\vec{b}\cdot\vec{c})\,\frac{K_2}{h}$ into equation 26, $2\,(\vec{a}\cdot\vec{b})\,L$ into equation 27 ... "

pg. 423: Equation (41): Here " $\frac{\mathfrak{M}}{M+m}$ " should be substituted for " $\frac{\mathfrak{M}}{M}$ " on the right side. Clearly the author ignores the " m " since $\frac{m}{M} << 1$.

pg. 423: The fifth line down from Equation (40) should read:

" of (39) having ... "

pg. 425: The next to the last line should read:

" ... Next, dotting $\frac{\vec{b}}{H}$ into equation 43a and $\frac{\vec{a}}{h}$ into ... "

pg. 426: The line above Equations (51) should read:

" Using (1), (2), (6), and (21), we observe ... "

pg. 426: The first equation of (51) apparently uses

$$\frac{dK_1}{dt} = 2K_1 \frac{1}{\Omega_\oplus} \frac{d\Omega_\oplus}{dt} = 2K_1 \frac{1}{C\Omega_\oplus} \frac{d(C\Omega_\oplus)}{dt} = \frac{2K_1}{H} \frac{dH}{dt}$$

This implies $\Omega_\oplus \frac{dC}{dt} << C \frac{d\Omega_\oplus}{dt}$

pg. 426: The equation above (55) resolves vector $\vec{T}$ along two independent sets of orthogonal coordinates; this procedure is ambiguous. Equations (69) - (74) show that the expression for $\vec{T}$ is really the sum of the lunar and solar torques, with the first three vectors being the lunar torque resolved along $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ and the last three vectors being the solar torque resolved along $(\vec{f}_1, \vec{f}_2, \vec{f}_3)$.

pg. 427: Equation (58): MacDonald (1964) has sign q' = sign $\frac{z - \alpha}{(1 - z^2)^{1/2}}$ using Goldreich's notation.

pg. 428: The right sides of the last two equations of (63) should read:

$$\text{``} \ldots = -\frac{2mA}{\pi a^6} q\, B(q) \sin 2\delta \text{''}$$

$$\text{``} \ldots = \frac{2mA}{\pi a^6} q' F(q) \sin 2\delta \text{''}$$

Confusion arises here because Goldreich corrects errors in Equations (42) and (44) of MacDonald (1964), but inadvertently includes "n" in the last two equations of (63). I must confess that I do not know if the signs of the two equations in my correction are right, since they depend on MacDonald's derivation, which I could not follow in places.

pg. 428: The author uses slightly different notation from Kaula (1964) in Equation (65); " m^* " is brought outside of " B_m " and written explicitly in Equation (64).

pg. 428: Following the notation of Kaula (1964), " q " has been set equal to zero in Equation (66).

pg. 429: The right sides of Equations (67) and (68) should both be multiplied by " m " (the lunar torque) or " $\mathfrak{M}$ " (the solar torque).

pg. 429: The second term of Equation (69) and of (71) should each be multiplied by " k_2 ".

TABLE 1

The angular speeds, phase lag angles, and amplitude factors for the seven tides are given. Adopted from Darwin (1880).

TABLE 1

Angular speed	$2(n-\Omega)$	$2n$	$2(n+\Omega)$	$n-2\Omega$	n	$n+2\Omega$	2Ω
Phase lag angle	$2f_1$	$2f$	$2f_2$	g_1	g	g_2	$2h$
Amplitude factor	F_1	F	F_2	G_1	G	G_2	H

TABLE 2

The critical angle ψ_c for various viscosities is given. ϵ is the distance from c_0, where $\sin 2g_1$ is zero, to where $\sin 2g_1 = \pm 1$.

TABLE 2

Viscosity (poises)	ψ_c (degrees)	$\frac{\epsilon}{c_0}$
10^{17}	8.5	8×10^{-3}
10^{18}	2.7	8×10^{-4}
10^{19}	0.85	8×10^{-5}
10^{20}	0.27	8×10^{-6}
10^{21}	0.085	8×10^{-7}

TABLE 3

Summary of computer data for the curves shown in Figures 15, 16, and 17. The computer program itself is given in Appendix C. The column labelled "ψ" refers to its value when the moon is at or near 3.83 earth radii distance from the earth. The column labelled "Time" gives the time required for the moon to move from 3.83 earth radii to 10 earth radii. The quantities Δt, NQ, $\Delta\xi$, and CRIT are explained in Chapter III, Section C.

TABLE 3

Figure number	Viscosity (poises)	ψ (degrees)	Time (Years)	Δt (sec $\times 10^{-9}$)	N Q	$\Delta\xi$	CRIT
15	10^{15}	3	570	2.5×10^{-6}	600	5×10^{-4}	0.05
15	10^{16}	3	3600	2.5×10^{-5}	600	5×10^{-4}	0.05
15	10^{17}	3	3.6×10^{4}	2.5×10^{-4}	600	5×10^{-4}	0.05
16	10^{18}	2.68	3.6×10^{5}	5×10^{-4}	600	2.5×10^{-4}	0.05
16	10^{19}	0.85	3.6×10^{6}	5×10^{-3}	600	2.5×10^{-4}	0.05
16	10^{20}	0.268	3.6×10^{7}	5×10^{-2}	600	2.5×10^{-4}	0.05
16	10^{21}	0.085	3.6×10^{8}	2.5×10^{-2}	2400	2.5×10^{-4}	0.05
17	10^{18}	1	3.6×10^{5}	5×10^{-4}	1200	1.25×10^{-4}	0.025
17	10^{18}	2	3.6×10^{5}	5×10^{-4}	1200	1.25×10^{-4}	0.025
17	10^{18}	3	3.5×10^{5}	5×10^{-4}	1200	1.25×10^{-4}	0.025
17	10^{21}	1	3.6×10^{8}	1×10^{-2}	1200	1.25×10^{-4}	0.025
17	10^{21}	2	3.5×10^{8}	1×10^{-2}	1200	1.25×10^{-4}	0.025
17	10^{21}	3	3.5×10^{8}	1×10^{-2}	1200	1.25×10^{-4}	0.025

TABLE 4

The important quantities used in this work are listed.

TABLE 4

Symbol	Description	Numerical value
a	mean radius of the earth	6.37×10^{8} cm
b	L_M/ξ	$7.21 \times 10^{40}\ \frac{\text{g-cm}^2}{\text{sec}}$
c	earth-moon distance	—
c_0	3.83 earth radii	2.44×10^{9} cm
e	eccentricity of the lunar orbit	—
$2f_1$	lag angle of the tide with speed $2(n - \Omega)$	—
g	gravitational acceleration at the surface of the earth	980.7 cm/sec^2
g	lag angle of the tide with speed n	—
$\mathfrak{g}$	$\frac{2}{5}\ \frac{g}{a}$	6.16×10^{-7} sec^{-2}
g_1	lag angle of the tide with speed $n - 2\Omega$	—
i	angle between the invariable plane and the earth's equatorial plane	—
j	angle between the invariable plane and the moon's orbital plane	—
k	$\frac{C\ \Omega_0\ c_0}{G\ M\ m}$	1.14×10^{4} sec
m	mass of the moon	7.35×10^{25} g
n	angular velocity of the earth	—
r	radial distance measured from the center of the earth	—
t	time	—
x	$c - c_0$	—
C	polar moment of inertia of the earth	8.11×10^{44} g-cm^2

TABLE 4 (Continued)

Symbol	Description	Numerical value
G	universal gravitational constant	$6.67 \times 10^{-8}\ \frac{\text{cm}^3}{\text{g-sec}^2}$
I	angle between the ecliptic and the earth's proper plane	—
$I_{/}$	angle between the earth's proper plane and equatorial plane	—
J	angle between the moon's proper plane and orbital plane	—
$J_{/}$	angle between the ecliptic and the moon's proper plane	—
L_E	rotational angular momentum of the earth	—
L_M	orbital angular momentum of the earth-moon system	—
L_T	total angular momentum of the earth-moon system	$34.2 \times 10^{40}\ \frac{\text{g-cm}^2}{\text{sec}}$
M	mass of the earth	5.98×10^{27} g
R	disturbing function	—
T	absolute temperature of the earth	—
ϵ	$\left[\zeta\left(3\frac{\Omega_0}{c_0} - \frac{b}{2c_0C}\right)\right]^{-1}$	$7.98 \times 10^{14}\ \frac{c_0}{\nu}$
ζ	$\frac{19\nu}{2ga\rho}$	$2.76 \times 10^{-12}\ \nu$
κ	$\sin\frac{1}{2}(i+j)$	—
λ	Ω/n	—
ξ	$\left(\frac{c}{c_0}\right)^{1/2}$	—

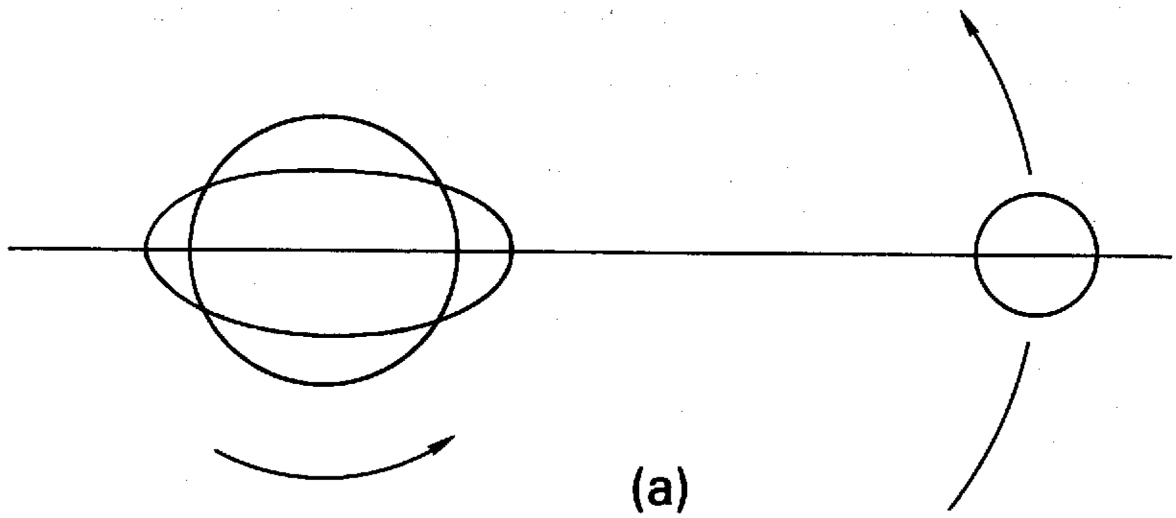
TABLE 4 (Continued)

Symbol	Description	Numerical value
π	$\cos \frac{1}{2}(i + j)$	—
ρ	density of the earth	5.5 g/cm^3
σ	displacement of the earth's surface	—
τ	$\frac{Gm}{c^3} = \frac{\tau_0}{\xi^6}$	$\tau_0 = 3.37 \times 10^{-10}\ sec^{-2}$
ν	viscosity of the earth	—
ψ	angle between the moon's orbital plane and earth's equatorial plane = $i + j$	—
ψ_c	critical angle = $\sqrt{\sin 4 f_1}$	—
Ω	orbital angular velocity of the moon	—

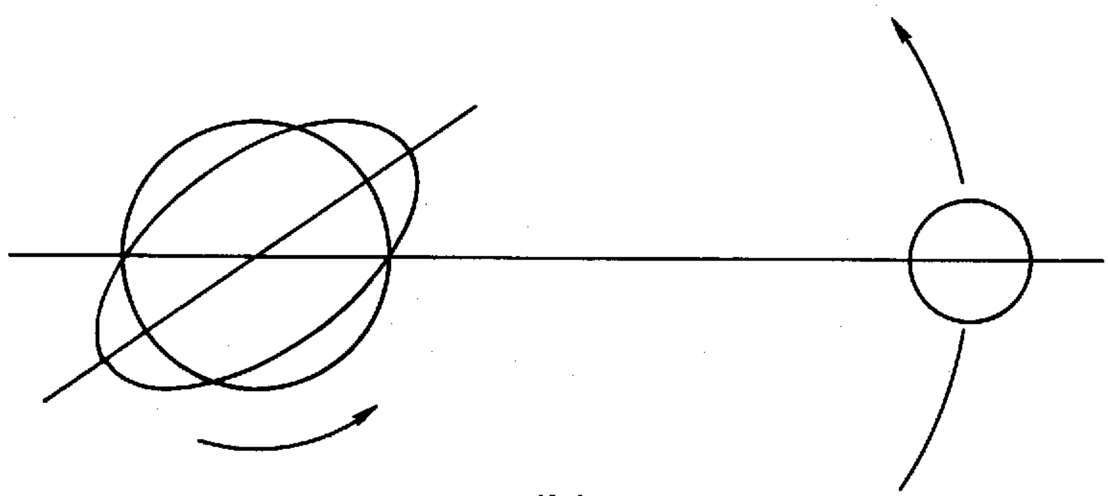
FIGURE 1

(a) The earth and its attendant tidal bulge is shown on the left and the moon on the right in the figure. The moon orbits in the equatorial plane of the earth in the same direction that the earth rotates. No friction is present.

