Steady-State Decoupling and Design of Linear Multivariable Systems

Research Report
Grant No. NGR 05-017-010
June 1972 - June 1974

G.J. Thaler
Investigator

Jen-Yen Huang
Research Assistant

This report is prepared for Ames Research Center,
NASA, Moffett Field, California 94035
Principal Investigator: D.D. Siljak

The University of Santa Clara  •  California
STeady-State Decoupling and Design
of
Linear Multivariable Systems

Research Report
Grant No. NGR 05-017-010
June 1972 - June 1974

G. J. Thaler
Investigator

Jen-Yen Huang
Research Assistant

This report is prepared for Ames Research Center,
NASA, Moffett Field, California 94035
Principal Investigator: D. D. Siljak
ABSTRACT

This report consists of three parts:

Part I: Steady State Decoupling
Part II: Stability and Design
Part III: Application to STOL Aircraft

Part I presents a constructive criterion for decoupling the steady states of a linear time-invariant multivariable system. This criterion consists of a set of inequalities which, when satisfied, will cause the steady states of a system to be decoupled. It turns out that pure integrators in the loops play an important role. Stability analysis and a new design technique for such systems are given in Part II. A new and simple connection between single-loop and multivariable cases is found. This makes possible the application of the existing single-loop methods to multivariable cases. These results are then applied in Part III to the compensation design for NASA STOL C-8A aircraft. Both steady-state decoupling and stability are justified through computer simulations.
NOMENCLATURE

1. \( r \) \text{nxl input vector}
2. \( y \) \text{nxl output vector}
3. \( H(s) \) \text{closed-loop transfer function matrix}
4. \( G_p(s) \) \text{nxm plant matrix}
5. \( G_c(s) \) \text{mxn compensator matrix}
6. \( T_p \) \text{nxm type number matrix of the nxm plant} \( G_p(s) \)
7. \( T_c \) \text{mxn type number matrix of the mxn compensator} \( G_c(s) \)
8. \( P_g \) \text{poles of any transfer function} \( g(s) \)
9. \( Z_g \) \text{zeros of any transfer function} \( g(s) \)
10. \( \text{Max}\{\ldots\} \) \text{maximum value among all the elements in the brackets}
11. \( \text{LCD}\{\ldots\} \) \text{least common denominator of the elements in the brackets}
12. \( \text{LCM}\{\ldots\} \) \text{least common multiplier of the elements in the brackets}
13. \( (G_p)_{ij} \) \text{cofactor of the ijth element of} \( G_p \)
14. \( (I+G_pG_c)_{ij} \) \text{cofactor of the ijth element of} \( I+G_pG_c \)
15. \( \det(I+G_pG_c) \) \text{determinant of the matrix} \( I+G_pG_c \)
16. \( G \left( i_1, \ldots, i_k \right) \) \( \left( j_1, \ldots, j_k \right) \) \text{minor of matrix} \( G \text{ formed from rows} i_1, \ldots, i_k \text{ and columns} j_1, \ldots, j_k \)
# TABLE OF CONTENTS

## PART I: STEADY-STATE DECOUPLING

1. **INTRODUCTION** ........................................... 1
2. **FUNDAMENTAL RESULTS** .................................... 4
3. **SIMPLE CASES** ........................................... 11
   3.1 **2x2 Case** ........................................... 11
   3.2 **3x3 Case** ........................................... 32
4. **GENERAL nxm CASE** ....................................... 39

## PART II: STABILITY AND DESIGN

5. **INTRODUCTION TO PART II** ............................... 51
6. **DESIGN OF CLOSED-LOOP SYSTEMS WITH 2x2 PLANT AND**
   **DIAGONAL GC** ............................................. 52
   6.1 **Characteristic Equation** ............................. 53
   6.2 **Connection Between Single-Loop and Multi-**
       **variable Cases** ....................................... 55
   6.3 **Identification of Poles and Zeros** .................. 58
   6.4 **Design Example** ....................................... 64
7. **DESIGN OF CLOSED-LOOP SYSTEMS WITH 3x3 PLANT AND**
   **DIAGONAL GC** ............................................. 76
   7.1 **General** ............................................... 76
   7.2 **Identification of Poles and Zeros** .................. 85
   7.3 **Design Example** ....................................... 93
PART III: APPLICATION TO STOL AIRCRAFT

