PARAMETER ESTIMATION SUPPLEMENT

to the

Mission Analysis Evaluation and Space Trajectory Operations Program
MAESTRO

N 74-16541

PARAMETER ESTIMATION
SUPPLEMENT TO THE MISSION ANALYSIS EVALUATION AND SPACE TRAJECTORY OPERATIONS PROGRAM (Analytical Mechanics Associates, Inc.) 99 F HC $7.00

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August 1973

Prepared by

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for

NASA Goddard Space Flight Center
Greenbelt, Maryland
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INTRODUCTION

This Parameter Estimation Supplement describes the PEST computer program and gives instructions for its use in determination of lunar gravitation field coefficients.

PEST was developed under Contract NAS 5-11900 for use in the RAE-B lunar orbiting mission as a means of lunar field recovery. The observations processed by PEST are short-arc osculating orbital elements. These observations are the end product of an orbit determination process obtained with another program. PEST's end product is a set of harmonic coefficients, $C_{nm}$ and $S_{nm}$, to be used in long-term prediction of the lunar orbit. PEST employs some novel techniques in its estimation process, notably a "square-root" batch estimator and linear variational equations in the orbital elements (both osculating and mean) for measurement sensitivities.

Section 1 describes the program's capabilities, giving operating instructions and input/output examples. This section supplements the User's Manual for the MAESTRO program (Ref. 1). PEST utilizes MAESTRO routines for its trajectory propagation. Section 2 describes PEST's program structure and contains descriptions of those of its subroutines which are not common to MAESTRO. The MAESTRO subroutines are described in the Programmer's Manual for MAESTRO (Ref. 2). Section 3 contains some of the theoretical background information for the estimation process. Appendix A contains a derivation of linear variational equations for the Method 7 elements.
I. PEST PROGRAM DESCRIPTION (User's Guide)

1.1 Highlights and Capabilities

The parameter estimation (PEST) program is a weighted least squares batch processor which estimates

a. coefficients of the lunar gravitational field
b. initial orbital elements
c. solar pressure coefficients.

The measurements used in the estimation are short-arc determinations of the orbital elements relative to the Moon. PEST can process all or any subset of the residuals in the elements. The elements can be osculating or averaged values of either classical form or Gaussian.

Methods 4 or 5: \( p, e, M, \omega, i, \Omega \)

Method 7: \( p, e \sin \omega, e \cos \omega, \omega + f, i, \Omega \)

Method 8: \( p, e \sin \omega, e \cos \omega, \omega + M, i, \Omega \)

The measurement data can be

a. read in from magnetic tape
b. read in on cards
c. simulated by the program itself

Measurement sensitivities can be generated either by the secant method or by integration of linear variational equations. The sensitivities may be "frozen" at some iteration of the estimation process, thus saving the computation time needed to recompute sensitivities. The estimated state increment may be included with a gain factor other than unity to combat nonlinearity problems. Selected parameters may be held fixed during the estimation while they are being "considered" in the calculation of the state increment.
1.2 Operating Instructions

Figure 1.1 illustrates the order in which PEST accepts and expects input (card) data. The READ statements occur in only two subroutines, INPUTF and FILTER. INPUTF reads data into INPUT common, reading three (address, value) pairs per card with a \((1X, 3(I5, D15.8))\) format. A blank or zero in the first address field causes a return from INPUTF.

The first or only card read by FILTER is to identify the parameters to be estimated and to flag the primary options of this capability. The next card is a measurement selector. FILTER next expects measurement data. If an end of file is encountered at this point, the (only) case will be executed. If another case is to follow the current one, an \(F10.2\) zero must be read when FILTER expects a measurement. Control then is returned to the main program and subsequently to INPUTF before returning to FILTER.

1. INPUT common data
2. Blank or 16 zero to exit from INPUTF
3. IX array card(s) to identify state variables and primary options
   * 4. Measurement selector card
   * 5. Considered parameters card
   * 6. Measured elements and their epochs
   ** 7. Blank or \(F10.2\) zero to signal "data for next case following"
   ** 8. INPUT common data for case 2
   ** 9. Steps 2-7 for case 2, etc.

* optional
** needed only if more than one case is to be run
Figure 1.1 Input Data Read Logic
INPUT Common Data to Consider

General

1044   MODE    Flag to invoke PEST. Set to 8.
1030   KOUT    Output flag. Set to -1.
1033   KRW     Read-write flag for logical 22. Usually 1.
1049   KSAD    Shadow flag. Set to 0.

Trajectory Propagation

1036   KMETH   Propagation method flag. May be 4, 5, 7, or 8 unless KVAR = 1.

4      TFINAL  Final time (not used in estimation, where measurements control stop).

180    DELT    Integration step size (see KMETH).

1068   NINT    Number of intervals used in averaging.

1069   NORD    Number of ordinates used in averaging.

450    TOUT    Print table (set it larger than last measurement time).

460    STEP    Time interval for simulated measurements or when KRW is zero.

Field Model

1035   MODLEM  Lunar field model flag (3 for JPL 15, 8; 1 for L1).

1094   NMOD    Order of zonal harmonic model.

1095   MMOD    Order of tesseral harmonic model.

1084   KSOLP   Solar pressure flag (must be nonzero if SP is estimated).

Epoch

50-55   Input epoch to which all times are referred.

Measurement Weights

56-76   Measurement weighting matrix (upper half of symmetric matrix).
Initial Conditions

1019 KINPT Input coordinate flag (see KAVST).
30-35 ELM1 Initial elements (see KAVST).
1098 KAVST Averaging and initial condition flag.
   0 neither input elements nor measurements are averaged.
   1 average input elements if KMETH is 5 or 8.
   2 average measurements but not initial elements.
   3 average both initial elements and measurements.
   -1 average nothing, but set initial elements to first measurement.
   -2 average measurements, set initial elements to first measurement.

Sensitivity Calculation

1096 KVAR Sensitivity method flag. 0 for secant, 1 for variational.
1064 NGROPT Number of trials for which sensitivities are to be recomputed.
1086 KZIP Flag for ignoring implicit term in variational equations.

Specification of State Variables and Options

The IX array is read in by means of one or more records (cards) written in a 20I3 format. Reading of these records is terminated when the (N+1)st element of IX is negative or zero, or when five records (100 elements) have been read. Valid identification numbers are:

Field coefficients IX(i) = 2-271 except MOD(IX(i), 16) = 0 or when IX(i) lies outside the field specified by NMOD, MMOD.

Initial conditions IX(i) = 273-278.
Solar pressure IX(i) = 279, 280.

The first nonpositive identifier controls the operational logic:

IX(1) = 0 Generate simulated measurement data.
IX(N+1) = 0 Estimate the state if N is greater than zero.
IX(N+1) = -1 Compute measurement residuals and (if N is greater than 0) compute/print measurement sensitivities without actually estimating a new state.
The field identifiers can be obtained with the assistance of Table 1.1. Find the appropriate coefficient in the table and add its row-number (shown on the right) to the number shown above its column. This sum is the identifier to use in the IX array. The order in which you specify these numbers is now immaterial — IX will be sorted appropriately by the program.

Examples:

\[0\] Simulated data will be generated from time zero to TFINAL by STEP.
\[2\ -1\] Residuals and sensitivity to \(C_{20}\) will be printed.
\[34274\ 0\] Estimate \(C_{22}\) and either \(e_0\) or \(e_0\cos\omega_0\), depending on KMETH.

**Measurement Selector Card**

The measurement selector (IM) card is read by means of a 20I3 format. The first six words are the numbers (1-6) of the elements to be used in processing. A zero in the field will exclude the corresponding element. (1, 2, 3, 0, 5, 6 excludes \(\omega+f\) residuals if Method 7, \(\omega+M\) if Method 8, is used, or \(\omega\) if Methods 4 or 5 are used in propagation.) The next number is the number of observations to be processed. If this is greater than the number of observations to be found on tape, only the smaller number will be processed, of course. The next number dictates IQ, the weighting matrix treatment. If IQ is 1, \(Q\) is I; if IQ is 2, \(Q\) is constant; and if IQ is 3, \(Q\) is variable and is read in from the tape with the observation. In this latter case, the input weighting matrix is assumed to describe the uncertainty in the Cartesian state and will be converted appropriately by the program to an element weighting matrix. If \(Q\) is to be treated as a constant, it must be read in (locations 56 through 76) as an element weighting matrix, as it will not be converted by the program. In either case, \(Q\) must be positive definite.

ITMAX, the next indicator on the card, limits the number of iterations allowed in estimation if convergence is not achieved. The estimator will quit after passing through the data span ITMAX+1 times. The next indicator on the card is IGAIN,
| Field Identifier Aid | 0 | 16 | 32 | 48 | 64 | 80 | 96 | 112 | 128 | 144 | 160 | 176 | 192 | 208 | 224 | 240 | 256 |
|----------------------|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| c_{1,0}              | c_{1,1} | s_{1,1} | s_{2,1} | s_{3,1} | s_{4,1} | s_{5,1} | s_{6,1} | s_{7,1} | s_{8,1} | s_{9,1} | s_{10,1} | s_{11,1} | s_{12,1} | s_{13,1} | s_{14,1} | s_{15,1} |
| c_{2,0}              | c_{2,1} | c_{2,2} | s_{2,2} | s_{3,2} | s_{4,2} | s_{5,2} | s_{6,2} | s_{7,2} | s_{8,2} | s_{9,2} | s_{10,2} | s_{11,2} | s_{12,2} | s_{13,2} | s_{14,2} | s_{15,2} |
| c_{3,0}              | c_{3,1} | c_{3,2} | c_{3,3} | s_{3,3} | s_{4,3} | s_{5,3} | s_{6,3} | s_{7,3} | s_{8,3} | s_{9,3} | s_{10,3} | s_{11,3} | s_{12,3} | s_{13,3} | s_{14,3} | s_{15,3} |
| c_{4,0}              | c_{4,1} | c_{4,2} | c_{4,3} | c_{4,4} | s_{4,4} | s_{5,4} | s_{6,4} | s_{7,4} | s_{8,4} | s_{9,4} | s_{10,4} | s_{11,4} | s_{12,4} | s_{13,4} | s_{14,4} | s_{15,4} |
| c_{5,0}              | c_{5,1} | c_{5,2} | c_{5,3} | c_{5,4} | c_{5,5} | s_{5,5} | s_{6,5} | s_{7,5} | s_{8,5} | s_{9,5} | s_{10,5} | s_{11,5} | s_{12,5} | s_{13,5} | s_{14,5} | s_{15,5} |
| c_{6,0}              | c_{6,1} | c_{6,2} | c_{6,3} | c_{6,4} | c_{6,5} | c_{6,6} | s_{6,6} | s_{7,6} | s_{8,6} | s_{9,6} | s_{10,6} | s_{11,6} | s_{12,6} | s_{13,6} | s_{14,6} | s_{15,6} |
| c_{7,0}              | c_{7,1} | c_{7,2} | c_{7,3} | c_{7,4} | s_{7,4} | s_{8,4} | s_{9,4} | s_{10,4} | s_{11,4} | s_{12,4} | s_{13,4} | s_{14,4} | s_{15,4} | s_{16,4} | s_{17,4} | s_{18,4} |
| c_{8,0}              | c_{8,1} | c_{8,2} | c_{8,3} | c_{8,4} | c_{8,5} | c_{8,6} | c_{8,7} | c_{8,8} | c_{8,9} | c_{8,10} | c_{8,11} | c_{8,12} | c_{8,13} | c_{8,14} | c_{8,15} | c_{8,16} |
| c_{9,0}              | c_{9,1} | c_{9,2} | c_{9,3} | c_{9,4} | c_{9,5} | c_{9,6} | c_{9,7} | c_{9,8} | c_{9,9} | c_{9,10} | c_{9,11} | c_{9,12} | c_{9,13} | c_{9,14} | c_{9,15} | c_{9,16} |
| c_{10,0}             | c_{10,1} | c_{10,2} | c_{10,3} | c_{10,4} | c_{10,5} | c_{10,6} | c_{10,7} | c_{10,8} | c_{10,9} | c_{10,10} | c_{10,11} | c_{10,12} | c_{10,13} | c_{10,14} | c_{10,15} | c_{10,16} |
| c_{11,0}             | c_{11,1} | c_{11,2} | c_{11,3} | c_{11,4} | c_{11,5} | c_{11,6} | c_{11,7} | c_{11,8} | c_{11,9} | c_{11,10} | c_{11,11} | c_{11,12} | c_{11,13} | c_{11,14} | c_{11,15} | c_{11,16} |
| c_{12,0}             | c_{12,1} | c_{12,2} | c_{12,3} | c_{12,4} | c_{12,5} | c_{12,6} | c_{12,7} | c_{12,8} | c_{12,9} | c_{12,10} | c_{12,11} | c_{12,12} | c_{12,13} | c_{12,14} | c_{12,15} | c_{12,16} |
| c_{13,0}             | c_{13,1} | c_{13,2} | c_{13,3} | c_{13,4} | c_{13,5} | c_{13,6} | c_{13,7} | c_{13,8} | c_{13,9} | c_{13,10} | c_{13,11} | c_{13,12} | c_{13,13} | c_{13,14} | c_{13,15} | c_{13,16} |
| c_{14,0}             | c_{14,1} | c_{14,2} | c_{14,3} | c_{14,4} | c_{14,5} | c_{14,6} | c_{14,7} | c_{14,8} | c_{14,9} | c_{14,10} | c_{14,11} | c_{14,12} | c_{14,13} | c_{14,14} | c_{14,15} | c_{14,16} |
| c_{15,0}             | c_{15,1} | c_{15,2} | c_{15,3} | c_{15,4} | c_{15,5} | c_{15,6} | c_{15,7} | c_{15,8} | c_{15,9} | c_{15,10} | c_{15,11} | c_{15,12} | c_{15,13} | c_{15,14} | c_{15,15} | c_{15,16} |
the reciprocal gain factor on the incremental state to be added to the state between trials. Thus, if IGAIN = 2, only half of the estimated increment will be added. LTOL, the next number on the card, controls the tolerance on the iteration as follows: TOL = 1.D–LTOL. The next and last number is an indicator for the "consider" option. If ICON = 0, nothing happens in this regard. If ICON ≠ 0, another card is expected to identify the "considered" parameters. The "considered" parameters are identified by the same code as the IX array.

Input of Measured Elements

The current means for input of measured elements is by means of DATA5. Each record contains 8 words written in an 8F10.2 format. These words are:

- YYMMDD Year, month, and day of epoch (710824 is 8/24/71)
- HHHMMSS Hour, minute, second UMT (183025.5 is 18:30:25.5)
- A Semi-major axis (km)
- E Eccentricity
- F True Anomaly (deg)
- W Argument of pericenter (deg)
- I Inclination (deg)
- O Longitude of the ascending node (deg)

If KMETH is 5 or 8 and KAVST is -2, 2, or 3, the input measurements are considered to be osculating and selenographic and will be averaged and converted to mean of 1950.0 before being written onto logical 22. If KMETH is 5 or 8 but KAVST is -1, 0, or 1, the input measurements (and epochs) will be considered to be already averaged and converted to mean of 1950.0, so will be simply converted to KMETH variables before being written onto 22. If KMETH is 4 or 7 (KAVST should then be -1, 0, or 1), the input measurements will be considered to be osculating and selenographic and will be converted to KMETH variables, mean of 1950.0. The longitude parameter for KMETH = 8 is treated as ω+f outside the propagation routines when computing residuals and sensitivities.
Input Measurements from GTDS

The GTDS measurements are expected on magnetic tape in the form of

| YYMMDD | Year, month, and day of epoch |
| HHMMSS | Hour, minute, and second of epoch |
| X       | Cartesian position components (km), selenocentric, Equator of 1950.0 |
| Y       |                                      |
| Z       |                                      |
| \dot{X} | Cartesian velocity components (km/sec), selenocentric, Equator of 1950.0 |
| \dot{Y} |                                      |
| \dot{Z} |                                      |

Definition of Peripheral Units

The parameter estimation program uses logical units 21, 22, and 23 as sequential datasets which must be defined in the JCL of the job setup. Logical 21 is used for input measurement data from GTDS. (Two words of epoch and six components of Cartesian state, EE50 in an 8D23.15 format, requiring a record length of 188 bytes.) Logical 22 is used for storage of the measured time and elements. It therefore requires a record length (LRECL) of 60 (8*7+4) or greater and a block-size of J*LRECL+4. Logical 23 is used as a scratch dataset for storage of the weighted sensitivities. It requires a record length of 48*N+4 or greater, where N is the number of state variables included in the case.

