STABILITY REGION MAXIMIZATION
BY DECOMPOSITION-AGGREGATION METHOD

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The purpose of this work is to considerably improve the estimates of the stability regions by formulating and resolving a proper maximization problem. The solution of the problem provides the best estimate of the maximal value of the structural parameter and at the same time yields the optimum comparison system, which can be used to determine the degree of stability of the Skylab. The analysis procedure is completely computerized, resulting in a flexible and powerful tool for stability considerations of large-scale linear as well as nonlinear systems.
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| DESCRIPTION OF THE COMPUTER PROGRAM |
Nomenclature

\( I_1, I_2, I_3 = \) principal moment of inertia of body about \( i \) coordinate \( I^*_1 + 2Mr^2 \), \( I^*_2 + 2Mr^2 \), respectively

\( I^*_i = \) principal moment of inertia of rigid core body about \( i \) body - fixed coordinate

\( k_i = \) stiffness coefficient characterizing nonrotating boom stiffness

\( m = \) tip mass of boom

\( T_i = \) applied torque about \( i \) coordinate

\( t = \) time

\( (\cdot) = \frac{\text{d}}{\text{d}t} = \) differentiation with respect to real time \( t \)

\( u_i = u^I_i - u^{II}_i = \) skew symmetric mode of elastic deformations

\( u^m_i = \) displacement of \( m \) tip mass from spinning steady state in \( i \) direction \((m = I, II)\)

\( w_i = \) perturbation (about spinning steady state) velocity about \( i \) coordinate

\( r_2 = \) steady-state boom dimension in 2-axis direction from center of mass to tip mass

\( r_3 = \) the asymmetry in the setting of the booms

\( K_1, K_2 = \) ratios of inertia \((I_2-I_3)/I_1\) and \((I_3-I_1)/I_2\) respectively
\[ \alpha = \frac{1 + K_1}{1 - K_2} = \frac{I_2}{I_1} = \text{ratio of inertia} \quad \frac{I_2}{I_1} \]

\[ \gamma = \frac{2m r_2^2}{I_1} = \text{dimensionless inertia ratio} \]

\[ \Delta_i = \frac{d_i}{m \Omega} = \text{dimensionless damping ratio} \]

\[ u_i = \frac{u_i}{2 r_2} = \text{general skew symmetric coordinate} \]

\[ \sigma_i^2 = \frac{k_i}{m \Omega^2} = \text{dimensionless natural frequency coefficient of boom} \]

\[ \xi = \frac{r_3}{r_2} = \text{dimensionless length ratio} \]

\[ \tau = \Omega t = \text{dimensionless time} \]

\[ \Omega = \text{steady-state spin rate about 3 axis} \]

\[ v_i = \frac{w_i}{\Omega} = \text{dimensionless wobble ratio} \quad (i = 1, 2, 3) \]

\[ (\cdot)' = \frac{d}{d\tau} = \text{differentiation with respect to} \quad \tau \]

\[ \text{subscript} \quad i = \text{index referring to three body-fixed coordinates} \]

\[ (i = 1, 2, 3) \]

\[ \gamma_3 = \frac{2m r_2^2}{I_3} = \text{dimensionless inertia ratio} \]

\[ \beta = \frac{I_3}{I_1} = \text{dimensionless inertia ratio} \]

\[ \phi_i = \text{angular rotations about} \quad i^{\text{th}} \text{coordinate axis} \]
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1. INTRODUCTION

This report is a direct continuation of the decomposition-aggregation stability analysis of the spinning Skylab presented in reference [1]. Therefore, for the mathematical basis, more detailed derivations of the state model, and computer implementations of the decomposition-aggregation analysis, this report will rely heavily on [1].

In the preceding report [1], the linear equations of the Skylab motion, which include both the passive stabilization by extendable booms with tip masses and the active stabilization by control torques about the body fixed axes, were decomposed into two sets of equations describing the wobble motion and the spin motion. Then, two sets of equations were treated as subsystems which were interconnected by a coupling parameter representing the asymmetry in the booms' setting. Stability properties of each subsystem were aggregated into a single quadratic Liapunov function. The vector Liapunov function was formed which had subsystem Liapunov functions as components. A linear second order comparison system was constructed in terms of the vector Liapunov function. Stability conditions of the aggregate comparison system provided estimates of the coupling parameter. Such an investigation is motivated by the fact that the mathematical model of the system is of high dimension and a straightforward analysis would become bogged down in the welter of detail requiring an excessive computer storage and time to complete the investigation. The multi-level decomposition-aggregation approach [2, 3] offers to solve the stability problems "piece-by-piece" and not only make more economical the computer use, but also reduce the liability of the errors in the analysis. Furthermore, by decomposing the system into parts that have important physical meaning, the decomposition-aggregation approach yields significant structural information about the behavior of the system, which is not generally available in a straightforward stability investigation.
The decomposition-aggregation stability analysis is based upon the Liapunov stability theory, and it is inherently conservative. Therefore, the stability region of the structural parameter obtained in [1] is relatively small.

The purpose of this work is to considerably improve the estimates of the stability regions by formulating and resolving a proper maximization problem [4, 5]. The solution of the problem provides the best estimate of the maximal value of the structural parameter and at the same time yields the optimum comparison system, which can be used to determine the degree of stability of the Skylab. The analysis procedure is completely computerized resulting in a flexible and powerful tool for stability considerations of large-scale linear as well as nonlinear systems.

The research reported herein was performed by S. M. Cuk under the supervision of D. D. Siljak, and was used as a part of S. M. Cuk's M.S.E.E. Thesis, at the Electrical Engineering and Computer Science Department, University of Santa Clara.
2. PASSIVE STABILIZATION

In this section, we will consider the passive stabilization of the spinning Skylab by extendable booms attached to the body of the vehicle. The linear vector state equation describing the vehicle

$$S: \quad x'(\tau) = P x(\tau) \tag{2.1}$$

is obtained in [1] from the linearized equations of motion derived in [4]. The symbols in equations (2.2-3) are introduced in the Nomenclature.

An important feature of equations (2.2-3) is that when $\Gamma_3 = 0$, they become uncoupled into two sets of equations: the wobble motion $(w_1, w_2, u_3)$ described by (2.2); and spin motion $(w_3, u_1, u_2)$ described by (2.3). The influence of the asymmetry in the arrangements of the booms ($\Gamma_3 \neq 0$) will be treated as the coupling parameter between the two motions. In the decomposition-aggregation
analysis, each motion represents a subsystem, and the coupling parameter can be made to appear explicitly in the interconnections among the two subsystems.

Passive control equations (2.2-3) can be rewritten as follows:

\[ v_1' - K_1 v_2 + \gamma (\mu_3' + \mu_3) - \xi \gamma (2\mu_1' + \mu_2' - \mu_2) = 0 \]

\[ -K_2 \alpha v_1 + \alpha v_2 + \xi (\mu_1' - \mu_1 - 2\mu_2) = 0 \]

\[ v_1 + v_2 + v_3' + \Delta_3 v_3' + (\sigma_3^2 + 1) \mu_3 = 0 \] (2.4)

\[ \xi (v_1 + v_2') + \mu_1' + \Delta_1 \mu_1 + \sigma_1^2 \mu_1 - 2\mu_2' = 0 \]

\[ \xi (-v_1' + v_2) + 2\mu_1' + \mu_2' + \Delta_2 \mu_2 + (\sigma_2^2 - 1) \mu_2 = 0 \] (2.5)

where the notation is again as in the Nomenclature. The dimensionless parameter \( \xi = \Gamma_3/\Gamma_2 \) is the coupling parameter between the two sets of equations (2.4) and (2.5).

The state space representation (2.1) of the over-all system \( S \) described by (2.4-5), is obtained by choosing the state 8-vector \( x(\tau) \) as

\[ x(\tau) = (v_1 \ v_2 \ \mu_3' \ \mu_1' \ \mu_3 \ \mu_1' \ \mu_2' \ \mu_2')^T \] (2.6)

The system \( S \) of equation (2.1) can be decomposed into two interconnected subsystems described by

wobble motion \( S_1: x_1'(\tau) = P_1 x_1(\tau) + \xi^2 Q_{11}(\xi) x_1(\tau) + \xi Q_{12}(\xi) x_2(\tau) \) (2.7)

spin motion \( S_2: x_2'(\tau) = P_2 x_2(\tau) + \xi Q_{21}(\xi) x_1(\tau) + \xi^2 Q_{22}(\xi) x_2(\tau) \) (2.8)
where the state vectors $x(t), x_1(t), x_2(t)$ of the system $S$ and the two subsystems $S_1$ and $S_2$ are

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_1(t) = (v_1 v_2 v_3 v_4)^T$$

$$x_2(t) = (v_1 v_2 v_3 v_4)^T,$$

(2.9)

In (2.7-8), the $4 \times 4$ matrices $P_1$ and $P_2$ correspond to the "free" subsystems $S_1$ and $S_2$, and the $4 \times 4$ matrices $Q_{11}(\xi), Q_{12}(\xi), Q_{21}(\xi), \text{ and } Q_{22}(\xi)$ represent the interconnections between the two subsystems.

In order to extract the subsystem matrices $P_1$ and $P_2$ independent of the coupling parameter $\xi$ and obtain the decomposition of (2.7-8), it was necessary to use the following identities:

$$\frac{1}{\alpha - \xi^2 \gamma} \equiv \frac{1}{\alpha} + \frac{\xi^2 \gamma}{\alpha(\alpha - \xi^2 \gamma)}$$

$$\frac{1}{\gamma - 1 + \xi^2} \equiv \frac{1}{\gamma - 1} + \frac{\xi^2 \gamma}{(\gamma - 1)(\gamma - 1 + \xi^2 \gamma)}$$

(2.10)

The matrices of (2.7-8) are:

$$P_1 = \begin{bmatrix}
0 & p_{12}^1 & p_{13}^1 & p_{14}^1 \\
p_{21}^1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & p_{42}^1 & p_{43}^1 & p_{44}^1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
p_{12}^1 = - p_{42}^1 = (K_1 + \gamma)/(1 - \gamma) \\
p_{13}^1 = \gamma \sigma_3^2/(1 - \gamma) \\
p_{14}^1 = \Delta_3 \gamma/(1 - \gamma) \\
p_{21}^1 = K_2 \\
p_{42}^1 = - (\sigma_3^2 + 1 - \gamma)/(1 - \gamma) \\
p_{44}^1 = - \Delta_3/(1 - \gamma)
\end{bmatrix}$$
\[ P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ p_{21}^2 & p_{22}^2 & 0 & p_{24}^2 \\ 0 & 0 & 0 & 1 \\ 0 & p_{42}^2 & p_{43}^2 & p_{44}^2 \end{bmatrix} \]

\[ p_{21}^2 = \sigma_1^2 \]

\[ p_{22}^2 = -\Delta_1 \]

\[ p_{24}^2 = 2 \]

\[ p_{42}^2 = -2 \]

\[ p_{43}^2 = -(\sigma_2^2 - 1) \]

\[ p_{44}^2 = -\Delta_2 \]

\[
Q_{11}(\xi) = \begin{bmatrix} 0 & q_{12}^{11} & q_{13}^{11} & q_{14}^{11} \\ q_{21}^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q_{42}^{11} & q_{43}^{11} & q_{44}^{11} \end{bmatrix}
\]

\[ q_{11}^{11} = -q_{42}^{11} = \frac{(2\gamma + K_1 - 1)}{(1 - \gamma)(1 - \gamma - \xi^2 \gamma)} \]

\[ q_{12}^{11} = q_{13}^{11} = -q_{43}^{11} = \frac{\gamma^2 \Delta_3}{(1 - \gamma)(1 - \gamma - \xi^2 \gamma)} \]

\[ q_{14}^{11} = -q_{44}^{11} = \frac{\gamma^2 \Delta_3}{(1 - \gamma)(1 - \gamma - \xi^2 \gamma)} \]

\[ q_{21}^{11} = \frac{(K_2 + 1)}{\alpha - \xi^2 \gamma} \]

\[
Q_{12}(\xi) = \begin{bmatrix} 0 & q_{12}^{12} & q_{13}^{12} & 0 \\ q_{21}^{12} & q_{22}^{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q_{12}^{12} & q_{13}^{12} & 0 \end{bmatrix}
\]

\[ q_{12}^{12} = -q_{42}^{12} = \frac{2\gamma}{(1 - \gamma - \xi^2 \gamma)} \]

\[ q_{13}^{12} = -q_{43}^{12} = \frac{-\gamma}{(1 - \gamma - \xi^2 \gamma)} \]

\[ q_{21}^{12} = \frac{\gamma (\sigma_1^2 + 1)}{(\alpha - \xi^2 \gamma)} \]

\[ q_{22}^{12} = \frac{\gamma \Delta_1}{(\alpha - \xi^2 \gamma)} \]
The structural configuration of the system $S$ as composed of the two subsystems $S_1$ and $S_2$ and the interconnections between them through the coupling parameter $\xi$ can then be depicted as in Fig. 2.1a.

It is obvious that the system of Fig. 2.1a becomes that of Fig. 2.1b when $\xi = 0$. When $\xi \to \infty$, the system of Fig. 2.1a is again decoupled into the two subsystems shown in Fig. 2.1c, because the interconnection matrices $\xi Q_{12}(\xi)$, $\xi Q_{21}(\xi)$ and $\xi^2 Q_{11}(\xi)$, $\xi^2 Q_{22}(\xi)$ become zero and constant matrices, respectively.
The subsequent stability analysis shows that the free subsystems \( S_1 \) and \( S_2 \) \( (\xi = 0) \) are stable. It is easy to check, however, that the decoupled subsystems in Fig. 2.1c are unstable. Therefore, our main objective is to determine the best estimate \( \xi^0 \) of the maximum allowable value of \( \xi \) which lies between the two extremes \( \xi = 0 \) and \( \xi = \infty \), and for which the overall system of Fig. 2.1a is stable.

On the basis of the Skylab physical characteristics the matrices \( Q_{ij}(\xi) \) \( (i, j = 1, 2) \), can be made independent of \( \xi \) and denoted by \( Q_{ij} \). This is accomplished by neglecting the term \( \xi^2 \gamma = 0.197\xi^2 \) with respect to the terms \( 1 - \gamma = 0.803 \) and \( \alpha = 5.52 \) since \( \xi \ll 1 \).

