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**DYNAMICS  
OF FLEXIBLE SPINNING SATELLITES  
WITH RADIAL WIRE ANTENNAS**

**R. W. LONGMAN  
J. V. FEDOR**

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Greenbelt, Maryland

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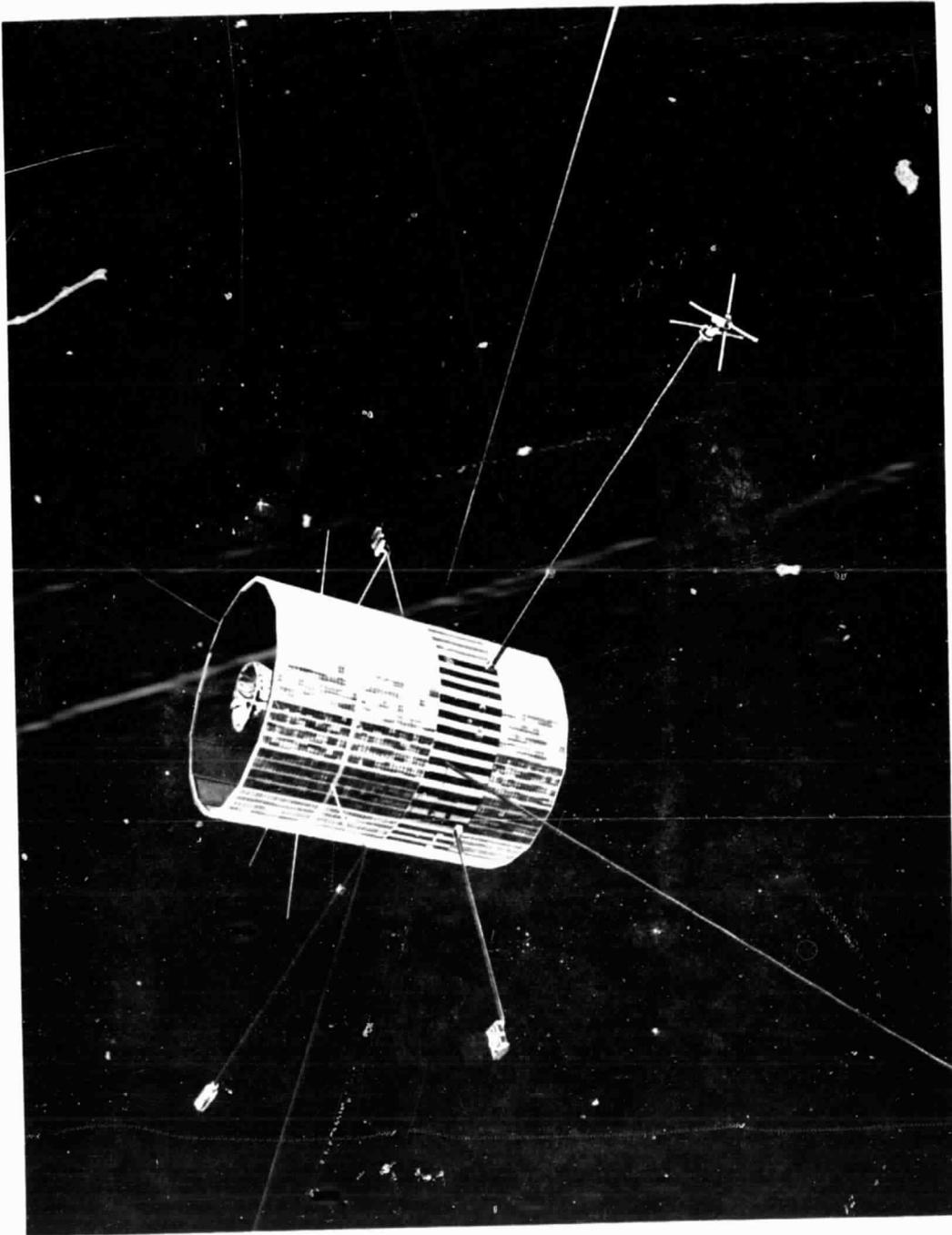


Figure 1. IMP-J Spacecraft

## DYNAMICS OF FLEXIBLE SPINNING SATELLITES WITH RADIAL WIRE ANTENNAS

### Abstract

A dynamic analysis is presented for a spin stabilized spacecraft employing four radial wire antennas with tip masses, a configuration first employed in the IMP-J spacecraft. The use of wires in place of the usual booms represents the ultimate in weight reduction at the expense of flexibility. The satellite is modelled as a 14 degree of freedom system, and the linearized equations of motion are found. The lowest order vibrational modes and natural frequencies of the gyroscopically coupled system are then determined. Because the satellite spin rate is decreased by antenna deployment, a spin-up maneuver is needed. The response of the time varying mode equations during spin-up is found, for the planar modes, in terms of Bessel functions and a Struve function of order  $-1/4$ . Because tables of the latter are not readily available, the particular solution is expressed in various forms including an infinite series of Bessel functions and a particularly useful asymptotic expansion. An error formula for the latter is derived showing that it gives good accuracy. Also, a simple approximation to the complementary function is obtained using the WKB method, and the phase error in the approximation is shown to be small.

# DYNAMICS OF FLEXIBLE SPINNING SATELLITES WITH RADIAL WIRE ANTENNAS

## Introduction

A dynamic analysis is presented for a spin stabilized spacecraft employing four radial wire antennas with tip masses. Most satellite designs use booms which are made as rigid as possible within weight and storage limitations. The design analyzed here represents a logical extreme of weight reduction, in which the booms are replaced by wires; and correspondingly it represents an extreme in flexibility since the wires have essentially no bending stiffness. An effective bending stiffness is supplied by the spin of the spacecraft.

Among the advantages of the wires over the boom configuration are that the wires are lighter, and the deployment devices simpler. In a spin stabilized spacecraft the deployment decreases the rotation rate. Hence, a spin-up maneuver is required. Here again the wires have advantages. Less fuel is needed in the spin-up, creating an additional savings in weight, and there are no problems of buckling at the root during the maneuver. Prior to planning of the IMP-J spacecraft (see Figure 1) for which this study was made, probably the major deterrent to the use of the radial wire configuration was the apparent complication in predicting dynamic behavior, together with a fear of the effects of extreme flexibility on spacecraft behavior and stability. These difficulties are more apparent than real for a space vehicle with high spin rate. As is evident from the present analysis, the radial wire problem is an order of magnitude more tractable than the problem with nonzero bending stiffness.

For purposes of the analysis, the satellite is modelled as a symmetric rigid body with four radial spherical physical pendulums. Thus, the model predicts the behavior of only the lowest order vibrational modes in which the wires behave as rigid bodies hinged at their attachment points. These modes definitely predominate at high spin rates, as in the case of the IMP-J. This vehicle is designed to spin at least at 23 rpm at all times. The wires when fully extended are 200 ft. long, and weigh  $1/2$  gram per foot with small 3 gram tip masses at their ends. Note that the distributed mass of the wire is much more important than the tip masses, so that a simple pendulum model cannot be substituted for the present physical pendulum model with distributed centrifugal loading.

The quadratic approximation to the Lagrangian is obtained, and from it the linearized equations of motion for the 14 degree of freedom system. A

transformation of variables gives these equations in a form where the vibrational mode shapes and their natural frequencies can be recognized. The presence of gyroscopic coupling complicates this process, and produces an eighth order characteristic equation which must be factored to get some of the system natural frequencies. This problem is then reduced to one of finding roots of a cubic equation, and an approximate method based on root locus techniques from control theory is used to obtain the roots in the case of the IMP-J. Implicit in this solution for the mode shapes and mode frequencies, is the general solution of the differential equations of motion of the satellite for any given initial conditions.

The spin-up maneuver needed in antenna deployment is a planar operation. In order to analyze the behavior of the wires during spin-up, the differential equations for the planar mode variables are obtained, linearized about the rigid body spin-up solution. Although these equations are linear, they have time varying coefficients, and thus present some difficulties in obtaining their general solutions. By a transformation into the complex plane, the gyroscopically coupled modes, as well as the other planar modes, are all shown to be governed by one fundamental equation. By proper transformations of both the dependent and independent variables the analytic solution of this equation is obtained in terms of Bessel functions of order  $\pm 1/4$  and a Struve function of order  $-1/4$ .