1.3 Input/Output Examples

Figure 1.2 shows the input which resulted in the output of Figure 1.3. The INPUTF data is abbreviated in Figure 1.2, but fully printed in Figure 1.3. The important numbers of the INPUTF data are:
### Figure 1.2 Estimation Input

<table>
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<td>31</td>
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<td>32</td>
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<td></td>
<td>OTHER INPUT DATA</td>
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### DATA CARDS FOR CASE 1

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<th>33</th>
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<th>35</th>
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<td>30 0.2838000000D 04</td>
<td>0.3557000000D 02</td>
<td>0.1235210000D 03</td>
<td>C4 31 C.35700CCCD-02</td>
<td>35</td>
<td>C.12699CCCD03</td>
<td>36</td>
</tr>
<tr>
<td>33 0.1846999000D 03</td>
<td>C6 0.1161588000D 04</td>
<td>0.1973000000D 04</td>
<td>34</td>
<td>C.3CCCC000000D 02</td>
<td>35</td>
<td>C.4000000000D 02</td>
</tr>
<tr>
<td>50 0.6000000000D 01</td>
<td>51 0.3570000000D 02</td>
<td>0.0 0.0 0.0</td>
<td>30 0.2838000000D 04</td>
<td>0.0 0.0 0.0</td>
<td>31</td>
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**INITIAL JULIAN DATE 2441664.1352**

**INITIAL CONDITIONS**

IN EARTH MEAN EQUATOR AND EQUINOX OF 1950

6/30/73 15 HRS 14 MINS 43.000 SECS

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**INITIAL METHOD 7 ELEMENTS AT TIME = 0.**
Figure 1.3 2937.66919 -0.00981 0.01592 5.71579 0.67350 2.73629

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ESTIMATED STATE INCREMENT (10 OBS)

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INITIAL METHOD 7 ELEMENTS AT TIME = 0.

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ESTIMATED STATE INCREMENT (10 OBS)

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TRIAL NO. 2

INITIAL METHOD 7 ELEMENTS AT TIME = 0.
Figure 1.3 (cont.)

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ESTIMATED STATE INCREMENT (10 OBS)

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PENALTY FUNCTION $D^*(Q**-1)*D^Y = 0.3357D-09$

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INITIAL METHOD 7 ELEMENTS AT TIME = 0

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ESTIMATED STATE INCREMENT (10 OBS)

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PENALTY FUNCTION $D^*(Q**-1)*D^Y = 0.7622D-11$

SQUARE-ROOT INFORMATION MATRIX

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</table>
MODE is 8 to signal parameter estimation

Output key is negative to suppress extra output

Zero signals no print of input common

3 signifies the JPL 15, 8 field is estimated

4 means the field is truncated at n=4

4 means the field is truncated at m=4

Method 7 propagation is used (Gauss' elements, no averaging)

The integration time step is 20 minutes

Initial conditions (time and elements) are taken from the measurement tape when KAVST = -1

KVAR is 1 to use linear variational equations for measurement sensitivities

KZIP is 1, but has no significance when 1036 is 7

The zero in column 6 of Figure 1.2 causes a return from INPUTF and continuation to FILTER where the next card is read. This card is to select parameters to be estimated. For the case shown, these were all of the gravitational coefficients of a 3, 3 field plus four initial elements. The significance of the numbers is indicated in the printout shown in Figure 1.3. The next card says that residuals in all six Method 7 elements are to be processed in the estimation. It also says that only the first 10 sets of elements on the measurement file are to be processed, that no weighting is to be done (IQ=1), that no more than three iterations are to take place, that unit gain is to be put on the included state increment, that the convergence tolerance is $10^{-8}$, and that no parameters are to be "considered." The floating-point zero which makes up the last card shown in Figure 1.2 indicates that there is no card-input measurement data for this case.

Referring now to Figure 1.3 under the heading "Parameter Estimation," we see eight columns of data. These are the measurements. Each row contains the date in the first two columns (YYMMDD, HHMMSS), then three components each of position
and velocity (selenocentric, EE50). The data shown here were generated by the GTDS program and placed on unit 21 for processing by PEST. They represented osculating sets generated by perturbing certain elements of the JPL 15, 8 field truncated to 4, 4. It was the intent of this PEST run to determine which coefficients had been perturbed and by how much. The Method 7 initial conditions are printed, followed by the residuals in Method 7 elements. The residuals are in kilometers and radians, while time is in days from epoch. The estimated state increment tells how much change is to be made in the estimated parameters from the residuals just printed. The penalty function printed out next is $\mathbf{Y} Q^{-1} \mathbf{Y}$ for all of the observations. This is the function to be minimized by the (weighted) least squares procedure. The estimation process takes three trials to converge because the RMS state increment is greater than $10^{-8}$ for the first two trials.

We may notice that the penalty function has reduced from 0.5796 to $7.622 \times 10^{-11}$ after three iterations. The square-root information matrix is printed out in its triangular form (folded by format after 6 rows). This matrix provides information on the observability of the various parameters being estimated. The post-estimation gravitational coefficient values are printed out finally. These indicate that $C(2, 0)$, $S(2, 1)$, $C(3, 0)$, and $S(3, 1)$ were perturbed (by comparing with the initially estimated field).

Although the initial orbital element estimates are not printed, their increments for the three trials add to zero (essentially), so we can assume that the residuals were entirely due to the four perturbed coefficients. The fact that the residuals (and penalty function) on the last trial were so near to zero indicates that no parameters other than those estimated were perturbed. The measurement sensitivities for this process were computed only on the first trial, then "frozen" for the remaining trials without corrupting convergence appreciably.

Figure 1.4 shows input for a two-case run. No output is shown for this run. The first case is a simulation of measurements. Initial elements are taken
Figure 1.4 Input for Simulation and Residual Print
from locations 30-35 in INPUT common, then converted according to KMETH (1036) for propagation. The time step for measurements (location 460) must also be provided for this case. The first zero in column 6 causes escape from INPUTF. The zero in column 3 tells FILTER that a simulation is to be performed. The floating-point zero is to "end" measurement data input from cards and signal another case following. The next 16 zero is read by INPUTF at the beginning of case 2, causing a return to FILTER. Notice that more INPUT cards could have been read in before leaving INPUTF. FILTER reads the next card. The -1 in column 12 tells FILTER not to call WLS, but to compute and print residuals. The 2, 18, and 49 indicate that sensitivities of the measurements to C(2, 0), C(2, 1), and S(2, 1) are to be printed. The only necessary quantities on the measurement selector card for this case are M and IQ. The last zero, which signals "no more" measurement data cards, is omitted since this is the last case of the run.
II. PEST PROGRAM DESCRIPTION (Programmer's Guide)

This section supplements Reference 2 by describing the PEST program insofar as it differs from MAESTRO.

2.1 Program Structure

Figure 2.1 is an overview of the structure of PEST and a flow chart of the main program or driver. The input and initialization functions for trajectory propagation are identical to those of MAESTRO. The primary difference is that PEST includes a path for MODE = 8, the parameter estimation mode. A lunar lifetime mode (6) is retained from the MAESTRO capabilities in order to verify the characteristics of an estimated lunar field. For further elaboration of PEST's structure, see first the flow chart for FILTER (Section 2.3) and then that for WLS.

2.2 Subroutine Cross-Reference and COMMON Blocks Descriptions

Table 2.1 lists the subroutines which make up the PEST program and defines their functions or purposes. It does not include those subroutines which are in common with MAESTRO.

Figure 2.2 is a cross-reference map for the parameter estimation portion of PEST. For further details, see Figures 3.1 and 3.3 of Reference 2.

Table 2.2 lists the COMMON blocks referred to by PEST subroutines. Those COMMON blocks marked by asterisks are found only in PEST — not in MAESTRO. The PEST-only COMMON blocks are described in Table 2.3. The others are described in Reference 2.
PEST Main Program Flow Chart

ENTER

Clear COMMON blocks

INPUTF
Read input data

no more data → STOP

CONTROl

SETUP
Initialize COMMON blocks and state

Set compute interval table

MODE

=6

FOWARD
Compute lunar lifetime case

=8

FILTER
Initialize for estimation

WLS
Estimate parameters from observational data

Figure 2.1 PEST Program Overview
<table>
<thead>
<tr>
<th>Subroutine</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAIN</td>
<td>Initiates and drives PEST</td>
</tr>
<tr>
<td>ACCEL</td>
<td>Determines spacecraft acceleration</td>
</tr>
<tr>
<td>AVEQNS</td>
<td>Averages derivatives of the elements over one revolution</td>
</tr>
<tr>
<td>AVSTRT</td>
<td>Averages osculating elements and variational equations</td>
</tr>
<tr>
<td>EQNS</td>
<td>Determines derivatives of the state for RKSEVN</td>
</tr>
<tr>
<td>FIELD2</td>
<td>Evaluates central aspherical disturbing acceleration and its gradient</td>
</tr>
<tr>
<td>FILTER</td>
<td>Initiates the parameter estimation process</td>
</tr>
<tr>
<td>GRAV</td>
<td>Evaluates disturbing acceleration from noncentral bodies</td>
</tr>
<tr>
<td>MATMPY</td>
<td>Matrix multiplication routine</td>
</tr>
<tr>
<td>MODEL</td>
<td>Controls state and sensitivity propagation</td>
</tr>
<tr>
<td>OSCAR</td>
<td>Averages osculating elements</td>
</tr>
<tr>
<td>OUTPUT</td>
<td>Prints trajectory and sensitivity information</td>
</tr>
<tr>
<td>PHOUSE</td>
<td>Performs matrix reduction by Householder's algorithm</td>
</tr>
<tr>
<td>PUTINV</td>
<td>Inverts a packed upper triangular matrix</td>
</tr>
<tr>
<td>REDATA</td>
<td>Initializes and reads measurement file</td>
</tr>
<tr>
<td>RKSEVN</td>
<td>Seventh-order Runge-Kutta integration routine</td>
</tr>
<tr>
<td>SHIMMY</td>
<td>Evaluates derivative for linear variational equations</td>
</tr>
<tr>
<td>SOLVE</td>
<td>Solves weighted least squares equation for state increment</td>
</tr>
<tr>
<td>SORDERD</td>
<td>Sorts and prints parameters to be estimated</td>
</tr>
<tr>
<td>SPN2</td>
<td>Evaluates Legendre polynomials for FIELD2</td>
</tr>
<tr>
<td>SQRUT</td>
<td>Computes triangular square root of P-D symmetric matrix</td>
</tr>
<tr>
<td>TIMEC</td>
<td>Controls logic for trajectory propagation</td>
</tr>
<tr>
<td>TUMULT</td>
<td>Multiplies by a packed upper triangular matrix</td>
</tr>
<tr>
<td>UPDATX</td>
<td>Updates state with the estimated increment</td>
</tr>
<tr>
<td>WLS</td>
<td>Weighted least squares estimation routine</td>
</tr>
</tbody>
</table>
Figure 2.2 Parameter Estimation Cross-Reference Map

- REDATA - DATE, OSCAR, ORBIT, TRMN, MAPMPY
  - MODEL - FOWARD, ORBIT, TRMN
    - REDATA
    - MODEL
      - SQRUT
      - PUTINV
      - TUMULT
      - PHOUSE
      - SOLVE
      - UPDATX
  - OSCAR - M50LEQ, ORBIT, MVTRN, TRMN, PLANET, INTEG
  - SORDID, ORBIT, TRMN, DATE, M50LEQ, MVTRN, CALEND
<table>
<thead>
<tr>
<th>COMMON</th>
<th>PEST Subroutines Referring to this COMMON</th>
</tr>
</thead>
<tbody>
<tr>
<td>BLANK</td>
<td>OUTPUT</td>
</tr>
<tr>
<td>AVG</td>
<td>MAIN, AVEQNS</td>
</tr>
<tr>
<td>CETBL2</td>
<td>MAIN</td>
</tr>
<tr>
<td>CETBL3</td>
<td>MAIN</td>
</tr>
<tr>
<td>*CNSDR</td>
<td>FILTER, UPDATX</td>
</tr>
<tr>
<td>CNTRL</td>
<td>MAIN, ACCEL, AVEQNS, AVSTR, EQNS, FIELD2, GRAV, OUTPUT, RKSEVN, SHIMMY, TIMEC</td>
</tr>
<tr>
<td>*COMIC</td>
<td>REDATA, UPDATX</td>
</tr>
<tr>
<td>CONST</td>
<td>MAIN, AVEQNS, AVSTR, EQNS, FIELD2, FILTER, GRAV, MODEL, OSCAR, OUTPUT, REDATA, SHIMMY, TIMEC, WLS</td>
</tr>
<tr>
<td>*DUMY</td>
<td>MAIN, AVEQNS, EQNS, FIELD2, GRAV</td>
</tr>
<tr>
<td>*DXCOM</td>
<td>MODEL, PHOUSE, REDATA, SOLVE, UPDATX, WLS</td>
</tr>
<tr>
<td>FIELDM</td>
<td>MAIN, FIELD2, FILTER, MODEL, SHIMMY, SORDII, UPDATX</td>
</tr>
<tr>
<td>*FIELDX</td>
<td>ACCEL, FIELD2, GRAV, OUTPUT, SHIMMY</td>
</tr>
<tr>
<td>GRAVITY</td>
<td>ACCEL, AVEQNS, EQNS, FIELD2, GRAV, OUTPUT</td>
</tr>
<tr>
<td>*HELMO</td>
<td>FILTER, MODEL, REDATA</td>
</tr>
<tr>
<td>*IMCOM</td>
<td>FILTER, MODEL, REDATA, UPDATX, WLS</td>
</tr>
<tr>
<td>INPUT</td>
<td>MAIN, ACCEL, AVEQNS, AVSTR, EQNS, FIELD2, FILTER, GRAV, MODEL, OSCAR, OUTPUT, REDATA, RKSEVN, SHIMMY, SORDII, TIMEC, UPDATX, WLS</td>
</tr>
<tr>
<td>INPUTS</td>
<td>MAIN</td>
</tr>
<tr>
<td>INTER</td>
<td>TIMEC</td>
</tr>
<tr>
<td>INTVAR</td>
<td>MAIN, AVEQNS, AVSTR, EQNS, FIELD2, GRAV, OSCAR, OUTPUT, RKSEVN, SHIMMY, TIMEC</td>
</tr>
<tr>
<td>*INTVRX</td>
<td>MAIN, AVEQNS, AVSTR, EQNS, FILTER, MODEL, OSCAR, OUTPUT, REDATA, RKSEVN, SHIMMY, TIMEC, WLS</td>
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<tr>
<td><strong>IXCOM</strong></td>
<td><strong>PERT</strong></td>
</tr>
<tr>
<td>MAIN, AVSTRT, FIELD2, FILTER, MODEL, OUTPUT, PHOUSE, REDATA, SHIMMY, SOLVE, SORDID, UPDATX, WLS</td>
<td>MAIN, ACCEL, AVEQNS, EQNS, FIELD2, GRAV, SHIMMY, TIMEC</td>
</tr>
<tr>
<td><strong>YCOM</strong></td>
<td>FILTER, MODEL, REDATA, WLS</td>
</tr>
<tr>
<td>Block</td>
<td>Length (DP words)</td>
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<tr>
<td>-----------</td>
<td>-------------------</td>
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<td>COMIC</td>
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<td>FIELDX</td>
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<td>HELMO</td>
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<td></td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
IQ

Weighting treatment key

ITMAX

Maximum iterations allowed

ID

Observation counter \((1 \leq ID \leq M)\)

IT

Iteration counter \((1 \leq IT \leq ITMAX)\)

YHAT

Communicates time, measurement, estimate, and weight

TI

Current time

YHAT(6)

Estimated elements

TF

Next measurement time

Y(6)

Measured elements

Q(36)

Measurement weighting matrix
2.3 Subroutine Descriptions

Documentation is included here for those subroutines which are uniquely required by PEST. PEST requires many of the subroutines of MAESTRO, but since these are described in detail in Reference 2, only differences between MAESTRO and PEST versions of the common routines will be discussed. Table 2.4 categorizes the subroutines.

### TABLE 2.4

**PEST/MAESTRO SUBROUTINES**

<table>
<thead>
<tr>
<th>PEST Subroutines</th>
<th>Modified MAESTRO Subroutines</th>
</tr>
</thead>
<tbody>
<tr>
<td>FILTER</td>
<td>MAIN</td>
</tr>
<tr>
<td>MODEL</td>
<td>ACCEL</td>
</tr>
<tr>
<td>OSCAR</td>
<td>AVEQNS</td>
</tr>
<tr>
<td>HOUSE</td>
<td>AVSTRT</td>
</tr>
<tr>
<td>PUTINV</td>
<td>EQNS</td>
</tr>
<tr>
<td>REDATA</td>
<td>FIELD2</td>
</tr>
<tr>
<td>SHIMMY</td>
<td>GRAV</td>
</tr>
<tr>
<td>SOLVE</td>
<td>MATMPY</td>
</tr>
<tr>
<td>SORDID</td>
<td>OUTPUT</td>
</tr>
<tr>
<td>SQRUT</td>
<td>RKSEVN</td>
</tr>
<tr>
<td>TUMULT</td>
<td>SPNM</td>
</tr>
<tr>
<td>UPDATX</td>
<td>TIMEC</td>
</tr>
<tr>
<td>WLS</td>
<td></td>
</tr>
</tbody>
</table>

The primary modification to the MAESTRO subroutines is to increase the size of INTVRX common from 13 to 1213 to permit calculation of 100 sets of linear variational equations. Other differences are noted below.

