NON-LINEAR BEHAVIOR OF FIBER COMPOSITE LAMINATES

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**Non-linear Behavior of Fiber Composite Laminates**

The non-linear behavior of fiber composite laminates which results from lamina non-linear characteristics was examined. The analysis uses a Ramberg-Osgood representation of the lamina transverse and shear stress strain curves in conjunction with deformation theory to describe the resultant laminate non-linear behavior.

A laminate having an arbitrary number of oriented layers and subjected to a general state of membrane stress was treated. Parametric results and comparison with experimental data and prior theoretical results are presented.

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**Key Words**
- Fiber composite laminates
- Non-linear behavior
- Lamina non-linear characteristics
- Ramberg-Osgood representation
- Deformation theory
- Parametric results
- Experimental data
- Prior theoretical results

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**Abstract**

The non-linear behavior of fiber composite laminates which results from lamina non-linear characteristics was examined. The analysis uses a Ramberg-Osgood representation of the lamina transverse and shear stress strain curves in conjunction with deformation theory to describe the resultant laminate non-linear behavior.

A laminate having an arbitrary number of oriented layers and subjected to a general state of membrane stress was treated. Parametric results and comparison with experimental data and prior theoretical results are presented.
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<tr>
<td>2a, 2b</td>
<td>Laminate dimension;</td>
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<tr>
<td>c</td>
<td>Fiber volume fraction;</td>
</tr>
<tr>
<td>C_{ijkl}</td>
<td>Stiffness matrix;</td>
</tr>
<tr>
<td>E_1</td>
<td>Elastic Young's modulus;</td>
</tr>
<tr>
<td>E_A</td>
<td>Young's modulus in fiber direction;</td>
</tr>
<tr>
<td>E_T</td>
<td>Young's modulus in transverse direction;</td>
</tr>
<tr>
<td>F</td>
<td>Function of stress ( \sigma_{ij} );</td>
</tr>
<tr>
<td>G_1_G</td>
<td>Elastic shear modulus of matrix;</td>
</tr>
<tr>
<td>G_A</td>
<td>Axial shear modulus;</td>
</tr>
<tr>
<td>G_S_A</td>
<td>Effective secant shear modulus of composite;</td>
</tr>
<tr>
<td>ijk_l</td>
<td>Subscripts ranging from 1 to 3;</td>
</tr>
<tr>
<td>k, m, n</td>
<td>Ramberg-Osgood parameters for the matrix;</td>
</tr>
<tr>
<td>I_1, 2, 3, 4, 5</td>
<td>Stress invariants;</td>
</tr>
<tr>
<td>J_2</td>
<td>( s_{ij} s_{ij}/2 );</td>
</tr>
<tr>
<td>K</td>
<td>Number of laminae in a laminate;</td>
</tr>
<tr>
<td>L, L</td>
<td>Matrices, also quadratic function of stresses;</td>
</tr>
<tr>
<td>M, N</td>
<td>Ramberg-Osgood parameters for the composite;</td>
</tr>
<tr>
<td>r, ( \theta ), z</td>
<td>Cylindrical coordinate system;</td>
</tr>
<tr>
<td>s_ij</td>
<td>Stress deviator;</td>
</tr>
<tr>
<td>S_ij</td>
<td>Compliance matrix;</td>
</tr>
<tr>
<td>S'_ij</td>
<td>Elastic compliance matrix;</td>
</tr>
<tr>
<td>S''_ij</td>
<td>Inelastic compliance matrix;</td>
</tr>
<tr>
<td>t</td>
<td>Laminate thickness;</td>
</tr>
<tr>
<td>T_i</td>
<td>Surface tractions;</td>
</tr>
<tr>
<td>T_{q\beta}</td>
<td>Constant edge forces per unit length;</td>
</tr>
<tr>
<td>u_1, 2, 3</td>
<td>Displacements in ( x_{1, 2, 3} ) directions, respectively;</td>
</tr>
<tr>
<td>( \tilde{u}_i )</td>
<td>Admissible displacement field;</td>
</tr>
<tr>
<td>U_\varepsilon</td>
<td>Strain energy;</td>
</tr>
<tr>
<td>U_c</td>
<td>Complimentary energy;</td>
</tr>
<tr>
<td>U_p</td>
<td>Potential energy;</td>
</tr>
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SYMBOLS CONTINUED

\( V \) - Volume;
\( W^e \) - Strain energy density;
\( W^o \) - Complimentary energy density;
\( x_{1,2,3} \) - Fixed coordinate directions;
\( x_{(k)} \) - Local coordinate system for the kth lamina;
\( \alpha, \beta \) - Subscripts ranging from 1 to 2;
\( \gamma \) - Shear strain;
\( \epsilon \) - Strain;
\( \bar{\epsilon}_{ij} \) - Average strain tensor;
\( \epsilon'_{\alpha\beta} \) - Elastic strains;
\( \epsilon''_{\alpha\beta} \) - Inelastic strains;
\( \sigma \) - Stress;
\( \sigma_Y \) - Nominal composite yield stress;
\( \sigma' \) - Nominal matrix yield stress;
\( \sigma^o \) - Stress \( \sigma^o_{\alpha\beta} \) at the edges;
\( \bar{\sigma}_{ij} \) - Average stress tensor;
\( \sigma_{ij} \) - Applied stress;
\( \tau \) - Shear stress;
\( \tau_Y \) - Nominal composite yield shear stress;
\( \tau_o \) - Average shear stress in composite;
\( \tau' \) - Nominal matrix yield shear stress;
\( \nu_A \) - Poisson's ratio; and
\( \theta_k \) - Reinforcement angle.
SUMMARY

The non-linear stress-strain behavior of fiber composite laminates has been analyzed to define the relationship between laminate behavior and the non-linear stress-strain characteristics of unidirectional composites. The resulting analysis has been programmed to yield an efficient computerized design and analysis tool.

The approach utilized herein was to adopt a Ramberg-Osgood representation of the non-linear stress-strain behavior and to utilize deformation theory as an adequate representation of the material nonlinearities. The problem was viewed on two levels. First, the relationship between the constituent properties and the stress-strain response of a unidirectional fiber composite material was studied. For this problem, the primary attention was directed toward axial shear behavior, and an expression was established relating the composite average-stress/average-strain curve to the fiber moduli and the matrix non-linear stress-strain curve. Second level of approach is to treat the interrelationship between the properties of the unidirectional layers and those of the laminate. For this case, the starting point is a non-linear stress-strain curve for transverse stress and for axial shear and a linear stress-strain relation for stress in the fiber direction. The non-linear lamina stress-strain curves can be modeled by proper selection of the Ramberg-Osgood parameters. In the present study, with this as a starting point, an interaction expression was formulated to account for simultaneous application of axial shear and transverse stress.

A laminate having an arbitrary number of oriented layers and subjected to a general state of membrane stress was treated. Parametric results and comparison with experimental data and prior theoretical results are presented.
1. **INTRODUCTION**

A basic requirement for the engineer designing with fiber composite materials is a definition of the stiffness and strength of these materials under a variety of loading conditions, including cases for which experimental materials properties data are not available. For this purpose, it is necessary that he have at his disposal reasonably accurate procedures to predict these mechanical properties. Existing analyses can predict the elastic behavior of a laminated composite quite well when the elastic properties of the unidirectional materials from which it is made are known. However, the situation has been much more complicated and much less satisfactory with regard to the inelastic stiffness and strength of a laminate. The present program was undertaken to develop a computerized analysis of the inelastic behavior of fiber composite laminates which could be used as a design tool. The results of this study and comparisons of these results with experimental data are presented in this report.

It is essential to recognize that the utilization of fiber composite materials in structural design involves the incorporation of material design into the structural design process. This is illustrated clearly by the fact that the gross material properties of a fiber composite laminate change when any change is made in the laminate ply orientations. Even when the designer considers a material formed from a particular combination of fiber and matrix materials, there remains a large number of geometric variables associated with the laminate design. Thus, in the preliminary design phase, experimental material properties data will generally be too limited. In the case of elastic properties, sufficient capability to synthesize the necessary properties exists. This procedure generally starts with the definition of the elastic properties of unidirectional fiber composite materials. These can, of course, be determined experimentally. Also, when such data are not available, they can be estimated using a variety of analytical techniques. These

2.
latter are generally referred to as micromechanics analyses. For example, a set of relatively simple relations for predicting the moduli of unidirectional reinforced composites are presented in [1]. Alternate micromechanics approaches are described in [2] to [4]. A review of these methods is presented in [5]. With these properties available, it is assumed that the individual laminae are homogeneous and anisotropic. A laminate analysis is carried out in a straightforward fashion following methods originally developed for such materials as plywood, and more recently extended to the more general cases associated with fiber composite laminates (e.g., [6] to [8]).

However, contemporary fiber composite materials generally consist of elastic brittle fibers such as glass, boron or graphite in relatively soft matrix materials such as epoxy or aluminum. For these matrix materials it is reasonable to anticipate that at a certain loading state the matrix will begin to exhibit inelastic effects. This results in non-linear relations between structural loads and deformations. These inelastic effects can, of course, be expected to have a significant effect upon failure of the laminate. It is quite clear that adequate definition of these failure conditions are essential to achieve structural designs of high reliability.

In the present study, a non-linear laminate analysis has been developed which can provide realistic assessments of the stresses and strains in the various laminae and of the inelastic stiffnesses of the laminate at any stress level. This information can be used for assessment of such effects as structural stability or structural stress distributions. The stress distributions in the laminae and the laminates can also be utilized for the development of more realistic failure criteria.

Inelastic matrix behavior can be classified broadly as either time dependent or time independent. Time dependent behavior is called viscoelastic if linear and creep if non-linear. Polymeric matrices such as epoxy do exhibit such behavior. In
the case of metallic matrix materials, such as aluminum, time-dependent effects are generally negligible unless elevated temperature conditions are considered. The present study is concerned with time independent non-linear matrix behavior which is of significance for both polymeric and metallic matrices. Throughout this paper the expression "inelastic" is used to describe this time independent mechanical behavior. The method of approach to these problems is similar to that of the elastic analysis. Thus, it is necessary to determine, first, the inelastic properties of the unidirectional fiber composite materials. This can be done experimentally or by micromechanics methods. Given this information, a method to determine stresses and strains in an inelastic laminate is then devised. The problem is complicated by the fact that the inelastic stress-strain relations are non-linear.

A limited number of pertinent investigations can be found in the literature. Hill [4] considered, in approximate fashion, a limited aspect of inelastic behavior of a uniaxially reinforced material: the case of stress in fiber direction combined with isotropic transverse stress. Petit and Waddoups [6] devised an incremental method for laminate analysis in which it was assumed that in single laminae there is no interaction of stress components in different directions as far as lamina deformation is concerned. This assumption is restrictive, and also their incremental laminate analysis scheme is unduly complicated. Adams [7] used a finite element technique for numerical analysis of unidirectional materials in the form of periodic fiber arrays under conditions of plane strain. Huang [8] gave an approximate analysis for transverse inelastic behavior for a unidirectional material in plane strain, but it is difficult to assess the validity of the approximations introduced.

A detailed analysis of the inelastic laminate problem has been given by Foye and Baker [9]. Using finite element methods, they computed the inelastic effective properties of unidirectional rectangular and square arrays of elastic fibers 4.
in inelastic matrix. These properties were then used in an inelastic laminate analysis. The analysis is based on incremental plasticity theory and is, unfortunately, very complicated and requires a great deal of computer time. The results obtained are, however, of great importance for comparison with results predicted by more simplified theories, such as the one which will be given in the present work.

The body of this report is divided into four major sections. In the first, consideration is given to the behavior of unidirectional fiber composite materials. This requires: a definition of the appropriate form of the inelastic stress-strain relations; some consideration of the relationship between composite properties and constituent properties; and a definition of the appropriate form of the interaction between various stress components. The basic objective in this phase of the report was to define appropriate constitutive relations for the individual lamina which can be used in the non-linear laminate analysis. Further, there is a desire to gain some insight into the influence of the particular constituent properties upon the lamina stress-strain relations. In this phase of the study, it is found useful to characterize the unidirectional material with the aid of Ramberg-Osgood stress-strain relations.

In the next section of the report, the analysis of the inelastic behavior of laminates is described. Here, a procedure for incorporating the non-linear constitutive relations into an analysis which defines the state of stress in the individual laminae under an arbitrary set of external loads, is defined. Analyses are developed for the case of symmetric laminates subjected to membrane loading. The equations which are developed uniquely define the desired laminate internal average stress distribution under a given set of membrane loads. Governing equations, however, are non-linear and require numerical solution procedures. An efficient algorithm has been defined which enables computer solution to be achieved for arbitrary
laminates at minimal cost. The solution is obtained by application of the Newton-Raphson method.

In the final section, the computerized analysis which has been developed is applied to series of problems. The first group presents comparisons with various analytical results from the more complex analyses of Ref. [6] and [9]. The second group of numerical results presents comparisons between theoretical results from the present model and available experimental data. The third group of results provides several parametric studies to gain insight into those factors which contribute significantly to the non-linear behavior of fiber composite laminates. Also, computations have been made to provide a preliminary assessment of combined load effects including comparisons with limited experimental data.

Details of the various analytical developments, as well as descriptions of the computer program, are presented in appendices to the report.

The principal result of the present program is a computer program which provides a simple engineering tool which can be used for the parametric study of the influence of material properties upon laminate performance. This laminate analysis capability can be used by the structural designer to define design allowable stresses and to aid in the selection of fiber composite materials for structural applications. A comparison of the present results with the limited amount of available experimental data shows good agreement. There are, however certain cases in which the agreement is not good, particularly as the laminate loading approaches failure. The results of the present analytical method agree well with the results for those problems for which more exact and more complex analytical results exist.
2. NON-LINEAR STRESS-STRAIN RELATIONS OF UNIAXIAL FIBER REINFORCED MATERIALS

2.1 General Form of Stress-Strain Relations

An effective stress-strain relation of a composite material is defined as a relation between average stress $\bar{\sigma}_{ij}$ and average strain $\bar{\varepsilon}_{ij}$. Here and in the following latin indices range over 1, 2, and 3. If the composite is elastic the general effective stress-strain relation takes the form

$$\bar{\sigma}_{ij} = C_{ijkl} \bar{\varepsilon}_{kl} \quad (2.1.1)$$

where $C_{ijkl}$ are the effective elastic moduli which are material constants and are thus independent of stress or strain. Thus, (2.1.1) is a linear relation between average stress and strain.

If the composite is subject to symmetries the form of (2.1.1) simplifies. For a uniaxial FRM the most important cases of symmetry are transverse isotropy, around fiber direction, and square array (square symmetry). In these cases the stress-strain relations (2.1.1) for transverse isotropy assume the form:

$$\bar{\sigma}_{11} = C_{11} \bar{\varepsilon}_{11} + C_{12} \bar{\varepsilon}_{22} + C_{13} \bar{\varepsilon}_{33}$$
$$\bar{\sigma}_{22} = C_{12} \bar{\varepsilon}_{11} + C_{22} \bar{\varepsilon}_{22} + C_{23} \bar{\varepsilon}_{33}$$
$$\bar{\sigma}_{33} = C_{12} \bar{\varepsilon}_{11} + C_{23} \bar{\varepsilon}_{22} + C_{22} \bar{\varepsilon}_{33}$$
$$\bar{\sigma}_{12} = 2C_{44} \bar{\varepsilon}_{12} \quad (2.1.2)$$
$$\bar{\sigma}_{23} = 2C_{55} \bar{\varepsilon}_{23}$$
$$\bar{\sigma}_{31} = 2C_{44} \bar{\varepsilon}_{31}$$

and

$$C_{55} = (C_{22} - C_{23})/2 \quad (2.1.3)$$

In (2.1.2-3) 1 indicates direction and 2, 3 perpendicular directions transverse to 1.

In the event of inelastic matrix and elastic fibers, the situation is much more complicated since the stress-strain
relation are nonlinearity and history dependent. In no case is stress proportional to strain so that superposition of effects is not valid, and in order to determine current strain it is not sufficient to know current stress but it is necessary to know precisely the variation of stress which preceded its current value. Thus, for a material in a known state of combined shear and uniaxial tension, the state of strain is different if: (a) tension is first applied and then the shear, (b) shear is first applied and then the tension- (c) tension and shear are applied simultaneously. For this reason stress-strain relations must be presented in incremental form. That is, strain increment is related to stress and stress increment. This complicates matters enormously. However, it is known that in the case of proportional loading, that is, all stresses at a point grow simultaneously in a fixed ratio to one another, incremental theory can be integrated into the much simpler total or deformation theory for which current strain is completely determined by current stress.

Deformation theories have a wider range of validity than proportional loading. Comparison of numerous detailed solutions carried out both incrementally and by much simpler deformation theory show surprising agreement in many cases, and Budiansky [10] has shown that deformation theory can also be valid for "neighboring" loading paths.

In the present work, we are concerned with composites which are subjected to some external load. If it is supposed that the various external load components grow proportionally, this does not necessarily imply that the components of stress at a typical internal point also grow proportionally. It is, however, felt that the manner of growth of these internal stress components cannot deviate severely from proportional loading if external loading is proportional. Consequently, deformation type stress-strain relations are assumed for the matrix.

This assumption results in considerable simplification. It will be seen that it yields results which are extremely 8.
close to the ones obtained in [9] on the basis of the much more complicated incremental theory.

It is shown in Appendix A that for elastic fibers and an inelastic matrix described by deformation type theory, the effective stress-strain relations for a transversely isotropic or square symmetric FRM are:

\[
\begin{align*}
\bar{\varepsilon}_{11} &= S_{11} \overline{\sigma}_{11} + S_{12} \overline{\sigma}_{22} + S_{13} \overline{\sigma}_{33} \\
\bar{\varepsilon}_{22} &= S_{12} \overline{\sigma}_{11} + S_{22} \overline{\sigma}_{22} + S_{23} \overline{\sigma}_{33} \\
\bar{\varepsilon}_{33} &= S_{12} \overline{\sigma}_{11} + S_{23} \overline{\sigma}_{22} + S_{22} \overline{\sigma}_{33} \\
\bar{\varepsilon}_{12} &= 2S_{44} \overline{\sigma}_{12} \\
\bar{\varepsilon}_{23} &= 2S_{55} \overline{\sigma}_{23} \\
\bar{\varepsilon}_{13} &= 2S_{44} \overline{\sigma}_{13}
\end{align*}
\]  

(2.1.4)

and

\[S_{55} = (S_{22} - S_{23})/2\]  

(2.1.5)

The coefficients \(S_{11}, S_{12}, \) etc. are the effective inelastic compliances of the material and are functions of the average stresses, or rather of certain invariants of the average stress tensor.

We are here primarily concerned with thin uniaxially reinforced laminae which are in a state of plane stress. Let \(x_1\) denote fiber direction, \(x_2\) direction transverse to fibers in lamina plane, and \(x_3\) direction perpendicular to lamina, Figure 1. Then the plane stress condition is expressed by:

\[\overline{\sigma}_{13} = \overline{\sigma}_{23} = \overline{\sigma}_{33} = 0\]  

(2.1.6)

Equus. (2.1.4) then assume the form:

\[
\begin{align*}
\bar{\varepsilon}_{11} &= S_{11} \overline{\sigma}_{11} + S_{12} \overline{\sigma}_{22} \\
\bar{\varepsilon}_{22} &= S_{12} \overline{\sigma}_{11} + S_{22} \overline{\sigma}_{22} \\
\bar{\varepsilon}_{12} &= 2S_{44} \overline{\sigma}_{12}
\end{align*}
\]  

(2.1.7)

9.
Note that $\bar{e}_{33}$ does not vanish. It is however of no interest for present purposes.

The inelastic compliances in (2.1.7) are functions of the stresses $\bar{\sigma}_{11'}, \bar{\sigma}_{22'}, \bar{\sigma}_{12'}$.

It is convenient to split the strains in (2.1.7) into elastic strains $\bar{\varepsilon}_{\alpha \beta}$, and inelastic strains $\bar{\varepsilon}''_{\alpha \beta}$. Thus:

$$\bar{\varepsilon}_{\alpha \beta} = \bar{\varepsilon}'_{\alpha \beta} + \bar{\varepsilon}''_{\alpha \beta}$$

(2.1.8)

where here and in the following greek indices range over 1, 2. The elastic strains are recovered after unloading of the composite and are related to the stresses by elastic stress-strain relations. Thus:

$$\bar{\varepsilon}'_{11} = S_{11} \bar{\sigma}_{11} + S_{12} \bar{\sigma}_{22}$$
$$\bar{\varepsilon}'_{22} = S_{12} \bar{\sigma}_{11} + S_{22} \bar{\sigma}_{22}$$
$$\bar{\varepsilon}'_{12} = 2S_{44} \bar{\sigma}_{12}$$

(2.1.9)

where

$$S_{11} = \frac{1}{E_A}$$
$$S_{12} = -\frac{v_A}{E_A}$$
$$S_{22} = \frac{1}{E_T}$$
$$S_{44} = \frac{1}{4G_A}$$

(2.1.10)

Here $E_A$ is the effective Young's modulus in fiber direction, $v_A$ - the associated effective Poisson's ration, $E_T$ - the effective Young's modulus transverse to fibers and $G_A$ - axial effective shear modulus, related to 1-2 shear.