(b) Friction is present. The diagrams are not to scale.



(a)

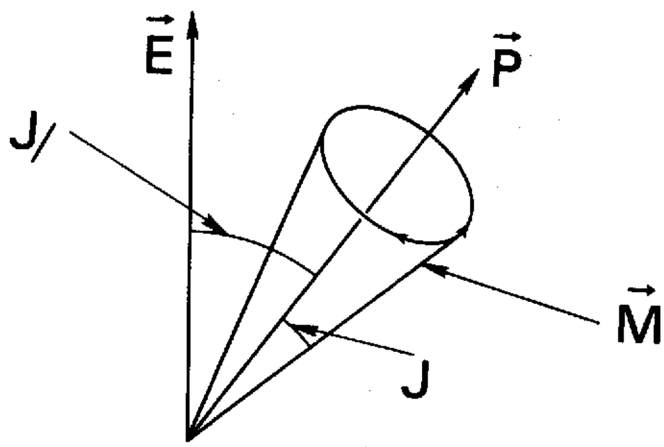


(b)

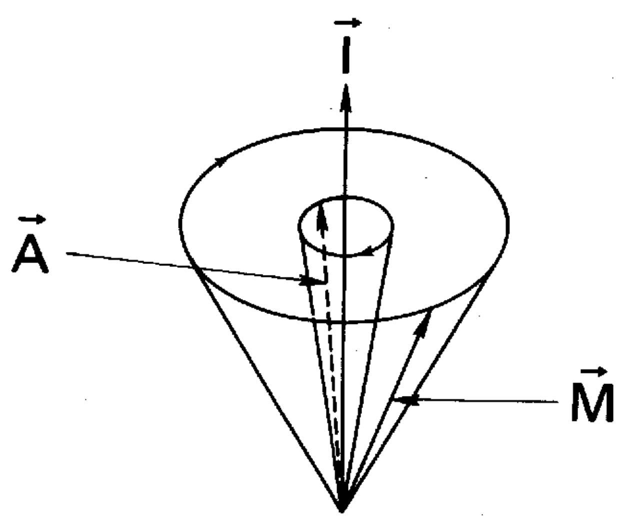
FIGURE 2

(a) $\vec{E}$ is a vector normal to the ecliptic, $\vec{P}$ normal to the proper plane of the satellite, and $\vec{M}$ normal to the plane of the satellite's orbit. $\vec{M}$ sweeps out a cone about $\vec{P}$. J is the angle between $\vec{M}$ and $\vec{P}$, and $J_/$ is the angle between $\vec{E}$ and $\vec{P}$.

(b) $\vec{I}$ is normal to the invariable plane of the planet-satellite system, $\vec{A}$ normal to the planet's equatorial plane, and $\vec{M}$ normal to the satellite's orbital plane. $\vec{A}$ and $\vec{M}$ sweep out cones about $\vec{I}$ when solar influence is negligible, with all three vectors lying in a single plane.



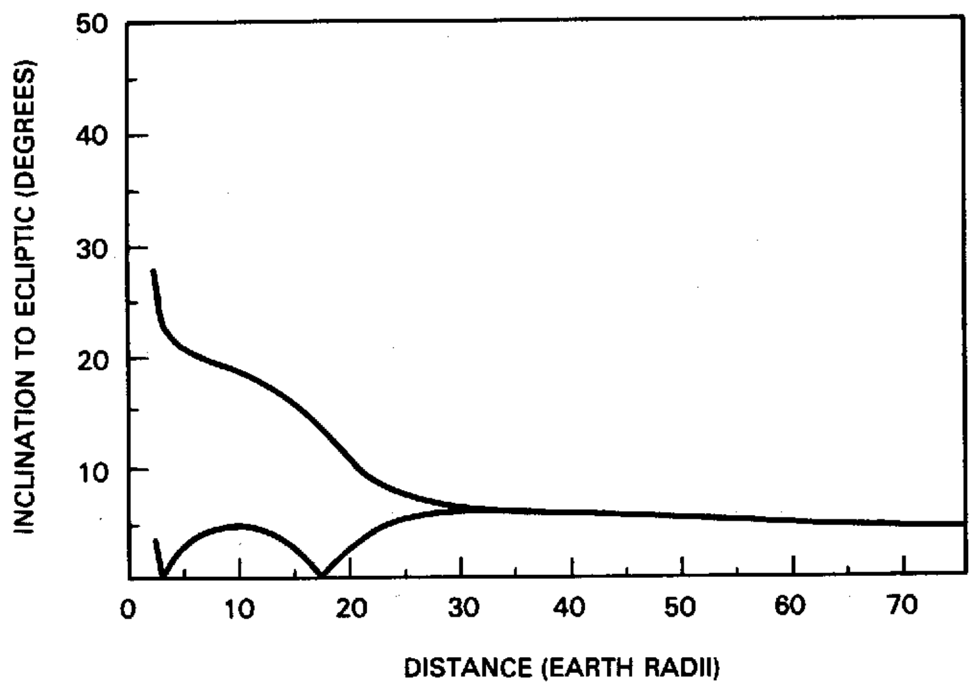
(a)



(b)

FIGURE 3

Figure 7 of Goldreich (1966), showing the inclination of the moon's orbital plane to the ecliptic. Precession of the lunar orbit causes the inclination to oscillate between the two branches of the curve.



INCLINATION TO ECLIPTIC (DEGREES)
50
40
30
20
10
0
10
20
30
40
50
60
70
DISTANCE (EARTH RADII)

FIGURE 4

(a) $\vec{E}$ is normal to the ecliptic, $\vec{P}$ normal to the moon's proper plane. $J_{/}$ is small compared to J so that the two vectors are nearly parallel and the inclination of the moon's orbital plane to the ecliptic is nearly constant.

(b) $J_{/}$ becomes appreciable so that the orbital plane clearly does not maintain a constant inclination to the ecliptic.

(c) $\vec{E}$ lies in the surface of the cone swept out by the vector normal to the lunar orbit and $J_{/} = \dot{J}$.

(d) $\vec{E}$ falls outside the cone.

The diagrams are schematic only.

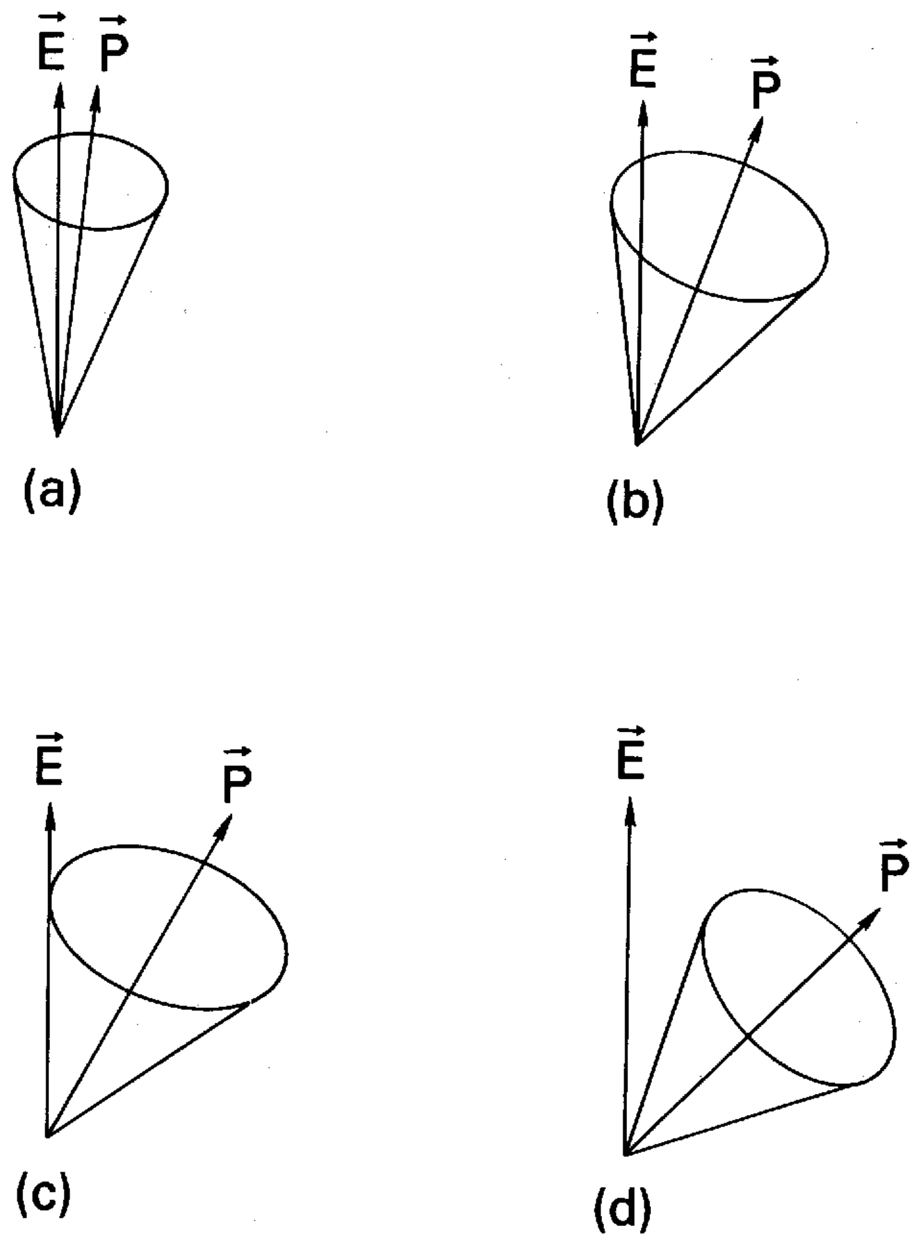
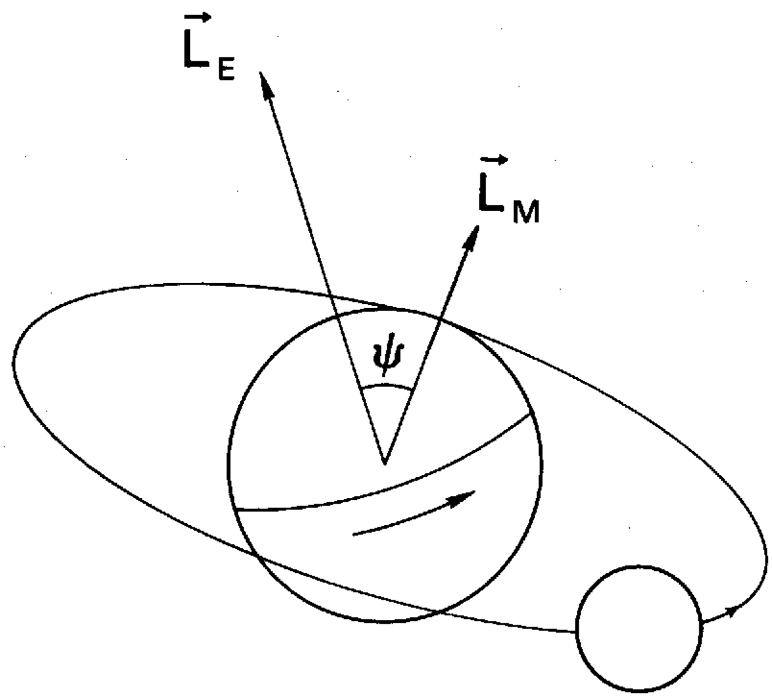

$\vec{E}$ $\vec{P}$
(a)
$\vec{E}$ $\vec{P}$
(b)
$\vec{E}$ $\vec{P}$
(c)
$\vec{E}$ $\vec{P}$
(d)

FIGURE 5

The upper diagram shows the moon orbiting about the earth. $\vec{L}_M$ is the orbital angular momentum of the system and is perpendicular to the moon's orbital plane. $\vec{L}_E$ is the rotational angular momentum of the earth and lies along the earth's axis, perpendicular to the equatorial plane. ψ is the angle between the orbital and equatorial planes. The lower diagram shows the angular momentum triangle. $\vec{L}_T$ is the total angular momentum of the system. The magnitudes of $\vec{L}_E$ and $\vec{L}_M$ are denoted by L_E and L_M, respectively.