- **MAIN**
  - FILTER is called when MODE = 8

- **ACCEL**
  - Contains FIELDX common and initializes DVDX2 array for linear variational equations (LVE) computation
AVEQNS  Averages linear variational equations as well as elements
AVSTRT  Averages initial LVE as well as initial elements
EQNS    Calls SHIMMY in computation of LVE
FIELD2  Computes acceleration sensitivities to gravitational coefficients for LVE
GRAV    Computes gradient of non-central-body acceleration for LVE
MATMPY  Includes a \( C=AB^T \) call (MATMPT)
OUTPUT  Includes extra output for LVE
RKSEVN  Has larger internal storage for INTVRX variables
SPNM    Generates Legendre polynomials a row-at-a-time for FIELD2
TIMEC   Only differences are for larger INTVRX common
SUBROUTINE FILTER

Calling Sequence:

CALL FILTER

Purpose:

FILTER initializes the estimation process and controls (1) simulated data generation and (2) nonestimation sensitivity and residual calculations.

Common Blocks Required:

CNSDR, CONST, FIELDM, HELMO, IMCOM, INPUT, INTVRX, IXCOM, STATE, YCOM

Subroutines Called:

CALEND, DATE, MODEL, MVTRN, M50LEQ, ORBIT, OSCAR, REDATA, SORDID, TRMN, WLS

Input/Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>PI, PI2</td>
<td>1</td>
<td>CONST(2,3)</td>
<td>$\pi, 2\pi$</td>
</tr>
<tr>
<td>I</td>
<td>GM</td>
<td>12</td>
<td>CONST(5)</td>
<td>Planetary gravitational constants</td>
</tr>
<tr>
<td>I</td>
<td>ZONL</td>
<td>288</td>
<td>FIELDM(1)</td>
<td>Lunar field coefficients</td>
</tr>
<tr>
<td>O</td>
<td>ELMO</td>
<td>6</td>
<td>HELMO(1)</td>
<td>Estimated orbital elements</td>
</tr>
<tr>
<td>O</td>
<td>H</td>
<td>606</td>
<td>INTVRX(1)</td>
<td>Measurement sensitivities</td>
</tr>
<tr>
<td>I</td>
<td>TFINAL</td>
<td>1</td>
<td>INPUT(4)</td>
<td>Final time</td>
</tr>
<tr>
<td>I</td>
<td>COV</td>
<td>21</td>
<td>INPUT(56)</td>
<td>Input weighting matrix</td>
</tr>
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<td>I</td>
<td>A(197)</td>
<td>2</td>
<td>INPUT(197)</td>
<td>Solar pressure coefficients</td>
</tr>
<tr>
<td>I</td>
<td>STEP</td>
<td>1</td>
<td>INPUT(460)</td>
<td>Simulated measurement time increment</td>
</tr>
<tr>
<td>O</td>
<td>KRASH</td>
<td>1</td>
<td>INPUT(1032)</td>
<td>Integration stop type key</td>
</tr>
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<td>KMETH</td>
<td>1</td>
<td>INPUT(1036)</td>
<td>Integration method key</td>
</tr>
<tr>
<td>I/O</td>
<td>Symbolic Name</td>
<td>Program Dimension</td>
<td>Common Block</td>
<td>Dimension</td>
</tr>
<tr>
<td>-----</td>
<td>---------------</td>
<td>-------------------</td>
<td>--------------</td>
<td>-----------</td>
</tr>
<tr>
<td>O</td>
<td>IX</td>
<td>100</td>
<td>IXCOM(1)</td>
<td>Estimated state identifiers</td>
</tr>
<tr>
<td>O</td>
<td>N</td>
<td>1</td>
<td>IXCOM(101)</td>
<td>Number of state components</td>
</tr>
<tr>
<td>O</td>
<td>MI</td>
<td>1</td>
<td>IXCOM(102)</td>
<td>Number of observables at each measurement</td>
</tr>
<tr>
<td>O</td>
<td>Y</td>
<td>6</td>
<td>YCOM(9)</td>
<td>Propagated elements</td>
</tr>
<tr>
<td>O</td>
<td>Q</td>
<td>36</td>
<td>YCOM(15)</td>
<td>Measurement weighting matrix</td>
</tr>
<tr>
<td>O</td>
<td>ICNSDR</td>
<td>20</td>
<td>CNSDR(1)</td>
<td>&quot;Considered&quot; parameter identifiers</td>
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<td>DJO</td>
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<td>INPUT(46)</td>
<td>Initial Julian date</td>
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<tr>
<td>I</td>
<td>KVAR</td>
<td>1</td>
<td>INPUT(1096)</td>
<td>Key for sensitivity generation method</td>
</tr>
<tr>
<td>I</td>
<td>KAVST</td>
<td>1</td>
<td>INPUT(1098)</td>
<td>Key for averaging elements, setting I,C,'s</td>
</tr>
<tr>
<td>O</td>
<td>NQ</td>
<td>1</td>
<td>INTVRX(1213)</td>
<td>Number of sensitivity elements</td>
</tr>
<tr>
<td>I</td>
<td>IM</td>
<td>6</td>
<td>IMCOM(1)</td>
<td>Measurement selector array</td>
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<td>I</td>
<td>IGAIN</td>
<td>1</td>
<td>IMCOM(7)</td>
<td>State increment inclusion number</td>
</tr>
<tr>
<td>I</td>
<td>LTOL</td>
<td>1</td>
<td>IMCOM(8)</td>
<td>Estimation iteration tolerance number</td>
</tr>
<tr>
<td>O</td>
<td>ID</td>
<td>1</td>
<td>IXCOM(107)</td>
<td>Measurement counter (set for REDATA)</td>
</tr>
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<td>O</td>
<td>IT</td>
<td>1</td>
<td>IXCOM(108)</td>
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</tr>
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<td>O</td>
<td>EJO</td>
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<td>STATE(26)</td>
<td>Ephemeris time epoch</td>
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<td>O</td>
<td>M</td>
<td>1</td>
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<td>Number of measurements to be processed</td>
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<td>O</td>
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<td>IXCOM(106)</td>
<td>Maximum iterations in WLS</td>
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<td>I</td>
<td>X</td>
<td>6</td>
<td>STATE(1)</td>
<td>Initial estimated Cartesian state</td>
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<tr>
<td>O</td>
<td>TI</td>
<td>1</td>
<td>YCOM(1)</td>
<td>Previous measurement time</td>
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<tr>
<td>O</td>
<td>TF</td>
<td>1</td>
<td>YCOM(8)</td>
<td>Current measurement time</td>
</tr>
</tbody>
</table>
Discussion

FILTER performs three basic functions

1. It initializes for WLS, the estimator
2. It contains logic for generating a simulated data file.
3. It contains logic for reading the data file, printing measurements, residuals and sensitivities.

The IX array is read in from the input unit. It is this array which determines which function FILTER will perform:

- IX(1) ≤ 0 , N = 0 , generate simulated data file
- IX(1) > 0 , IX(N+1) 0 , call the estimator, WLS
- IX(1) > 0 , IX(N+1) < 0 , read data file, compute and print
FILTER FLOW CHART

Read in identifiers, IX and count them (N)

0 IX(1) ≠ 0

Read measurement array, IM and count them (M1)

ICON ≠ 0

Read "considered" identifiers, ICNSDR

0 N ≠ 0

Call SORDID to sort and print state elements to be estimated

310 KAVST <0 350

Convert input Cartesian state to mean (orbit) or averaged (OSCAR) elements and store them in ELMO
Read cards for measurement input data

\[ \text{DAT} = 0 \]

Convert input calendar date to Julian date, input selenographic elements to Cartesian EE50

Write calendar date and Cartesian EE50 state on logical unit 21

\[ \text{N} > 0 \]

Simulated Measurement Loop
 Call MODEL to update elements.
 Convert time from epoch to calendar date and elements to Cartesian EE50 state

Write calendar date and Cartesian EE50 state on logical unit 21

\[ \text{TFINAL} \leq \text{Tfinal} \]

Return
SIX(N+1)

No-Estimation Propagation Loop
Call REDATA to read 21 for measurement data.
Call MODEL to update elements.
Compute and print residuals.
Print measurement sensitivities.

Parameter Estimation Loop
Convert weighting matrix.
Call WLS to perform the estimation process.
Call SORDID to print the resulting parameters.

Return

Return
SUBROUTINE MODEL

Calling Sequence: CALL MODEL

Purpose: MODEL computes the estimated measurement (orbital elements) and secant partials of the measurement with respect to state at a specified time.

Common Blocks Required: CONST, DXCOM, FIELDM, HELMO, IMCOM, INPUT, INTVRX, IXCOM, STATE, YCOM

Subroutines Required: FOWARD, ORBIT, TRMN

Input/Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
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</thead>
<tbody>
<tr>
<td>I</td>
<td>GM</td>
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<td>CONST(5)</td>
<td>Gravitational constant array (GM)</td>
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<td>QH</td>
<td>5650</td>
<td>DXCOM(201)</td>
<td>Measurement sensitivities to state</td>
</tr>
<tr>
<td>I/O</td>
<td>ZONL</td>
<td>16,18</td>
<td>FIELDM(1)</td>
<td>Lunar field harmonic coefficients</td>
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<tr>
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<td>6</td>
<td>HELMO(1)</td>
<td>Estimated elements</td>
</tr>
<tr>
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<td>EODINK</td>
<td>6</td>
<td>HELMO(7)</td>
<td>Initial element increments for partials</td>
</tr>
<tr>
<td>I/O</td>
<td>H</td>
<td>606</td>
<td>INTVRX(1)</td>
<td>Measurement sensitivities to state</td>
</tr>
<tr>
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<td>INTVRX(1213)</td>
<td>Sensitivity computation key</td>
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<td>1</td>
<td>INPUT(4)</td>
<td>Integration stop time</td>
</tr>
<tr>
<td>I/O</td>
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<td>INPUT(198)</td>
<td>Solar radiation pressure coefficients</td>
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<td>1</td>
<td>INPUT(1036)</td>
<td>Integration method type</td>
</tr>
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<td>I</td>
<td>KVAR</td>
<td>1</td>
<td>INPUT(1096)</td>
<td>Sensitivity computation type key</td>
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<td>I</td>
<td>IX</td>
<td>N</td>
<td>IXCOM(1)</td>
<td>State component identifier array</td>
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<td>I</td>
<td>N</td>
<td>1</td>
<td>IXCOM(101)</td>
<td>Number of state components</td>
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<tr>
<td>I/O</td>
<td>Symbolic Name</td>
<td>Program Dimension</td>
<td>Common Block</td>
<td>Definition</td>
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<td>--------------</td>
<td>------------</td>
</tr>
<tr>
<td>I</td>
<td>NQH</td>
<td>1</td>
<td>IXCOM(104)</td>
<td>Number of square-root matrix elements</td>
</tr>
<tr>
<td>O</td>
<td>X</td>
<td>6</td>
<td>STATE(1)</td>
<td>Cartesian state of spacecraft</td>
</tr>
<tr>
<td>I</td>
<td>X(36)</td>
<td>1</td>
<td>STATE(36)</td>
<td>Solar pressure coefficient</td>
</tr>
<tr>
<td>O</td>
<td>T</td>
<td>1</td>
<td>STATE(10)</td>
<td>Time</td>
</tr>
<tr>
<td>O</td>
<td>ELM</td>
<td>6</td>
<td>STATE(14)</td>
<td>Orbital elements of spacecraft</td>
</tr>
<tr>
<td>I</td>
<td>TI</td>
<td>1</td>
<td>YCOM(1)</td>
<td>Entry time</td>
</tr>
<tr>
<td>O</td>
<td>YHAT</td>
<td>6</td>
<td>YCOM(2)</td>
<td>Estimated measurement (elements)</td>
</tr>
<tr>
<td>I</td>
<td>TF</td>
<td>1</td>
<td>YCOM(8)</td>
<td>Exit or measurement time</td>
</tr>
<tr>
<td>I</td>
<td>IM</td>
<td>6</td>
<td>IMCOM(1)</td>
<td>Measurement identifiers</td>
</tr>
</tbody>
</table>

Discussion

MODEL calls FOWARD to propagate the estimated orbital elements, ELMO, from TI to TF. Secant partials of the estimated measurement, YHAT = ELMO, are computed in H and in QH beginning with QH(NQH+1). The reason for double storage is that the QH array is destroyed by the estimation process in subroutine PHOUSE, while H is used to store initial conditions for integration between measurements. That is

\[
H_{ij} = \begin{cases} 
\frac{\delta YHAT_i(T)}{\delta C_j} & \text{for gravitational or solar pressure coefficients, } C_j \\
\frac{\delta YHAT_i(T)}{\delta ELMO_j(0)} & \text{for initial condition states, } ELMO_j(0) 
\end{cases}
\]

and is computed from

\[
H(TF) = \frac{YHAT(TF, \hat{S} + \Delta) - YHAT(TF, \hat{S})}{\Delta} = \frac{YHAT(TF, \hat{S} + \Delta) - ELMO(TF)}{\Delta}
\]
where \( \hat{S} \) is the estimated state and \( \Delta \) is a small change in \( \hat{S} \). Rather than carrying along \( YHAT(T,\hat{S}+\Delta) \) from measurement to measurement, \( YHAT(TI,\hat{S}+\Delta) \) is reconstructed from \( H \) to initiate integration over the next interval.

\[
YHAT(TI,\hat{S}+\Delta) = ELMO(TI) + H(TI) * \Delta
\]

If KVAR is nonzero, linear variational equations replace secant partials as the means for computing measurement sensitivities. These sensitivities reside in \( H \) (INTVRX) upon return from FOWARD.

The QH array (DXCOM) is upper-loaded from \( H \) according to the measurement identification array, IM. \( H \) is viewed as a 6xN array, while QH is viewed as an M1 x6 array, although both are programmed as single-dimensioned arrays.
MODEL Flow Chart

Initial Elements
ELMO

Save
ELMI = ELMO
ELM = ELMO

Call ORBIT to compute initial Cartesian state

Call FOWARD(2) to advance ELM to final time

Save
ELMO = ELM
YHAT = ELM

Computation of Secant Partials

DO Loop L = 1, N

(Initial element)
DINK = EODINK

(Gravity Coeff.)
1. Save the coeff.
2. DINK = 1*coeff.
3. Coeff. = Coeff + DINK

(273, 278)

(LX(L))

(Solar Pressure)
1. Save the coeff.
2. DINK = 1*coeff.
3. Coeff. = Coeff + DINK

(279, 280)

ELM = ELM + H*DINK

T = TI

Call ORBIT to compute initial Cartesian state

Call FOWARD(2) to advance ELM to final time

Compute H
H = (ELM - ELMO)/DINK

Restore gravity or solar pressure coeff. to saved value

Load H = QH

RETURN

=1 KVAR

=0

RETURN

NQ = 0

RETURN
SUBROUTINE OSCAR

Calling Sequence:

CALL OSCAR(KIN, TI, EI, TO, EO)

Purpose:

OSCAR converts osculating input (elements or Cartesian state) to averaged elements.

Common Blocks Required:

CONST, INPUT, INTVAR, INTVRX, STATE

Subroutines Called:

INTEG, MVTRN, M50LEQ, ORBIT, PLANET

Input/Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>KIN</td>
<td>1</td>
<td>Call LIST</td>
<td>Input type key (8: Cartesian; other: elements)</td>
</tr>
<tr>
<td>I</td>
<td>TI</td>
<td>1</td>
<td>Call LIST</td>
<td>Osculating reference time from epoch</td>
</tr>
<tr>
<td>I</td>
<td>EI</td>
<td>6</td>
<td>Call LIST</td>
<td>Input osculating elements</td>
</tr>
<tr>
<td>O</td>
<td>TO</td>
<td>1</td>
<td>Call LIST</td>
<td>Output reference time from epoch</td>
</tr>
<tr>
<td>O</td>
<td>EO</td>
<td>6</td>
<td>Call LIST</td>
<td>Averaged elements output</td>
</tr>
<tr>
<td>I</td>
<td>RAD, PI2</td>
<td>1</td>
<td>CONST(1)</td>
<td>Angular constants</td>
</tr>
<tr>
<td>I</td>
<td>GM</td>
<td>12</td>
<td>CONST(5)</td>
<td>Gravitational constants</td>
</tr>
<tr>
<td>I</td>
<td>DJO</td>
<td>1</td>
<td>INPUT(46)</td>
<td>Julian date at epoch</td>
</tr>
<tr>
<td>I</td>
<td>KMETH</td>
<td>1</td>
<td>INPUT(1036)</td>
<td>Trajectory propagation method key</td>
</tr>
<tr>
<td>I</td>
<td>XX</td>
<td>6</td>
<td>STATE(1)</td>
<td>Input osculating Cartesian state</td>
</tr>
<tr>
<td>I/O</td>
<td>X, H</td>
<td>1</td>
<td>INTVAR(1)</td>
<td>Integration time and step</td>
</tr>
<tr>
<td>I/O</td>
<td>Z</td>
<td>6</td>
<td>INTVRX(1)</td>
<td>Intermediate osculating elements</td>
</tr>
</tbody>
</table>
Discussion

OSCAR is very similar to AVSTRT, but may be called at times other than epoch to average osculating sets other than the estimate. It is used to average "measured" osculating sets for use with averaged propagation methods in estimation. The osculating elements are integrated forward over one mean orbital period (found by iteration). The integral is then divided by that period to render the averaged elements, which are taken to be referred to the center of the integration interval.
OSCAR

Save certain parameters.  
Reset.

