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STABILITY REGION MAXIMIZATION BY DECOMPOSITION-AGGREGATION METHOD

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The purpose of this work is to considerably improve the estimates of the stability regions by formulating and resolving a proper maximization problem. The solution of the problem provides the best estimate of the maximal value of the structural parameter and at the same time yields the optimum comparison system, which can be used to determine the degree of stability of the Skylab. The analysis procedure is completely computerized, resulting in a flexible and powerful tool for stability considerations of large-scale linear as well as nonlinear systems.

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TABLE OF CONTENTS

1. INTRODUCTION 1

2. PASSIVE STABILIZATION 3

3. ACTIVE STABILIZATION 11

4. MAXIMIZATION OF THE STABILITY REGION 20

5. OPTIMUM COMPARISON SYSTEM 46

6. CONCLUSION 50

REFERENCES 52

APPENDIX: 53

DESCRIPTION OF THE COMPUTER PROGRAM

Nomenclature

I_1, I_2, I_3	= principal moment of inertia of body about i coordinate $I_1^* + 2m\Gamma_2^2$, I_2^*, $I_3^* + 2m\Gamma_2^2$, respectively
I_i^*	= principal moment of inertia of rigid core body about i body - fixed coordinate
k_i	= stiffness coefficient characterizing nonrotating boom stiffness
m	= tip mass of boom
T_i	= applied torque about i coordinate
t	= time
$\dot{(\ )} = d/dt$	= differentiation with respect to real time t
$u_i = u_i^I - u_i^{II}$	= skew symmetric mode of elastic deformations
u_i^m	= displacement of m tip mass from spinning steady state in i direction $(m = I, II)$
w_i	= perturbation (about spinning steady state) velocity about i coordinate
Γ_2	= steady-state boom dimension in 2-axis direction from center of mass to tip mass
Γ_3	= the asymmetry in the setting of the booms
K_1, K_2	= ratios of inertia $(I_2-I_3)/I_1$ and $(I_3-I_1)/I_2$ respectively

$\alpha = \frac{1 + K_1}{1 - K_2} = \frac{I_2}{I_1}$	= ratio of inertia $\frac{I_2}{I_1}$
$\gamma = 2m\Gamma_2^2/I_1$	= dimensionless inertia ratio
$\Delta_i = d_i/m\Omega$	= dimensionless damping ratio
$\mu_i = u_i/2\Gamma_2$	= general skew symmetric coordinate
$\sigma_i^2 = k_i/m\Omega^2$	= dimensionless natural frequency coefficient of boom
$\xi = \frac{\Gamma_3}{\Gamma_2}$	= dimensionless length ratio
$\tau = \Omega t$	= dimensionless time
Ω	= steady-state spin rate about 3 axis
$\nu_i = \frac{w_i}{\Omega}$	= dimensionless wobble ratio (i = 1, 2, 3)
$(\acute{\ }) = d/d\tau$	= differentiation with respect to τ
subscript i	= index referring to three body-fixed coordinates (i = 1, 2, 3)
$\gamma_3 = 2m\Gamma_2^2/I_3$	= dimensionless inertia ratio
$\beta = \frac{I_3}{I_1}$	= dimensionless inertia ratio
ϕ_i	= angular rotations about i-th coordinate axis

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1. INTRODUCTION

This report is a direct continuation of the decomposition-aggregation stability analysis of the spinning Skylab presented in reference [1]. Therefore, for the mathematical basis, more detailed derivations of the state model, and computer implementations of the decomposition-aggregation analysis, this report will rely heavily on [1].

In the preceeding report [1], the linear equations of the Skylab motion, which include both the passive stabilization by extendable booms with tip masses and the active stabilization by control torques about the body fixed axes, were decomposed into two sets of equations describing the wobble motion and the spin motion. Then, two sets of equations were treated as subsystems which were interconnected by a coupling parameter representing the asymmetry in the booms setting. Stability properties of each subsystem were aggregated into a single quadratic Liapunov function. The vector Liapunov function was formed which had subsystem Liapunov functions as components. A linear second order comparison system was constructed in terms of the vector Liapunov function. Stability conditions of the aggregate comparison system provided estimates of the coupling parameter. Such an investigation is motivated by the fact that the mathematical model of the system is of high dimension and a straightforward analysis would become bogged down in the welter of detail requiring an excessive computer storage and time to complete the investigation. The multi-level decomposition-aggregation approach [2, 3] offers to solve the stability problems "piece-by-piece" and not only make more economical the computer use, but also reduce the liability of the errors in the analysis. Furthermore, by decomposing the system into parts that have important physical meaning, the decomposition-aggregation approach yields significant structural information about the behavior of the system, which is not generally available in a straightforward stability investigation.

The decomposition-aggregation stability analysis is based upon the Liapunov stability theory, and it is inherently conservative. Therefore, the stability region of the structural parameter obtained in [1] is relatively small.

The purpose of this work is to considerably improve the estimates of the stability regions by formulating and resolving a proper maximization problem [4, 5]. The solution of the problem provides the best estimate of the maximal value of the structural parameter and at the same time yields the optimum comparison system, which can be used to determine the degree of stability of the Skylab. The analysis procedure is completely computerized resulting in a flexible and powerful tool for stability considerations of large-scale linear as well as nonlinear systems.

The research reported herein was performed by S. M. Cuk under the supervision of D. D. Siljak, and was used as a part of S. M. Cuk's M.S.E.E. Thesis, at the Electrical Engineering and Computer Science Department, University of Santa Clara.

2. PASSIVE STABILIZATION

In this section, we will consider the passive stabilization of the spinning Skylab by extendable booms attached to the body of the vehicle. The linear vector state equation describing the vehicle

$$S: \quad x'(\tau) = Px(\tau) \tag{2.1}$$

is obtained in [1] from the linearized equations of motion

$$\text{wobble motion} \begin{cases} I_1 \dot{w}_1 + (I_3 - I_2)\,\Omega w_2 + m\Gamma_2\,(\ddot{u}_3 + \Omega^2 u_3) \\ -m\Gamma_3\,(2\Omega\dot{u}_1 + \ddot{u}_2 - \Omega^2 u_2) = 0 \\ (I_1 - I_3)\,\Omega w_1 + I_2 \dot{w}_2 + m\Gamma_3\,(\ddot{u}_1 - \Omega^2 u_1 - 2\Omega\dot{u}_2) = 0 \\ 2m\Gamma_2\,(\dot{w}_1 + \Omega w_2) + m\ddot{u}_3 + d_3\,\dot{u}_3 + (k_3 + m\Omega^2)\,u_3 = 0 \end{cases} \tag{2.2}$$

$$\text{spin motion} \begin{cases} 2m\Gamma_3\,(\Omega w_1 + \dot{w}_2) + m\ddot{u}_1 \\ + d_1\,\dot{u}_1 + k_1\,u_1 - 2m\Omega\dot{u}_2 = 0 \\ 2m\Gamma_3\,(-\dot{w}_1 + \Omega w_2) + 2m\Omega\dot{u}_1 + m\ddot{u}_2 \\ + d_2\,\dot{u}_2 + (k_2 - m\Omega^2)\,u_2 = 0\ , \end{cases} \tag{2.3}$$

derived in [4]. The symbols in equations (2.2-3) are introduced in the Nomenclature.

An important feature of equations (2.2-3) is that when $\Gamma_3 = 0$, they become uncoupled into two sets of equations: the wobble motion (w_1, w_2, u_3) described by (2.2); and spin motion (w_3, u_1, u_2) described by (2.3). The influence of the asymmetry in the arrangements of the booms $(\Gamma_3 \neq 0)$ will be treated as the coupling parameter between the two motions. In the decomposition-aggregation

analysis, each motion represents a subsystem, and the coupling parameter can be made to appear explicitly in the interconnections among the two subsystems.

Passive control equations (2.2-3) can be rewritten as follows:

$$\nu_1' - K_1\nu_2 + \gamma(\mu_3'' + \mu_3) - \xi\gamma(2\mu_1' + \mu_2'' - \mu_2) = 0$$

$$-K_2\alpha\nu_1 + \alpha\nu_2' + \xi\gamma(\mu_1'' - \mu_1 - 2\mu_2') = 0$$

$$\nu_1' + \nu_2 + \mu_3'' + \Delta_3\mu_3' + (\sigma_3^2 + 1)\,\mu_3 = 0 \tag{2.4}$$

$$\xi(\nu_1 + \nu_2') + \mu_1'' + \Delta_1\mu_1' + \sigma_1^2\mu_1 - 2\mu_2' = 0$$

$$\xi(-\nu_1' + \nu_2) + 2\mu_1' + \mu_2'' + \Delta_2\,\mu_2' + (\sigma_2^2 - 1)\,\mu_2 = 0\ , \tag{2.5}$$

where the notation is again as in the Nomenclature. The dimensionless parameter $\xi = \Gamma_3/\Gamma_2$ is the coupling parameter between the two sets of equations (2.4) and (2.5).

The state space representation (2.1) of the over-all system S described by (2.4-5), is obtained by choosing the state 8-vector $x(\tau)$ as

$$x(\tau) = (\nu_1\ \nu_2\ \mu_3\ \mu_3'\ \mu_1\ \mu_1'\ \mu_2\ \mu_2')^T\ . \tag{2.6}$$

The system S of equation (2.1) can be decomposed into two interconnected subsystems described by

$$\begin{matrix}\text{wobble}\\ \text{motion}\end{matrix}\quad S_1:\ x_1'(\tau) = P_1x_1(\tau) + \xi^2Q_{11}(\xi)\ x_1(\tau) + \xi Q_{12}(\xi)\ x_2(\tau) \tag{2.7}$$

$$\begin{matrix}\text{spin}\\ \text{motion}\end{matrix}\quad S_2:\ x_2'(\tau) = P_2x_2(\tau) + \xi Q_{21}(\xi)\ x_1(\tau) + \xi^2Q_{22}(\xi)\ x_2(\tau) \tag{2.8}$$

where the state vectors $x(\tau)$, $x_1(\tau)$, $x_2(\tau)$ of the system S and the two subsystems S_1 and S_2 are

$$x(\tau) = \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} ; \quad \begin{aligned} x_1(\tau) &= (\nu_1 \ \nu_2 \ \mu_3 \ \mu_3')^T \\ x_2(\tau) &= (\mu_1 \ \mu_1' \ \mu_2 \ \mu_2')^T \end{aligned} , \tag{2.9}$$

In (2.7-8), the 4×4 matrices P_1 and P_2 correspond to the "free" subsystems S_1 and S_2, and the 4×4 matrices $Q_{11}(\xi)$, $Q_{12}(\xi)$, $Q_{21}(\xi)$, and $Q_{22}(\xi)$ represent the interconnections between the two subsystems.

In order to extract the subsystem matrices P_1 and P_2 independent of the coupling parameter ξ and obtain the decomposition of (2.7-8), it was necessary to use the following identities:

$$\begin{aligned} \frac{1}{\alpha - \xi^2\gamma} &\equiv \frac{1}{\alpha} + \frac{\xi^2\gamma}{\alpha(\alpha-\xi^2\gamma)} \\ \frac{1}{\gamma - 1 + \xi^2} &\equiv \frac{1}{\gamma - 1} + \frac{\xi^2\gamma}{(1-\gamma)(\gamma-1+\xi^2\gamma)} \end{aligned} \tag{2.10}$$

The matrices of (2.7-8) are:

$$P_1 = \begin{bmatrix} 0 & p_{12}^1 & p_{13}^1 & p_{14}^1 \\ p_{21}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & p_{42}^1 & p_{43}^1 & p_{44}^1 \end{bmatrix} \quad \begin{aligned} p_{12}^1 &= - p_{42}^1 = (K_1 + \gamma)/(1 - \gamma) \\ p_{13}^1 &= \gamma \ \sigma_3^2/(1 - \gamma) \\ p_{14}^1 &= \Delta_3 \ \gamma/(1 - \gamma) \\ p_{21}^1 &= K_2 \\ p_{43}^1 &= - (\sigma_3^2 + 1 - \gamma)/(1 - \gamma) \\ p_{44}^1 &= - \Delta_3/(1 - \gamma) \end{aligned}$$

$$P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ p_{21}^2 & p_{22}^2 & 0 & p_{24}^2 \\ 0 & 0 & 0 & 1 \\ 0 & p_{42}^2 & p_{43}^2 & p_{44}^2 \end{bmatrix} \qquad \begin{aligned} p_{21}^2 &= -\sigma_1^2 \\ p_{22}^2 &= -\Delta_1 \\ p_{24}^2 &= 2 \\ p_{42}^2 &= -2 \\ p_{43}^2 &= -(\sigma_2^2 - 1) \\ p_{44}^2 &= -\Delta_2 \end{aligned}$$

$$Q_{11}(\xi) = \begin{bmatrix} 0 & q_{12}^{11} & q_{13}^{11} & q_{14}^{11} \\ q_{21}^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q_{42}^{11} & q_{43}^{11} & q_{44}^{11} \end{bmatrix} \qquad \begin{aligned} q_{12}^{11} &= -q_{42}^{11} = \frac{(2\gamma + K_1 - 1)}{(1-\gamma)(1-\gamma-\xi^2\gamma)} \\ q_{13}^{11} &= -q_{43}^{11} = \frac{\gamma^2\sigma_3^2}{(1-\gamma)(1-\gamma-\xi^2\gamma)} \\ q_{14}^{11} &= -q_{44}^{11} = \frac{\gamma^2\Delta_3}{(1-\gamma)(1-\gamma-\xi^2\gamma)} \\ q_{21}^{11} &= \frac{(K_2 + 1)}{\alpha-\xi^2\gamma} \end{aligned}$$

$$Q_{12}(\xi) = \begin{bmatrix} 0 & q_{12}^{12} & q_{13}^{12} & 0 \\ q_{21}^{12} & q_{22}^{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q_{12}^{12} & q_{13}^{12} & 0 \end{bmatrix} \qquad \begin{aligned} q_{12}^{12} &= -q_{42}^{12} = 2\gamma/(1-\gamma-\xi^2\gamma) \\ q_{13}^{12} &= -q_{43}^{12} = -\gamma/(1-\gamma-\xi^2\gamma) \\ q_{21}^{12} &= \gamma(\sigma_1^2 + 1)/(\alpha - \xi^2\gamma) \\ q_{22}^{12} &= \gamma\Delta_1/(\alpha - \xi^2\gamma) \end{aligned}$$

$$Q_{21}(\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q_{21}^{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q_{42}^{21} & q_{43}^{21} & q_{44}^{21} \end{bmatrix} \qquad \begin{aligned} q_{21}^{21} &= -(1+K_2)\alpha/(\alpha-\xi^2\gamma) \\ q_{42}^{11} &= (2\alpha+K_1-1)/(1-\gamma-\xi^2\gamma) \\ q_{43}^{21} &= \gamma\sigma_3^2/(1-\gamma-\xi^2\gamma) \\ q_{44}^{21} &= \gamma\Delta_3/(1-\gamma-\xi^2\gamma) \end{aligned}$$

$$Q_{22}(\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q_{21}^{22} & q_{22}^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q_{42}^{22} & q_{43}^{22} & 0 \end{bmatrix} \qquad \begin{aligned} q_{21}^{22} &= -\gamma(\sigma_1^2+1)/(\alpha-\xi^2\gamma) \\ q_{21}^{22} &= -\gamma\Delta_1/(\alpha-\xi^2\gamma) \\ q_{42}^{22} &= 2\gamma/(1-\gamma-\xi^2\gamma) \\ q_{43}^{22} &= -\gamma/(1-\gamma-\xi^2\gamma) \end{aligned} \tag{2.11}$$

The structural configuration of the system S as composed of the two subsystems S_1 and S_2 and the interconnections between them through the coupling parameter ξ can then be depicted as in Fig. 2.1a.