8. COMPENSATION DESIGN FOR STOL C-8A AIRCRAFT WITH STEADY-STATE DECOUPLING ........................................ 107
   8.1 General ........................................ 107
   8.2 Design Procedure .................................... 108

9. CONCLUSIONS ................................................ 120
   9.1 Comparison of the Result in Part I to That of State Variable Feedback Approach ........................................ 120
   9.2 Possible Generalization of the System Configuration ........................................ 121
   9.3 Stability and Design .................................... 122

REFERENCES ....................................................... 125
APPENDIX A: FORMULAE (4.1) AND (4.2) ........................................ 127
APPENDIX B: CHARACTERISTIC EQUATION ........................................ 128
APPENDIX C: ROOT LOCUS GAIN ........................................ 144
PART I

STEADY-STATE DECOUPLING
1. INTRODUCTION

Considerable research has been done on the decoupling of linear multivariable systems (e.g. see (1)). Such decoupling, referred to as total decoupling in this report, requires the system to be characterized by a nonsingular, diagonal transfer function matrix, and in general, linear state variable feedbacks have been employed.

The advantage of total decoupling is obvious, however, due to the restriction of having a diagonal transfer function matrix, less freedom should be expected when stability of the system is concerned.

This loss of freedom can be recovered to some extent by requiring only the steady states to be decoupled. Loosely speaking, a steady-state decoupled system is one in which changes in each input (i-th) are reflected in a corresponding output and only that output, when steady state is reached. Thus, different from total decoupling described above, mutual interactions are allowed during the transient period (but only during this period).

Necessary and sufficient conditions for decoupling the steady states of a system via linear state variable feedback were obtained by Wolovich (2).

His result, in terms of transfer function matrix representation is as follows:
A system characterized by an \((n \times m)\) proper rational transfer function matrix, \(G_p(s)\) having no poles at the origin \((s = 0)\), can be steady-state decoupled (via linear state variable feedback or perhaps some other less ambitious scheme) if and only if

\[
\rho(G_p(0)) = n \tag{1.1}
\]

where \(\rho(G_p(0))\) denotes rank of the matrix \(G_p(s)\) as \(s\) approaches zero.

However, it is found that if classical cascade feedback compensation other than linear state variable feedback is used, the rank condition (1.1) is no more necessary. Furthermore, the precluded poles at the origin are allowed. Actually, such poles are very helpful for decoupling the steady states of a multivariable system. Therefore, significant advantages over the linear state variable approach can be obtained through classical feedback configuration which then is obviously not "less ambitious".

The constructive criterion for steady-state decoupling will be derived in this part of the report. It will be shown that this criterion consists of \(n(n-1)\) inequalities (\(n\) is the number of outputs of the given plant), with the type numbers of the compensator transfer functions as unknowns. These unknowns are chosen to satisfy the inequalities and hence achieve a
steady-state decoupling scheme. Fundamental mathematical relations are derived in Chapter 2. Two simple applications for 2 x 2 and 3 x 3 cases are given in Chapter 3. Finally, the general case is considered in Chapter 4. Direct comparison of the result to that of the state variable approach is included in Chapter 9, which marks the end of this report.

The research reported herein was included in the Jen-Yen Huang M.S.E.E. thesis at the Department of Electrical Engineering and Computer Science, University of Santa Clara, Santa Clara, California. The thesis was supervised by G. J. Thaler, U.S. Naval Postgraduate School, Monterey, California.
2. FUNDAMENTAL RESULTS

The system under consideration in this thesis is shown below in Figure 2.1:

![Diagram](image)

**FIGURE 2.1**

Where $G_{p}(nxm)$ characterizes the given m-input, n-output plant, $G_{c}(mxn)$ is the n-input, m-output compensator to be designed. N unity feedbacks are used and complete controllability and observability are assumed (3), (4), to assure the complete description of the system by transfer function matrices. $r, y$ are the $nx1$ input and output vectors, respectively.