8

KIN

≠8

Selenographic elements come in.  
Convert them to Cartesian EE50

20

Convert Cartesian state to 
appropriate elements

90

Initialize for averaging.  
Save integration variables.

Changed initial conditions, 
so call INTEG(-1)

100

Call INTEG to take one R-K step.  
Compute period of average orbit, 
compare it with the current time

Time is greater than \( \tau + \text{TOL} \),  
Compute step size to make \( T = \tau \) 
using Newton-Raphson formula.  
Re-initialize at previous time,  
continue

No

Save time and elements

\[ |T - \tau| < \text{TOL} \]?

Yes

\[ T > \tau - \text{TOL} \]?

No

Yes

Divide elements by \( T \) to average.  
Set output time to \( T/2 \).  
Restore saved parameters.

RETURN
SUBROUTINE PHOUSE

Calling Sequence: CALL PHOUSE


Common Blocks Required: DXCOM, IXCOM

Subroutines Called: None

Input / Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>N</td>
<td>1</td>
<td>IXCOM(101)</td>
<td>Columns of input and output matrix</td>
</tr>
<tr>
<td>I</td>
<td>MI</td>
<td>1</td>
<td>IXCOM(102)</td>
<td>Rows in excess of N</td>
</tr>
<tr>
<td>I</td>
<td>NQH</td>
<td>1</td>
<td>IXCOM(104)</td>
<td>N * (N+1)/2, non-zero elements of N x N</td>
</tr>
<tr>
<td>I/O</td>
<td>QH, WT</td>
<td>NQH+N*M1</td>
<td>DXCOM(1)</td>
<td>Square-root information matrix (packed upper-triangular)</td>
</tr>
</tbody>
</table>

Limitations: QH is singly-dimensioned as follows

\[
QH = ( w_{11}, w_{12}, w_{22}, w_{13}, w_{23}, \ldots, w_{mn}, h_{11}, h_{21}, \ldots, h_{m1}, h_{12}, h_{22}, \ldots, h_{m,n}, h_{NQH} \) 
\]

The input QH array is replaced by the output array. M1 must not exceed 6 and NQH+N*M1 must not exceed 105 unless program dimensions are changed.

Method

PHOUSE is a special-purpose implementation of Householder's Reduction Algorithm (1,2) for reducing a matrix to upper-triangular form. It is used in the weighted least squares
process to accumulate the information matrix, $H^T Q^{-1} H$, in upper-triangular square-root form. In this discussion, we can let $Q^{-1} = I$ without loss of generality.

Let us assume that at some point in a WLS process we have an upper-triangular SRIM (square-root information matrix), $W$, which is $N \times N$. That is, $W$ contains $N \times (N+1)/2$ potentially non-zero elements above the diagonal and zeros below. $W$ also contains all of the information from $k-1$ measurements.

$$W^T_{k-1} W_{k-1} = \sum_{i=1}^{k-1} H_i^T H_i$$  \hspace{1cm} (1)

In the equation, $H_i$ is the sensitivity of measurement to state at the $i$-th measurement.

We now consider what happens to $W$ when we add another measurement, whose sensitivity matrix is $H_k$. The new SRIM, $W_k$, must obey

$$W^T_k W_k = W^T_{k-1} W_{k-1} + H_k^T H_k$$  \hspace{1cm} (2)

which could be written

$$W^T_k W_k = \begin{bmatrix} W^T_{k-1} & H_k^T \end{bmatrix} \begin{bmatrix} W_{k-1} \\ H_k \end{bmatrix}$$  \hspace{1cm} (3)

We may look at the reduction problem then as the problem of finding an orthogonal matrix, $R$, such that

$$\begin{bmatrix} W_k \\ 0 \end{bmatrix} = R \begin{bmatrix} W_{k-1} \\ H_k \end{bmatrix}$$  \hspace{1cm} (4)

because

$$\begin{bmatrix} W_k \\ 0 \end{bmatrix}^T \begin{bmatrix} W_k \\ 0 \end{bmatrix} = \begin{bmatrix} W_{k-1} \\ H_k \end{bmatrix}^T R^T R \begin{bmatrix} W_{k-1} \\ H_k \end{bmatrix}$$  \hspace{1cm} (5)

is identical to equation (3) for orthogonal $R$. Let us now simplify the notation by dropping the subscripts and renaming so that equation (4) becomes (6).

$$W = RA$$  \hspace{1cm} (6)
Householder's choice of $R$ is the product of a sequence of reflection matrices (which are symmetric and orthogonal).

$$R = R^{(n)} R^{(n-1)} \ldots R^{(1)}$$

(Introduction of new indices is unavoidable)

$R^{(k)}$ is defined by

$$R^{(k)} = I - \beta_k W^{(k)} W^{(k)\top}$$

where $I$ is the $(n+m)$ identity, $W^{(k)}$ is a column vector to be developed, and

$$\beta_k = \frac{2}{\|W^{(k)}\|^2}.$$ 

The derivation of $W^{(k)}$ and $R^{(k)}$ is inductive. If $A$ of equation (6) is partitioned by columns, $a_1$, as in equation (10)

$$A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$$

we note from equation (2) that the first element of $W$ must be $\pm a_1$. We define

$$\alpha_1 = \left| a_1 \right| \text{sign } a_1.$$ (11)

If we choose $R^{(1)}$ to zero the subdiagonal elements of $W^{(1)}$, where $W^{(k)}$ is defined by equation (12),

$$W^{(k)} = R^{(k)} R^{(k-1)} \ldots R^{(1)} A = R^{(k)} W^{(k-1)}$$

we may solve for $W^{(k)}$ (which turns out to be the $k$-th column of $W^{(k)}$).

$$R^{(1)} a_1 = a_1 \beta \frac{W^{(1)} W^{(1)\top}}{a_1} = \alpha_1 e_1$$

In equation (13), $e_1$ is the first unit basis vector. The unit basis vector $e_1$ has 1 as its $i$-th component and zeroes elsewhere. We may solve (13) for $W^{(1)}$,

$$W^{(1)} = \frac{a_1 + \alpha_1 e_1}{\beta a_1\top W^{(1)}}.$$ (14)

but we do not know how to compute the denominator of (14). The length of $W^{(k)}$ in equation (8) is arbitrary, however, due to the definition of $\beta_k$, so we can simply define its length to be such that $\beta a_1\top W^{(1)} = 1$. Then we have equation (15).
\[ W^{(1)} = a_1 + \alpha_1 e_1. \] (15)

Having now solved for \( W^{(1)} \), we could form \( W^{(0)} \) from equation (12) by making the association, \( W^{(0)} = A \). The first column of \( W^{(1)} \) is zero, except for the first element, which is equal to \(-\alpha_1\). We could now repeat the process, equation (12), by looking for \( R^{(2)} \) which would zero the subdiagonal elements of the second column of \( W^{(2)} \) without disturbing the first column. We can preserve the first column of \( W^{(1)} \) under the transformation, \( R^{(2)} \), by setting the first element of \( W^{(2)} \) to zero. Then we have

\[
R^{(2)} W^{(1)} = W^{(1)} - \beta_2 W^{(2)} W^{(1)} W^{(1)} = W^{(1)}
\] (16)

The construction of \( W^{(2)} \) is otherwise identical to that of \( W^{(1)} \). Proceeding inductively, we can arrive at the following set of equations.

\[
\alpha_k = \sqrt{\sum_{i=k}^{m} \left[ W^{(k)}_{ik} \right]^2 \cdot \text{sign} \left[ W^{(k)}_{kk} \right]} \] (17)

\[
\beta_k = \frac{1}{\alpha_k \left( \alpha_k + W^{(k)}_{kk} \right)}
\] (18)

\[
W_{1i}^{(k)} = \begin{cases} 
0 ; & i < k \\
\alpha_k + W^{(k)}_{kk} ; & i = k \\
W^{(k)}_{ik} ; & i = k+1, n+m 
\end{cases}
\] (19)

\[
Y_{1i}^{(k)} = \begin{cases} 
0 ; & i < k \\
1 ; & i = k \\
\beta_k W^{(k)} W^{(k)}_{ik} ; & i = k+1, n+m 
\end{cases}
\] (20)

\[ W_{i}^{(k)} \text{ in equations (20) and(21) is the } i\text{-th column of } W^{(k)}. \]

\[ W_{i}^{(k+1)} = W_{i}^{(k)} - Y_{i}^{(k)} W^{(k)} ; \ i = k+1, n+m \] (21)

Congratulations to you if you have followed the development to this point. It was not easy to write or type it, either.
In coding this routine, some advantage has been taken of the fact that the initial matrix is partly upper triangular at entry. That is, $W_{k-1}$ of equation (4) is upper-triangular to begin with, so the dimension of $W_{(k)}$ in equation (19) needs only to be $m+1$ and the inner product of equation (20) needs to sum only $m$ multiplications. These advantages can be significant if $n$ is large.

References


SUBROUTINE PUTINV

Calling Sequence: CALL PUTINV (B, C, N)

Purpose: PUTINV inverts a packed upper triangular matrix.

Common Blocks Required: None

Subroutines Called: None

Input/Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Communication</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>B</td>
<td>N(N+1)/2</td>
<td>Call List</td>
<td>Input matrix (P.U.T.)</td>
</tr>
<tr>
<td>O</td>
<td>C</td>
<td>N(N+1)/2</td>
<td>Call List</td>
<td>Output matrix (inverse of B, P.U.T.)</td>
</tr>
<tr>
<td>I</td>
<td>N</td>
<td>I</td>
<td>Call List</td>
<td>Rows or columns of upper triangular matrix</td>
</tr>
</tbody>
</table>

Note: B and C are single-dimensioned arrays containing the potentially non-zero elements of an upper triangular matrix.

\[ B = (b_{11}, b_{12}, b_{22}, b_{13}, \ldots, b_{nn}) \]
Method

The input matrix, \( B \), and its inverse, \( C \), are treated as upper triangular arrays.

\[
B = \begin{bmatrix}
  b_{11} & b_{12} & b_{13} & \text{etc.} \\
  0 & b_{22} & b_{23} & \text{etc.} \\
  0 & 0 & b_{33} & \text{etc.} \\
  \text{etc.}
\end{bmatrix}
\]

They must obey \( CB = I \), which is equivalent to the following set of equations.

\[
\begin{align*}
  c_{11} b_{11} &= 1 \\
  c_{22} b_{22} &= 1 \\
  c_{11} b_{12} + c_{12} b_{22} &= 0 \\
  c_{33} b_{33} &= 1 \\
  c_{11} b_{13} + c_{12} b_{23} + c_{13} b_{33} &= 0 \\
  c_{22} b_{23} + c_{23} b_{33} &= 0 \\
  \text{etc.}
\end{align*}
\]

\[
\begin{align*}
  c_{11} &= 1/b_{11} \\
  c_{22} &= 1/b_{22} \\
  c_{12} &= -c_{22} c_{11} b_{12} \\
  c_{33} &= 1/b_{33} \\
  c_{13} &= -c_{33} (c_{11} b_{13} + c_{12} b_{23}) \\
  c_{23} &= -c_{33} (c_{22} b_{23}) \\
  \text{etc.}
\end{align*}
\]

\[
\sum_{j=1}^{n} c_{ij} b_{jk} = \delta_{ik}
\]

\[
\begin{align*}
  c_{kk} &= 1/b_{kk} \\
  c_{jk} &= -c_{kk} \left( \sum_{i=j}^{k-1} c_{ji} b_{ik} \right)
\end{align*}
\]

\( B \) and \( C \) are treated as single - dimensioned arrays, so the programming requires calculation of the appropriate single indices which correspond to the double indices used in the development.
SUBROUTINE REDATA

Calling Sequence:
CALL REDATA

Purpose:
REDATA reads the data file and loads measurements into common. It also initializes certain variables at the beginning of the case and at the beginning of each iteration.

Common Blocks Required:
COMIC, CONST, DXCOM, HELMO, IMCOM, INPUT, INTVRX, IXCOM, STATE, YCOM

Subroutines Called:
DATE, MATMPT, MATMPY, ORBIT, OSCAR, TRMN

Input/Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I/O</td>
<td>ELM1</td>
<td>6</td>
<td>COMIC(1)</td>
<td>Estimated initial elements</td>
</tr>
<tr>
<td>O</td>
<td>QH</td>
<td>5650</td>
<td>DXCOM(201)</td>
<td>Initial sensitivity matrix for WLS</td>
</tr>
<tr>
<td>I/O</td>
<td>ELM0</td>
<td>6</td>
<td>HELMO(1)</td>
<td>Initial elements for integrator</td>
</tr>
<tr>
<td>O</td>
<td>EODINK</td>
<td>6</td>
<td>HELMO(7)</td>
<td>Initial element increments for secant partials</td>
</tr>
<tr>
<td>O</td>
<td>H</td>
<td>606</td>
<td>INTVRX(1)</td>
<td>Initial sensitivity matrix for MODEL</td>
</tr>
<tr>
<td>I</td>
<td>N</td>
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<td>IXCOM(101)</td>
<td>Number of states</td>
</tr>
<tr>
<td>I</td>
<td>M1</td>
<td>1</td>
<td>IXCOM(102)</td>
<td>Number of observables at each measurement</td>
</tr>
<tr>
<td>O</td>
<td>M</td>
<td>1</td>
<td>IXCOM(104)</td>
<td>Number of measurements</td>
</tr>
<tr>
<td>I</td>
<td>NQH</td>
<td>1</td>
<td>IXCOM(105)</td>
<td>Number of elements in SRIM, N(N+1)/2</td>
</tr>
<tr>
<td>I</td>
<td>IQ</td>
<td>1</td>
<td>IXCOM(106)</td>
<td>Weighting flags</td>
</tr>
<tr>
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<td>ID</td>
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<td>IXCOM(108)</td>
<td>Measurement counter</td>
</tr>
<tr>
<td>I</td>
<td>IT</td>
<td>1</td>
<td>IXCOM(109)</td>
<td>Iteration counter</td>
</tr>
<tr>
<td>O</td>
<td>TI</td>
<td>1</td>
<td>YCOM(1)</td>
<td>Most recent measurement time</td>
</tr>
</tbody>
</table>
**Discussion**

Subroutine REDATA performs two functions: (1) loading the measurement from the data file and (2) initialization of certain program variables. The data file (logical 22) is read once on each entry with an unformatted binary read, expecting a time and six elements on each record. When an end-of-file is encountered, \( M \), the number to signal "no more data — estimate the state now" is sent to WLS. The initialization functions are:

1. Load the estimated initial element vector with elements developed from inputs at the beginning of the case.
2. Load the integrator's initial element vector, ELMO, and the estimated initial measurement vector, YHAT, with the estimated initial elements at the beginning of each iteration or pass through the data.
3. Rewind the logical unit on which the data reside at the beginning of each iteration.
4. At the beginning of each iteration, set the measurement sensitivity matrix \((H\) and part of \(QH\)) to zero — or to one in the case of the sensitivity of an initial element measurement to an initial element state. The secant increment is also set when an initial element is to be estimated. The increment is \(1/100,000\) of the element's initial value.

**Initialization Logic** (see flow chart)

Input measurement data is assumed to reside on unit 21 in the form of calendar date plus Cartesian state (EE50). The state may be regarded as osculating or averaged. It will be converted to processing form depending upon the propagation method, KMETH, and the averaging key, KAVST. The time from epoch and the converted state (elements of the KMETH type) are written onto logical unit 22. If KMETH is 4 or 7, the measurements need be converted only once, but if KMETH is 5 or 8 (averaging), the measurements will be re-averaged at each iteration with the current estimated lunar gravitational field.

The weighting matrix, \(Q\), used in weighted least squares estimation under the IQ=3 option can be converted from Cartesian EE50 variables to method KMETH variables by REDATA. It is expected that COVIN will be read in from logical 21 if IQ=3, and that COVIN will be the upper (or lower) half of a covariance matrix in the Cartesian state. This is loaded into a filled-out, symmetrical \(Q\) and converted to elements by

\[
Q_E = \left(\frac{\partial E}{\partial X}\right) Q_X \left(\frac{\partial E}{\partial X}\right)^T
\]

The required partial derivatives are computed by the secant method. \(Q_E\) is written onto logical 22 for potential use by subroutines WLS and SQRUT.