It is obvious that the system of Fig. 2.1a becomes that of Fig. 2.1b when $\xi = 0$. When $\xi \to \infty$, the system of Fig. 2.1a is again decoupled into the two subsystems shown in Fig. 2.1c, because the interconnection matrices $\xi Q_{12}(\xi)$, $\xi Q_{21}(\xi)$ and $\xi^2 Q_{11}(\xi)$, $\xi^2 Q_{22}(\xi)$ become zero and constant matrices, respectively.

The subsequent stability analysis shows that the free subsystems S_1 and S_2 $(\xi = 0)$ are stable. It is easy to check, however, that the decoupled subsystems in Fig. 2.1c are unstable. Therefore, our main objective is to determine the best estimate ξ^0 of the maximum allowable value of ξ which lies between the two extremes $\xi = 0$ and $\xi = \infty$, and for which the overall system of Fig. 2.1a is stable.

On the basis of the Skylab physical characteristics the matrices $Q_{ij}(\xi)$ $(i, j = 1, 2)$, can be made independent of ξ and denoted by Q_{ij}. This is accomplished by neglecting the term $\xi^2\gamma = 0.197\xi^2$ with respect to the terms $1 - \gamma = 0.803$ and $\alpha = 5.52$ since $\xi << 1$.

After this simplification, the numbers $\bar{\xi}_{ij}(i = 1, 2)$ of the norm of the coupling matrices Q_{ij} can be computed using

$$\bar{\xi}_{ij} = [\Lambda(Q_{ij}^T Q_{ij})]^{1/2}, \; i, j = 1, 2 \tag{2.12}$$

The subsystem Liapunov functions v_1, v_2 are chosen as

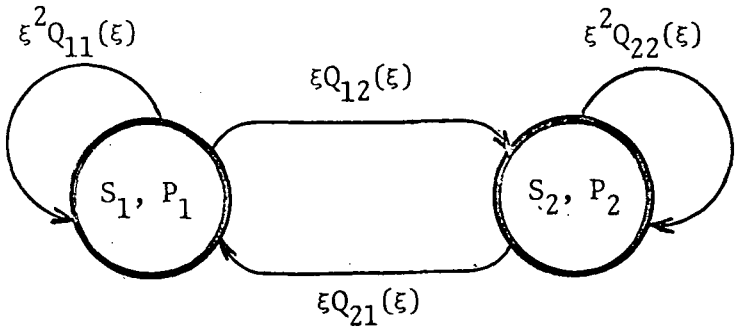
$$v_i(x) = (x_i^T H_i x_i)^{1/2}, \; i = 1, 2 \tag{2.13}$$

and the differential inequality

$$\dot{v} \leq Av \tag{2.14}$$

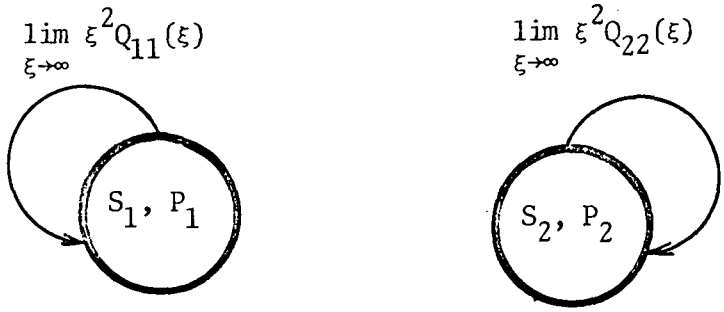
is formed following [1-3], where $v = (v_1 \; v_2)^T$ is the vector Liapunov function and the aggregate 2×2 matrix $A = (a_{ij})$ is defined as

$$A = \begin{bmatrix} -\frac{1}{2}\frac{\lambda(G_1)}{\Lambda(H_1)} + \xi^2 \frac{\Lambda(H_1)}{\lambda(H_1)} & \frac{\bar{\xi}_{12}|\xi|\Lambda(H_1)}{\lambda^{1/2}(H_1)\,\lambda^{1/2}(H_2)} \\ \frac{\bar{\xi}_{21}|\xi|\Lambda(H_2)}{\lambda^{1/2}(H_1)\,\lambda^{1/2}(H_2)} & -\frac{1}{2}\frac{\lambda(G_2)}{\Lambda(H_2)} + \xi^2 \frac{\Lambda(H_2)}{\lambda(H_2)} \end{bmatrix} \tag{2.15}$$



(a) $\xi \neq 0$

(b) $\xi = 0$



(c) $\xi = \infty$

Fig. 2.1 Structural Decomposition

In (2.15), λ and Λ denote minimum and maximum eigenvalues of the indicated matrices, respectively. The subsystem Liapunov functions in (2.13) are obtained by solving the Liapunov matrix equation

$$P_i^T H_i + H_i P_i = -G_i \, , \; i = 1, 2 \tag{2.16}$$

using the direct method of solution. From [2, 3], the conditions

$$a_{11} < 0 \; , \; \det A > 0 \; , \tag{2.17}$$

are necessary and sufficient for stability of A in (2.15), and sufficient for stability of the overall system (2.1). For every choice of positive definite symmetric matrices G_1, G_2 in (2.16) inequalities (2.17) yield an estimate of the stability region of the coupling parameter ξ . The special choice of $G_1 = G_2 = I$ in (2.16) produces positive definite matrices H_1, H_2 and establishes stability of the decoupled subsystems S_1, S_2 when $\xi = 0$ [1].

The range of the coupling parameter ξ obtained in [1] is small due to the conservativeness of the stability procedure. However, the maximum estimate of ξ could be considerably increased by a proper choice of the matrices G_i, $i = 1, 2$, in (2.16). A meaningful optimization problem can be formulated as the maximization of ξ over all matrices G_1, G_2 . In Section 4, the corresponding optimization problem is formulated and resolved. [4,5]

3. ACTIVE STABILIZATION

In order to inertially fix the axis of the Skylab pointed at the sun in presence of disturbance torques, attitude control torques must be applied to the vehicle [4]. The control torques depend on error signals that are proportional to the angle between the inertially fixed axis and the solar vector. Sun sensors and rate gyros on the present Skylab can readily provide the signals ϕ_1, ϕ_2, w_1 and w_2 needed for control.

Again, the linear vector state equation is considered

$$S: \quad x'(\tau) = Px(\tau) \ , \tag{3.1}$$

which is obtained in [1] from the linearized equations of motion

$$\text{wobble motion} \begin{cases} I_1\dot{w}_1 + (I_3 - I_2)\Omega w_2 + m\Gamma_2(\ddot{u}_3 + \Omega^2 u_3) \\ -m\Gamma_3 (2\Omega\dot{u}_1 + \ddot{u}_2 - \Omega^2 u_2) = T_1 \\ (I_1 - I_3)\Omega w_1 + I_2 \dot{w}_2 + m\Gamma_3 (\ddot{u}_1 - \Omega^2 u_1 - 2\Omega\dot{u}_2) = T_2 \\ 2m\Gamma_2 (\dot{w}_1 + \Omega w_2) + m\ddot{u}_3 + d_3 \dot{u}_3 + (k_3 + m\Omega^2) u_3 = 0 \end{cases} \tag{3.2}$$

$$\text{spin motion} \begin{cases} I_3\dot{w}_3 - m\Gamma_2 (\ddot{u}_1 - 2\Omega\dot{u}_2) = T_3 \\ 2m\Gamma_3 (\Omega w_1 + \dot{w}_2) - 2m\Gamma_2\dot{w}_3 + m\ddot{u}_1 \\ + d_1 \dot{u}_1 + k_1 u_1 - 2m\Omega\dot{u}_2 = 0 \\ 2m\Gamma_3 (-\dot{w}_1 + \Omega w_2) - 4m\Gamma_2\Omega w_3 + 2m\Omega\dot{u}_1 + m\ddot{u}_2 \\ + d_2 \dot{u}_2 + (k_2 - m\Omega^2) u_2 = 0 \ , \end{cases} \tag{3.3}$$

derived in [4] using the linear control law

$$T = \alpha\phi + \beta\omega \ . \tag{3.4}$$

In (3.4), $T = [T_1\ T_2\ T_3]^T$ is the vector of control torques; $\phi = [\phi_1\ \phi_2\ \phi_3]^T$ is the vector of angular rotations; $\omega = [w_1\ w_2\ w_3 + \Omega]^T$ is the vector of angular velocities; α, β are 3×3 matrices

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad \beta = \begin{bmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ 0 & 0 & \beta_{33} \end{bmatrix} ; \tag{3.5}$$

and kinematic relationships are

$$\omega = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \dot{\phi} + \begin{bmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \phi \ . \tag{3.6}$$

The control law in this study is chosen as

$$\alpha_{12} = I_1\Omega^2 \ , \qquad \text{all other } \alpha_{ij} = 0$$

$$\beta_{11} = I_1\Omega\delta \ , \quad \beta_{33} = -\ I_1\Omega\rho, \ \text{all other } \beta_{ij} = 0 \tag{3.7}$$

so that the normalized control torques $v = [v_1\ v_2\ v_3]^T = [T_1/I_1\Omega^2\ \ T_2/I_1\Omega^2\ \ T_3/I_1\Omega^2]^T$ are

$$v_1 = (\varepsilon + \delta)\phi_2 - \delta\ \phi_1'$$

$$v_2 = 0$$

$$v_3 = \rho\ \phi_3' \tag{3.8}$$

Referring to equations (3.2) and (3.3), the control torque T_1 is used to stabilize the subsystem S_1 (wobble motion), and the torque T_3 is used to stabilize the subsystem S_2 (spin motion).

In (3.8), ε, δ, ρ are control parameters to be selected in the stabilization process.

Upon introducing these transformations the linearized equations of motion become:

$$\phi_1'' - (1+K_1)\phi_2' - K_1\phi_1 - \gamma(\mu_3''+\mu_3) + \xi\gamma(2\mu_1'+\mu_2''-\mu_2) + (\varepsilon+\delta)\phi_2 - \delta\phi_1' = 0$$

wobble motion

$$(1+K_1)\phi_1' + K_2\alpha\phi_2 + \alpha\phi_2'' - \xi\gamma(\mu_1''-\mu_1-2\mu_2') = 0$$

$$-\phi_1'' - \phi_1 + \mu_3'' + \Delta_3\mu_3' + (\sigma_3^2 + 1)\mu_3 = 0 \qquad (3.9)$$

$$\beta\phi_3'' + \gamma(\mu_1'' - 2\mu_2') + \rho\phi_3' = 0$$

spin motion

$$-2\xi\phi_1' - \xi\phi_2'' + \phi_3'' + \mu_1'' + \Delta_1\mu_1' + \sigma_1^2\mu_1 - 2\mu_2' + \xi\phi_2 = 0$$

$$\xi\phi_1'' - 2\xi\phi_2' - \xi\phi_1 + 2\phi_3' + 2\mu_1' + \mu_2'' + \Delta_2\mu_2' + (\sigma_2^2 - 1)\mu_2 = 0 \,. \qquad (3.10)$$

The state space representation of the overall system S described by (3.9-10), is obtained by choosing the state 11-vector $x(\tau)$ as

$$x(\tau) = (\phi_1, \phi_2, \mu_3, \phi_1', \phi_2', \mu_3', \phi_3', \mu_1, \mu_1', \mu_2, \mu_2')^T \,. \qquad (3.11)$$

The system S of equation (3.1) can be decomposed into two interconnected subsystems described by:

wobble motion $\quad S_1: \; x_1'(\tau) = P_1\, x_1(\tau) + \xi^2 Q_{11}(\xi) x_1(\tau) + \xi Q_{12}(\xi) x_2(\tau)$ (3.12)

spin motion $\quad S_2: \; x_2'(\tau) = P_2\, x_2(\tau) + \xi Q_{21}(\xi) x_1(\tau) + \xi^2 Q_{22}(\xi) x_2(\tau)$ (3.13)

using the same procedure outlined in the previous section.

The state vectors $x(\tau)$, $x_1(\tau)$, $x_2(\tau)$ of the system S and two subsystems S_1 and S_2 are

$$x(\tau) = \begin{bmatrix} x_1(\tau) \\ \\ x_2(\tau) \end{bmatrix} ; \quad \begin{aligned} x_1(\tau) &= (\phi_1\ \phi_2\ \mu_3\ \phi_1'\ \phi_2'\ \mu_3')^T \\ x_2(\tau) &= (\phi_3'\ \mu_1\ \mu_1'\ \mu_2\ \mu_2')^T \end{aligned} \tag{3.14}$$

In (3.12-13) the 6×6 and 5×5 matrices P_1, P_2 correspond to the subsystems S_1 and S_2 and 6×6, 6×5, 5×6 and 5×5 matrices $Q_{11}(\xi)$, $Q_{12}(\xi)$, $Q_{21}(\xi)$, $Q_{22}(\xi)$ represent the interconnections between the two subsystems:

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ p_{41}^1 & p_{42}^1 & p_{43}^1 & p_{44}^1 & p_{45}^1 & p_{46}^1 \\ 0 & p_{52}^1 & 0 & p_{54}^1 & 0 & 0 \\ p_{61}^1 & p_{62}^1 & p_{63}^1 & p_{64}^1 & p_{65}^1 & p_{66}^1 \end{bmatrix}$$

$p^1_{41} = (K_1+\gamma)/(1-\gamma)$

$p^1_{42} = p^1_{62} = -(\varepsilon+\delta)/(1-\gamma)$

$p^1_{43} = -\gamma\sigma_3^2/(1-\gamma)$

$p^1_{44} = p^1_{64} = \delta/(1-\gamma)$

$p^1_{45} = p^1_{65} = (1+K_1)/(1-\gamma)$

$p^1_{46} = -\gamma\Delta_3/(1-\gamma)$

$p^1_{52} = -K_2$

$p^1_{54} = K_2-1$

$p^1_{61} = (1+K_1)/(1-\gamma)$

$p^1_{63} = -(\sigma_3^2 + 1-\gamma)/(1-\gamma)$

$p^1_{66} = -\Delta_3/(1-\gamma)$

$$P_2 = \begin{bmatrix} p^2_{11} & p^2_{12} & p^2_{13} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ p^2_{31} & p^2_{32} & p^2_{33} & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -2 & p^2_{54} & p^2_{55} \end{bmatrix}$$