The inelastic, permanent, strains then have the form:

$$\bar{\varepsilon}''_{11} = S''_{11} \bar{\sigma}_{11} + S''_{12} \bar{\sigma}_{22}$$
$$\bar{\varepsilon}''_{22} = S''_{12} \bar{\sigma}_{11} + S''_{22} \bar{\sigma}_{22}$$
$$\bar{\varepsilon}''_{12} = 2S''_{44} \bar{\sigma}_{12}$$

(2.1.11)

where

$$S''_{2\beta} = S''_{2\beta} (\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{12})$$

(2.1.12)
In order to further simplify the stress-strain relations (2.1.11-.12), some specific features of FRM will be taken into account. In such materials, the fibers are by an order of magnitude stiffer than the matrix (for the case of boron and/or graphite fiber in an epoxy matrix the ratio of fiber to matrix Young's modulus can be in excess of 100). The stiffness ratio becomes larger in the inelastic range since the matrix loses stiffness (i.e., flows) while the fibers retain their stiffness. It is, therefore, clear that the stress $\sigma_{11}$ in fiber direction is practically carried by the fibers alone, with insignificant matrix contribution.

On the other hand, the transverse stress $\sigma_{22}$ and the shear stress $\sigma_{12}$ are primarily carried by the matrix with little fiber contribution.

It follows that inelastic behavior of the FRM is produced primarily by $\sigma_{22}$ and $\sigma_{12}$ while inelastic behavior for $\sigma_{11}$ load can be neglected.

The foregoing comments are summarized into two basic assumptions:

(a) the inelastic strains $\epsilon_{22}^{\prime\prime}$ and $\epsilon_{12}^{\prime\prime}$ are not functions of $\sigma_{11}$

(b) the inelastic strain $\epsilon_{11}^{\prime\prime}$ always vanishes.

On the basis of these assumptions, the stress-strain relations (2.1.11-.12) simplify to:

$$\epsilon_{11}^{\prime\prime} = 0$$

$$\epsilon_{22}^{\prime\prime} = S_{22}^{\prime\prime} (\sigma_{22}, \sigma_{12}) \, \sigma_{22}$$

$$\epsilon_{12}^{\prime\prime} = 2S_{44}^{\prime\prime} (\sigma_{22}, \sigma_{12}) \, \sigma_{12}$$

(2.1.13)
2.2 Plane Stress-Strain Relations in Ramberg-Osgood Form

A convenient representation of non-linear one dimensional stress-strain relations has been given by Ramberg and Osgood [11]. For uniaxial stress, for example:

\[ \varepsilon = \frac{\sigma}{E_1} \left[ 1 + k \left( \frac{\sigma}{\sigma_y} \right)^m \right] \]  

(2.2.1)

where \( E_1 \) represents the elastic Young's modulus, and \( k, \sigma', \) and \( m \) are three parameters to be obtained by curve fitting. The parameter \( \sigma' \) is sometimes called nominal yield stress. Equation (2.2.1) represents a family of curves with initial slope \( E_1 \), and monotonically decreasing slope with increasing \( \sigma \). The curves flatten out with increasing \( m \) (Fig. 2). Without loss of generality (2.2.1) can be written in the form:

\[ \varepsilon = \frac{\sigma}{E_1} \left[ 1 + \left( \frac{\sigma}{\sigma_y} \right)^m \right] \]  

(2.2.2)

which will be used from now on. Similarly, a stress-strain curve in shear can be represented in the form:

\[ \gamma = \frac{\tau}{G_1} \left[ 1 + \left( \frac{\tau}{\tau_y} \right)^n \right] \]  

(2.2.3)

where \( G_1 \) is the elastic shear modulus.

It should be emphasized that (2.2.2-.3) are valid only for one dimensional cases. The question of the generalization to general states of stress and strain has no unique answer. One common used form is isotropic \( J_2 \) deformation theory [12].

Next, we consider the case of effective or macroscopic stress-strain relations for the special case of a uniaxially reinforced material in which the matrix is non-linear, with stress-strain relations in Ramberg-Osgood form.

Consider, for example, the case of uniaxial average stress \( \bar{\sigma}_{22} \) in direction transverse to fibers, all other average stresses vanish. It then follows from (2.1.7) that:

\[ \bar{\varepsilon}_{22} = S_{22} \left( \bar{\sigma}_{22} \right) \bar{\sigma}_{22} \]  

(2.2.4)
Similarly, if the only nonvanishing average stress is $\bar{\sigma}_{12}$, the shear stress-strain relation of the composite is:

$$\bar{\varepsilon}_{12} = 2S_{44} (\bar{\sigma}_{12}) \bar{\sigma}_{12}$$  \hspace{1cm} (2.2.5)

Evidently the inelastic effective compliances $S_{22}$ and $S_{44}$ are functions of the parameters of the inelastic Ramberg-Osgood stress-strain relations of the matrix, of the elastic properties of the fibers and of the internal geometry of the composite. Actual prediction is a very difficult problem. Such problems will be considered in limited fashion in the next paragraph.

Just as matrix stress-strain relations are represented in Ramberg-Osgood form, the same type of curve fitting can also be applied for the effective stress-strain relation of the composite. Thus (2.2.2-.3) are written in the form:

$$\varepsilon_{22} = \frac{\bar{\sigma}_{22}}{\bar{E}_T} [1 + (\frac{\bar{\sigma}_{22}}{\bar{\sigma}_y})^{M-1}] \hspace{1cm} (a)$$

$$\varepsilon_{12} = \frac{\bar{\sigma}_{12}}{2\bar{G}_A} [1 + (\frac{\bar{\sigma}_{12}}{\bar{\tau}_y})^{N-1}] \hspace{1cm} (2.2.6)$$

Where $\bar{E}_T$ is the effective transverse elastic Young's modulus $\bar{G}_A$ - effective axial elastic shear modulus and $\bar{\sigma}_y$, $\bar{\tau}_y$, $M$ and $N$ are curve fitting parameters which are in general quite different from the corresponding Ramberg-Osgood matrix parameters.

A question of fundamental and of practical importance is the form of the stress-strain relations for the case of plane stress, taking into account interaction among the various stress components. It should be noted in this respect that (2.2.6) are special stress-strain relations when $\bar{\sigma}_{22}$ or $\bar{\sigma}_{12}$ act only by themselves.

It is recalled that equations (2.1.13) represent the inelastic parts of the strains for plane stress-strain relations for FRM with stiff fibers. It is shown in Appendix B
that the Ramberg-Osgood form of such plane stress-strain relations is as follows:

\[
\begin{align*}
\varepsilon_{11}'' & = 0 \\
\varepsilon_{22}'' & = \frac{\bar{o}_{22}}{E_T} \left[ \left( \frac{\bar{o}_{22}}{\sigma_Y} \right)^2 + \left( \frac{\bar{o}_{12}}{\tau_Y} \right)^2 \right] \frac{M-1}{2} \\
\varepsilon_{12}'' & = \frac{\bar{o}_{12}}{2G_A} \left[ \left( \frac{\bar{o}_{22}}{\sigma_Y} \right)^2 + \left( \frac{\bar{o}_{12}}{\tau_Y} \right)^2 \right] \frac{N-1}{2}
\end{align*}
\]

(2.2.7)

The parameters \(E_T, G_A, \sigma_Y, \tau_Y, M, N\) in (2.2.7) are those of the one dimensional stress-strain relations (2.2.6) which may be regarded as experimentally (or perhaps theoretically) known.

The inelastic parts of the strains are given by (2.1.9-.10), and the total strains are then given by adding equations (2.2.7) and (2.1.9).

Equations (2.2.7) have been compared with computed numerical results given in [9]. Reasonable agreement was obtained. Comparisons for the interaction cases of transverse stress, \(\bar{o}_{22}'\), versus transverse strain, \(\varepsilon_{22}'\), in the presence of axial shear stress, \(\bar{o}_{12}'\), and axial shear stress, \(\bar{o}_{12}'\), versus axial shear strain, \(\gamma_{12}'\), are shown in Figures 3 and 4 respectively (in both cases \(\bar{o}_{22}'/\bar{o}_{12}' = 8/3\)). It is seen that the agreement is fair for transverse stress-strain relations (Fig. 3) and very good for the shear stress-strain relations (Fig. 4).

Figures 3 and 4 also show the stress-strain relations obtained from Eqs. (2.2.7) for one dimensional transverse tension \(\bar{o}_{22}'\), and axial shear, \(\bar{o}_{12}'\), respectively.
2.3 Axial Shear Stress-Strain Relation

This paragraph is concerned with the problem of prediction of a one dimensional effective axial shear stress-strain relation of a uniaxial FRM in terms of matrix and fiber properties and the internal geometry of the composite.

The main reason for concentrating on the axial shear problem is that the inelastic effect is predominant in axial shear for which significant nonlinearity of the stress-strain response is obtained (e.g., Figure 4). The effect in fiber direction is practically non-existent as has indeed been assumed above, and is relatively small in transverse stress which is shown by the small curvature of the stress-strain relation in this case (e.g., Figure 3).

On the basis of all this, it can indeed be assumed as first approximation that the nonlinearity of the uniaxial FRM is limited to axial shear alone.

Consider a uniaxially reinforced lamina which is subjected to pure axial shear, Figure 5, on its surface. The boundary conditions are:

\[
\begin{align*}
  x_3 &= \pm \frac{t}{2} & \sigma_{31} &= \sigma_{32} = \sigma_{33} = 0 \\
  x_2 &= \pm b & \sigma_{12} &= \tau_o & \sigma_{22} &= \sigma_{23} = 0 \\
  x_1 &= \pm a & \sigma_{12} &= \tau_o & \sigma_{11} &= \sigma_{13} = 0
\end{align*}
\]

(2.3.1)

It may be shown that under such load the only nonvanishing average stress in the composite is:

\[
\bar{\sigma}_{12} = \tau_o
\]

(2.3.2)

It would seem at first that, given the complexity of the internal geometry of the composite, the state of stress at any interior matrix or fiber point is generally three dimensional. Surprisingly enough, however, this is not so and the only nonvanishing stress components in the interior of the composites
are the shear stresses \( \sigma_{12} \) and \( \sigma_{13} \), which are moreover functions of \( x_2 \) and \( x_3 \) only. Thus, the interior state of stress is:

\[
\begin{align*}
\sigma_{12} &= \sigma_{12} (x_2, x_3) \\
\sigma_{13} &= \sigma_{13} (x_2, x_3) \\
\sigma_{11} &= \sigma_{22} = \sigma_{33} = \sigma_{23} = 0
\end{align*}
\]

(2.3.3)

The validity of equations (2.3.3) for the case of an elastic composite has been proved in [5]. Their validity for the present much more general inelastic case will be shown elsewhere.

The effective stress-strain relation of the composite in axial shear is defined by:

\[
\ddot{\varepsilon}_{12} = \frac{\varepsilon_{12}}{2G_A^s} = \frac{\varepsilon_{12}}{2G_A^s}
\]

(2.3.4)

\[
G_A^s = G_A^s(\sigma_{12}) = G_A^s(\tau_0)
\]

where \( G_A^s \) is the effective secant shear modulus of the material. The nonlinearity of the stress-strain relation is expressed by the fact that \( G_A^s \) function of the applied stress.

It is seen that in order to determine \( G_A^s \) it is necessary to compute the average shear strain \( \ddot{\varepsilon}_{12} \) for given applied shear stress. This is a formidable problem even with the simplification (2.3.3) and we shall content ourselves with a brief outline of its formulation. To simplify matters, the fibers shall be assumed to be ideally rigid relative to the matrix. This is a very accurate assumption for the case of Boron and Graphite Fibers. There is no difficulty to extend the formulation to the case of non-rigid elastic fibers.

In view of (2.3.3) the problem is two dimensional and need only be considered in a typical \( x_2, x_3 \) section. In the matrix domain:

16.
\[
\frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \quad (2.3.5a)
\]
\[
\varepsilon_{12} = \frac{\sigma_{12}}{2G} [1 + \left(\frac{\tau}{\tau_y}\right)^n - 1] \quad (2.3.5b)
\]
\[
\varepsilon_{13} = \frac{\sigma_{13}}{2G} [1 + \left(\frac{\tau}{\tau_y}\right)^n - 1] \quad (2.3.5c)
\]
\[
\tau = \sqrt{\sigma_{12}^2 + \sigma_{13}^2} \quad (2.3.6)
\]
\[
\varepsilon_{12} = \frac{1}{2} \frac{\partial u_1}{\partial x_2} \quad (2.3.7a)
\]
\[
\varepsilon_{13} = \frac{1}{2} \frac{\partial u_1}{\partial x_3} \quad (2.3.7b)
\]
\[
u_1 = u_1 (x_2, x_3) \quad (2.3.8)
\]

and, \( u_1 = 0 \) at fiber/matrix interface.

Here equ. (2.3.5) is the only surviving equilibrium equation, (2.3.6) are Ramberg-Osgood stress-strain relations for isotropic \( J_2 \) theory (2.3.7) are usual strain-displacement relations in which \( u_2 \) and \( u_3 \) do not enter since it may be shown that they are not functions of \( x_1 \) and (2.3.8) expresses the ideal rigidity of the fibers.

Equis. (2.3.5–8) must be solved subject to boundary condition (2.3.1). If this is done the strain \( \varepsilon_{12} \) is known everywhere and can be averaged to obtain \( G_A^s \) from (2.3.4).

The problem is exceedingly difficult because of the non-linearity introduced by the stress-strain relations (2.3.6). There is very little hope to solve it analytically for any kind of fiber geometry. It should therefore be handled by numerical methods for fiber arrangements and fiber shapes of engineering interest.

Another way to approach an analytically intractable problem such as the present one is by variational techniques.
In this fashion, approximations or bounds for quantities of interest are obtained by methods which are much simpler than bonafide solution of the problem. Such variational methods have been extensively used for determination of effective elastic moduli of FRM (e.g., [5]).

In the course of the present work, it has been found that variational methods can also be used for inelastic problems such as the present one to obtain bounds on effective secant moduli. The main ingredients of the method are:

(a) Construction of an extremum principle in terms of an energy integral such that the true energy is the minimum of the integral.

(b) Expression of the true energy in terms of effective secant modulus.

(c) Establishment of admissible fields to obtain a value of the energy integral which is larger than the true energy, thus obtaining a bound for \( G^s_A \).

The work involves complicated developments and derivations which are given in Appendix C. Here only the end result for a lower bound on \( G^s_A \) will be given for a special geometry of FRM which is known as composite cylinder assemblage. This geometry has been described in detail in [1, 5] and consists of an assemblage of composite cylinders of variable sizes which are joined together so as to fill the whole volume of the composite. In order to fill the whole volume, composite cylinders vary from finite to infinitesimal size. This geometry has been used to advantage for elastic FRM to obtain simple expressions for effective elastic moduli which are well verified by experiment [1, 5]. In the present case only a lower bound on \( G^s_A \) has been obtained for the case in which the exponent \( n \) in matrix stress-strain relations is \( n=3 \).

It has been found that with this exponent and proper choice of \( \tau_y \), epoxy shear stress-strain relations can be well described. The result for the lower bound is:

18.
\[
G_A^S > G_A^S(-) = \frac{G \frac{1+c}{1-c}}{1 + \left( \frac{\tau_y}{\tau'_y \tau_0} \right)^2 \frac{3+13c+c^2+c^3}{3(1+c)^3}}
\]  

(2.3.9)

where

\( c \) - volume fraction of fibers
\( G \) - elastic (initial) matrix shear modulus
\( \tau'_y \) - Ramberg-Osgood matrix stress parameter, and
\( \tau_0 \) - applied shear stress.

It follows from (2.3.4) that:

\[
\bar{\varepsilon}_{12} \leq \frac{\tau_0}{2G_A^S(-)}
\]

(2.3.10)

In other words, with the lower bound on \( G_A^S \) an upper bound on \( \bar{\varepsilon}_{12} \) variation with \( \tau_0 \) is obtained.

If (2.3.10) is explicitly written in terms of (2.3.9) it assumes the form:

\[
\bar{\varepsilon}_{12} \leq \frac{\tau_0}{2G \frac{1+c}{1-c}} \left[ 1 + \left( \frac{\tau_y}{\tau'_y \tau_0} \right)^2 \frac{3+13c+c^2+c^3}{3(1+c)^3} \right]
\]

(2.3.11)

Recalling that for the composite cylinder assemblage with rigid fibers the axial elastic shear modulus \( G_A \) is given in [1, 5] as:

\[
G_A = G \frac{1+c}{1-c}
\]

(2.3.12)
and comparing (2.3.11) with (2.2.6) with choice of exponent \( N=3 \) (which is the same as matrix exponent), it is seen that:

\[
\frac{\tau_y^2}{\gamma_y} \geq \frac{3(1+c)^3}{3+13c+c^2+c^3} \tag{2.3.13}
\]

The prediction of (2.3.11) has been compared with numerical results obtained in [9]. Figure 6 shows the variation of the right side of (2.3.11) in comparison with the results obtained in [9] for a fiber volume fraction, \( c=0.5 \). Since results of [9] were for boron fibers in epoxy matrix, the rigid fiber approximation is accurately valid. It is seen that the results are reasonably close. It should be noted that the geometry of [9] is a rectangular fiber array which is quite different from the composite cylinder assemblage geometry.

The results defined by (2.3.12) and (2.3.13) used in equation (2.2.6) yield the result plotted in non-dimensional form in Fig. 7. The shear strains are normalized with respect to the matrix elastic strain, \( \gamma_{ye} \), at the yield stress, \( \tau_y \):

\[
\gamma_{ye} = \frac{\tau_y}{G} \tag{2.3.14}
\]

It is natural to also consider the establishment of an upper bound on \( G_A^S \). Unfortunately, however, this is a matter of formidable difficulty for the reason that inversion of (2.3.6) to express stresses in terms of strains leads very complicated expressions. Further discussion of this difficulty is given in Appendix C.
3. **ANALYSIS OF NON-LINEAR LAMINATES**

3.1 **Formulation**

The general problem to be investigated in the present chapter is as follows: given the inelastic stress-strain relations of uniaxially reinforced laminae determined theoretically or experimentally, and a laminate composed of such laminae and loaded on its edges by uniformly distributed loads in the plate of the laminate:

(a) What are the stresses in the various laminae?

(b) What is the macroscopic strain response of the laminate to the loads?

This problem has been extensively investigated for elastic laminates, and the results obtained will serve as important guidelines for the present much more complicated problem. It is therefore very helpful to first briefly review the theory of elastic laminates.

Let the laminate be referred to a fixed system of coordinates $x_1, x_2, x_3$ as shown in Figure 8. This will henceforth be referred to as the **laminate coordinate system**.

Any lamina, $k$th say, in the laminate will be referred to its material system of coordinates $x_1^{(k)}, x_2^{(k)}, x_3$ where $x_1^{(k)}$ is in fiber direction, $x_2^{(k)}$ perpendicular to fiber direction and $x_3$ is the same as the laminate $x_3$, Figure 8. The reinforcement angle $\theta_k$ is defined by:

\[ \theta = \theta (x_1, x_1^{(k)}) = \theta (x_2, x_2^{(k)}) \]  

(3.1.1)

Let it be assumed that the laminae are in states of plane stress. It will be later explained under what conditions this is true. Then the stress-strain relations of a single lamina referred to its material coordinate system are written in the forms:

\[ \varepsilon_{\alpha \beta}^{(k)} = S_{\alpha \beta}^{(k)} \sigma^{(k)} \]  

(a)

(3.1.2)

\[ \varepsilon_{\gamma \delta}^{(k)} = S_{\gamma \delta}^{(k)} \sigma^{(k)}. \]  

(b)
where (3.1.2a) is in tensor notation with range of subscripts 1, 2 and (3.1.2b) is in matrix notation. It should be noted that (3.1.2) represent the stress-strain relations (2.1.9 - .10), i.e.,

\[
\begin{align*}
\varepsilon_{11}^{(k)} &= \frac{\sigma_{11}^{(k)}}{E_A^{(k)}} - \frac{\nu A}{E_A^{(k)}} \sigma_{22}^{(k)} \\
\varepsilon_{22}^{(k)} &= -\frac{\nu A}{E_A^{(k)}} \sigma_{11}^{(k)} + \frac{\sigma_{22}^{(k)}}{E_T^{(k)}} \\
\varepsilon_{12}^{(k)} &= \frac{\sigma_{12}^{(k)}}{2G_A^{(k)}}
\end{align*}
\]

(3.1.3)

Let a laminate of rectangular form, Figure 8, be loaded by a uniform edge stress:

\[
\begin{align*}
\sigma_{11}(+a, x_2) &= \sigma_{11} \nonumber \\
\sigma_{12}(+a, x_2) &= \sigma_{12} \\
\sigma_{12}(x_1, +b) &= \sigma_{12} \nonumber \\
\sigma_{22}(x_1, +b) &= \sigma_{22}
\end{align*}
\]

(3.1.4)

The elasticity solution of the laminate must satisfy the following requirements:

(a) Equilibrium of stresses,
(b) Traction continuity at laminae interfaces,
(c) Boundary conditions (3.1.4), and
(d) Displacement continuity at laminae interfaces.