$\vec{L}_E$
$\vec{L}_M$
ψ

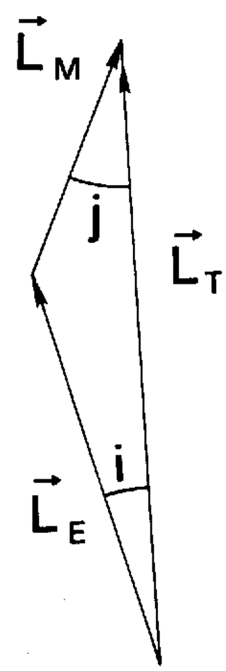

$\vec{L}_M$
j
$\vec{L}_T$
i
$\vec{L}_E$

FIGURE 6

The rotational angular velocity of the earth n and the orbital angular velocity of the moon Ω are shown as a function of earth-moon distance. The orbit of the moon lies in the equatorial plane of the earth. The dashed line is the Roche limit and the dotted line is the distance c_0 where $n = 2\Omega$ (3.83 earth radii).

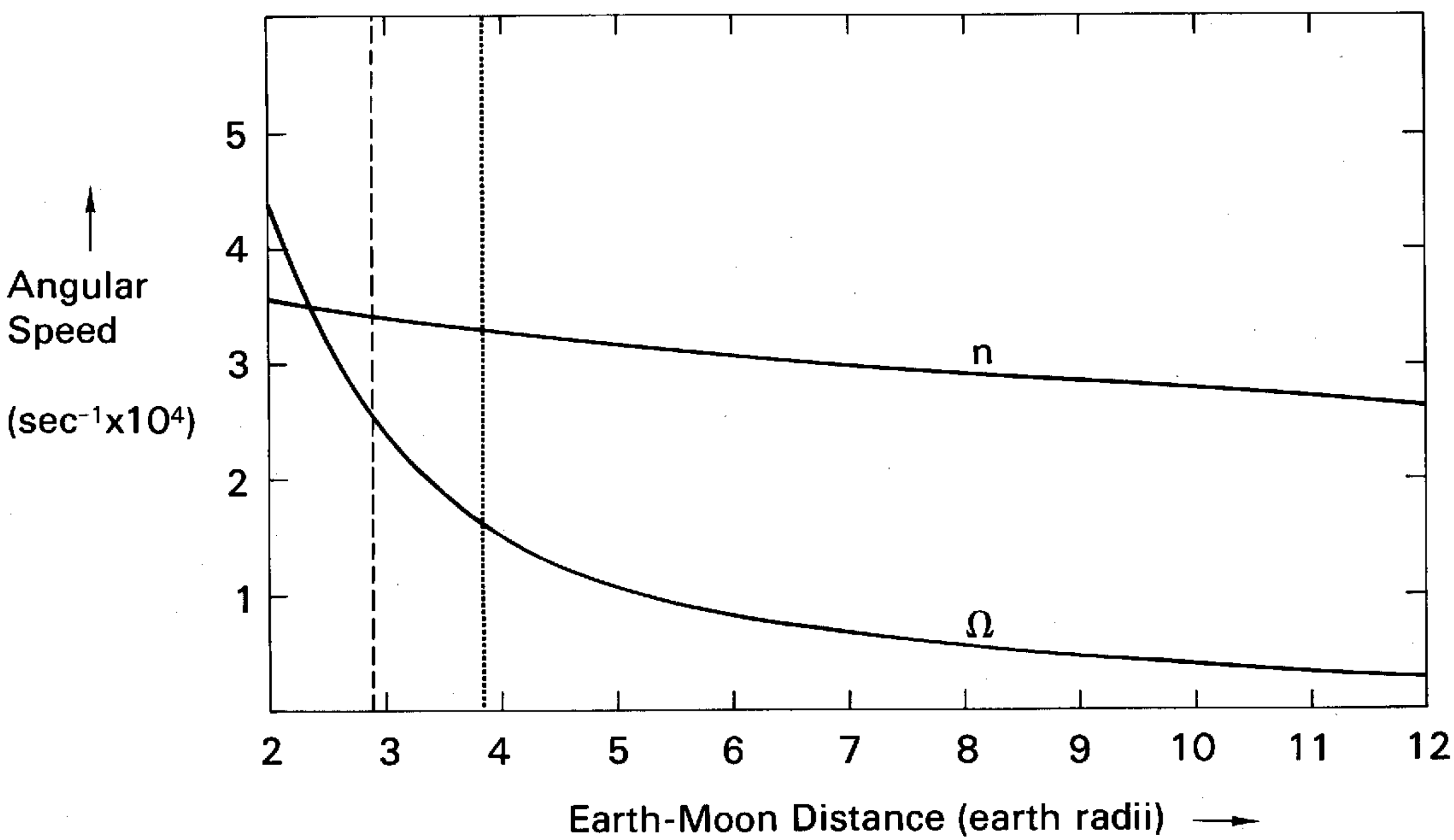

Angular Speed
(sec-1x104)
1
2
3
4
5
n
Ω
2
3
4
5
6
7
8
9
10
11
12
Earth-Moon Distance (earth radii)

FIGURE 7

The angular speeds of the three principal tides are shown as a function of earth-moon distance. Speeds n, 2 (n - Ω), and n - 2Ω correspond to the K_1, M_2, and O tides, respectively. The orbit of the moon lies in the equatorial plane of the earth. The dashed line is the Roche limit and the dotted line is the distance c_0 where n = 2Ω (3.83 earth radii).

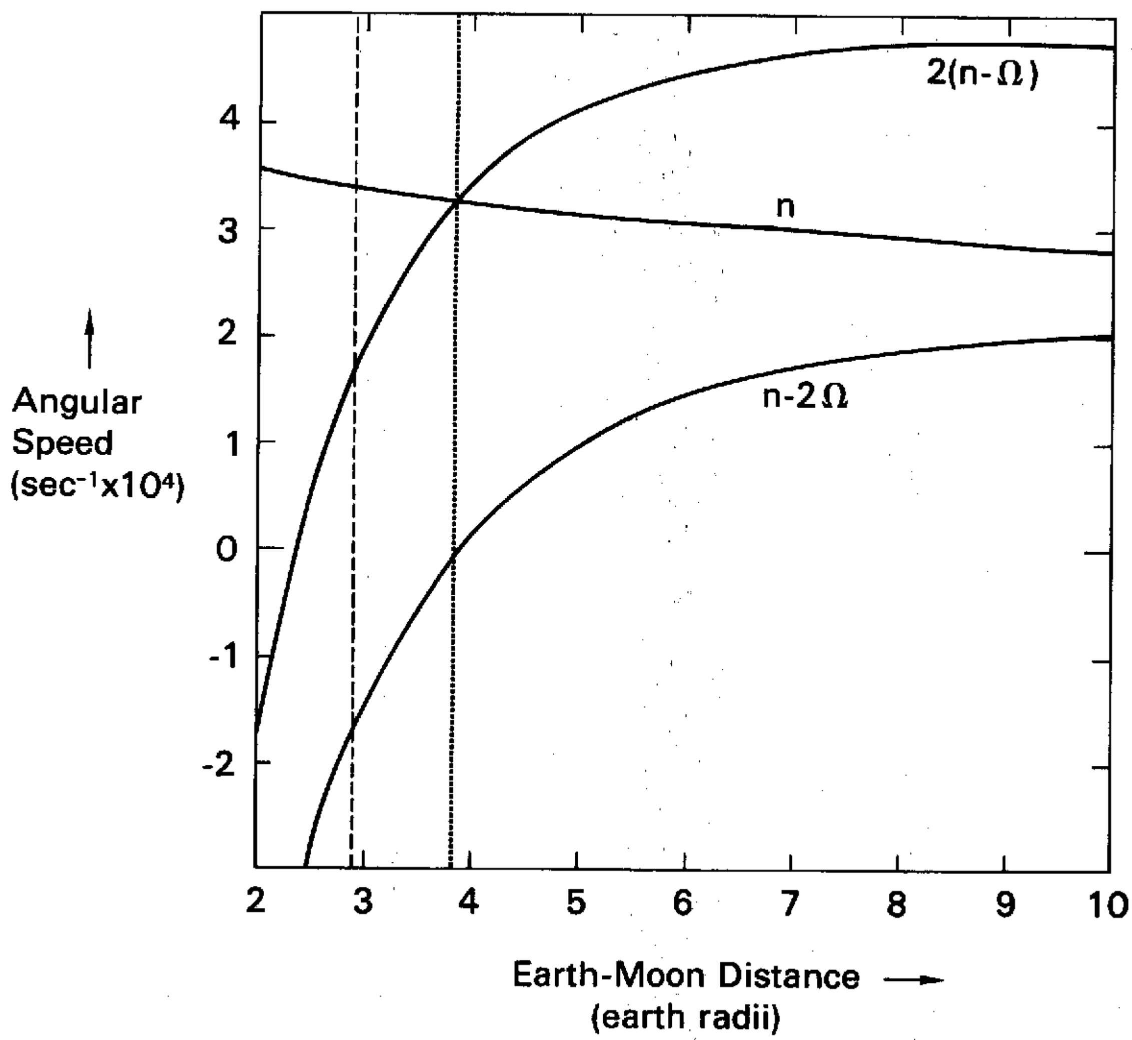

2(n-Ω)
n
n-2Ω
4
3
2
1
0
-1
-2
Angular
Speed
(sec⁻¹x10⁴)
2
3
4
5
6
7
8
9
10
Earth-Moon Distance
(earth radii)

FIGURE 8

$\lambda = \Omega / n$ as a function of earth-moon distance. The orbit of the moon lies in the equatorial plane of the earth.