$p^2_{11} = -\rho/(\beta-\gamma)$

$p^2_{12} = \gamma\sigma_1^2/(\beta-\gamma)$

$p^2_{13} = \gamma\Delta_1/(\beta-\gamma)$

$p^2_{31} = \rho/(\beta-\gamma)$

$p^2_{32} = -\beta\sigma_1^2/(\beta-\gamma)$

$p^2_{33} = -\beta\Delta_1/(\beta-\gamma)$

$p^2_{54} = -(\sigma_2^2-1)$

$p^2_{55} = -\Delta_2$

$$Q_{11}(\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ q_{41}^{11} & q_{42}^{11} & q_{43}^{11} & q_{44}^{11} & q_{45}^{11} & q_{46}^{11} \\ 0 & q_{52}^{11} & 0 & q_{54}^{11} & 0 & 0 \\ q_{61}^{11} & q_{62}^{11} & q_{63}^{11} & q_{64}^{11} & q_{65}^{11} & q_{66}^{11} \end{bmatrix}$$

$$q_{41}^{11} = q_{61}^{11} = \frac{\gamma(K_1+2\gamma-1)}{(1-\gamma)(1-\gamma-\xi^2\gamma)} \qquad q_{45}^{11} = q_{65}^{11} = \frac{(K_1+2\gamma-1)\gamma}{(1-\gamma)(1-\gamma-\xi^2\gamma)}$$

$$q_{42}^{11} = q_{62}^{11} = -\frac{\gamma(\varepsilon+\delta)}{(1-\gamma)(1-\gamma-\xi^2\gamma)} \qquad q_{46}^{11} = q_{66}^{11} = -\frac{\gamma^2\Delta_3}{(1-\gamma)(1-\gamma-\xi^2\gamma)}$$

$$q_{43}^{11} = q_{63}^{11} = -\frac{\gamma^2\sigma_3^2}{(1-\gamma)(1-\gamma-\xi^2\gamma)} \qquad q_{52}^{11} = -\frac{\gamma(K_2+1)}{\alpha(1-\gamma_3)-\xi^2\gamma}$$

$$q_{44}^{11} = q_{54}^{11} = \frac{\gamma\delta}{(1-\gamma)(1-\gamma-\xi^2\gamma)} \qquad q_{54}^{11} = -\frac{\gamma(3-K_2)}{\alpha(1-\gamma_3)-\xi^2\gamma}$$

$$Q_{12}(\xi) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{41}^{12} & 0 & 0 & q_{44}^{12} & q_{45}^{12} \\ q_{51}^{12} & q_{52}^{12} & q_{53}^{12} & 0 & 0 \\ q_{61}^{12} & 0 & 0 & q_{64}^{12} & q_{65}^{12} \end{bmatrix}$$

$$q_{41}^{12} = q_{61}^{12} = \frac{2\gamma}{1-\gamma-\xi^2\gamma} \qquad q_{51}^{12} = \frac{\gamma_3\rho}{\alpha(1-\gamma_3)-\xi^2\gamma}$$

$$q_{44}^{12} = q_{64}^{12} = \frac{\gamma\sigma_2^2}{1-\gamma-\xi^2\gamma} \qquad q_{52}^{12} = -\frac{\gamma(1-\gamma_3+\sigma_1^2)}{\alpha(1-\gamma_3)-\xi^2\gamma}$$

$$q_{45}^{12} = q_{65}^{12} = \frac{\gamma\Delta_2}{1-\gamma-\xi^2\gamma} \qquad q_{53}^{12} = -\frac{\gamma\Delta_1}{\alpha(1-\gamma_3)-\xi^2\gamma}.$$

$$Q_{22}(\xi) = \begin{bmatrix} q_{11}^{22} & q_{12}^{22} & q_{13}^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{31}^{22} & q_{32}^{22} & q_{33}^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{51}^{22} & 0 & 0 & q_{54}^{22} & q_{55}^{22} \end{bmatrix}$$

$$q_{11}^{22} = -\frac{\rho\gamma_3^2}{(1-\gamma_3)[\alpha(1-\gamma_3)-\xi^2\gamma]} \qquad q_{33}^{22} = -\frac{\gamma\Delta_1}{(1-\gamma_3)[\alpha(1-\gamma_3)-\xi^2\gamma]}$$

$$q_{12}^{22} = \frac{\gamma\gamma_3(\sigma_1^2+1-\gamma_3)}{(1-\gamma_3)[\alpha(1-\gamma_3)-\xi^2\gamma]} \qquad q_{51}^{22} = -\frac{2\gamma}{1-\gamma-\xi^2\gamma}$$

$$q_{13}^{22} = \frac{\gamma\gamma_3\Delta_1}{(1-\gamma_3)[\alpha(1-\gamma_3)-\xi^2\gamma]} \qquad q_{54}^{22} = -\frac{\gamma\sigma_2^2}{(1-\gamma-\xi^2\gamma)}$$

$$q_{31}^{22} = \frac{\rho\gamma}{\beta(1-\gamma_3)[\alpha(1-\gamma_3)-\xi^2\gamma]} \qquad q_{55}^{22} = -\frac{\gamma\Delta_2}{1-\gamma-\xi^2\gamma}$$

$$q_{32}^{22} = -\frac{\gamma(\sigma_1^2+1-\gamma_3)}{(1-\gamma_3)[\alpha(1-\gamma_3)-\xi^2\gamma]}$$

$$Q_{21}(\xi) = \begin{bmatrix} 0 & q_{12}^{21} & 0 & q_{14}^{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{32}^{21} & 0 & q_{34}^{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ q_{51}^{21} & q_{52}^{21} & q_{53}^{21} & q_{54}^{21} & q_{55}^{21} & q_{56}^{21} \end{bmatrix}$$

$$q_{12}^{21} = \frac{\gamma_3\alpha(K_2+1)}{(1-\gamma_3)-\xi^2\gamma} \qquad q_{52}^{21} = \frac{\varepsilon+\delta}{1-\gamma-\xi^2\gamma}$$

$$q_{14}^{21} = \frac{\gamma_3(2\alpha-1+K_1)}{\alpha(1-\gamma_3)-\xi^2\gamma} \qquad q_{53}^{21} = \frac{\gamma\sigma_3^2}{1-\gamma-\xi^2\gamma}$$

$$q_{32}^{21} = \frac{\alpha(K_2+1)}{\alpha(1-\gamma_3)-\xi^2\gamma} \qquad q_{54}^{21} = -\frac{\delta}{1-\gamma-\xi^2\gamma}$$

$$q_{34}^{21} = \frac{2\alpha-K_1-1}{\alpha(1-\gamma_3)-\xi^2\gamma} \qquad q_{55}^{21} = \frac{1-K_1-2\gamma}{1-\gamma-\xi^2\gamma}$$

$$q_{51}^{21} = \frac{1-K_1-2\gamma}{1-\gamma-\xi^2\gamma} \qquad q_{56}^{21} = \frac{\gamma\Delta_3}{1-\gamma-\xi^2\gamma} \tag{3.15}$$

The following identity relationships were used in order to get the subsystem matrices P_1 and P_2 independent of the coupling parameter ξ :

$$\frac{1}{\alpha(1-\gamma_3)-\xi^2} \equiv \frac{1}{\alpha(1-\gamma_3)} + \frac{\xi^2\gamma}{\alpha(1-\gamma_3)[\alpha(1-\gamma_3)-\xi^2\gamma]}$$

$$\frac{1}{1-\gamma-\xi^2} \equiv \frac{1}{1-\gamma} + \frac{\xi^2\gamma}{(1-\gamma)(1-\gamma-\xi^2\gamma)} \tag{3.16}$$

The graphical interpretation of the interconnected subsystem S_1 and S_2 is the same as in Fig. 2.1 of the previous section

Again, on the basis of the physical characteristic of the Skylab given at the beginning of the report, the matrices $Q_{ij}(\xi)(i, j = 1, 2)$ of (3.15) can be made independent of ξ and denoted by Q_{ij}. This is obtained by neglecting the term $\xi^2\gamma = 8.5 \times 10^{-4}$ with respect to the terms

$$1-\gamma = 0.803 \quad \text{and} \quad \alpha(1-\gamma_3) = 5.33 \ . \tag{3.17}$$

Using the following specific values of the control parameters ε, δ, ρ:

$$\varepsilon = 2.0 \qquad \delta = -1.0 \qquad \rho = 1.0 \tag{3.18}$$

the same computational algorithm as in the Passive Control case can be applied to the stability analysis of the Active Control as shown in [1].

The choice of the 6×6 and 5×5 identity matrices for the G matrices of the first and second subsystem results in 6×6 and 5×5 positive definite matrices H_1, H_2 and establishes the global asymptotic stability of the subsystems.

The interval of the coupling parameter ξ obtained in [1], for which the overall system is globally exponentially stable, is relatively small due to the following reasons:

1. The inherent conservativeness of the stability analysis;
2. The choice of the matrices G_1, G_2 is not the "best" regarding the maximum value of ξ.

As in the Passive Control case, the solution to the second problem (and, therefore reducing conservativeness of the analysis) lies in the solution to the corresponding maximization problem formulated and solved in the following Section 4.

4. MAXIMIZATION OF THE STABILITY REGION

Now, several comments mentioned in previous sections about the conservativeness of the obtained result can be clarified. First, up to this point, there has been no effort made to obtain the maximum possible value ξ_m of ξ. Secondly, there is a considerable flexibility in the choice of the matrices $G_i (i = 1, 2)$ which are only constrained to be symmetric and positive definite. This freedom in choice, however, if not used in an optimal way, might show as the big disadvantage. Furthermore, the following analysis will reveal the remarkable sensitivity of the size of stability region (that is ξ_m) on the choice of G_1, G_2 matrices.

Therefore, the following question naturally arises: How to choose subsystem Liapunov functions such that the corresponding comparison stability inequalities will not be overly sufficient. In other words, instead of having the comparison system as constructed in [1], that is

$$\dot{v}_1 \leq (-0.96 \times 10^{-4} + 186.75\xi^2)v_1 + 1954.26|\xi|v_2$$

$$\dot{v}_2 \leq 392.98|\xi|v_1 + (-11.54 \times 10^{-4} + 573.86\xi^2)v_2$$

$$A = \begin{bmatrix} -0.96 \times 10^{-4} + 186.75\xi^2 & 1954.26|\xi| \\ 392.98|\xi| & -11.54 \times 10^{-4} + 573.86\xi^2 \end{bmatrix} \tag{4.1}$$

we would prefer the following comparison system

$$\dot{v}_1 \le (-\theta_1^o + \theta_{11}^o \xi^2) v_1 + \theta_{12}^o |\xi| v_2$$

$$\dot{v}_2 \le \theta_{21}^o |\xi| v_1 + (-\theta_2^o + \theta_{22}^o \xi^2) v_2$$

$$A^o = \begin{bmatrix} -\theta_1^o + \theta_{11}^o \xi^2 & \theta_{12}^o |\xi| \\ \theta_{21}^o |\xi| & -\theta_2^o + \theta_{22}^o \xi^2 \end{bmatrix} \tag{4.2}$$

in which the nonnegative numbers θ_1^o, θ_{11}^o, θ_{12}^o, θ_{21}^o, θ_2^o, θ_{22}^o dependent on G_1, G_2, are chosen in such a way that

$$A^o - A \le 0 \ , \ \forall A \in H$$

where H is the class of matrices which satisfy conditions (2.17).

Because v_1, v_2 are Liapunov functions for the subsystems, $v_1 \ge 0$, $v_2 \ge 0$ and the system (4.2) is then better than (4.1) as the comparison system since (4.2) implies (4.1) but the opposite is not true. Furthermore (4.2) gives better exponential estimate than (4.1). Obviously the system (4.2) will allow larger variation of ξ for which the comparison system will be still stable.

By introducing the norms of the coupling matrices and the absolute values of the coupling parameter ξ we already "washed out" the fine structure of the interconnections, but with the bad choice of the subsystem Liapunov functions and therefore with the corresponding estimates η_{i1}, η_{i2}, η_{i3}, η_{i4} $(i = 1, 2)$ we could even worsen the situation.

In order to enlarge the stability region with respect to the coupling parameter ξ we can formulate the following optimization problem, that is the problem of nonlinear mathematical programming:

Problem A.

$$\max_{\{G_1,\, G_2\}} |\xi|$$

subject to the constraints: $P_i^T H_i + H_i P_i = -G_i \;,\; i = 1, 2$

$$\forall A \in H \tag{4.3}$$

Here $\{G_1, G_2\}$ is the set of all symmetric positive definite matrices for the two subsystems. Then for every choice of G_1, G_2, the corresponding Liapunov matrix equation should be solved for H_1, H_2 and then the estimates η_{i1}, η_{i2}, η_{i3}, η_{i4} can be computed from $\lambda(H_i)$, $\Lambda(H_i)$, $\lambda(G_i)$, $(i = 1, 2)$ [1]. Together with computed norms of the coupling matrices, these estimates would constitute the comparison matrix A. Checking the inequalities (2.17)for different values of ξ would lead to a maximum ξ for that particular choice of G_1, G_2 . Since the optimization is carried out over $\{G_1, G_2\}$ this problem would be even more difficult than that of attacking the overall system directly by the same nonlinear programming approach.

But following the same stream of ideas, we can reformulate the problem in the following way: If we neglect the higher order terms since the physical coupling parameter ξ is assumed to be small of the order 10^{-2} we obtain the following comparison matrix A

$$A = \begin{bmatrix} -\frac{1}{2}\frac{\lambda(G_1)}{\Lambda(H_1)} & \bar{\xi}_{12}|\xi| \frac{\Lambda(H_1)}{\sqrt{\lambda(H_1)\,\lambda(H_2)}} \\ \bar{\xi}_{21}|\xi| \frac{\Lambda(H_2)}{\sqrt{\lambda(H_1)\,\lambda(H_2)}} & -\frac{1}{2}\frac{\lambda(G_2)}{\Lambda(H_2)} \end{bmatrix} \tag{4.4}$$

This actually corresponds to retaining only cross-coupling terms in (2.7) and (2.8) and neglecting the self coupling terms as the second order effect.

Now, the first stability inequality $a_{11} < 0$ is automatically satisfied and the second one gives

$$\xi^2 < \frac{1}{4} \frac{\lambda(G_1)\ \lambda(H_1)\ \lambda(G_2)\ \lambda(H_2)}{\bar{\xi}_{12}\bar{\xi}_{21}\ \Lambda^2(H_1)\ \Lambda^2(H_2)} \tag{4.5}$$

Since the norms $\bar{\xi}_{12}$, $\bar{\xi}_{21}$ of the intercoupling matrices Q_{12} and Q_{21} are constant independent of the choice of matrices G_1 , G_2 the problem is then reduced to:

Problem B:

$$\max_{\{G_1,\ G_2\}} \frac{1}{4} \frac{\lambda(G_1)\ \lambda(H_1)\ \lambda(G_2)\ \lambda(H_2)}{\Lambda^2(H_1)\ \Lambda^2(H_2)}$$

subject to the constraints: $P_i^T H_i + H_i P_i = -G_i$, $i = 1, 2$ (4.6)

Since the choices of G_1 and G_2 are independent of each other, the problem can be further decomposed into two problems of the same structure, that is:

Problem C:

$$\max_{G_i} \frac{1}{2} \frac{\lambda(G_i)\ \lambda(H_i)}{\Lambda^2(H_i)}$$

subject to the constraint: $P_i^T H_i + H_i P_i = -G_i$, $i = 1, 2$ (4.7)

The numerical solution of this problem may be still of the same kind of

difficulty as that of solving the problem directly without decomposition using mathematical programming techniques (for example, interior point-penalty function approach). That is, to maximize ξ subject to the constraint that the matrix P in equation (2.1) is stable. Therefore, nothing would be gained by decomposition-aggregation approach. The attempt to solve Problem C analytically involves, however, two rather serious difficulties:

1. G_i and H_i are related through the Liapunov matrix equation (4.7) in a quite complicated manner.
2. Dependence of the minimum and maximum eigenvalue of a symmetric positive definite matrix on its elements is numerical in nature.