It is assumed that the stresses in any lamina are constant, but different in the different laminae. The condition (a) is satisfied within any lamina. Since the assumed lamina stresses are plane there are no traction components on laminae interfaces. Therefore (b) is satisfied.

The boundary conditions (3.1.4) cannot be strictly satisfied in each lamina but only in an average sense. To do this lamina stresses \(\sigma_{\alpha\beta}^{(k)}\) referred to lamina material coordinates...
are transformed to laminate axes. The stresses in the \( k \)th lamina referred to laminate axes are denoted \( (k)\sigma_{\alpha\beta} \). The transformation is given by:

\[
(k)\sigma_{11} = \sigma_{11}^{(k)} \cos^2 \theta_k + \sigma_{22}^{(k)} \sin^2 \theta_k - 2\sigma_{12}^{(k)} \cos \theta_k \sin \theta_k \\
(k)\sigma_{22} = \sigma_{22}^{(k)} \sin^2 \theta_k + \sigma_{11}^{(k)} \cos^2 \theta_k + 2\sigma_{12}^{(k)} \cos \theta_k \sin \theta_k \\
(k)\sigma_{12} = \sigma_{11}^{(k)} \sigma_{22}^{(k)} \sin \theta_k \cos \theta_k + \sigma_{12}^{(k)} (\cos^2 \theta_k - \sin^2 \theta_k)
\]

or in matrix notation:

\[
(k)\sigma = \varphi(k)\sigma(k)
\] (3.1.6)

Let the edges of the laminate be loaded by constant forces per unit length \( T_{11}, T_{22}, T_{12} \) and define the stresses (3.1.4) as edge averages over the laminate thickness \( h \):

\[
\sigma_{11}^0 = \frac{T_{11}}{h} \\
\sigma_{22}^0 = \frac{T_{22}}{h} \\
\sigma_{12}^0 = \frac{T_{12}}{h}
\] (3.1.7)

Equilibrium requires that:

\[
\sum_{k=1}^{K} \sigma_{11}^{(k)} = \sigma_{11}^0 \\
\sum_{k=1}^{K} \sigma_{22}^{(k)} = \sigma_{22}^0 \\
\sum_{k=1}^{K} \sigma_{12}^{(k)} = \sigma_{12}^0
\] (3.1.8)

where \( K \) is the number of laminae. Written in terms of stresses \( \sigma_{\alpha\beta}^{(k)} \) using (3.1.6), we have:

\[
\sum_{k=1}^{K} \varphi(k) \sigma_{\alpha\beta}^{(k)} = \sigma_{\alpha\beta}^0
\] (3.1.9)

where \( \sigma_{\alpha\beta}^0 \) denotes the stresses \( \sigma_{\alpha\beta}^0 \) at the edges.
Replacement of the boundary conditions (3.1.4) by (3.1.6) is an approximation of Saint Venant type. Thus, there must be expected edge perturbations (among them interlaminar shear) on the stresses predicted by laminate theory.

Equations (3.1.8) are three equations for the 3K stresses $\sigma_{a\beta}$, $\sigma_{a\beta}$, $\sigma_{a\beta}$ in the laminae. There are needed an additional $3(K-1)$ equations which are provided by displacement continuity at lamina interfaces, requirement (d).

Since the stresses in each laminae are by hypothesis uniform, so are the strains. Therefore, displacement continuity is ensured if the lamina strains in adjacent laminae, referred to laminae coordinate system are the same. Thus:

\[
\begin{align*}
(k)\varepsilon_{11} &= (k+1)\varepsilon_{11} & \forall k = 1, 2, \ldots, K \\
(k)\varepsilon_{22} &= (k+1)\varepsilon_{22} & \forall k = 1, 2, \ldots, K \\
(k)\varepsilon_{12} &= (k+1)\varepsilon_{12}
\end{align*}
\] (3.1.10)

Equations (3.1.10) are the additional required $3(K-1)$ equations. They will be written in terms of laminae stresses $\sigma_{a\beta}$ referred to laminae material axes. To do this it is noted that:

\[
\begin{align*}
(k)\varepsilon &= \vartheta(k)\varepsilon(k) \\
(k)\varepsilon_{11} &= \vartheta(k)\varepsilon_{11} \\
(k)\varepsilon_{22} &= \vartheta(k)\varepsilon_{22} \\
(k)\varepsilon_{12} &= \vartheta(k)\varepsilon_{12}
\end{align*}
\]

which is just a transformation of (3.1.6). From (3.1.2b):

\[
\begin{align*}
(k)\varepsilon &= \vartheta(k)\bar{\varepsilon}(k) \\
(k)\varepsilon_{11} &= \vartheta(k)\bar{\varepsilon}_{11} \\
(k)\varepsilon_{22} &= \vartheta(k)\bar{\varepsilon}_{22} \\
(k)\varepsilon_{12} &= \vartheta(k)\bar{\varepsilon}_{12}
\end{align*}
\] (3.1.11)

and inserting the last result in (3.1.10):

\[
\begin{align*}
\vartheta(k)\bar{\varepsilon}(k)\sigma(k) &= \vartheta(k+1)\bar{\varepsilon}(k+1)\sigma(k+1) & \forall k = 1, 2, \ldots, K
\end{align*}
\] (3.1.12)

Equations (3.1.9) and (3.1.12) are 3K linear equations for the 3K stresses in an elastic laminate, with K layers.

It should be carefully noted that the analysis given above is based on plane stress conditions in individual laminae. This
is a valid assumption if:

(a) The loads on the laminate are statically equivalent to in-plane forces (membrane forces) and produce neither bending nor twisting moments, and

(b) The laminate has a certain stacking sequence of laminae which defines a so called balanced or symmetric laminate.

This stacking sequence is an arrangement in which the laminate has a middle plane of geometrical and of material symmetry. The laminae are arranged in pairs with respect to the plane of symmetry. The laminae of such pair have equal thicknesses, same distances from middle plane, and are of the same material with same angles of reinforcement.

In a non-symmetric laminate application of membrane forces will in general produce bending and twisting of laminae and thus a plane state of stress will not be realized. The symmetric laminate is, however, sufficiently versatile to cover most cases of practical interest.

Let it now be assumed that the laminate is inelastic but still fulfills the conditions of symmetry and pure membrane loading. In this case the only equations which necessarily change in the preceding development are the stress-strain relations of the laminae, (3.1.2), which must be replaced by inelastic laminae stress-strain relations are given by (2.1.7) where the compliances are now functions of the stresses. These compliances now replace the elastic compliances in (3.1.2) which thus become non-linear.

It is convenient for later purposes to rewrite (3.1.2) in the inelastic case in different form. To do this the strains $\varepsilon^{(k)}_{\alpha\beta}$ are first split into elastic strains (2.1.9) and inelastic strains (2.1.11). Preceeding to (3.1.12) this equation assumes the form:

$$
\begin{align*}
\frac{\vartheta_{(k+1),1l(k+1)}}{S} \sigma_{(k+1)} - \frac{\vartheta_{(k),1l(k)}}{S} \sigma_{(k)} &= \\
- \frac{\vartheta_{(k+1),1l(k+1)}}{S} \sigma_{(k+1)} + \frac{\vartheta_{(k),1l(k)}}{S} \sigma_{(k)} \\
k = 1, 2 \ldots \ldots k
\end{align*}
$$

(3.1.13)
where

\[ S_{\text{el}}^{(k)} \] - elastic compliance matrix of kth layer

\[ S_{\text{inel}}^{(k)} \] - inelastic part of compliance matrix of kth layer

\[ S^{(k)} = S_{\text{el}}^{(k)} + S_{\text{inel}}^{(k)} \]  \hspace{1cm} (3.1.14)

Equations (3.1.13) are now written out in component form with notation (2.1.10), (2.1.12) for compliances:

\[ \sigma_{11}^{(k+1)} (S_{11}^{(k+1)} \cos^2 \theta_{k+1} + S_{12}^{(k+1)} \sin^2 \theta_{k+1}) + \sigma_{22}^{(k+1)} (S_{12}^{(k+1)} \cos \theta_{k+1} \sin \theta_{k+1}) = \sigma_{11}^{(k)} (S_{11}^{(k)} \cos^2 \theta_{k} + S_{12}^{(k)} \sin^2 \theta_{k}) - 4 \sigma_{12}^{(k)} S_{44}^{(k)} \cos \theta_{k+1} \sin \theta_{k+1} + \sigma_{22}^{(k)} (S_{12}^{(k)} \cos^2 \theta_{k} + S_{22}^{(k)} \sin^2 \theta_{k}) \]

\[ = - \sigma_{11}^{(k+1)} (S_{11}^{(k+1)} \cos^2 \theta_{k+1} + S_{12}^{(k+1)} \sin^2 \theta_{k+1}) + \sigma_{22}^{(k+1)} (S_{12}^{(k+1)} \cos \theta_{k+1} \sin \theta_{k+1}) - 4 \sigma_{12}^{(k+1)} S_{44}^{(k+1)} \cos \theta_{k+1} \sin \theta_{k+1} + \sigma_{22}^{(k)} (S_{12}^{(k)} \cos^2 \theta_{k} + S_{22}^{(k)} \sin^2 \theta_{k}) + \sigma_{12}^{(k)} (S_{44}^{(k)} \cos \theta_{k} \sin \theta_{k}) \]  \hspace{1cm} (3.1.15)
\[ \begin{align*}
\sigma^{(k+1)}_{11} (S^1_{11} \sin^2 \theta_{k+1} + S^1_{12} \cos^2 \theta_{k+1}) \\
+ \sigma^{(k+1)}_{22} (S^1_{22} \sin^2 \theta_{k+1} + S^1_{22} \cos^2 \theta_{k+1}) \\
+ 4 \sigma^{(k+1)}_{44} S^1_{44} \cos \theta_{k+1} \sin \theta_{k+1} \\
+ \sigma^{(k)}_{11} (S^1_{11} \sin^2 \theta_k + S^1_{12} \cos^2 \theta_k) \\
+ \sigma^{(k)}_{22} (S^1_{22} \sin^2 \theta_k + S^1_{22} \cos^2 \theta_k) \\
+ 4 \sigma^{(k)}_{44} S^1_{44} \cos \theta_k \sin \theta_k \\
&= \sigma^{(k+1)}_{11} (S^1_{11} \sin^2 \theta_{k+1} + S^1_{12} \cos^2 \theta_{k+1} - S^1_{11} \sin^2 \theta_k + S^1_{12} \cos^2 \theta_k) \\
+ \sigma^{(k+1)}_{22} (S^1_{22} \sin^2 \theta_{k+1} + S^1_{22} \cos^2 \theta_{k+1} - S^1_{22} \sin^2 \theta_k + S^1_{22} \cos^2 \theta_k) \\
+ 4 \sigma^{(k+1)}_{44} S^1_{44} \cos \theta_{k+1} \sin \theta_{k+1} + 4 \sigma^{(k)}_{44} S^1_{44} \cos \theta_k \sin \theta_k \\
+ \sigma^{(k)}_{22} (S^1_{12} \sin^2 \theta_k + S^1_{22} \cos^2 \theta_k - S^1_{12} \sin^2 \theta_k + S^1_{22} \cos^2 \theta_k) \\
&= (3.1.16)
\end{align*} \]

\[ \begin{align*}
\sigma^{(k+1)}_{11} (S^1_{11} - S^1_{22}) \sin \theta_{k+1} \cos \theta_{k+1} \\
+ \sigma^{(k+1)}_{22} (S^1_{22} - S^1_{11}) \sin \theta_{k+1} \cos \theta_{k+1} \\
+ 2 \sigma^{(k+1)}_{44} S^1_{44} \cos \theta_{k+1} \sin \theta_{k+1} \\
+ \sigma^{(k)}_{22} (S^1_{12} - S^1_{22}) \sin \theta_k \cos \theta_k \\
+ \sigma^{(k)}_{22} (S^1_{22} - S^1_{12}) \sin \theta_k \cos \theta_k + 2 \sigma^{(k)}_{44} S^1_{44} \cos \theta_k \sin \theta_k \\
+ \sigma^{(k+1)}_{22} (S^1_{12} - S^1_{22}) \sin \theta_{k+1} \cos \theta_{k+1} \\
&= \sigma^{(k+1)}_{22} (S^1_{12} - S^1_{22}) \sin \theta_{k+1} \cos \theta_{k+1} - S^1_{12} \sin \theta_k \cos \theta_k \\
+ 2 \sigma^{(k+1)}_{44} S^1_{44} \cos \theta_{k+1} \sin \theta_{k+1} \\
+ \sigma^{(k)}_{22} (S^1_{12} - S^1_{22}) \sin \theta_k \cos \theta_k \\
+ \sigma^{(k)}_{22} (S^1_{22} - S^1_{12}) \sin \theta_k \cos \theta_k \\
+ 2 \sigma^{(k+1)}_{44} S^1_{44} \cos \theta_{k+1} \sin \theta_{k+1} \\
&= (3.1.17)
\end{align*} \]

\[ k = 1, 2 \ldots k-1 \]
To these must be adjoined equations (3.1.9) which are written here in components:

\[ \sum_{k=1}^{k} (\sigma_{11}^{(k)} \cos^2 \theta_k + \sigma_{22}^{(k)} \sin^2 \theta_k - 2\sigma_{12}^{(k)} \cos \theta_k \sin \theta_k) t_k = \sigma_{11} \]  

(a)

\[ \sum_{k=1}^{k} (\sigma_{11}^{(k)} \sin \theta_k + \sigma_{22}^{(k)} \cos \theta_k + 2\sigma_{12}^{(k)} \cos \theta_k \sin \theta_k) t_k = \sigma_{22} \]  

(b)

\[ \sum_{k=1}^{k} (\sigma_{11}^{(k)} - \sigma_{22}^{(k)}) \cos \theta_k \sin \theta_k + \sigma_{12}^{(k)} (\cos^2 \theta_k - \sin^2 \theta_k) t_k = \sigma_{12} \]  

(c)

\[ (3.1.18) \]

We now consider special cases of interest. In the first case the inelastic laminae strains have the form (2.1.13). Then the right side of (3.1.15-.17) simplifies by setting:

\[ S_{11}^{(k)} = S_{11}^{(k+1)} = S_{12}^{(k)} = S_{12}^{(k+1)} = 0 \]

\[ S_{22}^{(k)} = S_{22}^{(k)} (\sigma_{22}^{(k)}, \sigma_{12}^{(k)}) \]

\[ S_{22}^{(k+1)} = S_{22}^{(k+1)} (\sigma_{22}^{(k+1)}, \sigma_{12}^{(k+1)}) \]

\[ S_{44}^{(k)} = S_{44}^{(k)} (\sigma_{22}^{(k)}, \sigma_{12}^{(k)}) \]

\[ S_{44}^{(k+1)} = S_{44}^{(k+1)} (\sigma_{22}^{(k+1)}, \sigma_{12}^{(k+1)}) \]  

(3.1.19)

Once the stresses in the laminae have been obtained the strains in the laminae, referred to laminate axes, are determined from (3.1.11). Since the strains in all laminae are the same when referred to the laminate coordinate system, these 28.
are also the average laminate strains and thus determine the inelastic response of the laminate.

In the simplest case the lamina material is assumed to be inelastic in shear only. In that event we have in addition to (3.1.19):

\[ S_{22}^{(k)} = S_{22}^{(k+1)} = 0 \] (3.1.20)

and for Ramberg-Osgood presentation of inelastic part of shear compliance:

\[
2S_{44}^{11(k)} = \frac{1}{2G_A} \left( \frac{\sigma_{12}^{(k)}}{\tau_f^{(k)}} \right) N_{k-1} \]

\[
2S_{44}^{11(k+1)} = \frac{1}{2G_A} \left( \frac{\sigma_{12}^{(k+1)}}{\tau_f^{(k+1)}} \right) N_{k+1-1} \] (3.1.21)

In Ramberg-Osgood representation (2.2.1) the inelastic parts of the compliances assume forms such as:

\[
S_{22}^{11(k)} = \frac{1}{E_T^{(k)}} \left[ \left( \frac{\sigma_{22}^{(k)}}{\sigma_y^{(k)}} \right)^2 + \left( \frac{\sigma_{12}^{(k)}}{\tau_f^{(k)}} \right)^2 \right]^{1/2}(N_{k-1}) \]

\[
2S_{44}^{11(k)} = \frac{1}{2G_A^{(k)}} \left[ \left( \frac{\sigma_{22}^{(k)}}{\sigma_y^{(k)}} \right)^2 + \left( \frac{\sigma_{12}^{(k)}}{\tau_f^{(k)}} \right)^2 \right]^{1/2}(N_{k-1}) \] (3.1.22)

\[
S_{22}^{11(k+1)} = \frac{1}{E_T^{(k+1)}} \left[ \left( \frac{\sigma_{22}^{(k+1)}}{\sigma_y^{(k+1)}} \right)^2 + \left( \frac{\sigma_{12}^{(k+1)}}{\tau_f^{(k+1)}} \right)^2 \right]^{1/2}(N_{k+1-1}) \]

\[
2S_{44}^{11(k+1)} = \frac{1}{2G_A^{(k+1)}} \left[ \left( \frac{\sigma_{22}^{(k+1)}}{\sigma_y^{(k+1)}} \right)^2 + \left( \frac{\sigma_{12}^{(k+1)}}{\tau_f^{(k+1)}} \right)^2 \right]^{1/2}(N_{k-1}) \]
3.2 Method of Solution

The equations which define the laminae stresses are (3.1.9) and (3.1.13) in condensed form, or equivalently, (3.1.15-3.1.17), (3.1.18) in full form. To explain the solution method it is simpler to write in terms of the condensed form.

Define the matrices:

\[ L^{1}(k+1) = \theta^{(k+1)} \sigma_{l}^{(k+1)} \]
\[ L^{11}(k+1) = \theta^{(k+1)} \sigma_{ll}^{(k+1)} \]
\[ L^{1}(k) = \theta^{(k)} \sigma_{l}^{(k)} \]
\[ L^{11}(k) = \theta^{(k)} \sigma_{ll}^{(k)} \]

Then equs. (3.1.13) assume the form:

\[ L^{1}(k+1) \sigma_{l}^{(k+1)} - L^{1}(k) \sigma_{l}^{(k)} = -L^{11}(k+1) \sigma_{ll}^{(k+1)} + L^{11}(k) \sigma_{ll}^{(k)} \] (3.2.2)

to which are adjoined equs. (3.1.9) which are here rewritten:

\[ \Sigma_{k=1}^{K} \theta^{(k)} \sigma_{l}^{(k)} = \sigma^{o} \] (3.2.3)

The equations may be solved numerically by an iteration method which proceeds as follows: Consider equs. (3.2.2-3) with the right side of (3.2.2) zero. This defines a set of stresses \( \sigma_{o}^{(k)} \) given by:

\[ L^{1}(k+1) \sigma_{o}^{(k+1)} - L^{1}(k) \sigma_{o}^{(k)} = 0 \] (a)

\[ k = 1,2, \ldots, k=1 \] (3.2.4a)

\[ \Sigma_{k=1}^{K} \theta^{(k)} \sigma_{o}^{(k)} = \sigma^{o} \] (b)

Since (3.2.4a) contains only elastic compliances \( S_{l}^{(k)} \) it is seen that the equations are linear and define the stresses in an elastic laminate. Now insert the stresses \( \sigma_{o}^{(k)} \) into the right side of (3.2.2) and define the stresses \( \sigma_{l}^{(k)} \) by:

30.
Equs. (3.2.5) defines (hopefully) a new approximation \( \sigma_1(k) \) which is the solution of a set of linear equations. The stresses in square brackets in the right side of (3.2.5) are to emphasize the stress dependence of the non-linear parts of the compliances.

The procedure just initiated can be repeated indefinitely. In general:

\[
L^{1(k+1)} \sigma_1(k+1) - L^{1(k)} \sigma_1(k) = - L^{11(k+1)} \left[ \frac{\sigma_0(k+1)}{\ell} \right] \sigma_1(k) \\
+ L^{11(k)} \left[ \frac{\sigma_0(k)}{\ell} \right] \sigma_1(k)
\]

\[k=1 \sum_{k=1}^{K} \theta(k) \sigma_1(k) = \sigma^0 \quad (3.2.6)\]

This iteration procedure is quite easy to carry out with aid of a computer. It replaces the solution of a set of non-linear equations by solution of a sequence of linear equations, provided of course, that convergence is obtained.

It should be noted that the first iteration step does not necessarily have to start with equs. (3.2.4a), i.e., with zero right side of (3.2.2). Any stresses \( \sigma_1(k) \) which fulfill (3.2.4b) can be used to start the iteration with (3.2.5) and continuing with the general iteration relation (3.2.6).

It is desired to obtain a laminate solution for only one load system \( \sigma^0 \) then it would seem most logical to start with (3.2.4). But suppose there is a sequence of loadings \( \Delta \sigma \), \( 2\Delta \sigma \), \( 3\Delta \sigma \), ... \( n\Delta \sigma \). Suppose that a solution for \( (n-1) \Delta \sigma \) has been obtained and that a solution for \( n\Delta \sigma \) is desired. One possibility is to multiply all stresses due to the load \( (n-1) \Delta \sigma \) by the factor \( n/(n-1) \). The stresses thus obtained certainly
also satisfy (3.2.6b) because of the linearity of these equations. They will generally be reasonable starting values \( \sigma^{(k)} \) for the iteration.