After this simplification, the numbers \( \xi_{ij} \) \( (i = 1, 2) \) of the norm of the coupling matrices \( Q_{ij} \) can be computed using

\[
\xi_{ij} = \left[ A(Q^T_{ij} Q_{ij}) \right]^{1/2}, \quad i, j = 1, 2
\]  

(2.12)

The subsystem Liapunov functions \( v_1, v_2 \) are chosen as

\[
 v_1(x) = (x_{11}^T x_{11})^{1/2}, \quad i = 1, 2
\]

(2.13)

and the differential inequality

\[
 \dot{v} \leq Av
\]

(2.14)

is formed following \([1-3]\), where \( v = (v_1, v_2)^T \) is the vector Liapunov function and the aggregate \( 2 \times 2 \) matrix \( A = (a_{ij}) \) is defined as

\[
 A = \begin{bmatrix}
 \frac{1}{2} \frac{\lambda(G_1)}{\lambda(H_1)} + \xi^2 \frac{\lambda(H_1)}{\lambda(H_1)} & \frac{\xi_{12}}{\lambda^{1/2}(H_1)} \frac{\lambda(G_1)}{\lambda^{1/2}(H_1)} \\
 \frac{\xi_{21}}{\lambda^{1/2}(H_1)} \frac{\lambda(G_2)}{\lambda^{1/2}(H_2)} & -\frac{1}{2} \frac{\lambda(G_2)}{\lambda(H_2)} + \xi^2 \frac{\lambda(H_2)}{\lambda(H_2)}
\end{bmatrix}
\]

(2.15)
Fig. 2.1 Structural Decomposition
In (2.15), \( \lambda \) and \( \Lambda \) denote minimum and maximum eigenvalues of the indicated matrices, respectively. The subsystem Lyapunov functions in (2.13) are obtained by solving the Lyapunov matrix equation

\[
P_i^T H_i + H_i P_i = -G_i, \quad i = 1, 2
\]

(2.16)

using the direct method of solution. From [2, 3], the conditions

\[a_{ii} < 0, \quad \det A > 0,\]

(2.17)

are necessary and sufficient for stability of \( A \) in (2.15), and sufficient for stability of the overall system (2.1). For every choice of positive definite symmetric matrices \( G_1, G_2 \) in (2.16) inequalities (2.17) yield an estimate of the stability region of the coupling parameter \( \xi \). The special choice of \( G_1 = G_2 = I \) in (2.16) produces positive definite matrices \( H_1, H_2 \) and establishes stability of the decoupled subsystems \( S_1, S_2 \) when \( \xi = 0 \) [1].

The range of the coupling parameter \( \xi \) obtained in [1] is small due to the conservativeness of the stability procedure. However, the maximum estimate of \( \xi \) could be considerably increased by a proper choice of the matrices \( G_i, i = 1, 2, \) in (2.16). A meaningful optimization problem can be formulated as the maximization of \( \xi \) over all matrices \( G_1, G_2 \). In Section 4, the corresponding optimization problem is formulated and resolved. [4, 5]
3. ACTIVE STABILIZATION

In order to inertially fix the axis of the Skylab pointed at the sun in presence of disturbance torques, attitude control torques must be applied to the vehicle [4]. The control torques depend on error signals that are proportional to the angle between the inertially fixed axis and the solar vector. Sun sensors and rate gyros on the present Skylab can readily provide the signals \( \phi_1, \phi_2, w_1 \) and \( w_2 \) needed for control.

Again, the linear vector state equation is considered

\[
S: \quad x'(\tau) = Px(\tau),
\]

which is obtained in [1] from the linearized equations of motion

\[
\begin{align*}
\text{wobble motion} \quad & \begin{cases}
I_1 \ddot{w}_1 + (I_3 - I_2) \omega \dot{w}_2 + m \dot{r}_2 (\ddot{u}_3 + \Omega^2 u_3) \\
-m \dot{r}_3 (2 \ddot{u}_1 + \ddot{u}_2 - \Omega^2 u_2) = T_1
\end{cases} \\
(I_1 - I_3) \omega \dot{w}_1 + I_2 \dot{w}_2 + m \dot{r}_3 (\ddot{u}_1 - \Omega^2 u_1 - 2 \ddot{u}_2) = T_2 \\
2m \dot{r}_2 (\dot{w}_1 + \Omega \dot{w}_2) + m \ddot{u}_3 + d_3 \ddot{u}_3 + (k_3 + m \Omega^2) u_3 = 0
\end{align*}
\]  

\[
\begin{align*}
\text{spin motion} \quad & \begin{cases}
I_3 \ddot{w}_3 - m \dot{r}_2 (\ddot{u}_1 - 2 \ddot{u}_2) = T_3 \\
2m \dot{r}_3 (\omega \dot{w}_1 + \dot{w}_2) - 2m \dot{r}_2 \dot{w}_3 + m \ddot{u}_1 \\
+ d_1 \dot{u}_1 + k_1 u_1 - 2m \ddot{u}_2 = 0
\end{cases} \\
2m \dot{r}_3 (-\dot{w}_1 + \Omega w_2) - 4m \dot{r}_2 \omega \dot{w}_3 + 2m \ddot{u}_1 + m \ddot{u}_2 \\
+ d_2 \dot{u}_2 + (k_2 - m \Omega^2) u_2 = 0,
\end{align*}
\]  

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derived in [4] using the linear control law

\[ T = \alpha \phi + \beta \omega. \] (3.4)

In (3.4), \( T = [T_1 \ T_2 \ T_3]^T \) is the vector of control torques; \( \phi = [\phi_1 \ \phi_2 \ \phi_3]^T \) is the vector of angular rotations; \( \omega = [w_1 \ w_2 \ w_3 + \Omega]^T \) is the vector of angular velocities; \( \alpha, \beta \) are \( 3 \times 3 \) matrices

\[
\alpha = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & 0 \\
\alpha_{21} & \alpha_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \beta = \begin{bmatrix}
\beta_{11} & \beta_{12} & 0 \\
\beta_{21} & \beta_{22} & 0 \\
0 & 0 & \beta_{33}
\end{bmatrix};
\] (3.5)

and kinematic relationships are

\[
\omega = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} \phi + \begin{bmatrix}
0 & \Omega & 0 \\
-\Omega & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \phi. \] (3.6)

The control law in this study is chosen as

\[
\alpha_{12} = I_1 \Omega^2, \quad \text{all other } \alpha_{ij} = 0
\]

\[
\beta_{11} = I_1 \Omega \delta, \quad \beta_{33} = -I_1 \Omega \rho, \quad \text{all other } \beta_{ij} = 0
\] (3.7)

so that the normalized control torques \( v = [v_1 \ v_2 \ v_3]^T = \left[ T_1/I_1 \Omega^2 \ T_2/I_1 \Omega^2 \ T_3/I_1 \Omega^2 \right]^T \) are

\[
v_1 = (\epsilon + \delta) \phi_2 - \delta \phi_1,
\]

\[
v_2 = 0
\]

\[
v_3 = \rho \phi_3
\] (3.8)
Referring to equations (3.2) and (3.3), the control torque $T_1$ is used to stabilize the subsystem $S_1$ (wobble motion), and the torque $T_3$ is used to stabilize the subsystem $S_2$ (spin motion).

In (3.8), $\varepsilon$, $\delta$, $\rho$ are control parameters to be selected in the stabilization process.

Upon introducing these transformations the linearized equations of motion become:

$$
\begin{align*}
\phi''_1 - (1+K_1)\phi'_2 - K_1\phi_1 - \gamma(\mu'_3 + \mu_3) + \xi(2\mu'_1 + \mu_2') + (\varepsilon + \delta)\phi_2 - \delta\phi'_1 &= 0 \\
(1+K_1)\phi'_1 + K_2\alpha_2 + \alpha_3 - \xi(\mu'_1 - \mu_1 - 2\mu'_2) &= 0 \\
-\phi''_1 + \phi'_2 + \mu'_3 + \Delta_3\mu_3 + (\sigma_3^2 + 1)\mu_3 &= 0 \\
8\phi''_3 + \gamma(\mu'_1 - 2\mu'_2) + \rho\phi_3 &= 0 \\
-2\xi\phi'_1 - \xi\phi'_2 + \phi'_3 + \mu'_1 + \Delta_1\mu_1 + \sigma_1^2\mu_1 - 2\mu'_2 + \xi\phi_2 &= 0 \\

\end{align*}
$$

The state space representation of the overall system $S$ described by (3.9-10), is obtained by choosing the state 11-vector $x(\tau)$ as

$$x(\tau) = (\phi_1, \phi_2, \mu_3, \phi'_1, \phi'_2, \mu'_3, \phi'_3, \mu'_1, \mu'_2, \mu'_2)^T.$$  (3.11)

The system $S$ of equation (3.1) can be decomposed into two interconnected subsystems described by:
wobble motion \[ S_1: x_1'(\tau) = P_1 x_1(\tau) + \xi^2 Q_{11}(\xi) x_1(\tau) + \xi Q_{12}(\xi) x_2(\tau) \] (3.12)

spin motion \[ S_2: x_2'(\tau) = P_2 x_2(\tau) + \xi Q_{21}(\xi) x_1(\tau) + \xi^2 Q_{22}(\xi) x_2(\tau) \] (3.13)

using the same procedure outlined in the previous section.

The state vectors \( x(\tau), x_1(\tau), x_2(\tau) \) of the system \( S \) and two subsystems \( S_1 \) and \( S_2 \) are

\[
\begin{bmatrix}
  x_1(\tau) \\
  x_2(\tau)
\end{bmatrix}
= \begin{bmatrix}
  (\phi_1 \phi_2 \mu_3 \phi'_1 \phi'_2 \mu'_3)^T \\
  (\phi'_3 \mu_1 \mu_2 \mu'_2)^T
\end{bmatrix}
\]

In (3.12-13) the \( 6 \times 6 \) and \( 5 \times 5 \) matrices \( P_1, P_2 \) correspond to the subsystems \( S_1 \) and \( S_2 \) and \( 6 \times 6, 6 \times 5, 5 \times 6 \) and \( 5 \times 5 \) matrices \( Q_{11}(\xi), Q_{12}(\xi), Q_{21}(\xi), Q_{22}(\xi) \) represent the interconnections between the two subsystems:

\[
P_1 = \begin{bmatrix}
  0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 1 \\
  p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & 1 \\
  0 & p_{52} & 0 & p_{54} & 0 & 0 & 1 \\
  p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & 1
\end{bmatrix}
\]
\[ P_{41} = \frac{(K_1 + \gamma)}{(1 - \gamma)} \]
\[ P_{42} = P_{62} = \frac{-e \delta}{(1 - \gamma)} \]
\[ P_{43} = \gamma \sigma_3^2/(1 - \gamma) \]
\[ P_{44} = P_{64} = \delta/(1 - \gamma) \]
\[ P_{45} = P_{65} = \frac{(1 + K_1)}{(1 - \gamma)} \]
\[ P_{46} = -\gamma \Delta_3/(1 - \gamma) \]

\[ P_{11} = -\frac{s}{\beta - \gamma} \]
\[ P_{12} = \gamma \sigma_1^2/(\beta - \gamma) \]
\[ P_{13} = \gamma \Delta_1/(\beta - \gamma) \]
\[ P_{31} = s/(\beta - \gamma) \]

\[ P_{2} = \begin{bmatrix}
  P_{11}^2 & P_{12}^2 & P_{13}^2 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  P_{31}^2 & P_{32}^2 & P_{33}^2 & 0 & 2 \\
  0 & 0 & 0 & 0 & 1 \\
  -2 & 0 & -2 & P_{54}^2 & P_{55}^2 \\
\end{bmatrix} \]

\[ P_{11}^2 = -\frac{s}{\beta - \gamma} \]
\[ P_{12}^2 = \gamma \sigma_1^2/(\beta - \gamma) \]
\[ P_{13}^2 = \gamma \Delta_1/(\beta - \gamma) \]

\[ P_{52} = -K_2 \]
\[ P_{54} = K_2^{-1} \]
\[ P_{61} = \frac{(1 + K_1)}{(1 - \gamma)} \]
\[ P_{63} = -\frac{(\sigma_3^2 + 1 - \gamma)}{(1 - \gamma)} \]
\[ P_{66} = -\Delta_3/(1 - \gamma) \]
\[
q_{11}(\varepsilon) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
q_{41}^{11} & q_{42}^{11} & q_{43}^{11} & q_{44}^{11} & q_{45}^{11} & q_{46}^{11} \\
0 & q_{52}^{11} & 0 & q_{54}^{11} & 0 & 0 \\
q_{61}^{11} & q_{62}^{11} & q_{63}^{11} & q_{64}^{11} & q_{65}^{11} & q_{66}^{11}
\end{bmatrix}
\]

\[
q_{41}^{11} = q_{61}^{11} = \frac{\gamma(K_1 + 2\gamma - 1)}{(1-\gamma)(1-\gamma - \varepsilon^2\gamma)}
\]

\[
q_{42}^{11} = q_{62}^{11} = -\frac{\gamma(\varepsilon + \delta)}{(1-\gamma)(1-\gamma - \varepsilon^2\gamma)}
\]

\[
q_{43}^{11} = q_{63}^{11} = -\frac{\gamma^2\sigma_3^2}{(1-\gamma)(1-\gamma - \varepsilon^2\gamma)}
\]

\[
q_{44}^{11} = q_{54}^{11} = \frac{\gamma\delta}{(1-\gamma)(1-\gamma - \varepsilon^2\gamma)}
\]

\[
q_{45}^{11} = q_{65}^{11} = \frac{K_1 + 2\gamma - 1}{(1-\gamma)(1-\gamma - \varepsilon^2\gamma)}
\]

\[
q_{52}^{11} = \frac{\gamma^2\Delta_3}{(1-\gamma)(1-\gamma - \varepsilon^2\gamma)}
\]

\[
q_{52}^{11} = -\frac{\gamma K_2 + 1}{\alpha(1-\gamma_3 - \varepsilon^2\gamma)}
\]

\[
q_{52}^{11} = -\frac{\gamma^3 - K_2}{\alpha(1-\gamma_3 - \varepsilon^2\gamma)}
\]

\[
q_{12}(\varepsilon) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
q_{41}^{12} & 0 & 0 & q_{44}^{12} & q_{45}^{12} & q_{62}^{12} \\
q_{51}^{12} & q_{52}^{12} & q_{53}^{12} & 0 & 0 & 0 \\
q_{61}^{12} & 0 & 0 & q_{64}^{12} & q_{65}^{12}
\end{bmatrix}
\]