Despite these drawbacks, we will follow the second route because a better look at (4.7) in some special case of subsystems will suggest how to solve the problem in general.

In order to illustrate how the conservativeness of the result can be strongly affected by the choice of the matrices $G_i (i = 1, 2)$, the following simple example is analytically treated.

Let us suppose that the subsystem matrices P_1 and P_2 to be given instead by (2.11) are

$$P_i = -I \quad (i = 1, 2) \tag{4.8}$$

where I is, in this case, a 4×4 identity matrix; that would be the case when the subsystem states are completely decoupled. The Liapunov matrix equation can then be directly solved in matrix form:

$$-I^T H_i - H_i I = -G_i \quad \text{or}$$

$$H_i = \frac{1}{2} G_i \qquad (i = 1, 2) \tag{4.9}$$

From this equation the relationship between the eigenvalues of H_i and G_i is

$$\lambda(H_i) = \frac{1}{2}\lambda(G_i)$$

$$\Lambda(H_i) = \frac{1}{2}\Lambda(G_i) \quad (i = 1, 2) \tag{4.10}$$

Problem C is then reduced to the following problem:

$$\max_{G_i} \left[\frac{\lambda(G_i)}{\Lambda(G_i)}\right]^2 \quad (i = 1, 2) \tag{4.11}$$

This reveals how the final result very strongly depends on the ratio of minimum to maximum eigenvalue of the chosen matrix $G_i (i = 1, 2)$. Actually having in mind original *Problem B*, dependence is to the fourth power of this ratio when $G_1 = G_2$. For example, the choice of $G_1 = G_2$ with $\lambda(G_1)/\Lambda(G_1) = 10^{-2}$ would give 10^{-8} as the result.

In this special case of subsystem matrices, expression (4.11) not only discovers the sensitivity of the problem, but also the explicit solution to the optimization problem. From (4.11) it is obvious that the maximum is obtained when $\lambda(G_i)/\Lambda(G_i) = 1$, $(i = 1, 2)$. This is the case if and only if $G_1 = g_1 I$, $G_2 = g_2 I$, $(g_1, g_2 > 0)$ giving $H_1 = \frac{g_1}{2} I$, $H_2 = \frac{g_2}{2} I$ and the maximum value 1 . Therefore, the identity matrices constitute the solution set to the *Problem B* in this special case of subsystems.

But it would be misleading to conclude that identity matrix I as a choice for G_i $(i = 1, 2)$ is the best in general as quite conservative results in reference [1] point out. Though this choice of matrix G_i is very common and the easiest one, it will be shown that it is far from an optimum one.

The preceding example, however, indicates the way to approach the solution of quite formidable *Problem C*. Instead of looking at the problem in the original state space coordinates, we will first transform it to the domain of canonical coordinates, but in a special way. We will find the canonical transformations T_1 and T_2 for both subsystems S_1 and S_2 which will in case of complex eigenvalues $\sigma_i \pm j\omega_i$ $(i = 1, 2)$ of P_1 reduce it to the following canonical form:

$$\tilde{P}_1 = \begin{bmatrix} \sigma_1 & \omega_1 & 0 & 0 \\ -\omega_1 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_2 & \omega_2 \\ 0 & 0 & -\omega_2 & \sigma_2 \end{bmatrix} \tag{4.12}$$

or in the case of mixed real and complex eigenvalues of P_2, $\sigma_1 \pm j\omega_1$, σ_2, σ_3 to the canonical form:

$$\tilde{P}_2 = \begin{bmatrix} \sigma_1 & \omega_1 & 0 & 0 \\ -\omega_1 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_3 \end{bmatrix} \tag{4.13}$$

The structure chosen for the canonical form induces a canonical basis of real vectors even when the eigenvalues of P_1 and P_2 are complex and therefore T_1 and T_2 are always real.

In (4.12) and (4.13) we have chosen 4×4 matrices only because of its relation to the Passive Control case and in order to simplify presentation. However, the next development and the solution of the optimization *Problem C*

will be carried out in general for an $n \times n$ matrix P_1 and $q \times q$ matrix P_2. The only assumption which will be made is that either all the eigenvalues of the subsystem matrices are of multiplicity one, or that the order of their multiplicity is the same as the number of linearly independent characteristic vectors corresponding to that multiple eigenvalue. This is, however, quite a realistic assumption.

Let us now choose the sign "~" (tilda) to distinguish canonical basis, canonical forms, and transformed coupling matrices from the corresponding original one.

By introducing the change of basis:

$$x_1 = T_1\tilde{x}_1 \quad \text{and} \quad x_2 = T_2\tilde{x}_2 \tag{4.14}$$

or

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \tilde{x} \tag{4.15}$$

into the system description:

$$\begin{matrix}\text{wobble}\\ \text{motion}\end{matrix}\ S_1: \quad x_1' = P_1x_1 + \xi^2Q_{11}x_1 + \xi Q_{12}x_2 \tag{4.16}$$

$$\begin{matrix}\text{spin}\\ \text{motion}\end{matrix}\ S_2: \quad x_2' = P_2x_2 + \xi Q_{21}x_1 + \xi^2Q_{22}x_2 \tag{4.17}$$

we obtain in canonical coordinates:

$$\begin{matrix}\text{wobble}\\ \text{motion}\end{matrix}\ \tilde{S}_1: \quad \tilde{x}_1' = T_1^{-1}P_1T_1\tilde{x}_1 + \xi^2T_1^{-1}Q_{11}T_1\tilde{x}_1 + \xi T_1^{-1}Q_{12}T_2\tilde{x}_2 \tag{4.18}$$

$$\begin{matrix}\text{spin}\\ \text{motion}\end{matrix}\ \tilde{S}_2: \quad \tilde{x}_2' = T_2^{-1}P_2T_2\tilde{x}_2 + \xi T_2^{-1}Q_{21}T_1\tilde{x}_1 + \xi^2T_2^{-1}Q_{22}T_2\tilde{x}_2 \tag{4.19}$$

or the following:

$$\text{wobble motion } \tilde{S}_1: \quad \tilde{x}_1' = \tilde{P}_1\tilde{x}_1 + \xi^2\tilde{Q}_{11}\tilde{x}_1 + \xi\tilde{Q}_{12}\tilde{x}_2 \tag{4.20}$$

$$\text{spin motion } \tilde{S}_2: \quad \tilde{x}_2' = \tilde{P}_2\tilde{x}_2 + \xi\tilde{Q}_{21}\tilde{x}_1 + \xi^2\tilde{Q}_{22}\tilde{x}_2 \tag{4.21}$$

Let us now show how the corresponding Liapunov matrix equations are changed under these transformations:

Since

$$\tilde{P}_i = T_i^{-1}P_iT_i \quad \text{and} \quad P_i^TH_i + H_iP_i = -G_i \quad (i = 1, 2) \tag{4.22}$$

we obtain

$$\tilde{P}_i^TT_i^TH_iT_i + T_i^TH_iT_i\tilde{P}_i = -T_i^TG_iT_i \quad (i = 1, 2) \tag{4.23}$$

or by introducing

$$\tilde{H}_i = T_i^TH_iT_i\ , \ \tilde{G}_i = T_i^TG_iT_i \quad (i = 1, 2) \tag{4.24}$$

we finally get

$$\tilde{P}_i^T\tilde{H}_i + \tilde{H}_i\tilde{P}_i = -\tilde{G}_i \quad (i = 1, 2) \tag{4.25}$$

The transformations in (4.24) do not change the positive definiteness character of $\tilde{H}_i$ and $\tilde{G}_i$ and we arrive at Liapunov matrix equations (4.25) in transformed domain. Note, however, that in general, matrices T_i $(i = 1, 2)$ are not orthogonal since $T_i^TT_i \neq I$ and eigenvalues of G_i and $\tilde{G}_i$ as well as of H_i and $\tilde{H}_i$ $(i = 1, 2)$ are distinct. Therefore, after solving the *Problem C* in transformed space, it is not possible to come back to the original description and obtain the same estimate of the coupling parameter. The whole procedure should be then carried out in transformed space.

By retaining the cross-coupling terms only and neglecting self-coupling terms as the second order effect in (4.20) and (4.21) we arrive at:

Problem $\tilde{C}$:

$$\max_{\tilde{G}_i} \frac{1}{2} \frac{\lambda(\tilde{G}_i)\,\lambda(\tilde{H}_i)}{\Lambda(\tilde{H}_i)\,\Lambda(\tilde{H}_i)}$$

subject to: $\tilde{P}_i^T\tilde{H}_i + \tilde{H}_i\tilde{P}_i = -\tilde{G}_i \quad (i = 1, 2)$ (4.26)

Since the two problems (for i = 1, 2) are identical, we will now leave out index i for the simplicity of presentation and use $\tilde{P}$ as a canonical representation for either subsystem.

Before attempting to solve *Problem* $\tilde{C}$ we will prove some preliminary results.

To be concise, we will define for the $n \times n$ matrix $G = (g_{ij})$ an $p \times p$ principal submatrix G_p as

$$G_p = G\begin{pmatrix} i_1 i_2 \cdots i_p \\ i_1 i_2 \cdots i_p \end{pmatrix} = \begin{bmatrix} g_{i_1 i_1} & g_{i_1 i_2} & \cdots & g_{i_1 i_p} \\ g_{i_2 i_1} & g_{i_2 i_2} & \cdots & g_{i_2 i_p} \\ \vdots & & & \\ g_{i_p i_1} & g_{i_p i_2} & \cdots & g_{i_p i_p} \end{bmatrix} \tag{4.27}$$

where $1 \le i_1 < i_2 \ldots < i_p \le n$, and $1 \le p \le n$.

We will now state and prove the following theorem we need in order to solve *Problem* $\tilde{C}$ and known as the Inclusion Principle [5].

Theorem 4.1: Let $\lambda(G)$ *and* $\Lambda(G)$ *be the minimum and the maximum eigenvalue of an* $n \times n$ *symmetric matrix* G *and* $\lambda(G_p)$ *and* $\Lambda(G_p)$ *the minimum and maximum eigenvalues of any* $p \times p$ *principal submatrix* G_p . *Then for any*

$1 \leq p \leq n$

$$\Lambda(G) \geq \Lambda(G_p)$$

$$\lambda(G) \leq \lambda(G_p) \tag{4.28}$$

Proof. First we recall the variational description of the minimum and maximum eigenvalue of a symmetric matrix G

$$\Lambda(G) = \max_{x \neq 0} \frac{x^T G x}{x^T x}$$

$$\lambda(G) = \min_{x \neq 0} \frac{x^T G x}{x^T x} \tag{4.29}$$

where $x^T = (x_1, x_2, \ldots, x_n)$. We obtain the vector x^p from the vector x by simple vanishing its components $x_i = 0$ for all $i \neq i_1, i_2, \ldots, i_p$ and $1 \leq i \leq n$, that is $x^{pT} = (0, x_{i_1}, \ldots, x_{i_2}, 0, \ldots, x_{i_p}, \ldots, 0)$.

Since the maximum on a set is no less than the maximum on any of its subsets, we have the following:

$$\Lambda(G) = \max_{x \neq 0} \frac{x^T G x}{x^T x} \geq \max_{x^p \neq 0} \frac{x^{pT} G x^p}{x^{pT} x^p} = \Lambda(G_p) \tag{4.30}$$

Similarly the minimum on a set is no greater than the minimum on any of its subsets and therefore

$$\lambda(G) = \min_{x \neq 0} \frac{x^T G x}{x^T x} \leq \min_{x^p \neq 0} \frac{x^{pT} G x^p}{x^{pT} x^p} = \lambda(G_p) \tag{4.31}$$

This completes the proof.

Note however, that this theorem is valid for more general case of Hermitian matrices, but for our purposes, it suffices to consider symmetric matrices since our whole analysis is in the real domain.

As a consequence of the preceding theorem, we have the following:

Let a symmetric matrix G be partitioned into blocks $G = (G_{ij})$, such that G_{ii} are $n_i \times n_i$ matrices, $(i, j = 1, 2, \ldots, k)$, $\sum_{i=1}^{k} n_i = n$. Denote by G_D the block matrix obtained from G by taking $G_{ij} \equiv 0$ (zero matrix) for $i \neq j$. Then

$$\Lambda(G) \geq \max_i \Lambda(G_{ii}) = \Lambda(G_D) \quad , \quad 1 \leq i \leq k \tag{4.32}$$

$$\lambda(G) \leq \min_i \lambda(G_{ii}) = \lambda(G_D) \quad , \quad 1 \leq i \leq k \tag{4.33}$$

Using this result we can state and prove the following theorem.

<u>Theorem 4.2</u>: *When the canonical representation* $\tilde{P}$ *of the subsystem* $n \times n$ *matrix* P *with complex eigenvalues* $\sigma_i \pm j\omega_i$, $i = 1, 2, \ldots, m$ *and real eigenvalues* σ_i, $i = m + 1, \ldots, k$, *such that* $\sigma_i < 0$, $(i = 1, \ldots, k)$, $n = m + k$ *is*

$$\tilde{P} = \begin{bmatrix} \sigma_1 & \omega_1 & & & & & & \\ -\omega_1 & \sigma_1 & & & & & \bigcirc & \\ & & \ddots & & & & & \\ & & & \sigma_m & \omega_m & & & \\ & & & -\omega_m & \sigma_m & & & \\ & & & & & \sigma_{m+1} & & \\ & \bigcirc & & & & & \ddots & \\ & & & & & & & \sigma_k \end{bmatrix} \tag{4.34}$$

then the solution $\tilde{G}^*$ *of the Problem* $\tilde{C}$ *is*

$$\tilde{G}^* = 2g \begin{bmatrix} |\sigma_1| & & & & & & & \\ & |\sigma_1| & & & & & \bigcirc & \\ & & \ddots & & & & & \\ & & & |\sigma_m| & & & & \\ & & & & |\sigma_m| & & & \\ & & & & & |\sigma_{m+1}| & & \\ & \bigcirc & & & & & \ddots & \\ & & & & & & & |\sigma_k| \end{bmatrix} \tag{4.35}$$

where g *is an arbitrary real positive number, and the value of the maximum ratio is*

$$\frac{1}{2} \quad \frac{\lambda(\tilde{G}^*)\ \lambda(\tilde{H}^*)}{\Lambda^2(\tilde{H}^*)} = \min_i |\sigma_i| \qquad 1 \le i \le k \tag{4.36}$$

Proof. In the proof we will again use "decomposition type" approach. First we split the matrix $\tilde{P}$ into quasi-diagonal block form, that is, we represent $\tilde{P}$ in (4.34) as $\tilde{P} = (\tilde{P}_{ij})$, where

$$\tilde{P}_{ii} = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \text{ for } i = 1, \ldots, m \quad \tilde{P}_{ii} = \sigma_i \text{ for } i = m+1, \ldots, k$$

$$\tilde{P}_{ij} \equiv 0 \text{ for } i \neq j\ , \ (i, j = 1, 2, \ldots, k) \tag{4.37}$$

If we now use the same partitioning pattern for the matrices $\tilde{H}$ and $\tilde{G}$ from the Liapunov matrix equation

$$\tilde{P}^T\tilde{H} + \tilde{H}\tilde{P} = -\tilde{G} \tag{4.38}$$

we obtain the system of Liapunov-like matrix equations:

$$\tilde{P}_{ii}^T\tilde{H}_{ii} + \tilde{H}_{ii}\tilde{P}_{ii} = -\tilde{G}_{ii} \qquad i = 1, 2, \ldots, k \tag{4.39}$$

$$\tilde{P}_{ii}^T \tilde{H}_{ij} + \tilde{H}_{ij} \tilde{P}_{jj} = -\tilde{G}_{ij} \quad i < j \ , \ (i, j = 1, 2, \ldots , k) \tag{4.40}$$

The well known result that the matrix equations (4.40) have the unique solution for the unknown matrices $\tilde{H}_{ij}$ is that $\lambda_r(\tilde{P}_{ii}) + \lambda_\ell(\tilde{P}_{jj}) \neq 0$ for $i < j$, $(i, j = 1, 2, \ldots , k)$ and $r, \ell \in \{1, 2\}$. This is really the case since the subsystem matrix $\tilde{P}$ is supposed to be stable. Therefore, if $\tilde{G}_{ij} \equiv 0$, $i < j$, $(i, j = 1, 2, \ldots , k)$ then the unique solution of (4.40) is $\tilde{H}_{ij} \equiv 0$, $i < j$, $(i, j = 1, 2, \ldots , k)$.