The number of measurements written onto 22 will be the smaller of \(M\) (input maximum number) and the number found on logical 21.
REDATA Flow Chart (I)

ID>1

ID

ID=1

IT

IT>1

IT=1

First trial
Initialize

Set ELMO = ELMI
YHAT = ELMI

Rewind data unit
Set TF = T₀

Initialize H and EODINK

100

Set TI = TF
Read next time and measurement (TF, Y)

No EOF

EOF

YES

Set ID = M

RETURN

Initialize estimated initial elements from input

Load initial elements from (updated) estimate

Load measurement from data file

Set flag to indicate end of data
READATA FLOW CHART (II)
(Initialization Logic)

1. If \( T \neq 1 \), go to \( F \) NOAVG \( T \)
2. REWIND 21
3. REWIND 22
4. \( IVRED = IVRED + 1 \)
5. If \( IVRED > M \)
6. Read 21 for measured date and state
7. EOF
8. Convert date to time from epoch and state to KMETH type elements
9. If \( Q \neq 3 \)
10. Load and convert weighting matrix \( Q \)
11. Write time, elements \( Q \) on logical unit 22
12. If \( KAVST \neq 1 \)
13. Read initial elements, \( ELMO \), from 22
14. Set initial elements for estimator, \( ELMI = ELMO \)

15. \( M \)
SUBROUTINE SHIMMY

Calling Sequence: CALL SHIMMY (EM, D, ELMR, R)

Purpose: SHIMMY calculates the rates in the linear variational equations for the variables \( p, e \cos(\omega), e \sin(\omega), \omega + f, i, \Omega \)

Common Blocks Required: CONST, CNTRL, FIELDM, FIELDX, INPUT, INTVAR, INTVRX, IXCOM, PERT.

Subroutines required: MATMPY, ROTATE.

Input / Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>D</td>
<td>3, 3</td>
<td>Calling</td>
<td>Transformation matrix from M50 to position vector coordinates.</td>
</tr>
<tr>
<td>I</td>
<td>DFDCX</td>
<td>3, 100</td>
<td>FIELDX (10)</td>
<td>Partials of F w.r.t. harmonic coefficients</td>
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<tr>
<td>I</td>
<td>DVDX2</td>
<td>3, 3</td>
<td>FIELDX (1)</td>
<td>Gradient matrix</td>
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<tr>
<td>I</td>
<td>ELMR</td>
<td>12</td>
<td>Calling</td>
<td>Orbital elements with sines and cosines</td>
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<tr>
<td>I</td>
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<td>6, 3</td>
<td>Calling</td>
<td>Matrix which relates the disturbing acceleration to the orbital element rates</td>
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<tr>
<td>I</td>
<td>GM</td>
<td>12</td>
<td>CONST (5)</td>
<td>Gravitational constants</td>
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<tr>
<td>I</td>
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<td>Number of coefficients to be studied</td>
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<tr>
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<td>QATES</td>
<td>600</td>
<td>INTVRX (601)</td>
<td>The rates of the linear variational equations</td>
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<tr>
<td>I</td>
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<td>Distance to S/C</td>
</tr>
<tr>
<td>I</td>
<td>RCART</td>
<td>3</td>
<td>PERT (1)</td>
<td>Disturbing acceleration of S/C in mean of 50</td>
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### Input / Output (continued)

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<th>PERT (4)</th>
<th>Disturbing acceleration of S/C in orbital coordinates</th>
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</thead>
<tbody>
<tr>
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<td>SELNEQ</td>
<td>3, 3</td>
<td>FIELDM (289)</td>
<td>Transformation matrix from SELNEQ=Qᵀ to inertial selenographic coordinates.</td>
</tr>
<tr>
<td>I</td>
<td>Z</td>
<td>600</td>
<td>INTVRX (1)</td>
<td>Linear variational equation variables</td>
</tr>
</tbody>
</table>

**Theory:**

The equations for the time rates of the orbital elements are represented by:

\[
\dot{E} = F(E, C, \mathbf{r}(E), t),
\]

where \(\mathbf{r}(E)\) represents an implicit dependence upon the position/velocity vector. These equations may be differentiated to yield equations which represent the effects of small variations in the gravity coefficients upon the subsequent history of the orbital elements:

\[
\frac{\partial}{\partial C} \left( \frac{d}{dt} \frac{\partial E}{\partial C} \right) = \left( \frac{\partial F}{\partial E} \right) \left( \frac{\partial \mathbf{r}}{\partial E} \right) \mathbf{r} + \frac{\partial E}{\partial C} + \left( \frac{\partial F}{\partial C} \right),
\]

where the normal brackets ( ) represent explicit differentiation. The equations of motion can be written in matrix form as follows:

\[
\begin{bmatrix}
\dot{E} \\
\end{bmatrix}_{6\times1} = 
\begin{bmatrix}
N \\
M(E) \\
\end{bmatrix}_{6\times3}
\begin{bmatrix}
R \\
S \\
\omega \\
\end{bmatrix}_{3\times1} = 
\begin{bmatrix}
D(E) \\
Q(t) \\
\nabla V \\
\end{bmatrix}_{3\times3}
\begin{bmatrix}
\nabla V \\
\end{bmatrix}_{3\times1} = F
\]

where \(D(E)\) is the transformation matrix from the inertial frame to the radial, circumferential, orbit normal orbital coordinate system, \(Q(t)\) is the rotation from selenographic to inertial coordinates, and \(\nabla V\) is the gradient of the selenocentric potential function. The vector \(N\) contains the unperturbed (Keplerian) part of the rate of the true anomaly. The 6x6 matrix in curly brackets is evaluated in two parts made up of the explicit partial derivative matrix,
\[
\frac{\partial (\dot{p}, \dot{h}, \dot{k}, \dot{u}, \dot{i}, \dot{\Omega})}{\partial (p, h, k, u, i, \Omega)} \quad \text{and the product} \quad \left( \frac{\partial F}{\partial \dot{r}} \right) \left( \frac{\partial r}{\partial E} \right),
\]

where

\[
\left( \frac{\partial F}{\partial \dot{r}} \right) = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} Q \end{bmatrix} \begin{bmatrix} J_B \end{bmatrix} \begin{bmatrix} Q^T \end{bmatrix}
\]

and

\[
\frac{\partial r}{\partial E} = \frac{\partial}{\partial E} \left[ D^T \begin{bmatrix} \frac{\partial}{\partial 0} \end{bmatrix} \right] = \left( \frac{\partial}{\partial E} \right) \begin{bmatrix} D^T \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial 0} \end{bmatrix} + \begin{bmatrix} D^T \end{bmatrix} \frac{\partial}{\partial E} \begin{bmatrix} \frac{\partial}{\partial 0} \end{bmatrix}
\]

The last three columns of \( \frac{\partial F}{\partial \dot{r}} \) contain zeros because it is assumed that no velocity-dependent forces are acting. Therefore, the last three rows of \( \frac{\partial r}{\partial E} \) are not required and set to zero. The 3 X 3 matrix \( J_B \) is the Jacobian matrix of the disturbing potential with respect to the body-fixed cartesian coordinates.

Finally, the matrix \( \frac{\partial F}{\partial C} \) gives the explicit partials of the planetary equations with respect to the gravity coefficients.

Description: The equations discussed above are of the form \( Z = AZ + B \).

SHIMMY calculates the matrices A and B, and then uses the input value of Z to calculate the new value of (QATES). The matrices Z and QATES are dimensioned 6 X 100, and represent the quantities \( \dot{E} / \partial C \) and \( \dot{E} / \partial C \) respectively, where each column of the matrices corresponds to a different harmonic coefficient of the field. The input quantity NB determines how many coefficients are to be studied, with a maximum of one hundred.

The matrix B represents the quantity \( \frac{\partial F}{\partial C} \), discussed above, which may be written as

\[
\left( \frac{\partial F}{\partial C} \right) = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} Q \end{bmatrix} \left[ \frac{\partial \nabla V}{\partial C} \right],
\]

where \( M \) is a 6 X 3 matrix which represents the relationship between the perturbing acceleration and the rates of the orbital elements. The matrix is calculated in EQNS, and stored in EM. The transformation matrices \( \begin{bmatrix} D \end{bmatrix} \) & \( \begin{bmatrix} Q \end{bmatrix} \) are discussed in the theory section and are stored in the 3 x 3 matrices D and SELNEQ respectively. Both of these arrays are calculated outside of SHIMMY. The matrix \( \frac{\partial \nabla V}{\partial C} \) is calculated in FIELD2 and stored in DFDCX. For further discussion
of this matrix and the order in which the elements are stored, please see subroutine FIELD2.

The matrix A represents the quantity $\frac{\partial F}{\partial E}$, and may be expressed as

$$\left(\frac{\partial^2 F}{\partial E^2}\right) + \left(\frac{\partial F}{\partial E}\right) \frac{\partial^2 F}{\partial E \partial \mathbf{r}}.$$ Both of these quantities represent the product of several matrices, as indicated in the theory section above. In addition to the matrices already used for the calculation of B, the quantities discussed below must be available for the calculation of A. The matrix D2VDX2, calculated in FIELD2, contains the gradient of the force field and is used in the calculation of $\frac{\partial F}{\partial \mathbf{r}}$. The vector RSW represents the disturbing acceleration on the S/C in orbital coordinates, and the vector RCART represents the same disturbing acceleration in inertial coordinates. The matrix $\left(\frac{\partial F}{\partial E}\right)$, i.e. $\left(\frac{\partial \mathbf{p}}{\partial \mathbf{r}} \frac{\partial \mathbf{h}}{\partial \mathbf{r}} \frac{\partial \mathbf{k}}{\partial \mathbf{r}} \frac{\partial \mathbf{u}}{\partial \mathbf{r}} \frac{\partial \mathbf{i}}{\partial \mathbf{r}} \frac{\partial \mathbf{n}}{\partial \mathbf{r}}\right)$, is calculated column by column internally through successive matrix multiplications and stored in DFDEX. The matrix and vector multiplications are done in subroutine MATMPY and ROTATE. Once these calculations are completed, the vector Z is inserted into the equation and the vector QATES is output.
SHIMMY FLOW CHART

Enter

Calculate $\frac{\partial F}{\partial E}$ column by column

Calculate $\frac{\partial F}{\partial \dot{r}}$

Calculate $\frac{\partial \dot{r}}{\partial E}$

$DFDEX = \frac{\partial F}{\partial E} \frac{\partial F}{\partial \dot{r}} \frac{\partial \dot{r}}{\partial E}$

Rotate DFDCX to inertial frame

QATES = $(DFDEX) \cdot (Z) +$ rotated DFDCX

Return
SUBROUTINE SOLVE

Calling Sequence: CALL SOLVE

Purpose: SOLVE solves the equation $W^T WX = Y$ for $X$, given the vector, $Y$, and the upper-triangular matrix, $W$.

Common Blocks Required: DXCOM, IXCOM

Subroutines Called: None

Input / Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>QH</td>
<td>NQH, max 105</td>
<td>DXCOM(1)</td>
<td>Square-root information matrix, $W$</td>
</tr>
<tr>
<td>I</td>
<td>IQY</td>
<td>N, max 10</td>
<td>DXCOM(106)</td>
<td>Weighted residual vector, $Y$</td>
</tr>
<tr>
<td>O</td>
<td>DX</td>
<td>N, max 10</td>
<td>DXCOM(116)</td>
<td>Incremental state vector, $X$</td>
</tr>
<tr>
<td>I</td>
<td>N</td>
<td>1</td>
<td>IXCOM(101)</td>
<td>Number of states</td>
</tr>
</tbody>
</table>

Note: The square-root information matrix, QH, is the singly-dimensioned representation of a doubly-dimensioned packed upper-triangular matrix, $W$.

$$QH = (W_{11}, W_{12}, W_{22}, W_{13}, W_{23}, ..., W_{nn}, + unused \text{ words})$$

Method

The weighted least squares estimation equation can be stated in the form

$$(W^T W) X = Y$$

where $W$ is the upper-triangular square-root information matrix, $X$ is the estimated state increment to be solved for and $Y$ is the weighted residual vector. An upper triangular matrix is a square matrix which has only zeroes below its main diagonal. One obvious solution to the stated problem would be to form $W^T W$, invert it, and multiply $Y$ by it to obtain $X$. This subroutine uses another method, capitalizing on the fact that $W$ is upper-triangular. We first solve the problem
\[ \mathbf{W}^T \mathbf{B} = \mathbf{Y} \]

for the dummy vector, \( \mathbf{B} \), by forward substitution. We then solve
\[ \mathbf{WX} = \mathbf{B} \]
by backward substitution to obtain \( \mathbf{X} \).

Forward substitution:
\[
\begin{align*}
\mathbf{W}^T \mathbf{B} &= \mathbf{Y} \\
\mathbf{w}_{11} \mathbf{b}_1 &= \mathbf{y}_1 \\
\mathbf{w}_{12} \mathbf{b}_1 + \mathbf{w}_{22} \mathbf{b}_2 &= \mathbf{y}_2 \\
&\vdots\\n\sum_{i=1}^{j} \mathbf{w}_{ij} \mathbf{b}_i &= \mathbf{y}_j
\end{align*}
\]

Solution
\[
\begin{align*}
\mathbf{b}_1 &= \mathbf{y}_1 / \mathbf{w}_{11} \\
\mathbf{b}_2 &= (\mathbf{y}_2 - \mathbf{w}_{12} \mathbf{b}_1) / \mathbf{w}_{22} \\
&\vdots\\n\mathbf{b}_j &= (\mathbf{y}_j - \sum_{i=1}^{j-1} \mathbf{w}_{ij} \mathbf{b}_i) / \mathbf{w}_{jj}
\end{align*}
\]

Backward Substitution:
\[
\begin{align*}
\mathbf{WX} &= \mathbf{B} \\
\mathbf{w}_{nn} \mathbf{x}_n &= \mathbf{b}_n \\
\mathbf{w}_{n-1,n-1} \mathbf{x}_{n-1} + \mathbf{w}_{n-1,n} \mathbf{x}_n &= \mathbf{b}_{n-1} \\
&\vdots\\n\sum_{i=j}^{n} \mathbf{w}_{ij} \mathbf{x}_i &= \mathbf{b}_j
\end{align*}
\]

Solution
\[
\begin{align*}
\mathbf{x}_n &= \mathbf{b}_n / \mathbf{w}_{nn} \\
\mathbf{x}_{n-1} &= (\mathbf{b}_{n-1} - \mathbf{w}_{n-1,n} \mathbf{x}_n) / \mathbf{w}_{n-1,n-1} \\
&\vdots\\n\mathbf{x}_j &= (\mathbf{b}_j - \sum_{i=j+1}^{n} \mathbf{w}_{ji} \mathbf{x}_i) / \mathbf{w}_{jj}
\end{align*}
\]

Since the zeroes of \( \mathbf{W} \) do not enter the solution, the potentially non-zero elements of \( \mathbf{W} \) are stored in a singly-dimensioned array. The coding synthesizes the doubly-dimensioned indices to effect the above procedure as shown.
SUBROUTINE SORDID

Calling Sequence:  CALL SORDID

Purpose:  SORDID sorts the parameter identification array, IX, for FIELD2's required order and prints it out.

Common Blocks Required:  FIELDM, INPUT, IXCOM, STATE

Subroutines Called:  None

Input/Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
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<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>ZONL</td>
<td>16</td>
<td>FIELDM(1)</td>
<td>Zonal harmonic coefficients</td>
</tr>
<tr>
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<td>TSRL</td>
<td>16, 17</td>
<td>FIELDM(17)</td>
<td>Longitudinal harmonic coefficients</td>
</tr>
<tr>
<td>I</td>
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<td>1</td>
<td>FIELDM(298)</td>
<td>Maximum modeled N</td>
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<td>I</td>
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<td>Maximum modeled M</td>
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<td>IXCOM(1)</td>
<td>Parameter identification array</td>
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<td>Nonzero elements of IX</td>
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<td>Solar pressure coefficient (280)</td>
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<td>I</td>
<td>SPC</td>
<td>1</td>
<td>STATE(36)</td>
<td>Solar pressure coefficient (279)</td>
</tr>
</tbody>
</table>
Discussion

The IX array specifies the parameters to be estimated according to the code:

\[
\text{IX(I)} = \begin{cases} 
1 & \rightarrow 272 \text{ Gravitational parameters} \\
273 & \rightarrow 278 \text{ Initial elements} \\
279, 280 & \text{ Solar pressure coefficients}
\end{cases}
\]

The code for the gravitational parameters, \( C_{nm} \) and \( S_{nm} \), is that IX(I) shall be the element number of the ZONL, TSRL array shown in the following figure. SORDID checks that the input values for IX are valid. That is, IX numbers must lie between 1 and 280 and must not indicate coefficients outside the prescribed model (NMOD, MMOD). Furthermore, subroutine FIELD2 expects a particular order for the selected coefficients in order to efficiently evaluate sensitivities when the option for linear variational equations is selected. That order is \(((C(N,M),S(N,M)), M=1, MMOD), N=1, NMOD)\), where coefficients may be omitted, but must not be out of order. SORDID therefore puts the IX array into the order expected by FIELD2 even if the user failed to do so himself. Following the sorting procedure, SORDID prints out the IX number and (for gravitational parameters) the location number in the input array as well as the values residing in the location when SORDID was called. SORDID is called before and after parameter estimation, so parameter values at these times are printed.
<table>
<thead>
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<tbody>
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<td>s1.1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>c8.8</td>
<td>s8.8</td>
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<td></td>
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<td>c9.0</td>
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<td></td>
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<td>c9.9</td>
<td>s9.9</td>
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<td></td>
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<td>c10.0</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>c10.10</td>
<td>s10.10</td>
<td></td>
<td></td>
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<td>c11.11</td>
<td>s11.11</td>
<td></td>
<td></td>
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<td>11</td>
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<tr>
<td>c12.0</td>
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<td></td>
<td>c12.12</td>
<td>s12.12</td>
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<td>12</td>
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<tr>
<td>c13.0</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>c13.13</td>
<td>s13.13</td>
<td></td>
<td></td>
<td></td>
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<td>13</td>
</tr>
<tr>
<td>c14.0</td>
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<td>c14.14</td>
<td>s14.14</td>
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<td>14</td>
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<tr>
<td>c15.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>c15.15</td>
<td>s15.15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>15</td>
</tr>
</tbody>
</table>
SORT IX into ascending numerical order

Check for IX lying outside the available model

SORT gravitational part of IX into FIELD2's order

DO 1,N

Compute location of input array corresponding to IX(I)

Print IX(I), name, value and input location number

RETURN
SUBROUTINE SQRUT

Calling Sequence: CALL SQRUT (A, SA, N)

Purpose: SQRUT computes a square root of a positive semi-definite symmetric matrix.