This method of iteration to obtain a solution was found to work well for many sample problems; however, there were cases in which the solution did not converge. Attempts to modify the recurrence relations to overcome this problem met with only partial success. Thus, an alternate procedure for solution was defined. The solution was obtained by application of the Newton-Raphson method.

The set of \( 3K \) nonlinear equations represented by equs. (3.2.2-.3) may be presented in the form:

\[
F_n (\sigma_{ij}^k) = 0 \quad n = 1, 2 \ldots 3K
\]  

(3.2.7)

The function \( F_i \) is expanded in a Taylor series about an arbitrary set of initial stresses which may be taken as the solutions of the elasticity problem. Considering only two terms of the series, it is found that

\[
F_i = F_i^0 + \frac{\partial F_i^0}{\partial \sigma_{mn}^k} \Delta \sigma_{mn}^k = 0
\]  

(3.2.8)

or

\[
\sigma_{ij}^k = \sigma_{ij}^0 - \left[ \frac{\partial F_m^0}{\partial \sigma_{ij}^k} \right]^{-1} \frac{\partial F_m^0}{\partial \sigma_{ij}^k}
\]  

(3.2.9)

where \( \sigma_{ij}^k \) is the corrected solution obtained from the assumed solution \( \sigma_{ij}^0 \). Using \( \sigma_{ij}^k \) as the initial guess, the process is repeated until the result is obtained within a desired accuracy. A recurrence form of equation (3.2.9) to obtain the stresses at \( t+1 \) cycle from \( t \) cycle can be constructed as follows:

32.
After the stresses $\sigma_{ij}^k$ are obtained for all layers of the laminate, strains for any layer $k$ in terms of laminae axes can be computed using equs. (3.1.3). Strains in terms of the laminate axes can be obtained using the strain transformation law.

This analysis has been developed into an efficient computer program. A description of the program including a listing, is presented in Appendix E.
3.3 Numerical Results

The computer program which has been developed under the present study has been utilized in the analysis of a variety of different composite laminates. The initial studies using the computerized analysis were directed at presenting a comparison between the results of the present analysis and those of previous analyses, notably that of Ref. 9. (The present results were also compared to available experimental data, primarily those of Ref. 6 which had also been used for comparison with the analytical results in Ref. 9.) The objective of this phase of the numerical study was to determine whether the present results, which can be obtained with minimal computer usage, compare well with those of the more exact and complex analytical results in Ref. 9. The results of this comparison are highly encouraging, as will be shown below, and support the utilization of the present analysis as an efficient design tool.

In the second phase of the design numerical studies, consideration was given to examining the sensitivity of laminate results to individual properties of the layers. These parametric studies are presented for several classes of typical laminates.

A series of laminates of boron/epoxy composites for which experimental data had been obtained in Ref. 6 were examined analytically in Ref. 9. In Figures 9 to 15, results of the present analytical method are added to the comparison of experimental results of [6] and analytical results of [9]. For example, in Fig. 9, the experimental stress-strain curve for a 0-90 boron/epoxy laminate is compared to the analytical results obtained in Ref. 9 and in the present analysis. Both analytical results coincide; both show slightly less inelastic strain than the experiment. The solid point on the curve indicates the stress level at which fiber fracture is computed to occur in one of the layers of the laminate.

The shear stress-strain curve used in the present analysis was the best fit Ramberg-Osgood curve having an exponent n=3.
The values of modulus and yield stress obtained from the least squares fit are shown on the figure. A similar result is shown for the unidirectional tension +45° laminate in Fig. 10. Here it is seen that the two analytical curves are similar, although the agreement is not as close as in Fig. 9. Experimental data reflect a substantially higher degree of inelasticity than either analytical result. The present analysis shows a higher degree of inelastic strain at the higher stress level than that of Ref. 9. However, the reverse is true in the comparison of the two analytical results shown in Fig. 11 for a ±30° laminate. The present results were obtained with a linear stress-strain curve in the transverse direction within each of the layers. The computations were made in this fashion because the transverse stress-strain curve of Ref. 9 does not show a significant degree of inelasticity.

Figure 12 presents results for the case of a quasi-isotropic laminate (0/+45/90) of boron/epoxy. Both the present result and that of Ref. 9 show a relatively insignificant amount of inelasticity. Again, the experimental data show a greater inelastic effect. Here the predicted failure strain level is in good agreement with the experimental failure strain level; however, there is a significant difference in the failure stress level. A similar result is presented in Fig. 13 for the quasi-isotropic laminate formed from the 0/+60° configuration.

Computations performed for the present study for laminates having fibers in several directions, including the loading direction, for a simple unidirectional load have shown a relatively small amount of inelastic strain. Another example of this is presented in Fig. 14 for a 0/+45° laminate. Here, however, the agreement of all the analytical methods and the experimental method is very good.

The final comparison taken from Ref. 9 is presented in Fig. 15 for a laminate having fibers in three different directions and a tensile load applied at some intermediate angle. The present analysis agrees reasonably well with the results
Ref. 9. The discrepancy between the failure load predicted on the basis of fiber failure and the experimentally observed failure stress is quite substantial. It is possible that failure in laminate of this type could result from shearing or transverse stresses within the individual layers, and thus, not be a result of tension in the fiber failure. This mode of failure has not been treated in the present computer program. The mode of failure observed experimentally is not known to the authors.

The experimentally measured response of a multidirectional laminate to an applied shear stress has been reported in Ref. 13. Comparison of the experimental result with the theory of Ref. 9 was presented in Ref. 14. Computations for this case, made using the present analysis and the prior analytical result (Ref. 14), are compared to the experimental result in Fig. 16. Again, correlation between the two analytical results is good, agreement between analytical and theoretical results is reasonably good with the experimental observation showing higher inelastic strains and lower tangent shear moduli at the very high stress levels.

The conclusion of these comparisons with analytical and experimental data seem to justify the adoption of the present computer program as a useful engineering tool for the design and analysis of composite laminates. However, it appears that further study of the failure region is required.

Parametric study of the influence of various laminate geometric and mechanical properties has also been explored. Fig. 17 shows the results obtained for a 0/+45° laminate indicating that the inelastic response in the transverse direction can become significant at higher stress levels. Failure due to fiber fracture under a transverse stress applied to the laminate occurs at strain levels larger than those plotted in Fig. 17. In the quasi-isotropic laminate having four fiber directions, (0/+45/90) the degree of inelasticity in the longitudinal and transverse directions is of course the same and is
in both cases very small. It is to be anticipated, on an intuitive basis, that the maximum degree of inelastic response would be observed for a stress applied midway between two of the fiber directions on this quasi-isotropic laminate. The stress-strain curve for this latter case is also shown in Fig. 18. Although the inelastic strains for this case are not significant there is a large difference in the predicted failure stress levels based on stress in the fiber direction for the two cases. It is worthwhile to emphasize that the quasi-isotropic laminate need not be isotropic in its strength characteristics.

Because of the directional strength characteristics interesting effects may be expected for combined stress cases. Some results of the exploration of this question are presented in Fig. 19 where the four direction quasi-isotropic laminate is subjected to combined stress state with respect to a 22-1/2° axis of symmetry. This laminate shows high strength under both the unidirectional load and shear load by itself. The combined stress case for equal values of applied shear stress and axial stress results in fiber failure, and therefore, laminate failure, at a substantially lower stress. The stress-strain curve prior to failure is not affected significantly by the presence of combined stress. The quasi-isotropic laminate having fibers in three directions (0/+60) is examined in Fig. 20. The sensitivity of this laminate to the Ramberg-Osgood parameters for the individual ply had little effect upon the stress-strain result. Indeed as an extreme example of this variation all laminates stiffnesses except the axial stiffness were equal to zero. Enforcement of the Kirchhoff-Love plate assumptions for this case results in the so-called netting analysis. The response for this netting case, which is linear, is shown by the dashed curve in Fig. 20. Even with this extreme assumption, matrix inelasticity does not introduce a significant amount of inelastic strain. Experimental data for comparison with this result are not easily available, however Ref. 17 does present a stress-strain curve for this case which shows a transverse failure stress for the
quasi-isotropic $0/+60^\circ$ laminate which is about 60% of the failure stress in the axial direction. Also, the inelastic strain at failure is approximately 30% larger than the elastic strain associated with the failure stress level. The netting analysis result presented here suggests that in order to obtain such a strain, one might have to consider that the axial stiffness, either in tension, compression or both; or that other effects not considered in the conventional laminate analysis, such as interlaminar or transverse shear deformations, might contribute significantly to the overall laminate deformation.

The influence of the characteristic stress levels for transverse stress and axial shear of the unidirectional layer of a boron/epoxy material is examined in Fig. 21. The measure of this effect is taken to be the influence upon the stress-strain curve for the unidirectional tension of $+30^\circ$ laminate. The strong sensitivity to the characteristic axial shear stress $\tau_y$ and the relative insensitivity to the transverse characteristic stress $\sigma_y$ for the R-O representations is illustrated in the figure. A similar comparison made for a boron/aluminum laminate of the same geometry subjected to uniaxial applied stress is shown in Fig. 22. Similar sensitivities are observed for this case. Boron/aluminum laminate response under transverse applied stress with the same values of the Ramberg-Osgood parameters is shown in Fig. 23. Here the fiber failure criterion did not come into play and thus the computations were extended to rather large strains in matrix. It is clear, that for this case, the failure criterion based on other stress-strain components is required. The examination of the computer print-out permits one to terminate the stress-strain curves at some stress level prior to fiber fracture depending upon the choice of the failure criterion. This can be done rather readily. The choice of the failure criterion is discussed in Appendix D.

The lamina properties for boron/aluminum are used to
analyze a 0°/±30° laminate under combined loading. These results are shown in Fig. 24. Axial stress-strain curves are presented for varying ratios of axial shear stress to axial tensile stress.
4. CONCLUDING REMARKS

Current approaches to the definition of design allowable stress for advanced fiber composite laminates are based upon the utilization of extremely conservative criteria. These limit the laminate to stress levels below which no significant damage of any kind occurs. The utilization of overly conservative design criteria can negate much of the potential for effective design utilizing advanced composite materials. The heterogeneous nature of these materials is such that a variety of possible damage modes exist. Thus, matrix cracking or yielding, fiber fracture, debonding, and other inelastic effects can all occur in local regions at relatively low average stress levels. These nonuniform and nonlinear effects greatly complicate the problem of establishing reliable design allowables. In the present program, the problem of nonlinear laminate behavior resulting from nonlinearities in the behavior of the matrix material was studied. The objective of the program was to develop an understanding of the inelastic behavior of composite laminates and to develop a computer program which will be used as an engineering tool in the design of fiber composite laminated structures.

The method of approach utilized herein was to adopt a Ramberg-Osgood representation of the nonlinear stress-strain behavior and to utilize deformation theory as an adequate representation of the material nonlinearities. The problem was viewed on two levels. First, the relationship between the constituent properties and the stress-strain response of a unidirectional fiber composite material was studied. For this problem, the primary attention herein was directed toward the axial shear behavior, in as much as experimental data had indicated that it is this type of load which results in the most significant nonlinearities in material behavior. For this case, an expression was established relating the composite average-stress/average-strain curve to the fiber moduli and the matrix nonlinear stress-strain curve. This expression, which was developed as a lower bound, was found to give good agreement with the more exact results obtained by 40.
applying incremental plasticity theory and using a numerical finite element analysis to the assessment of the material behavior (Ref. 9).

The second level of approach treats the interrelationship between the properties of the unidirectional layers and those of the laminate. For this case, one may consider that the starting point is a nonlinear stress-strain curve for transverse stress, and for axial shear stress, alone, and a linear stress-strain relation for stress in the fiber direction. The nonlinear lamina stress-strain curves can be modeled by proper selection of the Ramberg-Osgood parameters.

In the present study, unlike other formulations an interaction expression was formulated to account for simultaneous application of axial shear and transverse stress. A laminate having an arbitrary number of oriented layers, and subjected to a general state of membrane stress, was treated. The results of this analysis were programmed into an efficient computer routine for numerical evaluation of arbitrary laminates. Results obtained show good agreement with those of previous complex numerical methods utilizing incremental plasticity theory.

Certain limitations connected with this program should also be discussed. First, deformation type stress-strain relations have been used; hence, it is implicit in this result that the stress and strain values obtained for any given set of loads are functions only of those loads and not of the loading history. On the other hand, if points are computed for intermediate values of loads, following different load paths, then different intermediate conditions will be obtained. Thus, the question is raised as to what is the accuracy of the results obtained for paths which do not yield proportional loading. It is known that for local proportional loading, the deformation theory result is the same as that for the incremental theory. In the laminate, local proportional loading does not exist, in general, even when the external loading is proportional. However, the assumption is made that the deformation theory will yield an approximation which is satisfactory to generate a
rational engineering tool. This can only be assessed by comparison with an exact analysis, or since this does not exist for the case of arbitrary loading paths, perhaps by comparison with experimental data.

Comparisons of the present results with experimental data tend to show moderately good agreement. There are, however, cases in which experimental results show a higher degree of inelastic strain than predicted by the present analysis. These experimental data are quite limited and may be insufficient for drawing conclusions in this regard.

The question of failure criteria incorporated into the present analysis required further consideration. The present analysis obtains more accurate representations of the stress components in the individual layers than have been obtained from elastic analyses. Hence, the use of these stress components in any failure criteria should represent an improvement in failure prediction.

In addition to a description of the methods of analysis, and of the numerical comparisons which have been carried out, the present report also presents a description of the computer program for study of nonlinear behavior of laminates in sufficient detail to permit the utilization of this program by others.
APPENDIX A

SYMmetry SIMplification oF NON-lINEAR STRESS-STRAIN RELATIONS

The most general inelastic stress-strain relations of the deformation type are of the form

\[ \varepsilon_{ij} = S_{ijkl} \sigma_{kl} \]  \hspace{1cm} (1)

where \( S_{ijkl} \) are functions of the stresses. Let it be assumed that the material is transversely isotropic with \( x_1 \) axis of symmetry. Any rotation about \( x_1 \) changes \( \varepsilon_{ij} \) and \( \sigma_{ij} \) into \( \varepsilon'_{ij} \) and \( \sigma'_{ij} \). Then the condition of transverse isotropy demands that

\[ \varepsilon'_{ij} = S_{ijkl} \sigma'_{kl} \]  \hspace{1cm} (2)

where \( S_{ijkl} \) in (1) and (2) are the same. To fulfill this last requirement it is necessary that \( S_{ijkl} \) be functions of stresses only through stress expressions which are invariant for rotations about the \( x_1 \) axis. There are five such invariants and they are given by, \[15\]

\[ I_1 = \sigma_{11} \]
\[ I_2 = \sigma_{22} + \sigma_{33} \]
\[ I_3 = \sigma_{12}^2 + \sigma_{13}^2 \]
\[ I_4 = 1/2(\sigma_{22} - \sigma_{33})^2 + 2\sigma_{23}^2 \]
\[ I_5 = 1/2(\sigma_{22} - \sigma_{33})(\sigma_{12}^2 - \sigma_{13}^2) + 2\sigma_{12}\sigma_{13}\sigma_{23} \]  \hspace{1cm} (3)
Thus

\[ S_{ijkl} = S_{ijkl} (I_1, I_2, I_3, I_4, I_5) \]  

(4)

It follows that for rotations around the \( x_1 \) axis of symmetry the \( S_{ijkl} \) behave as constants. Consequently, the symmetry reduction of (1) to transverse isotropy is just as in elasticity.

The reduction may be performed in following fashion: For rotation of angle \( \theta \) about the \( x_1 \) axis, the stress tensor \( \sigma_{ij} \) transforms into \( \sigma'_{ij} \) in the following fashion

\[
\begin{align*}
\sigma'_{11} &= \sigma_{11} \\
\sigma'_{22} &= \frac{1}{2} (\sigma_{22} + \sigma_{33}) + \frac{1}{2} (\sigma_{22} - \sigma_{33}) \cos 2\theta + \sigma_{23} \sin 2\theta \\
\sigma'_{33} &= \frac{1}{2} (\sigma_{22} + \sigma_{33}) - \frac{1}{2} (\sigma_{22} - \sigma_{33}) \cos 2\theta - \sigma_{23} \sin 2\theta \\
\sigma'_{23} &= \frac{1}{2} (\sigma_{33} - \sigma_{22}) \sin 2\theta + \sigma_{23} \cos 2\theta \\
\sigma'_{12} &= \sigma_{12} \cos \theta + \sigma_{13} \sin \theta \\
\sigma'_{13} &= -\sigma_{12} \sin \theta + \sigma_{13} \cos \theta
\end{align*}
\]  

(5)

The same transformation relations obviously also hold for strains. If the transformed stresses and strains are introduced into (2) then coefficients of \( \cos 2\theta, \sin 2\theta, \cos \theta \) and \( \sin \theta \) and remaining terms independent of \( \theta \) must be equal. These equalities result in relations among the various components which reduce the stress-strain law to the form (2.1.4-5) from Chapter 2 of this report. (Average stresses and strains appear in the latter but this obviously makes no differences in the derivation.)
The purpose of the present appendix is to arrive at equs. (2.2.7). For convenience in writing, overbars on stresses and strains will be omitted.

The present development is guided by isotropic $J_2$ theory for deformation type plastic stress-strain relations. The basic assumption of this theory in the isotropic case is that the plastic strains have the form

$$\varepsilon_{ij} = f(J_2) s_{ij}$$  \hspace{1cm} (1)

where $s_{ij}$ is the stress deviator and

$$J_2 = \frac{1}{2} s_{ij} s_{ij}$$  \hspace{1cm} (2)

is its second invariant.

It is instructive to work out the form of (1) for Ramberg-Osgood type stress-strain relations. Suppose that in pure shear the stress-strain relation is

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G} \left[ 1 + \frac{\sigma_{12}}{\tau_y} \right]^{n-1}$$ \hspace{1cm} (3)

Now in pure shear it follows from (2) that

$$J_2 = \sigma_{12}^2$$

Therefore (3) can be written in the form

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G} \left[ 1 + \left( \frac{\sigma_{12}}{\tau_y} \right)^{n-1} \right]$$ \hspace{1cm} (4)

which is in the form (1). Consequently, in the general case of three dimensional stress and strain

$$\varepsilon_{ij} = \frac{s_{ij}}{2G} \left[ 1 + \left( \frac{\sqrt{J_2}}{\tau_y} \right)^{n-1} \right]$$ \hspace{1cm} (5)
It should be emphasized that there is nothing fundamental about (1). It is an assumption which states that the plastic strains can be represented by the stress deviator components multiplied by a function of a quadratic expression in the stresses which is \( J_2 \). The choice of \( J_2 \) for a quadratic expression is not arbitrary but may be arrived at by isotropy arguments.

In an anisotropic material it may be assumed by generalization that plastic strains are given by

\[
\varepsilon_{ij}^p = s_{ij}^p f(L)
\]

(6)

Where \( L \) is some general quadratic function of the stresses. This assumption will form the basis of the present development.

Consider the stress-strain relations (2.1.13). It is assumed that \( s_{22}^p \) and \( s_{44}^p \) functions of the most general quadratic form in \( \sigma_{22} \) and \( \sigma_{12} \).

Thus

\[
s_{22}^p = s_{22}^p (A\sigma_{22}^2 + B\sigma_{12} + C\sigma_{12}^2)
\]

\[
s_{44}^p = s_{44}^p (A\sigma_{22}^2 + B\sigma_{12} + C\sigma_{12}^2)
\]

(7)

It should be noted that the material reacts in same fashion to positive or negative shear stress, therefore also in same fashion to some \( \sigma_{22} \) together with positive or negative shear stress. However, the middle term in the quadratic changes sign with shear stress. Therefore, this term should be omitted.

Now rewrite (7) in form

\[
s_{22}^p = \frac{1}{E} f_{22} \left( \alpha^2 \sigma_{22}^2 + \beta^2 \sigma_{12}^2 \right)
\]

\[
s_{44}^p = \frac{1}{2G} f_{44} \left( \alpha^2 \sigma_{22}^2 + \beta^2 \sigma_{12}^2 \right)
\]

(8)
where $f_{22}$ and $f_{44}$ are nondimensional functions and $\alpha$ and $\beta$ have dimensions of reciprocal of stress. If $\bar{\sigma}_{12} = 0$ the first of (8) assumes the form

$$S_{22}'' = \frac{1}{E_T} f_{22} (\alpha^2 \bar{\sigma}_{22}^2)$$

(9)

For one dimensional $\bar{\sigma}_{22}'$, from the Ramberg-Osgood stress-strain relation (2.2.6a)

$$S_{22}'' = \frac{1}{E_T} \frac{\bar{\sigma}_{22}}{(\sigma_y)^{M-1}}$$

which can be written as

$$S_{22}'' = \frac{1}{E_T} \left[ (\bar{\sigma}_{22})^2 \right] \frac{M-1}{2}$$

(10)

It follows from (8) and (10) that

$$\alpha^2 = \frac{1}{\sigma_y}$$

(11)

and the function of $f_{22}$ is determined as $(M-1)/2$ power.