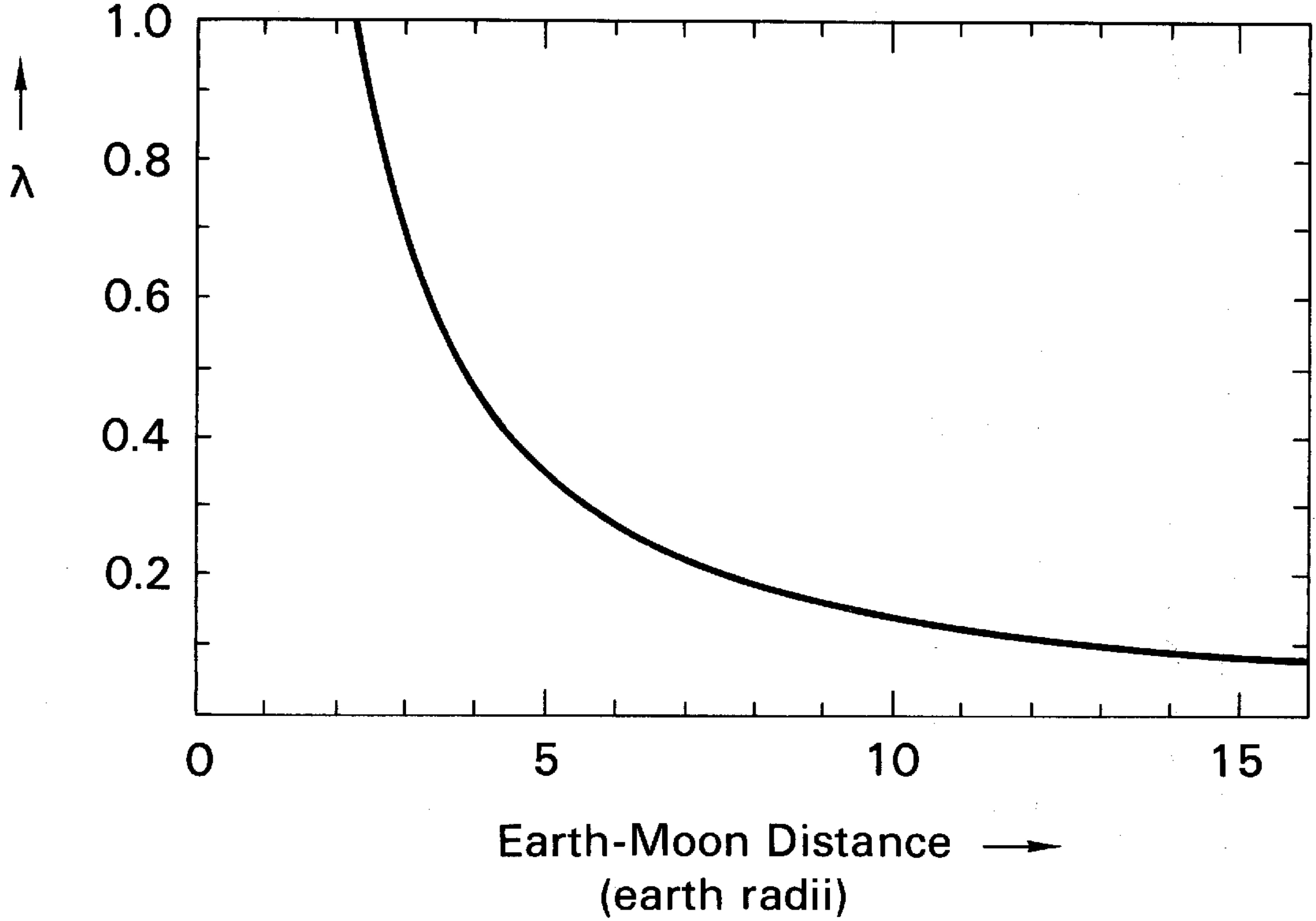
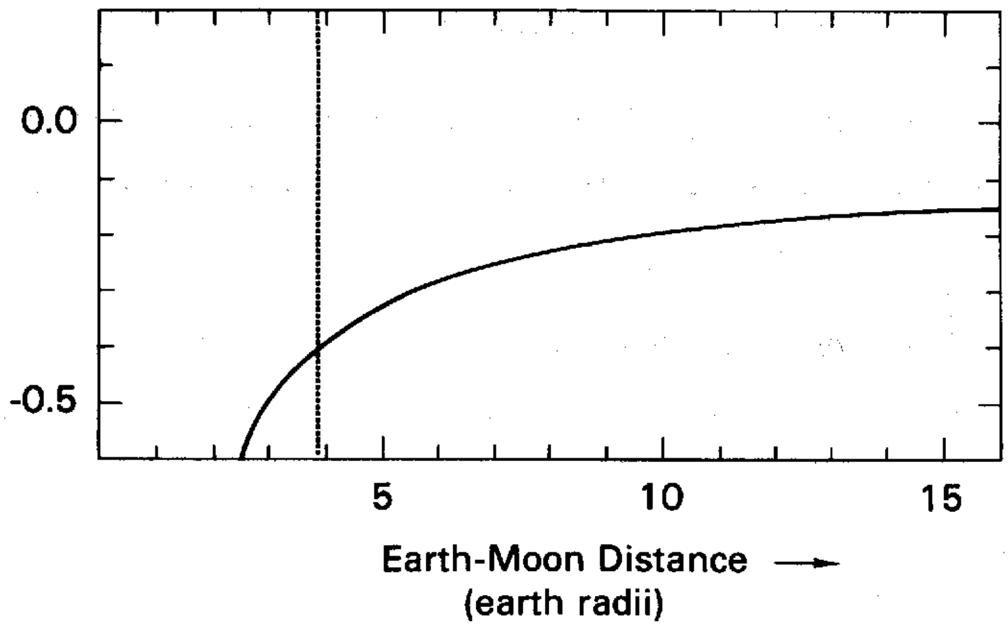

1.0
0.8
0.6
0.4
0.2
λ
0
5
10
15
Earth-Moon Distance
(earth radii)

FIGURE 9

(a) Expression (III-14) divided by $n\zeta$ as a function of earth-moon distance in the limit of low viscosity. The moon's orbit lies in the equatorial plane of the earth.

(b) Expression (III-14) multiplied by $n\zeta$ as a function of earth-moon distance in the limit of high viscosity. The moon's orbit lies in the equatorial plane of the earth. The function is discontinuous at c_0.


0.0
-0.5
5
10
15
Earth-Moon Distance
(earth radii)

(a)

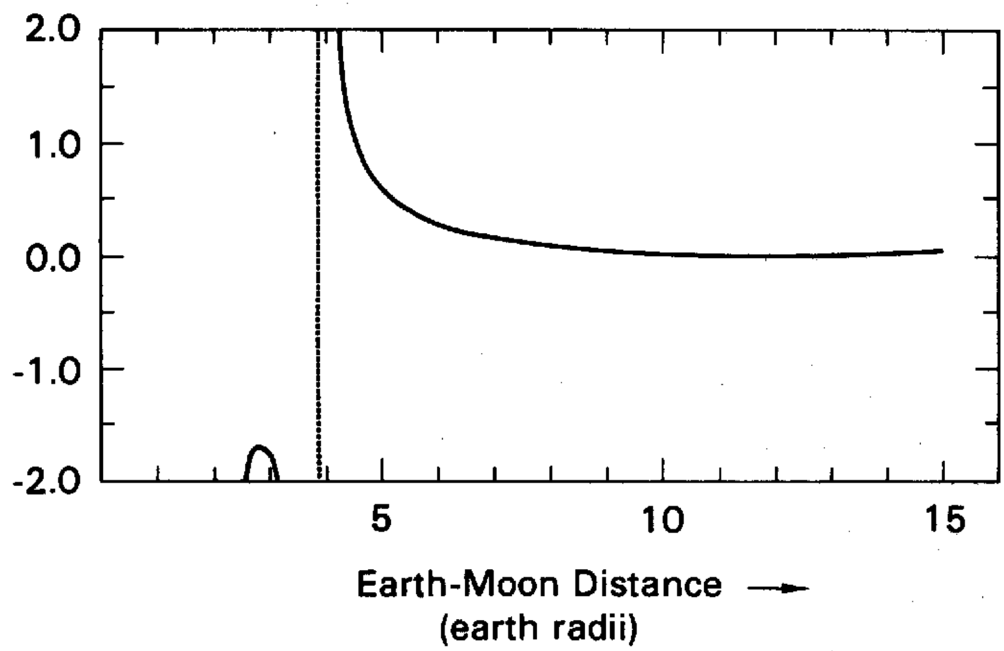

2.0
1.0
0.0
-1.0
-2.0
5
10
15
Earth-Moon Distance
(earth radii)

FIGURE 10

Sin $2g_1$ as a function of x for large viscosities ($\gg 10^{15}$ poises). The function reaches its extreme values at $-\epsilon$ and $+\epsilon$.

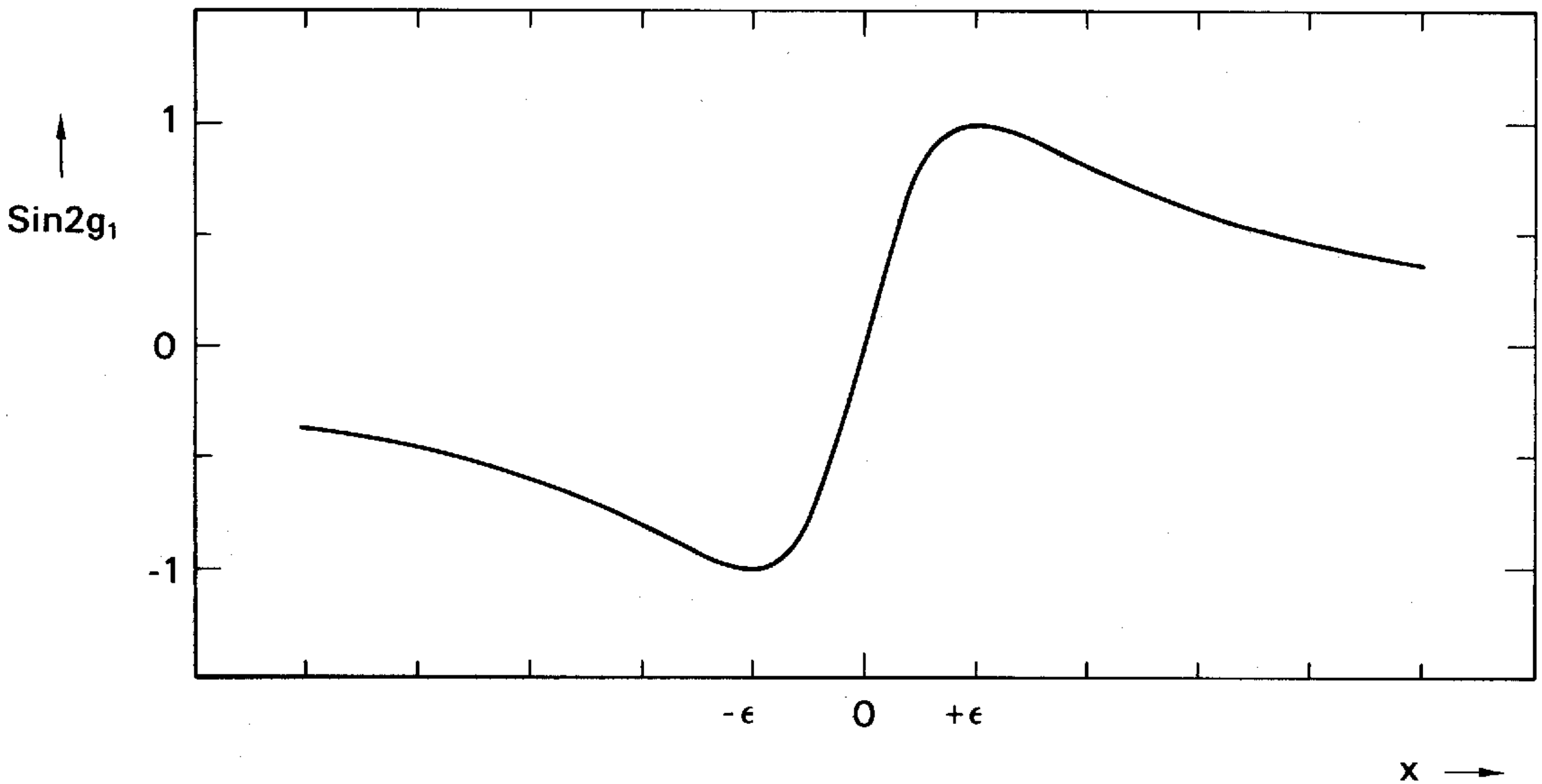

Sin2g$_1$
1
0
-1
-ϵ
0
+ϵ
x

FIGURE 11

The inclination ψ as a function of x for two different initial values of ψ for a viscosity of 10^{20} poises. In both cases ψ must fall below ψ_c at $x = -\epsilon$ (marked by the dot with the arrow) before the moon can pass to the outer regions. The solid line is discussed in the text. The dotted line shows different initial starting conditions. The lines are not displaced for clarity.