Common Blocks Required: None

Subroutines Called: None

Input / Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>A</td>
<td>(N, N)</td>
<td>Call List</td>
<td>Input matrix, covariance type</td>
</tr>
<tr>
<td>O</td>
<td>SA</td>
<td>N*(N+1)/2</td>
<td>Call List</td>
<td>Square root, packed upper-triangular</td>
</tr>
<tr>
<td>I</td>
<td>N</td>
<td>1</td>
<td>Call List</td>
<td>Matrix dimension</td>
</tr>
</tbody>
</table>

Method

Given a matrix , A, we seek an upper-triangular matrix, S, such that

\[ S^T S = A \]

A recursive algorithm for computing S is

\[
\begin{align*}
    s_{ii} &= \sqrt{a_{ii} - \sum_{j=1}^{i-1} s_{ij}^2} ; \quad i=1, n \\
    s_{ji} &= \begin{cases} 
        0 & ; \quad j < i \\
        \frac{1}{s_{ii}} \left( a_{ji} - \sum_{k=1}^{i-1} s_{jk} s_{ik} \right) & ; \quad j=i+1, n
    \end{cases}
\end{align*}
\]

The output matrix, SA, is "packed" as a singly-dimensioned array.

\[ SA = ( s_{11}, s_{12}, s_{22}, s_{13}, s_{23}, \ldots, s_{nn} ) \]
SUBROUTINE TUMULT

Calling Sequence: CALL TUMULT (A, B, C, NR, NC, IT)

Purpose: TUMULT multiplies a packed upper-triangular matrix (or its transpose) times a rectangular matrix.

Common Blocks Required: None

Subroutines Called: None

Input / Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>A</td>
<td>NR*(NR+1)/2</td>
<td>Call List</td>
<td>Packed upper-triangular matrix</td>
</tr>
<tr>
<td>I</td>
<td>B</td>
<td>NR * NC</td>
<td>Call List</td>
<td>Rectangular matrix or vector</td>
</tr>
<tr>
<td>O</td>
<td>C</td>
<td>NR * NC</td>
<td>Call List</td>
<td>Product matrix</td>
</tr>
<tr>
<td>I</td>
<td>NR</td>
<td>1</td>
<td>Call List</td>
<td>Rows of B</td>
</tr>
<tr>
<td>I</td>
<td>NC</td>
<td>1</td>
<td>Call List</td>
<td>Columns of B</td>
</tr>
<tr>
<td>I</td>
<td>IT</td>
<td>1</td>
<td>Call List</td>
<td>Transpose flag, IT = {0 : C = AB, 1 : C = A^T \cdot B}</td>
</tr>
</tbody>
</table>

Discussion

This subroutine uses the fact that A is upper-triangular to avoid unnecessary multiplications by zeroes in the product AB or A^T \cdot B. A is packed as a singly-dimensioned array.

\[ A = (a_{11}, a_{12}, a_{22}, a_{13}, \ldots, a_{nr, nr}) \]
SUBROUTINE UPDATX

Calling Sequence: CALL UPDATX

Purpose: UPDATX adds the estimated state increment to the appropriate program arrays (estimated state).

Common Blocks Required: CNSDR, COMIC, DXCOM, FIELDM, IMCOM, INPUT, IXCOM, STATE

Subroutines Called: None

Input/Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I/O</td>
<td>ELMI</td>
<td>6</td>
<td>COMIC(1)</td>
<td>Estimated initial elements</td>
</tr>
<tr>
<td>I/O</td>
<td>DX</td>
<td>N</td>
<td>DXCOM(116)</td>
<td>Incremental state (upper-loaded)</td>
</tr>
<tr>
<td>I/O</td>
<td>X</td>
<td>288</td>
<td>FIELDM(1)</td>
<td>Lunar gravitational field coefficients</td>
</tr>
<tr>
<td>I</td>
<td>ICNSDR</td>
<td>20</td>
<td>CNSDR(1)</td>
<td>Considered parameter identifiers</td>
</tr>
<tr>
<td>I/O</td>
<td>A</td>
<td>1000</td>
<td>INPUT(1)</td>
<td>Input array</td>
</tr>
<tr>
<td>I</td>
<td>IX</td>
<td>N</td>
<td>IXCOM(1)</td>
<td>State component identifiers (upper-loaded)</td>
</tr>
<tr>
<td>I</td>
<td>N</td>
<td>1</td>
<td>IXCOM(101)</td>
<td>Number of incremental states</td>
</tr>
<tr>
<td>I</td>
<td>IGAIN</td>
<td>1</td>
<td>IMCOM(7)</td>
<td>Reciprocal gain factor</td>
</tr>
<tr>
<td>I/O</td>
<td>S(36)</td>
<td>1</td>
<td>STATE(36)</td>
<td>Solar pressure coefficient</td>
</tr>
</tbody>
</table>
Discussion

Each incremental state component in $DX$ is identified by the corresponding element of $IX$. The program arrays are incremented by $DX$ with a unit gain as follows, according to $IX$.

<table>
<thead>
<tr>
<th>If $IX(I)$ is</th>
<th>$DX(I)$ is</th>
<th>$DX$ is added to</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \leq IX(I) \leq 271$</td>
<td>a gravitational coefficient</td>
<td>$X[IX(I)]$ in FIELDM</td>
</tr>
<tr>
<td>$273 \leq IX(I) \leq 278$</td>
<td>an initial orbital element</td>
<td>$ELMI[IX(I)-272]$ in COMIC</td>
</tr>
<tr>
<td>$279$</td>
<td>a solar pressure coefficient</td>
<td>$X(36)$ in STATE</td>
</tr>
<tr>
<td>$280$</td>
<td>a solar pressure coefficient</td>
<td>$A(198)$ in INPUT</td>
</tr>
</tbody>
</table>

A provision is made for adding the state increment to the estimate with an assigned gain (limit). If $IGAIN$ is input as a number greater than unity,

$$DX = DX/IGAIN$$

Another provision permits exclusion of any incrementing of selected parameters. That is, if any $IX(I)$ is found in the ICNSDR array of "considered" parameters, then the state increment $DX(I)$ corresponding to $IX(I)$ will be set zero. This treatment has the effect of changing the weighting matrix without changing certain parameters during the estimation process.

No flow chart is deemed necessary for this routine.
SUBROUTINE WLS

Calling Sequence: CALL WLS

Purpose: WLS is the driver for a weighted least squares estimation process.

Common Blocks Required: CONST, DXCOM, IMCOM, INPUT, INTVRX, IXCOM, YCOM

Subroutines Called: MODEL, PHOUSE, PUTINV, REDATA, SOLVE, SQRUT, TUMULT, UPDATX

Input/Output

<table>
<thead>
<tr>
<th>I/O</th>
<th>Symbolic Name</th>
<th>Program Dimension</th>
<th>Common Block</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>QH</td>
<td>5650</td>
<td>DXCOM(201)</td>
<td>Square root information matrix</td>
</tr>
<tr>
<td>O</td>
<td>HQY</td>
<td>100</td>
<td>DXCOM(101)</td>
<td>Weighted residual vector</td>
</tr>
<tr>
<td>O</td>
<td>DX</td>
<td>100</td>
<td>DXCOM(1)</td>
<td>Estimated state increment</td>
</tr>
<tr>
<td>I/O</td>
<td>KVAR</td>
<td>1</td>
<td>INPUT(1096)</td>
<td>Sensitivity computation type key</td>
</tr>
<tr>
<td>I</td>
<td>KMETH</td>
<td>1</td>
<td>INPUT(1036)</td>
<td>Integration method key</td>
</tr>
<tr>
<td>I</td>
<td>NGROPT</td>
<td>1</td>
<td>INPUT(1064)</td>
<td>Gradient treatment key</td>
</tr>
<tr>
<td>I</td>
<td>N</td>
<td>1</td>
<td>IXCOM(101)</td>
<td>Number of state components</td>
</tr>
<tr>
<td>I</td>
<td>M1</td>
<td>1</td>
<td>IXCOM(102)</td>
<td>Number of observations at each measurement</td>
</tr>
<tr>
<td>I</td>
<td>M</td>
<td>1</td>
<td>IXCOM(103)</td>
<td>Number of measurements</td>
</tr>
<tr>
<td>O</td>
<td>NQH</td>
<td>1</td>
<td>IXCOM(104)</td>
<td>Number of elements in packed SRIM</td>
</tr>
<tr>
<td>I</td>
<td>IQ</td>
<td>1</td>
<td>IXCOM(105)</td>
<td>Weighing type flag 1, no weighting 2, constant Q 3, variable Q</td>
</tr>
<tr>
<td>I</td>
<td>ITMAX</td>
<td>1</td>
<td>IXCOM(106)</td>
<td>Maximum number of iterations</td>
</tr>
<tr>
<td>I/O</td>
<td>Symbolic Name</td>
<td>Program Dimension</td>
<td>Common Block</td>
<td>Definition</td>
</tr>
<tr>
<td>-----</td>
<td>---------------</td>
<td>-------------------</td>
<td>--------------</td>
<td>------------</td>
</tr>
<tr>
<td>O</td>
<td>ID</td>
<td>1</td>
<td>IXCOM(107)</td>
<td>Measurement number counter</td>
</tr>
<tr>
<td>O</td>
<td>IT</td>
<td>1</td>
<td>IXCOM(108)</td>
<td>Iteration counter</td>
</tr>
<tr>
<td>I</td>
<td>YHAT</td>
<td>6</td>
<td>YCOM(2)</td>
<td>Estimated measurement counter</td>
</tr>
<tr>
<td>I</td>
<td>IGAIN, LTOL</td>
<td>1</td>
<td>IMCOM(7, 8)</td>
<td>Iteration convergence tolerance numbers</td>
</tr>
<tr>
<td>I</td>
<td>IM</td>
<td>6</td>
<td>IMCOM(1)</td>
<td>Measurement selector array</td>
</tr>
<tr>
<td>I</td>
<td>Y</td>
<td>6</td>
<td>YCOM(9)</td>
<td>Measurement vector</td>
</tr>
<tr>
<td>I</td>
<td>Q</td>
<td>36</td>
<td>YCOM(15)</td>
<td>Measurement weighting vector</td>
</tr>
</tbody>
</table>

Discussion

WLS contains the gross logic for a weighted least squares estimation procedure, using a square-root information matrix formulation. The usual form for the WLS estimator is

\[
\left( \sum_{i=1}^{m} H_i^T Q_i^{-1} H_i \right) \hat{\Delta} = \sum_{i=1}^{m} H_i^T Q_i^{-1} \tilde{Y}_i
\]

where \( \tilde{Y}_i \) is the measurement residual vector \((Y - YHAT)\) at the i-th measurement, \( Q_i^{-1} \) is the measurement weighting matrix, \( H_i \) is the measurement sensitivity to changes in the state and \( \hat{\Delta} \) is the estimated state increment to be determined. In the square-root formulation the information matrix (bracketed term) is not formed per se. Instead, a square-root information matrix, \( W \), is accumulated, where \( W \) obeys

\[
W^T W = \sum_{i=1}^{m} H_i^T Q_i^{-1} H_i
\]
See the description of W's computation in subroutine PHOUSE. At each measurement, 
$Q_i^{1/2} H_1$ is formed. This product is loaded into the QH array, beginning at location 
NQH+1 for use by PHOUSE in forming W. The weighted residual vector, HQY is 
accumulated as indicated by the following equation.

$$HQY = \sum_{i=1}^{m} (Q_i^{-1/2} H_1)^T (Q_i^{-1/2} \tilde{Y}_i)$$

When HQY and W have been accumulated for M measurements, subroutine SOLVE is 
called to solve

$$(W^T W) \hat{dX} = HQY$$

for $\hat{dX}$ which then is used by UPDATX to increment the estimated state.

WLS performs ITMAX iterations unless convergence, defined by

$$\sqrt{\sum_{i=1}^{N} DX_i} \leq \frac{IGAIN}{10 \cdot LTOL}$$

occurs first. During iterations after the first, the sensitivities (and, therefore, 
the square-root information matrix) may be held constant at their values on the 
NGROPT-th iteration. The weighted sensitivities, $Q_i^{1/2} H$, are written on logical 
23 as they are computed, then read back from 23 as they are needed. Of course, 
provision is made to avoid redundant computation of sensitivities by setting KVAR 
and NQ to zero.

When less than 6 elements are used for estimation of state, the residual vector is 
upper-packed with the necessary M1 components and the weighting matrix is upper-
left-packed as specified by the IM array.
WLS Flow Chart

\[ IT = 0 \quad \text{(IT = trial no.)} \]

100

**INITIALIZE**

ID = 0
IT = IT + 1
QH = 0
HQY = 0

\[ \text{(ID = data point no.)} \]

200

ID = ID + 1
CALL REDATA

Bring in ID-th measurement, \( Y \)

ID = 1
ID \geq M

\[ \hat{Y} = \text{ELMI} \]

CALL MODEL

Bring in estimated measurement, \( \hat{Y} \)

**COMPUTE RESIDUAL**

\[ DY = Y - \hat{Y} \]

\[ IQ = 2 \quad \text{and} \quad ID = 1 \]

\[ IQ = 2, \quad Q = \text{Constant} \]

\[ IQ = 3, \quad Q = \text{variable} \]

CALL SQRUT
CALL PUTINV

Replace weighting matrix by its square-root inverse

CALL TUMULT(SQ, DY, DY)
CALL TUMULT(SQ, QH, QH)

Weight residuals, \( DY \), andpartials, \( QH \)

\[ HQY = \Sigma (QH)^T \cdot DY \]

Accumulate weighted residuals in \( HQY \)

CALL PHOUSE

Accumulate square-root information matrix

ID < M, more data to come

CALL SOLVE

Compute estimated state increment

CALL UPDATX

Increment the estimated state

\[ [DX] > \text{TOLA}, \text{try again} \]

\[ ID < \text{TOLA} \quad \rightarrow \text{RETURN} \]

\[ [IDX] > \text{TOLA}, \text{try again} \]

\[ ID > \text{ITMAX} \quad \rightarrow \text{RETURN} \]

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II. THEORETICAL BACKGROUND

3.1 Definition of Symbols

Capital letters will denote arrays (vectors or matrices) unless otherwise stated. Scalars will be denoted by lower case. A subscripted scalar will usually denote a particular element of an array. For example, \( W = \{ w_{ij} \} \).

- \( X, \{ x_i \} \) State vector (gravitational coefficients, initial orbital elements, solar pressure coefficients)
- \( Y, \{ y_i \} \) Measurement vector (orbital elements)
- \( H, \{ h_{ij} \} \) Sensitivity matrix (of measurement to state)
- \( Q, \{ q_{ij} \} \) Measurement weighting matrix (error covariance matrix)
- \( n \) Number of state elements to be solved for
- \( k \) Number of observables in one measurement (usually 6)
- \( m \) Number of measurements processed
- \( t \) Time
- \( E, \{ e_i \} \) Orbital elements (measurement or a function of state and time)
- \( J, \{ j_i \} \) Zonal harmonic coefficients
- \( C, \{ c_{nm} \} \) Tesseral harmonic coefficients in cosine of longitude
- \( S, \{ s_{nm} \} \) Tesseral harmonic coefficients in sine of longitude
- \( d \) An operator implying a variation of its operand
- \( \hat{X} \) Estimated state
- \( \hat{Y} \) Estimated measurement, \( Y(\hat{X}, t) \)
- \( \tilde{Y} \) Measurement residual, \( Y - \hat{Y} \)
- \( W, \{ w_{ij} \} \) Square root information matrix, upper triangular
- ( ) Superior numbers in parentheses refer to Bibliography
3.2 Weighted Least Squares Parameter Estimation

The fundamental weighted least squares estimation procedure will be derived with no attempt at rigor.