In similar fashion, when $\bar{\sigma}_{22} = 0$, the second of (8) assumes the form

$$S_{44}'' = \frac{1}{2G_T} f_{44} (\beta^2 \bar{\sigma}_{12}^2)$$

(12)

From the Ramberg-Osgood relation (2.2.6b) for one dimensional $\bar{\sigma}_{12}$

$$S_{44}'' = \frac{1}{2G_T} \frac{\bar{\sigma}_{12}}{(\tau_y)^{N-1}}$$

which can be written as

$$S_{44}'' = \frac{1}{2G_T} \left[ (\bar{\sigma}_{12})^2 \right] \frac{N-1}{2}$$

(13)

It follows from (12) and (13) that

$$\beta^2 = \frac{1}{\tau_y^2}$$

(14)

47.
and the function \( f_{44} \) is determined as \((N-1)/2\) power.

Consequently (8) now assumes the form

\[
\begin{align*}
\sigma_{y}'' &= \frac{1}{E_1} \left[ \frac{\bar{\sigma}_{22}^2}{\bar{\sigma}_y} + \frac{\bar{\sigma}_{12}^2}{\bar{r}_y} \right] \frac{N-1}{2} \\
\sigma_{y}'' &= \frac{1}{2G_T} \left[ \frac{\bar{\sigma}_{22}^2}{\bar{\sigma}_y} + \frac{\bar{\sigma}_{12}^2}{\bar{r}_y} \right] \frac{N-1}{2}
\end{align*}
\]

(15)

Then (2.2.7) follows from (15) and (2.1.13).
APPENDIX C

1. EXTREMUM PRINCIPLES OF DEFORMATION THEORY OF PLASTICITY

i. Principle of Minimum Potential Energy

Let

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]  \hspace{1cm} (1.1)

where \( C_{ijkl} \) are functions of the strains. The strain energy density is defined by the path dependent integral

\[ w^e = \int_{\varepsilon=0}^{\varepsilon} \sigma_{ij}(\varepsilon)d\varepsilon_{ij} \]  \hspace{1cm} (1.2)

where \( \varepsilon \) is a concise notation for \( \varepsilon_{ij} \). The strain energy \( U^e \) of a body of volume \( V \) is defined by

\[ U^e = \int_V w^e dV \]  \hspace{1cm} (1.3)

Let the surface of the body be subjected to the boundary conditions

\[ u_i(S) = u^o_i \text{ on } S_u \]  \hspace{1cm} (1.4)

\[ T_i(S) = T^o_i \text{ on } S_T \]

and let the body forces vanish. The potential energy \( U_p \) is defined by

\[ U_p = \int_V w^e dV - \int_S T^o_i n_i \]  \hspace{1cm} (1.5)

Define an admissible displacement field \( \tilde{u}_i(x) \) by

\[ \tilde{u}_i = u^o_i \text{ on } S_u \]

\[ \tilde{u}_i(x) \text{ continuous everywhere} \]  \hspace{1cm} (1.6)

Associated with \( \tilde{u}_i \) are the strains \( \tilde{\varepsilon}_{ij} \) derived from it by the usual relations.
Define $\tilde{W}^e$ by

$$\tilde{W}^e = \int \delta_{ij} \, d\tilde{\varepsilon}_{ij}$$

where

$$\delta_{ij} = C_{ijkl} (\tilde{\varepsilon}) \tilde{\varepsilon}_{kl}$$

Define

$$\tilde{U}_p = \int \tilde{W}^e \, dv - \int_{S_T} T^o \tilde{u}_i \, ds$$

The principle of minimum potential energy for the present case then states that

$$\tilde{U}_p < \tilde{U}_p, (1.10)$$

equality taking place if and only if

$$\tilde{u}_i = u_i$$

In the event that displacements are prescribed over the entire surface, the surface integral in (1.9) vanishes. Then the principle reduces to that of minimum strain energy

$$\tilde{U}_e < U^e$$

ii. Principle of Minimum Complementary Energy

Let

$$\varepsilon_{ij} = S_{ijkl} (\sigma) \sigma_{kl}$$

where $S_{ijkl}$ are stress dependent compliances

50.
Define the complementary energy density $W^\circ$ by the path dependent integral
\[ W^\circ = \int_{\sigma=0} e_{ij} \delta \sigma_{ij} \] (1.13)

Let the surface of the body be subjected to the boundary conditions (1.4) and let the body forces vanish. The complementary energy $U_C$ is defined by
\[ U_C = \int_V W^\circ dV - \int_S T_i u^\circ_i dS \] (1.14)

Define an admissible stress field $\tilde{\sigma}_{ij}$ by the following requirements
\[ \tilde{\sigma}_{ij}, j = 0 \]
\[ \tilde{T}_i = \tilde{\sigma}_{ij} n_j \text{ continuous everywhere} \] (1.15)
\[ T_i(S) = T^\circ_i \text{ on } S_T \]

Define the complementary energy functional $U_C$ by
\[ \tilde{U}_C = \int_V \tilde{W} dV - \int_S \tilde{T}_i u^\circ_i dS \] (1.16)

where $\tilde{W}^\circ = \int_{\tilde{\sigma}} \tilde{e}_{ij} \tilde{\sigma}_{ij} d\tilde{\sigma}_{ij} \] (1.17)

$\tilde{e}_{ij} = S_{ijkl} (\tilde{\sigma}) \tilde{\sigma}_{kl}$

Then the principle of minimum complementary energy states that
\[ U_C \geq U^\circ \] (1.18)
equality occurring if and only if
\[ \tilde{\sigma}_{ij} = \sigma_{ij} \]

If tractions are prescribed over the entire surface, $S_u = 0$, the principle reduces to
\[ \tilde{U}^\circ \geq U^\circ \] (1.19)
For proof of these principles see e.g. [16]. An interesting application to obtain approximate solutions has been given in [17].

iii. Specialization of the Principles to Axial Shear with Ramberg-Osgood Stress-Strain Relations

In the case of axial shear of a uniaxially fiber reinforced material the only surviving stresses are

\[ \sigma_{12} = \tau_2 \quad \sigma_{13} = \tau_3 \]  

(1.20)

where 1 indicates fiber direction. Denote the associated shear strains by

\[ \varepsilon_{12} = \varepsilon_2 \quad \varepsilon_{13} = \varepsilon_3 \]  

(1.21)

Then the generalized Ramberg-Osgood stress-strain relations, Appendix B, (5) assume in the present case the form

\[ \varepsilon_2 = \frac{\tau_2}{2G} \left[ 1 + \left( \frac{\tau_2}{\tau_y} \right)^n \right] \]

\[ \varepsilon_3 = \frac{\tau_3}{2G} \left[ 1 + \left( \frac{\tau_3}{\tau_y} \right)^n \right] \]

(1.22)

\[ \tau = \sqrt{\tau_2^2 + \tau_3^2} \quad \sqrt{J_2} \]

In the present case

\[ \sigma_{ij} \varepsilon_{ij} = 2(\tau_2 \varepsilon_2^2 + \tau_3 \varepsilon_3^2) \]  

(1.23)

52.
Inserting (1.22) into (1.23) and using the relation
\[ \tau d\tau = \tau_2 d\tau + \tau_3 d\tau \]
it is easily shown that
\[ \sigma_{ij} d\varepsilon_{ij} = \frac{\tau}{G} \left[ 1 + n \left( \frac{\tau}{\gamma} \right)^{n-1} \right] d\tau \] (1.24)

To compute \( W^e \) as defined by (1.2) it is necessary to integrate (1.24) from zero to some state of strain \( \varepsilon_2, \varepsilon_3 \). But it should be noted that (1.24) is expressed in terms of the variable \( \tau \) only. Now \( \tau \) can be expressed in terms of strains in following fashion. Define
\[ \varepsilon = \left( \varepsilon_2^2 + \varepsilon_3^2 \right)^{1/2} \] (1.25)
It follows at once from (1.22) that
\[ \varepsilon = \frac{\tau}{2G} \left[ 1 + \left( \frac{\tau}{\gamma} \right)^{n-1} \right] \] (1.26)

This relation defines \( \tau \) as a function of \( \varepsilon \). Consequently, \( W^e \) assumes the form
\[ W^e = \frac{1}{G} \int_0^\tau \tau [1 + n \left( \frac{\tau}{\gamma} \right)^{n-1}] d\tau \]
which is easily integrated to yield
\[ W^e = \frac{\tau^2}{2G} \left[ 1 + \frac{2n}{n+1} \left( \frac{\tau}{\gamma} \right)^{n-1} \right] \] (1.27)
\[ \tau = \tau (\varepsilon) \]

According to (1.3) the strain energy \( U^e \) is then given by the volume integral of (1.27). Note however that it is very difficult to express \( U^e \) in terms of strains since this requires the solution of (1.26) for \( \tau \) in terms of \( \varepsilon \). In general it is not possible to do this analytically. This places a severe
limitation on the use of the principle of minimum potential energy or of minimum strain-energy with Ramberg-Osgood stress-strain relations.

Next we consider the principle of minimum complementary energy for axial shear. Since there are only shear stresses \( \tau_2, \tau_3 \) and shear strains \( \varepsilon_2, \varepsilon_3 \) the integrand in \( W^\sigma \), (1.13), is given by

\[
\varepsilon_{ij} \sigma_{ij} = 2(\varepsilon_2 d\tau_2 + \varepsilon_3 d\tau_3)
\]

(1.28)

It follows from (1.22-23) that (1.28) is given by

\[
\varepsilon_{ij} \sigma_{ij} = \frac{\tau}{G} [1 + (\frac{\tau}{\tau_y})^{n-1}] d\tau
\]

Integration of this expression from 0 to \( \tau \) yields

\[
W^\sigma = \frac{\tau^2}{2G} \left[ 1 + \frac{2}{n+1} \left( \frac{\tau}{\tau_y} \right)^{n-1} \right]
\]

(1.29)

Expression (1.29) now enters as the integral into the volume integral of \( U_C \), (1.14).

We now examine the meaning of an admissible stress field \( \tilde{\tau}_2, \tilde{\tau}_3 \) in the present case. The only surviving equilibrium equation is

\[
\frac{\partial \tilde{\tau}_2}{\partial x_2} + \frac{\partial \tilde{\tau}_3}{\partial x_3} = 0
\]

(1.30)

The traction components are

\[
\tilde{T}_1 = \tilde{\tau}_2 n_2 + \tilde{\tau}_3 n_3
\]

\[
\tilde{T}_2 = \tilde{\tau}_2 n_1
\]

\[
\tilde{T}_3 = \tilde{\tau}_3 n_1
\]

(1.31)

We shall be concerned with cylindrical boundaries in fiber reinforced materials whose generator is in \( x_1 \) direction. On such a surface \( n_1 = 0 \). Therefore the only surviving traction component on such a surface is

\[
\tilde{T}_1 = \tilde{\tau}_n = \tilde{\tau}_2 n_2 + \tilde{\tau}_3 n_3
\]

(1.32)
Consequently an admissible stress system $\tilde{\tau}_3$, $\tilde{\tau}_3$ must satisfy (1.30) and the value $\tau^o_n$ of $\tilde{\tau}_n$ wherever prescribed on the boundary.

The complementary energy functional (1.16) assumes the form

$$
\tilde{U}_C = \int_V \tilde{W}^o dV - \int_S \tilde{\tau}_1 u^o_1 dS \quad (a)
$$

$$
\tilde{W}^o = \frac{\tilde{\tau}^2}{2G} \left[ 1 + \frac{2}{n+1} \left( \frac{\tilde{\tau}}{\tilde{\tau}_y} \right)^{n-1} \right] \quad (b)
$$

$$
\bar{\tau} = \sqrt{\tilde{\tau}_2^2 + \tilde{\tau}_3^2} \quad (c)
$$
2. LOWER BOUND FOR AXIAL SHEAR MODULUS

Consider a uniaxially reinforced lamina which is subjected to axial shear $\tau_0$ in the 1-2 plane on its boundary, fig. 5. By the average stress theorem, of Ref. 5,

$$\bar{\sigma}_{12} = \tau_0 \quad \text{(2.1)}$$

and all other average stresses vanish.

By the average theorem of virtual work, of Ref. 5,

$$\int_V \epsilon_{ij} \overline{d\sigma}_{ij} = \int_V \overline{\epsilon}_{ij} \overline{d\sigma}_{ij} \quad \text{(2.2)}$$

Since the only nonvanishing average stress in the present case is (2.1) we have

$$\overline{\epsilon}_{ij} \overline{d\sigma}_{ij} = 2\overline{\epsilon}_{12} \overline{d\tau}_0 \quad \text{(2.3)}$$

The complementary energy of the body is given by (14) of Appendix A. The surface integral vanishes however in the present case since no displacements are prescribed on the boundary. Now

$$U_C = \int_V \overline{W}_0 dV = \int_V \int_0^{\overline{\epsilon}_{ij}} \overline{d\sigma}_{ij} dV = \int_V ^{\overline{\epsilon}_{ij}} \overline{d\sigma}_{ij} dV \quad \text{(2.4)}$$

$$= \int_V \int_0^{\overline{\epsilon}_{ij}} \overline{\epsilon}_{ij} \overline{d\sigma}_{ij} dV \int_0^{\overline{\epsilon}_{12}} \overline{d\tau}_0 \quad \text{(2.4)}$$

The last equality following from (2.2, 3).

By definition the effective secant modulus $G^s_A$ is given by

$$\overline{\epsilon}_{12} = \frac{\overline{\sigma}_{12}}{G^s_A (\overline{\sigma}_{12})} = \frac{\tau_0}{2G^s_A (\tau_0)} \quad \text{(2.5)}$$

Hence (2.4) assumes the form

$$U_C = \int_0^{\tau_0} \frac{\tau_0 \overline{\epsilon}_{12} d\tau_0}{G^s_A (\tau_0)} \quad \text{(2.6)}$$

In order to find a bound on $G^s_A$ it will be necessary to find a bound on (2.6) by use of the principle of minimum complementary energy.
It is assumed that the fibers are infinitely rigid in comparison to the matrix. Therefore at fiber/matrix interface

\[ u_1 = 0 \]  

and the only contribution to the complementary energy is from the matrix. Thus, the surface integral in (1.33a) vanishes and it can be written as

\[ U_C = \int_{V_m}^{} \tilde{W}^g \, dV \]  

(2.8)

where \( V_m \) is the matrix volume.

Furthermore, by (2.3.3) the actual stresses are functions of \( x_2, x_3 \) only. It is therefore natural to also choose admissible stresses as functions of \( x_2, x_3 \). Thus \( \tilde{W}^g \) in (1.33) becomes a function of \( x_2, x_3 \) only and therefore without loss of generality (1.33a) can be taken over unit length in fiber direction. Thus it can be written

\[ U_C = \int_{A_m}^{} \tilde{W}^g (x_2, x_3) \, dx_2 \, dx_3 \]  

(2.9)

In order to construct an admissible stress system it is necessary to devise a geometrical model for a uniaxially reinforced material. In past analyses of FRM two kinds of models have been successfully treated: Periodic arrays of identical circular fibers have been analyzed numerically with the aid of computers and the composite cylinder assemblage model has been treated analytically \([1,5]\) yielding simple closed results. Since the present treatment is to be analytical the composite cylinder assemblage model will be used. A detailed description of the model has been given in \([5]\). Suffice it to say here that the model represents a cylindrical specimen of a fiber reinforced material as an assemblage of composite cylinders of different sizes which fill the space in the limit. In each composite cylinder the inner cylinder is a fiber and the outer shell is matrix material.
In all cylinders the ratios of fiber to matrix shell radius are the same, (figure 26).

It is recalled that an admissible stress system must satisfy equilibrium and boundary conditions. An obvious possibility for such an admissible field are the stresses of the elastic solution since they certainly satisfy the required conditions. These stresses are the same in any composite cylinder of the assemblage and are given in cylindrical coordinates by (see [5])

\[ \bar{\tau}_{rz} = \bar{\tau}_{r} = \frac{\tau_{0}}{1+c} (1 + \frac{a^2}{r^2}) \cos \theta \] (2.10)

\[ \bar{\tau}_{\theta z} = \bar{\tau}_{\theta} = -\frac{\tau_{0}}{1+c} (1 - \frac{a^2}{r^2}) \sin \theta \]

where \( c \) is the volume fraction of fibers, \( a \) is the radius of any fiber and \( r, \theta \) are polar coordinates, fig. 26.

Since \( \bar{\tau} \) as expressed by (1.33c) is an invariant with respect to rotations about \( x, y, z \) we have also

\[ \bar{\tau}^2 = \bar{\tau}_{r}^2 + \bar{\tau}_{\theta}^2 \] (2.11)

Substituting (2.10) into (2.11) yields

\[ \bar{\tau}^2 = \frac{\tau_{0}^2}{1+c} \left( 1 + \frac{1}{\rho^2} + \frac{1}{\rho^4} \right) \cos \theta \] (2.12)

where

\[ \rho = \frac{r}{a} \]

(2.13)

To simplify the analysis the exponent \( n \) in (1.22) will be assigned the value

\[ n = 3 \] (2.14)

It has been found that with this value of \( n \), experimentally obtained shear stress-strain relations of epoxy can be quite accurately represented with proper choice of \( \tau \). Recalling (1.33), (2.9) then assumes the form

\[ U_C = \frac{1}{2G} \int_{A_m} \bar{\tau}^2 \left( 1 + \frac{1}{2} \left( \frac{\bar{\tau}}{Y} \right)^2 \right) dA \] (2.15)

58.
where $G$ is the matrix elastic shear modulus. Let the assemblage consist of $K$ composite cylinder. Define $\tilde{U}_C^k$ for the $k$th composite cylinder by

$$
\tilde{U}_C^k = \frac{1}{2G} \int_{A_{mk}} \tilde{\tau}^2 \left[ 1 + \frac{1}{2} \left( \frac{\tilde{\tau}}{\tau_y} \right)^2 \right] dA
$$

where $A_{mk}$ is the matrix area $a_k \leq r \leq b_k$ in the $k$th composite cylinder. Then

$$
\tilde{U}_C = \sum_{k=1}^{K} \tilde{U}_C^k
$$

Since $\tilde{\tau}^2$ has been expressed in polar coordinates, (2.12), it is convenient to also evaluate (2.16) in the same coordinates. Using the variable $\rho$ we have

$$
\tilde{U}_C = \frac{1}{2G} \int_0^\beta \int_0^{2\pi} \tilde{\tau}^2 \left[ 1 + \left( \frac{\tilde{\tau}}{\tau_y} \right)^2 \right] \rho d\rho d\theta
$$

where

$$
\beta = \frac{b_k}{a_k}
$$

which by construction has the same value in all composite cylinders. Note also that the volume fraction of fibers $c$ is given by

$$
c = \left( \frac{a_k}{b_k} \right)^2 = \frac{1}{\beta^2}
$$

Substituting (2.12) into (2.18) and carrying out the integration we have

$$
\tilde{U}_C^k = \frac{\pi b_k^2}{2G} \tau_y^2 \left[ \frac{1-c}{1+c} + \left( \frac{\tau}{\tau_y} \right)^2 \frac{3+10c-12c^2-c^3}{6(1+c)^4} \right]
$$

59.
where (2.20) has been used. It is seen that \( fb_k^2 \) is the area of the cross section of the \( k \)th composite cylinder and the parenthesis has the same value for all composite cylinders. Therefore, if (2.21) is inserted into (2.17) we find

\[
\dot{U}_C = \frac{A}{2G} \tau_o^2 \left[ \frac{1-\nu}{1+c} + \frac{(\tau_o^2 + 3+10\sigma-12\sigma^2-c^2)}{6(1+c)} \right] \tag{2.22}
\]

Let (2.22) be written

\[
\dot{U}_C = \frac{A}{T_o} \int_0^{T_o} \frac{\dot{U}_C}{\dot{T}_o} \, d\tau_o \tag{2.23}
\]

Without loss of generality (2.6) can be evaluated for unit height of cylindrical specimen. Thus

\[
\dot{U}_C = \frac{A}{T_o} \int_0^{T_o} \frac{\tau_o d\tau_o}{G_A^s(\tau_o)} \tag{2.24}
\]

Now introduce (2.23) and (2.24) into the minimum complementary inequality (1.18). Thus

\[
\int_0^{T_o} \left[ \frac{1}{A} \frac{d\dot{U}_C}{d\tau_o} - \frac{\tau_o}{G_A^s(\tau_o)} \right] d\tau_o \geq 0 \tag{2.25}
\]

Since the integral is positive for all values of \( \tau_o \), the integrand must also be positive for all values of \( \tau_o \). It follows that

\[
G_A^s(\tau_o) \geq \frac{A\tau_o}{d\dot{U}_C/d\tau_o} = \frac{G_A^s}{A(-)} \tag{2.26}
\]

where the extreme right denotes lower bound on the secant modulus \( G_A^s \). Substituting (2.22) into (2.26) and rearranging we find the lower bound (2.3.9) of Chapter 2.