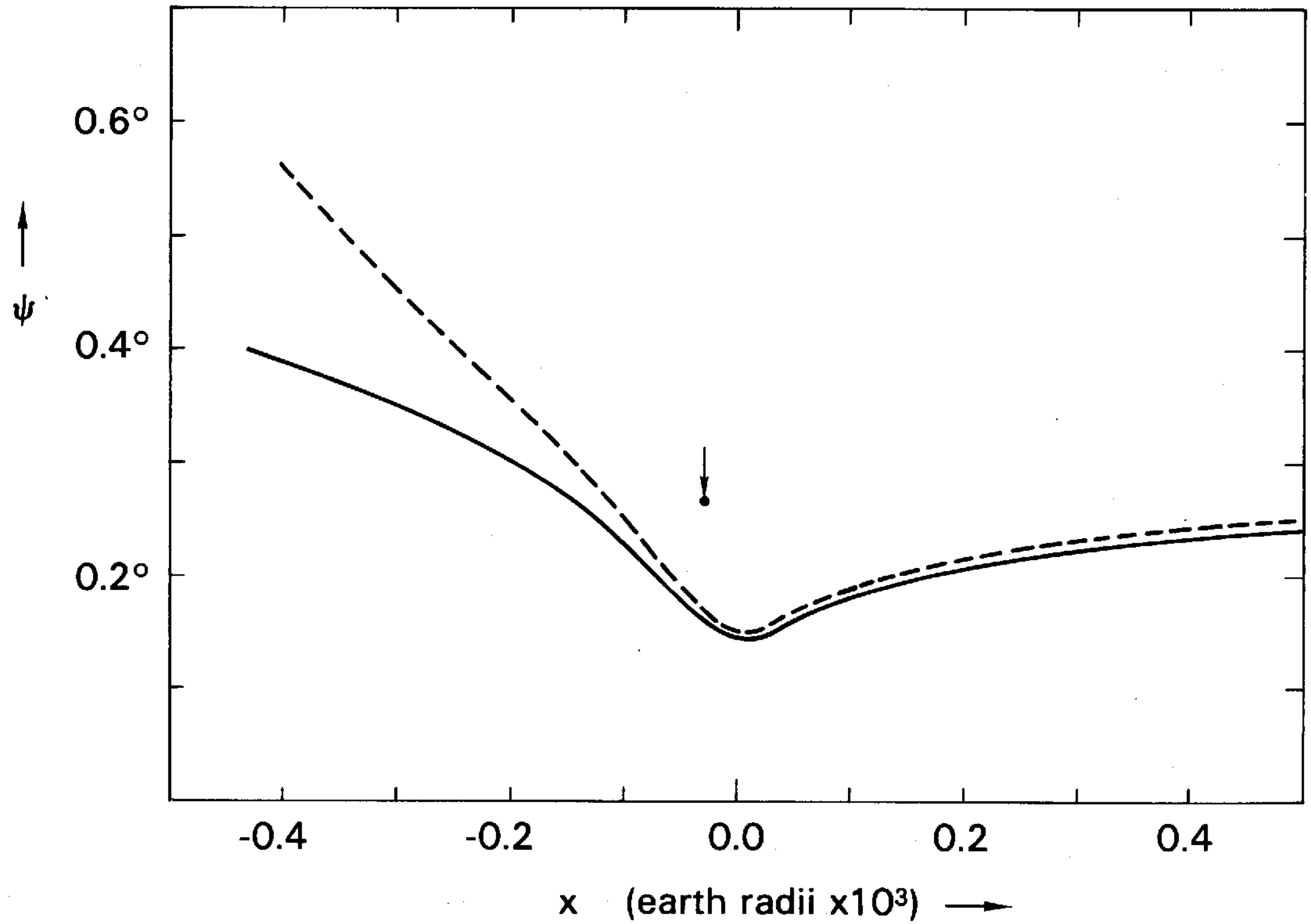

0.6°
0.4°
0.2°
ψ
-0.4
-0.2
0.0
0.2
0.4
x (earth radii x10³)

FIGURE 12

$\psi^2 \sin 2g_1$, $\sin 4f_1$, and $\sin 2g_1$ as functions of x for the case of the solid line shown in the previous figure. Sin $2g_1$ is not to scale; it is reduced by a factor of 10^5 compared to the other two functions. Sin $4f_1$ is nearly constant.

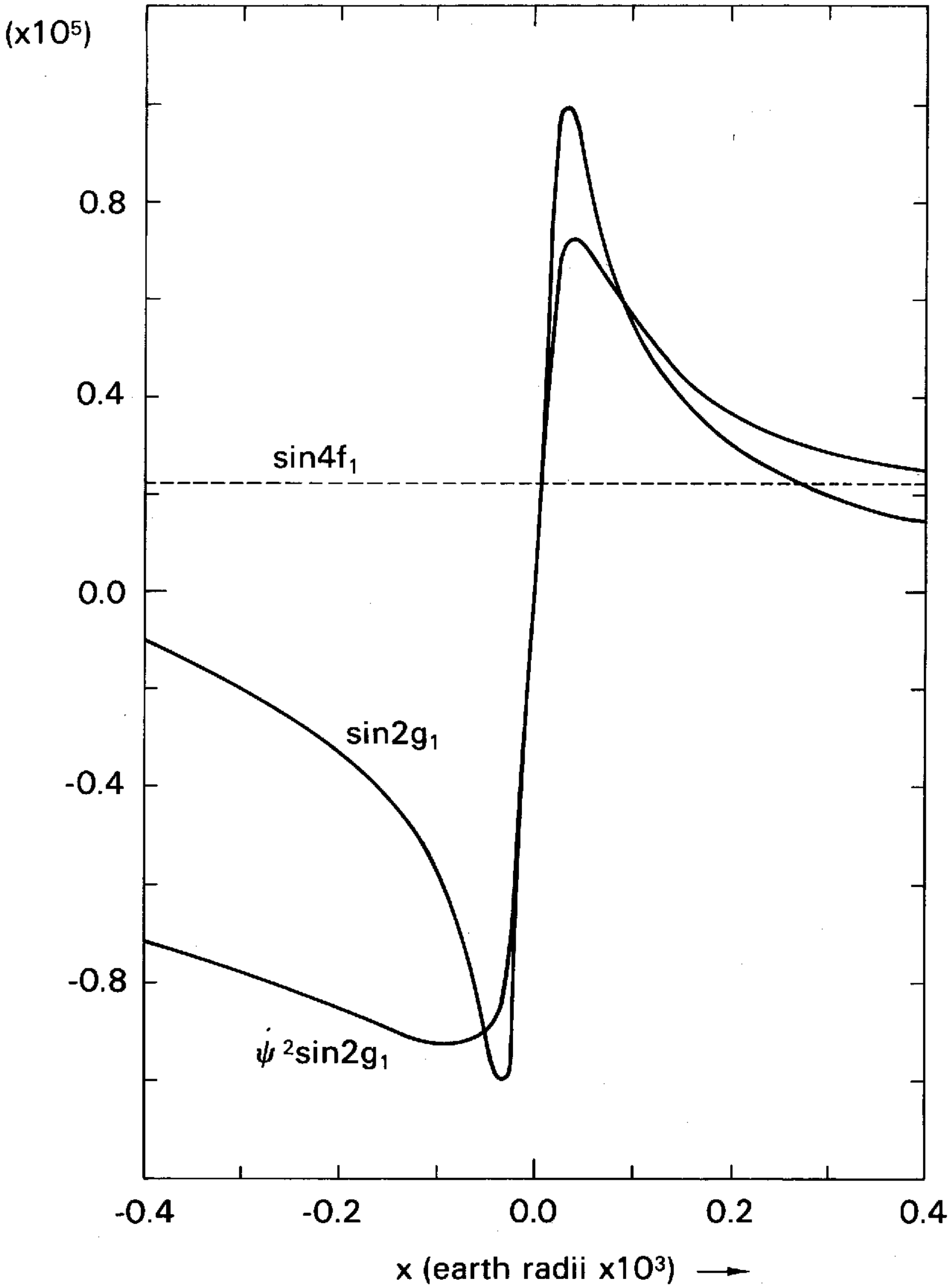

(x10^5)
0.8
0.4
0.0
-0.4
-0.8
sin4f1
sin2g1
ψ̇ 2sin2g1
-0.4
-0.2
0.0
0.2
0.4
x (earth radii x10^3)

FIGURE 13

$\frac{d\xi}{dt}$ for the case of the solid line shown in Figure 11.

