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INTRODUCTION

In this report, three coding schemes designed for improved data communication are evaluated. In Part A, four block codes are evaluated relative to a quality function, which is a function of both the amount of data rejected and the error rate.

Part B is an evaluation of the Viterbi Maximum Likelihood Decoding Algorithm as a decoding procedure. This evaluation is obtained by simulating the system on a digital computer.

In Part C, Short Constraint Length Rate 1/2 'Quick-Look' Codes are studied, and their performance is compared to general nonsystematic codes.
PART A
PERFORMANCE OF BLOCK CODES

I. Introduction:

Although the use of error control coding techniques in digital space communication systems has become fairly routine in recent years, there still exists a great deal of uncertainty as to the actual effectiveness of coding in achieving more reliable communication. The reason for this is to be found in the fact that the commonly used performance parameters do not take into account all the pertinent aspects of the coded transmission system. Thus, for example, the widely used Probability of Word Error criterion totally ignores the possibility that the decoder may incorporate some degree of data rejection. Likewise, the minimum distance criterion, another popular measure of code performance, is completely independent of the decoding algorithm and several other important system factors.

As a consequence of this state of affairs, it is virtually impossible to compare, say, a coding system with error correction and data rejection to one with error correction alone, using any of the existing criteria of performance, and it is therefore of value to define and evaluate measures which incorporate most, if not all, of the quantities affecting the overall system reliability. This is the objective of the present work.

II. Definition of Performance Measure:

For the simple types of block codes normally employed in space communication systems, the complexity of the encoder and decoder is of little consequence, since the use of integrated circuit technology allows the construction of the basic components in an inexpensive fashion. Furthermore, the complexity is essentially independent of the particular code-decoder used.
The processing speed is generally a function of the type of logic used and the technology in the construction of the integrated circuits. Although one could probably obtain cost figures as a function of processing speed, the importance of these costs in the overall system considerations is difficult to assess. Also, as with complexity, processing speed is not a strong function of the code-decoder combination.

Thus, the important factors determining the overall coding system performance are:

1. The accuracy of the data after decoding,
2. The amount of data rejected by the decoder,
3. The amount of redundancy in the code, and
4. The relative importance of data accuracy, data rejection, and data transmission rate.

Let us consider a situation in which $N$ blocks of received digits from a binary $(n, k)$ block code are to be decoded. The decoder generally rejects $N-X$ blocks, leaving $X$ blocks after decoding, of which $Y$ are correct. (See Figure 1)

![Figure 1. General Decoder Configuration](image)

The amount of data passed by the decoder is measured by the quantity

$$F_1 = \frac{1}{N} \mathbb{E}(X),$$
the accuracy of the data after decoding is measured by the quantity
\[ F_2 = \frac{E(Y)}{E(X)}, \]
and the amount of redundancy in the code is measured by the quantity
\[ F_3 = \frac{k}{n} = \frac{\text{number of data digits per block}}{\text{total number of digits per block}}, \]
Here \( E \) is the usual expectation operator.

We also define a quantity \( 0 \leq \alpha \leq 1 \) which measures the relative importance of data accuracy and data rejection.

As an overall measure of performance of the code-decoder combination, we then take quantity
\[ F = 1 - \alpha F_1^{\alpha} F_2^{1-\alpha}, \]
as a function of the energy per information bit-to-noise ratio, \( E_b/N_0 \).

When the \( N \) blocks are transmitted independently of each other and are treated as such by the decoder, \( 1 - F \) reduces to the probability of word rejection. For a decoder with no data rejection, \( F_2 \) becomes the probability of correct decoding. Thus, in the two limiting cases \( \alpha = 0 \) and \( \alpha = 1 \), \( F \) reduces to the probability of word error and word rejection, respectively.

III. Evaluation of F for Hamming Type Block Codes Over the Binary Symmetric Channel:

We assume that \( N \) blocks are transmitted independently and with equal probability over a binary symmetric channel whose digit error probability is \( p = 1-q \). The codes of interest are of two types: The standard \((n, k)\) Hamming code described by the parity check matrix
whose columns are all \(2^m - 1\) nonzero binary \(m\)-tuples (\(m\) any integer greater than 2), and a modified Hamming code whose parity check matrix differs from the above only in having an additional row of ones on top. The first code has block length \(n = 2^m - 1\), \(k = n - m\) information digits and minimum distance 3 and is thus able to correct all single errors. The second code has the same block length, \(k = n - m - 1\) and minimum distance 4 and can be decoded in either a single-error-correcting, double-error-detecting mode or a triple-error-detecting mode.

For both codes, the first step in decoding a received block \(v = (v_1, v_2, \ldots, v_n)\) consists of determining its syndrome. This is a binary \((n-k)\)-tuple given by

\[ s = vH^T \]

where \(T\) denotes matrix transposition and the multiplication and addition operations are modulo 2.

We now consider four cases, including, for purposes of comparison, the uncoded transmission of data blocks of length \(n\).

Case 1. No Coding - \((n, n)\) Code

Decoding Rule: Pass every block unchanged

Evidently, \(X = E(X) = N\) and since a block is correct at the decoder output if and only if it is correct at the input, we have \(E(Y) = N^n\)
Hence $F_1 = 1$; $F_2 = q^n$
and
$F = 1 - q^n(1-a)$

Since $F_3 = 1$, the relation between the channel error probability $p$ and $E_b/N_0$ is $p = \frac{1}{2} - \text{erf} \sqrt{2E_b/N_o}$

Case 2. Single-Error-Correcting Hamming Code

Decoding Rule: If the syndrome is zero, pass the block. If the syndrome is not equal to zero, assume a single error has occurred, correct it, and then pass the block.

Again, $E(X) = N$. For $E(Y)$ we have

$E(Y) = N \{ \text{Probability that a block has no error or a single error before decoding} \}
= N \{ q^n + nq^{n-1} p \}$

Therefore, $F_3 = \frac{k}{n}$, $F_1 = 1$, $F_2 = q^n + nq^{n-1} p$
and

$F = 1 - (q^n + npq^{n-1})$

where $p = \frac{1}{2} - \text{erf} \sqrt{2k/n \ E_b/N_o}$


Decoding Rule: If the syndrome is zero, pass the block. If the first digit and at least one of the remaining digits in the syndrome are one, assume a single error has occurred, correct it, and then pass the block. For all other syndromes, reject the block.

We have $F_3 = \frac{k-1}{n}$,
Case 4. Triple-Error-Detecting Hamming Code

Decoding Rule: If the syndrome is zero, pass the block. Otherwise, reject the block.

Here, $F_3 = \frac{k-1}{n}$,

$$F_1 = \frac{E(X)}{N} = \{\text{Probability that a block has zero syndrome or the first and at least one of the remaining digits equal one}\}$$

$$\frac{n-1}{2} = \sum_{i=0}^{n-2} A_{2i} q^{n-2i} p^{2i} + [(2i^n + 1) - A_{2i} + 1] q^{n-2i + 1} p^{2i + 1}$$

where $A_j$ is the number of codewords of weight $j$ of the Single-Error-Correcting Hamming Code,

and for $F_2$ we obtain

$$F_2 = \frac{E(Y)}{E(X)} = \frac{1}{F_1} \{\text{Probability that a received block is correct or has a single error}\}$$

The Hamming code weight spectra required for Cases 3 and 4 may be obtained as the coefficients of the polynomial.
\[ f(x) = \frac{1}{n+1} \left( (1+x)^n + n(1+x)^{\frac{n-1}{2}} (1-x)^{\frac{n+1}{2}} \right) \]

where \( A_i \) is the coefficient of \( x^i \).

IV. Evaluating the Performance Criterion:

A Fortran language program, reproduced in Appendix A, was written to evaluate the function \( F \) for the four cases described above. The program calculates \( F \) for 101 equally spaced values of \( E_b/N_0 \) ranging from 2 db to 10 db and all values of redundancy from \( m = 3 \) to \( m = 10 \).