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\[
\begin{align*}
q_{41} &= q_{61} = \frac{2\gamma}{1-\gamma-\xi^2\gamma} \\
q_{44} &= q_{64} = \frac{\gamma\sigma_2^2}{1-\gamma-\xi^2\gamma} \\
q_{45} &= q_{65} = \frac{\gamma\Delta_2}{1-\gamma-\xi^2\gamma} \\
q_{51} &= \frac{\gamma_3^0}{\alpha(1-\gamma_3)^2\gamma} \\
q_{52} &= \frac{\gamma(1-\gamma_3+\sigma_1^2)}{\alpha(1-\gamma_3)^2\gamma} \\
q_{53} &= \frac{\gamma\Delta_1}{\alpha(1-\gamma_3)^2\gamma}
\end{align*}
\]

\[
Q_{22}(\xi) = \begin{bmatrix}
q_{11} & q_{12} & q_{13} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
q_{31} & q_{32} & q_{33} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
q_{51} & 0 & 0 & q_{54} & q_{55}
\end{bmatrix}
\]

\[
\begin{align*}
q_{11} &= -\frac{\rho\gamma_3^2}{(1-\gamma_3)[\alpha(1-\gamma_3)^2\gamma]} \\
q_{12} &= \frac{\gamma\gamma_3(\sigma_1^2+1-\gamma_3)}{(1-\gamma_3)[\alpha(1-\gamma_3)^2\gamma]} \\
q_{13} &= \frac{\gamma\gamma_3\Delta_1}{(1-\gamma_3)[\alpha(1-\gamma_3)^2\gamma]} \\
q_{31} &= \frac{\rho\gamma}{\beta(1-\gamma_3)[\alpha(1-\gamma_3)^2\gamma]} \\
q_{32} &= \frac{\gamma(\sigma_1^2+1-\gamma_3)}{(1-\gamma_3)[\alpha(1-\gamma_3)^2\gamma]} \\
q_{33} &= \frac{\gamma\Delta_2}{(1-\gamma_3)^2\gamma} \\
q_{34} &= \frac{\gamma\sigma_2^2}{(1-\gamma_3)^2\gamma} \\
q_{35} &= \frac{\gamma\Delta_1}{(1-\gamma_3)^2\gamma} \\
q_{51} &= \frac{\gamma_3^0}{\alpha(1-\gamma_3)^2\gamma} \\
q_{52} &= \frac{\gamma(1-\gamma_3+\sigma_1^2)}{\alpha(1-\gamma_3)^2\gamma} \\
q_{53} &= \frac{\gamma\Delta_1}{\alpha(1-\gamma_3)^2\gamma}
\end{align*}
\]
The following identity relationships were used in order to get the sub-system matrices $P_1$ and $P_2$ independent of the coupling parameter $\xi$:

$$\frac{1}{\alpha(1-\gamma_3) - \xi^2} = \frac{1}{\alpha(1-\gamma_3)} + \frac{\xi^2 \gamma}{\alpha(1-\gamma_3)[\alpha(1-\gamma_3) - \xi^2 \gamma]}$$

$$\frac{1}{1-\gamma - \xi^2} = \frac{1}{1-\gamma} + \frac{\xi^2 \gamma}{(1-\gamma)(1-\gamma - \xi^2 \gamma)}$$
The graphical interpretation of the interconnected subsystem $S_1$ and $S_2$ is the same as in Fig. 2.1 of the previous section.

Again, on the basis of the physical characteristic of the Skylab given at the beginning of the report, the matrices $Q_{ij}(\xi)(i, j = 1, 2)$ of (3.15) can be made independent of $\xi$ and denoted by $Q_{ij}$. This is obtained by neglecting the term $\xi^2 = 8.5 \times 10^{-4}$ with respect to the terms $1 - \gamma = 0.803$ and $\alpha(1 - \gamma_3) = 5.33$. (3.17)

Using the following specific values of the control parameters $\epsilon, \delta, \rho$:

$$
\epsilon = 2.0 \quad \delta = -1.0 \quad \rho = 1.0
$$

the same computational algorithm as in the Passive Control case can be applied to the stability analysis of the Active Control as shown in [1].

The choice of the $6 \times 6$ and $5 \times 5$ identity matrices for the $G$ matrices of the first and second subsystem results in $6 \times 6$ and $5 \times 5$ positive definite matrices $H_1, H_2$ and establishes the global asymptotic stability of the subsystems.

The interval of the coupling parameter $\xi$ obtained in [1], for which the overall system is globally exponentially stable, is relatively small due to the following reasons:

1. The inherent conservativeness of the stability analysis;
2. The choice of the matrices $G_1, G_2$ is not the "best" regarding the maximum value of $\xi$.

As in the Passive Control case, the solution to the second problem (and, therefore reducing conservativeness of the analysis) lies in the solution to the corresponding maximization problem formulated and solved in the following Section 4.
4. MAXIMIZATION OF THE STABILITY REGION

Now, several comments mentioned in previous sections about the conservativeness of the obtained result can be clarified. First, up to this point, there has been no effort made to obtain the maximum possible value $\xi_m$ of $\xi$. Secondly, there is a considerable flexibility in the choice of the matrices $G_i (i = 1, 2)$ which are only constrained to be symmetric and positive definite. This freedom in choice, however, if not used in an optimal way, might show as the big disadvantage. Furthermore, the following analysis will reveal the remarkable sensitivity of the size of stability region (that is $\xi_m$) on the choice of $G_1, G_2$ matrices.

Therefore, the following question naturally arises: How to choose subsystem Lyapunov functions such that the corresponding comparison stability inequalities will not be overly sufficient. In other words, instead of having the comparison system as constructed in [1], that is

\[
\begin{align*}
\dot{y}_1 &\leq (-0.96 \times 10^{-4} + 186.75\xi^2)v_1 + 1954.26|\xi|v_2 \\
\dot{y}_2 &\leq 392.98|\xi|v_1 + (-11.54 \times 10^{-4} + 573.86\xi^2)v_2
\end{align*}
\]

\[
A = \begin{bmatrix}
-0.96 \times 10^{-4} + 186.75\xi^2 & 1954.26|\xi| \\
392.98|\xi| & -11.54 \times 10^{-4} + 573.86\xi^2
\end{bmatrix}
\]

we would prefer the following comparison system
\[
\dot{v}_1 \leq (-\theta_0^0 + \theta_{11}^0 \xi^2) v_1 + \theta_{12}^0 |\xi| v_2
\]
\[
\dot{v}_2 \leq \theta_{21}^0 |\xi| v_1 + (-\theta_0^0 + \theta_{22}^0 \xi^2) v_2
\]
\[
A^o = \begin{bmatrix}
-\theta_0^0 + \theta_{11}^0 \xi^2 & \theta_{12}^0 |\xi| \\
\theta_{21}^0 |\xi| & -\theta_0^0 + \theta_{22}^0 \xi^2
\end{bmatrix}
\] (4.2)

in which the nonnegative numbers \( \theta_0^0, \theta_{11}^0, \theta_{12}^0, \theta_{21}^0, \theta_2^0, \theta_{22}^0 \) dependent on \( G_1, G_2 \), are chosen in such a way that

\[
A^o - A \leq 0 , \forall A \in \mathcal{H}
\]

where \( \mathcal{H} \) is the class of matrices which satisfy conditions (2.17).

Because \( v_1, v_2 \) are Liapunov functions for the subsystems, \( v_1 \geq 0 \), \( v_2 \geq 0 \) and the system (4.2) is then better than (4.1) as the comparison system since (4.2) implies (4.1) but the opposite is not true. Furthermore (4.2) gives better exponential estimate than (4.1). Obviously the system (4.2) will allow larger variation of \( \xi \) for which the comparison system will be still stable.

By introducing the norms of the coupling matrices and the absolute values of the coupling parameter \( \xi \) we already "washed out" the fine structure of the interconnections, but with the bad choice of the subsystem Liapunov functions and therefore with the corresponding estimates \( n_{i1}, n_{i2}, n_{i3}, n_{i4} (i = 1, 2) \) we could even worsen the situation.

In order to enlarge the stability region with respect to the coupling parameter \( \xi \), we can formulate the following optimization problem, that is the problem of nonlinear mathematical programming:
Problem A.

\[
\max_{\{G_1, G_2\}} |\xi|
\]

subject to the constraints: \[P_i^T H_i + H_i P_i = -G_i, \ i = 1, 2\]

\[\forall A \in H\] (4.3)

Here \(\{G_1, G_2\}\) is the set of all symmetric positive definite matrices for the two subsystems. Then for every choice of \(G_1, G_2\), the corresponding Liapunov matrix equation should be solved for \(H_1, H_2\) and then the estimates \(\eta_{i1}, \eta_{i2}, \eta_{i3}, \eta_{i4}\) can be computed from \(\lambda(H_1), \lambda(H_2), \lambda(G_i), (i = 1, 2)\) [1]. Together with computed norms of the coupling matrices, these estimates would constitute the comparison matrix \(A\). Checking the inequalities (2.17) for different values of \(\xi\) would lead to a maximum \(\xi\) for that particular choice of \(G_1, G_2\). Since the optimization is carried out over \(\{G_1, G_2\}\) this problem would be even more difficult than that of attacking the overall system directly by the same nonlinear programming approach.

But following the same stream of ideas, we can reformulate the problem in the following way: If we neglect the higher order terms since the physical coupling parameter \(\xi\) is assumed to be small of the order \(10^{-2}\) we obtain the following comparison matrix \(A\)

\[
A = 
\begin{bmatrix}
-\frac{1}{2} \frac{\lambda(G_1)}{\lambda(H_1)} & \bar{\xi}_{12} |\xi| \frac{\lambda(H_1)}{\sqrt{\lambda(H_1)^2 + \lambda(H_2)^2}} \\
\bar{\xi}_{21} |\xi| \frac{\lambda(H_2)}{\sqrt{\lambda(H_1)^2 + \lambda(H_2)^2}} & -\frac{1}{2} \frac{\lambda(G_2)}{\lambda(H_2)}
\end{bmatrix}
\] (4.4)

This actually corresponds to retaining only cross-coupling terms in (2.7) and (2.8) and neglecting the self coupling terms as the second order effect.
Now, the first stability inequality $a_{11} < 0$ is automatically satisfied and the second one gives

$$
\tilde{\xi}^2 < \frac{1}{4} \frac{\lambda(G_1) \lambda(H_1) \lambda(G_2) \lambda(H_2)}{\tilde{\xi}_{12} \tilde{\xi}_{21} \lambda^2(H_1) \lambda^2(H_2)}
$$

(4.5)

Since the norms $\tilde{\xi}_{12}, \tilde{\xi}_{21}$ of the intercoupling matrices $Q_{12}$ and $Q_{21}$ are constant independent of the choice of matrices $G_1, G_2$ the problem is then reduced to:

**Problem B:**

\[
\max_{\{G_1, G_2\}} \frac{1}{4} \frac{\lambda(G_1) \lambda(H_1) \lambda(G_2) \lambda(H_2)}{\lambda^2(H_1) \lambda^2(H_2)}
\]

subject to the constraints: $p^T_{1i} H_i + H_i p_i = -G_{1i}, i = 1, 2$ (4.6)

Since the choices of $G_1$ and $G_2$ are independent of each other, the problem can be further decomposed into two problems of the same structure, that is:

**Problem C:**

\[
\max_{G_i} \frac{1}{2} \frac{\lambda(G_i) \lambda(H_i)}{\lambda^2(H_i)}
\]

subject to the constraint: $p^T_{1i} H_i + H_i p_i = -G_{1i}, i = 1, 2$ (4.7)

The numerical solution of this problem may be still of the same kind of
difficulty as that of solving the problem directly without decomposition using mathematical programming techniques (for example, interior point-penalty function approach). That is, to maximize $\xi$ subject to the constraint that the matrix $P$ in equation (2.1) is stable. Therefore, nothing would be gained by decomposition-aggregation approach. The attempt to solve Problem C analytically involves, however, two rather serious difficulties:

1. $G_i$ and $H_i$ are related through the Liapunov matrix equation (4.7) in a quite complicated manner.

2. Dependence of the minimum and maximum eigenvalue of a symmetric positive definite matrix on its elements is numerical in nature.

Despite these drawbacks, we will follow the second route because a better look at (4.7) in some special case of subsystems will suggest how to solve the problem in general.

In order to illustrate how the conservativeness of the result can be strongly affected by the choice of the matrices $G_i (i = 1, 2)$, the following simple example is analytically treated.

Let us suppose that the subsystem matrices $P_1$ and $P_2$ to be given instead by (2.11) are

$$P_i = -I \quad (i = 1, 2) \quad (4.8)$$

where $I$ is, in this case, a $4 \times 4$ identity matrix; that would be the case when the subsystem states are completely decoupled. The Liapunov matrix equation can then be directly solved in matrix form:

$$-I^T H_i - H_i I = -G_i \quad \text{or}$$

$$H_i = \frac{1}{2} G_i \quad (i = 1, 2) \quad (4.9)$$
From this equation the relationship between the eigenvalues of $H_i$ and $G_i$ is

$$
\lambda(H_i) = \frac{1}{2} \lambda(G_i) \\
\lambda(H_i) = \frac{1}{2} \lambda(G_i) \quad (i = 1, 2) \tag{4.10}
$$

Problem C is then reduced to the following problem:

$$
\max_{G_i} \left[ \frac{\lambda(G_i)}{\lambda(G_i)} \right]^2 \quad (i = 1, 2) \tag{4.11}
$$

This reveals how the final result very strongly depends on the ratio of minimum to maximum eigenvalue of the chosen matrix $G_i (i = 1, 2)$. Actually having in mind original Problem B, dependence is to the fourth power of this ratio when $G_1 = G_2$. For example, the choice of $G_1 = G_2$ with $\lambda(G_1)/\lambda(G_1) = 10^{-2}$ would give $10^{-8}$ as the result.

In this special case of subsystem matrices, expression (4.11) not only discovers the sensitivity of the problem, but also the explicit solution to the optimization problem. From (4.11) it is obvious that the maximum is obtained when $\lambda(G_i)/\lambda(G_i) = 1$, $(i = 1, 2)$. This is the case if and only if $G_1 = g_1 I$, $G_2 = g_2 I$, $(g_1, g_2 > 0)$ giving $H_1 = \frac{g_1}{2} I$, $H_2 = \frac{g_2}{2} I$ and the maximum value 1. Therefore, the identity matrices constitute the solution set to the Problem B in this special case of subsystems.