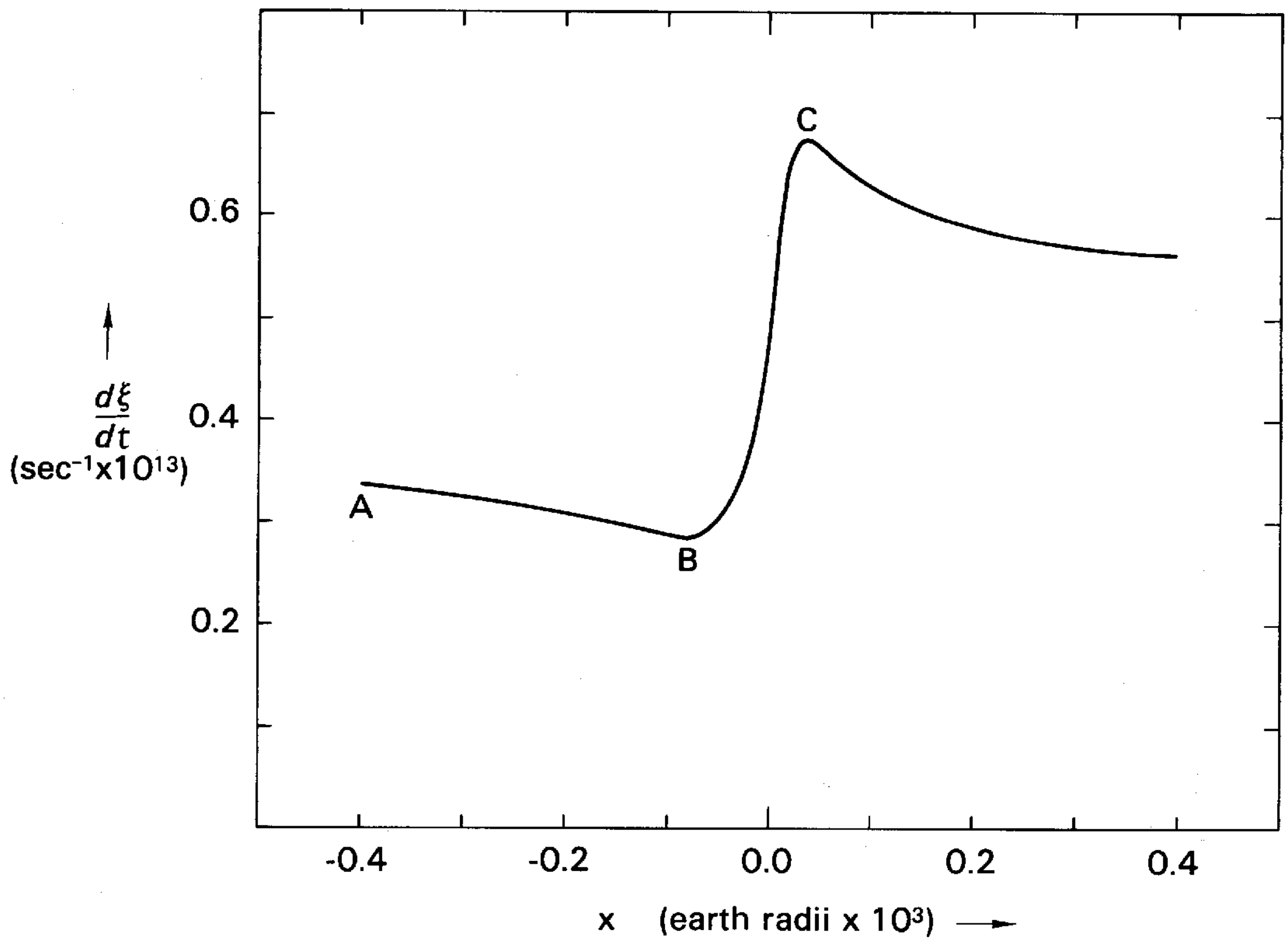
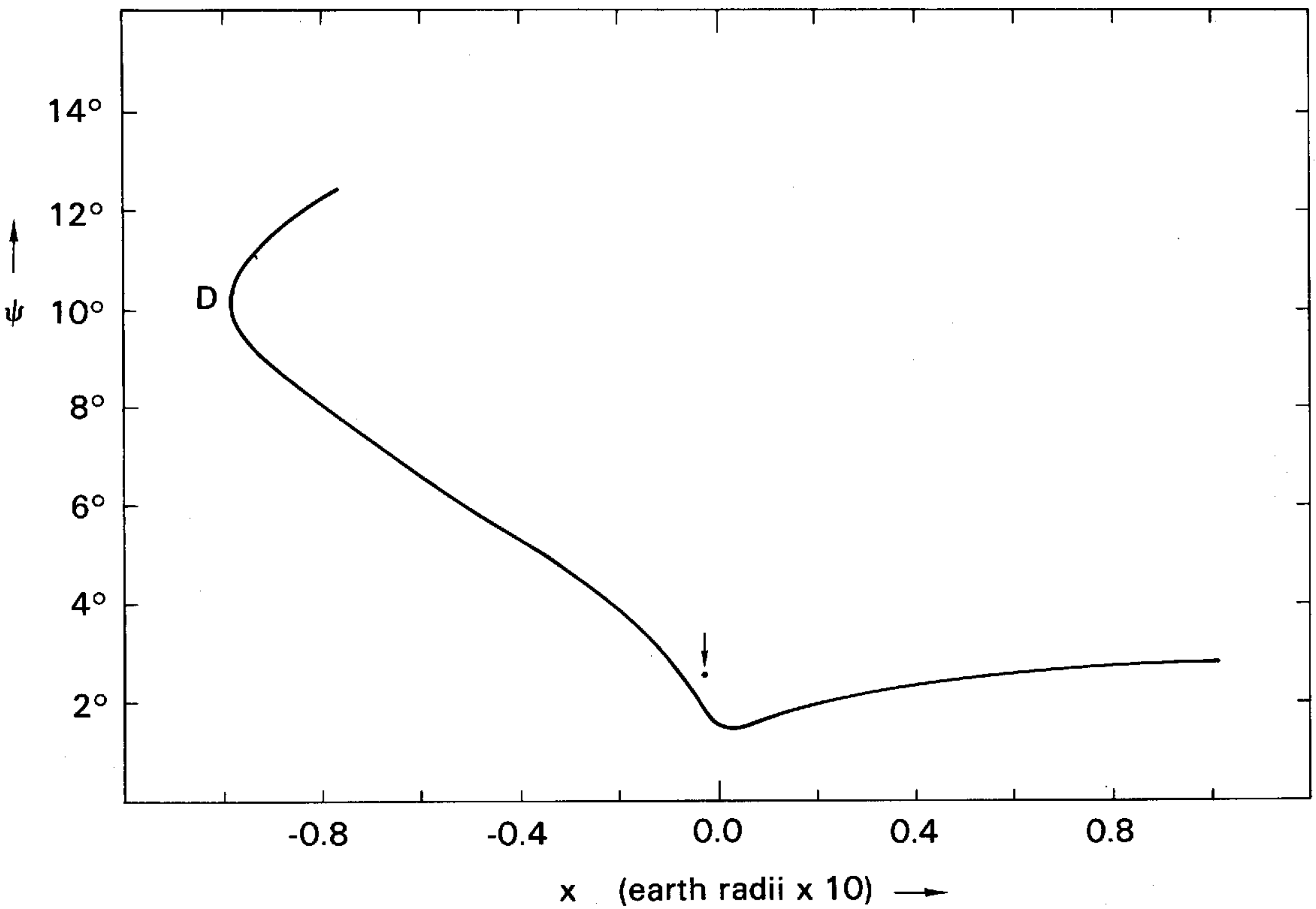

C
0.6
0.4
0.2
$\frac{d\xi}{dt}$
(sec^{-1}x10^{13})
A
B
-0.4
-0.2
0.0
0.2
0.4
x (earth radii x 10^3)

FIGURE 14

The inclination ψ as a function of x for 10^{18} poises for a large initial value of ψ. The moon moves toward the earth until it reaches point D. Thereafter it moves away from the earth. ψ must drop below the critical angle ψ_c (marked by the dot with the arrow) before the moon can pass into the outer regions.



14°
12°
10°
8°
6°
4°
2°
ψ
D
-0.8
-0.4
0.0
0.4
0.8
x (earth radii x 10)

FIGURE 15

The inclination ψ as a function of earth-moon distance for viscosities of 10^{15}, 10^{16}, and 10^{17} poises for an initial perturbation of 3° at c_0 (3.83 earth radii).

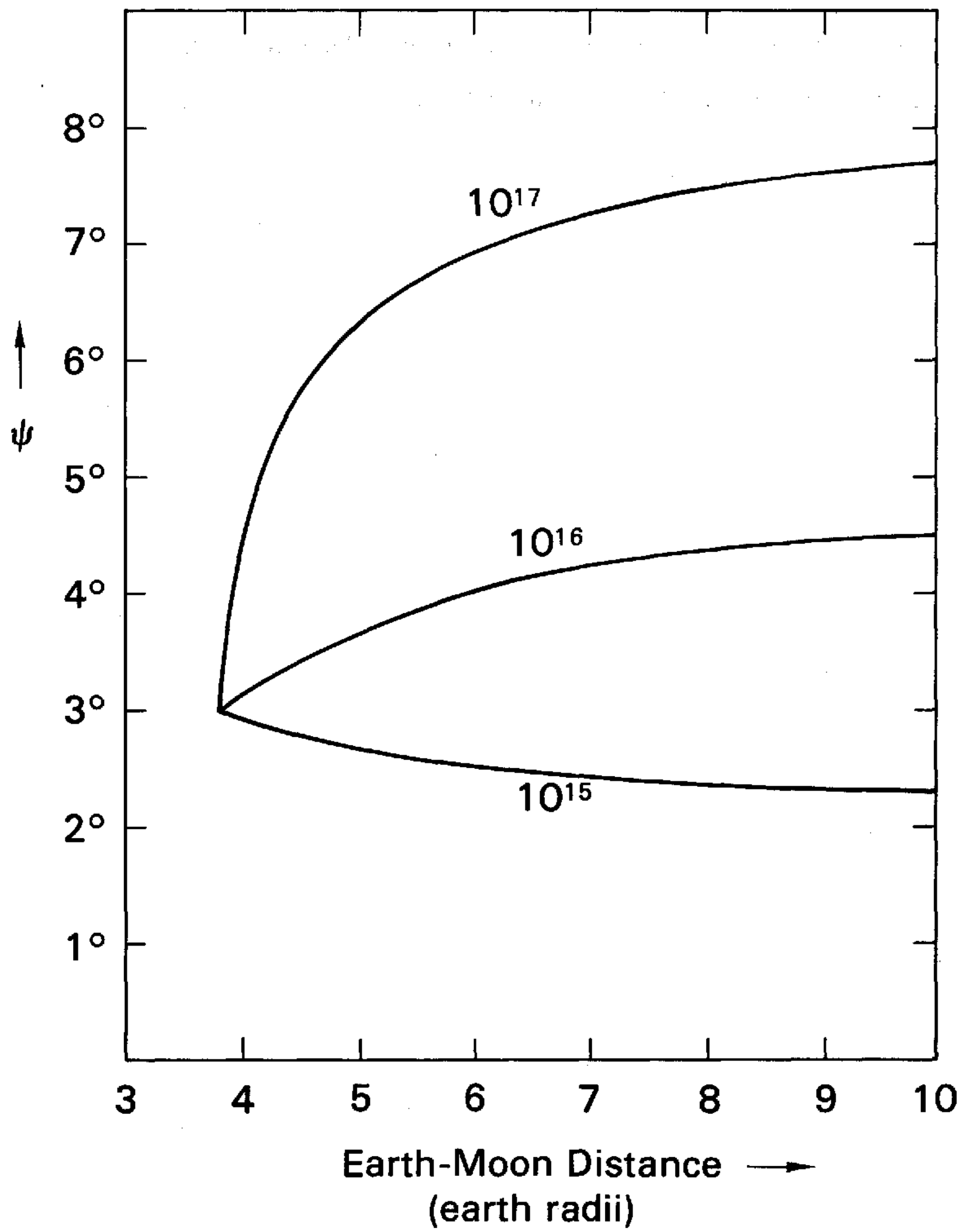
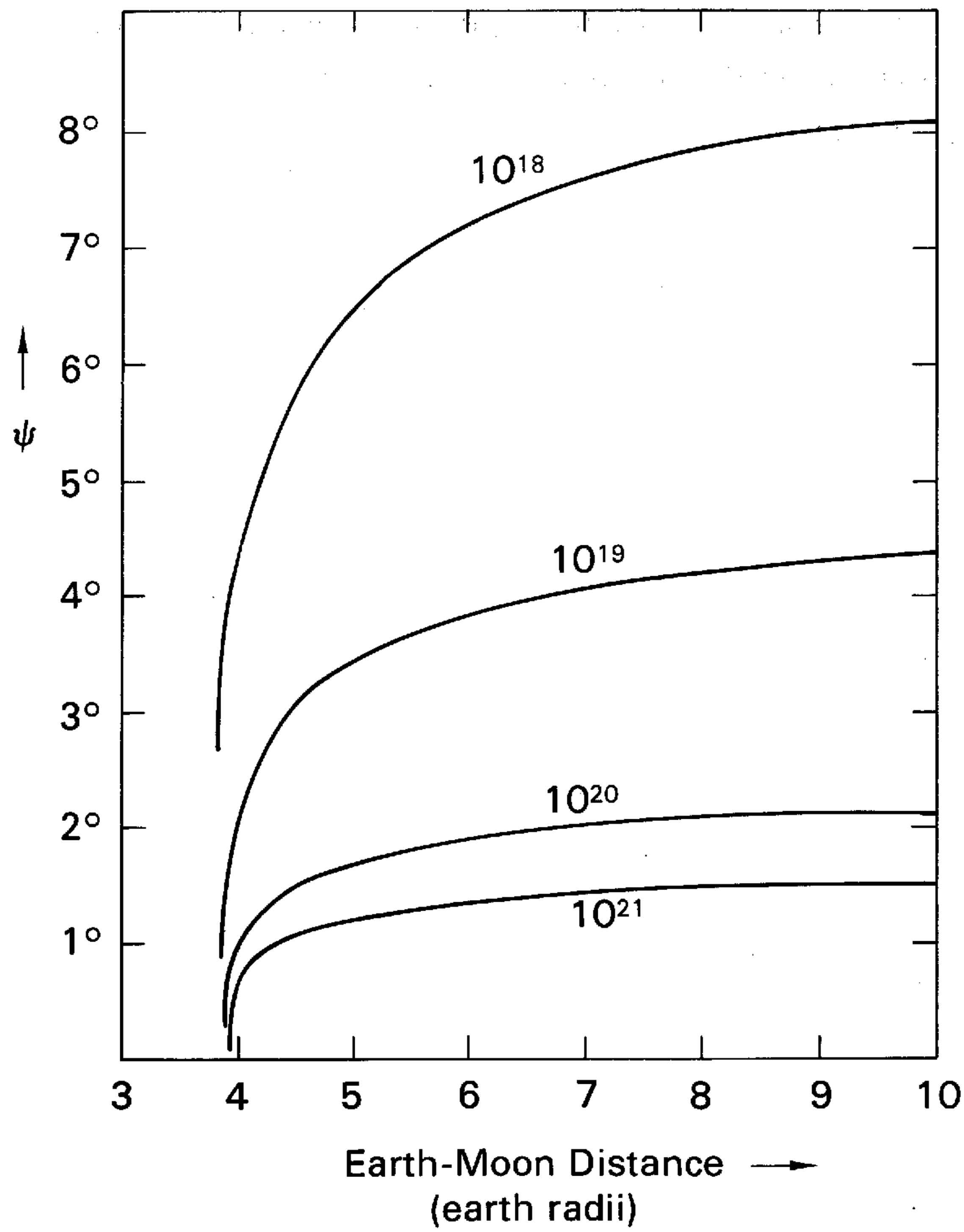

8°
7°
6°
5°
4°
3°
2°
1°
ψ
10^{17}
10^{16}
10^{15}
3
4
5
6
7
8
9
10
Earth-Moon Distance
(earth radii)

FIGURE 16

The inclination ψ as a function of earth-moon distance for viscosities of 10^{18}, 10^{19}, 10^{20}, and 10^{21} poises. In each case $\psi = \psi_c$ at $c_0 - \epsilon$.