A major part of the program is devoted to calculating the coefficients of the function

\[ f(X) = \frac{1}{n+1} \left( (1+X)^n + n(1+X)^{\frac{n-1}{2}} (1-X)^{\frac{n+1}{2}} \right) \]

which are used in Cases 3 and 4. The main difficulty in this computation is that some of them have magnitudes on the order of \( 10^{300} \) for large values of \( n \). Overflow on the IBM 370 occurs with numbers as small as \( 10^{77} \). To overcome this problem, most calculations are done using logarithms. Thus, for example, the logarithms of the coefficients of \( (1+X)^n \) are stored in an array called LGCOEF. Similarly, the components of \( (1+X)^{n-1/2} \) and \( (1+X)^{n+1/2} \) are stored in LCNM1 and LCNPI, respectively. Note that LCNPI contains the logarithms of the coefficients of \( (1+X)^{n+1/2} \) and not \( (1-X)^{n+1/2} \), since the latter has negative coefficients whose logarithms do not exist. The variable SIGN, which always equals \( \pm 1 \), is used to convert the coefficients of \( (1+X)^{n-1/2} \) to the coefficients of \( (1-X)^{n+1/2} \) when making calculations of \( f(X) \).

A special procedure is used throughout the program to achieve addition of these very large numbers. Obviously this addition cannot be achieved directly using logarithms. To illustrate this procedure consider adding the numbers \( A = 7.3147 \times 10^{298} \) and \( B = 2.1532 \times 10^{295} \) given the logarithms of these numbers.
ALOG = \log(A) = 298.864196 = 3.864196 + 295.

BLOG = \log(B) = 295.333084 = 0.333084 + 295.

Let Z = X+Y and ZLOG = \log(Z).

\[
ZLOG = \log(10^{3.864196} + 10^{0.333084}) + 295
\]
\[
= \log(7314.7 + 2.1532) + 295
\]
\[
= \log(7316.8532) + 295
\]
\[
= 3.864324 + 295
\]
\[
= 298.864324
\]

Thus, the log of the sum has been calculated using numbers no bigger than 7316.8532. Since the calculations on the IBM 370 have only 16 significant figures, numbers whose magnitudes differ by more than 10^{16} are not added by the above method. In this case the sum is set equal to the larger of the two numbers.

The coefficients of f(X) are calculated using the aforementioned techniques and stored in an array called RIGHT. The variable RINOM is set equal to the logs of certain binomial coefficients, and it is used in calculating terms of FICAS3 (F_1 for Case 3; i.e., F_1 for SEC-DED) of the form

\[
\left( \binom{n}{i} - A_i \right) q^{n-i} p^i
\]

These terms are stored in COEFC3.

In order to calculate the parameters for each case for any particular value of S/N, values for q and p, which are dependent on the code rate, must be evaluated. The dependence on code rate requires calculations of Q1 and P1, Q2 and P2, and Q3 and P3 for use with Cases 1, 2, and 3 and 4 respectively. Since Case 4 has the same code rate as Case 3, Q3 and P3 are applicable to both.

IV. Numerical Results and Conclusions:

In Figures 2-49, we have plotted the performance measure F as a function of the signal-to-noise ratio E_b/N_0 of the binary symmetric channel in db, for
$N=7 \quad \alpha=0.0$

**Legend:**
- □: NO CODING
- ○: SEC
- △: SEC-DED
- ✷: TED

**Figure 2**
\[ N = 7 \quad \alpha = 0.1 \]

**Figure 3**
\( N = 7 \quad \alpha = 0.3 \)
$N=7 \quad \alpha=0.5$

**FIGURE 5**
N=7  \( \alpha = 0.8 \)
N = 7 \quad \alpha = 1.0

FIGURE 7
N=15  \alpha=0.0

FIGURE 8
$N=15 \quad \alpha=0.1$
$N=15 \quad \alpha=0.3$
N = 15  \alpha = 0.5
N = 15  \alpha = 0.8
N = 15  \alpha = 1.0

FIGURE 13
N = 31 \quad \alpha = 0.0

FIGURE 14

22
$N=31 \quad \alpha=0.1$
$N = 31 \quad \alpha = 0.3$
$N = 31 \quad \alpha = 0.8$

**Figure 18**

26
$N=31 \quad \alpha=1.0$

FIGURE 19
N = 63 \hspace{1em} \alpha = 0.0
$N=63 \quad \alpha = 0.1$

**FIGURE 21**
$N=63 \quad \alpha=0.3$
N = 63  \alpha = 0.5

FIGURE 23
FIGURE 24

$N = 63 \quad \alpha = 0.8$
$N=63 \quad \alpha=1.0$

![Graph showing the relationship between S/N and noise level with two curves and markers indicating the noise levels at different S/N values.]

FIGURE 25
N = 127  \alpha = 0.0

FIGURE 26
$N = 127 \quad \alpha = 0.3$
$N = 127 \quad \alpha = 0.1$
N = 127  \quad \alpha = 0.5
$N=127 \quad \alpha=0.8$
$N=127 \quad \alpha=1.0$

FIGURE 31

$S/N (DB)$

$F$
$N=255 \quad \alpha=0.1$

FIGURE 33

41
$N = 255 \quad \alpha = 0.3$

Figure 34
\[ N = 255 \quad \alpha = 0.5 \]
$N=255 \quad \alpha=0.8$

FIGURE 36

44
N=255  \( \alpha = 1.0 \)

**Figure 37**

Diagram showing the relationship between S/N (DB) and some other measurement or parameter with markers indicating specific points.

Legend: □ AND ○ = 0
\[ N = 511 \quad \alpha = 0.0 \]

Figure 38
N = 511  \( \alpha = 0.1 \)
N=511  \alpha=0.3

FIGURE 40

48
$N = 511 \quad \alpha = 0.5$

![Graph](image-url)

**Figure 41**
N = 511 \quad \alpha = 0.8

FIGURE 42
N = 511  \alpha = 1.0

\[0.00001, 0.0001, 0.01, 0.1, 1.0\]

\[2.00, 4.00, 6.00, 8.00, 10.00, 12.00\]

FIGURE 43
N = 1023 \quad \alpha = 0.0

FIGURE 44
N = 1023  \alpha = 0.8

![Graph showing a curve with S/N (DB) on the x-axis and F on the y-axis.](image)
all four cases described above, and for \( n = 7, 15, 31, 63, 127, 255, 511, 1023 \), and \( \alpha = 0, 0.1, 0.3, 0.5, 0.8, 1.0 \). We draw the following conclusions. For the extreme values of \( \alpha \) the relative importance of each case remains fixed for all values of \( E_b/N_0 \). When \( \alpha = 0 \), indicating an interest only in the quality of the output, the relative ratings from best to worst are: TED, SEC-DED, SEC, No coding. As expected, when \( \alpha = 1 \), indicating an interest only in the quantity of output data, the relative ratings are just opposite to the \( \alpha = 0 \) cases. For \( \alpha = .5 \) and \( n \geq 15 \), the relative order also remains fixed: SEC, SEC-DED, No coding, TED. Notice that the extreme cases of large quantity of output achievable with no coding and high quality of output achievable with a TEC system are both given poor relative ratings for this value of \( \alpha \) showing no preference of quantity over quality or vice-versa. Also, for \( \alpha = .5 \) \( n = 7 \), no coding becomes preferable to SEC-DED at signal-to-noise ratios below 4.1 db. This would be due to the increased data rejection by a SEC-DED decoder as the noise becomes greater.

A preference for quality over quantity without total disinterest in the latter is explored by the \( \alpha = .1 \) and \( \alpha = .3 \) cases. No coding for these values of \( \alpha \) is never preferable to any other of the choices considered except TED. As \( S/N \) increases TED becomes relatively less desirable as triple errors become less likely and its low transmission rate becomes dominant. Similarly, as \( E_b/N_0 \) increases SEC-DED becomes less important than just SEC. However, as the block size increases, TED and SEC-DED become more important since the probabilities of the errors these decoders are designed to correct increase.

Finally, when quantity is somewhat preferred over quality as with \( \alpha = .8 \), as might be expected the relatively extreme quality achieved by TED is shown to be undesirable for all values of \( S/N \) tested since this quality is achieved at the expense of quantity. The SEC decoder which employs no data rejection yet achieves some degree of error correction is found to be the best of all.
four cases for all values of $E_b/N_0$ considered. At very low noise levels the error correcting properties of SEC-DED make it more desirable than no coding, while at high noise levels the data-rejection of SEC-DED make it less desirable than no coding. For example, for $n = 1023$, SEC-DED is preferable to no coding for $E_b/N_0$ from 4.4 to 6.9 db.