Because tables for the needed Struve function are not readily available, approximate solutions are obtained which give additional insight into the satellite behavior. The complementary function can be approximated using the WKB method, and the phase error is shown to remain small. The Struve function can be approximated using a Taylor series, but convergence is very slow. Somewhat faster convergence is obtained by deriving an expansion in terms of an infinite series of Bessel functions. However, it is shown that the most useful expression for the particular solution is in the form of an infinite asymptotic expansion which diverges for all finite time, but nevertheless gives very good accuracy when a small number of terms are used. An error formula is derived which expresses the error present in this approximation, and also gives a lower bound on the number of terms which can be included before the error begins to increase. Combining these approximate solutions gives the general solution for the planar response of the system under a spin-up torque.

### Quadratic Approximation to the Lagrangian

We wish to obtain the equations of motion linearized about a steady rigid body spin for the modal analysis, and linearized about a rigid body spin-up corresponding to a constant applied torque for the spin up analysis. The formulation used will obtain both sets of linearized equations simultaneously. It

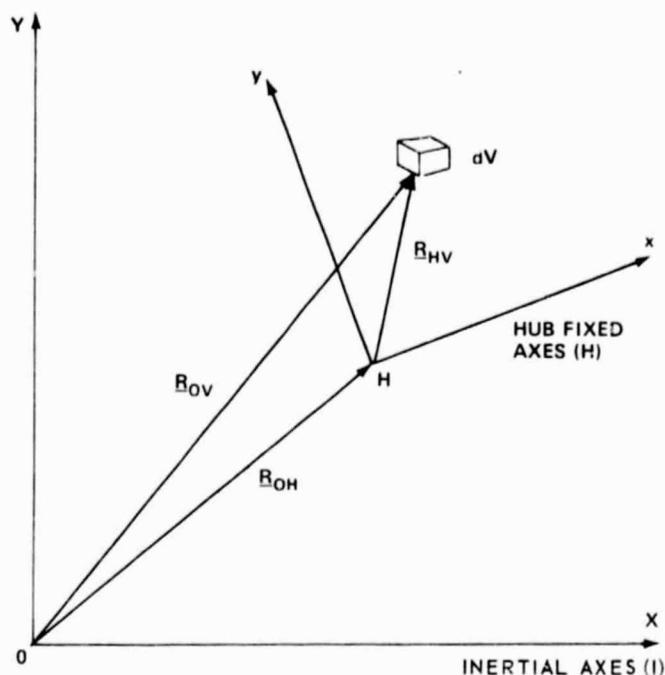


Figure 2. Inertial and Hub Fixed Coordinate Systems

suffices to use the quadratic approximation to the Lagrangian in order to obtain linearized equations of motion using Lagrange's equations. Since there is no potential energy we need only the quadratic approximation to the kinetic energy,  $T$ .

Figure 2 shows an inertially fixed coordinate system  $XYZ$  and coordinates  $xyz$  fixed in the symmetric hub of the spacecraft and centered at its center of mass. Let  $\underline{R}_{Ov}$  be the vector from the center of the inertially fixed coordinates to an arbitrary volume element  $dV$  of the spacecraft, and let  $\underline{R}_{OH}$  and  $\underline{R}_{Hv}$  be as shown. Then the velocity of the element  $dV$  relative to inertial space is  $d\underline{R}_{Ov}/dt|_I$ , the time derivative in inertial (I) axes. The spacecraft kinetic energy can be written

$$\begin{aligned}
 T = & \frac{1}{2} \iiint_{\text{Hub}} \rho \left( \frac{d\underline{R}_{Ov}}{dt} \Big|_I \cdot \frac{d\underline{R}_{Ov}}{dt} \Big|_I \right) dV + \frac{1}{2} \iiint_{\text{wires}} \rho \left( \frac{d\underline{R}_{Ov}}{dt} \Big|_I \cdot \frac{d\underline{R}_{Ov}}{dt} \Big|_I \right) dV \\
 & + \frac{1}{2} \sum_{\text{Tip Masses}} m_t \left( \frac{d\underline{R}_{Ov}}{dt} \Big|_I \cdot \frac{d\underline{R}_{Ov}}{dt} \Big|_I \right)
 \end{aligned} \tag{1}$$

where  $\rho$  is the density and  $m_t$  is the mass of a tip mass.

The angular velocity  $\omega$  of the hub axes relative to inertial space expressed in hub coordinates will be needed. It can be written as

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \cos \theta_2 \cos \theta_3 + \dot{\theta}_2 \sin \theta_3 \\ -\dot{\theta}_1 \cos \theta_2 \sin \theta_3 + \dot{\theta}_2 \cos \theta_3 \\ \dot{\theta}_1 \sin \theta_2 + \dot{\theta}_3 \end{bmatrix}$$

where  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  are 1, 2, 3 type Euler angles, i.e. a rotation through angle  $\theta_1$  in a right handed sense about X, followed by a rotation  $\theta_2$  about the resulting Y axis, and then a rotation  $\theta_3$  about the resulting Z axis gives a set of axes parallel to the hub xyz coordinate system.

To perform the linearization we can write

$$\dot{\theta}_3 = w_0(t) + \delta\dot{\theta}_3$$

$$\theta_3 = \psi(t) + \delta\theta_3$$

where  $\delta\theta_3$  and  $\delta\dot{\theta}_3$  are assumed small quantities. For the modal analysis we linearize about the steady rotation of a rigid body, so that  $w_0(t) = \omega_0$  a constant angular velocity about the spin axis, and  $\psi(t) = \omega_0 t$ . For the spin-up analysis we need to linearize about the values of  $\dot{\theta}_3$  and  $\theta_3$  corresponding to a constant torque Q applied about the spin axis, assuming the satellite to be rigid and to have moment of inertia  $I_3$  about that axis. Thus, we let

$$w_0(t) = \omega_0 + (Q/I_3) t$$

$$\psi(t) = \omega_0 t + \frac{1}{2} (Q/I_3) t^2$$

where  $\omega_0$  is the initial spin rate before spin-up.

The quadratic approximation to the angular velocity can now be written

$$\omega = \begin{bmatrix} 0 \\ 0 \\ w_0 \end{bmatrix} + \begin{bmatrix} \dot{\theta}_1 \cos \psi + \dot{\theta}_2 \sin \psi \\ -\dot{\theta}_1 \sin \psi + \dot{\theta}_2 \cos \psi \\ \delta \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} -\dot{\theta}_1 \delta \theta_3 \sin \psi + \dot{\theta}_2 \delta \theta_3 \cos \psi \\ -\dot{\theta}_1 \delta \theta_3 \cos \psi - \dot{\theta}_2 \delta \theta_3 \sin \psi \\ \dot{\theta}_1 \theta_2 \end{bmatrix} + \dots$$

$$= W_0 + W_1 + W_2 + \dots$$

Here and elsewhere the three dots indicate terms of higher order than quadratic. The other angles  $\theta_1$  and  $\theta_2$  are expanded about zero nominal values.

The true generalized coordinates for the center of mass of the hub will be the components X, Y, Z of  $\underline{R}_{OH}$  in inertial space. Their derivatives  $\dot{X}$ ,  $\dot{Y}$ ,  $\dot{Z}$  represent the components in coordinate system I of the inertial derivative  $d\underline{R}_{OH}/dt|_I$ . It will be convenient to define  $\dot{x}_H$ ,  $\dot{y}_H$ ,  $\dot{z}_H$  as the components of this vector in the hub H coordinate system (not to be confused with components of the derivative of  $\underline{R}_{OH}$  as seen in H coordinates,  $d\underline{R}_{OH}/dt|_H$ ).

The first term in Eq. (1) represents the kinetic energy of the hub, and can be written as the sum of the translational kinetic energy of the center of mass and the rotational kinetic energy about the center of mass:

$$\frac{1}{2} m_H (\dot{x}_H^2 + \dot{y}_H^2 + \dot{z}_H^2) + \frac{1}{2} I_{H1} (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_{H3} \omega_3^2$$

where  $m_H$  is the mass of the hub, and  $I_{H1}$ ,  $I_{H1}$ ,  $I_{H3}$  are the principle moments of inertia about x, y, and z axes respectively. The z axis is the axis of symmetry. The quadratic approximation for this is obtained by substituting the above quadratic approximation to  $\omega$ .