Suppose now that we picked some $\tilde{G}$. We denote by $\tilde{G}_D$ matrix obtained from $\tilde{G}$ by letting $\tilde{G}_{ij} \equiv 0$, $i < j$, $(i, j = 1, 2, \ldots , k)$, and similarly $\tilde{H}_D$ obtained from $\tilde{H}$. Since $\tilde{G}_{ij} \equiv 0$ implies $\tilde{H}_{ij} \equiv 0$ (for $i \neq j$), then using the results of Theorem 4.1, that is, (4.32) and (4.33)

$$\begin{aligned} \lambda(\tilde{G}) &\leq \lambda(\tilde{G}_D) \\ \lambda(\tilde{H}) &\leq \lambda(\tilde{H}_D) \\ \Lambda(\tilde{H}) &\geq \Lambda(\tilde{H}_D) \end{aligned} \tag{4.41}$$

From (4.41) we obtain the following inequality for the ratio $R(\tilde{G})$

$$R(\tilde{G}) = \frac{1}{2} \frac{\lambda(\tilde{G}) \ \lambda(\tilde{H})}{\Lambda(\tilde{H}) \ \Lambda(\tilde{H})} \leq \frac{1}{2} \frac{\lambda(\tilde{G}_D) \ \lambda(\tilde{H}_D)}{\Lambda(\tilde{H}_D) \ \Lambda(\tilde{H}_D)} = R(\tilde{G}_D) \tag{4.42}$$

because $\tilde{G}$, $\tilde{H}$ are positive definite matrices and therefore have real positive eigenvalues.

From (4.42) we conclude that the maximum of $R(\tilde{G})$ over $\tilde{G}$ will be obtained for $\tilde{G}_D$, which implies $\tilde{H}_D$. The proof is complete.

The equations (4.40) are then automatically satisfied and we should solve the following subproblem:

Subproblem S:

$$\max_{\{G_p\}} \frac{1}{2} \frac{\lambda(G_p)\ \lambda(H_p)}{\Lambda^2(H_p)}$$

where $\quad P_p^T H_p + H_p P_p = -G_p$

and $$P_p = \begin{bmatrix} \sigma_p & \omega_p \\ -\omega_p & \sigma_p \end{bmatrix} \tag{4.43}$$

Lemma 4.3. The solution to the subproblem S is

$$G_p^* = g_p \begin{bmatrix} 2|\sigma_p| & 0 \\ 0 & 2|\sigma_p| \end{bmatrix} \tag{4.44}$$

where g_p *is arbitrary positive number and the maximum value is* $|\sigma_p|$.

Proof. If we take $G_p = \begin{bmatrix} g_1 & g_2 \\ g_2 & g_3 \end{bmatrix}$ the *subproblem S* can be reformulated into

$$\max_{\substack{g_1,\ g_3 > 0 \\ g_1 g_3 - g_2^2 > 0}} \frac{\left(g_1 + g_3 - \sqrt{(g_1 - g_3)^2 + 4g_2^2}\right)\left(\frac{g_1 + g_3}{|\sigma_p|} - \frac{1}{\sqrt{\sigma_p^2 + \omega_p^2}} \sqrt{(g_1 - g_3)^2 + 4g_2^2}\right)}{\left(\frac{g_1 + g_3}{|\sigma_p|} + \frac{1}{\sqrt{\sigma_p^2 + \omega_p^2}} \sqrt{(g_1 - g_3)^2 + 4g_2^2}\right)^2} \tag{4.45}$$

The solution to (8.45) is obtained when $\sqrt{(g_1 - g_3)^2 + 4g_2^2} = 0$ or when $g_1 = g_3$ and $g_2 = 0$, the maximum value is $|\sigma_p|$ and the proof of Lemma is completed.

Using the result of this Lemma we can now proceed to prove Theorem 4.2.

From (8.42)

$$\max_{\tilde{G}} R(\tilde{G}) = \max_{\tilde{G}_D} R(\tilde{G}_D) = \max_{\tilde{G}_D} \frac{1}{2} \frac{\lambda(\tilde{G}_D)\ \lambda(\tilde{H}_D)}{\Lambda^2(\tilde{H}_D)} = \max_{\substack{\{\tilde{G}_{ii}\} \\ i=1,\ldots,k}} \frac{1}{2} \frac{\min_i \lambda(\tilde{G}_{ii})\ \min_i \lambda(\tilde{H}_{ii})}{[\max_i \Lambda(\tilde{H}_{ii})]^2} \tag{4.46}$$

The following inequalities can be derived from (4.46)

$$\max_{\substack{\{\tilde{G}_{ii}\} \\ i=1,2,\ldots,k}} \frac{1}{2} \frac{\min_i \lambda(\tilde{G}_{ii})\ \min_i \lambda(\tilde{H}_{ii})}{[\max_i \Lambda(\tilde{H}_{ii})]^2} \leq \max_{\tilde{G}_{jj}} \frac{1}{2} \frac{\lambda(\tilde{G}_{jj})\ \lambda(\tilde{H}_{jj})}{[\Lambda(\tilde{H}_{jj})]^2} \tag{4.47}$$

for every $j = 1, 2, \ldots, k$.

Since the constraining equations (4.39) are completely decoupled and using the result of Lemma 4.3

$$\max_{\tilde{G}_{jj}} \frac{1}{2} \frac{\lambda(\tilde{G}_{jj})\ \lambda(\tilde{H}_{jj})}{[\Lambda(\tilde{H}_{jj})]^2} = |\sigma_j| \qquad j = 1, \ldots, k \tag{4.48}$$

However, inequality (4.47) is satisfied for every $j = 1, 2, \ldots, k$, and therefore also for one which has minimum module of a real part of eigenvalue. Therefore,

$$\max_{\tilde{G}_D} R(\tilde{G}_D) \leq \min_j |\sigma_j| \qquad 1 \leq j \leq k \tag{4.49}$$

If we now can find such a $\tilde{G}_D^*$ that

$$R(\tilde{G}_D^*) = \min_j |\sigma_j| \qquad 1 \leq j \leq k \tag{4.50}$$

we have found a solution to *Problem* $\tilde{C}$.

This is, however, easy to accomplish simply by taking $\tilde{G}_D^* = (\tilde{G}_{ii}^*)$ where

$$\tilde{G}^*_{ii} = \begin{bmatrix} 2|\sigma_i| & 0 \\ 0 & 2|\sigma_i| \end{bmatrix}, \text{ which leads to } \tilde{H}^*_{ii} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for all } i = 1, 2,$$

$\ldots$, k . Therefore $\tilde{H}$ is an $n \times n$ identity matrix with $\lambda(\tilde{H}) = \Lambda(\tilde{H}) = 1$, $\lambda(\tilde{G}^*_D) = \min_j |\sigma_j|$ $(j = 1, 2, \ldots, k)$, and equality (4.50) is satisfied.

Note that $\tilde{G}^* = g \cdot \tilde{G}^*_D$ $(g > 0)$ is also a solution to the *Problem C* since the ratio $R(\tilde{G})$ is invariant under multiplication of a chosen $\tilde{G}$ with the positive scalar. This completes the proof of Theorem 4.2.

But, still one question remains unanswered. That is, whether the solution to *Problem* $\tilde{C}$ in the transform domain gives the greater value for the maximum allowable coupling parameters than the solution to the *Problem C* in the original domain? As it was noticed earlier, no direct correspondence has been established between the values $R(G)$ and $R(\tilde{G})$. Since $\tilde{G} = T^T G T$, and T nonsingular transformation matrix in general not orthogonal, $T^T T \neq I$, it was only certain that these values were different. One might naturally ask the question whether there exists such a G^* in original domain which would give greater value for the ratio $R(G^*)$ than the maximum value of the ratio $R(\tilde{G}^*) = \min_j |\sigma_j|$ $(j = 1, 2, \ldots, k)$ obtained in the canonical domain. We will answer this question negatively by proving the following theorem.

<u>*Theorem 4.4*</u>: *Let* G , $\tilde{G}$, *and* H , $\tilde{H}$, *be* $n \times n$ *symmetric positive definite matrices related by*

$$\begin{aligned} &\tilde{G} = T^T G T &&, \quad \tilde{H} = T^T H T \\ &P^T H + HP = -G &&, \quad \tilde{P}^T \tilde{H} + \tilde{H}\tilde{P} = -\tilde{G} \\ &\tilde{P} = T^{-1} P T &&, \quad T^T T \neq I \end{aligned} \tag{4.51}$$

where $\tilde{P}$ *is the canonical representation of the known subsystem matrix* P ,

and T *is corresponding normalized transformation. Then*

$$R(G) = \frac{1}{2}\ \frac{\lambda(G)\ \lambda(H)}{\Lambda^2(H)} < \frac{1}{2}\ \frac{\lambda(\tilde{G})\ \lambda(\tilde{H})}{\Lambda^2(\tilde{H})} = R(\tilde{G}) \tag{4.52}$$

Proof. Using the variational description of the minimum and maximum eigenvalue we have

$$\Lambda(H) = \max_{x\neq 0} \frac{x^T H x}{x^T x} = \max_{\tilde{x}\neq 0} \frac{\tilde{x}^T T^T H T \tilde{x}}{\tilde{x}^T T^T T \tilde{x}} = \max_{\tilde{x}\neq 0} \frac{\tilde{x}^T \tilde{H} \tilde{x}}{\tilde{x}^T T^T T \tilde{x}} \tag{4.53}$$

since $x = T\tilde{x}$ and T is nonsingular.

Now, from (4.53) we derive the following weak (greater than or equal) inequality

$$\Lambda(H) = \max_{\tilde{x}\neq 0} \frac{\tilde{x}^T \tilde{H} \tilde{x}}{\tilde{x}^T T^T T \tilde{x}} = \max_{\tilde{x}\neq 0} \frac{\dfrac{\tilde{x}^T \tilde{H} \tilde{x}}{\tilde{x}^T \tilde{x}}}{\dfrac{\tilde{x}^T T^T T \tilde{x}}{\tilde{x}^T \tilde{x}}} \geq \frac{\max\limits_{\tilde{x}\neq 0} \dfrac{\tilde{x}^T \tilde{H} \tilde{x}}{\tilde{x}^T \tilde{x}}}{\min\limits_{\tilde{x}\neq 0} \dfrac{\tilde{x}^T T^T T \tilde{x}}{\tilde{x}^T \tilde{x}}} \tag{4.54}$$

Since $x^T x$ is positive definite quadratic form so is $\tilde{x}^T T^T T \tilde{x}$ and $T^T T$ is symmetric positive definite matrix. Furthermore, due to normalization of $T = (t_1\ t_2, \ldots, t_n)$ where $t_i (i = 1, 2, \ldots, n)$ are column vectors, we have $t_i^T t_i = 1 \quad i = 1, 2, \ldots, n$.

Then

$$T^T T = \begin{bmatrix} 1 & t_1^T t_2 & t_1^T t_3 & \cdots & t_1^T t_n \\ t_2^T t_1 & 1 & t_2^T t_3 & \cdots & \\ \vdots & & 1 & & \\ & & & \ddots & \\ t_n^T t_1 & & & & 1 \end{bmatrix}$$

Now, since $T^TT \neq I$ at least some of the off-diagonal entries will be non-zero and direct application of the Theorem 4.1 gives

$$\lambda(T^TT) < 1$$
$$\Lambda(T^TT) > 1 \tag{4.55}$$

Using (4.55) we get

$$\Lambda(H) \geq \frac{\Lambda(\tilde{H})}{\lambda(T^TT)} > \Lambda(\tilde{H}) \tag{4.56}$$

Similarly, for the minimum eigenvalue

$$\lambda(H) = \min_{\tilde{x}\neq 0} \frac{\tilde{x}^T\tilde{H}\tilde{x}}{\tilde{x}^TT^TT\tilde{x}} \leq \frac{\max\limits_{\tilde{x}\neq 0} \dfrac{\tilde{x}^T\tilde{H}\tilde{x}}{\tilde{x}^T\tilde{x}}}{\max\limits_{\tilde{x}\neq 0} \dfrac{\tilde{x}^TT^TT\tilde{x}}{\tilde{x}^T\tilde{x}}} = \frac{\lambda(\tilde{H})}{\Lambda(T^TT)} < \lambda(\tilde{H}) \tag{4.57}$$

By the same procedure

$$\lambda(G) < \lambda(\tilde{G}) \tag{4.58}$$

Now from (4.56), (4.57) and (4.58), the inequality (4.52) directly follows and the theorem is proved.

Note, however, that in the very special case when $T^TT = I$ then $R(G) = R(\tilde{G})$.

Stated in words, for every choice of G in original space we can find $\tilde{G}$ in canonical space which would give greater (or in the very special case equal) value for ratio $R(\cdot)$. Consequently, it must be true also for the maximum values, that is

$$\max_{G} R(G) < \max_{\tilde{G}} R(\tilde{G}) \tag{4.59}$$

The inequality (4.59), however, does not imply that the matrices G^* and $\tilde{G}^*$ for which these maximums are obtained are related by $\tilde{G}^* = T^T G T$. Therefore, how to find G^* and $R(G^*)$ remains an open question, though, because of (4.59) we are not interested anymore in answering that question.