More insight into the nature of this function can be gained by looking at what is necessary to achieve a desired level of effectiveness. A typical example is shown in Table 1. Here the desired value of $F$ is set at .01. With no coding or SEC, a higher value of $E_b/N_0$ is required as quality becomes more preferable. However, with SEC-DED or TED a lower value of $E_b/N_0$ is required to achieve the same value of $F$ as emphasis is switched to quantity.

$$E_b/N_0 \text{ IN DB REQUIRED TO ACHIEVE } F = .01 \text{ FOR } n = 255$$

<table>
<thead>
<tr>
<th>Alpha</th>
<th>No Coding</th>
<th>SEC</th>
<th>SEC-DED</th>
<th>TED</th>
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<td>7.36</td>
<td>6.64</td>
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<td>.3</td>
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<td>.8</td>
<td>8.00</td>
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<td>7.28</td>
<td>8.96</td>
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</table>

TABLE 1
I. Introduction:

The Viterbi algorithm is a method for determining the most likely sequence of states of a time-discrete Markov process; and, as such it is an optimal method for decoding convolutional codes. An evaluation of the effectiveness of this algorithm as a decoding method is accomplished herein through simulation on an IBM 370 computer using a main program written in the Fortran language and three subroutines written in Assembler language.

II. The Simulation Procedure:

A block diagram of the simulation is shown in Figure 50. A pseudo-random number generator is used to independently generate binary source bits of equal probability and binary noise bits whose probabilities depend on the assumed channel characteristics. The source bits are encoded with the appropriate parity check bits in blocks of two. Each bit is added to a noise bit using modulo-2 arithmetic (simulating channel noise). The information noise bits and parity check noise bits are generated independently. The corrupted bits are then decoded using the Viterbi algorithm. Accuracy of the decoding algorithm is measured as

\[ E = \lim_{n \to \infty} E_n = \frac{\text{Number of correct information bits}}{\text{Total bits}} \]

where \( E_n \) is the ratio after information bits have been transmitted.

III. The Viterbi Algorithm:

Given an observed output sequence \( Z = (z_1, z_2, \ldots, z_k) \), the purpose of the Viterbi algorithm is to determine the most likely input sequence \( X = (x_0, \)
NOISE
(2 RANDOM GENERATORS)

SOURCE
(RANDOM NUMBER
GENERATOR)

SOURCE BITS

ENCODING
SIMULATION

ENCODED BITS

CORRUPTED
ENCODED BITS

VITERBI
DECODING
SIMULATION

DECODED
INFORMATION BITS

COMPARE

DETERMINE

\[ E_n = \frac{\text{# of correct bits decoded}}{\text{total # bits decoded}} \]

BLOCK DIAGRAM OF THE VITERBI ALGORITHM SIMULATION

FIGURE 50
The subscripts refer to the discrete time states. Since the process is assumed to be Markov, the probability of state $x_{i+1}$ depends only on the state $x_i$:

$$\text{i.e., } \Pr(x_{i+1} | X_0, x_1, \ldots, x_i) = \Pr(x_{i+1} | x_i).$$

The channel is assumed to be memoryless so that the observed output $z_i$ at time $i$ depends only on the transition from state $x_i$ to state $x_{i+1}$. This transition is symbolized as $t_i$. We want to determine the maximum a posteriori $\Pr(X, Z)$.

Because of the aforementioned Markov and memoryless assumptions:

$$\Pr(X, Z) = \Pr(X)\Pr(Z | X)$$

$$= \prod_{i=0}^{k-1} \Pr(x_{i+1} | x_i) \prod_{i=0}^{k-1} \Pr(z_i | t_i)$$

The Viterbi algorithm is a method of determining the shortest path between two points. We, therefore, assign a 'path length' between each pair of possible states from time $= i$ to time $= i + 1$. This length $\lambda(t_i)$ is defined as

$$\lambda(t_i) = -\ln \Pr(x_{i+1} | x_i) - \ln \Pr(z_i | t_i).$$

The total length for some input sequence $X$ would be

$$-\ln \Pr(X, Z) = \sum_{i=0}^{k-1} \lambda(t_i)$$

Since path length is a negative logarithm of the probability, the shortest (critical) path length between two points (i.e., the initial and final states) would be the one with the highest probability. This is the maximum a posteriori probability we are seeking. The Viterbi method of finding this critical path is based on the observation that at any given time $i$, each state $x_i$ has associated with it a shortest path to the initial state. This shortest sequence is called a survivor, designated $\hat{X}(x_i)$. The path length of survivor $\hat{X}(x_i)$ is designated...
gamma \( (x_i) \). Extending these survivors to time \( i + 1 \) requires merely adding the appropriate digit ("bit" for our purposes) to the existing survivor and adding the corresponding path length gamma \( (x_{i+1}, x_i) \) for comparison purposes in determining the survivors for each state at time \( i + 1 \). At the end of the sequence (time = \( k \)) the survivor corresponding to the state with the shortest survivor path length is optimal.

For the purpose of convolutional code decoding, the states correspond to the possible binary state permutations of a block of \( m \) shift registers. Assuming that for the source \( Pr(0) = Pr(1) = \frac{1}{2} \), it follows that for all possible transitions between states the term \( Pr(x_{i+1} | x_i) = \frac{1}{2} \); and, since it is a constant for all possible transitions, it may be ignored when calculating the optimal path. Thus, only the term \( Pr(z_i | t_i) \) is significant. For systematic codes, the observation \( z_i \) corresponds to both the information bit and the parity bit received as a block at time = \( i \). For non-systematic codes, \( z_i \) corresponds to a block containing a parity bit for each subgenerator polynomial. These probabilities are pre-calculated for each state and each possible received block before decoding begins.

Since we are concerned with a communications system with a semi-infinite number of bits transmitted, corresponding to a semi-infinite sequence, and since storing the resulting semi-infinite survivors is impractical, a limit must be placed on the number of bits stored as a survivor (i.e., the survivor length). Call this limit delta. Thus, at time = \( i \), a decision must be made concerning the bit at time = \( i - \delta \) (\( i \) minus delta). This survivor transaction becomes insignificant for delta large enough because survivors tend to converge to the same state nodes.
IV. The Simulation Program for Rate 1/2 Codes:

A Fortran language main program was used in conjunction with three custom written Assembler language subroutines. The main thing to be aware of when using the program is that the delta defined in the program is one greater than the corresponding delta as defined in the Viterbi algorithm (e.g., if you wish to get results for delta = 75, set delta = 76 in this program!). For each state at time i there are two possible states to which it can branch at time \(i + 1\) (one of which has an incoming 0 bit, the other has an incoming 1 bit). These possible transitions are stored in an array called NEXT. NEXT(1,1) corresponds to the branch of state 1 with an incoming 0; whereas, NEXT(1,2) corresponds to an incoming 1.

Probabilities which determine the path lengths are calculated prior to the main iterative loop. These calculations are done for all four possible two bit permutations corresponding to a received block containing an information bit and a parity check bit in the systematic case, or two parity check bits in the non-systematic case. Array POFZLN stores these predetermined path lengths. Thus in the main iterative loop the increase in the total path length \(\Gamma\) of each state can be determined by a simple table reference (i.e., POFZLN). Survivors are stored and saved by arrays SURVIV and SAVUR, while the corresponding path lengths are saved using arrays GAMMA and SAVE.

The appropriate subgenerator vectors are stored in the array GEN. In the case of a systematic code the second subgenerator is a 1 followed by \(m - 1\) zeros. For example, the subgenerators for an \(m = 5\) systematic code would be: 10000 and 11011. Note that the subgenerator 10000 merely generates the information bit. SHIFT2 saves the contents of the simulated encoding shift registers.
All random bit generation is done independently for each application. Source bits of equal probability are generated using a Scientific Subroutine Package member called RANDU. These source bits are stored in PRSOUR. The generated noise bits \( \Pr(0) = q \) are stored in PNOISE. Decoded bits are stored in PROUT for comparison with the original source bits in PRSOUR.

Many different counters are used to keep track of time states corresponding to source bits, noise bits, and output bits. IOUT determines the printed increments for \( E_n \). During the course of this research, IOUT was set equal to 1000 so that the accuracy \( E_n \) was printed out for \( n = 1000, 2000, 3000 \), etc. The \( n \) refers to the number of decoded bits and is called NDECOD within the program.

Read and punch statements are included to save the information necessary to restart the program where it left off. This feature is desirable to enable the programmer to check the convergence of the accuracy figures and compare them with other data in order to determine the desirability of decoding a greater number of bits. However, due to the fact that hexadecimal double precision accuracy used by the IBM 370 computer is equivalent to about 16.7 decimal digits, and the data cards are punched with decimal numbers, there is a slight loss in accuracy that is sometimes noticeable but generally insignificant.