In order to calculate the second and third terms of Eq. (1) we must determine the inertial velocity  $d\underline{R}_{OV}/dt|_I$  of a volume element of the wire or the tip mass. Using Coriolis law we can write

$$\frac{d\underline{R}_{OV}}{dt} \Big|_I = \frac{d\underline{R}_{OH}}{dt} \Big|_I + \frac{d\underline{R}_{HV}}{dt} \Big|_I = \frac{d\underline{R}_{OH}}{dt} \Big|_I + \frac{d\underline{R}_{HV}}{dt} \Big|_H + \underline{\omega} \times \underline{R}_{HV}$$

The term  $d\underline{R}_{HV}/dt|_H$  represents the rate of change of  $\underline{R}_{HV}$  as seen in H coordinates.

Figure 3 shows a volume element  $dV$  of the  $i^{\text{th}}$  wire. The variables  $a_i$  and  $b_i$  are the in-plane and out-of-plane angles specifying the deviation of the wire from the nominal position. The  $c_i$  represent the angle from the x hub axis to the  $i^{\text{th}}$  nominal wire position, and  $c_i$  is taken as zero. From the figure, the  $x, y, z$  components of the volume element position are easily written. Then the quadratic approximations to the matrix  $[\underline{R}_{HV}]_H$  of components of  $\underline{R}_{HV}$  in H coordinates can be written

$$[\underline{R}_{HV}]_H = (q + r) \begin{bmatrix} \cos c_i \\ \sin c_i \\ 0 \end{bmatrix} + q \begin{bmatrix} -a_i \sin c_i \\ a_i \cos c_i \\ b_i \end{bmatrix} - \frac{1}{2} q(b_i^2 + a_i^2) \begin{bmatrix} \cos c_i \\ \sin c_i \\ 0 \end{bmatrix} + \dots$$

$$= U_0 + U_1 + U_2 + \dots$$

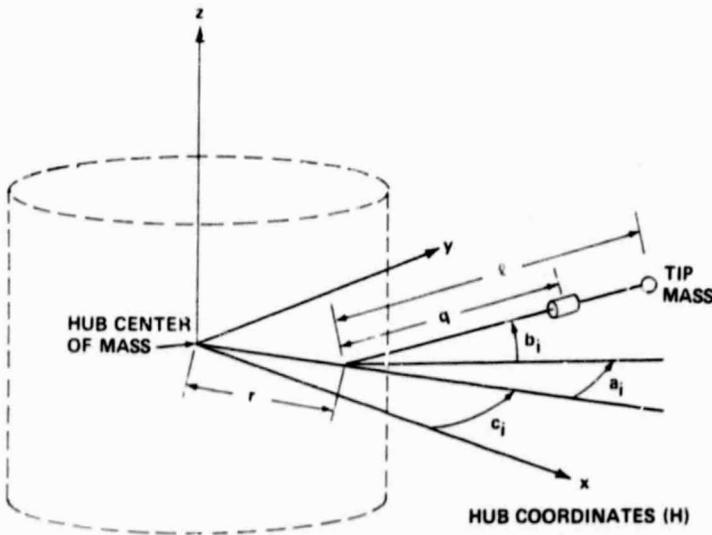


Figure 3. Variables Specifying Position of Wire Element

Similarly, the matrix  $[d\underline{R}_{HV}/dt|_H]_H$  of derivatives of x, y, z components is

$$[d\underline{R}_{HV}/dt|_H]_H = q \begin{bmatrix} -\dot{a}_i \sin c_i \\ \dot{a}_i \cos c_i \\ b_i \end{bmatrix} - q(b_i \dot{b}_i + a_i \dot{a}_i) \begin{bmatrix} \cos c_i \\ \sin c_i \\ 0 \end{bmatrix} + \dots$$

$$= \dot{U}_1 + \dot{U}_2 + \dots$$

Combining these results, the matrix  $[d\underline{R}_{OV}/dt|_I]_H$  of components in H coordinates can be written

$$[d\underline{R}_{OV}/dt|_I]_H = [\dot{x}_H \dot{y}_H \dot{z}_H]^T + \dot{u}_1 + \dot{u}_2 + (\tilde{W}_0 + \tilde{W}_1 + \tilde{W}_2) (U_0 + U_1 + U_2) + \dots$$

where superscript T indicates transpose, and the tilde indicates the matrix equivalent of a cross product (if the components of  $\tilde{W}_1$  are  $w_j$ , then the i, k component of the square matrix  $\tilde{W}_1$  is given by

$$\sum_{j=1}^3 \epsilon_{ijk} w_j \quad \text{where} \quad \epsilon_{ijk} = \frac{1}{2} (i - j) (j - k) (k - i).$$

The quadratic approximation to the kinetic energy, per unit mass, of the volume element is obtained by calculating the dot product

$$\frac{1}{2} [d\underline{R}_{OV}/dt|_I]_H^T [d\underline{R}_{OV}/dt|_I]_H$$

and retaining only terms through second order. Multiplication by  $m_i$  and substituting  $\ell$  for q (see Fig. 2) gives the kinetic energy of the  $i^{\text{th}}$  tip mass. Multiplication by the density per unit length of the wire,  $\rho$ , and integrating q from 0 to  $\ell$  gives the kinetic energy of the  $i^{\text{th}}$  wire. Adding these results for each wire and tip mass to the kinetic energy of the hub gives the quadratic approximation of the total kinetic energy

$$\begin{aligned}
T = & \frac{1}{2} M(\dot{x}_H^2 + \dot{y}_H^2 + \dot{z}_H^2) + I_3 w_0 (\delta \dot{\theta}_3 + \dot{\theta}_1 \theta_2) \\
& + \frac{1}{2} I_1 (\dot{\theta}_1^2 + \dot{\theta}_2^2) + \frac{1}{2} I_3 (\delta \dot{\theta}_3)^2 + M_1 (w_0 + \delta \dot{\theta}_3) \sum_{i=1}^4 \dot{a}_i \\
& + \frac{1}{2} M_3 w_0^2 \sum_{i=1}^4 a_i^2 + M_2 \dot{z}_H \sum_{i=1}^4 \dot{b}_i \\
& - \frac{1}{2} M_1 w_0^2 \sum_{i=1}^4 (a_i^2 + b_i^2) + \frac{1}{2} M_3 \sum_{i=1}^4 (\dot{a}_i^2 + \dot{b}_i^2) \\
& + M_2 \sum_{i=1}^4 [-\dot{x}_H (\dot{a}_i \sin c_i + w_0 a_i \cos c_i) + \dot{y}_H (\dot{a}_i \cos c_i - w_0 a_i \sin c_i)] \\
& + M_1 \sum_{i=1}^4 \{\dot{\theta}_1 [\dot{b}_i \sin(\psi + c_i) - w_0 b_i \cos(\psi + c_i)] \\
& - \dot{\theta}_2 [\dot{b}_i \cos(\psi + c_i) + w_0 b_i \sin(\psi + c_i)]\} + \dots
\end{aligned}$$

where

$$M = m_H + 4m + 4m_t$$

$$M_3 = \left(m_t + \frac{1}{3}m\right) \ell^2$$

$$M_2 = \left(m_t + \frac{1}{2} m\right) \ell$$

$$M_1 = M_3 + rM_2$$

$$I_1 = I_{H1} + \left[4 \left(m_t + \frac{1}{3} m\right) \ell^2 + 8 \left(m_t + \frac{1}{2} m\right) r\ell + 4(m_t + m) r^2\right]$$

$$I_3 = I_{H3} + \left[2 \left(m_t + \frac{1}{3} m\right) \ell^2 + 4 \left(m_t + \frac{1}{2} m\right) r\ell + 2(m_t + m) r^2\right]$$

$I_1, I_2,$  and  $I_3$  are the three principle moments of inertia of the entire spacecraft about the center of mass assuming it is rigid;  $I_{H1}, I_{H2}, I_{H3}$  are the corresponding inertias for the hub;  $r$  and  $\ell$  are defined in Figure 2;  $m$  is the mass of a wire,  $m_t$  that of a tip mass, and  $m_H$  that of the hub.