But it would be misleading to conclude that identity matrix $I$ as a choice for $G_i (i = 1, 2)$ is the best in general as quite conservative results in reference [1] point out. Though this choice of matrix $G_i$ is very common and the easiest one, it will be shown that it is far from an optimum one.
The preceding example, however, indicates the way to approach the solution of quite formidable Problem C. Instead of looking at the problem in the original state space coordinates, we will first transform it to the domain of canonical coordinates, but in a special way. We will find the canonical transformations $T_1$ and $T_2$ for both subsystems $S_1$ and $S_2$ which will in case of complex eigenvalues $\sigma_i + j\omega_i$ ($i = 1, 2$) of $P_1$ reduce it to the following canonical form:

$$
\hat{P}_1 = \begin{bmatrix}
\sigma_1 & \omega_1 & 0 & 0 \\
-\omega_1 & \sigma_1 & 0 & 0 \\
0 & 0 & \sigma_2 & \omega_2 \\
0 & 0 & -\omega_2 & \sigma_2
\end{bmatrix}
$$

(4.12)

or in the case of mixed real and complex eigenvalues of $P_2$, $\sigma_1 + j\omega_1$, $\sigma_2$, $\sigma_3$ to the canonical form:

$$
\hat{P}_2 = \begin{bmatrix}
\sigma_1 & \omega_1 & 0 & 0 \\
-\omega_1 & \sigma_1 & 0 & 0 \\
0 & 0 & \sigma_2 & 0 \\
0 & 0 & 0 & \sigma_3
\end{bmatrix}
$$

(4.13)

The structure chosen for the canonical form induces a canonical basis of real vectors even when the eigenvalues of $P_1$ and $P_2$ are complex and therefore $T_1$ and $T_2$ are always real.

In (4.12) and (4.13) we have chosen $4 \times 4$ matrices only because of its relation to the Passive Control case and in order to simplify presentation. However, the next development and the solution of the optimization Problem C
will be carried out in general for an $n \times n$ matrix $P_1$ and $q \times q$ matrix $P_2$. The only assumption which will be made is that either all the eigenvalues of the subsystem matrices are of multiplicity one, or that the order of their multiplicity is the same as the number of linearly independent characteristic vectors corresponding to that multiple eigenvalue. This is, however, quite a realistic assumption.

Let us now choose the sign "-' (tilda) to distinguish canonical basis, canonical forms, and transformed coupling matrices from the corresponding original one.

By introducing the change of basis:

$$x_1 = T_1 \tilde{x}_1 \quad \text{and} \quad x_2 = T_2 \tilde{x}_2$$

or

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \tilde{x}$$

into the system description:

\[
\begin{align*}
\text{wobble motion } S_1: & \quad \dot{x}_1' = P_1 x_1 + \xi^2 Q_{11} x_1 + \xi Q_{12} x_2 \\
\text{spin motion } S_2: & \quad \dot{x}_2' = P_2 x_2 + \xi Q_{21} x_1 + \xi^2 Q_{22} x_2
\end{align*}
\]

we obtain in canonical coordinates:

\[
\begin{align*}
\text{wobble motion } \tilde{S}_1: & \quad \ddot{x}_1 = T_1^{-1} P_1 T_1 \tilde{x}_1 + \xi^2 T_1^{-1} Q_{11} T_1 \tilde{x}_1 + \xi T_1^{-1} Q_{12} T_2 \tilde{x}_2 \\
\text{spin motion } \tilde{S}_2: & \quad \ddot{x}_2 = T_2^{-1} P_2 T_2 \tilde{x}_2 + \xi T_2^{-1} Q_{21} T_1 \tilde{x}_1 + \xi^2 T_2^{-1} Q_{22} T_2 \tilde{x}_2
\end{align*}
\]

or the following:
Let us now show how the corresponding Liapunov matrix equations are changed under these transformations:

Since

\[ \hat{\dot{x}}_1 = \ddot{\hat{p}}_1 + \xi \dot{Q}_{11} \hat{x}_1 + \xi \dot{Q}_{12} \hat{x}_2 \tag{4.20} \]

\[ \hat{\dot{x}}_2 = \ddot{\hat{p}}_2 + \xi \dot{Q}_{21} \hat{x}_1 + \xi \dot{Q}_{22} \hat{x}_2 \tag{4.21} \]

we obtain

\[ \dddot{\hat{p}}_i + \dddot{\hat{Q}}_i = -G_i (i = 1, 2) \tag{4.22} \]

or by introducing

\[ \dddot{\hat{H}}_i = \dddot{\hat{Q}}_i, \quad \dddot{\hat{G}}_i = \dddot{\hat{Q}}_i (i = 1, 2) \tag{4.24} \]

we finally get

\[ \dddot{\hat{p}}_i H_i + \dddot{\hat{Q}}_i \hat{p}_i = -G_i (i = 1, 2) \tag{4.25} \]

The transformations in (4.24) do not change the positive definiteness character of \( \hat{H}_i \) and \( \hat{G}_i \) and we arrive at Liapunov matrix equations (4.25) in transformed domain. Note, however, that in general, matrices \( T_i \) (i = 1, 2) are not orthogonal since \( T_i^T T_i \neq I \) and eigenvalues of \( G_i \) and \( \hat{G}_i \) as well as of \( H_i \) and \( \hat{H}_i \) (i = 1, 2) are distinct. Therefore, after solving the Problem C in transformed space, it is not possible to come back to the original description and obtain the same estimate of the coupling parameter. The whole procedure should be then carried out in transformed space.

By retaining the cross-coupling terms only and neglecting self-coupling terms as the second order effect in (4.20) and (4.21) we arrive at:
Problem C:

\[
\max_{\tilde{G}} \frac{1}{2} \frac{\lambda(\tilde{G}_1) \lambda(\tilde{H}_1)}{\lambda(\tilde{H}_1) \lambda(\tilde{H}_1)}
\]

subject to:

\[
\tilde{P}_1^T \tilde{H}_1 + \tilde{H}_1 \tilde{P}_1 = -\tilde{G}_1 \quad (i = 1, 2)
\] (4.26)

Since the two problems (for \(i = 1, 2\)) are identical, we will now leave out index \(i\) for the simplicity of presentation and use \(\tilde{P}\) as a canonical representation for either subsystem.

Before attempting to solve Problem C we will prove some preliminary results.

To be concise, we will define for the \(n \times n\) matrix \(G = (g_{ij})\) an \(p \times p\) principal submatrix \(G_p\) as

\[
G_p = G\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ i_1 & i_2 & \cdots & i_p \\ \vdots & \vdots & \ddots & \vdots \\ i_1 & i_2 & \cdots & i_p \end{pmatrix} = \begin{pmatrix} g_{i_1i_1} & g_{i_1i_2} & \cdots & g_{i_1i_p} \\ g_{i_2i_1} & g_{i_2i_2} & \cdots & g_{i_2i_p} \\ \vdots & \vdots & \ddots & \vdots \\ g_{i_pi_1} & g_{i_pi_2} & \cdots & g_{i_pi_p} \end{pmatrix}
\] (4.27)

where \(1 \leq i_1 < i_2 \ldots < i_p \leq n\), and \(1 \leq p \leq n\).

We will now state and prove the following theorem we need in order to solve Problem C and known as the Inclusion Principle [5].

**Theorem 4.1:** Let \(\lambda(G)\) and \(\Lambda(G)\) be the minimum and the maximum eigenvalue of an \(n \times n\) symmetric matrix \(G\) and \(\lambda(G_p)\) and \(\Lambda(G_p)\) the minimum and maximum eigenvalues of any \(p \times p\) principal submatrix \(G_p\). Then for any
\[1 \leq p \leq n\]

\[
\Lambda(G) \geq \Lambda(G_p)
\]

\[
\lambda(G) \leq \lambda(G_p)
\]  \hspace{1cm} (4.28)

**Proof.** First we recall the variational description of the minimum and maximum eigenvalue of a symmetric matrix \( G \)

\[
\Lambda(G) = \max_{x \neq 0} \frac{x^T G x}{x^T x}
\]

\[
\lambda(G) = \min_{x \neq 0} \frac{x^T G x}{x^T x}
\]  \hspace{1cm} (4.29)

where \( x^T = (x_1, x_2, \ldots, x_n) \). We obtain the vector \( x_p \) from the vector \( x \) by simple vanishing its components \( x_i = 0 \) for all \( i \neq i_1, i_2, \ldots, i_p \) and \( 1 \leq i \leq n \), that is \( x_p^T = (0, x_{i_1}, \ldots, x_{i_2}, 0, \ldots, x_{i_1}, \ldots, 0) \).

Since the maximum on a set is no less than the maximum on any of its subsets, we have the following:

\[
\Lambda(G) = \max_{x \neq 0} \frac{x^T G x}{x^T x} \geq \max_{x_p \neq 0} \frac{x_p^T G x_p}{x_p^T x_p} = \Lambda(G_p)
\]  \hspace{1cm} (4.30)

Similarly the minimum on a set is no greater than the minimum on any of its subsets and therefore

\[
\lambda(G) = \min_{x \neq 0} \frac{x^T G x}{x^T x} \leq \min_{x_p \neq 0} \frac{x_p^T G x_p}{x_p^T x_p} = \lambda(G_p)
\]  \hspace{1cm} (4.31)

This completes the proof.
Note however, that this theorem is valid for more general case of Hermitian matrices, but for our purposes, it suffices to consider symmetric matrices since our whole analysis is in the real domain.

As a consequence of the preceding theorem, we have the following:

Let a symmetric matrix $G$ be partitioned into blocks $G = (G_{ij})$, such that $G_{ii}$ are $n_i \times n_i$ matrices, $(i, j = 1, 2, \ldots, k)$, $\sum_{i=1}^{k} n_i = n$. Denote by $G_D$ the block matrix obtained from $G$ by taking $G_{ij} \equiv 0$ (zero matrix) for $i \neq j$. Then

$$\lambda(G) \geq \max_i \lambda(G_{ii}) = \lambda(G_D), \quad 1 \leq i \leq k \quad (4.32)$$

$$\lambda(G) \leq \min_i \lambda(G_{ii}) = \lambda(G_D), \quad 1 \leq i \leq k \quad (4.33)$$

Using this result we can state and prove the following theorem.

**Theorem 4.2:** When the canonical representation $\tilde{P}$ of the subsystem $n \times n$ matrix $P$ with complex eigenvalues $\sigma_i \pm j\omega_i$, $i = 1, 2, \ldots, m$ and real eigenvalues $\sigma_i$, $i = m + 1, \ldots, k$, such that $\sigma_i < 0$, $(i = 1, \ldots, k)$, $n = m + k$, is

$$\tilde{P} = \begin{bmatrix}
\sigma_1 & \omega_1 \\
-\omega_1 & \sigma_1 \\
& \ddots & \ddots & \ddots \\
& & & & \sigma_m \\
& & & & -\omega_m & \sigma_m \\
& & & & & \sigma_{m+1} & \ddots & \ddots \\
& & & & & & \ddots & \ddots & \ddots \\
& & & & & & & & \sigma_k
\end{bmatrix} \quad (4.34)$$

then the solution $\bar{G}^*$ of the Problem $\mathcal{C}$ is
where \( g \) is an arbitrary real positive number, and the value of the maximum ratio is

\[
\frac{1}{2} \frac{\lambda(G^*) \lambda(H^*)}{\lambda^2(H^*)} = \min_i |\sigma_i| \quad 1 \leq i \leq k
\]  

(4.36)

**Proof.** In the proof we will again use 'decomposition type' approach. First we split the matrix \( \tilde{P} \) into quasi-diagonal block form, that is, we represent \( \tilde{P} \) in (4.34) as \( \tilde{P} = (\tilde{P}_{ij}) \), where

\[
\tilde{P}_{ii} = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \text{ for } i = 1, \ldots, m \quad \tilde{P}_{ii} = \sigma_i \text{ for } i = m+1, \ldots, k \\
\tilde{P}_{ij} = 0 \text{ for } i \neq j, (i, j = 1, 2, \ldots, k)
\]  

(4.37)

If we now use the same partitioning pattern for the matrices \( \tilde{H} \) and \( \tilde{G} \) from the Liapunov matrix equation

\[
\tilde{P}^T \tilde{H} + \tilde{H} \tilde{P} = -\tilde{G}
\]  

(4.38)

we obtain the system of Liapunov-like matrix equations:

\[
\tilde{P}_{ii}^T \tilde{H}_{ii} + \tilde{H}_{ii} \tilde{P}_{ii} = -\tilde{G}_{ii} \quad i = 1, 2, \ldots, k
\]  

(4.39)
\[
\tilde{p}_{ij}^T \tilde{H}_{ij} + \tilde{H}_{ij} \tilde{p}_{jj} = -\tilde{g}_{ij}, \quad i < j, \quad (i, j = 1, 2, \ldots, k)
\]  
(4.40)

The well known result that the matrix equations (4.40) have the unique solution for the unknown matrices \( \tilde{H}_{ij} \) is that \( \lambda_r(\tilde{p}_{ii}) + \lambda_\lambda(\tilde{p}_{jj}) \neq 0 \) for \( i < j, \quad (i, j = 1, 2, \ldots, k) \) and \( r, \lambda \in \{1, 2\} \). This is really the case since the subsystem matrix \( \tilde{P} \) is supposed to be stable. Therefore, if \( \tilde{g}_{ij} = 0, \quad i < j, \quad (i, j = 1, 2, \ldots, k) \) then the unique solution of (4.40) is \( \tilde{H}_{ij} = 0, \quad i < j, \quad (i, j = 1, 2, \ldots, k) \).

Suppose now that we picked some \( \tilde{G} \). We denote by \( \tilde{G}_D \) matrix obtained from \( \tilde{G} \) by letting \( \tilde{G}_{ij} = 0, \quad i < j, \quad (i, j = 1, 2, \ldots, k) \), and similarly \( \tilde{H}_D \) obtained from \( \tilde{H} \). Since \( \tilde{G}_{ij} = 0 \) implies \( \tilde{H}_{ij} = 0 \) (for \( i \neq j \)), then using the results of Theorem 4.1, that is, (4.32) and (4.33)

\[
\begin{align*}
\lambda(\tilde{G}) &\leq \lambda(\tilde{G}_D) \\
\lambda(\tilde{H}) &\leq \lambda(\tilde{H}_D) \\
\Lambda(\tilde{H}) &\geq \Lambda(\tilde{H}_D)
\end{align*}
\]  
(4.41)

From (4.41) we obtain the following inequality for the ratio \( R(\tilde{G}) \)

\[
R(\tilde{G}) = \frac{1}{2} \frac{\lambda(\tilde{G}) \lambda(\tilde{H})}{\Lambda(\tilde{H}) \Lambda(\tilde{H})} \leq \frac{1}{2} \frac{\lambda(\tilde{G}_D) \lambda(\tilde{H}_D)}{\Lambda(\tilde{H}_D) \Lambda(\tilde{H}_D)} = R(\tilde{G}_D)
\]  
(4.42)

because \( \tilde{G}, \tilde{H} \) are positive definite matrices and therefore have real positive eigenvalues.

From (4.42) we conclude that the maximum of \( R(\tilde{G}) \) over \( \tilde{G} \) will be obtained for \( \tilde{G}_D \), which implies \( \tilde{H}_D \). The proof is complete.