As compared to (4.4) we have the comparison matrix $\tilde{A}$ in the canonical domain

$$\tilde{A} = \begin{bmatrix} -\frac{1}{2}\frac{\lambda(\tilde{G}_1)}{\Lambda(\tilde{H}_1)} & \tilde{\xi}_{12}|\xi| \frac{\Lambda(\tilde{H}_1)}{\sqrt{\lambda(\tilde{H}_1)\lambda(\tilde{H}_2)}} \\ \tilde{\xi}_{21}|\xi| \frac{\Lambda(\tilde{H}_2)}{\sqrt{\lambda(\tilde{H}_1)\ \lambda(\tilde{H}_2)}} & -\frac{1}{2}\frac{\lambda(\tilde{G}_2)}{\Lambda(\tilde{H}_2)} \end{bmatrix} \tag{4.60}$$

where $\tilde{\xi}_{12}$ and $\tilde{\xi}_{21}$ are norms of the transformed intercoupling matrices $\tilde{Q}_{12}$ and $\tilde{Q}_{21}$ respectively. Due to the sparse structure of the coupling matrices Q_{12}, Q_{21} and by taking normalized transformations T_1, T_2 we obtain

$$\tilde{\xi}_{12} < \bar{\xi}_{12} \quad , \quad \tilde{\xi}_{21} < \bar{\xi}_{21} \tag{4.61}$$

what the computer results confirm.

Similarly to (4.5), the second stability inequality gives

$$\xi^2 < \frac{1}{4} \frac{\lambda(\tilde{G}_1)\ \lambda(\tilde{H}_1)\ \lambda(\tilde{G}_2)\ \lambda(\tilde{H}_2)}{\tilde{\xi}_{12}\tilde{\xi}_{21}\ \Lambda^2(\tilde{H}_1)\ \Lambda^2(\tilde{H}_2)} \tag{4.62}$$

If we now choose $\tilde{G}_1^*$ and $\tilde{G}_2^*$ as proposed by Theorem 4.2 and use the result of Theorem 4.4, that is (4.59), together with (4.61), we find that the maximum stability region of the coupling parameter ξ is obtained for the comparison matrix

$$\tilde{A}_m = \begin{bmatrix} -|\sigma_{m1}| & \tilde{\xi}_{12}|\xi| \\ \tilde{\xi}_{21}|\xi| & -|\sigma_{m2}| \end{bmatrix} \tag{4.63}$$

where σ_{m1} and σ_{m2} are the eigenvalues with the minimum module of the real part for the subsystem matrices P_1 and P_2. The largest estimate for the stability region

$$\xi_m = \sqrt{\frac{\sigma_{m1} \quad \sigma_{m2}}{\tilde{\xi}_{12} \quad \tilde{\xi}_{21}}} \tag{4.64}$$

is obtained for the subsystem Liapunov functions

$$v_1 = ||\tilde{x}_1|| \ , \quad v_2 = ||\tilde{x}_2|| \tag{4.65}$$

Before presenting the numerical results which will show to what extent the results have been improved by this optimization approach, we will discuss some of the features introduced by the canonical transformations.

The crucial advantages of introducing the canonical transformation lie in the following:

1. The problem which did not seem to be tractable appeared in a manageable form in the canonical space, where it was "well defined" and solved.
2. Due to the relationships (4.59) and (4.61), the solution in the canonical space offered the solution to the original problem of maximizing the stability region of the parameter ξ.
3. The canonical transformation is introduced at the subsystem level. The subsystems are assumed to be of the low order.

The numerical problems imposed in canonical transformation are then of much less difficulty than numerically solving the problem using nonlinear mathematical programming algorithms.

4. The simple analytical expression (4.64) not only offers the practical way of computing maximum allowable ξ but more importantly, gives the significant insight into the existing trade-off between the degree of the subsystem stability and the strength of interconnections as compared to the overall system stability. This immediately suggests a possible design procedure based on introducing the controls on the subsystem level in such a way as to shift all subsystem poles as much to the left but without too much affecting the interconnections. These qualitative discussions and the existing trade-off will be made apparent in the numerical examples of Passive and Active Control case.

The computer program implementing this optimization procedure is given in the Appendix together with the obtained results for both Passive and Active Control case. The results are as follows:

PASSIVE CONTROL CASE

The two subsystem matrices are

$$P_1 = \begin{bmatrix} 0 & 0.0463 & 0.4387 & 0.0131 \\ 0.8478 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 \\ 0 & -1.0463 & -3.2253 & -0.0665 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 & 1.0 & 0 & 0 \\ -1.7865 & -0.0534 & 0 & 2.0 \\ 0 & 0 & 0 & 1.0 \\ 0 & -2.0 & -904.5314 & -1.2036 \end{bmatrix} \tag{4.66}$$

The canonical forms are

$$\tilde{P}_1 = \begin{bmatrix} -0.0005 & 0.2909 & 0 & 0 \\ -0.2909 & -0.0005 & 0 & 0 \\ 0 & 0 & -0.0327 & 1.7607 \\ 0 & 0 & -1.7607 & -0.0327 \end{bmatrix}$$

$$\tilde{P}_2 = \begin{bmatrix} -0.0266 & 1.3334 & 0 & 0 \\ -1.3334 & -0.0266 & 0 & 0 \\ 0 & 0 & -0.6019 & 30.1359 \\ 0 & 0 & -30.1359 & -0.6019 \end{bmatrix} \tag{4.67}$$

The computed norms of the transformed intercoupling matrices are

$$\tilde{\xi}_{12} = 0.6314 < 0.7766 = \bar{\xi}_{12}$$

$$\tilde{\xi}_{21} = 1.4545 < 1.8478 = \bar{\xi}_{21} \tag{4.68}$$

The comparison matrix $\tilde{A}_m$ is

$$\tilde{A}_m = \begin{bmatrix} -0.0005 & 0.6314|\xi| \\ 1.4545|\xi| & -0.0266 \end{bmatrix} \tag{4.69}$$

and the largest estimate of the stability region of ξ

$$|\xi| \leq \xi_m = 0.403 \times 10^{-2} \tag{4.70}$$

ACTIVE CONTROL CASE

The subsystem matrices are

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.0463 & -1.2456 & -0.4387 & -1.2456 & 1.0463 & -0.0131 \\ 0 & -0.8478 & 0 & -0.1521 & 0 & 0 \\ 1.0463 & -1.2456 & -3.2253 & -1.2456 & 1.0463 & -0.0665 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -0.1823 & 0.0642 & 0.0019 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.1823 & -1.8508 & -0.0553 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -2 & -904.5314 & -1.2036 \end{bmatrix} \tag{4.71}$$

The canonical forms are

$$\tilde{P}_1 = \begin{bmatrix} -0.1288 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.8408 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0783 & 1.7204 & 0 & 0 \\ 0 & 0 & -1.7024 & -0.0783 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0929 & 0.8990 \\ 0 & 0 & 0 & 0 & -0.8990 & -0.0929 \end{bmatrix}$$

$$\tilde{P}_2 = \begin{bmatrix} -0.1761 & 0 & 0 & 0 & 0 \\ 0 & -0.0306 & 1.3567 & 0 & 0 \\ 0 & -1.3567 & -0.0306 & 0 & 0 \\ 0 & 0 & 0 & -0.6019 & 30.1359 \\ 0 & 0 & 0 & -30.1359 & -0.6019 \end{bmatrix} \tag{4.72}$$

The computed norms of the transformed intercoupling matrices are

$$\tilde{\xi}_{12} = 14.2154 < 314.5269 = \bar{\xi}_{12} \tag{4.73}$$

$$\tilde{\xi}_{21} = 3.7497 > 2.7088 = \bar{\xi}_{21}$$

but

$$\tilde{\xi}_{12} \cdot \tilde{\xi}_{21} < \bar{\xi}_{12}\, \bar{\xi}_{21} \tag{4.74}$$

The comparison matrix $\tilde{A}_m$ is

$$\tilde{A}_m = \begin{bmatrix} -0.0783 & 14.2154|\xi| \\ 3.7497|\xi| & -0.0306 \end{bmatrix} \tag{4.75}$$

and the largest estimate of the stability region of ξ is

$$|\xi| \leq \xi_m = 0.6714 \times 10^{-2} \tag{4.76}$$

If we now compare the results of direct approach taken in reference [1] with the optimization approach, in both the Passive and the Active Control case, we find considerable improvement of more than 10^4 times. Besides that, random experimentation with different choices G_1, G_2 in the case without optimization, showed that chosen identity matrices for G_1 and G_2 appear to be the best.

These results reveal the significance of introducing the canonical transformation on the subsystem level. Besides the significantly improved results, the computational alternative offered by this optimization procedure seems to be very attractive and rather simple. There is no need for solving subsystem Liapunov matrix equations and computing the eigenvalues of the G , H matrices. These computational steps are replaced by the subsystem canonical transformations, whereas the other steps of computing the norms and the estimate of the stability region of the coupling parameter ξ remain unchanged.

The remarkable advantage of this approach is that the optimization problem does not require the numerical optimization procedure over all possible G_1 , G_2 , but is based on the similarity transformation of subsystem matrices to the properly chosen canonical form. Therefore, the low-order of the subsystems, the simplicity of the optimization procedure, and substantial enlargement of the stability region (more than 10^4 times) offer very attractive ways to circumvent the inherent over-sufficiency of decomposition-aggregation stability procedure via the Liapunov vector functions.

Furthermore, in the following section, these theoretical as well as computational results will be directly applied to obtain optimal comparison system as suggested at the beginning of this section.

5. OPTIMUM COMPARISON SYSTEM

As we were motivated at the beginning of the previous section, we are really looking for the comparison system as given by (5.2). Before trying to formulate the optimality conditions for obtaining optimal comparison system, we will investigate some of its pecularities.

Suppose that we have the following comparison system

$$\dot{v}_1 \leq -\alpha_{11}v_1 + \alpha_{12}v_2$$

$$\dot{v}_2 \leq \alpha_{21}v_1 - \alpha_{22}v_2$$

$$A = \begin{bmatrix} -\alpha_{11} & \alpha_{12} \\ \alpha_{21} & -\alpha_{22} \end{bmatrix} \tag{5.1}$$

where α_{ij} $(i, j = 1, 2)$ are nonnegative numbers and $v_1 \geq 0$, $v_2 \geq 0$. Due to linearity of the comparison system (5.1) after introducing the new variables $\tilde{v}_1$, $\tilde{v}_2$ by

$$v_1 = g_1\tilde{v}_1 \qquad v_2 = g_2\tilde{v}_2 \qquad (g_1, g_2 > 0) \tag{5.2}$$

we obtain the comparison system

$$\dot{\tilde{v}}_1 \leq -\alpha_{11}\tilde{v}_1 + \alpha_{12}g\tilde{v}_2$$

$$\dot{\tilde{v}}_2 \leq \frac{1}{g}\ \alpha_{21}\tilde{v}_1 - \alpha_{22}\tilde{v}_2 \tag{5.3}$$

where $g = \dfrac{g_2}{g_1} > 0$.

The comparison systems (5.1) and (5.3) are equivalent in the sense that one may be obtained from the other and vice versa by using (5.2). In terms

of subsystem Liapunov functions it amounts to arbitrarily choosing scalar multiples of some known $\tilde{v}_1$, $\tilde{v}_2$ as the new Liapunov functions.

If we consider α_{ij} (i, j = 1, 2) as the adjustable parameters depending on the matrices G_1 , G_2 as given in (5.4), we can formulate the following criterion for the optimality of the comparison system (5.4).

Problem D.

Find A^o such that

$$A^o - A \leq 0 \ , \ \forall A \in K \tag{5.4}$$

subject to constraints:

$$P_i^T H_i + H_i P_i = G_i \quad i = 1, 2$$

$$\lambda(G_1) = 2|\sigma_{m1}| \ , \ \lambda(G_2) = 2|\sigma_{m2}| \tag{5.5}$$

where K is the set of all matrices $\{G_1, G_2\}$ which are normalized by the condition (5.5). Here σ_{m1} , σ_{m2} are as defined by (5.63).

Having in mind the equivalence between the systems (5.1) and (5.3) we can now reformulate this problem into

Problem E.

$$\max_{G_i} \ \frac{1}{2} \ \frac{\lambda(G_i)}{\Lambda(H_i)} \quad , \quad i = 1, 2$$

and

$$\min_{\{G_1, G_2\}} \ \frac{\Lambda(H_1) \ \Lambda(H_2)}{\lambda(H_1) \ \lambda(H_2)}$$

subject to the constraints:

$$P_i^T H_i + H_i P_i = -G_i \qquad i = 1, 2$$

$$\lambda(G_1) = 2|\sigma_{m1}| \quad \lambda(G_2) = 2|\sigma_{m2}| \tag{5.6}$$

Following exactly the same steps as in solving *Problem C* and then *Problem* $\tilde{C}$ we arrive at the same solution $\tilde{G}_1^*$, $\tilde{G}_2^*$ for the *Problem E*. Therefore, the obtained comparison matrix $\tilde{A}_m$ in (4.63) is also the *optimal* comparison matrix A^o .

Furthermore, with this kind of approach there is no more need to neglect the self-coupling terms as was done in arriving at (4.4) but the complete system (4.20), (4.21) can be optimized by the same optimal choice of $\tilde{G}_1^*$, $\tilde{G}_2^*$. This is true, because additional optimality requirements

$$\min_{G_i} \frac{\Lambda(H_i)}{\lambda(H_i)} \qquad i = 1, 2 \tag{5.7}$$

are satisfied for the same optimal choice of $\tilde{G}_1^*$, $\tilde{G}_2^*$.

Consequently, the optimal comparison matrix for the Passive Control case, taking into account the self-coupling terms, is

$$A^o = \begin{bmatrix} -0.0005 + 0.766\xi^2 & 0.631|\xi| \\ 1.454|\xi| & -0.0266 + 0.144\xi^2 \end{bmatrix} \tag{5.8}$$

and for the Active Control case is

$$A^o = \begin{bmatrix} -0.078 + 0.165\xi^2 & 14.215|\xi| \\ 3.749|\xi| & -0.030 + 0.744\xi^2 \end{bmatrix} \tag{5.9}$$

Finally, we have succeeded in obtaining the optimal comparison systems for

the mathematical model description of the Passive and Active Control Case of the Spinning Skylab and at the same time we have maximized the stability region of the important structural parameter ξ which represents the ratio of the asymmetry of the boom settings and the length of the booms.

6. CONCLUSION

A decomposition-aggregation method is outlined for stability analysis of large-scale dynamic systems. The method takes advantage of special structural features of the complex systems to reduce the memory and computational time requirements. Furthermore, the method is suitable for accommodation of nonlinearities either in the subsystems or in their interconnections and provides the significant insight into the structural properties of complex dynamic systems.

The straightforward application of the Liapunov Direct Method to the stability analysis of high order multivariable dynamic systems though theoretically possible, involves formidable practical limitations. By decomposition of the system into a number of lower order subsystems, it has only been necessary to apply Liapunov Direct Method to the lower order subsystems. Therefore, the full advantage of the experience already accumulated in the construction of Liapunov functions for systems of comparatively small order can be taken. Consequently, this approach broadens the class of problems to which Liapunov Direct Method has practical application and also makes the treatment of large-scale systems feasible. However, the method is inherently conservative since a series of approximations are involved in obtaining the comparison inequalities and establishing sufficient conditions for stability. This may lead to overly restrictive (overly sufficient) conditions for stability as the results in reference [1] point out.

The decomposition-aggregation method was applied in reference [1] to the dynamic model of a spinning Skylab. After the model was decomposed into the wobble and spin subsystems, both the passive and the active control were considered. Such decomposition made an important structural parameter asymmetry in the boom settings to appear as an interconnection parameter of the two subsystems. Subsequent stability analysis was aimed at estimating the interval

of this coupling parameter for global exponential stability of the overall system. The estimates obtained in reference [1] turned out to be very conservative since the flexibility of the decomposition-aggregation method was not used to the full extent.