There naturally arises the question of the establishment of an upper bound. The difficulties involved have been discussed above: It is not in general possible to solve Ramberg-Osgood relations for stresses in terms of strains. It is
therefore not possible to analytically express the potential energy functional in terms of admissible strains.

A possibility to resolve the difficulty is to write inelastic stress-strain relations of type (1.22) in the form

\[ T = 2G\varepsilon_2 \left[ 1 - \left( \frac{\varepsilon}{\varepsilon_y} \right)^{a-1} \right] \]
\[ T = 3G\varepsilon_3 \left[ 1 - \left( \frac{\varepsilon}{\varepsilon_y} \right)^{a-1} \right] \quad (2.27) \]
\[ \varepsilon = \sqrt{\frac{\varepsilon_2^2 + \varepsilon_3^2}{2}} \]

where \( a \) and \( \varepsilon_y \) are to be determined by curve fitting. The minus sign in the parenthesis is due to the fact that the stress-strain curve is below a straight line with the initial slope.

It should be noted that (2.27) are not an inversion of (1.22). They are merely another form of approximation of actual stress-strain curves.

In principle the representation (2.27) can now be used in conjunction with the principle of minimum potential energy to establish an upper bound on \( G_f \) in same fashion as a lower bound has been established. It has however been found that in attempting to fit (2.27) to actual epoxy stress-strain curves a fractional exponent \( a \) was needed. This led to integrals of formidable difficulty in the evaluation of potential energy functionals. Therefore this approach has not been continued here.
APPENDIX D

FAILURE OF NON-LINEAR LAMINATES

It is expedient to separate the problem of the establishment of failure criteria of laminates into two separate problems:

(a) Establishment of failure criteria for uniaxially fiber reinforced material, i.e., laminae.

(b) Establishment of failure criteria of the laminate on the basis of laminae failure criteria.

A great deal of work has been done on problem (a). The problem has been approached in micro as well as macro-fashion. In micro-approach, it is attempted to predict failure on the basis of local analysis of the interior of the composite. Such an approach evidently encounters extreme difficulties. Although important work of fundamental nature has been done in this area, we shall not be concerned with it here since the work has not advanced to the stage of prediction of failure criteria under states of combined stress.

In the macro-approach, a failure criterion is heuristically postulated as some function of pertinent state variables (generally average stresses) which also contains undetermined parameters. These parameters are then to be determined in terms of experimentally accessible information.

We shall in the present discussion limit ourselves to states of plane stress. The simplest failure criterion is the so-called maximum stress criterion which states that failure occurs when either one of: stress in fiber direction, stress transverse to fibers, shear stress, reaches its critical value, these critical values being the same whether or not the stresses act simultaneously. In symbols the criterion is:

\[ \sigma_{11} = \sigma_A \]

or

\[ \sigma_{22} = \sigma_T \]

or

\[ \sigma_{12} = \tau_{AT} \]
where 1 is fiber direction and 2 is the transverse direction.

Generally, failure stresses $\sigma_A$ and $\sigma_T$ are different in tension and compression. This is known as Bauschinger effect. There is evidently no Bauschinger effect for the shear stress. The simplest generalization of (1) to account for Bauschinger effect would be to assume as failure criterion:

$$\sigma_{11} = \sigma_A^+ \quad \text{if} \quad \sigma_{11} > 0$$

$$\sigma_{11} = \sigma_A^- \quad \text{if} \quad \sigma_{11} < 0$$

$$\sigma_{22} = \sigma_T^+ \quad \text{if} \quad \sigma_{22} > 0$$

$$\sigma_{22} = \sigma_T^- \quad \text{if} \quad \sigma_{22} < 0$$

$$\sigma_{12} = \tau_{AT} \quad \text{all} \quad \sigma_{12}$$

whichever occurs first, where (+) and (-) superscripts denote failure stresses in tension and compression respectively. The main drawback of these simple criteria is in that they take no account of interaction effects.

The most commonly used criterion which takes into account interaction is of quadratic form. For plane stress it has the form

$$A_{11}\sigma_{11}^2 + A_{22}\sigma_{22}^2 + A_{12}\sigma_{11}\sigma_{22} + A_{44}\sigma_{12}^2 = 1 \quad (3)$$

Here, products of shear stress with normal stress have been omitted since the material cannot distinguish between positive and negative shear stress. Therefore, odd powers (one, in this case) of shear stress cannot appear.

Applying (3) to failure for stress in fiber direction alone, stress transverse to fiber direction alone, shear stress alone, in turn, it is seen at once that
The coefficient $A_{12}$ is troublesome since its determination requires a failure experiment under combined stress. Several authors have proposed to use failure experiments on off-axis specimens under uniaxial stress for the determination of $A_{12}$. See e.g. [18] for discussion.

The situation becomes more complicated if it is required to take into account Bauschinger effect, that is difference of failure stresses in tension and compression. One possibility to account for this effect is to assume that $A_{11}, A_{22}$ assume different values for tension and compression. The situation regarding $A_{12}$, however, becomes very awkward as it would have to assume four different values to account for four different possibilities of sign combination in biaxial stressing and

It is also possible to add linear terms to (3) in which case it would assume the form:

$$A_{11} \sigma_{11}^2 + A_{22} \sigma_{22}^2 + A_{12} \sigma_{11} \sigma_{22} + A_{44} \sigma_{12}^2 +$$

$$B_1 \sigma_{11} + B_2 \sigma_{22} = 1$$

Such a device was suggested by Hoffman [19]. In this case it is possible to determine values of $A_{11}, B_1, A_{22}, B_2$ to account for different tensile and compressive uniaxial failure stresses in fiber direction and transverse to it. But the difficulty of assigning four different values to $A_{12}$ remains, unfortunately.

In summary, the status of quadratic failure criteria has to date not been finalized. However, special versions of such criteria have been successfully fitted to experimental data.

It is of importance to realize that in the fiber reinforced materials used in practice failure predictions on the
basis of maximum stress criterion or quadratic failure criterion are not very different. This is due to the large ratios between strength in fiber direction and transverse and shear strengths and is easiest realized by considering the failure criteria as surfaces in $\sigma_{11}, \sigma_{22}, \sigma_{12}$ stress space. The maximum stress criterion is a very elongated rectangular parallelepiped while the quadratic failure criterion is an ellipsoid. For $\Lambda_{12}=0$, Fig. 25 shows this schematically on a cut in the $\sigma_{11}, \sigma_{22}$ plane. Thus it is seen that stress points on the two failure surfaces are close together for most parts of the surfaces.

The situation would be entirely different for a material in which $\sigma_A$, and $\sigma_T$ were of comparable magnitudes.

We shall now consider problem (b) i.e., the establishment of laminate failure criteria in terms of laminae failure criteria. The most conservative laminate failure criterion is to assume that once any lamina has failed the laminate has reached its ultimate load. There are cases of laminates in which all laminae would fail simultaneously and then this criterion would be justified. For example: a $\pm \theta$ laminate in which the external load direction bisects the angle between the fibers.

In most cases, however, a certain group of laminae will fail first and failure of remaining groups would require further increase of load. Therefore a more realistic alternative is to determine the load at which the first laminae group fails. At this state, the further carrying capacity of the laminate may be assumed to be given by the remaining undamaged laminae. The increase in load which fails another group of laminae is then determined. This process is continued until failure of all laminae has taken place.

Still another possibility is to assume that when a lamina has failed, certain of its stiffnesses reduce to zero. For example: suppose that a lamina or group of laminae has failed in shear. Such a failure implies a crack through the lamina in fiber direction. In that event, it is reasonable to assume
that the shear and transverse stiffnesses of the lamina are zero, but it still retains its stiffness in fiber direction. If, however, a lamina fails because of the stress in fiber direction the damage is so widespread that all of its stiffnesses will be negligible. According to the type of failures encountered analysis is continued for the damaged laminate with the new stiffness rearrangement. This process is continued until failure of all laminae has taken place. This method of analysis seems to be the most realistic but is also the most complicated.

In almost all of the practical strength analyses of laminates in the literature, according to any of the methods outlined above, the stresses used for failure criteria have been determined on the basis of elastic laminate analysis. With the present inelastic laminate analysis, more realistic stresses are available in a better assessment of laminate failure loads.
APPENDIX E
MSC-NOLIN COMPUTER PROGRAM

1. General Description of the Program

This is a computer program developed for the inelastic analysis of a laminate subject to any constant, arbitrary combination of in-plane loading. Details of the method of analysis and of the numerical solution, using the Newton-Raphson method, have been described in the body of this report. The essential features of the program are summarized below.

The primary capability of MSC-NOLIN is to compute laminae properties when the laminate loads are defined. There is also a limited capability to work with constituent properties, rather than laminae properties, as the input. Details of the input options are discussed subsequently. Basically, the inputs required are the stress-strain characteristics of the individual laminae for each of the three in-plane stress components applied separately. The stress-strain curves for transverse stress and for stress and for axial shear stress are defined by Ramberg-Osgood stress-strain curves. The parameters for these curves along with the laminae elastic constants are the required material property inputs.

It has been observed that axial shear stresses in individual laminae are a major, perhaps the major, source of non-linearities in laminate response. Therefore, several additional options have been included in the MSC-NOLIN to accommodate more detailed characterization of shear response. First, the laminae shear stress-strain response may be input in tabular form and a least squares fit to the data is automatically obtained for the R-O yield stress (limited to the use of an exponent, n=3). Secondly, the matrix shear stress-strain curve can be input along with fiber elastic properties and the laminae shear stress-strain curve will be computed. In this latter case, the laminae elastic constants are also computed.
The input specifies one of two options for the determination of the initial set of stresses to be used in the iteration at each value of applied load on the laminate. In one case the stresses found at one load are increased to the load for which the stresses were evaluated. In the other, and generally used option, the increment between the initial stresses used at the nth laminate load value and the actual stresses found for the (n-1)st load value bears the same relation to the ratio of those two load values as the similar relation computed at the previous load cycle, that is,

\[ \frac{\sigma_{ij}^{(n)} - \sigma_{ij}^{(n-1)}}{F_n / F_{n-1}} = \frac{\sigma_{ij}^{(n-1)} - \sigma_{ij}^{(n-2)}}{F_{n-1} / F_{n-2}} \]

The program contains a number of controls to define: the size and number of steps of loading at which computations are made; the maximum number of iterations to be permitted in the numerical solution; the desired accuracy to be obtained in convergence; the criteria for divergence of the solution in the iterative process to avoid the use of unnecessary execution time in the case of breakdown of the solution procedure. The program defines the failure of the laminate in a limited fashion, either on the basis of the maximum allowable stress in the fiber in tension or compression, or on the basis that the tangent modulus of the stress-strain curve of the laminate becomes less than a specified value. Failure due to shear or transverse stress are not included at this stage in the development of the program.

2. **Input**

The main features of input in this program are the following:

(a) Specify the number of laminates or problems to be solved.
(b) Define the geometrical properties of each layer;
(c) Define either the material properties of each layer or the properties of its constituents;
(d) Define either of the following for each layer:
   (i) yield stress in transverse direction and yield stress in shear;
   (ii) yield stress in transverse direction and a table of values defining shear stress-strain curve for the matrix plus a set of values of stresses to be used for the computation of yield stress in shear;
(e) Specify the type of Ramberg-Osgood relation to be used;
(f) Define the loadings; and
(g) Define the control parameters.

A guide to the preparation of input data for this program is given in section 4 below.

3. **Details of Output**

The output can be divided basically into two steps:

(a) **Output of Input Data:**

The first section of the output deals with the output of the input data. If the input is in the form of properties of constituents of the layer, it gives an output of the properties of the constituents first and then the computed value of the properties of the layer; otherwise, it gives output directly the properties of the layer.

(b) **Output of Stresses and Strains:**

For each set of loading, the computer prints the following:

1. value of the load applied;
2. number of iterations for convergence;
3. stresses for individual laminae with respect to principal elastic axes of the laminae; and
4. strains for individual laminae in terms of both laminae and laminate axes.
4. **Input Details for MSC-NOLIN**

(1) Read (I5) NSETS

NSETS: number of problems

(2) Read (I5) LAY

LAY: number of layers in this laminate analysis

(3) Read (I5) INP

INP: Option for reading in material properties

INP = 1; read in material properties of individual laminae;

INP = 2; compute properties of laminae from the properties of constituents.

(4) (a) If INP = 1

(i) Read (5D15.5) \( E_{11}, E_{22}, \mu_{12}, \mu_{21} \)

(ii) Read (5D15.5) \( G_{12}, SY, TY \)

(iii) Read (D15.5, I5) T, IANG

(b) If INP = 2

(i) Read (4D15.5) EF, MUF, GF, VF

(ii) Read (3D15.5) EM, MUM, GM

(iii) Read (I5) I2

If I 2 = 0: read in SY and TY

(i) Read (2D15.5) SY, TY

If I 2 = 1: TY is to be computed

(i) Read (5,1002) SYCE

(ii) Read ( 2 I 5) NUMT

NUMT = number of values in the table

(iii) Read (5D15.5) TAU (J), J=1, NUMT

(Table of shear stress values of matrix read in)

(iv) Read (5D15.5) GAM (J),J=1, NUMT

(Table of shear strain values of matrix read in)

(v) Read (5D15.5) SGL2 (J), J=2,11

(Table of shear stress values of laminae read in)
(5) Read (5D15.5) XN, XM

XN: exponent in nonlinear transverse stress-strain law;
XM: exponent in nonlinear shear stress-strain law.

(6) Read (5D15.5) SO11, SO22, SO12

SO11: applied stress in X-direction
SO22: applied stress in Y-direction
SO12: shear stress in XY

(7) Read (I5, D15.5) KSGM, SMLT

KSGM: total number of loading increments
SMLT: ratio of load increment to the initial load.

(8) Read (D15.5) STIFF

STIFF: tangent modulus of stress-strain curve in terms of the laminate axes; specify a value of STIFF below which the program will not run.

Read (D15.5) SGR

SGR: maximum allowable stress in the fiber in tension or compression

(9) Read (I 5, 2D15.5) IT, EPS, UPBD

IT: maximum number of iteration permitted in Newton-Raphson analysis
EPS: convergence criteria; (ratio of values of two successive iterations should be less than EPS)
UPBD: divergence criteria (solution will stop if ratio of two successive iterations is greater than 10⁻¹²)

(10) Read(I5) INMT

If INMT = 1, the program uses ratio of previous two solutions as the initial guess value iteration process;
If INMT = 2, the program uses extrapolated value of previous two solutions proportioned on the basis of stress ratio as the initial guess.
REFERENCES


Fig. 1 - Coordinate system for unidirectional fiber composite material.
Fig. 2 - Nondimensional Ramberg-Osgood stress-strain curves.
(e.g., 2.2.1 with $k = 0.4$).
MATERIAL PROPERTIES

\[ E_T = 12.8 \times 10^3 \text{ MN/m}^2 \]
\[ G_A = 3.56 \times 10^3 \text{ MN/m}^2 \]
\[ \gamma_A = 0.212 \]
\[ \sigma_y = 357 \text{ MN/m}^2 \]
\[ \tau_y = 111 \text{ MN/m}^2 \]
\[ n = m = 3 \]

Fig. 3 - Transverse normal \( \sigma - \varepsilon \) curves for unidirectional Boron/Epoxy (for normal loading and normal + shear loading).
Fig. 4 - Shear $\sigma$-$\epsilon$ curves for unidirectional Boron/Epoxy (for shear loading and normal + shear loading).

MATERIAL PROPERTIES

- $E_T = 12.8 \times 10^3$ MN/m$^2$
- $G_A = 3.56 \times 10^3$ MN/m$^2$
- $\gamma_A = 0.212$
- $\sigma_y = 357$ MN/m$^2$
- $\tau_y = 111$ MN/m$^2$
Fig. 5 - Unidirectional fiber composite material under axial shear stress.
Fig. 6 - Axial shear stress-strain curve for composite computed from matrix properties.
Fig. 7 - Generalized Ramberg-Osgood stress-strain curve in axial shear for composite and matrix (n=3).
Fig. 8 - Laminate coordinate system.
Fig. 9 - Tensile stress-strain curve with [0/90] B/Ep.
Fig. 10 - Tensile stress-strain curves for [+45] B/Ep.
Fig. 11 - Tensile stress-strain curve with [+30] B/Ep.
Fig. 12 - $0^\circ$ Tensile stress-strain curves for $[0/\pm 45/90]$ B/Ep.

PRESENT ANALYSIS

\[
G_{12} = 6.8 \times 10^3 \text{ MN/m}^2
\]

\[
\sigma_y = 112 \text{ MN/m}^2
\]
Fig. 13 - 0° Tensile stress-strain curves for [0/±60] B/Ep.
Fig. 14 - 0° Tensile stress-strain curves for [0_3/+45] B/Ep.

Experimental Results, Ref. 6

Present Analysis

\( G_{12} = 6.8 \times 10^3 \text{ MN/m}^2 \)
\( \sigma_y = 112 \text{ MN/m}^2 \)
Fig. 15 - 0° Tensile stress-strain behavior of [653/20/-70] B/Ep
Fig. 16 - Comparison of present results with experimental data for Glass/Epoxy 90/+18 tubes in torsion, and with the results of Ref. 9.
Fig. 17 - Stress-Strain curves for Axial, transverse and shear loading for [0°/+45°] laminate.
Fig. 18 - Four directional quasi-isotropic Boron/Epoxy plate under unidirectional tension in fiber direction and between fiber directions.
Fig. 19 - Combined stress effects on quasi-isotropic \([\pm 22.5/\pm 67.5]\) laminate.

**MATERIAL PROPERTIES**

- \(E_A = 198 \times 10^3 \text{ MN/m}^2\)
- \(E_T = 15.1 \times 10^3 \text{ MN/m}^2\)
- \(G_A = 4.13 \times 10^3 \text{ MN/m}^2\)
- \(\nu_A = 0.255\)
MATERIAL PROPERTIES:
\[ E_{11} = 1.98 \times 10^3 \text{ MN/m}^2 \]
\[ E_{22} = 15.1 \times 10^3 \]
\[ \mu_{12} = 0.255; \mu_{21} = 0.01938 \]
\[ G_{12} = 6.8 \times 10^3 \text{ MN/m}^2 \]

Fig. 20 - Effect of laminae inelasticity on transverse stress-strain curve for 0/±60 quasi-isotropic laminate.
Fig. 21 - Influence of laminae inelasticity upon axial tensile stress-strain curve of $\pm 30^\circ$ Boron/Epoxy laminate.

STRAIN ($\epsilon_{||}$)

STRESS ($\sigma_{||}$) MN/m²

$\sigma_y = 138$ MN/m²
$\tau_y = 34.4$ MN/m²

$\sigma_y = 68.9$ MN/m²
$\tau_y = 34.4$ MN/m²

B/E $\leq 30$
Fig. 22 - Influence of laminae inelasticity upon axial tensile stress-strain curve of $\pm 30^\circ$ Boron/Aluminum laminate.
Fig. 23 - 0° Tensile stress-strain curves for \([0_3/\pm 45] B/Ep\).
MATERIAL PROPERTIES

\( E_A = 230.8 \times 10^3 \text{ MN/m}^2 \)
\( E_T = 172.2 \times 10^3 \text{ MN/m}^2 \)
\( \nu_A = 0.25 \)
\( G_A = 55.9 \times 10^3 \text{ MN/m}^2 \)
\( \sigma_y = 68.9 \text{ MN/m}^2 \)
\( \tau_y = 34.4 \text{ MN/m}^2 \)
\( n = m = 3 \)

Fig. 24 - Stress-strain curves of Boron/Aluminum [0/+30] laminate under combined loading.
Fig. 25 - Comparison of failure criteria.