The equations (4.40) are then automatically satisfied and we should solve the following subproblem:
Subproblem $S$: 

$$\max_{\{G_p\}} \frac{1}{2} \frac{\lambda(G_p) \lambda(H_p)}{\lambda^2(H_p)}$$

where 

$$p^T p H_p + H_p p = -G_p$$

and 

$$p = \begin{bmatrix} \sigma_p & \omega_p \\ -\omega_p & \sigma_p \end{bmatrix} \quad (4.43)$$

Lemma 4.3. The solution to the subproblem $S$ is

$$G^*_p = g_p \begin{bmatrix} 2|\sigma_p| & 0 \\ 0 & 2|\sigma_p| \end{bmatrix} \quad (4.44)$$

where $g_p$ is an arbitrary positive number and the maximum value is $|\sigma_p|$. 

Proof. If we take $G_p = \begin{bmatrix} g_1 & g_2 \\ g_2 & g_3 \end{bmatrix}$, the subproblem $S$ can be reformulated into

$$\max_{g_1, g_3 > 0} \quad \frac{(g_1 + g_3 - \sqrt{(g_1 - g_3)^2 + 4g_2^2})}{2|\sigma_p|} \quad \frac{1}{\sqrt{\sigma_p^2 + \omega_p^2}} \quad \sqrt{(g_1 - g_3)^2 + 4g_2^2}$$

The solution to (4.45) is obtained when $\sqrt{(g_1 - g_3)^2 + 4g_2^2} = 0$ or when $g_1 = g_3$ and $g_2 = 0$, the maximum value is $|\sigma_p|$ and the proof of Lemma is completed.

Using the result of this Lemma we can now proceed to prove Theorem 4.2.
From (8.42)

\[
\max_{\tilde{G}} R(\tilde{G}) = \max_{\tilde{G}_D} R(\tilde{G}_D) = \max_{\tilde{G}_D} \frac{1}{2} \frac{\lambda(\tilde{G}_D) \lambda(H_{ii})}{\lambda^2(H_{Dii})} = \max_{\{G_{ii}\}} \frac{1}{2} \frac{\min_{i} \lambda(\tilde{G}_{ii}) \min_{i} \lambda(H_{ii})}{[\max_{i} \lambda(H_{ii})]^2}
\]

The following inequalities can be derived from (4.46)

\[
\max_{\{G_{ii}\}} \frac{1}{2} \frac{\min_{i} \lambda(\tilde{G}_{ii}) \min_{i} \lambda(H_{ii})}{[\max_{i} \lambda(H_{ii})]^2} \leq \max_{\tilde{G}_{jj}} \frac{1}{2} \frac{\lambda(\tilde{G}_{jj}) \lambda(H_{jj})}{[\lambda(H_{jj})]^2}
\]

for every \(j = 1, 2, \ldots, k\).

Since the constraining equations (4.39) are completely decoupled and using the result of Lemma 4.3

\[
\max_{\tilde{G}_{jj}} \frac{1}{2} \frac{\lambda(\tilde{G}_{jj}) \lambda(H_{jj})}{[\lambda(H_{jj})]^2} = |\sigma_j| \quad j = 1, \ldots, k
\]

(4.48)

However, inequality (4.47) is satisfied for every \(j = 1, 2, \ldots, k\), and therefore also for one which has minimum module of a real part of eigenvalue. Therefore,

\[
\max_{\tilde{G}_D} R(\tilde{G}_D) \leq \min_{j} |\sigma_j| \quad 1 \leq j \leq k
\]

(4.49)

If we now can find such a \(\tilde{G}_D^*\) that

\[
R(\tilde{G}_D^*) = \min_{j} |\sigma_j| \quad 1 \leq j \leq k
\]

(4.50)

we have found a solution to Problem \(\tilde{c}\).

This is, however, easy to accomplish simply by taking \(\tilde{G}_D^* = (\tilde{G}_{ii})\) where
\[
\tilde{G}_{ii}^* = \begin{bmatrix}
2|\sigma_i| & 0 \\
0 & 2|\sigma_i|
\end{bmatrix},
\]
which leads to
\[
\tilde{H}_{ii}^* = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
for all \(i = 1, 2, \ldots, k\). Therefore \(\tilde{H}\) is an \(n \times n\) identity matrix with \(\lambda(\tilde{H}) = \Lambda(\tilde{H}) = 1\), \(\lambda(\tilde{G}_D^*) = \min_j |\sigma_j| (j = 1, 2, \ldots, k)\), and equality (4.50) is satisfied.

Note that \(\tilde{G}^* = g \cdot \tilde{G}_D^* (g > 0)\) is also a solution to the Problem C since the ratio \(R(\tilde{G})\) is invariant under multiplication of a chosen \(\tilde{G}\) with the positive scalar. This completes the proof of Theorem 4.2.

But, still one question remains unanswered. That is, whether the solution to Problem C in the transform domain gives the greater value for the maximum allowable coupling parameters than the solution to the Problem C in the original domain? As it was noticed earlier, no direct correspondence has been established between the values \(R(G)\) and \(R(\tilde{G})\). Since \(\tilde{G} = T^TGT\), and \(T\) non-singular transformation matrix in general not orthogonal, \(T^T T \neq I\), it was only certain that these values were different. One might naturally ask the question whether there exists such a \(G^*\) in original domain which would give greater value for the ratio \(R(G^*)\) than the maximum value of the ratio \(R(G^*) = \min_j |\sigma_j| (j = 1, 2, \ldots, k)\) obtained in the canonical domain. We will answer this question negatively by proving the following theorem.

**Theorem 4.4**: Let \(G, \tilde{G}\), and \(H, \tilde{H}\), be \(n \times n\) symmetric positive definite matrices related by

\[
\tilde{G} = T^TGT, \quad \tilde{H} = T^THT
\]

\[
P^T_H + HP = -G, \quad \tilde{P}^T_H + \tilde{H}\tilde{P} = -\tilde{G}
\]

\[
\tilde{P} = T^{-1}PT, \quad T^T T \neq I
\]

where \(\tilde{P}\) is the canonical representation of the known subsystem matrix \(P\),

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and $T$ is corresponding normalized transformation. Then

$$ R(G) = \frac{1}{2} \frac{\lambda(G)}{\lambda^2(H)} \lambda(H) < \frac{1}{2} \frac{\lambda(\tilde{G})}{\lambda^2(\tilde{H})} = R(\tilde{G}) \quad (4.52) $$

**Proof.** Using the variational description of the minimum and maximum eigenvalue we have

$$ \lambda(H) = \max_{x \neq 0} \frac{x^T H x}{x^T x} = \max_{x \neq 0} \frac{\tilde{x}^T T \tilde{H} \tilde{x}}{\tilde{x}^T \tilde{x}} = \max_{x \neq 0} \frac{\tilde{x}^T \tilde{H} \tilde{x}}{\tilde{x}^T \tilde{x}} \quad (4.53) $$

since $x = T\tilde{x}$ and $T$ is nonsingular.

Now, from (4.53) we derive the following weak (greater than or equal) inequality

$$ \lambda(H) = \max_{x \neq 0} \frac{\tilde{x}^T \tilde{H} \tilde{x}}{\tilde{x}^T \tilde{x}} = \max_{x \neq 0} \frac{\tilde{x}^T \tilde{H} \tilde{x}}{\tilde{x}^T \tilde{x}} \geq \frac{\max_{x \neq 0} \frac{\tilde{x}^T \tilde{H} \tilde{x}}{\tilde{x}^T \tilde{x}}}{\min_{x \neq 0} \frac{\tilde{x}^T \tilde{T} \tilde{x}}{\tilde{x}^T \tilde{x}}} \quad (4.54) $$

Since $x^T x$ is positive definite quadratic form so is $\tilde{x}^T T \tilde{T} \tilde{x}$ and $T^T T$ is symmetric positive definite matrix. Furthermore, due to normalization of $T = (t_1, t_2, \ldots, t_n)$ where $t_i (i = 1, 2, \ldots, n)$ are column vectors, we have $\tilde{t}_1^T t_1 = 1$ $i = 1, 2, \ldots, n$.

Then

$$ T^T T = \begin{bmatrix}
1 & t_1^T t_2 & t_1^T t_3 & \cdots & t_1^T t_n \\
t_2^T t_1 & 1 & t_2^T t_3 & \cdots \\
\vdots & \vdots & 1 & \cdots & 1 \\
t_n^T t_1
\end{bmatrix} $$
Now, since $\mathbf{T}^T \neq \mathbf{I}$ at least some of the off-diagonal entries will be non-zero and direct application of the Theorem 4.1 gives

$$\lambda(\mathbf{T}^T) < 1$$

$$\lambda(\mathbf{T}^T) > 1$$ \hspace{1cm} (4.55)

Using (4.55) we get

$$\lambda(\mathbf{H}) > \frac{\lambda(\mathbf{H})}{\lambda(\mathbf{T}^T)} > \lambda(\mathbf{H})$$ \hspace{1cm} (4.56)

Similarly, for the minimum eigenvalue

$$\lambda(\mathbf{H}) = \min_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T \mathbf{H} \mathbf{x}}{\mathbf{x}^T \mathbf{T} \mathbf{T} \mathbf{x}} \leq \max_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T \mathbf{H} \mathbf{x}}{\mathbf{x}^T \mathbf{T} \mathbf{T} \mathbf{x}} = \frac{\lambda(\mathbf{H})}{\lambda(\mathbf{T}^T)} < \lambda(\mathbf{H})$$ \hspace{1cm} (4.57)

By the same procedure

$$\lambda(\mathbf{G}) < \lambda(\mathbf{\tilde{G}})$$ \hspace{1cm} (4.58)

Now from (4.56), (4.57) and (4.58), the inequality (4.52) directly follows and the theorem is proved.

Note, however, that in the very special case when $\mathbf{T}^T = \mathbf{I}$ then $\mathbf{R}(\mathbf{G}) = \mathbf{R}(\mathbf{\tilde{G}})$.

Stated in words, for every choice of $\mathbf{G}$ in original space we can find $\mathbf{\tilde{G}}$ in canonical space which would give greater (or in the very special case equal) value for ratio $\mathbf{R}(\cdot)$. Consequently, it must be true also for the maximum values, that is

$$\max_{\mathbf{G}} \mathbf{R}(\mathbf{G}) < \max_{\mathbf{\tilde{G}}} \mathbf{R}(\mathbf{\tilde{G}})$$ \hspace{1cm} (4.59)
The inequality (4.59), however, does not imply that the matrices \( \tilde{G}^* \) and \( \tilde{G}^\ast \) for which these maximums are obtained are related by \( \tilde{G}^* = T^T \tilde{G} T \). Therefore, how to find \( G^* \) and \( R(G^*) \) remains an open question, though, because of (4.59) we are not interested anymore in answering that question.

As compared to (4.4) we have the comparison matrix \( \tilde{A} \) in the canonical domain

\[
\tilde{A} = \begin{bmatrix}
\lambda(\tilde{G}_1) & \xi_{12} |\xi| & \Lambda(\tilde{H}_1) \\
-\frac{1}{2} \lambda(\tilde{H}_1) & \xi_{21} |\xi| & \frac{\Lambda(\tilde{H}_2)}{\sqrt{\lambda(\tilde{H}_1) \lambda(\tilde{H}_2)}} \\
\xi_{21} |\xi| & \frac{\Lambda(\tilde{H}_2)}{\sqrt{\lambda(\tilde{H}_1) \lambda(\tilde{H}_2)}} & -\frac{1}{2} \lambda(\tilde{G}_2)
\end{bmatrix}
\]  

(4.60)

where \( \xi_{12} \) and \( \xi_{21} \) are norms of the transformed intercoupling matrices \( \tilde{Q}_{12} \) and \( \tilde{Q}_{21} \) respectively. Due to the sparse structure of the coupling matrices \( Q_{12}, Q_{21} \) and by taking normalized transformations \( T_1, T_2 \) we obtain

\[
\xi_{12} < \bar{\xi}_{12}, \quad \xi_{21} < \bar{\xi}_{21}
\]

(4.61)

what the computer results confirm.

Similarly to (4.5), the second stability inequality gives

\[
\xi^2 < \frac{1}{4} \frac{\lambda(\tilde{G}_1) \lambda(\tilde{H}_1) \lambda(\tilde{G}_2) \lambda(\tilde{H}_2)}{\xi_{12}^2 \xi_{21}^2 \Lambda^2(\tilde{H}_1) \Lambda^2(\tilde{H}_2)}
\]

(4.62)

If we now choose \( \tilde{G}_1^* \) and \( \tilde{G}_2^* \) as proposed by Theorem 4.2 and use the result of Theorem 4.4, that is (4.59), together with (4.61), we find that the maximum stability region of the coupling parameter \( \xi \) is obtained for the comparison matrix
\[
\tilde{A}_m = \begin{bmatrix}
-|\sigma_{m1}| & \tilde{\xi}_{12}|\xi| \\
\tilde{\xi}_{21}|\xi| & -|\sigma_{m2}|
\end{bmatrix}
\] (4.63)

where \( \sigma_{m1} \) and \( \sigma_{m2} \) are the eigenvalues with the minimum module of the real part for the subsystem matrices \( P_1 \) and \( P_2 \). The largest estimate for the stability region

\[
\tilde{\xi}_m = \sqrt{\frac{\sigma_{m1} \sigma_{m2}}{\tilde{\xi}_{12} \tilde{\xi}_{21}}} \] (4.64)

is obtained for the subsystem Liapunov functions

\[
v_1 = ||\tilde{x}_1|| , \ v_2 = ||\tilde{x}_2|| \] (4.65)

Before presenting the numerical results which will show to what extent the results have been improved by this optimization approach, we will discuss some of the features introduced by the canonical transformations.

The crucial advantages of introducing the canonical transformation lie in the following:

1. The problem which did not seem to be tractable appeared in a manageable form in the canonical space, where it was "well defined" and solved.

2. Due to the relationships (4.59) and (4.61), the solution in the canonical space offered the solution to the original problem of maximizing the stability region of the parameter \( \xi \).

3. The canonical transformation is introduced at the subsystem level. The subsystems are assumed to be of the low order.
The numerical problems imposed in canonical transformation are then of much less difficulty than numerically solving the problem using nonlinear mathematical programming algorithms.

4. The simple analytical expression (4.64) not only offers the practical way of computing maximum allowable $\xi$ but more importantly, gives the significant insight into the existing trade-off between the degree of the subsystem stability and the strength of interconnections as compared to the overall system stability. This immediately suggests a possible design procedure based on introducing the controls on the subsystem level in such a way as to shift all subsystem poles as much to the left but without too much affecting the interconnections. These qualitative discussions and the existing trade-off will be made apparent in the numerical examples of Passive and Active Control case.