In this report, the problem of maximizing the stability region of the coupling parameter was formulated and resolved as a well-defined optimization problem [4]. The obtained optimality conditions turned out to be rather simple, requiring only the canonical transformation of the subsystem matrices as well as corresponding transformation of coupling matrices. Moreover, the possibility of introducing the controls at the subsystem level and still maintaining the insight into the effect on the overall stability region, provides a basis for possible design procedure. The substantial enlargement of the stability region (more than 10^4 times compared to nonoptimal procedure outlined in reference [1]) and the simplicity of the optimization procedure offers a very attractive way to overcome the inherent over-restrictiveness of the decomposition-aggregation stability procedure via the Liapunov vector functions. As shown in the Appendix, the entire optimization procedure can be suitably programmed for computer applications.

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APPENDIX

DESCRIPTION OF THE COMPUTER PROGRAM

The program for maximizing stability region of the coupling parameter ξ and obtaining the optimum comparison system is realized through the canonical transformation of the subsystem matrices to properly chosen canonical form.

The method used to compute the canonical transformation matrix T and reduce matrix P to its canonical form $\tilde{P}$ as given by (4.34) is as follows:

The coefficients of the characteristic equation are computed using Leverrier algorithm. Then, the roots of the characteristic equation are found using standard subroutines and matrix $\tilde{P}$ is formed. The system of linear equations

$$PT - T\tilde{P} = 0 \tag{A.1}$$

is then transformed into

$$[P \otimes I - I \otimes \tilde{P}]\ t = 0 \tag{A.2}$$

where the unknown vector t is formed from the columns of the matrix T and $\otimes$ denotes Kronecker product. Since $N \times N$ matrix T is not unique, first row is chosen arbitrarily and the corresponding system of $(N^2 - N) \times (N^2 - N)$ linear equations is solved to obtain the transformation matrix T.

Besides authors' own subroutines whose descriptions are presented here, two other IBM supplied subroutines were used: SIMQ for solving the system of linear equations and POLRT for finding the roots of the polynomial.

Subroutine ENORM

Purpose: Compute the transformed coupling matrix and its norm.

Usage: CALL ENORM (Q, T, TR, K, L, TINV, SIM, TEMP, EST)

Description of parameters:

Q - Original coupling matrix

T - First canonical transformation matrix of order $K \times K$

TR - Second canonical transformation matrix of order $L \times L$

K - Number of rows in matrix Q

L - Number of columns in matrix Q

TINV - Inverse matrix of the matrix T

SIM - Vector of eigenvalues of the matrix $(T^{-1}QTR)^T(T^{-1}QTR)$

TEMP - Temporary storage matrix

EST - Norm of the transformed coupling matrix

Remark: The computed transformed coupling matrix $T^{-1}QTR$ is stored in Q .

Method: The norm of the matrix Q is computed as the square root of the maximum eigenvalue of the matrix Q^TQ .

Subroutine CHVAL

Purpose: Compute the characteristic polynomial and the characteristic values of the given matrix. Then form the canonical matrix as given by (4.34).

Usage: CALL CHVAL (A, N, XCOF, COF, ROOTR, ROOTI, B, C, IER)

Description of parameters:

A - Name of input matrix of order $N \times N$

N - Order of input matrix

XCOF - Vector of $N + 1$ coefficients of the characteristic polynomial of matrix A ordered from smallest to largest power

COF - Working vector of length $N + 1$

ROOTR - Vector containing real parts of eigenvalues of A

ROOTI - Vector containing imaginary parts of eigenvalues of A

B - Canonical matrix of a matrix A

C - Temporary storage matrix

IER - Error code where:

IER = 0 No error

IER = 1 N less than one

IER = 2 N greater than 36

IER = 3 Unable to determine root with 500 iterations in 5 starting values

IER = 4 High order coefficient is zero

Subroutine CRONE

Purpose: From the matrices P and PS form the matrix $P \otimes I - I \otimes PS$ where $\otimes$ denotes Kronecker product.

Usage: CALL CRONE (P, PS, CRON, N)

Description of parameters:

P - Name of one input matrix

PS - Name of second input matrix

CRON - Name of output matrix $P \otimes I - I \otimes PS$ of order $N^2 \times N^2$

N - Order of matrices P and PS

Subroutine SLINE

Purpose: Find canonical transformation matrix by solving system of linear equations PT - TPS = 0 where P and PS are known $N \times N$ matrices.

Usage: CALL SLINE (CRON, XR, XC, N, KS)

Description of parameters:

CRON - Input matrix of the dependent system of homogeneous linear equations PT - TPS

XR - Input vector consisting of elements of the first row of transformation matrix T

XC - Computed vector of coefficients of the independent system of $(N^2 - N) \times (N^2 - N)$ linear equations obtained from PT - TPS = 0

N - Order of matrices P and PS

KS - Output digit

0 for a normal solution

1 for a singular set of equations

Method: Divide $N^2 \times N^2$ matrix CRONE into 4 block matrices where diagonal block matrices are of order $N \times N$ and $(N^2 - N) \cdot (N^2 - N)$. Use then second one as the coefficient matrix of the system of linear equations. Multiply then $(N^2 - N) \times N$ off-diagonal block matrix with the known $N \times 1$ vector XR to compute the coefficient vector of the $(N^2 - N) \times (N^2 - N)$ system of linear equations.

```
PAGE   1     09/25/73            JOB NO. 044103

// JOB T  0001

LOG DRIVE    CART SPEC    CART AVAIL   PHY DRIVE
  0000         0001         0001         0000

V2 M10   ACTUAL 16K  CONFIG 16K

// *

// *          PASSIVE AND ACTIVE STABILITY OF THE SPINNING SKYLAB

// *     MAXIMAZING THE REGION OF STABILITY WITH RESPECT TO COUPLING PARAMETA

// *

// FOR
*ONE WORD INTEGERS
*LIST SOURCE PROGRAM
      SUBROUTINE ENORM(Q,T,TR,K,L,TINV,SIM,TEMP,EST)
      DIMENSION LL(12),MM(12)
      DIMENSION Q(1),T(1),TR(1),TINV(1),SIM(1),TEMP(1)
      CALL MCPY(T,TINV,K,K,0)
      CALL MINV(TINV,K,DET,LL,MM)
      CALL GMPRD(TINV,Q,TEMP,K,K,L)
      CALL GMPRD(TEMP,TR,Q,K,L,L)
      CALL MATA(Q,SIM,L,L,0)
      CALL EIGEN(SIM,R,L,1)
      EST=SQRT(SIM(1))
      RETURN
      END

FEATURES SUPPORTED
 ONE WORD INTEGERS

CORE REQUIREMENTS FOR ENORM
 COMMON       0  VARIABLES      30  PROGRAM      98

RELATIVE ENTRY POINT ADDRESS IS 0020 (HEX)

END OF COMPILATION

// DUP

*STORE       WS  UA  ENORM
CART ID 0001   DB ADDR  58B0   DB CNT   0007

// EJECT
```

```
// FOR
*LIST SOURCE PROGRAM
*ONE WORD INTEGERS
      SUBROUTINE CHVAL(A,N,XCOF,COF,ROOTR,ROOTI,B,C,IER)
      DIMENSION A(1),B(1),C(1),XCOF(1),COF(1),ROOTR(1),ROOTI(1)
      ISUB(I,J)=(J-1)*N+I
      NN=N*N
      DO 10 I=1,NN
   10 B(I)=0.0
      DO 12 I=1,N
      J=I+(I-1)*N
   12 B(J)=1.0
      DO 20 I=1,N
      CALL GMPRD(A,B,C,N,N,N)
      TRACE=0.0
      DO 15 K=1,N
      J=K+(K-1)*N
   15 TRACE=TRACE+C(J)
      ROOTR(I)=TRACE
   20 CALL MCPY(C,B,N,N,0)
      COF(1)=1.0
      DO 40 K=1,N
      SUM=0.0
      DO 30 I=1,K
      L=K-I+1
   30 SUM=SUM+COF(L)*ROOTR(I)
   40 COF(K+1)=-SUM/K
      M=N+1
      DO 45 I=1,M
      J=N-I+2
   45 XCOF(I)=COF(J)
      CALL POLRT(XCOF,COF,N,ROOTR,ROOTI,IER)
      DO 50 I=1,N
   50 ROOTI(I)=-ROOTI(I)
      DO 60 I=1,N
      DO 60 J=1,N
      JJ=ISUB(I,J)
   60 B(JJ)=0.0
      DO 70 I=1,N
      II=ISUB(I,I)
      B(II)=ROOTR(I)
      IF(ROOTI(I)) 71,70,73
   73 IJ=ISUB(I,I+1)
      B(IJ)=ROOTI(I)
      GO TO 70
   71 IJ=ISUB(I,I-1)
      B(IJ)=ROOTI(I)
   70 CONTINUE
      RETURN
      END

FEATURES SUPPORTED
 ONE WORD INTEGERS

CORE REQUIREMENTS FOR CHVAL
 COMMON      0  VARIABLES     20  PROGRAM     528
```

```
PAGE    3     09/23/73

RELATIVE ENTRY POINT ADDRESS IS 002D (HEX)

END OF COMPILATION

// DUP

*STORE       WS  UA  CHVAL
CART ID 0001    DB ADDR  5BB7    DB CNT    0023

// EJECT
```

```
// FOR
*LIST SOURCE PROGRAM
*ONE WORD INTEGERS
      SUBROUTINE CRONE(P,PS,CRON,N)
      DIMENSION P(1),PS(1),CRON(1)
      ISUB(I,J)=(J-1)*N+I
      ISUBC(I,J)=(J-1)*N*N+I
      DO 20 K=1,N
      DO 20 L=1,N
      IP=ISUB(K,L)
      IF(K-L) 22,21,22
   21 DO 23 I=1,N
      DO 23 J=1,N
      II=(K-1)*N+I
      JJ=(L-1)*N+J
      NN=ISUBC(II,JJ)
      IPS=ISUB(J,I)
      IF(I-J) 24,25,24
   25 CRON(NN)=P(IP)-PS(IPS)
      GO TO 23
   24 CRON(NN)=-PS(IPS)
   23 CONTINUE
      GO TO 20
   22 DO 26 I=1,N
      DO 26 J=1,N
      II=(K-1)*N+I
      JJ=(L-1)*N+J
      NN=ISUBC(II,JJ)
      IF(I-J) 28,27,28
   27 CRON(NN)=P(IP)
      GO TO 26
   28 CRON(NN)=0.0
   26 CONTINUE
   20 CONTINUE
      RETURN
      END

FEATURES SUPPORTED
 ONE WORD INTEGERS

CORE REQUIREMENTS FOR CRONE
 COMMON      0  VARIABLES     12  PROGRAM    312

RELATIVE ENTRY POINT ADDRESS IS 0036 (HEX)

END OF COMPILATION

// DUP

*STORE      WS  UA  CRONE
CART ID 0001  DB ADDR  5BDA  DB CNT  0014

// EJECT
```

```
// FOR
*ONE WORD INTEGERS
*LIST SOURCE PROGRAM
      SUBROUTINE SLINE(CRON,XR,XC,N,KS)
      DIMENSION CRON(1),XR(1),XC(1)
      ISUB(I,J)=(J-1)*N+I
      NN=N*N
      M=N*(N-1)
      MN=M*N
      KK=MN*(N-1)
      DO 10 J=1,NN
      DO 10 I=1,M
      NI=(J-1)*NN+N+I
      NJ=(J-1)*M+I
   10 CRON(NJ)=CRON(NI)
      CALL GMPRD(CRON,XR,XC,M,N,1)
      DO 20 I=1,M
   20 XC(I)=-XC(I)
      DO 30 I=1,KK
      LL=MN+I
   30 CRON(I)=CRON(LL)
      CALL SIMQ(CRON,XC,M,KS)
      DO 40 I=1,M
      K=NN-I+1
      L=M-I+1
   40 XC(K)=XC(L)
      DO 50 I=1,N
   50 XC(I)=XR(I)
      DO 60 I=1,N
      DO 60 J=1,N
      IF(I-J) 61,60,60
   61 II=ISUB(I,J)
      JJ=ISUB(J,I)
      TEMP=XC(II)
      XC(II)=XC(JJ)
      XC(JJ)=TEMP
   60 CONTINUE
      RETURN
      END

FEATURES SUPPORTED
 ONE WORD INTEGERS

CORE REQUIREMENTS FOR SLINE
 COMMON      0  VARIABLES    18  PROGRAM   368

RELATIVE ENTRY POINT ADDRESS IS 0025 (HEX)

END OF COMPILATION

// DUP

*STORE      WS  UA  SLINE
CART ID 0001   DB ADDR  5BEE   DB CNT   0018

// EJECT
```

```
// FOR
*LIST SOURCE PROGRAM
*IOCS(CARD, 1403 PRINTER
*ONE WORD INTEGERS
      REAL I1,I2,I3,MASS,K1,K2
      DIMENSION P(36),PS(36),CRON(1296),XR(6),XC(36),TEMP(36),TINV(36)
      DIMENSION T(36),Q(36),SIM(21),COF(7),XCOF(7),ROOTR(6),ROOTI(6)
      DIMENSION SMIN(2)
C     READ THE PHYSICAL PARAMETARS OF THE SKYLAB
      READ(2,50) I1,I2,I3,G2,MASS,EK1,EK2,OMEGA
   50 FORMAT(8F10.0)
C     COMPUTE THE NORMALIZED PARAMETARS
      K1=(I2-I3)/I1
      K2=(I3-I1)/I2
      ALPHA=(1.0+K1)/(1.0-K2)
      BETA=I3/I1
      GAMA=(2.*MASS*G2*G2)/I1
      SIGS1=EK1/(MASS*OMEGA*OMEGA)
      SIGS2=EK2/(MASS*OMEGA*OMEGA)
      SIGM1=SQRT(SIGS1)
      SIGM2=SQRT(SIGS2)
      DEL1=0.04*SIGM1
      DEL2=0.04*SIGM2
      GAMA1=1.0-GAMA
      GAMA2=GAMA1*GAMA1
      GAMA3=GAMA/BETA
C
C     S U B S Y S T E M   A N A L Y S I S
C
      DO 83 K=1,2
C     READ  N - THE ORDER OF THE SUBSYSTEM
      READ (2,100) N
  100 FORMAT(I2)
      NN=N*N
      NNN=NN*NN
  101 FORMAT(8F10.0)
      DO 86 J=1,NN
   86 P(J)=0.0
      GO TO (41,42),K
C     COMPUTE THE ELEMENTS OF THE FIRST SUBSYSTEM MATRIX P1
   41 P(5)=(K1+GAMA)/GAMA1
      P(9)=(GAMA*SIGS1)/GAMA1
      P(13)=(GAMA*DEL1)/GAMA1
      P(2)=K2
      P(15)=1.0
      P(8)=-(K1+1.0)/GAMA1
      P(12)=(GAMA-SIGS1-1.0)/GAMA1
      P(16)=-DEL1/GAMA1
      GO TO 8
C     COMPUTE THE ELEMENTS OF THE SECOND SUBSYSTEM MATRIX P2
   42 P(5)=1.0
      P(2)=-SIGS1
      P(6)=-DEL1
      P(14)=2.0
      P(15)=1.0
      P(8)=-2.0
      P(12)=1.0-SIGS2
      P(16)=-DEL2
```

```
C       WRITE THE SUBSYSTEM MATRIX
      8 WRITE(5,160)
    160 FORMAT('1THE SYSTEM MATRIX IS',/)
        DO 6 I=1,N
      6 WRITE(5,200) (P(J),J=I,NN,N)
    200 FORMAT(1X,12F10.4)
C       COMPUTE THE CHARACTERISTIC EQUTION AND ITS ROOTS
        CALL CHVAL(P,N,XCOF,COF,ROOTR,ROOTI,PS,TEMP,IER)
        WRITE(5,105)
        IF(IER-3) 60,49,60
     49 WRITE(5,300)
    300 FORMAT(' UNABLE TO DETERMINE ROOT WITH 500 ITERATIONS')
        GO TO 90
     60 WRITE(5,400)
    400 FORMAT('0THE ROOTS OF THE CHARACTERISTIC EQUATION ARE',/)
        WRITE(5,450)
    450 FORMAT('     REAL PART   ',5X,'IMAGINARY PART',/)
        DO 80 I=1,N
     80 WRITE(5,500) ROOTR(I),ROOTI(I)
    500 FORMAT(F12.4,5X,F12.4,/)
        WRITE(5,250)
C       WRITE THE SIMILAR MATRIX
    250 FORMAT(' THE SIMILAR MATRIX IS',/)
        DO 40 I=1,N
     40 WRITE(5,200) (PS(J),J=I,NN,N)
C       FIND THE ROOT WITH THE MAXIMUM REAL PART
        SMIN(K)=ROOTR(1)
        DO 81 I=1,N
        IF(SMIN(K)-ROOTR(I)) 82,81,81
     82 SMIN(K)=ROOTR(I)
     81 CONTINUE
C       READ THE VECTOR XR TO DETERMINE THE UNIQUE
C       SIMILARITY TRANSFORMATION
        READ(2,101) XR
        CALL CRONE(P,PS,CRON,N)
        WRITE(5,105)
    105 FORMAT(///)
C       SOLVE THE SYSTEM OF LINEAR EQUATIONS AND
C       COMPUTE THE  SIMILARITY TRANSFORMATION MATRIX
        CALL SLINE(CRON,XR,XC,N,KS)
        IF(KS-1) 32,31,32
     31 WRITE(5,201)
    201 FORMAT(' SINGULAR CASE')
        GO TO 90
C       WRITE THE SIMILARITY TRANSFORMATION MATRIX
     32 WRITE(5,305)
    305 FORMAT(' THE SIMILARITY TRANSFORMATION MATRIX IS',/)
        DO 12 I=1,N
     12 WRITE(5,103) (XC(J),J=I,NN,N)
    103 FORMAT (10F12.4)
        WRITE(5,105)
        IF(K-1) 83,84,83
C       CHANGE THE ORDER OF THE FIRST SUBSYSTEM TO VARIABLE M
     84 M=N
        DO 85 J=1,NN
     85 T(J)=XC(J)
     83 CONTINUE
C
```

```
C     I N T E R C O N N E C T I O N   A N A L Y S I S
C
      MN=M*N
      MM=M*M
      DO 61 I=1,MM
C     COMPUTE THE TRANSFORMED SELFCOUPLING MATRIX Q11
   61 Q(I)=0.0
      Q(5)=GAMA*(2.0*GAMA+K1-1.0)/GAMA2
      Q(9)=SIGS1*GAMA*GAMA/GAMA2
      Q(13)=(DEL1*GAMA*GAMA)/GAMA2
      Q(2)=GAMA*(K2+1.0)/ALPHA
      Q(8)=-Q(5)
      Q(12)=-Q(9)
      Q(16)=-Q(13)
      CALL ENORM(Q,T,T,M,M,TINV,SIM,TEMP,EST1)
      WRITE(5,301)
  301 FORMAT('1THE TRANSFORMED SELFCOUPLING MATRIX Q11 IS',/)
      DO 21 I=1,M
   21 WRITE(5,103) (Q(J),J=I,MM,M)
      DO 62 I=1,MN
C     COMPUTE THE TRANSFORMED INTERCOUPLING MATRIX Q12
   62 Q(I)=0.0
      Q(5)=2.0*GAMA/GAMA1
      Q(9)=-GAMA/GAMA1
      Q(2)=GAMA*(SIGS1+1.0)/ALPHA
      Q(6)=GAMA*DEL1/ALPHA
      Q(8)=-Q(5)
      Q(12)=-Q(9)
       CALL ENORM (Q,T,XC,M,N,TINV,SIM,TEMP,EST2)
      WRITE(5,302)
  302 FORMAT(///,' THE TRANSFORMED INTERCOUPLING MATRIX Q12 IS',/)
      DO 23 I=1,M
   23 WRITE(5,103) (Q(J),J=I,MN,N)
C     COMPUTE THE TRANSFORMED INTERCOUPLING MATRIX Q21
      DO 63 I=1,MN
   63 Q(I)=0.0
      Q(2)=-(1.0+K2)
      Q(8)=(2.0*GAMA+K1-1.0)/GAMA1
      Q(12)=GAMA*SIGS1/GAMA1
      Q(16)=GAMA*DEL1/GAMA1
      CALL ENORM(Q,XC, T,N,M,TINV,SIM,TEMP,EST3)
      WRITE(5,303)
  303 FORMAT(///,' THE TRANSFORMED INTERCOUPLING MATRIX Q21 IS',/)
      DO 25 I=1,N
   25 WRITE(5,103) (Q(J),J=I,MN,M)
C     COMPUTE THE TRANSFORMED SELFCOUPLIN MATRIX Q22
      DO 64 I=1,NN
   64 Q(I)=0.0
      Q(2)=-GAMA*(SIGS1+1.0)/ALPHA
      Q(6)=-GAMA*DEL1/ALPHA
      Q(8)=2.0*GAMA/GAMA1
      Q(12)=-GAMA/GAMA1
      CALL ENORM(Q,XC,XC,N,N,TINV,SIM,TEMP,EST4)
      WRITE(5,304)
  304 FORMAT(///,' THE TRANSFORMED SELFCOUPLING MATRIX Q22 IS',/)
      DO 27 I=1,N
   27 WRITE(5,103) (Q(J),J=I,NN,N)
      WRITE(5,306)
```

```
  306 FORMAT(////,' THE FOUR NORMS OF THE COUPLING MATRICES ARE',/)
C     WRITE THE FOUR NORMS OF THE COUPLING MATRICES
      WRITE (5,108) EST1,EST2,EST3,EST4
  108 FORMAT(4E16.6)
      EST=EST2*EST3
      ZET2=SMIN(1)*SMIN(2)/EST
      ZETA=SQRT(ZET2)
      WRITE(5,105)
      WRITE(5,308) ZETA
  308 FORMAT('OTHE ESTIMATE OF THE COUPLING PARAMETAR IS ',E16.6)
   90 CALL EXIT
      END
```

```
FEATURES SUPPORTED
 ONE WORD INTEGERS
 IOCS

CORE REQUIREMENTS FOR
 COMMON        0  VARIABLES   3286  PROGRAM   1586

END OF COMPILATION

// XEQ
```