8°
7°
6°
ψ
5°
4°
3°
2°
1°
10^18
10^19
10^20
10^21
3
4
5
6
7
8
9
10
Earth-Moon Distance
(earth radii)

FIGURE 17

The inclination ψ as a function of earth-moon distance for 10^{18} poises (solid lines) and 10^{21} poises (dashed lines) for perturbations of 1°, 2°, and 3° at c_0 (3.83 earth radii).

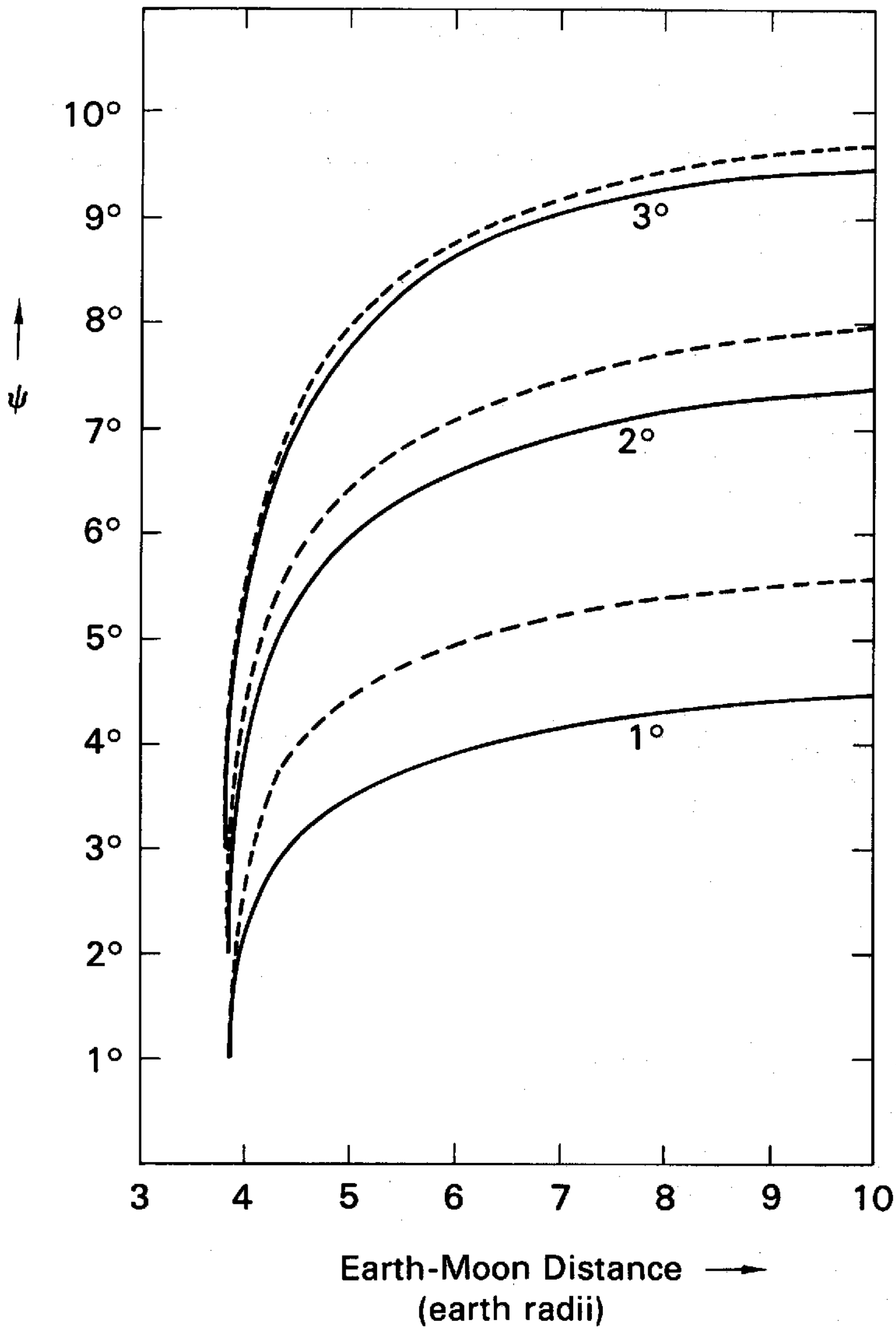

10°
9°
8°
7°
6°
5°
4°
3°
2°
1°
ψ
3°
2°
1°
3
4
5
6
7
8
9
10
Earth-Moon Distance
(earth radii)

FIGURE 18

The angle i as a function of earth-moon distance for 10^{18} poises (solid lines) and 10^{21} poises (dashed lines) for perturbations in ψ of 1°, 2°, and 3° at c_0 (3.83 earth radii).

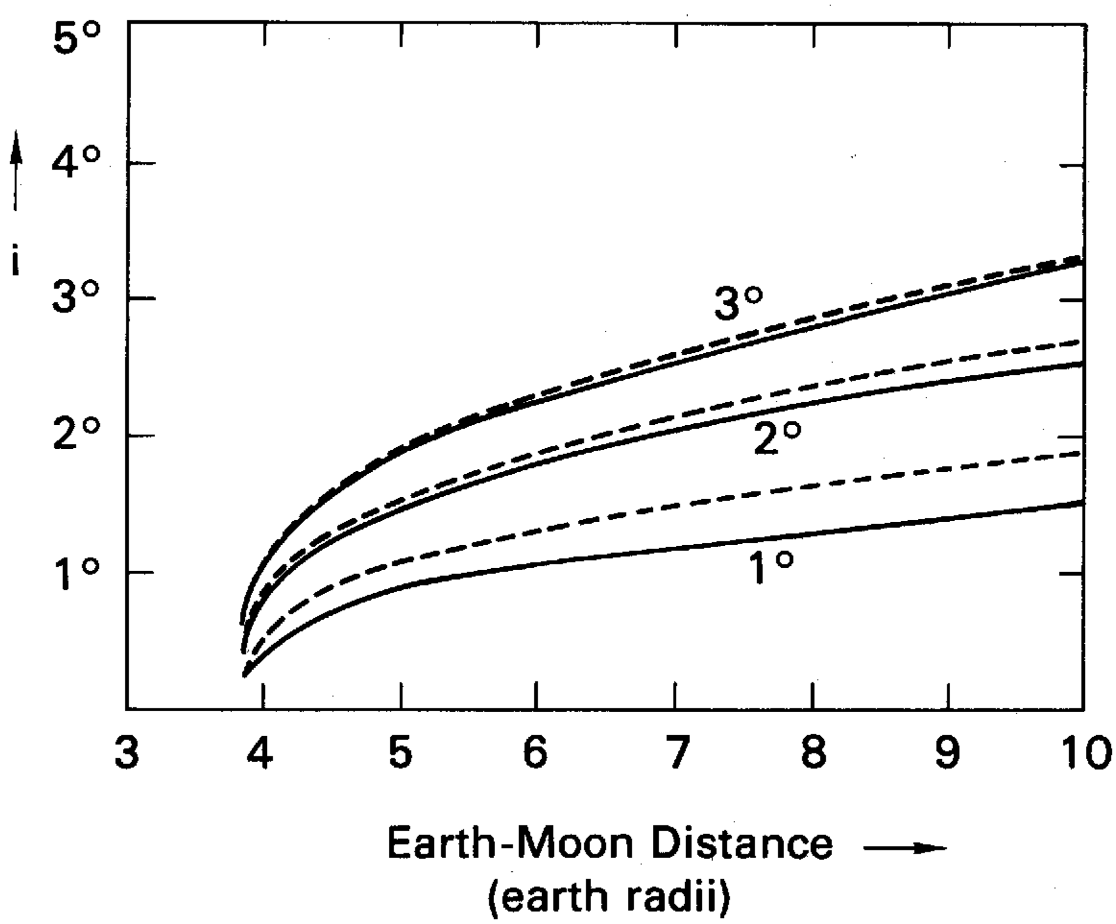
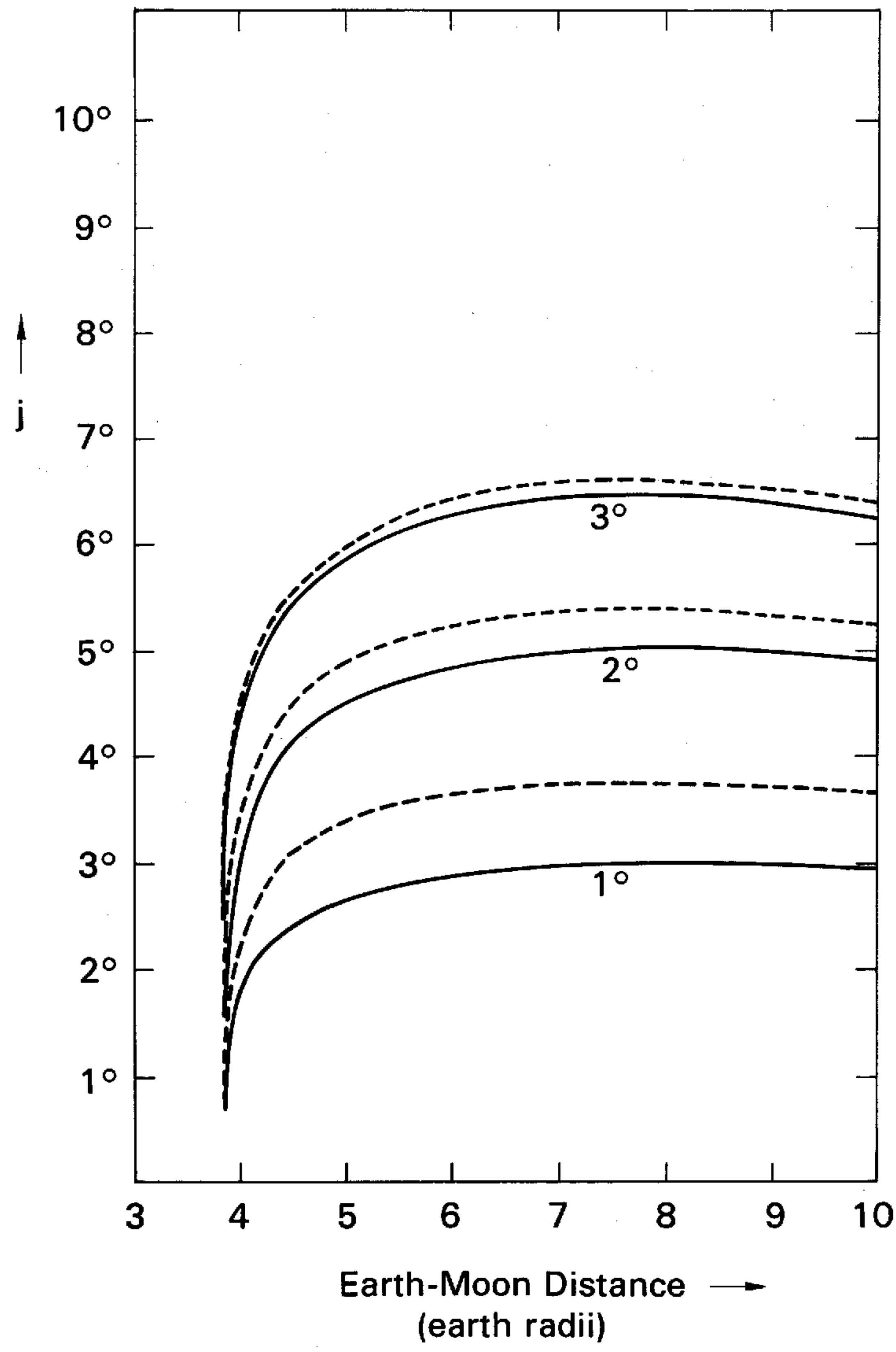

5°
4°
i
3°
2°
1°
3°
2°
1°
3
4
5
6
7
8
9
10
Earth-Moon Distance
(earth radii)

FIGURE 19

The angle j as a function of earth-moon distance for 10^{18} poises (solid lines) and 10^{21} poises (dashed lines) for perturbations in ψ of 1°, 2°, and 3° at c_0 (3.83 earth radii).