The computer program implementing this optimization procedure is given in the Appendix together with the obtained results for both Passive and Active Control case. The results are as follows:

**PASSIVE CONTROL CASE**

The two subsystem matrices are

$$
P_1 = \begin{bmatrix}
0 & 0.0463 & 0.4387 & 0.0131 \\
0.8478 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0 \\
0 & -1.0463 & -3.2253 & -0.0665
\end{bmatrix}
$$
The canonical forms are

\[
\begin{bmatrix}
0 & 1.0 & 0 & 0 \\
\text{-}1.7865 & \text{-}0.0534 & 0 & 2.0 \\
0 & 0 & 0 & 1.0 \\
0 & \text{-}2.0 & \text{-}904.5314 & \text{-}1.2036 \\
\end{bmatrix}
\]  

(4.66)

The computed norms of the transformed intercoupling matrices are

\[
\xi_{12} = 0.6314 < 0.7766 = \bar{\xi}_{12}
\]

\[
\xi_{21} = 1.4545 < 1.8478 = \bar{\xi}_{21}
\]  

(4.68)

The comparison matrix \( \tilde{A}_m \) is

\[
\tilde{A}_m = \begin{bmatrix}
-0.0005 & 0.6314|\xi| \\
1.4545|\xi| & -0.0266 \\
\end{bmatrix}
\]  

(4.69)

and the largest estimate of the stability region of \( \xi \)
\[|\xi| \leq \xi_m = 0.403 \times 10^{-2}\]  \hfill (4.70)

**ACTIVE CONTROL CASE**

The subsystem matrices are

\[
P_1 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0.0463 & -1.2456 & -0.4387 & -1.2456 & 1.0463 & -0.0131 \\
0 & -0.8478 & 0 & -0.1521 & 0 & 0 \\
1.0463 & -1.2456 & -3.2253 & -1.2456 & 1.0463 & -0.0665
\end{bmatrix}
\]

\[
P_2 = \begin{bmatrix}
-0.1823 & 0.0642 & 0.0019 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0.1823 & -1.8508 & -0.0553 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 \\
-2 & 0 & -2 & -0.9045314 & -1.2036
\end{bmatrix} \hfill (4.71)

The canonical forms are

\[
\bar{P}_1 = \begin{bmatrix}
-0.1288 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.8408 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.0783 & 1.7204 & 0 & 0 \\
0 & 0 & -1.7024 & -0.0783 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.0929 & 0.8990 \\
0 & 0 & 0 & 0 & -0.8990 & -0.0929
\end{bmatrix}
\]
The computed norms of the transformed intercoupling matrices are

\[
\tilde{\xi}_{12} = 14.2154 < 314.5269 = \tilde{\xi}_{12}
\]

\[
\tilde{\xi}_{21} = 3.7497 > 2.7088 = \tilde{\xi}_{21}
\]

but

\[
\tilde{\xi}_{12} \cdot \tilde{\xi}_{21} < \tilde{\xi}_{12} \tilde{\xi}_{21}
\]

The comparison matrix \( \tilde{A}_m \) is

\[
\tilde{A}_m = \begin{bmatrix}
-0.0783 & 14.2154 |\xi| \\
3.7497 |\xi| & -0.0306
\end{bmatrix}
\]

and the largest estimate of the stability region of \( \xi \) is

\[
|\xi| \leq \xi_m = 0.6714 \times 10^{-2}
\]

If we now compare the results of direct approach taken in reference [1] with the optimization approach, in both the Passive and the Active Control case, we find considerable improvement of more than \( 10^4 \) times. Besides that, random experimentation with different choices \( G_1, G_2 \) in the case without optimization, showed that chosen identity matrices for \( G_1 \) and \( G_2 \) appear to be the best.
These results reveal the significance of introducing the canonical transformation on the subsystem level. Besides the significantly improved results, the computational alternative offered by this optimization procedure seems to be very attractive and rather simple. There is no need for solving subsystem Liapunov matrix equations and computing the eigenvalues of the $G$, $H$ matrices. These computational steps are replaced by the subsystem canonical transformations, whereas the other steps of computing the norms and the estimate of the stability region of the coupling parameter $\xi$ remain unchanged.

The remarkable advantage of this approach is that the optimization problem does not require the numerical optimization procedure over all possible $G_1$, $G_2$, but is based on the similarity transformation of subsystem matrices to the properly chosen canonical form. Therefore, the low-order of the subsystems, the simplicity of the optimization procedure, and substantial enlargement of the stability region (more than $10^4$ times) offer very attractive ways to circumvent the inherent over-sufficiency of decomposition-aggregation stability procedure via the Liapunov vector functions.

Furthermore, in the following section, these theoretical as well as computational results will be directly applied to obtain optimal comparison system as suggested at the beginning of this section.
5. OPTIMUM COMPARISON SYSTEM

As we were motivated at the beginning of the previous section, we are really looking for the comparison system as given by (5.2). Before trying to formulate the optimality conditions for obtaining optimal comparison system, we will investigate some of its peculiarities.

Suppose that we have the following comparison system

\[
\begin{align*}
\dot{v}_1 &\leq -\alpha_{11} v_1 + \alpha_{12} v_2 \\
\dot{v}_2 &\leq \alpha_{21} v_1 - \alpha_{22} v_2
\end{align*}
\]

where \( \alpha_{ij} (i, j = 1, 2) \) are nonnegative numbers and \( v_1 \geq 0, v_2 \geq 0 \). Due to linearity of the comparison system (5.1) after introducing the new variables \( \tilde{v}_1, \tilde{v}_2 \) by

\[
\tilde{v}_1 = g_1 v_1 \quad \tilde{v}_2 = g_2 v_2 \quad (g_1, g_2 > 0)
\]

we obtain the comparison system

\[
\begin{align*}
\dot{\tilde{v}}_1 &\leq -\alpha_{11} \tilde{v}_1 + \alpha_{12} g \tilde{v}_2 \\
\dot{\tilde{v}}_2 &\leq \frac{1}{g} \alpha_{21} \tilde{v}_1 - \alpha_{22} \tilde{v}_2
\end{align*}
\]

where \( g = \frac{g_2}{g_1} > 0 \).

The comparison systems (5.1) and (5.3) are equivalent in the sense that one may be obtained from the other and vice versa by using (5.2). In terms
of subsystem Lyapunov functions it amounts to arbitrarily choosing scalar multiples of some known $\hat{v}_1$, $\hat{v}_2$ as the new Lyapunov functions.

If we consider $a_{ij}$ $(i, j = 1, 2)$ as the adjustable parameters depending on the matrices $G_1, G_2$ as given in (5.4), we can formulate the following criterion for the optimality of the comparison system (5.4).

**Problem D.**

Find $A^0$ such that

$$A^0 - A \leq 0, \forall A \in K$$

subject to constraints:

$$P_i^T H_i + H_i P_i = G_i, \quad i = 1, 2$$

$$\lambda(G_1) = 2|\sigma_{m1}|, \lambda(G_2) = 2|\sigma_{m2}|$$

where $K$ is the set of all matrices $\{G_1, G_2\}$ which are normalized by the condition (5.5). Here $\sigma_{m1}, \sigma_{m2}$ are as defined by (5.63).

Having in mind the equivalence between the systems (5.1) and (5.3) we can now reformulate this problem into

**Problem E.**

$$\max_{G_i} \frac{1}{2} \frac{\lambda(G_i)}{\lambda(H_i)}, \quad i = 1, 2$$

and

$$\min_{\{G_1, G_2\}} \frac{\lambda(H_1)}{\lambda(H_1)} \lambda(H_2)$$

subject to the constraints:
Following exactly the same steps as in solving Problem C and then Problem Ĉ we arrive at the same solution $\tilde{G}_1^*, \tilde{G}_2^*$ for the Problem Ė. Therefore, the obtained comparison matrix $\tilde{A}_m$ in (4.63) is also the optimal comparison matrix $A^o$.

Furthermore, with this kind of approach there is no more need to neglect the self-coupling terms as was done in arriving at (4.4) but the complete system (4.20), (4.21) can be optimized by the same optimal choice of $\tilde{G}_1^*, \tilde{G}_2^*$. This is true, because additional optimality requirements

$$\min_{G_i} \frac{\Lambda(H_i)}{\lambda(G_i)} \quad i = 1, 2$$

are satisfied for the same optimal choice of $\tilde{G}_1^*, \tilde{G}_2^*$.

Consequently, the optimal comparison matrix for the Passive Control case, taking into account the self-coupling terms, is

$$A^o = \begin{bmatrix}
-0.0005 + 0.766\xi^2 & 0.631|\xi| \\
1.454|\xi| & -0.0266 + 0.144\xi^2
\end{bmatrix}$$

and for the Active Control case is

$$A^o = \begin{bmatrix}
-0.078 + 0.165\xi^2 & 14.215|\xi| \\
3.749|\xi| & -0.030 + 0.744\xi^2
\end{bmatrix}$$

Finally, we have succeeded in obtaining the optimal comparison systems for
the mathematical model description of the Passive and Active Control Case of the Spinning Skylab and at the same time we have maximized the stability region of the important structural parameter $\xi$, which represents the ratio of the asymmetry of the boom settings and the length of the booms.
6. CONCLUSION

A decomposition-aggregation method is outlined for stability analysis of large-scale dynamic systems. The method takes advantage of special structural features of the complex systems to reduce the memory and computational time requirements. Furthermore, the method is suitable for accommodation of nonlinearities either in the subsystems or in their interconnections and provides the significant insight into the structural properties of complex dynamic systems.

The straightforward application of the Liapunov Direct Method to the stability analysis of high order multivariable dynamic systems though theoretically possible, involves formidable practical limitations. By decomposition of the system into a number of lower order subsystems, it has only been necessary to apply Liapunov Direct Method to the lower order subsystems. Therefore, the full advantage of the experience already accumulated in the construction of Liapunov functions for systems of comparatively small order can be taken. Consequently, this approach broadens the class of problems to which Liapunov Direct Method has practical application and also makes the treatment of large-scale systems feasible. However, the method is inherently conservative since a series of approximations are involved in obtaining the comparison inequalities and establishing sufficient conditions for stability. This may lead to overly restrictive (overly sufficient) conditions for stability as the results in reference [1] point out.

The decomposition-aggregation method was applied in reference [1] to the dynamic model of a spinning Skylab. After the model was decomposed into the wobble and spin subsystems, both the passive and the active control were considered. Such decomposition made an important structural parameter asymmetry in the boom settings to appear as an interconnection parameter of the two subsystems. Subsequent stability analysis was aimed at estimating the interval
of this coupling parameter for global exponential stability of the overall system. The estimates obtained in reference [1] turned out to be very conservative since the flexibility of the decomposition-aggregation method was not used to the full extent.

In this report, the problem of maximizing the stability region of the coupling parameter was formulated and resolved as a well-defined optimization problem [4]. The obtained optimality conditions turned out to be rather simple, requiring only the canonical transformation of the subsystem matrices as well as corresponding transformation of coupling matrices. Moreover, the possibility of introducing the controls at the subsystem level and still maintaining the insight into the effect on the overall stability region, provides a basis for possible design procedure. The substantial enlargement of the stability region (more than $10^4$ times compared to nonoptimal procedure outlined in reference [1]) and the simplicity of the optimization procedure offers a very attractive way to overcome the inherent over-restrictiveness of the decomposition-aggregation stability procedure via the Liapunov vector functions. As shown in the Appendix, the entire optimization procedure can be suitably programmed for computer applications.
REFERENCES


DESCRIPTION OF THE COMPUTER PROGRAM

The program for maximizing stability region of the coupling parameter \( \xi \) and obtaining the optimum comparison system is realized through the canonical transformation of the subsystem matrices to properly chosen canonical form.

The method used to compute the canonical transformation matrix \( T \) and reduce matrix \( P \) to its canonical form \( \tilde{P} \) as given by (4.34) is as follows:

The coefficients of the characteristic equation are computed using Leverrier algorithm. Then, the roots of the characteristic equation are found using standard subroutines and matrix \( \tilde{P} \) is formed. The system of linear equations

\[
PT - T\tilde{P} = 0
\]  

(A.1)

is then transformed into

\[
[P \otimes I - I \otimes \tilde{P}] t = 0
\]  

(A.2)

where the unknown vector \( t \) is formed from the columns of the matrix \( T \) and \( \otimes \) denotes Kronecker product. Since \( N \times N \) matrix \( T \) is not unique, first row is chosen arbitrarily and the corresponding system of \( (N^2 - N) \times (N^2 - N) \) linear equations is solved to obtain the transformation matrix \( T \).

Besides authors' own subroutines whose descriptions are presented here, two other IBM supplied subroutines were used: SIMQ for solving the system of linear equations and POLRT for finding the roots of the polynomial.
Subroutine ENORM

Purpose: Compute the transformed coupling matrix and its norm.

Usage: CALL ENORM (Q, T, TR, K, L, TINV, SIM, TEMP, EST)

Description of parameters:

\[ \begin{align*}
Q & \quad \text{Original coupling matrix} \\
T & \quad \text{First canonical transformation matrix of order } K \times K \\
TR & \quad \text{Second canonical transformation matrix of order } L \times L \\
K & \quad \text{Number of rows in matrix } Q \\
L & \quad \text{Number of columns in matrix } Q \\
TINV & \quad \text{Inverse matrix of the matrix } T \\
SIM & \quad \text{Vector of eigenvalues of the matrix } (T^{-1}QTR)^T(T^{-1}QTR) \\
TEMP & \quad \text{Temporary storage matrix} \\
EST & \quad \text{Norm of the transformed coupling matrix}
\end{align*} \]

Remark: The computed transformed coupling matrix \( T^{-1}QTR \) is stored in \( Q \).

Method: The norm of the matrix \( Q \) is computed as the square root of the maximum eigenvalue of the matrix \( QTQ \).

Subroutine CHVAL

Purpose: Compute the characteristic polynomial and the characteristic values of the given matrix. Then form the canonical matrix as given by (4.34).

Usage: CALL CHVAL (A, N, XCOF, COF, ROOTR, ROOTI, B, C, IER)

Description of parameters:

\[ \begin{align*}
A & \quad \text{Name of input matrix of order } N \times N \\
N & \quad \text{Order of input matrix} \\
XCOF & \quad \text{Vector of } N+1 \text{ coefficients of the characteristic polynomial of matrix } A \text{ ordered from smallest to largest power} \\
COF & \quad \text{Working vector of length } N+1 \\
ROOTR & \quad \text{Vector containing real parts of eigenvalues of } A \\
ROOTI & \quad \text{Vector containing imaginary parts of eigenvalues of } A
\end{align*} \]
**B** - Canonical matrix of a matrix A

**C** - Temporary storage matrix

**IER** - Error code where:

- **IER = 0** No error
- **IER = 1** $N$ less than one
- **IER = 2** $N$ greater than 36
- **IER = 3** Unable to determine root with 500 iterations in 5 starting values
- **IER = 4** High order coefficient is zero

**Subroutine CRONE**

**Purpose:** From the matrices $P$ and $PS$ form the matrix $P \bigotimes I - I \bigotimes PS$ where $\bigotimes$ denotes Kronecker product.