Three Assembler languages subroutines were written to expedite the execution of the program. These are COPYAR, SHIFT, and TESTBT. COPYAR simply copies SAVSUR into SURVIV. SHIFT is used to shift the survivor of a row in SURVIV, add a 0 or a 1, and transfer the resulting survivor to the row specified by NEXT in the array SAVSUR for the next time increment. TESTBT determines whether the bit at time \( k - \delta \) (Viterbi definition of \( \delta \)) is a 0 or a 1 in the survivor of the current state whose total path length is the least.
Typical decoding rates are shown in Table 2. The number of bits refers to the total number of both parity and information bits. To obtain the number of information bits decoded per second, multiply the rates shown in Table 2 by the code rate, which in all cases explored here is $\frac{1}{2}$. Note that decoding an $m = 7$ code is approximately twice as slow as decoding an $m = 6$ code since the latter has half the possible states of the former.

**TYPICAL DECODING RATES**

<table>
<thead>
<tr>
<th>$m$</th>
<th>delta</th>
<th>bits/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>31</td>
<td>578</td>
</tr>
<tr>
<td>5</td>
<td>52</td>
<td>506</td>
</tr>
<tr>
<td>6</td>
<td>31</td>
<td>324</td>
</tr>
<tr>
<td>6</td>
<td>59</td>
<td>275</td>
</tr>
<tr>
<td>7</td>
<td>27</td>
<td>165</td>
</tr>
<tr>
<td>7</td>
<td>59</td>
<td>145</td>
</tr>
</tbody>
</table>

**TABLE 2**

V. Program for Simulating A Viterbi Decoder
Read: MAXBIT, N, M, NROWS, NCOL, GEN(M), DELTA, STON, NXMTD, IXSOUR, IXNOIS(N).
Define: N2, K, K2, DELMI, IDEPTH, IDEPCP, IBIT, MAXDEC, NSTATE, Q.

Determine the two branches for each state: i.e., NEXT(I,1), NEXT(I,2).

Print initial values of IXSOUR and IXNOIS.

\[\text{NXMTD} = 0?\]

No

Initialize: ERRORS, NTOTAL, NCOUNT, NDECO, PARCK(N), SHIFT2(M), GAMMA(NSTATE), SURVIV(NSTATE, NCOL).

Define IOUT.
Initialize NPLOT.
Print STON and Q.
Establish P with a lower limit of 1.D-75.
Define: QLN, PLN.

Yes

Read: ERRORS, NTOTAL, PNOISE(NCOUNT, N), PRSOUR(NCOUNT), PARCK(N), GAMMA(NSTATE) SURVIV(IDEPCP, NCOL), SHIFT2(M).

Calculate POFZLN.
This array contains the predetermined branch lengths based on the probability that a given state is correct for the block of bits received. i.e., POFZLN(I, LOC) contains the probability, expressed as a path length that state I is correct for block number LOC which can be any of 2**N permutations.
Call random number generator and generate a source bit.

Increment: NXMTD, NCOUNT.

Store source bit in PRSOUR.

Shift source bit into SHIFT2.

Generate parity bits.

Add noise.

Determine the location LOC in POFZLN corresponding to the block of corrupted parity bits.

For each pair of states at time = NXMTD:
Determine GAMCHK at time NXMTD-1.
Determine and save GAMMA in array SAVE for state at time NXMTD.
Save new survivors in SAVSUR at time NXMTD.

Determine the state IMIN which has the shortest overall path length.
Determine GAMMAX, the maximum stored overall path length.

Reduce all values of Gamma by $10^{50}$.
Call COPYAR to copy SAVSUR into SURVIV.

NDECOD<0?

Yes

Call TESTBT to make a final decision on the bit at time NXMTD-(DELTA-1) based on survivor of state IMIN. Store this bit in PROUT.

No

Increment NDECOD and NTOTAL.

NXMTD>MAXDEC?

No

Yes

Set IOUT = NDECOD-1

Yes

Increment NDECOD and NTOTAL.

Determine cumulative number of noise bits on each block bit corresponding to NTOTAL information bits using PNOISE and store in PARCK.

Determine total errors for the decoded bits by comparing PROUT to PRSOUR.
Reset NCOUNT = DELM1 and NDECOD = 1.

Determine current accuracy and store in FPLT.

Transfer last DELM1 source into first DELM1 positions of PRSOUR.

Store noise counts in PNPLT.

Store NTOTAL in XMTPLT.

Transfer last DELM1 noise bits into first DELM1 positions of PNOISE.

If NXMTD > MAXDEC?

Print: M, GEN, DELTA, table headings, bits, accuracy, noise.

Punch data for restart.

STOP.
PROGRAM FOR SIMULATING A VITERBI DECODER

FOR STORAGE REASONS DELTA IN THIS PROGRAM SHOULD BE SET TO ONE MORE THAN THE CASE BEING SIMULATED, I.E. TO SIMULATE DELTA=13, SET DELTA=14.

THE FOLLOWING MINIMUM ARRAY DIMENSIONS MUST BE ALLOCATED:

PNOISE (IOUT+DELTA,N)
PRSOUR (IOUT+DELTA)
PROUT (IOUT)
PNPLOT (IOUT)
GEN (M,N)
SHIFT2 (M)
L (M)
NEXT (NSTATE,2)
PAR (4)
OUTPUT (N)
XMTPLT (NUMBER OF LINES PRINTED=(NUMBER OF INFORMATION BITS DECODED THIS RUN/IOUT))
FPLOT (SAME AS XMTPLT)
PARCK (N)
POFZLN (NSTATE,N2=2**N)
GAMMA (NSTATE)
SAVE (NSTATE)
IXNOIS (N)

SURVIV AND SAVSUR MUST BE DIMENSIONED EXACTLY (NROWS,NCOL) WHERE NROWS MUST BE AT LEAST 2**M AND NCOL MUST BE AT LEAST IDEPTH

INTEGER PARSUM
INTEGER*2 PNOISE(1064,3),PRSOUR(1064)
INTEGER GAMCHK,TEST
INTEGER*2 PROUT(1000)
INTEGER PNPLOT(1000,3)
INTEGER*2 GEN(8,3),SHIFT2(8),JKL, IOUT,M,K2,DELTA,DELTA1
INTEGER*2 DELMI,NSTATE,L(8),LR,NEXT(256,2),PAR(3),PARITY
INTEGER*2 OUTPUT(3),LOC,SOURCE
INTEGER SURVIV(256,2),SAVSUR(256,2)
INTEGER XMTPLT(1000)
INTEGER PARCK(3)
REAL*8 POFZLN(256,4),GAMMA(256),SAVE(256)
REAL*8 FPLOT(100)
INTEGER NTOTAL,ERRORS,NPLOT,NCOUNT,NDECD,NXMTD,MAXBIT,IXNOIS(3)
REAL*8 Q,P,DFLOAT,STONDERF,DSQRTQLN,PLN
REAL*8 OLOG,GAMMIN,GAMMAX