Let $H(s) = (h_{ij}(s))_{nxn}$ be the closed loop proper transfer function matrix, then by the above assumption, it characterizes the system completely, and we have:
\[ y(s) = H(s) \cdot r(s) \]

or

\[ y_i(s) = \sum_{j=1}^{n} h_{ij}(s) \cdot r_j(s) \]

\[ = h_{ii}(s) r_i(s) + \sum_{j=1, j \neq i}^{n} h_{ij}(s) r_j(s) \quad i=1, \ldots, n \quad (2.1) \]

By (2.1) and the Final Value Theorem, we have:

\[ \lim_{s \to 0} y_i(t) = \lim_{s \to 0} s y_i(s) \]

\[ = \lim_{s \to 0} s h_{ii}(s) r_i(s) + \lim_{s \to 0} s \sum_{j=1, j \neq i}^{n} h_{ij}(s) r_j(s) \quad (2.2) \]

\[ i=1, \ldots, n \]

Then the following formal definition can be given:

**DEFINITION:**

A system with the transfer function matrix \( H(s) \) is steady-state decoupled if and only if it is asymptotically stable and

\[ \lim_{s \to 0} s \sum_{j=1, j \neq i}^{n} h_{ij}(s) r_j(s) = 0 \quad \text{for all } i=1, \ldots, n \quad (2.3) \]

\[ ^1 \text{i.e., all the poles of the closed loop system lie in the open left half plane (Re(s)<0). This guarantees the application of the final value theorem.} \]
For systems as shown in Figure 2.1, it is well known that the closed loop transfer function matrix $H(2)$ can be expressed as

$$H = (I + G_p G_c)^{-1} G_p G_c$$

(2.4)

where $I$ is the nxn identity matrix.

(2.4) can be simplified further as follows:

$$H = (I + G_p G_c)^{-1} G_p G_c$$

$$= (I + G_p G_c)^{-1} (I + G_p G_c) - (I + G_p G_c)^{-1}$$

$$= I - (I + G_p G_c)^{-1}$$

(2.5)

(2.5) shows that the elements of $H$ depend in a very simple way on the cofactors of the elements of the matrix $I + G_p G_c$, i.e.,

$$h_{ii} = 1 - \frac{(I+G_p G_c)_{ii}}{\det(I+G_p G_c)} \quad i = 1, \ldots, n$$

(2.6)

$$h_{ij} = - \frac{(I+G_p G_c)_{ji}}{\det(I+G_p G_c)} \quad i, j = 1, \ldots, n \quad i \neq j$$

(2.7)

2The arguments will be dropped whenever no confusion exists.
where \( \det(I + G_p G_c) \) denotes the determinant of the \( nxn \) matrix \( I + G_p G_c \) and \( (I + G_p G_c)^{-1} \) denotes the cofactor of the \( ji \)th element of \( I + G_p G_c \).

Let the inputs to the system be polynomial inputs with only one term, e.g., step, ramp or parabola inputs, which are of primary importance. The Laplace transform of each input \( r_j(t) \), \( j = 1, \ldots, n \), is then

\[
 r_j(s) = L(r_j(t)) = \frac{r_j}{s^{k_j}} \tag{2.8}
\]

where \( r_j \) (without argument) is a constant, and \( k_j \) is a positive integer, e.g., if the \( j \)th input is a step then \( k_j=1 \), a ramp then \( k_j=2 \), etc.

Then, by (2.7), the steady-state decoupling criterion (2.3) becomes:

\[
\lim_{s \to 0} \sum_{j=1}^{n} S \cdot \frac{r_j}{s^{k_j}} \cdot \frac{-(I+G_p G_c)^{-1}_{ji}}{\det(I+G_p G_c)} = 0 \quad \text{for all } i=1, \ldots, n
\]

Thus, the following fundamental theorem for steady-state decoupling is developed:
THEOREM 2.1:

Assume that the given plant $G_p(n \times m)$ is stabilizable through the configuration of Figure 2.1, then the system is steady-state decoupled if and only if:

$$\lim_{s \to 0} \sum_{j=1}^{n} \frac{r_j}{s^{k_j-1}} \cdot \frac{(I+G_pG_c)_{ji}}{\det(I+G_pG_c)} = 0 \quad \text{for all } i = 1, \ldots, n$$

(2.9)

Theorem 2.1 can be simplified further for systems whose inputs are not fixed. This is desirable in most practical applications, for example, consider an aircraft as our plant $G_p$, the thrust, flap and elevator inputs must not be fixed in order to perform different functions.