### The Linearized Differential Equations

The linearized differential equations are obtained by applying Lagrange's equations to the above kinetic energy for each of the 14 generalized coordinates  $X, Y, Z, \theta_1, \theta_2, \theta_3, a_i, b_i$ . Note that  $X, Y, Z$  are cyclic so that the equations reduce to  $\partial T / \partial \dot{X} = C_1, \partial T / \partial \dot{Y} = C_2, \partial T / \partial \dot{Z} = C_3, C_k$  all constant. Furthermore, we can choose the  $I$  axes so that there is no momentum of the spacecraft relative to  $I$  at time  $t = 0$ . Assuming this has been done, the constant generalized momenta are zero,  $C_1 = C_2 = C_3 = 0$ .

Direct application of Lagrange's equations yields linear differential equations with variable coefficients. Some of this difficulty can be circumvented by keeping the  $\dot{x}_H, \dot{y}_H,$  and  $\dot{z}_H$  coordinates. We recognize that to a linear approximation

$$\begin{aligned} \frac{\partial T}{\partial \dot{x}_H} &= \frac{\partial T}{\partial \dot{X}} \frac{\partial \dot{X}}{\partial \dot{x}_H} + \frac{\partial T}{\partial \dot{Y}} \frac{\partial \dot{Y}}{\partial \dot{x}_H} + \frac{\partial T}{\partial \dot{Z}} \frac{\partial \dot{Z}}{\partial \dot{x}_H} \\ &= \frac{\partial T}{\partial \dot{X}} \cos \psi - \frac{\partial T}{\partial \dot{Y}} \sin \psi + \dots \end{aligned}$$

or  $\partial T / \partial \dot{x}_H = 0$ , and similarly for  $\dot{y}_H$  and  $\dot{z}_H$ .

The linearized equations of motion are then

$$M\dot{x}_H - M_2 \sum_{i=1}^4 (\dot{a}_i \sin c_i + w_0(t) a_i \cos c_i) = 0 \quad (2a)$$

$$M\dot{y}_H + M_2 \sum_{i=1}^4 (\dot{a}_i \cos c_i - w_0(t) a_i \sin c_i) = 0 \quad (2b)$$

$$M\dot{z}_H + M_2 \sum_{i=1}^4 \dot{b}_i = 0 \quad (2c)$$

$$I_1 \ddot{\theta}_1 + I_3 [w_0(t) \dot{\theta}_2 + \dot{w}_0(t) \theta_2] + M_1 \sum_{i=1}^4 [(\ddot{b}_i + w_0^2 b_i) \sin(\psi + c_i) - \dot{w}_0 b_i \cos(\psi + c_i)] = 0 \quad (2d)$$

$$I_1 \ddot{\theta}_2 - I_3 w_0(t) \dot{\theta}_1 - M_1 \sum_{i=1}^4 [(\ddot{b}_i + w_0^2 b_i) \cos(\psi + c_i) + \dot{w}_0 b_i \sin(\psi + c_i)] = 0 \quad (2e)$$

$$I_3 \delta \ddot{\theta}_3 + M_1 \sum_{i=1}^4 \ddot{a}_i = 0 \quad (2f)$$

$$M_3 \ddot{a}_i + M_1 \delta \ddot{\theta}_3 + r M_2 w_0^2 a_i + M_2 [(-\ddot{x}_H + w_0 \dot{y}_H) \sin c_i + (\ddot{y}_H + w_0 \dot{x}_H) \cos c_i] = -M_1 \dot{w}_0 \quad (2g)$$

$$M_3 \ddot{b}_i + M_2 \ddot{z}_H + M_1 w_0^2 b_i + M_1 [(\ddot{\theta}_1 + 2w_0 \dot{\theta}_2) \sin(\psi + c_i) + (-\ddot{\theta}_2 + 2w_0 \dot{\theta}_1) \cos(\psi + c_i)] = 0 \quad (2h)$$

where  $i = 1, 2, 3, 4$ .

### Mode Shapes and Natural Frequencies

We wish to find a change of variables which will uncouple the equations of motion, and thus determine the normal modes of vibration and their natural frequencies. Because of the spin of the spacecraft, we cannot actually expect complete decoupling. Certain modes will be gyroscopically coupled. It might be possible to determine the needed mode variables mathematically, but it is much easier to appeal to physical intuition to guide us. Figure 4 presents eight ways in which one would expect the system to oscillate. Associated with each mode shape is a mode variable which should behave sinusoidally

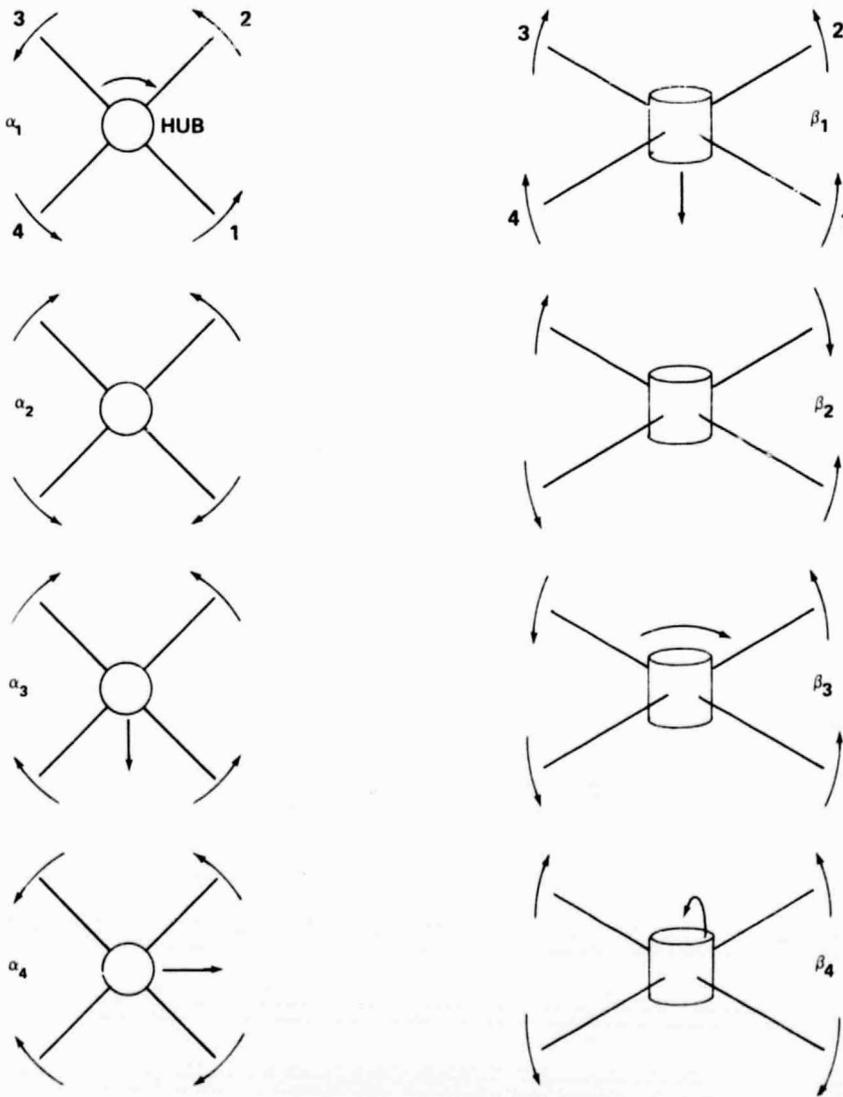


Figure 4. Vibrational Mode Shapes

$$\alpha_1 = \frac{1}{4} (a_1 + a_2 + a_3 + a_4) \quad \beta_1 = \frac{1}{4} (b_1 + b_2 + b_3 + b_4)$$

$$\alpha_2 = \frac{1}{4} (a_1 - a_2 + a_3 - a_4) \quad \beta_2 = \frac{1}{4} (b_1 - b_2 + b_3 - b_4)$$

$$\alpha_3 = \frac{1}{4} (a_1 + a_2 - a_3 - a_4) \quad \beta_3 = \frac{1}{4} (b_1 + b_2 - b_3 - b_4)$$

$$\alpha_4 = \frac{1}{4} (-a_1 + a_2 + a_3 - a_4) \quad \beta_4 = \frac{1}{4} (-b_1 + b_2 + b_3 - b_4)$$

Note that  $\alpha_2$  and  $\beta_2$  should be completely independent of other mode variables. Also,  $\alpha_1$  and  $\beta_1$  equations will couple with the rotation and translation equations, but will be independent of other mode variables. However, we expect  $\alpha_3$  and  $\alpha_4$  to be gyroscopically coupled, and  $\beta_3, \beta_4$  to be coupled with the precession and nutation of the hub.