MNPLT AND PNPLOT STORE INFO AND PARITY NOISE COUNTS RESPECTIVELY
MAXBIT IS THE MAXIMUM NUMBER OF BITS TO BE XMTD

READ1104,MAXBIT
FORMAT(17)
N IS BLOCK LENGTH
READ110G,N
N2=2**N
K IS THE NUMBER OF INFORMATION BITS REPRESENTED BY A BLOCK
OF LENGTH N.
THIS PROGRAM WILL ONLY SIMULATE CODES OF RATE 1/N.
THerefore K must always equal 1.
K=1
K2=2**K
C M IS CONSTRAINT LENGTH
READ1100,M
1103 FORMAT(I1)
READ1104, NROWS
READ1104, NCOL
C GEN IS SUBGENERATOR VECTOR
DO 1506 J=1,N
1506 READ1102, (GEN(I,J), I=1,M)
1102 FORMAT(I11)
PRINT698
698 FORMAT(*1,1*)
C DELTA=SURVIVOR LENGTH PLUS ONE
READ1106, DELTA
1106 FORMAT(I13)
C STON IS THE SIGNAL-TO-NOISE RATIO IN DECIBELS.
READ1108, STON
1108 FORMAT(D15.7)
G=.50D+.50C*DERF(OSQRT(DFLOAT(K)/DFLOAT(N)*10.DO**(STON/10.DO)))
C , NXMTD=TOTAL SOURCE BITS XMTD
READ2000,NXMTD
2000 FORMAT(D120)
C IXSOUR AND IXNOIS ARE RANDOM INTEGERS USED AS STARTING
C POINTS FOR THE RANDOM NUMBER GENERATOR.
READ2000, IXSOUR
DO 1510 I=1,N
1510 READ2000, IXNOIS(I)
C IDEPTH IS THE NUMBER OF WORDS REQUIRED TO STORE ONE SURVIVOR.
IDEPTH=DELM1/32+1
IDEPCP=NCOL+1-IDEPTH
IBIT=DELTA-32*(TIEPTH-1)
MAXDEC=MAXBIT+DELM1
M1=M+1
DO 9 I=1,M
9 L(I)=1
DO 16 I=M,NSTATE
DO 11 L(R)=1
11 IF(L(L(R)))11,14,11
14 L(L(R))=0
GO TO 15
15 L(L(R))=2
GO TO 16
C NEXT IS AN ARRAY CONTAINING THE NUMBERS CORRESPONDING
C TO THE 2 STATES THAT ANY GIVEN STATE CAN BRANCH TO.
16 NEXT(1,1)=1
DO 30 LM=2,M
30 NEXT(1,1)=NEXT(1,1)+2**1(LM-2)*1(LM)
17 NEXT(1,2)=NEXT(1,1)+2**1(M-1)
PRINT1138, IXSOUR
1138 FORMAT(120)
DO 1139 I=1,N
1139 PRINT1140, IXNOIS(I)
1140 FORMAT(*1,1*)
IF(NXMTD.EQ.0)GO TO 1300
READ2000,ERRORS
READ2000,NTOTAL
NCOUNT=DELM1
NDECODE=1
2001 FORMAT(7211)
DO 1550 J=1,N
1550 READ2001,(PNOISE(I,J),I=1,NCOUNT)
READ2001,(PRSOUR(I),I=1,NCOUNT)
DO 1551 I=1,N
1551 READ2003,PARCK(I)
DO 1302 I=1,NSTATE
READ2002,GAMMA(I)
2002 FORMAT(D23.16)
DO 1303 I=1,NSTATE
READ2000,SURVIV(I,J)
READ2001,(SHIFT2(I),I=1,M)
GO TO 1301
1300 CONTINUE
C INITIALIZE RANDU SOURCE AND NOISE
C ERRORS=NUMBER OF DECODING ERRORS
ERRORS=0
C NTOTAL=TOTAL BITS PRINTED
NTOTAL=1-DELTA
C NCOUNT PLACES OUTPUT BITS IN CORRECT VECTOR POSITION
NCOUNT=0
C NDECOD=NUMBER OF BITS DECODED
NDECOD=2-DELTA
C PARCK=TOTAL NUMBER OF PRINTED PARITY CHECK BITS CORRUPTED BY NOISE
DO 1552 I=1,N
1552 PARCK(I)=0
C SHIFT2=CONTENTS OF ENCODING SHIFT REGISTER
DO 1  I=1,M
1  SHIFT2(I)=0
GAMMA(I)=0.00
DO 60 I=2,NSTATE
60 GAMMA(I)=1.040
DO 61 I=1,NROWS
61 SURVIV(I,J)=0
1301 CONTINUE
C IOUT=NUMBER OF BITS PRINTED PER LINE
IOUT=1000
C NPLOT COUNTS NUMBER OF TIMES PRINTING ALGORITHM IS USED
NPLOT=0
PRINT998,STON
998 FORMAT(*'-**S/N = ',D15.7)
PRINT704,Q
704 FORMAT(*'+**T3Q,*Q= ',D15.7)
P=1.0000
IF(P.LT.1.0-75)P=1.0-75
QLN=-1.00*DLOG(Q)
PLN=-1.00*DLOG(P)
DO 49 I=1,M
49 L(I)=1
DO 50 I=1,NSTATE
DO 51 LR=1,M
50 L(LR)=0
GO TO 53
51 L(LR)=1
52 L(LR)=1
53 DO 1500 IBITNR=1,N
PARSUM=0
DO 20 II=1,M
1500 CONTINUE
20 PARSUM=PARSUM+GEN(II,IBITNR)*L(II)

73
PAR(IBITNR)=PARSUM-PARSUM/2*2
DO 21 I3=1,N
21 OUTPUT(I3)=1
DO 50 J=1,N2
DO 55 I4=1,N
I4MIN1=N+1-I4
IF OUTPUT(I4MIN1)55,56,55
55 OUTPUT(I4MIN1)=0
GO TO 57
56 OUTPUT(I4MIN1)=1
DO 50 IL77=1,N
IF(OUTPUT(IL77).EQ.PAR(IL77))POFZLN(I,J)=POFZLN(I,J)+QLN
IF(OUTPUT(IL77).NE.PAR(IL77))POFZLN(I,J)=POFZLN(I,J)+PLN
699 CALL RANDU(IXSOUR,YXSOUR,YSOUR)
IXSOUR=IYSOUR
SOURCE=0
IF(YXSOUR-0.5)1511,1512,1512
1512 SOURCE=1
NXMTD=NXMTD+1
NCOUNT=NCOUNT+1
PRSOR(NCOUNT)=SOURCE
DO 2 I=2,M
I1=M+2-I
2 SHIFT2(I1)=SHIFT2(I1-1)
SHIFT2(I1)=SOURCE
C PARITY=PARITY CHECK DIGIT
DO 1516 IBITNR=1,N
PARSUM=0
DO 1517 I1=1,M
1517 PARSUM=PARSUM+GEN(I1,IBITNR)*SHIFT2(I1)
1516 PAR(IBITNR)=PARSUM-PARSUM/2*2
DO 1530 I=1,N
PNOISE(NCOUNT,I)=0
CALL RANDU(IXNOIS,IYNOIS,YNOIS)
IXNOIS(I)=IYNOIS
IF(YNOIS-0.5)1531,1531,1530
1531 PNOISE(NCOUNT,I)=1
KPAR=PAR(I)
IF(KPAR.EQ.0)PAR(I)=1
IF(KPAR.EQ.1)PAR(I)=0
1530 CONTINUE
C LOC= LOCATION IN PROBABILITY MATRIX CORRESPONDING TO BLOCK RECEIVED
LOC=1
DO 1504 LOCSUM=1,N
1504 LOC=LOC+2**(N-LOCSUM)*PAR(LOCSUM)
C
C STATES ARE NUMBERED SUCH THAT 1 AND 2, 3 AND 4, ETC. ARE
C PAIRS THAT BRANCH TO THE SAME STATES.
C I.E. 0X AND 1X BOTH BRANCH TO X0 AND X1, WHERE X REPRESENTS
C A PERMUTATION OF M-1 BITS.
C
C GAMCHK IS THE STATE OF GIVEN PAIR OF STATES WHICH HAS THE SHORTEST
C TOTAL PATH LENGTH GAMMA.
C
C DO 100 I=1,NSTATE,2
GAMCHK = 1
IF (GAMMA(I) .GT. GAMMA(I+1)) GAMCHK = I+1
DO 101 I = 1, K2
101 SAVE (NEXT(I,J)) = GAMMA(GAMCHK) + PDFZLN(NEXT(I,J), LOC)
IROWC = NEXT(I,1)
IROW1 = NEXT(I,2)
CALL SHIFT (SURVIV, SAVSUR, NROWS, NCOL, GAMCHK, IROWC, IROW1)
CONTINUE
DO 139 I = 1, NSTATE
139 GAMMA(I) = SAVE(I)
142 GAMMIN = 1.0D70
GAMMAX = -1.0D70
IMIN = 1
DO 140 I = 1, NSTATE
IF (GAMMA(I) .GT. GAMMAX) GAMMAX = GAMMA(I)
IF (GAMMA(I) .GE. GAMMIN) GO TO 140
GAMMIN = GAMMA(I)
IMIN = I
CONTINUE
149 IF (GAMMAX .LE. 1.0D50) GO TO 150
DO 141 I = 1, NSTATE
141 GAMMA(I) = GAMMA(I) - 1.0D50
GO TO 142
150 CALL COPYAR (SAVSUR, SURVIV, NROWS, NCOL)
IF (INDECOD .LE. 1) GO TO 715
CALL TESTBT (SURVIV (IMIN, IDEPCP), I81T, TEST)
PROUT (INDECOD) = TEST
715 NDECOD = NDECOD + 1
NTOTAL = NTOTAL + 1
IF (NXMTD .GE. MAXDEC) GO TO 750
IF (INDECOD .GT. IOUT) GO TO 700
GO TO 699
CONTINUE
NPLOT = NPLOT + 1
DO 776 J = 1, N
DO 776 I = 1, IOUT
776 IF (PNQOISE(I,J) .EQ. 1) PARCK(J) = PARCK(J) + 1
777 IF (PNSOUR(I,J) .NE. PROUT(I)) ERRORS = ERRORS + 1
FPLOT(NPLOT) = DFLOAT(NTOTAL - ERRORS) / DFLOAT(NTOTAL)
DO 1561 I = 1, N
1561 PNPL0T(NPLOT, I) = PARCK(I)
XMTPLT(NPLOT) = NTOTAL
IF (NXMTD .GE. MAXDEC) GO TO 900
DO 778 J = 1, N
DO 778 I = 1, DELM1
778 PNQOISE(I,J) = PNQOISE(I, IOUT+I, J)
DO 783 I = 1, DELM1
793 PNSOUR(I) = PNSOUR(I, IOUT+I)
NQOUNT = DELM1
NDECOD = 1
GO TO 699
750 IOUT = NDECOD - 1
GO TO 700
CONTINUE
PRINT505, M
500 FORMAT ('-', 'CONSTRAINT LENGTH=', 13)
DO 1565 J = 1, N
1565 PRINT501, (GEN(1, J), I = 1, M)
501 FORMAT ('-', 'SUBGENERATOR=', 10I6)
PRINT1001, DELTA