98.
Fig. 26 - Composite Cylinder Assemblage
C INELASTIC LAMINATE ANALYSIS

C

C 0001  C IMPLICIT REAL*8(A-H,C-J)
C 0002  C DIMENSION E1(20),E2(20),V12(20),V21(20),G12(20),SY(20),
C 0003  C               S1(20),S2(20),S3(20),S4(20),T(20),
C 0004  C               A(20),SG(20),I,SM(20),SC(20)
C 0005  C COMMON /ANGREG/SMG,STIF
C 0007  C COMMON /RAFIN1/S11,S12,S21,E22
C 0008  C LOGICAL MSING,MSINGE

C INPUT
C
C NOTE: CONVENTION FOR V12 AND V21 ESTABLISHED BY FOLLOWING
C RELATIONSHIPS
C
C S12 = -V12/E11
C S21 = -V21/E22

C

C Print
C
C 0009  C WRITE(6,15001)
C 0010  C READ(5,1010) NSETS
C 0011  C DO 999 J = 1,NSETS
C 0012  C READ(5,1010) LAY
C 0013  C READ(5,1010) INP
C
C LAMINATE OUTPUT HEADING
C
C 0014  C WRITE(6,15001) J
C 0015  C WRITE(6,15011)
C 0016  C WRITE(6,15011) LAY
C 0017  C GO TO (20,25), INP
C
C INPUT 1:
C
C 0018  C 20 CONTINUE
C 0019  C DO 24 J = 1,LAY
C 0020  C READ(5,1002) E11(J),E22(J),V12(J),V21(J)
C 0021  C READ(5,1002) G12(J),SY(J),TY(J)
C 0022  C READ(5,1002) T(J),TANG(J)
C 0023  C 24 CONTINUE
C 0024  C GO TO 50
C
C INPUT 2:
C
C 0025  C 25 CONTINUE
C 0026  C WRITE(6,15011)
C 0027  C WRITE(6,15011)
C 0028  C WRITE(6,15011)
C 0029  C CALL INPLT2(E11,E22,V12,V21,G12,SY,TY,TANG)
C 0030  C READ(5,1002) T(J),TANG(J)
C 50. CONTINUE
C
C EQUATION PARAMETERS
READ(5,1022) XN, XM
C
C INITIAL LOADING
READ(5,1022) SO11, SG22, SG12
C
C INCORPORATION PARAMETERS
READ(5,1024) KSG, SMLT
C
C LAMINATE TEST PARAMETERS
READ(5,1023) STIFF
C
C REAS (5,1022) SGR
C
C CONTROL PARAMETERS
READ(5,1024) IT, EPS, UPED
READ(5,1013) INMT

C
C OUTPUT
WRITE(6,1513)
DO 60 i=1, LAY
WRITE(6,1515) 1, I, ANG(I), T(I), E(I), E22(I), V12(I), V21(I), G12(I), S(I), T(Y(I))
60 CONTINUE
WRITE(6,1516)
WRITE(6,1517) XM
WRITE(6,1518) XN
WRITE(6,1520)
WRITE(6,1993)
WRITE(6,1995)

C ANGLE REDUCTION ROUTINE
CALL ANGLE(LAY, TANG)

C
C INITIAL ASSIGNMENTS AND COMPUTATIONS
TT = 0.000
DO 100 I=1, LAY
TT = TT + T(I)
100 CONTINUE
DO 105 I=1, 20
SG(I) = 0.0000 00
SE(I) = 1.000
105 CONTINUE
SM11 = SMLT*SO11
SM22 = SMLT*SO22
SM12 = SMLT*SC12
KSG = 1

C
LY1 = LAY
N = LT1

C CONTINUE
DO 107 I = 1, LAY
SI(I) = 1.0000/E(I)
107 CONTINUE
0027      110 CONTINUE
  C      RETURN TO 110 AFTER INCORRECTING APPLIED LOAD OR
  C      FROM LAMINATE TESTS 3 OR 4
  C
  0028      MISING = .FALSE.
  0029      MISINGD = .FALSE.
  0030      NIT = 0
  C
  0031      DO 111 K = 1,20
  0032      DGK(K) = 0.000
  0033      SGCI(K,1) = 0.000
  0034      SG11(K,1) = 0.000
  0035      SUZ(K) = 0.000
  0036      DO 111 I = 1,20
  0037      UBI(K,1) = 0.000
  0038      ALK(I) = 0.000
  0039      111 CONTINUE
  
  0040      DO 112 I = 1,N
  0041      S2Z(I) = 1.000/E2Z(I)
  0042      S44(I) = 1.000/14.000*G12(I)
  0043      113 CONTINUE

C      AFTER 2ND INCORRECTION USE MULTIPlicative FACTOR
C      AS INITIAL STRESS SOLUTION ESTIMATE

0044      112 CONTINUE
  0045      IF(N, EQ, 13) GO TO 120
  0046      DO 110 I = 1,N,M
      C
  0047      SN5 = SIN5(I)
  0048      CS5 = COS5(I)
  0049      SNZ2 = SINZ2(I)
  0050      CSZ2 = COSZ2(I)

  0051      IF(N, EQ, 11) GO TO 113
  0052      SN5P = SIN5(I+1)
  0053      CS5P = COS5(I+1)
  0054      SNZ2P = SINZ2(I+1)
  0055      CSZ2P = COSZ2(I+1)

  0056      113 CONTINUE
  0057      A(I1, I ) = CS5*T(I)
  0058      A(I1, I+N ) = SN5*T(I)
  0059      A(I1, I+2*N ) = -SNZ2*T(I)

  0060      A(2, I ) = SNS*T(I)
  0061      A(2, I+N ) = CSN*T(I)
  0062      A(2, I+2*N) = -SNZ2*T(I)

  0063      A(3, I ) = SN2*T(I)/2.000
  0064      A(3, I+N ) = -SN2*T(I)/2.000
  0065      A(3, I+2*N) = CSZ2*I
  0066      1F16, EQ, 13) GO TO 116

  0067      A(3*1, I+1, I ) = -S11(I, I)*CS5 -S21(I, I)*SN5
  0068      A(3*1, I+1, I+N) = S11(I, I)*CS5 +S21(I, I)*SN5
  0069      A(3*1, I+1, 2*N) = -S12(I, I)*CS5 -S22(I, I)*SN5

  0070      A(3*1, I+1, I+1) = S12(I, I)*CS5*S22(I, I)*SN5
  0071      A(3*1, I+1, I+2*N) = 2.000*SN2*I*SNZ2
  0072      A(3*1, I+1, I+2*N) = -2.000*SN2*1*S2Z
END
C RETURN TO 125 FOR NEXT ITERATION STEP

0168 125 CONTINUE

0169 NIT = NIT + 1

0170 126 CONTINUE

0171 DO 127 J=1,LT3

0172 SG(I,J) = SG(I,J-1)

0173 DT(I,J) = 0.00 00

0174 127 CONTINUE

C

0175 130 CONTINUE

0176 DO 130 K=1,LT3

0177 DO 130 I=1,LT3

0178 DBK,L) = 0.00 00

0179 1305 CONTINUE

0180 IF(N,E1) LM1=1

0181 DO 131 J=1,LM1

0182 CALL NTPM (LAY,SG,F,G,H,G2,SY,TY,XN,XM,1)

0183 CALL NTRM (LAY,SG,F,G,H,G2,SY,TY,XN,XM,1+1)

C

0184 SNH = SIN(I1)

0185 CSS = COS(I1)

0186 SN2 = SIN(I2)

0187 CSS2 = COS(I2)

C

0188 IF(N,E1) GO TO 131

0189 SN1 = SIN(I1)

0190 CSS1 = COS(I1)

0191 SN2 = SIN(I2)

0192 CSS2 = COS(I2)

C

0193 131 CONTINUE

0194 DB1,I) = CSS*Y(I)

0195 DB1(I,N) = SN1*Y(I)

0196 DB1,2+2*N) = -SN2*Y(I)

0197 DB2,1) = SN1*Y(I)

0198 DB2,2+2*N) = CSS*Y(I)

0199 DB2,2+2*N) = SN2*Y(I)

0200 DB3,1) = SN1*Y(I)/2.

0201 DB3,2+2*N) = -SN2*Y(I)/2.

0202 DB3,2+2*N) = CSS*Y(I)

0203 IF(N,E1) GO TO 161

C

0204 DB3,2+1,1) = -F(I1,1)

0205 DB3,2+1,1) = -F(I2,1)

0206 DB3,2+1,1) = -F(I3,1)

0207 DB3,2+1,1) = -F(I4,1)

0208 DB3,2+1,1) = -F(I5,1)

0209 DB3,2+1,1) = -F(I6,1)

C

0210 DB3,2+2,1) = -G(I1,1)

0211 DB3,2+2,1) = -G(I2,1)

0212 DB3,2+2,1) = -G(I3,1)

0213 DB3,2+2,1) = -G(I4,1)

0214 DB3,2+2,1) = -G(I5,1)

0215 DB3,2+2,1) = -G(I6,1)

C

0216 DB3,2+3,1) = -P(I1,1)

0217 DB3,2+3,1) = -P(I2,1)

0218 DB3,2+3,1) = -P(I3,1)

0219 DB3,2+3,1) = -P(I4,1)
0265  S46(K) = (1.0000*SG12**(1.0*1.0*1.0))/4.0*SG12(K)
0266  7 CONTINUE
0267  DD 0 K=1,L1
0268  SNS = S1NS(K)
0269  CSS = CCSS(K)
0270  SN2 = S1NS2(K)
0271  CS2 = CCSS2(K)
0272  SN3P = S1NS(K+1)
0273  CSSP = CCSS(K+1)
0274  SN2P = S1NS2(K+1)
0275  CS2P = CCSS2(K+1)
0276  DC(3+3*K-2) = -(S11(K)*CSS+S21(K)*SNS)*SG(K,1) -
0277  SG(LAY=K,1)*S12(K)*CSS+S22(K)*SNS) + 2.*S44(K)*SN2*
0278  DC(3+3*K) = -(S11(K)*CSS+S21(K)*SNS)*SG(K,1) - (S12(K)*SNS +
0279  S22(K)*CSS)*SG(LAY=K,1) - 2.*S44(K)*SN2*
0280  DD 4 J=1,L3
0281  MX(J) = -DC(J)
0282  9 CONTINUE
0283  DD 1 I=1,L3
0284  DD 1 K=1,L3
0285  1 BT(I) = BT(I) + C8(I,K)*CC(K)
0286  DD 15 I=1,L3
0287  SG(I,1) = SG1(I,1) + BT(I)
0288  15 CONTINUE
0289  IF (RESSTRESS .LT. 0.00000) GO TO 166
0289  166 IF (RELATIVE STRESS .LT. 1.0) GO TO 166
0289  15 CONTINUE
0290  CALL CONVERGENCE CHECK
0291  CALL CCNV(LAY,SG,SG1,KSG,L12S,LV00)
0292  500 CONTINUE
0293  DD 540 I = 1,LAY
0294  SNS = S1NS(I)
0295  CSS = CCSS(I)
0296  SN2 = S1NS2(I)
0297  CS2 = CCSS2(I)
0298  P11(I) = S11(I)*SG1(I,1) + S12(I)*SG1(I+1,1)
0299  T125 = (SG1**2*N1,1)**2
FORTRAN IV: LEVEL 21

MAIN

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0300 TS25 = (SG11+N,1) * 1
0301 S125 = T12S + TS25
0302 P22(i) = S21(i)/SG11 + SG11+N,1/E22(i)-1,000 + S125*(iN)
0303 P12(i) = SG11*2*M,1/12 -6G12(i)1/1,000 + S125*[(LN-1)/2*1]
0304 520 CONTINUE
0305 EP11(i) = P11(i)*SNS + P22(i)*SNS - P12(i)*SNS
0306 EP22(i) = P11(i)*SNS + P22(i)*SNS + P12(i)*SNS
0307 EP12(i) = P11(i) - P22(i)*SNS/2.00 + P12(i)*SNS
0308 140 CONTINUE
0309 C LAPLACIAN TESTS
0310 CALL LAMSTILAY(SG,SGS*KSC,KSGM)
0311 C OUTPUT 11.
0312 WRITE(6,1527) S011
0313 WRITE(6,1527) S022
0314 WRITE(6,1527) S012
0315 WRITE(6,1533) H1T
0316 WRITE(6,1536)
0317 WRITE(6,1537)
0318 WRITE(6,1538)
0319 DO 750 I = 1,LAY
0320 WRITE(6,1550) I,S1G11,SG11+N,1,SG11+2*M,1
0321 750 CONTINUE
0322 WRITE(6,1590)
0323 C IF KSCG KSGC 790,790,760
0324 C MULTILACIC FATOR FOR INITIAL ESTIMATE OF SUCCEEDING
0325 C INCORRENTATION
0326 760 CONTINUE
0327 GO TO (782,766), LNT
0328 C EATORY OF PREVIOUS SOLUTIONS
0329 762 CONTINUE
0330 763 CONTINUE
0331 SG11 = SG11/SG11
0332 GO TO 765
0333 766 CONTINUE
0334 GO TO 770
0335 C
0336 764 CONTINUE
0337 DO 765 I = 1,I13
0338 IF KSCG(i) GO TO 770
0339 IF KSGC(i) GO TO 763
0340 VS(i) = KSC + 1
0341 LOCAL = VS(i)*(VS(i)-1)*(VS(i)-2)
0342 EP12(i) = 1.0000 - LOCAL*SGS(i)/SG11
0343 GO TO 770
0344 767 CONTINUE

52
0345      S1(1) = 1.000
0346      760 CONTINUE

C STORE STRESS AND STRAIN VALUES

0347      DD 775 I = 1, LAY
0348      SG5(I) = SG(I + 1)
0349      SG5(I+1) = SG(I + 2)
0350      SG5(I+2) = SG(I + 3)
0351      P51(I) = P11(I)
0352      P52(I) = P22(I)
0353      P51(I+1) = P12(I)
0354      P52(I+1) = P21(I)
0355      EPSI(I) = EPS(I)
0356      EPSI(I+1) = EPS(I+1)
0357      EPSI(I+2) = EPS(I+2)
0358      775 CONTINUE

C INCREMENT APPLIED LOADING

0359      K5G = K5G + 1
0360      S011 = S011 + S11
0361      S022 = S022 + S22
0362      S012 = S012 + S12
0363      GO TO 110
0364      790 CONTINUE
0365      500 CONTINUE
0366      599 CONTINUE
0367      WRITE(*,1999)

C

0368      1002 FORMAT (5D15.5)
0369      1010 FORMAT (5F15)
0370      1022 FORMAT (515.5,15)
0371      1024 FORMAT (15,5D15.5)

C

0372      1420 FORMAT (/* MATRIX IS SINGULAR OF RANK = '1',
           '1'; DETERMINANT = '1',D15.5)
0373      1425 FORMAT (6D15.5)

C

0374      1500 FORMAT (/* INELASTIC LAMINATE ANALYSIS'///))
0375      1504 FORMAT (1H1,*LAMINATE '1',12)
0376      1510 FORMAT (/* REMARK '' NUMBER OF LAYERS = '1',12)
0377      1513 FORMAT (/* LAYER THETA',7X,'1',15X,'E11',11X,'E22'11X,
           1 'V12',11X,'V21',11X,'G12',11X,'SIG Y1',9X,'TAU Y7')
0378      1515 FORMAT (14,5X,'12',2X,'C12',5X,'J2X,D12,5)
0379      1516 FORMAT (/* EQUATIONS PARAMEETERS'//)
0380      1517 FORMAT (/* EXponent M = '1',D12,5)
0381      1518 FORMAT (/* EXponent A = '1',D12,5)
0382      1520 FORMAT (/* LEADING INCAPETATION'//)
0383      1525 FORMAT (/* EXTERNAL APPLIED LOADING'//)
0384      1527 FORMAT (/* F X = '1',D13,5)
0385      1528 FORMAT (/* F Y = '1',D13,5)
0386      1525 FORMAT (/* F XY = '1',D13,5)
0387      1536 FORMAT (/*2X,'STRESS',3B1,5P1H1,3B1,5P1H1)
0388      1537 FORMAT (/*2X,'LAMINATE',5F5,2X,'LAYER AXES')
0389      1538 FORMAT (/* LAYER',4X,'SU',4X,'SVM Y,8X,'SVM XY',9X,'EPS X',
           1 'Y',8X,'EPS Y',8X,'EPS XY',11X,'EPS X',6X,'EPS Y',8X,
           2 'EPS XY')
0390      1550 FORMAT (14,5X,'213(3,5,4),3C13,5)
0391 1561 FORMAT (/3X,'MATERIAL PROPERTIES')
0392 1563 FORMAT (/34X,'F18.9',5X,'KAT=1X')
0393 1565 FORMAT (/1       LAYERS,3X,2(3X,F15.5),11X,'V',12X,'W',12X,'V',12X/) 
0394 1735 FORMAT (/1         SOLUTION,4P STRESS CONVERGENCE WITHIN 1A,
                     1        ITERATIONS/) 
0395 1581 FORMAT ('---------------------')
0396 1583 FORMAT ('---------------------')
0397 1586 FORMAT ('---------------------')
0398 1595 FORMAT (1HC)
0399 1599 FORMAT (1P1)
0400      STOP
0401         END
SUBROUTINE ANGLE(LAY, IANG)

C
C COMPUTING SIN AND COS
C

IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION SIN(20), COS(20), SIN2(20), COS2(20), IANG(20)

COMMON /ANGRED/SIN2, COS2, SIN, COS

GO TO 72 1 = 1, LAY

IANG2 = 2*IANG(I)

ANG = IANG(I)

ANG2 = IANG2

RAD = ANG / 3.141592653589793238462643383279502884197169399375

IF(IANG2 .LT. 0.0) GO TO 66

IF(IANG2 .NE. 90.0) GO TO 62

SING(I) = DCOS(I) * SGN(I)**2

COS(I) = DSIN(I) * SGN(I)

GO TO 72

62 CONTINUE

ISGN = IANG(I) / IAVAL

SGN = ISGN

64 CONTINUE

IF(IANG2 .LT. 0.0) GO TO 66

IF(IANG2 .NE. 45.0) GO TO 66

SIN2(I) = COS(I) * SGN(I)**2

COS2(I) = DSIN(I) * SGN(I)

GO TO 68

66 CONTINUE

IF(IANG2 .LT. 0.0) GO TO 66

IF(IANG2 .NE. 90.0) GO TO 62

RAD = ACOS(I) * SGN(RAD)**2

COS2(I) = - SGN(DSIN(I) * SGN(RAD))

GO TO 68

68 CONTINUE

SIN2(I) = DSIN(RAD)

COS2(I) = DCOS(RAD2)

GO TO 72

72 CONTINUE

RETURN

END
SUBROUTINE CONVP(LAY,SG,SGI,KS,(*,*))

C

IMPLICIT F (ALPHA,(*,*)

DIMENSION SG(20,11),SGI(21,11),DFS(20)

COMMON /CFPARM,EPS,UPBD

C

ICON = 1

A = LAY

LT3 = LAY+3

C CONVERGENCE CHECK

DO 375 J3=1,LT3

SUB = DABS(SG(J3,11))-DABS(SGI(J3,11))

IF(SG(J3,11)EQ.0.0001) GO TO 330

DIF(J3) = DABS(SUB/SG(J3,11))

GO TO 335

330 CONTINUE

DIF/J31 = SUB

335 CONTINUE

IF(DIF(J3),GT,EPS) GC TO 340

GO TO 375

C ITERATION CHECK

340 CONTINUE

IF(NEIT-JT).NE.0) GO TO 350

ICON = 3

16(DIF(J3).LE.UPBD) GO TO 375

ICON = 4

GO TO 375

C CONVERGENCE CHECK

350 IF(DIF(J3).LE.UPBD) GC TO 370

ICON = 4

GO TO 375

C CONVNTANCE

370 CONTINUE

ICON = 2

GO TO (380,390,382,386), ICON

C NCR-CONVNTANCE DUMP

380 CONTINUE

WRITE(6,17201)

WRITE(6,17222) EPS

GO TO 395

CONTINUE

WRITE(6,17301)

WRITE(6,17322) UPBD

395 CONTINUE

WRITE(6,17472)

DN 397 = LAY

WRITE(6,15501) 1,SGC(1,11),SGC(N,11),SGC(1+2*N,11),SGC(1,11),SGC(1+N,11),

1.0,SGC(1+2*N,11),DIF(1),DIF(1+N),DIF(1+2*N)

397 CONTINUE

RETURN 2
0046     400 RETURN
0047     500 RETURN

0048     1550 FORMAT ('4,1X,2(D13.5,+,Y),3D13.5')
0049     1720 FORMAT (' SOLUTION FOR STRESS DOES NOT CONVERGE:1')
0050     1722 FORMAT (' RELATIVE ERROR GT.15.5')
0051     1730 FORMAT (' SOLUTION FOR STRESS DIVERGES:1')
0052     1741 FORMAT ('/(IAX,'+','ITRAN','+','13,1'),2X,'(IAX,'+','ITRAN','+','13,1)')
0053     1742 FORMAT ('/,(AX,'+','SGM X'),8X,'(AX,'+','SGM Y'),8X,'(AX,'+','SGM XY'),11X,'(SGM X'),
                1 8X,'(SGM Y'),8X,'(SGM XY'),11X,'(REL X'),8X,'(REL Y'),8X,
                2 'REL XY')

C     END
FORTRAN IV C LEVEL 2.