Setting  $w_0 = \omega_0$ ,  $\psi = \omega_0 t$ ,  $c_i = (i - 1)\pi/2$ , and using mode variables, Eqs. (2a-c) and (2f) become

$$\dot{x}_H = (2M_2/M) [(\dot{\alpha}_3 + \dot{\alpha}_4) + \omega_0(\alpha_3 - \alpha_4)]$$

$$\dot{y}_H = - (2M_2/M) [(\dot{\alpha}_3 - \dot{\alpha}_4) - \omega_0(\alpha_3 + \alpha_4)]$$

$$\dot{z}_H = - (4M_2/M) \beta_1$$

$$\delta \ddot{\theta}_3 = - (4M_1/I_3) \ddot{\alpha}_1$$

Adding Eqs. (2g) for  $i = 1, 2, 3, 4$  and using the above  $\delta \ddot{\theta}_3$ , and similarly adding Eqs. (2h) and using  $\ddot{z}_H$  gives

$$[M_3 - (4M_1^2/I_3)] \ddot{\alpha}_1 + rM_2\omega_0^2\alpha_1 = 0$$

$$[M_3 - (4M_2^2/M)] \ddot{\beta}_1 + M_1\omega_0^2\beta_1 = 0$$

Therefore, the natural frequencies associated with these modes, normalized by the rotation rate  $\omega_0$ , are

$$\Omega_{\alpha_1} = \{rM_2/[M_3 - (4M_1^2/I_3)]\}^{1/2}$$

$$\Omega_{\beta_1} = \{M_1/[M_3 - (4M_2^2/M)]\}^{1/2}$$

Similarly, taking the linear combinations of Eqs. (2g) and (2h) indicated by the definitions of  $\alpha_2$  and  $\beta_2$  give

$$\Omega_{\alpha_2} = \{rM_2/M_3\}^{1/2}$$

$$\Omega_{\beta_2} = \{M_1/M_3\}^{1/2}$$

and

$$\ddot{\alpha}_2 + \Omega_{\alpha_2}^2 \omega_0^2 \alpha_2 = 0, \quad \ddot{\beta}_2 + \Omega_{\beta_2}^2 \omega_0^2 \beta_2 = 0.$$

Adding Eqs. (2g) as indicated by  $\alpha_3$  and  $\alpha_4$ , and using  $\dot{x}_H, \dot{y}_H$ , gives two equations

$$[M_3 - (2M_2^2/M)] \ddot{\alpha}_3 + [4M_2^2/M] \omega_0 \dot{\alpha}_4 + [rM_2 + (2M_2^2/M)] \omega_0^2 \alpha_3 = 0$$

$$[M_3 - (2M_2^2/M)] \ddot{\alpha}_4 - [4M_2^2/M] \omega_0 \dot{\alpha}_3 + [rM_2 + (2M_2^2/M)] \omega_0^2 \alpha_4 = 0$$

with skew-symmetric gyroscopic coupling of the first derivatives. The normalized frequencies associated with the roots of the characteristic equation for this system are

$$\Omega_{\alpha_3, \alpha_4} = \frac{1}{2} B \pm \left( \frac{1}{4} B^2 + C \right)^{1/2}$$

where

$$B = 4M_2^2 / (MM_3 - 2M_2^2)$$

$$C = (rMM_2 + 2M_2^2) / (MM_3 - 2M_2^2)$$

To uncouple the equations for  $\beta_3$  and  $\beta_4$ , obtained using Eqs. (2h), from the remaining equations we must use Eqs. (2d) and (2e). Setting  $w_0 = \omega_0$ ,  $\psi = \omega_0 t$ ,  $c_i = (i - 1)\pi/2$ , and using  $\beta_3$  and  $\beta_4$  to eliminate the  $b_i$ , the latter equations still contain time varying coefficients which complicate the elimination of  $\theta_1$  and  $\theta_2$  from Eqs. (2h). However, we note that when  $w_0 = \omega_0$  these two angles are cyclic coordinates. We then define a rotation of the derivative variables which will eliminate the explicit time dependence. Let

$$\eta_1 = \dot{\theta}_1 \cos(\omega_0 t) + \dot{\theta}_2 \sin(\omega_0 t)$$

$$\eta_2 = -\dot{\theta}_2 \sin(\omega_0 t) + \dot{\theta}_1 \cos(\omega_0 t)$$

and Eqs. (2d) and (2e) become

$$I_1 \dot{\eta}_1 + (I_3 - I_1) \omega_0 \eta_2 + 2M_1 [(\ddot{\beta}_3 + \ddot{\beta}_4) + \omega_0^2(\beta_3 + \beta_4)] = 0$$

$$I_1 \dot{\eta}_2 - (I_3 - I_1) \omega_0 \eta_1 - 2M_1 [(\ddot{\beta}_3 - \ddot{\beta}_4) + \omega_0^2(\beta_3 - \beta_4)] = 0$$

(Note that these same equations might have been obtained more directly by using quasicordinates  $\omega_1, \omega_2, \omega_3$  from the beginning.) The equations for  $\beta_3$  and  $\beta_4$  from Eqs. (2h) in terms of  $\eta_1$  and  $\eta_2$  are

$$M_3 \ddot{\beta}_3 + M_1 \omega_0^2 \beta_3 + \frac{1}{2} M_1 [(\dot{\eta}_1 - \dot{\eta}_2) + \omega_0(\eta_1 + \eta_2)] = 0$$

$$M_3 \ddot{\beta}_4 + M_1 \omega_0^2 \beta_4 + \frac{1}{2} M_1 [(\dot{\eta}_1 + \dot{\eta}_2) - \omega_0(\eta_1 - \eta_2)] = 0$$

The natural frequencies associated with  $\beta_3$  and  $\beta_4$  must be determined from the characteristic equation of the above system of four simultaneous equations:

$$\begin{aligned} & \{(s^2 + P\omega_0^2)(s^2 + K\omega_0^2) - 2PN(s^2 + \omega_0^2)(s^2 + K\omega_0^2)\}^2 \\ & + \{2PN\omega_0 s(s^2 + \omega_0^2)(K - 1)\}^2 = 0 \end{aligned}$$

where

$$P = M_1/M_3, \quad N = M_1/I_1, \quad \text{and} \quad K = (I_3 - I_1)/I_1.$$

It would appear that we must solve for the roots of an eighth order polynomial. The normalized frequencies  $\Omega_{\beta_3, \beta_4}$  are the positive roots  $\Omega$  of this equation when we set  $s = i\omega_0 \Omega$ . However, we note that the equation is then of the form  $x^2 - y^2 = (x - y)(x + y)$  so we can reduce the problem to finding the roots of two fourth order equations

$$\begin{aligned} & (\Omega^2 - P)(\Omega^2 - K^2) - 2PN(\Omega^2 - 1)(\Omega^2 - K) \\ & \pm 2PN\Omega(\Omega^2 - 1)(K - 1) = 0 \end{aligned}$$

However, from the form of these equations, if  $\Omega^*$  is a root of one, then  $-\Omega^*$  is a root of the other. Therefore, we need only solve for the four roots of one of the quartics and take their absolute values. We can still do better if we observe that the precession frequency (or its negative),  $\Omega = |K|$ , is a root of the quartic. We conclude that the four frequencies  $\Omega_{\beta_3, \beta_4}$  associated with  $\beta_3$  and  $\beta_4$  mode variables are

$$\Omega_{\beta_3, \beta_4} = |I_3 - I_1| / I_1$$

and the absolute values of the three roots of

$$(\Omega^2 - 1) [\Omega - (K - 2PN)/(1 - 2PN)] - k(\Omega + K) = 0$$

where  $k = (P - 1)/(1 - 2PN)$ . Approximate values for these three roots can be obtained easily using root locus techniques as shown below.