75
1001 FORMAT('-','DELTA=',13)
       PRINT602
602 FORMAT(130,'T2','BIT2','T19','ACCUR','T37','NOISE COUNT: BIT1,BIT2,
       BIT3, ETC. ')
600 PRINT604
604 PRINT612, XMTPLT(J), FPL0T(J), (PNPLOT(J), I=1,N)
       PRINT615, MAXBIT
1195 FORMAT(17,T73,'MAXBIT')
       PUNCH1135,N
1135 FORMAT(17,T73,'N')
       PUNCH1101,M
1101 FORMAT(17,T73,'M')
       PUNCH1136,NROWS
1136 FORMAT(17,T73,'NROWS')
       PUNCH1137,NCOL
1137 FORMAT(17,T73,'NCOL')
       DO 1566 J=1,N
1566 PUNCH1103,J, (GEN(I,J), I=1,M)
1103 FORMAT(23,13), T19,9I7
       PUNCH1107,DELTA
1107 FORMAT(17,T73,'DELTA')
       PUNCH1109,STON
1109 FORMAT(17,T73,'STON')
       PUNCH3000,NXMTD
3000 FORMAT(17,T73,'NXMTD')
       PUNCH3100,IXSOUR
3100 FORMAT(17,T73,'IXSOUR')
       DO 1567 J=1,N
1567 PUNCH3200,IXNOIS(J), J
3200 FORMAT(120,T73,'IXNOIS(*)')
       PUNCH3300,ERRORS
3300 FORMAT(17,T73,'ERRORS')
       PUNCH3400,NTOTAL
3400 FORMAT(17,T73,'NTOTAL')
       DO 1569 J=1,N
1569 PUNCH3101,J, (PNoise(IOUT+I,J), I=1,DELM1)
3101 FORMAT(17,T73,'PNoise(*)')
       PUNCH3201,IPSOUR(IOUT+I), I=1,DELM1
3201 FORMAT(17,T73,'IPSOUR(*)')
       DO 1570 J=1,N
1570 PUNCH3500,PARCK(J), J
3500 FORMAT(17,T73,'PARCK(*)')
       DO 4000 I=1,NSTATE
4000 PUNCH3002,GAMMA(I), I
3002 FORMAT(17,T73,'GAMMA(*)')
       DO 4001 I=1,NSTATE
4001 PUNCH3001,J, SURVIV(I,J)
3301 FORMAT(17,T73,'SURVIV(*)')
       STOP
END
TESTBIT PROGRAM
***TESTBIT***CHECKS VALUE OF ANY BIT IN A FULL WORD IN MAIN STORAGE.
* R1--ADDR OF ARGUMENT LIST
* C1(1) ADDR OF WORD
* 4(1) ADDR OF TESTBIT
* 8(1) ADDR FOR RETURN CCCE

L 2,C(1) LCAD ACCR CF WORD TO BE SHIFTED
L 4,C(1) LCAD ACCR CF TESTBIT
L 0,C(4) LCAD TESTBIT
LA 3,0 SET FOR COMPARE
L 4,C(2) LCAD TEST WORD
LA 5,32 LCAD 32 FOR SUBTRACTION
SR 5,0 FIND COMPLEMENT
SLL 4,C(5) SHIFT SC TEST BIT IS IN THE SIGN POSITION
CR 4,3 COMPARE RESULTS AGAINST ZERO
BNL NCTNEG
LA 3,1

NCTNEG ECU *
L 5,E(1) LCAD ACCR OF RETURN WORD
ST 3,C(5) STORE BIT IN RETURN WORD
STOP
ENC
**CCPYAR**

**PRCGRAP**

*****COPYARRAY***** COPIES AN ARRAY THROUGH THE USE OF THE MVCL INSTRUCTION.

* **INPUT:**
  * R1——ADDR OF ARGUMENT LIST
  * 0(1) ADDR OF INPUT ARRAY
  * 4(1) ADDR OF CUTPLT ARRAY
  * 8(1) ADDR OF NUMBER OF ROWS IN THE ARRAYS
  * 12(1) ADDR OF NUMBER OF COLUMNS IN THE ARRAYS
  *

**CCPYARRAY EQU** *

| L | 2,4(1) | ADDR OF ARRAYC |
| L | 4,C(1) | ADDR OF ARRAYI |
| L | 6,F(1) | LCAC ADDR OF NUMBER OF ROWS |
| L | 5,12(1) | LCAC ADDR OF NUMBER OF COLUMNS |
| L | 5,0(5) | LCAC NUMBER OF COLUMNS |
| MH | 5,2(6) | MLTIPLY NUMBER OF ROWS BY COLUMNS |
| SLA | 5,2 | (ROWS * COLUMNS) * 4 |
| LR | 3,5 | CCPY LENGTH IN R3 FOR MVCL |
| MVCL | 2,4 | CCPY ARRAY |
| STCP | | |
| ENC | | |

78
SHIFT PROGRAM

***COPYROWS*** CONTAINS THE LOGIC NECESSARY TO COPY AND SHIFT

* THE SPECIFIC ROWS AS SPECIFIED. COPYROWS WILL COPY RCWI
* INTO RCWG AND ROW1. SHIFT THESE ROWS TC THE LEFT ONE BIT,
* AND SET THE RIGHTMOST BIT OF RCWG AND ROW1 TC ZERO AND ONE
* RESPECTIVELY.

* INPUT:

* R1--ADDR OF ARGUMENT LIST
* 0(1) ADDR OF INPUT ARRAY (ARRAY1)
* 4(1) ADDR OF OUTPUT ARRAY (ARRAYC)
* 8(1) ADDR OF NUMBER OF ROWS
* 12(1) ADDR OF NUMBER OF COLUMNS
* 16(1) ADDR OF ROW1
* 20(1) ADDR OF ROWO
* 24(1) ADDR OF ROW1