**Usage:** CALL CRONE (P, PS, CRON, N)

**Description of parameters:**

- **P** - Name of one input matrix
- **PS** - Name of second input matrix
- **CRON** - Name of output matrix $P \bigotimes I - I \bigotimes PS$ of order $N^2 \times N^2$
- **N** - Order of matrices $P$ and $PS$

**Subroutine SLINE**

**Purpose:** Find canonical transformation matrix by solving system of linear equations $PT - TPS = 0$ where $P$ and $PS$ are known $N \times N$ matrices.

**Usage:** CALL SLINE (CRON, XR, XC, N, KS)

**Description of parameters:**

- **CRON** - Input matrix of the dependent system of homogeneous linear equations $PT - TPS$
- **XR** - Input vector consisting of elements of the first row of transformation matrix $T$
- **XC** - Computed vector of coefficients of the independent system of $(N^2 - N) \times (N^2 - N)$ linear equations obtained from $PT - TPS = 0$
N - Order of matrices P and PS

KS - Output digit

0 for a normal solution

1 for a singular set of equations

Method: Divide \( N^2 \times N^2 \) matrix CRONE into 4 block matrices where diagonal block matrices are of order \( N \times N \) and \( (N^2 - N)'(N^2 - N) \). Use then second one as the coefficient matrix of the system of linear equations. Multiply then \( (N^2 - N) \times N \) off-diagonal block matrix with the known \( N \times 1 \) vector XR to compute the coefficient vector of the \( (N^2 - N) \times (N^2 - N) \) system of linear equations.
PASSIVE AND ACTIVE STABILITY OF THE SPINNING SKYLAB

MAXIMIZING THE REGION OF STABILITY WITH RESPECT TO COUPLING PARAMETER

// FOR
*ONE WORD INTEGERS
*LIST SOURCE PROGRAM

SUBROUTINE ENORM(Q,T,TR,K,L,TINV,SIM,TEMP,EST)
    DIMENSION LL(12),MM(12)
    DIMENSION Q(I),T(I),TR(I),TINV(I),SIM(I),TEMP(I)
    CALL MCPY(T,TINV,K,K,0)
    CALL MINV(TINV,K,DET,LL,MM)
    CALL GMPRD(TINV,Q,TEMP,K,K,L)
    CALL GMPRD(TEMP,TR,K,K,L)
    CALL MATA(Q,SIM,L,L,0)
    CALL EIGEN(SIM,R,L,1)
    EST=SORT(SIM(1))
    RETURN
END

FEATURES SUPPORTED
ONE WORD INTEGERS

CORE REQUIREMENTS FOR ENORM
COMMON 0 VARIABLES 30 PROGRAM 98

RELATIVE ENTRY POINT ADDRESS IS 0020 (HEX)

END OF COMPILATION
FOR
LIST SOURCE PROGRAM
ONE WORD INTEGERS

SUBROUTINE CHVAL(A, N, XCOF, COF, ROOTR, ROOTI, B, C, IER)
DIMENSION A(1), B(1), C(1), XCOF(1), COF(1), ROOTR(1), ROOTI(1)
ISUB(I, J) = (J - 1) * N + I.
N*N = N * N

DO 10 I = 1, N
10 B(I) = 0.0
DO 12 I = 1, N
J = I + (I - 1) * N
12 B(J) = 1.0
DO 20 I = 1, N
20 CALL GMPRD(A, B, C, N, N, N)

TRACE = 0.0
DO 15 K = 1, N
J = K + (K - 1) * N
15 TRACE = TRACE + C(J)

ROOTR(I) = TRACE

DO 20 CALL MCPY(C, B, N, N, 0)
COF(1) = 1.0
DO 40 K = 1, N
SUM = 0.0
DO 30 I = 1, K
L = K - I + 1
30 SUM = SUM + COF(L) * ROOTR(I)
40 COF(K + 1) = -SUM / K
M = N + 1
DO 45 I = 1, M
J = N - I + 2
45 XCOF(I) = COF(J)
CALL POLRT(XCOF, COF, N, ROOTR, ROOTI, IER)

DO 50 I = 1, N
50 ROOTI(I) = -ROOTI(I)
DO 60 I = 1, N
DO 60 J = 1, N
JJ = ISUB(I, J)
60 B(JJ) = 0.0
DO 70 I = 1, N
II = ISUB(I, I)
B(II) = ROOTR(I)
70 CONTINUE
GO TO 71, 70, 73
73 IJ = ISUB(I, I + 1)
B(IJ) = ROOTI(I)
GO TO 70
71 IJ = ISUB(I, I - 1)
B(IJ) = ROOTI(I)
RETURN
END

FEATURES SUPPORTED
ONE WORD INTEGERS

CORE REQUIREMENTS FOR CHVAL
COMMON 0 VARIABLES 20 PROGRAM 528
RELATIVE ENTRY POINT ADDRESS IS 002D (HEX)

END OF COMPILATION

// DUP

*STORE WS, UA, CHVAL
CART ID 0001 DB ADDR 5BB7 DB CNT 0023

// EJECT
SUBROUTINE CRONE (P, PS, CRON, N)
DIMENSION P(I), PS(I), CRON(I)
ISUB(I, J) = (J-1)*N + I
ISUBC(I, J) = (J-1)*N*N + I
DO 20 K = 1, N
DO 20 L = 1, N
IP = ISUB(K, L)
IF (K - L) 22, 21, 22
21 DO 23 I = 1, N
DO 23 J = 1, N
II = (K-1)*N + I
JJ = (L-1)*N + J
NN = ISUBC(II, JJ)
IPS = ISUB(J, I)
IF (I - J) 24, 25, 24
24 CRON(NN) = -PS(IPS)
25 CRON(NN) = P(IP) - PS(IPS)
GO TO 23
23 CONTINUE
GO TO 20
22 DO 26 I = 1, N
DO 26 J = 1, N
II = (K-1)*N + I
JJ = (L-1)*N + J
NN = ISUBC(II, JJ)
IF (I - J) 27, 28, 27
27 CRON(NN) = P(IP)
GO TO 26
28 CRON(NN) = 0.0
26 CONTINUE
20 CONTINUE
RETURN
END

FEATURES SUPPORTED
ONE WORD INTEGERS

CORE REQUIREMENTS FOR CRONE
COMMON 0 VARIABLES 12 PROGRAM 312

RELATIVE ENTRY POINT ADDRESS IS 0036 (HEX)

END OF COMPILATION

// DUP

*STORE WS UA CRONE
CART ID 0001 DB ADDR 5BDA DB CNT 0014

// EJECT
FOR ONE WORD INTEGERS

SUBROUTINE SLINE(CRON,XR,XC,N,KS)
DIMENSION CRON(1),XR(1),XC(1)
ISUB(I,J)=(J-1)*N+I
NN=N*N
M=N*(N-1)
MN=M*N
KK=M*N*(N-1)
DO 10 J=1,NN
DO 10 I=1,M
NI=(J-1)*NN+N+I
NJ=(J-1)*M+I
10 CRON(NJ)=CRON(NI)
CALL GMPRD(CRON,XR,XC,M,N,1)
DO 20 I=1,M
20 XC(I)=-XC(I)
DO 30 I=1,KK
LL=M*N+I
30 CRON(I)=CRON(LL)
CALL SIMQ(CRON,XC,M,KS)
DO 40 I=1,M
K=NN-I+1
L=M-I+1
40 XC(K)=XC(L)
DO 50 I=1,N
50 XC(I)=XR(I)
DO 60 J=1,N
IF(I-J) 6L,60
6L I=ISUB(I,J)
JJ=ISUB(J,I)
TEMP=XC(I)
XC(I)=XC(JJ)
XC(JJ)=TEMP
60 CONTINUE
RETURN
END

FEATURES SUPPORTED
ONE WORD INTEGERS

CORE REQUIREMENTS FOR SLINE
COMMON 0 VARIABLES 18 PROGRAM 368
RELATIVE ENTRY POINT ADDRESS IS .0025 (HEX)
END OF COMPILATION

// DUP
*STORE WS UA SLINE
CART ID 0001 DB ADDR 5BEE DB CNT 0018
// EJECT
// FOR
*LIST SOURCE PROGRAM
*IOCSICARD, 1403 PRINTER
*ONE WORD INTEGERS
REAL I1, I2, I3, MASS, K1, K2
DIMENSION P(36), PS(36), CRON(1296), XR(6), XC(36), TEMP(36), TINV(36)
DIMENSION T(36), Q(36), SIM(21), COF(7), XCOF(7), ROOTR(6), ROOTI(6)
DIMENSION SMIN(2)
C READ THE PHYSICAL PARAMETERS OF THE SKYLAB
READ(2, 50) I1, I2, I3, G2, MASS, EK1, EK2, OMEGA
50 FORMAT(8F10.0)
C COMPUTE THE NORMALIZED PARAMETERS
K1 = (I2 - I3) / I1
K2 = (I3 - I1) / I2
ALPHA = (1.0 + K1) / (1.0 - K2)
BETA = I3 / I1
GAMA = (2.0 * MASS * G2 * G2) / I1
SIGS1 = EK1 / (MASS * OMEGA * OMEGA)
SIGS2 = EK2 / (MASS * OMEGA * OMEGA)
SIGM1 = SQRT(SIGS1)
SIGM2 = SQRT(SIGS2)
DEL1 = 0.04 * SIGM1
DEL2 = 0.04 * SIGM2
GAMA1 = 1.0 - GAMA
GAMA2 = GAMA1 * GAMAI
GAMA3 = GAMA / BETA

C SUBSYSTEM ANALYSIS
C
DO 83 K = 1, 2
C READ N - THE ORDER OF THE SUBSYSTEM
READ (2, 100) N
100 FORMAT(12)
NN = N * N
101 FORMAT(8F10.0)
DD 86 J = 1, NN
86 P(J) = 0.0
GO TO (41, 42), K
C COMPUTE THE ELEMENTS OF THE FIRST SUBSYSTEM MATRIX P1
41 P(5) = (K1 + GAMA) / GAMA1
P(9) = (GAMA * SIGS1) / GAMA1
P(13) = (GAMA * DEL1) / GAMA1
P(2) = K2
P(15) = 1.0
P(8) = -(K1 + 1.0) / GAMA1
P(12) = (GAMA - SIGS1 - 1.0) / GAMA1
P(16) = -DEL1 / GAMA1
GO TO 8
C COMPUTE THE ELEMENTS OF THE SECOND SUBSYSTEM MATRIX P2
42 P(5) = 1.0
P(2) = -SIGS1
P(6) = -DEL1
P(14) = 2.0
P(15) = 1.0
P(8) = -2.0
P(12) = 1.0 - SIGS2
P(16) = -DEL2
WRITE THE SUBSYSTEM MATRIX

WRITE(5,160)
160 FORMAT('THE SYSTEM MATRIX IS',/)
DO 6 I=1,N
6 WRITE(5,200) (P(J),J=I,NN,N)
200 FORMAT(1X,12F10.4)

COMPUTE THE CHARACTERISTIC EQUATION AND ITS ROOTS
CALL CHVAL(P,N,XCOF,COF,ROOTR,ROOTI,PS,TEMP,IER)
WRITE(5,105)
IF(IER-3) 60,49,60
49 WRITE(5,300)
300 FORMAT('UNABLE TO DETERMINE ROOT WITH 500 ITERATIONS')
GO TO 90
60 WRITE(5,400)
400 FORMAT('THE ROOTS OF THE CHARACTERISTIC EQUATION ARE',/)
WRITE(5,450)
450 FORMAT('REAL PART ',5X,'IMAGINARY PART',/)
DO 80 I=1,N
80 WRITE(5,500) ROOTR(I),ROOTI(I)
500 FORMAT(F12.4,5X,F12.4,/)
WRITE(5,250)
250 FORMAT('THE SIMILAR MATRIX IS',/)
DO 40 I=1,N
40 WRITE(5,200) (PS(J),J=I,NN,N)
WRITE(5,105)

FIND THE ROOT WITH THE MAXIMUM REAL PART
SMIN(K)=ROOTR(1)
DO 81 I=1,N
IF(SMIN(K)-ROOTR(I)) 82,81,81
82 SMIN(K)=ROOTR(I)
81 CONTINUE

READ THE VECTOR XR TO DETERMINE THE UNIQUE SIMILARITY TRANSFORMATION
READ(2,101) XR
CALL CRONE(P,PS,CRON,N)
WRITE(5,105)

SOLVE THE SYSTEM OF LINEAR EQUATIONS AND COMPUTE THE SIMILARITY TRANSFORMATION MATRIX
CALL SLINE(CRON,XR,XC,N,KS)
IF(KS-1) 32,31,32
31 WRITE(5,201)
201 FORMAT('SINGULAR CASE')
GO TO 90

WRITE THE SIMILARITY TRANSFORMATION MATRIX
32 WRITE(5,305)
305 FORMAT('THE SIMILARITY TRANSFORMATION MATRIX IS',/)
DO 12 I=1,N
12 WRITE(5,103) (XC(J),J=I,NN,N)
103 FORMAT(10F12.4)
WRITE(5,105)
IF(K-1) 83,84,83
84 CONTINUE

CHANGE THE ORDER OF THE FIRST SUBSYSTEM TO VARIABLE M
M=N
DO 85 J=1,NN
85 T(J)=XC(J)
83 CONTINUE
INTERCONNECTION ANALYSIS

MN = M * N
MM = M * M

DO 61 I = 1, MM
C
COMPUTE THE TRANSFORMED SELFCOUPLING MATRIX Q11

61 Q(1) = 0.0
Q(5) = GAMMA * ((2.0 * GAMMA + K1 - 1.0) / GAMMA2)
Q(9) = SIGS1 * GAMMA * GAMMA / GAMMA2
Q(13) = (DELI * GAMMA * GAMMA) / GAMMA2
Q(2) = GAMMA * (K2 + 1.0) / ALPHA
Q(8) = -Q(5)
Q(12) = -Q(9)
Q(16) = -Q(13)
CALL ENORM (Q, T, T, M, M, TINV, SIM, TEMP, EST1)
WRITE (5, 301)
301 FORMAT (' THE TRANSFORMED SELFCOUPLING MATRIX Q11 IS',/) DO 21 I = 1, M
21 WRITE (5, 103) (Q(J), J = I, MM, M)