#### Application of Results to IMP-J Spacecraft

The parameters of the IMP-J spacecraft are:  $M_t = 3$  gms,  $m = 100$  gms,  $\ell = 200$  ft,  $r = 2$  ft,  $M = 19.37$  slugs,  $I_{H3} = 117$  slug ft<sup>2</sup> (hard booms deployed), and the transverse inertias are approximated by  $I_{H1} = 76$  slug ft<sup>2</sup>. Observe that

$$P = 1 + (r/\ell) \left[ \left( m_t + \frac{1}{2} m \right) / \left( m_t + \frac{1}{3} m \right) \right]$$

and is near unity (actually 1.01459) because of the ratio  $(r/\ell)$ . The factor  $k$  is then small (0.0539). The above cubic equation can be written as

$$1 = \frac{k(\Omega + K)}{(\Omega^2 - 1)(\Omega - G)}$$

where  $G = (K - 2PN)/(1 - 2PN)$ . It is thus in the form needed for a root locus plot; there is one zero at  $-K$ , and three poles at  $\pm 1$ , and  $G$ , and we are interested in the roots for a small negative gain  $(-k)$ . For the IMP-J  $K = 0.875$ ,  $G = 0.540$ , and the root locus plot is shown in Figure 5. An approximate value for each root can be obtained by calculating the root sensitivity at  $k = 0$ , i.e. finding the rate of change of the root location with  $k$  at  $k = 0$ , and extrapolating along the tangent:

$$\Omega(k) \approx \Omega(0) + \left[ \frac{d\Omega}{dk} (0) \right] k$$

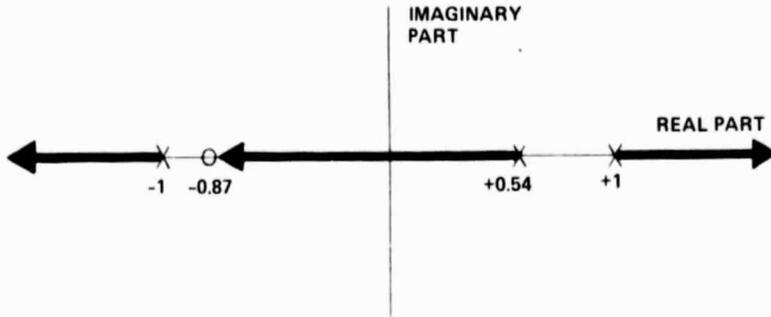


Figure 5. Root Locus Plot

where

$$\frac{d\Omega}{dk} = \frac{\Omega + K}{3\Omega^2 - 2G\Omega - 1 - k}$$

As a check on the result, the value of  $k$  corresponding to the approximate root can be calculated by substitution in the root locus equation after solving it for  $k$ . If the value is not as close as desired, iterating by making a small change in  $\Omega$  and recalculating  $k$  quickly gives the needed result.

Using this method to obtain the frequencies for  $\Omega_{\beta_3, \beta_4}$  and the formulas of the previous section for the other modes gives the following natural frequencies normalized by the spin rate  $\omega_0$ :

$$\Omega_{\alpha_1} = 0.2563$$

$$\Omega_{\alpha_2} = 0.1208$$

$$\Omega_{\alpha_3, \alpha_4} = 0.1236 \text{ and } 0.1225$$

$$\Omega_{\beta_1} = 1.007819$$

$$\Omega_{\beta_2} = 1.007267$$

$$\Omega_{\beta_3, \beta_4} = 0.8754, 1.002215, 1.09185, \text{ and } 0.4499.$$

For a spin of 23 rpm the period for the  $\alpha_1$  mode is roughly 10 sec., while the  $\alpha_2$ ,  $\alpha_3$  and  $\alpha_4$  modes all give approximately 21 sec. The out of plane modes give periods between 2 and 3 sec. with one exception of 5.8 sec. associated with one of the  $\beta_3, \beta_4$  frequencies.

### Fundamental Spin-Up Equation: Analytic Solution

We now turn our attention to the dynamic behavior of the satellite during the spin-up operation. From physical considerations it is clear that the mode  $\alpha_1$  is the most directly affected by the spin-up torque. We therefore examine this mode first.

The differential equation for  $\alpha_1$  is obtained by taking one fourth the sum of the four Eqs. (2g) and eliminating  $\delta\theta_3$  using Eq. (2f). The  $w_0(t)$  is taken as  $\omega_0 + (Q/I_3)t$ . The result is

$$\ddot{\alpha}_1 + \Omega_{\alpha_1}^2 \omega_0^2 (1 + \epsilon t)^2 \alpha_1 = -\epsilon (\omega_0 M_1 / r M_2) \Omega_{\alpha_1}^2$$

where  $\epsilon = Q/(I_3 \omega_0)$ . The spin-up torque for the IMP-J is 0.96 ft. lbs., which is small compared to the total vehicle inertia  $I_3$ , and therefore  $\epsilon$  is a small number. The  $\Omega_{\alpha_1}$  is the natural frequency normalized by the spin rate before the start of the spin-up,  $\omega_0$ . The differential equation can be written in an alternate form by defining a new time variable  $\tau$  and letting prime indicate differentiation with respect to  $\tau$

$$\tau = 1 + \epsilon t$$

$$\alpha_1'' + \bar{\Omega}_{\alpha_1}^2 \tau^2 \alpha_1 = -\epsilon [M_1 / (\omega_0 r M_2)] \bar{\Omega}_{\alpha_1}^2 \quad (3)$$

where  $\bar{\Omega}_{\alpha_1} = \Omega_{\alpha_1} \omega_0 / \epsilon$  (we will use analogous definitions for all other modes). Despite the simple appearance of this equation, it is difficult to obtain the solution in a form which is easily applied.

Using the change of variables from  $\alpha_1, \tau$  to  $J, \xi$  where

$$\xi = \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2$$

$$J(\xi) = (\tau)^{-1/2} \alpha_1(\tau)$$

the differential equation is transformed to

$$\xi^2 \frac{d^2 J}{d\xi^2} + \xi \frac{dJ}{d\xi} + \left[ \xi^2 - \frac{1}{16} \right] J = -\epsilon \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \right)^{5/4} [M_1 / (\omega_0 r M_2)] (\xi)^{3/4}$$

The complementary function for this equation is a linear combination of Bessel functions of order  $\pm 1/4$ , and hence the complementary function for  $\alpha_1$  is

$$\alpha_{1c} = c_1 \sqrt{\tau} J_{1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right) + c_2 \sqrt{\tau} J_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right) \quad (4)$$

( $c_1$  and  $c_2$  arbitrary constants). Bessel function tables for these fractional orders can be found in [1].

To obtain the particular solution, note that the Struve function of order  $\nu$ ,  $H_\nu(\xi)$ , is a particular solution of

$$\xi^2 \frac{d^2 J}{d\xi^2} + \xi \frac{dJ}{d\xi} + (\xi^2 - \nu^2) J = \frac{4 \left( \frac{1}{2} \xi \right)^{\nu+1}}{\sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right)}$$

(see [2]). By making the appropriate adjustment of constants the particular solution for  $\alpha_1$  can be written as

$$\alpha_{1P} = -\epsilon (\bar{\Omega}_{\alpha_1}/4)^{5/4} \left[ \sqrt{\tau} \Gamma \left( \frac{1}{4} \right) M_1 / (\omega_0 r M_2) \right] \sqrt{\tau} H_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right) \quad (5)$$

Tables of Struve functions of the needed order are not readily available. One can obtain an expression for  $H_{-1/4}$  in terms of a product of a fractional power of  $\xi$  times an infinite series in  $\xi$  by using the method of Frobenius on the differential equation for  $H_{-1/4}$  [2]. The resulting particular solution for  $a$  is

$$a_{1P} = -\epsilon \left[ \frac{\sqrt{\pi} \Gamma\left(\frac{1}{4}\right) M_1}{\omega_0 r M_2} \right] \left( \frac{1}{4} \bar{\Omega}_{a1} \tau \right)^2 \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{4} \bar{\Omega}_{a1} \tau^2 \right)^{2k}}{\Gamma\left(k + \frac{3}{2}\right) \Gamma\left(k + \frac{5}{4}\right)} \quad (6)$$

which represents an entire function.

