A NOTE ON PARALLEL AND PIPELINE COMPUTATION
OF FAST UNITARY TRANSFORMS

by

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ABSTRACT

This correspondence discusses the parallel and pipeline organization
of fast unitary transforms algorithms such as the Fast Fourier Transform
and points out the efficiency of a combined parallel-pipeline processor
of a transform such as the Haar transform in which \((2^n - 1)\) hardware
"butterflies" generate a transform of order \(2^n\) every computation cycle.

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Algorithms for all fast unitary transforms, such as the Fast Fourier transform (FFT), fast Walsh-Hadamard transform (FWT) and other fast unitary transform [1], require \( n \) stages of computation for transforms of order \( 2^n \). Each stage of computation can be in turn decomposed into at most \( 2^{n-1} \) "butterflies" [2], each performing a rotation by a matrix of order 2. Some or all of the butterflies at one stage of computation can operate in parallel (see [3], [4] for FFT) and fast unitary transforms have thus a greater potential in applications with the development of low cost parallel circuitry. For example, we show in Fig. 1a the FFT Cooley-Tukey algorithm of order 4 with 2 butterflies in each of its 2 stages of computation. If \( \tau \) seconds is the time required to perform a butterfly operation, each stage can be performed in \( \tau \) seconds with the highest possible degree of parallelism which uses \( 2^{n-1} \) butterflies. Thus, a transform of order \( 2^n \) can be performed in \( n\tau \) seconds as compared to \( n2^{n-1}\tau \) seconds with sequential computation (which requires only one butterfly).

If a number of successive transforms have to be computed, it is possible to increase further the throughput rate with several transformers working simultaneously, each operating on a different input vector and each possibly at a different stage of computation (see [5] for FFT); this is generally referred to as a pipeline organization. Parallel and pipeline organizations can be combined conveniently with \( n2^{n-1} \) (at most) butterflies working in parallel and one transform of order \( 2^n \) is obtained every \( \tau \) seconds on the average. Fig. 1b shows a possible organization of the FFT Cooley-Tukey algorithm of order 4. All stages of this pipeline algorithm are identical: the 2 first butterflies perform the first stage.
of Fig. 1a and the 2 last butterflies perform the second stage. The input vector is entered in the first 4 cells and its FFT transform obtained in the same cells after 2 cycles. This algorithm can be wired-in and will give the transform coefficients in any order but it requires a large amount of hardware and requires the access at its input of two sets of $n2^n$ storage cells.¹

Some transforms, however, do not require $2^{n-1}$ butterflies at each stage of computation and then a pipeline algorithm can be implemented with much less hardware. We consider now in particular a pipeline algorithm for the Fast Haar Transform (FHT). Although less known, the FHT is closely related to the FWT [6], has a fast algorithm [7], is certainly a transform of interest for signal encoding [8], [9] and other applications [10]. A pipeline-parallel algorithm for the FHT requires only $(2^n-1)$ butterflies and still produces a transform of order $2^n$ at every cycle. We show in Fig. 2a the Haar matrix of order 8 and in Fig. 2b a possible organization of the FHT of the same order. The number of butterflies decreases for successive stages and this is the property which can be exploited in a pipeline processor. In Fig. 3, we show a stage of a possible organization of the pipeline FHT of order 8.

Many other transforms can have similar pipeline algorithms with reduced amount of hardware: the Modified generalized discrete transforms [11], the WFH transforms [1], the Slant Haar transforms [12] and other generalized Slant transforms [13]. In all cases, the pipeline-parallel algorithm needed to perform a transform of order $2^n$ in one cycle is the total number of butterflies appearing in the flow diagram of the algorithm. By contrast, parallel processing requires the maximum number of butterflies needed at any stage.

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REFERENCES


FOOTNOTE

The computation can be also performed "in place" with $n^2$ storage cells only followed by cyclic shifts by $2^n$ cells.

CAPTIONS

Fig. 1a : FFT Cooley-Tukey Algorithm of order 4

Fig. 1b : Pipeline FFT Cooley-Tukey Algorithm of order 4

Fig. 2a : Haar matrix of order 8

Fig. 2b : Fast Haar Transform of order 8

Fig. 3 : Pipeline Fast Haar Transform of order 8.
Input vector
- \( V_0 \)
- \( V_1 \)
- \( V_2 \)
- \( V_3 \)

Output vector
- \( F_0 \)
- \( F_1 \)
- \( F_2 \)
- \( F_3 \)

\{ bit-reversal ordering \}

\begin{align*}
&V_0 \\
&V_1 \\
&V_2 \\
&V_3 \\
\end{align*}

(a)

New input vector for \( \text{p} \)th vector

Transform vector for \( \text{p}-2 \)th vector in bit-reversal order

\begin{align*}
&V_0, V_1, V_2, V_3 \\
\end{align*}

First stage intermediate vector for \( \text{p}-1 \)th vector

\begin{align*}
&A \\
&B \\
\end{align*}

\( A + B \)

\( b \)

\( b(A-B) \)

denotes a "butterfly"

(b)
\[ [H_8] = \frac{1}{\sqrt{8}} \]

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \\
\end{bmatrix}
\]

(a)

(b)
New input vector (p-th)

First stage intermediate results for (p-1)th vector

Second stage int. res. for (p-2)th

Partial transform coefficients for (p-2)th vector

Partial transform coefficients for (p-1)th vector

A

B stands for A - B