Assume that variations, $\text{d}X$, in the state relate to deviations, $\text{d}Y$, in the measurement by the linear relationship (3.1).

$$\text{d}Y = H \text{d}X \tag{3.1}$$

Presuming to have recorded the measurement vector, $Y$, and to have computed the measurement as a function of the state, $Y(X)$, one may compute the measurement residual, $\tilde{Y}$.

$$\tilde{Y} = Y - Y(X) = Y - \hat{Y} \tag{3.2}$$

If we were to hypothesize a change, $\text{d}X$, being made in the estimated state, we would expect that, under the linear assumption (3.1), the residual would become

$$\tilde{Y} = Y - (Y + H \text{d}X). \tag{3.3}$$

We seek that $\text{d}X$ which minimizes the quadratic function, $\tilde{Y}^T Q^{-1} \tilde{Y}$, where $Q$ is a symmetric, positive-definite weighting matrix. One ready derivation of the appropriate $\text{d}X$ is found by differentiation of the quadratic function with respect to $\text{d}X$ and equation of the result with zero,

$$-2 \tilde{Y}^T Q^{-1} H = 0 \tag{3.4}$$

followed by solution of the resulting equation for $\text{d}X$.

$$\text{d}X = (H^T Q^{-1} H)^{-1} H^T Q^{-1} (Y - \hat{Y}) \tag{3.5}$$

The matrix, $H^T Q^{-1} H$, is called the "information matrix." It must span the state space if it is to possess a unique inverse, although "pseudo-inverses" can always be computed to render solutions to (3.5). Iteration on (3.5) is performed by adding $\text{d}X$ to $\hat{X}$ and re-computing $H$ and $\hat{Y}$ with the sum before re-applying (3.5). For certain problems, convergence can be obtained without re-computing $H$ during the iteration.
3.3 Square-Root Solution

Certain advantages are to be obtained by using the "square root" form of (3.5). The square root information matrix (SRIM) is defined here as the upper triangular matrix, W, such that (3.6) holds.

\[ W^TW = H^TQ^{-1}H \]  \hspace{1cm} (3.6)

An upper triangular matrix is a square matrix characterized by having only zeroes below its main diagonal. Conversely, a lower triangular matrix such as \( W^T \) has only zeroes above its main diagonal. The factorization of the information matrix can be accomplished by Cholesky decomposition. In special cases, it can also be achieved by forming the square root information matrix rather than the information matrix itself, so that the information matrix need never be factored. The solution of (3.5) is obtained very efficiently, given W, by the method of Banachiewicz.

Beginning with (3.5) in the form of (3.7),

\[ (W^TW) \hat{d}X = H^TQ^{-1}Y = D \]  \hspace{1cm} (3.7)

we first solve for the dummy vector, B, of (3.8) by forward substitution.

\[ W^TB = D \]  \hspace{1cm} (3.8)

We then solve (3.9) for \( \hat{d}X \) by backward substitution.

\[ W \hat{d}X = B \]  \hspace{1cm} (3.9)

Denoting an element in the i-th row and j-th column of W by \( w_{ij} \), the forward substitution procedure is (3.10).

\[ b_i = d_i / w_{11} \]  \hspace{1cm} (3.10.1)

\[ b_2 = (d_2 - w_{12}b_1) / w_{22} \]  \hspace{1cm} (3.10.2)

\[ \vdots \]

\[ b_j = (d_j - \sum_{k=1}^{j-1} w_{ij}b_k) / w_{jj} \]  \hspace{1cm} (3.10.j)

The backward substitution procedure for n state variables is (3.11).

\[ dx_n = b_n / w_{nn} \]  \hspace{1cm} (3.11.1)

\[ dx_{n-1} = (b_{n-1} - w_{n-1,n}dx_n) / w_{n-1,n-1} \]  \hspace{1cm} (3.11.2)

\[ \vdots \]

\[ dx_j = (b_j - \sum_{k=j+1}^{n} w_{ij}dx_k) / w_{jj} \]  \hspace{1cm} (3.11.j)
This forward/backward substitution just described has the nice property that it provides \( \hat{dX} \) satisfying (3.7) without explicitly inverting either the information matrix or its square root, thus saving the storage required for the SRIM and its inverse. If both the SRIM and its inverse are required for another reason or if storage is no problem, the substitution method is not quite as advantageous as simple inversion. If it is required, the inverse of the triangular SRIM is very simply obtained by another substitution procedure. (2) For sequential information filtering, it is advantageous to retain the SRIM as opposed to replacing it in storage by its inverse.
3.4 Implementing the SRWLS Solution

If successive measurement errors are assumed to be uncorrelated, the right-hand side of (3.7) can be accumulated as a simple vector sum and the SRIM can be accumulated by Householder reduction. If we are trying to estimate \( n \) components of state and if we measure \( km \) quantities k-at-a-time, the dimensions of the component arrays of (3.7) are as shown in the following table.

<table>
<thead>
<tr>
<th>Array</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{dX} )</td>
<td>( n )</td>
</tr>
<tr>
<td>( \tilde{Y} )</td>
<td>( km )</td>
</tr>
<tr>
<td>( Q )</td>
<td>( km \times km )</td>
</tr>
<tr>
<td>( H )</td>
<td>( km \times n )</td>
</tr>
<tr>
<td>( W )</td>
<td>( n \times n )</td>
</tr>
</tbody>
</table>

If successive measurements are uncorrelated by the assumed weighting matrix, then

\[
Q^{-1} = \begin{bmatrix}
Q_1^{-1} & 0 & 0 & \cdots \\
0 & Q_2^{-1} & 0 & \cdots \\
0 & 0 & Q_3^{-1} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & Q_m^{-1}
\end{bmatrix}
\]

(3.12)

where \( Q_i^{-1} \) is a \( k \times k \) matrix which weights components of the \( i \)-th measurement against each other and against \( Q_j^{-1}, \ j \neq i \). The \( Q_i^{-1} \) are themselves positive definite and symmetric, possibly diagonal and possibly equal to each other. We can partition the residual into \( m \) \( k \)-vectors and \( H \) into \( m \) \((k \times n)\) matrices. If the residual vector for the \( i \)-th measurement is \( \tilde{Y}_i \) and if \( H_i \) is the \( k \times n \) sensitivity matrix for the \( i \)-th measurement, the right-hand side of (3.7) is the simple vector sum (3.13).

\[
H^T Q^{-1} \tilde{Y} = \sum_{i=1}^{m} H_i^T Q_i^{-1} \tilde{Y}_i
\]

(3.13)

The information matrix can also be accumulated as a sum.

\[
H^T Q^{-1} H = \sum_{i=1}^{m} H_i^T Q_i^{-1} H_i
\]

(3.14)
The implemented square root method eliminates about half of the multiplications implied in (3.14) by accumulating the SRIM rather than the information matrix. If $W_{i-1}$ is the upper triangular SRIM after $i-1$ measurements have been included, the SRIM after the $i$-th measurement must obey (3.15).

$$W_i^T W_i = W_{i-1}^T W_{i-1} + H_i^T Q_i^{-1} H_i$$

Equation (3.15) can be written in partitioned form as follows.

$$
\begin{bmatrix}
    W_i^T & 0 \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    W_i \\
    0
\end{bmatrix}
= 
\begin{bmatrix}
    W_{i-1}^T & H_i^T Q_i^{-1/2} \\
    H_i Q_i^{-1/2} H_i & Q_i^{-1/2}
\end{bmatrix}
\begin{bmatrix}
    W_{i-1} \\
    0
\end{bmatrix}
$$

The computation of $W_i$ consists, then, of finding a means for converting

$$
\begin{bmatrix}
    W_{i-1}^T \\
    Q_i^{-1/2} H_i
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
    W_i^T \\
    0
\end{bmatrix}
$$

where $W_{i-1}$ and $W_i$ are $n \times n$ upper triangular matrices, where $Q_i^{-1/2}$ and $0$ are $k \times n$ matrices and where $W_i$ satisfies (3.15). The process which has been implemented is called the Householder reduction algorithm. Since $Q_i^{-1/2}$ and $Q_i^{-1/2} H_i$ must be formed for the HRA, the vector sum in (3.13) can be programmed as in equation (3.18).

$$H_i^T Q_i^{-1/2} Y = \sum_{z=1}^{\infty} (Q_i^{-1/2} H_i)^T (Q_i^{-1/2} Y_i)$$

The SRIM is initialized to zero to characterize the lack of a priori information.
Application to Lunar Field Determination

The particular application of concern here is, of course, in determination of the harmonic coefficients of the lunar gravitational field. In this application, the state vector, $X$, consists of those coefficients we choose to solve for plus, optionally, the initial orbital elements (or Cartesian state) and one or more coefficients of the solar radiation pressure model. The state vector might have the following appearance.

\[
X = \begin{bmatrix}
  j_2 \\
c_{2,1} \\
s_{2,1} \\
0 \\
c_e \\
c_{sp}
\end{bmatrix}
\]  

(example)  

The measurement vector, $Y$, consists of elements of the lunar orbit.

\[
Y = \begin{bmatrix}
p \\
e \sin \omega \\
e \cos \omega \\
M + \omega \\
i \\
\Omega
\end{bmatrix}
\]  

These elements are to be mean elements of the lunar orbit at specified times, referred to the Earth's mean equator and equinox of 1950.0 and having been obtained from short-arc determinations using range and Doppler data. Sensitivities of $Y$ to $X$ are obtained by the Secant Method or, optionally, by integration of variational equations. Propagation of the estimated elements will be carried out by:

1. choosing an initial estimate, $\hat{X}$, of the state
2. choosing initial elements, probably as the first measurement which would not then be processed as a measurement,
3. integrating the averaged equations of motion across the time span of data.


REFERENCES


Appendix A
Derivation of Linear Variational Equations for the Method 7 Elements

This appendix contains the expressions for the partial derivative matrices needed in the calculation of the rates for the linear variational equations for the method 7 elements. The method by which these equations are averaged is given at the end of the development.

The equations of motion (the planetary equations in Gauss's form) are

\[
\begin{align*}
\dot{E} &= F(E, C, t) = F(E, r(E), C, t) \\
\dot{E} &= N(E) + M(E) \begin{bmatrix} R' \\ S' \\ W' \end{bmatrix}
\end{align*}
\]

or

\[
\dot{E} = \begin{bmatrix} \dot{p} \\ \dot{u} \\ \dot{k} \\ \dot{\mu} \\ \dot{\hat{u}} \\ \dot{\hat{n}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{4p}}{r^2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \end{bmatrix} \begin{bmatrix} R' \\ S' \\ W' \end{bmatrix}
\]

\[q = \sqrt{\frac{\mu}{p}}\]

\[
\left[ M \right] = \begin{bmatrix} 0 & \frac{2 q p}{(1 + \frac{r}{p}) \cos u + h \sin u) - q k \frac{r}{p} \sin u \cot i \\
q(-\cos u) & q \left(1 + \frac{r}{p}\right) \sin u + \frac{r}{p} h & -q k \frac{r}{p} \sin u \cot i \\
q \sin u & q \left(1 + \frac{r}{p}\right) \cos u + \frac{r}{p} k & q \frac{r}{p} \sin u \cot i \\
0 & 0 & q \frac{r}{p} \cos u \\
0 & 0 & q \frac{r}{p} \sin u \frac{r}{\sin i} \\
0 & 0 & q \frac{r}{p} \sin i \end{bmatrix}
\]
We wish to describe the effects of small variations in the parameters C on the subsequent motion of the elements. Taking the variation of (1) with respect to the C's gives

\[
\frac{\partial \mathbf{E}}{\partial C} = \frac{d}{dt} \frac{\partial \mathbf{F}}{\partial C} \approx \left\{ \left( \frac{\partial \mathbf{F}}{\partial \mathbf{E}} \right) + \frac{\partial \mathbf{F}}{\partial \mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial \mathbf{E}} \right\} \frac{\partial \mathbf{E}}{\partial C} + \left( \frac{\partial \mathbf{F}}{\partial C} \right)
\]

(2)

where the normal brackets () represent explicit differentiation and the 6-vector \( \mathbf{r} \) represents the cartesian position and velocity. The first column of the 6 x 6 matrix \( \left( \frac{\partial \mathbf{F}}{\partial \mathbf{E}} \right) \) is given by

\[
\left( \frac{\partial \mathbf{F}}{\partial \mathbf{p}} \right) = \frac{\partial \mathbf{M}}{\partial \mathbf{p}} \left[ \mathbf{R} \right] + \frac{\partial \mathbf{N}}{\partial \mathbf{p}} \mathbf{W}
\]

and similarly for \( \frac{\partial \mathbf{F}}{\partial h} \) and \( \frac{\partial \mathbf{F}}{\partial k} \). The expressions for these first three columns of the matrix \( \left( \frac{\partial \mathbf{F}}{\partial \mathbf{E}} \right) \) are given in tables A1, A2, and A3.

The disturbing acceleration vector, \( \mathbf{R}' S' W' \), is dependent upon the angular elements \( u, i, \) and \( \Omega \) as well as the time, viz:

\[
\begin{bmatrix}
\mathbf{R}' \\
\mathbf{S}' \\
\mathbf{W}'
\end{bmatrix} =
\left[
\begin{array}{c}
\mathbf{D} \\
3\times3
\end{array}
\right]
\left[
\begin{array}{c}
\mathbf{Q} \\
3\times3
\end{array}
\right]
\left[
\begin{array}{c}
\mathbf{VV} \\
3\times1
\end{array}
\right]
\]

where \( \mathbf{D} \) is the transformation matrix from inertial coordinates to radial, circumferential, orbit normal coordinates. The matrix \( \mathbf{D} \) and its derivatives with respect to \( u, i, \) and \( \Omega \) are given in table A7. The matrix \( \mathbf{Q} \) represents the transformation from central planet body-fixed coordinates to the inertial integration frame. \( \mathbf{VV} \) represents the disturbing acceleration due to the central planet non-sphericity and \( \mathbf{V} \) is the familiar spherical harmonic representation of the disturbing potential:

\[
\mathbf{V} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{R^n}{r^{n+1}} \mathbf{P}_n^m (\sin \beta) (C_{nm} \cos m \lambda + S_{nm} \sin m \lambda)
\]

(3)

where \( r, \beta, \) and \( \lambda \) are the body-fixed radius, latitude, and longitude of the spacecraft, \( R \) is the central planet's mean equatorial radius, and \( \mathbf{P}_n^m \) are the associated Legendre polynomials. The gradient indicates differentiation with
respect to central-planet-fixed coordinates.

Because the matrix D is a function of i, Ω, and u, the explicit derivatives in equation (2) must contain terms involving \( \frac{\partial D}{\partial u} \), \( \frac{\partial D}{\partial i} \), and \( \frac{\partial D}{\partial \Omega} \). These expressions are given in tables A4, A5, and A6 respectively.

The disturbing acceleration itself, \( \vec{\mathbf{V}} \), is treated as a function of the cartesian position vector and is differentiated in the usual way, viz:

\[
\frac{\partial \vec{\mathbf{V}}}{\partial \vec{r}} = \left[ \frac{\partial \vec{V}}{\partial \vec{r}} \right] \left[ \frac{\partial \vec{r}}{\partial \vec{r}} \right] = \left[ \frac{\partial \vec{V}}{\partial \vec{r}} \right] \left[ \begin{array}{c} r \end{array} \right]^T
\]

where \( \vec{r} \) represents the cartesian position vector in a frame fixed in the central planet. \( \vec{r} \) is just the cartesian vector equivalent of the central body-fixed radius, \( r \), latitude, \( \beta \), and longitude \( \lambda \) of the disturbing function given in equation (3). It is assumed that there are no velocity-dependent forces and the second term of (2) is

\[
\left( \frac{\partial F}{\partial \vec{r}} \right) = \left[ M \right] \left[ D \right] \left[ \begin{array}{c} Q J_3 \quad \mathbf{Q}^T \quad + J_{3rd} + J_{SP} \end{array} \right]
\]

where \( J_{3rd} \) is the Jacobian of the third body disturbing potential and \( J_{SP} \) is the gradient of the solar pressure disturbing force. Other disturbing forces should be treated in the same way.