DO 62 I = 1, MM
C
COMPUTE THE TRANSFORMED INTERCOUPLING MATRIX Q12

62 Q(1) = 0.0
Q(5) = 2.0 * GAMMA / GAMMA1
Q(9) = -GAMMA / GAMMA1
Q(2) = GAMMA * (SIGS1 + 1.0) / ALPHA
Q(6) = GAMMA * DEL1 / ALPHA
Q(8) = -Q(5)
Q(12) = -Q(9)
CALL ENORM (Q, T, XC, M, N, TINV, SIM, TEMP, EST2)
WRITE (5, 302)
302 FORMAT (' THE TRANSFORMED INTERCOUPLING MATRIX Q12 IS',/) DO 23 I = 1, M
23 WRITE (5, 103) (Q(J), J = I, MM, M)

C
COMPUTE THE TRANSFORMED INTERCOUPLING MATRIX Q21

DO 63 I = 1, MM
63 Q(1) = 0.0
Q(2) = (1.0 + K2)
Q(8) = (2.0 * GAMMA + K1 - 1.0) / GAMMA1
Q(12) = GAMMA * SIGS1 / GAMMA1
Q(16) = GAMMA * DEL1 / GAMMA1
CALL ENORM (Q, XC, T, N, M, TINV, SIM, TEMP, EST3)
WRITE (5, 303)
303 FORMAT (' THE TRANSFORMED INTERCOUPLING MATRIX Q21 IS',/) DO 25 I = 1, N
25 WRITE (5, 103) (Q(J), J = I, MN, N)

C
COMPUTE THE TRANSFORMED SELFCOUPLING MATRIX Q22

DO 64 I = 1, NN
64 Q(1) = 0.0
Q(2) = -GAMMA * (SIGS1 + 1.0) / ALPHA
Q(6) = -GAMMA * DEL1 / ALPHA
Q(8) = 2.0 * GAMMA / GAMMA1
Q(12) = -GAMMA / GAMMA1
CALL ENORM (Q, XC, XC, N, N, TINV, SIM, TEMP, EST4)
WRITE (5, 304)
304 FORMAT (' THE TRANSFORMED SELFCOUPLING MATRIX Q22 IS',/) DO 27 I = 1, N
27 WRITE (5, 103) (Q(J), J = I, NN, N)
WRITE (5, 306)
C
WRITE THE FOUR NORMS OF THE COUPLING MATRICES
WRITE (5,108) EST1,EST2,EST3,EST4

108 FORMAT(4E16.6)
EST=EST2*EST3
ZET2=SMIN(1)*SMIN(2)/EST
ZETA=SQRT(ZET2)
WRITE(5,105)
WRITE(5,308) ZETA

308 FORMAT('THE ESTIMATE OF THE COUPLING PARAMETER IS ',E16.6)
90 CALL EXIT
END

FEATURES SUPPORTED
ONE WORD INTEGERS
IOCS

CORE REQUIREMENTS FOR
COMMON VARIABLES 3286 PROGRAM 1586

END OF COMPILATION

// XEQ
THE SYSTEM MATRIX IS

\[
\begin{bmatrix}
0.0000 & 0.0463 & 0.4387 & 0.0131 \\
0.8478 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 1.0000 \\
0.0000 & -1.0463 & -3.2253 & -0.0665
\end{bmatrix}
\]

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE

<table>
<thead>
<tr>
<th>REAL PART</th>
<th>IMAGINARY PART</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0005</td>
<td>0.2909</td>
</tr>
<tr>
<td>-0.0005</td>
<td>-0.2909</td>
</tr>
<tr>
<td>-0.0327</td>
<td>1.7607</td>
</tr>
<tr>
<td>-0.0327</td>
<td>-1.7607</td>
</tr>
</tbody>
</table>

THE SIMILAR MATRIX IS

\[
\begin{bmatrix}
-0.0005 & 0.2909 & 0.0000 & 0.0000 \\
-0.2909 & -0.0005 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -0.0327 & 1.7607 \\
0.0000 & 0.0000 & -1.7607 & -0.0327
\end{bmatrix}
\]

THE SIMILARITY TRANSFORMATION MATRIX IS

\[
\begin{bmatrix}
0.3000 & 0.3000 & 0.1000 & 0.1000 \\
0.8723 & -0.8757 & 0.0472 & -0.0490 \\
-0.2888 & 0.2935 & -0.3919 & 0.4200 \\
-0.0852 & -0.0842 & -0.7267 & -0.7039
\end{bmatrix}
\]
THE SYSTEM MATRIX IS

\[
\begin{pmatrix}
0.0000 & 1.0000 & 0.0000 & 0.0000 \\
-1.7865 & -0.0534 & 0.0000 & 2.0000 \\
0.0000 & 0.0000 & 0.0000 & 1.0000 \\
0.0000 & -2.0000 & -904.5314 & -1.2036 \\
\end{pmatrix}
\]

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE

<table>
<thead>
<tr>
<th>REAL PART</th>
<th>IMAGINARY PART</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0266</td>
<td>1.3334</td>
</tr>
<tr>
<td>-0.0266</td>
<td>-1.3334</td>
</tr>
<tr>
<td>-0.6019</td>
<td>30.1359</td>
</tr>
<tr>
<td>-0.6019</td>
<td>-30.1359</td>
</tr>
</tbody>
</table>

THE SIMILAR MATRIX IS

\[
\begin{pmatrix}
-0.0266 & 1.3334 & 0.0000 & 0.0000 \\
-1.3334 & -0.0266 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -0.6019 & 30.1359 \\
0.0000 & 0.0000 & -30.1359 & -0.6019 \\
\end{pmatrix}
\]

THE SIMILARITY TRANSFORMATION MATRIX IS

\[
\begin{pmatrix}
0.3000 & 0.3000 & 0.0030 & 0.0030 \\
-0.4080 & 0.3920 & -0.0939 & 0.0894 \\
0.0009 & -0.0008 & -0.0468 & 0.0446 \\
0.0011 & 0.0012 & -1.3186 & -1.4377 \\
\end{pmatrix}
\]
The transformed self-coupling matrix \( Q_{11} \) is

\[
\begin{bmatrix}
-0.3406 & 0.3652 & -0.0792 & 0.0853 \\
-0.3632 & 0.3407 & -0.0869 & 0.0772 \\
-0.1372 & 0.1210 & -0.0332 & 0.0270 \\
-0.1084 & 0.1260 & -0.0247 & 0.0299 \\
\end{bmatrix}
\]

The transformed intercoupling matrix \( Q_{12} \) is

\[
\begin{bmatrix}
-0.2826 & 0.3063 & -0.0517 & 0.0495 \\
-0.3164 & 0.2690 & -0.0517 & 0.0488 \\
-0.1219 & 0.0925 & -0.0190 & 0.0178 \\
-0.0870 & 0.1090 & -0.0171 & 0.0165 \\
\end{bmatrix}
\]

The transformed intercoupling matrix \( Q_{21} \) is

\[
\begin{bmatrix}
0.6862 & 0.6933 & 0.2291 & 0.2307 \\
-0.6932 & -0.6864 & -0.2307 & -0.2291 \\
0.3536 & -0.3269 & 0.0848 & -0.0738 \\
0.3431 & -0.3698 & 0.0797 & -0.0865 \\
\end{bmatrix}
\]

The transformed self-coupling matrix \( Q_{22} \) is

\[
\begin{bmatrix}
0.0354 & 0.0388 & 0.0000 & 0.0007 \\
-0.0369 & -0.0374 & -0.0002 & -0.0004 \\
0.0717 & -0.0674 & 0.0122 & -0.0116 \\
0.0737 & -0.0721 & 0.0128 & -0.0122 \\
\end{bmatrix}
\]

The four norms of the coupling matrices are

\[
\begin{align*}
0.766883\times10^0 & \quad 0.631457\times10^0 & \quad 0.145454\times10^1 & \quad 0.144751\times10^0 \\
\end{align*}
\]

The estimate of the coupling parameter is \( 0.403822\times10^{-2} \)
The system matrix is

\[
\begin{pmatrix}
0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\
0.0463 & -1.2456 & -0.4387 & -1.2456 & 1.0463 & -0.0131 \\
0.0000 & -0.8478 & 0.0000 & -0.1521 & 0.0000 & 0.0000 \\
1.0463 & -1.2456 & -3.2253 & -1.2456 & 1.0463 & -0.0665
\end{pmatrix}
\]

The roots of the characteristic equation are

<table>
<thead>
<tr>
<th>Real Part</th>
<th>Imaginary Part</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1288</td>
<td>0.0000</td>
</tr>
<tr>
<td>-0.8408</td>
<td>0.0000</td>
</tr>
<tr>
<td>-0.0783</td>
<td>1.7204</td>
</tr>
<tr>
<td>-0.0783</td>
<td>-1.7204</td>
</tr>
<tr>
<td>-0.0929</td>
<td>0.8990</td>
</tr>
<tr>
<td>-0.0929</td>
<td>-0.8990</td>
</tr>
</tbody>
</table>

The similar matrix is

\[
\begin{pmatrix}
-0.1288 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.8408 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & -0.0783 & 1.7204 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & -1.7204 & -0.0783 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0929 & 0.8990 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.8990 & -0.0929
\end{pmatrix}
\]

The similarity transformation matrix is

\[
\begin{pmatrix}
1.0000 & 1.0000 & 0.1000 & 0.1000 & 1.0000 & 1.0000 \\
0.0226 & 0.0822 & -0.0112 & 0.0133 & 0.9183 & 0.6392 \\
0.3635 & 0.4949 & 1.1220 & 0.1349 & 0.1836 & 0.0278 \\
-0.1288 & -0.8408 & -0.1798 & 0.1642 & -0.9919 & 0.8061 \\
-0.0029 & -0.0691 & -0.0220 & -0.0204 & -0.6600 & 0.7662 \\
-0.0468 & -0.4162 & -0.3200 & 1.9198 & -0.0420 & 0.1625
\end{pmatrix}
\]

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THE SYSTEM MATRIX IS

\[
\begin{pmatrix}
-0.1823 & 0.0642 & 0.0019 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\
0.1823 & -1.8508 & -0.0553 & 0.0000 & 2.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\
-2.0000 & 0.0000 & -2.0000 & -904.5314 & -1.2036 \\
\end{pmatrix}
\]

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE

<table>
<thead>
<tr>
<th>REAL PART</th>
<th>IMAGINARY PART</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1761</td>
<td>0.0000</td>
</tr>
<tr>
<td>-0.0306</td>
<td>1.3567</td>
</tr>
<tr>
<td>-0.0306</td>
<td>-1.3567</td>
</tr>
<tr>
<td>-0.6019</td>
<td>30.1359</td>
</tr>
<tr>
<td>-0.6019</td>
<td>-30.1359</td>
</tr>
</tbody>
</table>

THE SIMILARITY MATRIX IS

\[
\begin{pmatrix}
-0.1761 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & -0.0306 & 1.3567 & 0.0000 & 0.0000 \\
0.0000 & -1.3567 & -0.0306 & 0.0000 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & -0.6019 & 30.1359 \\
0.0000 & 0.0000 & 0.0000 & -30.1359 & -0.6019 \\
\end{pmatrix}
\]

THE SIMILARITY TRANSFORMATION MATRIX IS

\[
\begin{pmatrix}
1.0000 & 0.0500 & 0.0500 & 0.0000 & 0.0000 \\
0.0978 & -0.8765 & 1.2306 & -0.0011 & 0.0019 \\
-0.0172 & -1.6427 & -1.2270 & -0.0588 & -0.0349 \\
-0.0021 & 0.0035 & 0.0026 & -0.0293 & -0.0173 \\
0.0003 & -0.0036 & 0.0047 & 0.5413 & -0.8749 \\
\end{pmatrix}
\]
The transformed self-coupling matrix \( Q_{11} \) is:

\[
\begin{pmatrix}
-0.3595 & -0.1923 & -0.1468 & -0.1009 & -0.1452 & -0.9725 \\
0.3918 & 0.1893 & 0.1559 & 0.1159 & 0.1537 & 1.1068 \\
-0.0440 & -0.0154 & -0.0163 & -0.0147 & -0.0159 & -0.1378 \\
-0.0560 & 0.0084 & -0.0156 & -0.0277 & -0.0146 & -0.2486 \\
-0.0347 & -0.0496 & -0.0205 & -0.0007 & -0.0209 & -0.0217 \\
0.0126 & 0.0533 & 0.0146 & -0.0099 & 0.0155 & -0.0739
\end{pmatrix}
\]

The transformed intercoupling matrix \( Q_{12} \) is:

\[
\begin{pmatrix}
0.0189 & 0.9356 & 1.1327 & -9.0227 & -5.8354 \\
-0.0196 & -1.0610 & -1.2005 & 9.9113 & 6.4102 \\
0.0018 & 0.1310 & 0.1249 & -1.1356 & -0.7345 \\
0.0010 & 0.2315 & 0.1166 & -1.6091 & -1.0409 \\
0.0034 & 0.0269 & 0.1616 & -0.7517 & -0.4860 \\
-0.0030 & 0.0621 & -0.1180 & 0.1376 & 0.0888
\end{pmatrix}
\]

The transformed intercoupling matrix \( Q_{21} \) is:

\[
\begin{pmatrix}
-0.0068 & -0.0732 & -0.0281 & 0.0269 & 0.2685 & 0.3834 \\
0.0793 & 0.5719 & 0.1435 & -0.1338 & 0.0645 & -1.0840 \\
0.0588 & 0.4134 & 0.1050 & -0.0968 & 0.0251 & -0.7966 \\
0.4949 & 0.1576 & 0.1819 & 0.1691 & 0.1474 & 1.5636 \\
-0.8110 & -0.0792 & -0.2584 & -0.3311 & -0.2659 & -3.0315
\end{pmatrix}
\]

The transformed self-coupling matrix \( Q_{22} \) is:

\[
\begin{pmatrix}
0.0000 & -0.0000 & 0.0001 & -0.0000 & -0.0000 & 0.0000 \\
0.0014 & -0.0373 & 0.0507 & -0.0053 & -0.0033 & 0.0000 \\
0.0009 & -0.0280 & 0.0350 & 0.0076 & 0.0050 & -0.0049 \\
-0.0037 & -0.4056 & -0.2924 & 3.1543 & 2.0403 & 0.0000 \\
0.0067 & 0.6740 & 0.5100 & -5.3337 & -3.4499 & 0.0000
\end{pmatrix}
\]

The four norms of the coupling matrices are:

\[
0.165203E+01 \quad 0.142154E+02 \quad 0.374971E+01 \quad 0.744500E+01
\]

The estimate of the coupling parameter is \( 0.671487E-02 \).

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NASA-Langley, 1974
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