* COPYROWS EQU *

LA 7,0
L 12,8(1) LCAC ADDR CF NUMBER OF ROWS
L 12,0(12) LCAC NUMBER OF ROWS
SLA 12,2 MULTIPLY BY FOUR FOR DISPLACEMENT
L 6,0(1) LCAC ADDR CF ARRAY1
L 8,12(1) LCAC ADDR CF NUMBER OF COLUMNS
L 8,0(18) LOAD NUMBER OF COLUMNS
L 2,4(1) LCAC ADDR CF ARRAYC
LR 3,8 COPY NUMBER OF COLUMNS FROM R8
BCTR 3,0 SUBTRACT ONE FROM NUMBER OF COLUMNS
L 4,8(1) LCAC ADDR CF NUMBER OF ROWS
L 3,24(1) COMPUTE ROWS*(COLUMNS-1)
LR 5,3 ***
LR 10,3 ***
LR 11,3 ***
L 3,16(1) ***
L 4,20(1) ***
L 5,24(1) ***
L 3,0(3) ***
L 4,0(4) ***
L 5,0(5) ***
BCTR 3,0 ***
BCTR 4,0 ***
BCTR 5,0 ***
AR 9,2 ***
AR 10,4 ***
AR 11,5 ***
SLA 9,2 ***
SLA 10,2 ***
SLA 11,2 ***
AR 9,6 ***
AR 10,2 ***
AR 11,2 ***
MVC 0(4,10),0(9) COPY LAST WORD OF RCWI TO ROWO
LA 0,0 INPUT FOR RCWG
LA 2,10 INPUT FOR SLIDE
BAL 14,Slice BRAND TO SLIDE
MVC 0(4,11),0(9) COPY LAST WORD OF RCWI TO ROW1
LA 0,1 INPUT FOR ROW1
LA 2,11 INPUT FOR SLIDE
BAL 14,Slice BRAND TO SLIDE
BCTR 8,0 DECREMENT COLUMN COUNT BY ONE
CR 8,7 COMPARE COLUMN COUNT AGAINST ZER
BE THRU IF EQUAL ENC
PART C
DECACCR EQU *
SR 9,12 ***
SR 10,12 ** DECREMENT ACDRS BY RCW MEMBER DIST
SR 11,12 *
MVC 0(4,10),0(9) "CPY NEXT RCWI MEMBER TO RCWO"
LR 2,10 "SET UP FCR SLICE--CARRY BIT ALREADY SET"
BAL 14,SLICE "BRANCH TO SLICE"
MVC 0(4,11),0(10) "COPY RCWO MEMBER INTO RCW1 MEMBER"
BCT 8,DECACCR "DECREMENT AND TEST COLUMN COUNT"
T=RU EQU *
STCP
ARGLSTAD CS F
***SLICE***SHIFTS ONE FULL WORD IN MAIN STORAGE TO THE LEFT ONE BIT
* SETTING THE RIGHTMOST BIT IN THE WORD AS INDICATED BY RC. THE
* ADDR OF THE WORD TO BE SHIFTED IS CONTAINED IN R2 AND THE CARRY
* CARRY BIT IS PLACED IN RC.
*
SLICE EQU *
STM 3,4,SAVEREGS "SAVE WORKING REGISTERS"
LA 4,C "LCAD ZER; FCR COMPARE"
L 3,0(2) "COPY WORD TO BE SHIFTED"
CR 3,4 "CHECK FCR CARRY"
SLL 3,1 "SHIFT OVER ONE Bit"
BAL CARRYO "BRANCH ACCORDINGLY"
CARRY1 EQU *
LA 4,1 "WORD NEGATIVE, THEREFORE SET CARRY BIT"
CARRYO AR 3,0 "SHIFT IN BIT BY ADDITION"
LR 0,4 "STORE CARRY VALUE"
ST 3,0(2) "REPLACE SHIFTED WORD"
LM 3,4,SAVEREGS "RESTORE WORKING REGISTERS"
BR 14 "RETURN TO CALL--BAL 14,SLIDE"
SAVEREGS CS 2F
END
I. Introduction:

A binary convolutional code of constraint length $K$ and rate $R = \frac{1}{2}$ is completely specified by a set of two generators which in transform notation have the form

$$G(j)(D) = g_0(j) + g_1(j) D + g_2(j) D^2 + \ldots + g_{K-1}(j) D^{K-1} (j = 1, 2)$$

with coefficients from GF(2). (Throughout we assume the codes are nondegenerate, i.e., at least one of $g_0(1)$ and $g_0(2)$ are at least one of $g_{K-1}(1)$ and $g_{K-1}(2)$ are one). If

$$I(D) = i_0 + i_1 D + i_2 D^2 + \ldots$$

is a sequence of binary information digits, then the result of applying $I(D)$ to the encoder is

$$T(j)(D) = I(D) G(j)(D) = t_0(j) + t_1(j) D + t_2(j) D^2 + \ldots (j = 1, 2)$$

so that for each information digit $i_k$ the encoder produces a block of two digits $[t_k(1), t_k(2)]$ that are functions of $i_k$ and the previous $K-1$ information digits. The linear sequential circuit that performs this operation consists of a shift register whose $K$ stages are connected to two modulo-2 adders in accordance with the coefficients of $G(1)(D)$ and $G(2)(D)$, respectively. The outputs of the adders at time $k$ then constitute the block $[t_k(1), t_k(2)]$. For convenience we denote the sequence of these blocks by $T(D)$.

In certain situations such as system check-out it is desirable to be able to recover the information sequence from the encoded sequence. Massey and Sain (1968) have shown that this is possible if and only if the code is noncatastrophic, i.e., if and only if

$$\gcd \{G(1)(D), G(2)(D)\} = D^2$$
for some \( l \geq 0 \). In this case, there always exists a linear sequential circuit that produces \( I(D) \) with a delay of exactly \( L \) digits for any integer \( L \geq l \) and it is completely described by two generator polynomials \( P(1) (D) \) and \( P(2) (D) \) that satisfy

\[
P(1) (D) G(1) (D) + P(2) (D) G(2) (D) = D^l
\]

To illustrate these ideas we consider the code

\[
G(1) (D) = 1 + D + D^2 + D^3 + D^6
\]
\[
G(2) (D) = 1 + D^2 + D^3 + D^5 + D^6
\]

This code has a constraint length \( K = 7 \) and its circuit realization is shown in Figure 51. If the input sequence is

\[
1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1
\]

then \( T(1) (D) \) and \( T(2) (D) \) are given by

\[
1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ ...
\]

and

\[
1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ ...
\]

respectively, and the encoder output sequence will be

\[
1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ ...
\]

Since \( G(1) (D) \) and \( G(2) (D) \) are relatively prime, an inverse circuit with delay zero exists and we may easily prove that \( P(1) (D) \) and \( P(2) (D) \) are given by

\[
P(1) (D) = 1 + D + D^2 + D^3 + D^4
\]
\[
P(2) (D) = D^2 + D^4
\]

Two versions of the circuit realization are shown in Figure 52.

Suppose now that the encoder output sequence \( T(D) \) is transmitted over a noisy channel prior to its inversion. Then, of course, the resulting sequence \( I(D) \) will generally not be a perfect match of the original information sequence.
In fact, Massey and Costello (1971) have shown that over the binary symmetric channel and at high signal-to-noise ratios the probability of an error in $\hat{I}(D)$ is related to the probability of error in the channel by

$$p_{\hat{I}} = A p_{BSC}$$

where $A$ is the error amplification factor given by

$$A = W[P^{(1)}(D)] + W[P^{(2)}(D)]$$

and $W[P^{(i)}(D)]$ denotes the Hamming weight of $P^{(i)}(D)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{encoding_circuit}
\caption{Encoding Circuit for the Code}
\end{figure}

In our example above $A$ has the value 7, so that an error in $\hat{I}(D)$ is seven times more likely than an error in the channel. This is quite obvious from Figure 52b. For a single error in the channel will, as it propagates through
FIGURE 52a. INVERSE CIRCUIT FOR THE CODE
1 1 1 1 0 0 1
1 0 1 1 0 1 1

FIGURE 52b. ALTERNATE INVERSE CIRCUIT FOR THE CODE
1 1 1 1 0 0 1
1 0 1 1 0 1 1

1(D) = ODD NUMBERED DIGITS
the circuit, produce 7 errors in the output of the adder, assuming that the channel errors are spaced far enough apart.

For low signal-to-noise ratios the simple reasoning leading to (1) no longer applies and the value of the error amplification must be determined empirically. Figure 53 shows the result for the code in the above example.

Consider next the system configuration of Figure 54.

At high signal-to-noise ratios a well designed decoder will be able to correct the overwhelming majority of the errors introduced in the channel and deliver an essentially perfect copy of \( I(D) \). If we then compare this output with that of the encoder inverse we obtain an indication of the signal-to-noise ratio in the channel.

With the binary symmetric channel, for example, we can get a good estimate of \( P^\dagger \) by computing the ratio of the number of ones in which the outputs of the decoder and the encoder inverse differ to the total number of digits processed. Using (1) we are then able to determine the value of \( P_{BSC} \).

The surprising fact is that this scheme also works for low signal-to-noise ratios, where the decoder output also includes errors, and produces a one-to-one relationship between \( P_{BSC} \) and the measured quantity, which we denote by \( \hat{P}_{BSC} \).

Figure 55 shows the simulation results for the code in our previous example, the binary symmetric channel and a 32 bit path length Viterbi decoder.