LAMST

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0001 SUBROUTINE LAMST(LAY,S,G,S,G,M)

C

0002 IMPLICIT REAL (A-H,O-Z)

0003 DIMENSION EPSL1(201,201),EPSL2(201,201),SG(201,1),SG(201)

C

0004 COMMON /LPAEY/SG,I2,STIF

C

0005 TEST 1: STIFFNESS TEST

0006 55C CONTINUE

0007 IF(KSG .GT. 0.99) GO TO 560

0008 IF(ABS(SG1 .LT. 0.0001) .OR. (ABS(SG1) .GT. EPSL1(1)))

0009 IF(RATIO .LT. STIF) GO TO 677

0010 C

0011 TEST 2: TEST FOR BOUND CN SGM X

0012 C

0013 560 CONTINUE

C

0014 WRITE(6,1555)

0015 RETURN

C

0016 677 LFAIL = 1

0017 KSG = KSGM

C

0018 GO TO 700

C

0019 675 LFAIL = 2

C

0020 KSG = KSGM

C

0021 GO TO 700

C

0022 700 CONTINUE

C

0023 GO TO (701,703), LFAIL

C

0024 701 WRITE(6,1450)

C

0025 GO TO 725

C

0026 703 WRITE(6,1452)

C

0027 GO TO 725

C

0028 725 WRITE(6,1455)

C

0029 1450 FORMAT (/" LAMINATE HAS FAILED: STIFFNESS TEST FAILURE")

C

0030 1452 FORMAT (/" LAMINATE HAS FAILED: SG X EXCEEDS MAXIMUM")

C

0031 1455 FORMAT (/" AT FAILURE")

C

0032 1995 FORMAT (1LC)

C

0033 RETURN

C

0034 END
IMPLICIT REAL(A-H,C-Z)
DOUBLE PRECISION DABS

DIMENSION S11(20), S12(20), S21(20), E22(20), G12(20),
      S1N(20), CS5(20), S1N2(20), S1S(20), CS120, CY, XM, 1,
      F(3,20), G(3,20), H(3,20), T(20), SY(20)

COMMON /RAE/ S11, S12, S21, E22,
     /ANGF6/ S1N2, CS2, SN5, CS5

N = LAY
T12S = (SG1*I2*N1,1/TY(1))**2
T12D = SG1*I2*N1,1/TY(1)**2
T22S = (SG1*I2*N1,1/TY(1))**2
T22D = SG1*I2*N1,1/TY(1)**2

VAL = SY(1)=1,000,000
PAT = SG1*I2*N1,1, SY(1)**2

RAT = S(LSIRATE)
RAT = 0,00 00
C12S = RAT/SY(1)
C12T = SG1*I2*N1,1, SY(1)**2

T12S = SG1*I2*N1,1, SY(1)**2
S12S = T12S + T22S

SNS = S1S(11)
CS5 = CS5(1)
S1N2 = S1N2(1)
CS2 = CS2(1)

F(1,1) = S11(11)*CS5 + S21(11)*SNS
F(2,1) = S12111*CS5 + SNS/E22(11) + SNS/E22(11)*S12**((XN-1.)/2.,)

1 + SNS*S12*S(S(XN-1./2.),E22(11))**2

2 + SNS*S12*S(S(XN-1./2.),E22(11))**2

2 + SNS*S12*S(S(XN-1./2.),E22(11))**2

G(1,1) = S11(11)*CS5 + S21(11)*CS5
G(2,1) = SN5*S12(11) + CS5/E22(11) + S12**((XN-1.)/2.)**2

1 + CS5*(XN-1.)/2.)**2/E22(11)*S12**((XN-1.)/2.)**2

2 + CS5*(XN-1.)/2.)**2/E22(11)*S12**((XN-1.)/2.)**2

2 + CS5*(XN-1.)/2.)**2/E22(11)*S12**((XN-1.)/2.)**2

H(1,1) = (S11(11)-S21(11))*SN2/2.
H(2,1) = S12111*SN2/2. - SN2/2.E22(11) - S12**((XN-1.)/2.)*SN2
      + ((XN-1.)/2.)**2/E22(11)**2

1 + ((XN-1.)/2.)*S12**((XN-1.)/2.)*SN2/2.E22(11)**2

2 + ((XN-1.)/2.)*S12**((XN-1.)/2.)*SN2/2.E22(11)**2

2 + (XN-1.)/2.)**2/E22(11)*S12**((XN-1.)/2.)**2

RETURN
END
SUBROUTINE RESET(LT3,SG,RVAL)

IMPLICIT REAL (A-H,O-Z)

DIMENSION SG(20,1)

SGMX = DABS(SG(1,1))

DO 319 K=2,LT3

IF(CAPS(SG(K,1)) .GT. SGMX) SGMX = DABS(SG(K,1))

314 CONTINUE

DO 319 K=1,LT3

RAT = CAPS(SG(K,1))/SGMX

IF(RAT .LT. RVAL) SG(K,1) = 0.00 00

319 CONTINUE

RETURN

END
IMPLICIT  LOGICAL(A-Z)

DOUBLE PRECISION DSM,GM

DIMENSION F11(20),F22(20),V12(20),G12(20),GAM(20),TAU(20),TY12(20)

INTEGER I,J,K,M,N,S,U,UU,V

READ(5,1002) EF,MUF,GF,VF
READ(5,1002) EM,MUM,GM

VM = 100D0 - VF

TRM1 = (1.000+VF)/(1.000-VF) + GM

WRITE(6,156) I,EF,MUF,GF,VF,EM,MUM,GM,VF

IF(12,6,1) GO TO 30

PEAD(5,1002) SY(1) T(1)

G12(I) = TRM1

GO TO 60

30 CONTINUE

READ(5,1002) T(1)
READ(5,1005) KMT
READ(5,1002) (TAU(U),J=1,KMT)

READ(5,1002) (GAM(J),J=1,U)

READ(5,1002) (GAM(J),J=2,U)

C

COMPUTE TY FROM TABLE

SUM6 = 0.000
SUM5 = 0.000
SUM3 = 0.000
DO 40 J=1,KMT
SUM5 = SUM5 + TAU(U)**6
SUM3 = SUM3 + TAU(U)**4

SUM3 = SUM3 + GAM(U) = TAU(U)**3

40 CONTINUE

TAU = DSCRT(SUMS6/10**SUMS3-SUMS4)

C

COMPUTE SY AND G12

SG12(I) = 0.000
SUM6 = 0.000
SUM5 = 0.000
SUM3 = 0.000

TRM2 = (3.000+13.000*VF)/(3.000*(1.000+VF)**3) * VF**2/
L = (1.000+1.000*VF)**3/3.000

GO TO 50 J=1,11
GSA(J) = TRM1/(SG12(J)/TAU(J)**2*TRM2*1.000)
SUM6 = SUM6 + SG12(J)**6
SUM5 = SUM5 + SG12(J)**4*GSA(J)/GSA(J)-1.000)

50 CONTINUE

SY(I) = DSCRT(SUMS6/SUMS4)

G12(I) = GSA(I)

C

GO CONTINUE

P1 = 3.1615900
RF = EM/12.000*(1.000+MUF)*(1.000-2.000*MUF)

KM = EM/12.000*(1.000+M*MUF)*(1.000-2.000*MUM)

M = (RF*P1+0.000*VF*VUM) + 2.000**KM*(P1-4.000+VF)**VUM/

1 = (RF*P1+0.000*VF) + 2.000**KM*(P1+MUM+2.000*VF)

KK = KM/2.000*(P1**4.000+P1)/4.000 +
FORTRAN IV C LEVEL 21

1  \( 4.000 \cdot M/(P1+4.000 \cdot P1) \cdot M1 \)

0048 \( E11(1) = E11 \cdot V + E11 \cdot (1.000 \cdot V) \)

0049 \( E12(1) = \mu \cdot E12 \cdot V + \mu \cdot E12 \cdot (1.000 \cdot V) \)

0050 \( E21(1) = \mu \cdot E21 \cdot V + \mu \cdot E21 \cdot (1.000 \cdot V) \)

0051 \( E22(1) = 2.00 \cdot E22 \cdot M11 \cdot M11 \cdot (1.000 \cdot M23) \cdot E11(1) \cdot (4.000 \cdot M2) \)

0052 \( V12(1) = V12(1) \cdot E22(1) \cdot E11(1) \)

0053 1002 FORMAT (5E15.5)

0054 1005 FORMAT (5E15.5)

0055 1967 FORMAT (14,2F7.4, 4D11.4, 2X)

0056 C RETURN

0057 END
SUBROUTINE MMULDA, A, B, C

DIMENSION A(NRDWA, NCOLA), B(NCOLA, NCOLB), C(NRDWA, NCOLB)

DOUBLE PRECISION A, B, C, X

DO 20 I = 1, N
   DO 10 J = 1, NP
      X = 0.
      DO 10 K = 1, NA
      10   X = X + A(I, K)*B(K, J)
   20   C(I, J) = X

RETURN
END
<table>
<thead>
<tr>
<th>LIN</th>
<th>CODE</th>
<th>COMMENT</th>
<th>LANG</th>
<th>LINES</th>
</tr>
</thead>
<tbody>
<tr>
<td>0059</td>
<td>TEMP=K(L)</td>
<td></td>
<td></td>
<td>1200059</td>
</tr>
<tr>
<td>0060</td>
<td>A(K,L)=A(l,1)</td>
<td></td>
<td></td>
<td>1200060</td>
</tr>
<tr>
<td>0061</td>
<td>120 A(I,1)=TEMP</td>
<td></td>
<td></td>
<td>1200061</td>
</tr>
<tr>
<td>0062</td>
<td>DO 150 L = 1, N</td>
<td></td>
<td></td>
<td>1200062</td>
</tr>
<tr>
<td>0063</td>
<td>TEMP=K(M)</td>
<td></td>
<td></td>
<td>1200063</td>
</tr>
<tr>
<td>0064</td>
<td>A(L,M)=A(L,1)</td>
<td></td>
<td></td>
<td>1200064</td>
</tr>
<tr>
<td>0065</td>
<td>130 A(L,1)=TEMP</td>
<td></td>
<td></td>
<td>1200065</td>
</tr>
<tr>
<td>0066</td>
<td>I=1+K</td>
<td></td>
<td></td>
<td>1200066</td>
</tr>
<tr>
<td>0067</td>
<td>I=S(K)+M</td>
<td></td>
<td></td>
<td>1200067</td>
</tr>
<tr>
<td>0068</td>
<td>14C CONTINUE</td>
<td></td>
<td></td>
<td>1200068</td>
</tr>
<tr>
<td>0069</td>
<td>I=40.RANKS</td>
<td></td>
<td></td>
<td>1200069</td>
</tr>
<tr>
<td>0070</td>
<td>RETURN</td>
<td></td>
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<td>1200070</td>
</tr>
<tr>
<td>0071</td>
<td>END</td>
<td></td>
<td></td>
<td>1200071</td>
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</table>
### Laminate 1

#### Sample Problem

**Number of Layers:** 3

<table>
<thead>
<tr>
<th>Layer</th>
<th>Theta</th>
<th>T</th>
<th>E11</th>
<th>E22</th>
<th>V12</th>
<th>V21</th>
<th>GL2</th>
<th>SIG Y</th>
<th>TAU Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>45</td>
<td>0.333300 00</td>
<td>0.297550 08</td>
<td>0.210550 07</td>
<td>0.455000 00</td>
<td>0.193000-01</td>
<td>0.987000 06</td>
<td>0.100000 05</td>
<td>0.500000 04</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>0.333300 00</td>
<td>0.297550 08</td>
<td>0.210550 07</td>
<td>0.455000 00</td>
<td>0.193000-01</td>
<td>0.987000 06</td>
<td>0.100000 05</td>
<td>0.500000 04</td>
</tr>
<tr>
<td>3</td>
<td>68</td>
<td>0.333300 00</td>
<td>0.297550 08</td>
<td>0.210550 07</td>
<td>0.455000 00</td>
<td>0.193000-01</td>
<td>0.987000 06</td>
<td>0.100000 05</td>
<td>0.500000 04</td>
</tr>
</tbody>
</table>

**Elevation Parameters**

- Exponent M = 0.300000 01
- Exponent N = 0.300000 01

**Loading Incrementation**

**External Applied Loading**

- Fx = 0.500000 04
- Fy = 0.500000 04
- Fxy = 0.500000 04

Solution for Stress converges within 2 iterations

<table>
<thead>
<tr>
<th>Layer</th>
<th>SGM x</th>
<th>SGM y</th>
<th>SGM xy</th>
<th>EPS x</th>
<th>EPS y</th>
<th>EPS xy</th>
<th>EPS x</th>
<th>EPS y</th>
<th>EPS xy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.11659G 05</td>
<td>0.40950D 03</td>
<td>0.69030U 03</td>
<td>0.41473D 03</td>
<td>-0.30259D 03</td>
<td>0.47092D-03</td>
<td>-0.41473D-03</td>
<td>0.52699D-03</td>
<td>0.35866D-03</td>
</tr>
<tr>
<td>2</td>
<td>0.11659G 05</td>
<td>-0.40950D 03</td>
<td>0.69030U 03</td>
<td>0.41473D 03</td>
<td>-0.30259D 03</td>
<td>0.47092D-03</td>
<td>0.41473D-03</td>
<td>-0.30259D-03</td>
<td>0.47092D-03</td>
</tr>
<tr>
<td>3</td>
<td>0.11659G 05</td>
<td>-0.40950D 03</td>
<td>0.69030U 03</td>
<td>0.41473D 03</td>
<td>-0.30259D 03</td>
<td>0.47092D-03</td>
<td>0.52699D-03</td>
<td>-0.41485D-03</td>
<td>-0.35866D-03</td>
</tr>
</tbody>
</table>

**External Applied Loading**

- Fx = 0.100000 05
- Fy = 0.100000 05
- Fxy = 0.100000 05

Solution for Stress converges within 2 iterations

<table>
<thead>
<tr>
<th>Layer</th>
<th>SGM x</th>
<th>SGM y</th>
<th>SGM xy</th>
<th>EPS x</th>
<th>EPS y</th>
<th>EPS xy</th>
<th>EPS x</th>
<th>EPS y</th>
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<th>$\mathbf{F}_y$</th>
<th>$\mathbf{F}_{xy}$</th>
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**Solution for Stress Converges Within 2 Iterations**

#### Stress (Layer Axes)

<table>
<thead>
<tr>
<th>Layer</th>
<th>$\sigma_{x}$</th>
<th>$\sigma_{y}$</th>
<th>$\sigma_{xy}$</th>
<th>$\epsilon_{x}$</th>
<th>$\epsilon_{y}$</th>
<th>$\epsilon_{xy}$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.237490 04</td>
<td>-0.160400 09</td>
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<td>0.458470 04</td>
<td>0.143320 02</td>
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<td>0.127460 02</td>
<td>0.143320 02</td>
<td>0.119800 02</td>
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#### Strain (Layer Axes)

<table>
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<tr>
<th>Layer</th>
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<td>1</td>
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**Solution for Stress Converges Within 2 Iterations**

#### Stress (Layer Axes)

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#### Strain (Layer Axes)

<table>
<thead>
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### STRESS

<table>
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### STRAIN (LAMINATE AXES)

<table>
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<td>1</td>
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<td>3</td>
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### STRAIN (LAYER AXES)

<table>
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<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
</tbody>
</table>

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### EXTERNAL APPLIED LOADING

- **F_X = 0.36366005**
- **F_Y = 0.00000000**
- **F_XY = 0.36366005**

SOLUTION: FOR STRESS CONVERGES WITHIN 2 ITERATIONS

### EXTERNAL APPLIED LOADING

- **F_X = 0.36366005**
- **F_Y = 0.00000000**
- **F_XY = 0.36366005**

SOLUTION: FOR STRESS CONVERGES WITHIN 2 ITERATIONS

---

### EXTERNAL APPLIED LOADING

- **F_X = 0.36366005**
- **F_Y = 0.00000000**
- **F_XY = 0.36366005**

SOLUTION: FOR STRESS CONVERGES WITHIN 2 ITERATIONS

---

### EXTERNAL APPLIED LOADING

- **F_X = 0.36366005**
- **F_Y = 0.00000000**
- **F_XY = 0.36366005**

SOLUTION: FOR STRESS CONVERGES WITHIN 2 ITERATIONS

---

### EXTERNAL APPLIED LOADING

- **F_X = 0.36366005**
- **F_Y = 0.00000000**
- **F_XY = 0.36366005**

SOLUTION: FOR STRESS CONVERGES WITHIN 2 ITERATIONS

---

### EXTERNAL APPLIED LOADING

- **F_X = 0.36366005**
- **F_Y = 0.00000000**
- **F_XY = 0.36366005**

SOLUTION: FOR STRESS CONVERGES WITHIN 2 ITERATIONS

---
### Solution for Stress Converges Within 2 Iterations

#### Stress

<table>
<thead>
<tr>
<th>Layer</th>
<th>SGX X</th>
<th>SGX Y</th>
<th>SGX XY</th>
<th>EPS X</th>
<th>EPS Y</th>
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<tbody>
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<td>0</td>
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<td>0.431104</td>
<td>0.385500</td>
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<td>0.385500</td>
<td>0.293500</td>
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#### Strain (Layer Axes)

<table>
<thead>
<tr>
<th>Layer</th>
<th>SGX X</th>
<th>SGX Y</th>
<th>SGX XY</th>
<th>EPS X</th>
<th>EPS Y</th>
<th>EPS XY</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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### External Applied Loading

- F X = 0.400000 00
- F Y = 0.400000 00
- F XY = 0.400000 00

---

### Solution for Stress Converges Within 2 Iterations

#### Stress

<table>
<thead>
<tr>
<th>Layer</th>
<th>SGX X</th>
<th>SGX Y</th>
<th>SGX XY</th>
<th>EPS X</th>
<th>EPS Y</th>
<th>EPS XY</th>
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<tbody>
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#### Strain (Layer Axes)

<table>
<thead>
<tr>
<th>Layer</th>
<th>SGX X</th>
<th>SGX Y</th>
<th>SGX XY</th>
<th>EPS X</th>
<th>EPS Y</th>
<th>EPS XY</th>
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</thead>
<tbody>
<tr>
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<td>0.443200</td>
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</tr>
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<td>0.443200</td>
<td>0.443200</td>
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### External Applied Loading

- F X = 0.500000 00
- F Y = 0.500000 00
- F XY = 0.500000 00

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### Solution for Stress Converges Within 2 Iterations

#### Stress

<table>
<thead>
<tr>
<th>Layer</th>
<th>SGX X</th>
<th>SGX Y</th>
<th>SGX XY</th>
<th>EPS X</th>
<th>EPS Y</th>
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<tbody>
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#### Strain (Layer Axes)

<table>
<thead>
<tr>
<th>Layer</th>
<th>SGX X</th>
<th>SGX Y</th>
<th>SGX XY</th>
<th>EPS X</th>
<th>EPS Y</th>
<th>EPS XY</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>-0.435800</td>
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### External Applied Loading

<table>
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<td>0.4624 U0</td>
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<tr>
<td>Fxy</td>
<td>0.30 U0</td>
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</tbody>
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### Solution for Stress Converges Within 2 Iterations

<table>
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<tr>
<th>Layer</th>
<th>Eps x (Laminate Axes)</th>
<th>Eps y (Laminate Axes)</th>
<th>Eps xy (Laminate Axes)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5450 U0 -0.4229 U0 -0.3470 U0</td>
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<td>0.5024 U0 -0.3529 U0 -0.3470 U0</td>
<td>0.5447 U0 -0.3483 U0 -0.3470 U0</td>
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<tr>
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<td>0.5028 U0 -0.3529 U0 -0.3470 U0</td>
<td>0.5447 U0 -0.3483 U0 -0.3470 U0</td>
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### External Applied Loading

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</tr>
<tr>
<td>Fxy</td>
<td>0.66 U0</td>
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</tbody>
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### Solution for Stress Converges Within 2 Iterations

<table>
<thead>
<tr>
<th>Layer</th>
<th>Eps x (Laminate Axes)</th>
<th>Eps y (Laminate Axes)</th>
<th>Eps xy (Laminate Axes)</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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### External Applied Loading

<table>
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<tr>
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<tr>
<td>Fxy</td>
<td>0.63 U0</td>
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### Solution for Stress Converges Within 2 Iterations

<table>
<thead>
<tr>
<th>Layer</th>
<th>Eps x (Laminate Axes)</th>
<th>Eps y (Laminate Axes)</th>
<th>Eps xy (Laminate Axes)</th>
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<tbody>
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</tr>
<tr>
<td>LAYER</td>
<td>SUM X</td>
<td>SUM Y</td>
<td>EPS X</td>
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<td>0.501620</td>
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</table>

**LAMINATE HAS FAILED: SUM X EXCEEDS MAXIMUM**

**AT FAILURE**

**EXTERNAL APPLIED LOADING**

- \( P_X = 0.700000 \) \( \text{G} \)
- \( P_Y = 0.400000 \) \( \text{G} \)
- \( P_{XY} = 0.700000 \) \( \text{G} \)

**SOLUTION FOR STRESS CONVERGENCE WITHIN 2 ITERATIONS**

**STRAIN (LAMINATE AXES)**

<table>
<thead>
<tr>
<th>LAYER</th>
<th>SUM X</th>
<th>SUM Y</th>
<th>EPS X</th>
<th>EPS Y</th>
<th>EPS XY</th>
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<tbody>
<tr>
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<tr>
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<td>0.106620</td>
<td>0.324660</td>
<td>0.510400</td>
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<td>3</td>
<td>0.141610</td>
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<td>0.546000</td>
<td>0.696000</td>
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</tbody>
</table>

**STRAIN (LAMINA AXES)**

<table>
<thead>
<tr>
<th>LAYER</th>
<th>SUM X</th>
<th>SUM Y</th>
<th>EPS X</th>
<th>EPS Y</th>
<th>EPS XY</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>-0.106620</td>
<td>0.324660</td>
<td>0.510400</td>
<td>0.957000</td>
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<tr>
<td>2</td>
<td>0.106620</td>
<td>0.324660</td>
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<td>0.596000</td>
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<td>0.141610</td>
<td>0.518370</td>
<td>0.521440</td>
<td>0.546000</td>
<td>0.696000</td>
</tr>
</tbody>
</table>

**NASA Langley, USA**
"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."
—National Aeronautics and Space Act of 1958

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