Note that  $\bar{\Omega}_{a1}$  is a large-number, and that the values of  $\tau$  of interest are greater than 1. Hence, although the above series converges for all  $\tau$ , the number of terms needed to calculate a good approximation to  $a_{1P}$  is quite large. A series with improved convergence properties can be formed using a Bessel function expansion. For any value of  $\mu$  not equal to a negative integer or zero [2]

$$\left( \frac{1}{2} \xi \right)^\mu = \sum_{k=0}^{\infty} \frac{(\mu + 2k) \Gamma(\mu + k)}{k!} J_{\mu+2k}(\xi)$$

Then the driving term to the Bessel equation form of our differential equation can be rewritten giving

$$\xi^2 \frac{d^2 J}{d\xi^2} + \xi \frac{dJ}{d\xi} + \left[ \xi^2 - \frac{1}{16} \right] J = -\epsilon (\bar{\Omega}_{a1})^{5/4} \left[ \frac{M_1}{\sqrt{2} \omega_0 r M_2} \right] \sum_{k=0}^{\infty} \frac{\left(2k + \frac{3}{4}\right) \Gamma\left(k + \frac{3}{4}\right)}{k!} J_{3/4+2k}(\xi)$$

Substitute a solution of the form  $J = \sum a_n J_{3/4+2n}(\xi)$ , and observe that the left hand side of the equation equals  $[\mu^2 - 1/16] J_\mu$  when  $J_\mu$  is substituted for  $J$ . The coefficients  $a_n$  are obtained immediately by equating coefficients of Bessel functions of like orders

$$\alpha_{1P} = - \frac{\epsilon (\bar{\Omega}_{a1})^{5/4} M_1}{\sqrt{2} \omega_0 r M_2} \sum_{k=0}^{\infty} \frac{\left(2k + \frac{3}{4}\right) \Gamma\left(k + \frac{3}{4}\right)}{k! \left[\left(\frac{3}{4} + 2k\right)^2 - \frac{1}{16}\right]} \sqrt{\tau} J_{3/4+2k} \left(\frac{1}{2} \bar{\Omega}_{a1} \tau^2\right) \quad (7)$$

This expression represents the exact solution and has better convergence properties than the power series solution. However, we will find that for most purposes an approximate solution in the form of an asymptotic expansion is most useful.

### WKB Approximation

The complementary function  $\alpha_{1c}$  found above appears not only in the  $\alpha_1$  mode, but in other modes as well. For purposes of making rapid calculations, as well as to give a somewhat better intuitive feel for the mode response, an approximate solution is generated by the WKB method [3].

The WKB functions

$$W_{\pm}(\tau) = [f(\tau)]^{-1/4} \exp \left[ \pm i \int_1^{\tau} \sqrt{f(\eta)} d\eta \right]$$

are easily shown to be exact solutions to

$$W_{\pm}'' + (f(\tau) + g(\tau)) W_{\pm} = 0$$

$$g(\tau) = \frac{1}{4} \frac{f''}{f} - \frac{5}{16} \left(\frac{f'}{f}\right)^2$$

and therefore approximate solutions to  $\alpha_1'' + f(\tau) \alpha_1 = 0$  provided  $g(\tau)$  is small compared to  $f(\tau)$ . For the present problem we can take

$$W_{\pm} = (1/\sqrt{\tau}) \exp \left[ \pm i \frac{1}{2} \bar{\Omega}_{a1} \tau^2 \right]$$

and the complementary function can then be written as

$$a_{1c}(\tau) = (A/\sqrt{\tau}) \cos\left(\frac{1}{2}\bar{\Omega}_{a1}\tau^2 + \delta\right) \quad (8)$$

where  $A$  and  $\delta$  are arbitrary constants. The function  $g(\tau) = -3/(4\tau^2)$  is small compared to  $f(\tau) = \bar{\Omega}_{a1}^2\tau^2$  because of the presence of a factor  $1/\epsilon$  in  $\bar{\Omega}_{a1}$ .

Applying the well known asymptotic form (for large  $\tau$ ) of the Bessel functions would give the same result, and therefore accuracy is good for properly chosen  $\delta$  when  $\tau$  is large. The error in this WKB solution when it is carried in to small values of  $\tau$  can be investigated by writing

$$a_{1c} = C_+(\tau) W_+(\tau) + C_-(\tau) W_-(\tau)$$

where the variation of the constants  $C_{\pm} = (A/2)e^{\pm i\delta}$  accounts for the difference between the true solution and the WKB approximation. The rate of change of  $C_{\pm}$  with  $\tau$  is given by [3]

$$\frac{dC_{\pm}}{d\tau} = \pm \frac{3i}{8\bar{\Omega}_{a1}\tau^3} \{C_{\pm} + C_{\mp} \exp[\mp i\bar{\Omega}_{a1}\tau^2]\}$$

If the  $C_{\pm}$  do not change much, we can calculate the change  $\Delta C_{\pm}$  by integrating

$$\Delta C_{\pm} = \int_1^{\infty} \frac{\pm 3iAe^{\pm i\delta}}{16\bar{\Omega}_{a1}\tau^3} \{1 + \exp[\mp i\bar{\Omega}_{a1}\tau^2 \mp 2i\delta]\} d\tau$$

treating  $\delta$  as a constant. The second term in the brackets is bounded by one in magnitude, so that we obtain an approximate expression

$$|\Delta C_{\pm}/C_{\pm}| < 3/(8\bar{\Omega}_{a1}) = 3\epsilon/(8\omega_0\bar{\Omega}_{a1})$$

showing that the relative error is small even for  $\tau$  as small as  $\tau = 1$ . Thus, no significant phase error in  $\delta$  accumulates in our approximate solution even after an arbitrarily large number of wavelengths.

### Asymptotic Expansion Solution

An approximation to the particular solution  $\alpha_{1P}$  can be obtained in the form of an asymptotic expansion. We formally substitute

$$\alpha_{1P} = \sum_{n=0}^{\infty} c_n \tau^{-n}$$

into the differential equation and solve for the coefficients  $c_n$ . By the ratio test, the resulting series is seen to diverge for all finite  $\tau$ . However, for properly chosen  $N$ , the series truncated to  $N$  terms forms an asymptotic expansion which will be shown to give good accuracy

$$\alpha_{1P} \sim -\frac{\epsilon M_1}{\omega_0 r M_2 \tau^2} \left[ 1 + \sum_{n=1}^N \frac{(-1)^n \prod_{k=0}^{n-1} (4k+2)(4k+3)}{\bar{\Omega}_{\alpha 1}^{2n} \tau^{4n}} \right] \quad (9)$$

Denote the right hand side of this equation by  $\alpha_{1N}$ , and determine the differential equation for which it is a particular solution by calculating  $\alpha_{1N}'' + \bar{\Omega}_{\alpha 1}^2 \tau^2 \alpha_{1N}$ . It is seen that the forcing function differs from the desired forcing function by the addition of a term depending on  $\tau^{-4(N+1)}$ . Then the differential equation satisfied by the error,  $\Delta\alpha = \alpha_{1P} - \alpha_{1N}$ , is found by subtracting the equation for  $\alpha_{1N}$  from Eq. (3)

$$\Delta\alpha'' + \bar{\Omega}_{\alpha 1}^2 \tau^2 \Delta\alpha = (-1)^N \left( \frac{\epsilon M_1}{\omega_0 r M_2} \right) \frac{\prod_{k=0}^N (4k+2)(4k+3)}{\bar{\Omega}_{\alpha 1}^{2N} \tau^{4N+4}}$$