From Figure 53 it is clear that if one attempts to reconstruct the original information sequence at the channel output without benefit of decoding, it is desirable to have a code with as low a value of error amplification as possible. The best in this regard are the so-called systematic codes for which one of the \( p_i(D) \) is one and the other equals zero, resulting in \( A = 1 \). Unfortunately, the error correcting capability of these codes is markedly inferior to that of certain nonsystematic codes when used in conjunction with sequential or maximum likelihood decoding algorithms.
FIGURE 53. ERROR AMPLIFICATION FACTOR FOR THE CODE
1 1 1 1 0 0 1
1 0 1 1 0 1 1

FIGURE 54. CHANNEL NOISE MEASURING SYSTEM
FIGURE 55. MEASURED VERSUS ACTUAL CHANNEL BIT ERROR PROBABILITY FOR CODE

1 1 1 1 0 0 1
1 0 1 1 0 1 1
For nonsystematic codes the lowest possible value of $A$ is 2 and is attained by the so-called quick-look codes [2]. Our purpose in this note is to investigate their relevant characteristics and in the process we obtain a number of interesting and practically useful results. Since our primary motivation is the application of quick-look codes to Viterbi decoding, we restrict consideration to constraint lengths less than eight.

II. Quick-Look Codes:

We define a rate $\frac{1}{2}$ quick-look code as any code in which the two generators differ in exactly one coefficient. Then

$$G^{(1)}(D) + G^{(2)}(D) = D^L$$

for some $0 < L < K - 1$ and an inverse circuit with delay $L$ and error amplification factor $A = 2$ is given by

$$p^{(1)}(D) = p^{(2)}(D) = 1$$

This, of course, amounts to nothing more than the modulo-2 addition of $T^{(1)}(D)$ and $T^{(2)}(D)$. Hence the word 'Quick-Look' [2].

Since we are dealing with nondegenerate codes only, it follows easily that all quick-look codes have

$$\gcd\left[G^{(1)}(D), G^{(2)}(D)\right] = 1$$

Thus, there always exists an inverse with delay zero, which is generally different from the quick-look inverse if $L > 0$.

For example, when $L = 1$, the zero delay inverse takes the form

$$p^{(1)}(D) = \frac{1 + G^{(j)}(D)}{D} \quad (i \neq j)$$

and its error amplification factor at high signal-to-noise ratios is
\[ A = W[G(1)(D)] + W[G(2)(D)] - 2 \]

For \( L = 2 \) the zero delay inverse becomes

\[ p(i)(D) = \frac{1 + (1 + a_1 D) G(j)(D)}{D^2} \quad i \neq j \]

Here \( A \) has the same value as above if \( a_1 = 0 \) and is a function of the coefficients of \( G(1)(D) \) and \( G(2)(D) \) if \( a_1 = 1 \).

As a concrete example, consider the constraint length 5 code

\[
G(1)(D) = 1 + D + D^2 + D^4 \\
G(2)(D) = 1 + D + D^4
\]

Clearly, \( L = 2 \) and the quick-look inverse circuit takes either of the forms in Figure 56.

The inverse circuit with zero delay is given by

\[
p(1)(D) = 1 + D^2 + D^3 \\
p(2)(D) = D + D^2 + D^3
\]

and Figure 57 shows the two alternate configurations for this case. Note that the error amplification factor increases from 2 to 6 over the quick-look inverse.

III. Maximum Free Distance Quick-Look Codes:

One commonly accepted measure of the performance of a convolutional code in conjunction with sequential or maximum likelihood decoding algorithms is free distance. For the codes under consideration here this is simply the smallest nonzero number of ones in the set of semi-infinite output sequences of the encoder.

Our objective is to find quick-look codes of constraint lengths \( 3 \leq K \leq 7 \), with as large a free distance as possible.
To narrow the search for such codes we first note that the maximum free distance of any rate 1/2 noncatastrophic convolutional code is bound by

\[ d_f \leq \begin{cases} 
K + 2; & 3 \leq K \leq 6 \\
K + 3; & K = 7 
\end{cases} \]

and that there always exists a code for which equality holds (Larsen, 1973).

Second, since the input sequence 100 ... produces as output sequence from each modulo-2 adder of the encoder the coefficients of the respective generator polynomial, the free distance of any code is evidently bounded by

\[ d_f \leq W[G^{(1)}(D)] + W[G^{(2)}(D)] \]

Finally, if \( G^*(D) \) denotes the reciprocal polynomial of \( G(D) \), then the codes \( G^{(1)}(D), G^{(2)}(D) \)

and

\( G^{(1)*}(D), G^{(2)*}(D) \)
FIGURE 57. ZERO DELAY INVERSE CIRCUITS FOR THE CODE

1 1 1 0 1
1 1 0 0 0 1
are equivalent. This follows readily from the relation

\[ [I(D) G(D)]^* = I^* (D) G^* (D) \]

and the fact that the weights of a polynomial and its reciprocal are identical.

We can, therefore, restrict our search to quick-look codes with delay
\( L \leq \lfloor K/2 \rfloor \) and an appropriate number of ones in the generator polynomials (the square brackets denote the integer part).

Now let \( L = 0 \). Then each 1 in the input sequence \( I(D) \) will produce a 1 in the output sequence \( T(D) \) as it enters the encoder shift register and in addition the last 1 in \( I(D) \) will produce two 1's in \( T(D) \) as it enters the last stage of the encoder. Therefore,

\[ W[T(D)] \geq 2 + W[I(D)] \]

and it follows that in testing whether a code has free distance less than \( d_f \), only input sequences with fewer than \( d_f - 2 \) ones need to be considered.

Since Bahl and Jelinek (1971) have shown that without loss of generality input sequences with zero-runs of length \( K - 2 \) or more may likewise be ignored, it follows that the length of the input sequences that must be tested does not exceed

\[ (d_f - 4)(K - 2) + 1 \]

For \( L > 0 \), the first 1 in \( I(D) \) produces two 1's in \( T(D) \) as it enters the encoder and another 1 as it enters the \( (L + 1) \)st stage of the encoder. Every subsequent 1 in \( I(D) \) likewise produces a 1 in \( T(D) \) as it enters the \( (L + 1) \)st stage. In addition, the last 1 in \( T(D) \) results in two 1's in \( T(D) \) as it enters the last stage of the encoder. Thus, the total number of ones in the output sequence satisfies

\[ W[T(D)] \geq 4 + W[I(D)] \]
and we can restrict consideration to input sequences with fewer than \( d_f - 4 \) ones and length no larger than
\[
(d_f - 6) (K - 2) + 1
\]
Using these principles we tested all quick-look codes of constraint length \( 3 \leq K \leq 7 \). Table 3 summarizes our results. Note that for \( 3 \leq K \leq 6 \) the best quick-look codes are comparable to the best general nonsystematic codes, whereas for \( K = 7 \) the free distance of the best quick-look codes is one less than the maximum achievable.

We also remark that the quick-look codes with \( L = 0 \) are uniformly inferior to those with \( L > 0 \), a result that reinforces the notion that among the best codes of a class there is always one whose generators possess complementarity (Bahl and Jelinek, 1972).

Although under normal circumstances free distance is a good indicator of a code's error correcting capability, this measure nevertheless depends only on the code and thus completely ignores the nature of the channel and the decoding algorithm. Even with the channel and decoder fixed, differences in the weight spectra of two codes with the same free distance can give rise to different decoder bit error rates.

For these reasons we have computed the decoder bit error rates of selected codes from Table 3 used over the binary symmetric channel and in conjunction with a Viterbi maximum likelihood decoding algorithm of 32 bit decoder path lengths. The results are presented in Figure 58. Note that these quick-look codes compare favorably to the best nonsystematic codes obtained in [7] and the complementary codes given by Jelinek and Bahl (1969).

In Figure 59 we show the error amplification factor for the same set of codes as above, as a function of the signal-to-noise ratio of a binary symmetric channel.
Finally, Figure 60 presents the relationship between actual and measured channel bit error rates for the same codes, the binary symmetric channel and a 32 bit path length Viterbi decoder.
FIGURE 58. VITERBI MAXIMUM LIKELIHOOD DECODING PERFORMANCE $\Delta = 32$
FIGURE 59. ERROR AMPLIFICATION FACTOR FOR CODES #1, 2, 3, 7

FIGURE 60. MEASURED VERSUS ACTUAL CHANNEL BIT ERROR PROBABILITY FOR CODES #1, 2, 3, 7
IV. References:


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