The remaining term in the curly brackets is the matrix representing the transformation from orbital elements to the cartesian position vector. Since

\[
\vec{r} = \left[ D \right]^T \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}, \text{ and } r = \frac{p}{1 + h \sin u + k \cos u},
\]

the first column of \( \frac{\partial \vec{r}}{\partial E} \) is

\[
\frac{\partial \vec{r}}{\partial E} = \frac{\partial D}{\partial \rho} \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} + D^T \begin{bmatrix} \frac{\partial r}{\partial \rho} \\ 0 \\ 0 \end{bmatrix},
\]

and similarly for the other elements. The total matrix is given by
The remaining term in equation (2) is the explicit derivative of the "plant" with respect to the parameters to be estimated. This differentiation is accomplished for the spherical harmonic coefficients by evaluation of the disturbing function with the desired coefficient set to unity and the other coefficients set to zero. This is possible because the harmonic coefficients enter the equations linearly. For efficiency purposes, this procedure is carried out internally in the subroutine (FIELD2) that calculates the disturbing acceleration due to the central planet non-sphericity. The mechanics of this operation are explained in the FIELD2 subroutine description of Reference 2.

Averaging the Linear Variational Equations

In order to average the variational equations we must change variables in the formulation because the short-periodic parts of the derivatives of the fast variable with respect to \( \theta \) and \( \kappa \) are not small. To change from \( u \) to \( \Xi \) in the formulation of the linear variational equations, we simply calculate the two additional terms of the matrix \( M \) and their derivatives.

In the first and second columns of the fourth row of \( M \) write

\[
M_{41} = \sqrt{\mu} \left[ \frac{Q}{e} \left( k \cos u + h \sin u \right) - 2\sqrt{1-e^2} \cdot \frac{r^2}{p} \right]
\]
and

\[ M_{42} = -\sqrt{\frac{p}{\mu}} \left[ \frac{Q}{e} \left( k \sin u - h \cos u \right) \left( 1 + \frac{r}{p} \right) \right] \]

in place of the zeros given earlier when the function \( F_4 \) was \( u \) instead of \( \Xi \).

The derivative matrices are modified similarly and we add to the tables A1 through A6 the following terms.

\[ \left( \frac{\partial M}{\partial p} \right)_{41} = \frac{1}{2\sqrt{\mu p}} \left[ \frac{Q}{e} \left( k \cos u + h \sin u \right) - 2\sqrt{1 - e^2} \left( \frac{r}{p} \right) \right] 
\]

\[ \left( \frac{\partial M}{\partial p} \right)_{42} = -\frac{1}{2\sqrt{\mu p}} \left[ \frac{Q}{e} \left( k \sin u - h \cos u \right) \left( 1 + \frac{r}{p} \right) \right] 
\]

\[ \left( \frac{\partial M}{\partial h} \right)_{41} = \sqrt{\frac{p}{\mu}} \left[ \frac{d}{dh} \left( \frac{Q}{e} \right) \left( k \cos u + h \sin u \right) + \frac{Q}{e} \sin u + 2 \left( \frac{r}{p} \right) \frac{h}{\sqrt{1 - e^2}} \right. 
\]
\[ \left. - 2\sqrt{1 - e^2} \frac{d}{dh} \left( \frac{r}{p} \right) \right] 
\]

\[ \left( \frac{\partial M}{\partial h} \right)_{42} = \sqrt{\frac{p}{\mu}} \left[ \frac{d}{dh} \left( \frac{Q}{e} \right) \left( k \sin u - h \cos u \right) \left( 1 + \frac{r}{p} \right) - \frac{Q}{e} \cos u \left( 1 + \frac{r}{p} \right) \right. 
\]
\[ \left. + \frac{Q}{e} \left( k \sin u - h \cos u \right) \frac{d}{dh} \left( \frac{r}{p} \right) \right] 
\]

\[ \left( \frac{\partial M}{\partial k} \right)_{41} = \sqrt{\frac{p}{\mu}} \left[ \frac{d}{dk} \left( \frac{Q}{e} \right) \left( k \cos u + h \sin u \right) + \frac{Q}{e} \cos u 
\]
\[ + 2 \left( \frac{r}{p} \right) \frac{k}{\sqrt{1 - e^2}} - 2\sqrt{1 - e^2} \frac{d}{dk} \left( \frac{r}{p} \right) \right] 
\]

\[ \left( \frac{\partial M}{\partial k} \right)_{42} = \sqrt{\frac{p}{\mu}} \left[ \frac{d}{dk} \left( \frac{Q}{e} \right) \left( k \sin u - h \cos u \right) \left( 1 + \frac{r}{p} \right) + \frac{Q}{e} \sin u \left( 1 + \frac{r}{p} \right) 
\]
\[ + \frac{Q}{e} \left( k \sin u - h \cos u \right) \frac{d}{dk} \left( \frac{r}{p} \right) \right] 
\]

\[ \left( \frac{\partial M}{\partial u} \right)_{41} = \sqrt{\frac{p}{\mu}} \left[ \frac{Q}{e} \left( h \cos u - k \sin u \right) - 2\sqrt{1 - e^2} \frac{d}{du} \left( \frac{r}{p} \right) \right] 
\]
\[
\left( \frac{\partial M}{\partial u} \right)_{42} = -\sqrt{\frac{\mu}{\rho}} \left[ \frac{Q}{e} (k \cos u + h \sin u)(1 + \frac{r}{p}) + \frac{Q}{e} (k \sin u - h \cos u) \frac{d}{dh} \left( \frac{r}{p} \right) \right]
\]

where

\[
\frac{Q}{e} = -\frac{1}{\sqrt{1-e^2} + 1}, \quad \frac{d}{dh} \left( \frac{Q}{e} \right) = -\frac{d}{de} \left( \frac{Q}{e} \right) \cdot \frac{h}{\sqrt{1-e^2}}, \quad \frac{d}{dk} \left( \frac{Q}{e} \right) = -\frac{d}{de} \left( \frac{Q}{e} \right) \frac{k}{\sqrt{1-e^2}},
\]

and

\[
\frac{d}{de} \left( \frac{Q}{e} \right) = -\frac{e}{\sqrt{1-e^2}} \left( \frac{Q}{e} \right)^2.
\]

Also

\[
\frac{d}{dh} \left( \frac{r}{p} \right) = -\sin u \left( \frac{r}{p} \right)^2
\]

\[
\frac{d}{dk} \left( \frac{r}{p} \right) = -\cos u \left( \frac{r}{p} \right)^2
\]

\[
\frac{d}{du} \left( \frac{r}{p} \right) = (k \sin u - h \cos u) \left( \frac{r}{p} \right)^2,
\]

and

\[
\frac{r}{p} \neq f(p) \text{ so } \frac{d}{dp} \left( \frac{r}{p} \right) = 0.
\]

Now, with the above modifications, we have written the linear variational equations for a system whose fast variable is \( \Xi = \omega + M \) instead of \( u = \omega + f \) but the right hand side is still in terms of \( u \). The solution to these equations will involve \( \frac{\lambda u}{\partial C} \) rather than \( \frac{\lambda u}{\partial C} \) and we must convert by writing

\[
\frac{\lambda u}{\partial C} = \frac{\lambda \Xi}{\partial C} \cdot \frac{\lambda u}{\lambda \Xi} = \frac{\lambda \Xi}{\partial C} \cdot \frac{\sqrt{\mu p}}{n r^2}
\]

whenever \( \frac{\lambda u}{\partial C} \) is needed.

The only remaining modification is to change the term \( \sqrt{\mu p / r^2} \) to \( \sqrt{\mu / a^3} \) in accordance with the equations for the fast parts of the variables \( u \) and \( \Xi \) respectively.

The fourth equation now reads

\[
\dot{\Xi} = \sqrt{\frac{\mu (1-e^2)^2}{p^3}} + \sum_{j=1}^{3} M_{4j} (RSW)_{j}
\]  

(4)
and, if we say that \( n = \sqrt{\mu/a^3} \),

\[
\frac{\partial \tilde{n}}{\partial C} = \frac{\partial n}{\partial p} \frac{\partial p}{\partial C} + \frac{\partial n}{\partial h} \frac{\partial h}{\partial C} + \frac{\partial n}{\partial k} \frac{\partial k}{\partial C} + \frac{3}{3} \sum_{j=1}^{3} M_{4j} (R^N) \frac{\partial C}{\partial C}.
\]

The first three terms average to

\[
\frac{\partial \tilde{n}}{\partial C} = \frac{\partial n}{\partial p} \frac{\partial p}{\partial C} + \frac{\partial n}{\partial h} \frac{\partial h}{\partial C} + \frac{\partial n}{\partial k} \frac{\partial k}{\partial C}
\]

and this large part of the variation of the fast variable can be added after the averaging integration with the help of

\[
\frac{\partial \tilde{n}}{\partial p} = -\frac{3}{2} \frac{n}{(1-e^2)} , \quad \frac{\partial \tilde{n}}{\partial h} = -\frac{3hn}{(1-e^2)} , \quad \frac{\partial \tilde{n}}{\partial k} = -\frac{3kn}{(1-e^2)} .
\]

The variational equations then, are averaged in a way that is exactly analogous to the averaging of the method 8 variables. After the numerical averaging of the perturbation terms, the derivative of the average mean motion is added to the rate of the fourth variable and the averaging of the mean motion is taken outside the numerical averaging algorithm.
Table A.1

\[
\begin{bmatrix}
0 & 3 \sqrt{\frac{p}{u}} & \left( \frac{\tau}{p} \right) \\
-\frac{\cos u}{2 \sqrt{\mu p}} & \left( \frac{1 + \frac{\tau}{p}}{2} \right) \sin u + \frac{\tau}{p} h & 0 \\
\frac{\sin u}{2 \sqrt{\mu p}} & \left( \frac{1 + \frac{\tau}{p}}{2} \right) \cos u + \frac{\tau}{p} k & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
R' \\
S' \\
W' \\
0
\end{bmatrix}
\]

\[
\begin{align*}
8 - \gamma & = \left( \frac{\partial \tau}{\partial p} \right) \\
& = 0 + \frac{\tau}{p} \sin u + \cot i - \frac{3}{2} \frac{\mu}{\sqrt{p}} / r^2
\end{align*}
\]
\[ \frac{\partial F}{\partial h} = \begin{bmatrix} 0 & 2 \cdot q \cdot p \cdot (-\sin u) \left( \frac{r}{p} \right)^2 \\ 0 & q \left[ (-\sin^2 u) \left( \frac{r}{p} \right)^2 + \frac{r}{p} \cdot -\sin u \left( \frac{r}{p} \right)^2 \right] + q \cdot k \cdot \sin^2 u \left( \frac{r}{p} \right)^2 \cot i \\ 0 & q \sin u \cot i \left[ \frac{r}{p} + h \cdot (-\sin u) \left( \frac{r}{p} \right)^2 \right] \\ 0 & 0 & q \cdot (-\sin^2 u) \left( \frac{r}{p} \right)^2 \cot i + 2 \sqrt{\frac{\mu}{p}} \cdot \sin u \frac{r}{\sin i} \end{bmatrix} \]

Note: \( q = \frac{\sqrt{p}}{\sqrt{\mu}} \)
Table A.3

\[
\begin{bmatrix}
0 & 2pq (- \cos u) \left( \frac{r}{p} \right)^2 \\
0 & q \left[ (- \cos u \sin u) \left( \frac{r}{p} \right)^2 - \cos u \left( \frac{r}{p} \right)^2 \right] \\
0 & q \left[ (- \cos^2 u) \left( \frac{r}{p} \right)^2 + \frac{r}{p} - k \cos u \left( \frac{r}{p} \right)^2 \right] \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
\frac{\partial F'}{\partial k} - 10 =
\begin{bmatrix}
0 & -q \sin u \cot i \left( \frac{r}{p} \right) \\
0 & q h \cot i \sin u \left( - \cos u \left( \frac{r}{p} \right)^2 \right) \\
0 & q \cos^2 u \left( \frac{r}{p} \right)^2 \\
0 & -q \frac{\sin u \cos u \left( \frac{r}{p} \right)^2}{\sin i} \\
0 & + \frac{\partial N}{\partial k}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \frac{2}{p} \cos u \\
0 & 0 & \frac{r}{p} \\
0 & 0 & 0
\end{bmatrix}
\]
\[ \frac{\partial F}{\partial u} = \frac{\partial M}{\partial u} DQV + M \frac{\partial D}{\partial u} QQV \]

\[
\begin{pmatrix}
0 & 2qp (k \sin u - h \cos u) \left( \frac{r}{p} \right)^2 & 0 \\
q \sin u & q \left[ (k \sin u - h \cos u) \left( \frac{r}{p} \right)^2 \sin u + (1 + \frac{r}{p}) \cos u + h(k \sin u - h \cos u) \left( \frac{r}{p} \right)^2 \right] & -qk \cot i \frac{\partial}{\partial u} \left( \frac{r}{p} \sin u \right) \\
q \cos u & q \left[ (k \sin u - h \cos u) \left( \frac{r}{p} \right)^2 \cos u - (1 + \frac{r}{p}) \right] + k(k \sin u - h \cos u) \left( \frac{r}{p} \right)^2 & qh \cot i \frac{\partial}{\partial u} \left( \frac{r}{p} \sin u \right) \\
0 & 0 & -q \cot i \frac{\partial}{\partial u} \left( \frac{r}{p} \sin u \right) \\
0 & 0 & q \frac{\partial}{\partial u} \left( \frac{r}{p} \cos u \right) \\
0 & 0 & q \frac{1}{\sin i} \frac{\partial}{\partial u} \left( \frac{r}{p} \sin u \right)
\end{pmatrix}
\]

where
\[
\frac{\partial}{\partial u} \left( \frac{r}{p} \sin u \right) = (k \sin u - h \cos u) \left( \frac{r}{p} \right)^2 \sin u + \frac{r}{p} \cos u
\]
\[
\frac{\partial}{\partial u} \left( \frac{r}{p} \cos u \right) = (k \sin u - h \cos u) \left( \frac{r}{p} \right)^2 \cos u - \frac{r}{p} \sin u
\]
\[ \frac{\partial F}{\partial i} = \frac{\partial M}{\partial i} DQ \left[ \overrightarrow{V} \right] + M \frac{\partial D}{\partial i} Q \left[ \overrightarrow{V} \right] \]

\[ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \frac{q k \left( \frac{L}{p} \right) \sin u}{\sin^2 i} \\
0 & 0 & -\frac{q \left( \frac{r}{p} \right) h \sin u}{\sin^2 i}
\end{bmatrix} \]

Note: \( \frac{\partial N}{\partial i} = 0 \)

\[ \frac{\partial M}{\partial i} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \frac{q \left( \frac{r}{p} \right) \sin u}{\sin^2 i} \\
0 & 0 & -\frac{q \left( \frac{r}{p} \right) \sin u}{\sin^2 i}
\end{bmatrix} \]

\[ \left( \frac{\partial F}{\partial \Omega} \right) = \left[ \frac{\partial M}{\partial \Omega} \right] \left[ \begin{array}{c}
\frac{r}{S} \\
\frac{r}{W}
\end{array} \right] + \left[ \frac{\partial N}{\partial \Omega} \right] + M \left[ \frac{\partial D}{\partial \Omega} \right] Q \left[ \overrightarrow{V} \right] \]

\[ \frac{\partial M}{\partial \delta} = 0, \quad \frac{\partial N}{\partial \delta} = 0 \]
Table A.7

\[ D = \begin{bmatrix}
  \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
  -\sin \phi & \cos \phi & 0 \\
  \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta
\end{bmatrix} \]

\[ \frac{\partial D}{\partial \theta} = \begin{bmatrix}
  \cos \phi & -\sin \phi & 0 \\
  -\sin \phi & -\cos \phi & 0 \\
  \cos \phi & \sin \phi & 0
\end{bmatrix} \]

\[ \frac{\partial D}{\partial \phi} = \begin{bmatrix}
  0 & 0 & -\sin \theta \\
  0 & 0 & \cos \theta \\
  \cos \theta & \sin \theta & 0
\end{bmatrix} \]

\[ H' = \begin{bmatrix}
  -\sin \phi & \cos \phi & 0 \\
  -\cos \phi & -\sin \phi & 0 \\
  \sin \phi & \cos \phi & 0
\end{bmatrix} \]

\[ \begin{align*}
  \left( \frac{\partial N}{\partial \rho} \right)_4 &= -\frac{3}{2} \sqrt{\frac{\mu}{\rho}} \frac{1}{r^2} \\
  \left( \frac{\partial N}{\partial k} \right)_4 &= 2 \sqrt{\frac{\mu}{\rho}} \frac{\cos u}{r} \\
  \left( \frac{\partial N}{\partial h} \right)_4 &= 2 \sqrt{\frac{\mu}{\rho}} \frac{\sin u}{r} \\
  \left( \frac{\partial N}{\partial u} \right)_4 &= 2 \sqrt{\frac{\mu}{\rho}} \left( -k \sin u + h \cos u \right)
\end{align*} \]

all other elements are zero.

\[ \text{si} = \sin i, \ \text{ci} = \cos i, \ \text{so} = \sin \Omega, \ \text{co} = \cos \Omega, \ \text{su} = \sin u, \ \text{cu} = \cos u. \]