The error in the asymptotic expansion is that particular solution  $\Delta\alpha$  satisfying  $\Delta\alpha = \Delta\alpha' = 0$  at  $\tau = 1$ . Denote the driving function in the above equation by  $F(\tau)$ . Then the desired particular solution is found by variation of parameters to be

$$\Delta\alpha = \int_1^{\tau} [\phi_1(\xi) \phi_2(\tau) - \phi_1(\tau) \phi_2(\xi)] [F(\xi)/W(\xi)] d\xi$$

where  $\phi_1$  and  $\phi_2$  are the two linearly independent solutions to the homogeneous equation

$$\phi_1 = \sqrt{\tau} J_{1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right) \quad \phi_2 = \sqrt{\tau} J_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right)$$

and  $W$  is the Wronskian. Then  $\phi_1'' + \bar{\Omega}_{\alpha_1}^2 \tau^2 \phi_1 = 0$ , and  $\phi_2'' + \bar{\Omega}_{\alpha_1}^2 \tau^2 \phi_2 = 0$ ; multiplying the first by  $\phi_2$ , the second by  $\phi_1$ , and subtracting gives an expression which when integrated shows that the Wronskian is constant. Let  $1/2 \bar{\Omega}_{\alpha_1}^2 \tau^2 = 1$ , and use the fact that the Wronskian of  $J_\nu(z)$ ,  $J_{-\nu}(z)$  is  $-2 \sin(\nu\pi)/(\pi z)$  from [2], to determine the constant

$$\begin{aligned} W &= \bar{\Omega}_{\alpha_1} \tau^2 \left[ J_{1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right) J'_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right) - J'_{1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right) J_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{\alpha_1} \tau^2 \right) \right] \\ &= 2 [J_{1/4}(1) J'_{-1/4}(1) - J'_{1/4}(1) J_{-1/4}(1)] \\ &= 2^{3/2}/\pi \end{aligned}$$

The Bessel functions are bounded by [2]

$$|J_{1/4}(x)| \leq 1 \quad |J_{-1/4}(x)| \leq 1 / \left[ \left| \frac{1}{2} x \right|^{1/4} \Gamma\left(\frac{3}{4}\right) \right]$$

Therefore, the error in the asymptotic expansion solution is bounded by

$$\begin{aligned}
|\Delta\alpha| &\leq \int_1^\tau [|\phi_1(\xi)| |\phi_2(\tau)| + |\phi_1(\tau)| |\phi_2(\xi)|] \cdot [|\mathbf{F}(\xi)| / \mathbf{W}] d\xi \\
&\leq \int_1^\tau \sqrt{\tau\xi} \left[ \left| J_{1/4} \left( \frac{1}{2} \bar{\Omega}_{a1} \xi^2 \right) \right| \left| J_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{a1} \tau^2 \right) \right| \right. \\
&\quad \left. + \left| J_{1/4} \left( \frac{1}{2} \bar{\Omega}_{a1} \tau^2 \right) \right| \left| J_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{a1} \xi^2 \right) \right| \right] [|\mathbf{F}(\xi)| / \mathbf{W}] d\xi \\
&\leq \frac{\epsilon^{2N+1.25} \pi M_1 \prod_{k=0}^N (4k+2)(4k+3)}{2\omega_0^{2N+1.25} r M_2 \bar{\Omega}_{a1}^{2N+0.25} \Gamma\left(\frac{3}{4}\right)} \left\{ \frac{1}{4N+2.5} \left[ 1 - \frac{1}{\tau^{4N+2.5}} \right] \right. \\
&\quad \left. + \frac{\sqrt{\tau}}{4N+3} \left[ 1 - \frac{1}{\tau^{4N+3}} \right] \right\}
\end{aligned}$$

Because  $\epsilon$  is quite small in the present application it is found that using three terms in the asymptotic expansion gives very good accuracy.

#### Planar Mode Response to Spin-Up

Because the spin-up maneuver is a planar one, we now obtain the dynamic response of the remaining planar modes under the spin-up torque. Taking the appropriate linear combination of Eqs. (2g), letting  $w_0(t) = \omega_0 + (Q/I_3)t$ , and changing from  $t$  to  $\tau$  as the independent variable gives

$$\alpha_2'' + \bar{\Omega}_{a2}^2 \tau^2 \alpha_2 = 0$$

and the corresponding solution is

$$\alpha_2 = C_3 \sqrt{\tau} J_{1/4} \left( \frac{1}{2} \bar{\Omega}_{a2} \tau^2 \right) + C_4 \sqrt{\tau} J_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{a2} \tau^2 \right)$$

where  $\bar{\Omega}_{a_2} = \Omega_{a_2} \omega_0 / \epsilon$ . The differential equations for the coupled modes  $a_3$  and  $a_4$  are found by taking the appropriate linear combinations of Eqs. (2g), using Eqs. (2a) and (2b) with  $c_1 = (i - 1)\pi/2$  and the above  $w_0(t)$ , and changing from  $t$  to  $\tau$

$$a_3'' + B\tau a_4' + C\tau^2 a_3 + \frac{1}{2} B a_4 = 0$$

$$a_4'' - B\tau a_3' + C\tau^2 a_4 - \frac{1}{2} B a_3 = 0$$

where

$$B = \frac{4M_2^2 \omega_0}{\epsilon(MM_3 - 2M_2^2)}$$

$$C = \frac{\omega_0(rMM_2 + 2M_2^2)}{\epsilon^2(MM_3 - 2M_2^2)}$$

Note that this time there is not only gyroscopic coupling of the first derivatives, but coupling in the undifferentiated terms as well. In order to obtain an analytic solution to this system of equations, use a complex valued change of variables to write the system as a single equation

$$a'' - iB\tau a' + \left\{ C\tau^2 - \frac{1}{2} iB \right\} a = 0$$

where  $a = a_3 + ia_4$ . A second change of variables  $a(\tau) = v(\tau)p(\tau)$ , where

$$p(\tau) = \exp \left[ \frac{1}{4} iB\tau^2 \right]$$

is chosen to eliminate the first derivative term in the differential equation, gives

$$v''(\tau) + \left\{ \frac{1}{4} B^2 + C \right\} \tau^2 v(\tau) = 0$$

which is the fundamental equation which has already been solved. Then the solutions for  $\alpha_3$  and  $\alpha_4$ , in terms of real valued arbitrary constants, are both of the form

$$\alpha_3, \alpha_4 = \left[ c_5 \cos\left(\frac{1}{4} B \tau^2\right) + c_6 \sin\left(\frac{1}{4} B \tau^2\right) \right] \sqrt{\tau} J_{1/4} \left( \frac{1}{2} \left( \frac{1}{4} B^2 + C \right)^{1/2} \tau^2 \right) + \left[ c_7 \cos\left(\frac{1}{4} B \tau^2\right) + c_8 \sin\left(\frac{1}{4} B \tau^2\right) \right] \sqrt{\tau} J_{-1/4} \left( \frac{1}{2} \left( \frac{1}{4} B^2 + C \right)^{1/2} \tau^2 \right) \quad (10)$$

A simple calculation shows that the WKB approximation is

$$\alpha_3, \alpha_4 = \frac{A_1}{\sqrt{\tau}} \cos\left(\frac{1}{2} \bar{\Omega}_{\alpha_3} \tau^2 + \delta_1\right) + \frac{A_2}{\sqrt{\tau}} \cos\left(\frac{1}{2} \bar{\Omega}_{\alpha_4} \tau^2 + \delta_2\right)$$

where  $A_1, A_2, \delta_1$ , and  $\delta_2$  are arbitrary constants determined by initial conditions.

Although this section is concerned with the planar modes, it is interesting to note that two of the out of plane modes are governed by the same fundamental spin-up equation. By a derivation closely paralleling the derivation of the mode frequencies for  $\beta_1$  and  $\beta_2$ , the spin-up equations are found to be

$$\beta_1'' + \bar{\Omega}_{\beta_1}^2 \tau^2 \beta_1 = 0$$

$$\beta_2'' + \bar{\Omega}_{\beta_2}^2 \tau^2 \beta_2 = 0$$

with the corresponding solutions

$$\beta_1 = c_9 \sqrt{\tau} J_{1/4} \left( \frac{1}{2} \bar{\Omega}_{\beta_1} \tau^2 \right) + c_{10} \sqrt{\tau} J_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{\beta_1} \tau^2 \right)$$

$$\beta_2 = c_{11}\sqrt{\tau} J_{1/4} \left( \frac{1}{2} \bar{\Omega}_{\beta_2} \tau^2 \right) + c_{12}\sqrt{\tau} J_{-1/4} \left( \frac{1}{2} \bar{\Omega}_{\beta_2} \tau^2 \right)$$

### References

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