10°
9°
8°
j
7°
6°
5°
4°
3°
2°
1°
3°
2°
1°
3
4
5
6
7
8
9
10
Earth-Moon Distance
(earth radii)

FIGURE 20

The inclination J of the moon's orbital plane to its proper plane for various formulations of tidal friction. The dashed line is derived from Goldreich (1966), where the three principal lag angles are equal to each other. The dotted line is Darwin's result for low viscosities ($\ll 10^{15}$ poises). The upper solid line shows J for a perturbation in ψ of 3° at c_0 (3.83 earth radii) for a viscosity of 10^{18} poises. The lower solid line shows J for a perturbation of 2.5° in ψ at c_0 (3.83 earth radii) for a viscosity of 10^{18} poises. The dashed line, dotted line, and lower solid line all give the present value of J at the present distance of 60 earth radii.

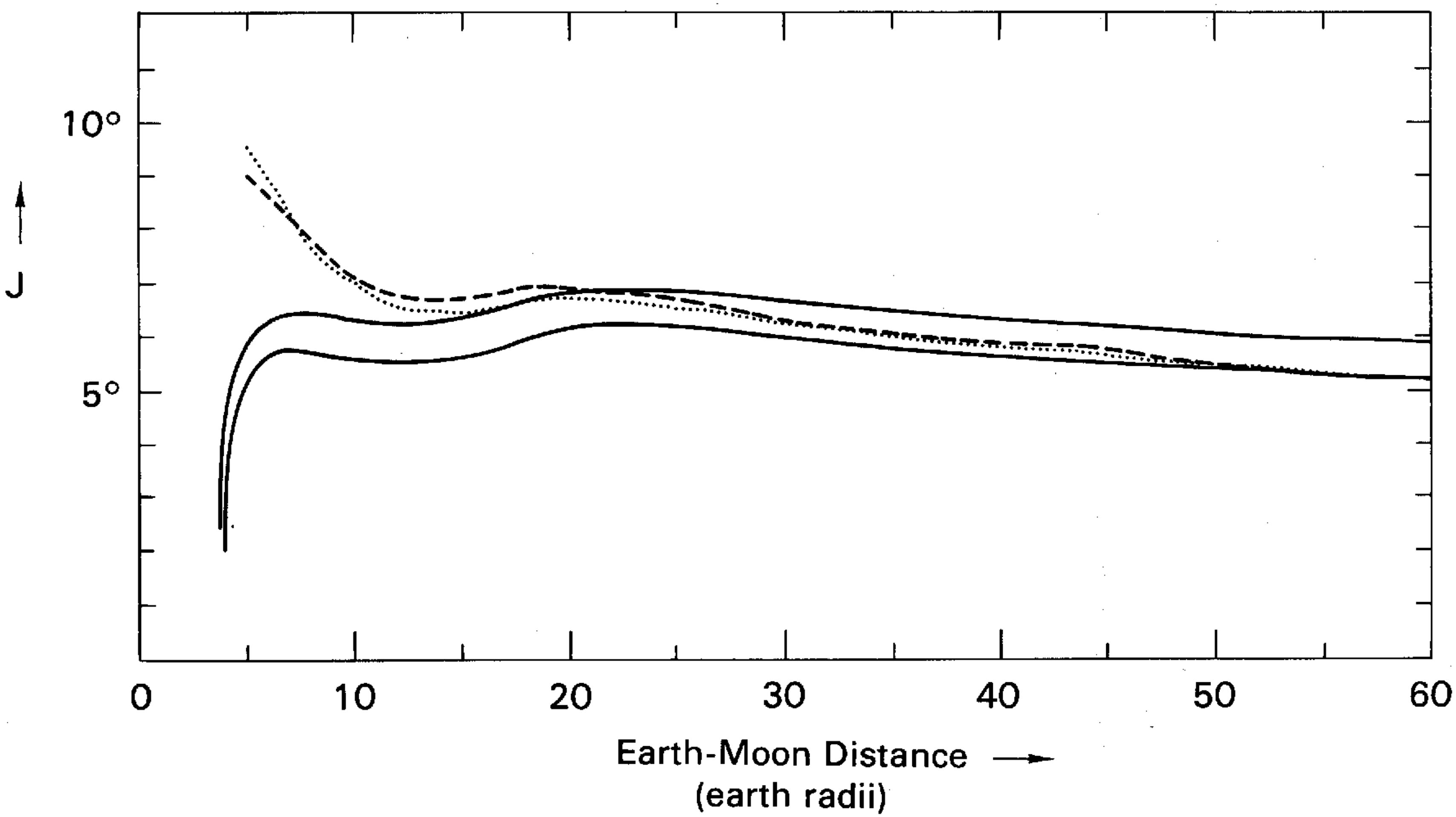
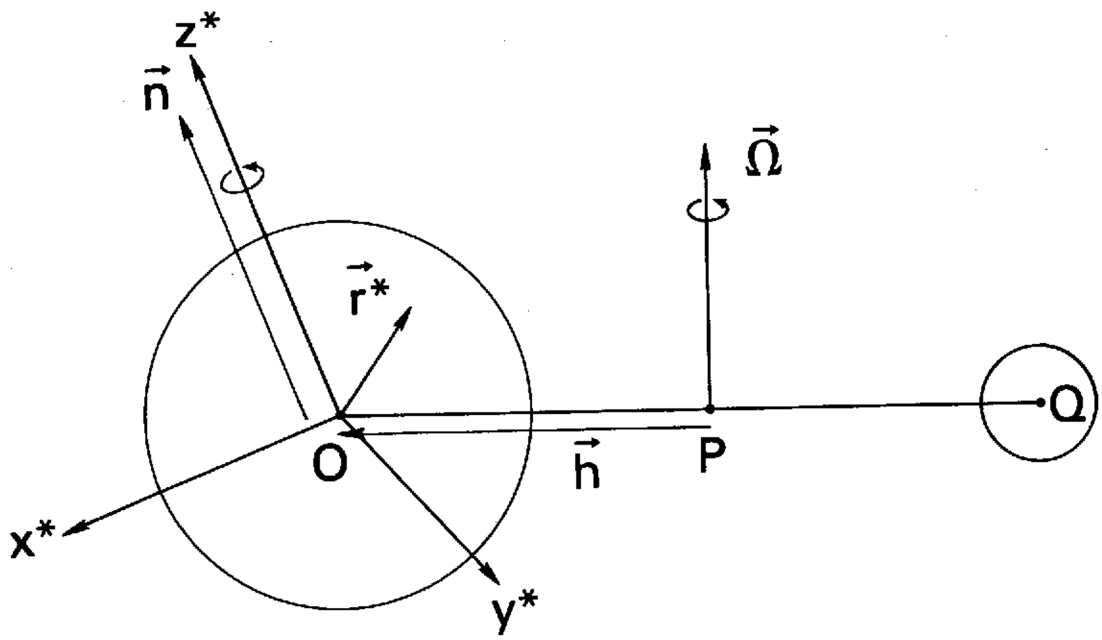

10°
5°
J
0
10
20
30
40
50
60
Earth-Moon Distance
(earth radii)

FIGURE 21

The point O is the center of mass of the earth and Q the center of mass of the moon. The earth and moon circle P, the center of mass of the earth-moon system, with angular velocity $\vec{\Omega}$. The earth rotates about the z* axis with angular velocity $\vec{n}$. Vectors $\vec{n}$ and $\vec{h}$ are displaced for clarity. $\vec{r}$ and $\vec{r}^*$ are the position vectors of the moon and mass element, respectively. Θ is the angle between $\vec{r}$ and $\vec{r}^*$.


z*
n
Ω
r*
O
h
P
Q
x*
y*

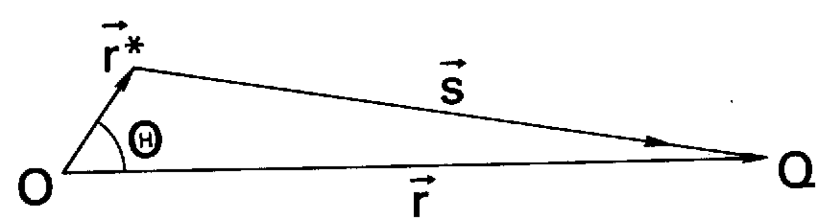
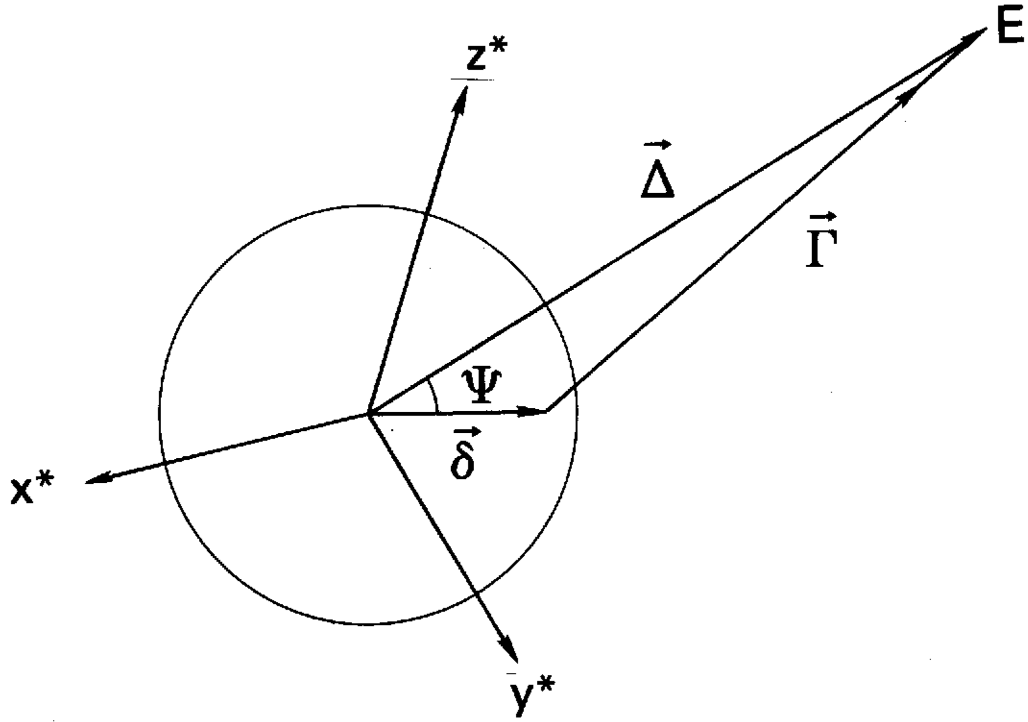

r*
s
Θ
O
r
Q

FIGURE 22

The position vector of the exterior point E is $\vec{\Delta}$. $\vec{\delta}$ is the position vector of a mass element in the earth. The angle between $\vec{\Delta}$ and $\vec{\delta}$ is Ψ.



z*
E
$\vec{\Delta}$
$\vec{\Gamma}$
Ψ
$\vec{\delta}$
x*
y*

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