THE SYSTEM MATRIX IS

0.0000	0.0463	0.4387	0.0131
0.8478	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0000
0.0000	-1.0463	-3.2253	-0.0665

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE

REAL PART	IMAGINARY PART
-0.0005	0.2909
-0.0005	-0.2909
-0.0327	1.7607
-0.0327	-1.7607

THE SIMILAR MATRIX IS

-0.0005	0.2909	0.0000	0.0000
-0.2909	-0.0005	0.0000	0.0000
0.0000	0.0000	-0.0327	1.7607
0.0000	0.0000	-1.7607	-0.0327

THE SIMILARITY TRANSFORMATION MATRIX IS

0.3000	0.3000	0.1000	0.1000
0.8723	-0.8757	0.0472	-0.0490
-0.2888	0.2935	-0.3919	0.4200
-0.0852	-0.0842	-0.7267	-0.7039

THE SYSTEM MATRIX IS

0.0000	1.0000	0.0000	0.0000
-1.7865	-0.0534	0.0000	2.0000
0.0000	0.0000	0.0000	1.0000
0.0000	-2.0000	-904.5314	-1.2036

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE

REAL PART	IMAGINARY PART
-0.0266	1.3334
-0.0266	-1.3334
-0.6019	30.1359
-0.6019	-30.1359

THE SIMILAR MATRIX IS

-0.0266	1.3334	0.0000	0.0000
-1.3334	-0.0266	0.0000	0.0000
0.0000	0.0000	-0.6019	30.1359
0.0000	0.0000	-30.1359	-0.6019

THE SIMILARITY TRANSFORMATION MATRIX IS

0.3000	0.3000	0.0030	0.0030
-0.4080	0.3920	-0.0939	0.0894
0.0009	-0.0008	-0.0468	0.0446
0.0011	0.0012	-1.3186	-1.4377

THE TRANSFORMED SELFCOUPLING MATRIX Q11 IS

-0.3406	0.3652	-0.0792	0.0853
-0.3632	0.3407	-0.0869	0.0772
-0.1372	0.1210	-0.0332	0.0270
-0.1084	0.1260	-0.0247	0.0299

THE TRANSFORMED INTERCOUPLING MATRIX Q12 IS

-0.2826	0.3063	-0.0517	0.0495
-0.3164	0.2690	-0.0517	0.0488
-0.1219	0.0925	-0.0190	0.0178
-0.0870	0.1090	-0.0171	0.0165

THE TRANSFORMED INTERCOUPLING MATRIX Q21 IS

0.6862	0.6933	0.2291	0.2307
-0.6932	-0.6864	-0.2307	-0.2291
0.3536	-0.3269	0.0848	-0.0738
0.3431	-0.3698	0.0797	-0.0865

THE TRANSFORMED SELFCOUPLING MATRIX Q22 IS

0.0354	0.0388	0.0000	0.0007
-0.0369	-0.0374	-0.0002	-0.0004
0.0717	-0.0674	0.0122	-0.0116
0.0737	-0.0721	0.0128	-0.0122

THE FOUR NORMS OF THE COUPLING MATRICES ARE

0.766883E 00 0.631457E 00 0.145454E 01 0.144751E 00

THE ESTIMATE OF THE COUPLING PARAMETAR IS 0.403822E-02

THE SYSTEM MATRIX IS

0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	1.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
0.0463	-1.2456	-0.4387	-1.2456	1.0463	-0.0131
0.0000	-0.8478	0.0000	-0.1521	0.0000	0.0000
1.0463	-1.2456	-3.2253	-1.2456	1.0463	-0.0665

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE

REAL PART	IMAGINARY PART
-0.1288	0.0000
-0.8408	0.0000
-0.0783	1.7204
-0.0783	-1.7204
-0.0929	0.8990
-0.0929	-0.8990

THE SIMILAR MATRIX IS

-0.1288	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	-0.8408	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	-0.0783	1.7204	0.0000	0.0000
0.0000	0.0000	-1.7204	-0.0783	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	-0.0929	0.8990
0.0000	0.0000	0.0000	0.0000	-0.8990	-0.0929

THE SIMILARITY TRANSFORMATION MATRIX IS

1.0000	1.0000	0.1000	0.1000	1.0000	1.0000
0.0226	0.0822	-0.0112	0.0133	0.9183	0.6392
0.3635	0.4949	1.1220	0.1349	0.1836	0.0278
-0.1288	-0.8408	-0.1798	0.1642	-0.9919	0.8061
-0.0029	-0.0691	-0.0220	-0.0204	-0.6600	0.7662
-0.0468	-0.4162	-0.3200	1.9198	-0.0420	0.1625

THE SYSTEM MATRIX IS

-0.1823	0.0642	0.0019	0.0000	0.0000
0.0000	0.0000	1.0000	0.0000	0.0000
0.1823	-1.8508	-0.0553	0.0000	2.0000
0.0000	0.0000	0.0000	0.0000	1.0000
-2.0000	0.0000	-2.0000	-904.5314	-1.2036

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE

REAL PART	IMAGINARY PART
-0.1761	0.0000
-0.0306	1.3567
-0.0306	-1.3567
-0.6019	30.1359
-0.6019	-30.1359

THE SIMILAR MATRIX IS

-0.1761	0.0000	0.0000	0.0000	0.0000
0.0000	-0.0306	1.3567	0.0000	0.0000
0.0000	-1.3567	-0.0306	0.0000	0.0000
0.0000	0.0000	0.0000	-0.6019	30.1359
0.0000	0.0000	0.0000	-30.1359	-0.6019

THE SIMILARITY TRANSFORMATION MATRIX IS

1.0000	0.0500	0.0500	0.0000	0.0000
0.0978	-0.8765	1.2306	-0.0011	0.0019
-0.0172	-1.6427	-1.2270	-0.0588	-0.0349
-0.0021	0.0035	0.0026	-0.0293	-0.0173
0.0003	-0.0036	0.0047	0.5413	-0.8749

THE TRANSFORMED SELFCOUPLING MATRIX Q11 IS

-0.3595	-0.1923	-0.1468	-0.1009	-0.1452	-0.9725
0.3918	0.1893	0.1559	0.1159	0.1537	1.1068
-0.0440	-0.0154	-0.0163	-0.0147	-0.0159	-0.1378
-0.0580	0.0084	-0.0156	-0.0277	-0.0146	-0.2486
-0.0347	-0.0496	-0.0205	-0.0007	-0.0209	-0.0217
0.0126	0.0533	0.0146	-0.0099	0.0155	-0.0739

THE TRANSFORMED INTERCOUPLING MATRIX Q12 IS

0.0189	0.9356	1.1327	-9.0227	-5.8354
-0.0196	-1.0610	-1.2005	9.9113	6.4102
0.0018	0.1310	0.1249	-1.1356	-0.7345
0.0010	0.2315	0.1166	-1.6091	-1.0409
0.0034	0.0269	0.1616	-0.7517	-0.4860
-0.0030	0.0621	-0.1180	0.1376	0.0888

THE TRANSFORMED INTERCOUPLING MATRIX Q21 IS

-0.0068	-0.0732	-0.0281	0.0269	0.2685	0.3804
0.0793	0.5719	0.1435	-0.1338	0.0645	-1.0848
0.0588	0.4134	0.1050	-0.0968	0.0251	-0.7966
0.4949	0.1576	0.1815	0.1691	0.1474	1.5636
-0.8110	-0.0792	-0.2584	-0.3311	-0.2659	-3.0315

THE TRANSFORMED SELFCOUPLING MATRIX Q22 IS

0.0000	-0.0000	0.0001	-0.0000	-0.0000
0.0014	-0.0373	0.0507	-0.0053	-0.0033
0.0009	-0.0280	0.0350	0.0076	0.0050
-0.0037	-0.4056	-0.2924	3.1543	2.0403
0.0067	0.6740	0.5100	-5.3337	-3.4499

THE FOUR NORMS OF THE COUPLING MATRICES ARE

0.165203E 01 0.142154E 02 0.374971E 01 0.744500E 01

THE ESTIMATE OF THE COUPLING PARAMETAR IS 0.671487E-02

NASA-Langley, 1974

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