Thus, for inputs with arbitrary constants $r_j$, we have

THEOREM 2.2:

Assume that the given plant $G_p(n \times m)$ is stabilizable through the configuration of Figure 2.1, and that the constants $r_j$ in all the inputs $r_j(t), j = 1 \ldots, n$ are arbitrary, then the system is steady-state decoupled if and only if
\[
\lim_{s \to \infty} \frac{1}{S(k_j - 1)} \cdot \frac{(I + G_p G_c)_{ji}}{\det(I + G_p G_c)} = 0
\]

for all \(i, j=1, \ldots, n\) and \(i \neq j\) \hspace{1cm} (2.10)

**PROOF:**

a) **Necessity:**

Suppose there exists \(i^*, j^*\) such that (2.10) is not true, then, by choosing \(r_{j^*}(t)\) as the only non-zero input, we have

\[
\lim_{s \to 0} \sum_{j=1}^{n} \frac{r_j}{S(k_j - 1)} \cdot \frac{(I + G_p G_c)_{ji}}{\det(I + G_p G_c)} = 0
\]

\hspace{1cm} \text{for } j \neq i^* \hspace{1cm} (2.11)

\[
= \lim_{s \to 0} \frac{r_{j^*}}{S(k_{j^*} - 1)} \cdot \frac{(I + G_p G_c)_{j^*i^*}}{\det(I + G_p G_c)}
\]

\hspace{1cm} \text{for } j^* \neq i^* \hspace{1cm} (2.12)

By our hypothesis, (2.12) is non-zero, hence (2.11) is non-zero, then by Theorem 2.1, the system is not steady-state decoupled.

b) **Sufficiency:**

Since (2.9) is simply a sum of (2.10) for different values of \(i, j\), if (2.10) is true, (2.9) is obviously true, hence the proof.

Q.E.D.
Note that by adjusting the value of \( k_j (=1,2,3,\ldots) \) associated with the jth input, both Theorem 2.1 and Theorem 2.2 can be applied to systems whose inputs are either all of the same type (e.g. all inputs are steps) or hybrid (e.g. input 1 is step, input 2 is ramp, input 3 is parabolic, etc.)

Both Theorem 2.1 and 2.2 are in neat mathematical forms. However, they cannot be applied directly, since our objective is to determine specifically what to put in the matrix \( G_c \) as the compensator functions in order to decouple the steady states of the system. Therefore, further result than (2.9) and (2.10) is necessary.

Direct approach, which utilizes the expansions of both the determinant and the cofactors of a matrix, is used. A general result will be given in Chapter 4. Before going into the general problem, however, two simple cases are treated first in the following chapter.
3. SIMPLE CASES

In this chapter, 2-input, 2-output and 3-input, 3-output plants, both compensated by diagonal $G_c$ using the feedback configuration Figure 2.1, will be considered.

Details for the $2 \times 2$ case are presented in Section 3.1. Then, in Section 3.2, the outline and results for the $3 \times 3$ case are given.

3.1 $2 \times 2$ CASE

For a given 2-input, 2-output plant,

$$G_p = \begin{bmatrix} g_{p11} & g_{p12} \\ g_{p21} & g_{p22} \end{bmatrix}$$

if the diagonal compensator matrix

$$G_c = \begin{bmatrix} g_{c11} & 0 \\ 0 & g_{c22} \end{bmatrix}$$

is used, the system configuration in Figure 2.1 becomes
Since

\[ G_p G_c = \begin{bmatrix} g_{p11} g_{c11} & g_{p12} g_{c22} \\ g_{p21} g_{c11} & g_{p22} g_{c22} \end{bmatrix} \]

we have

\[
\text{det}(I + G_p G_c) = 1 + g_{p11} g_{c11} + g_{p22} g_{c22} + g_{p11} g_{p22} g_{c11} g_{c22}
\]

\[-g_{p12} g_{p21} g_{c11} g_{c22} \]

(3.1)

\[
(I + G_p G_c)_{12} = -g_{p21} g_{c11}
\]

(3.2)

\[
(I + G_p G_c)_{21} = -g_{p12} g_{c22}
\]

(3.3)
By Theorem 2.2, for arbitrary constants in both of the inputs \( r_1(t) \) and \( r_2(t) \), the system is steady-state decoupled if and only if:

\[
\lim_{s \to 0} \frac{1}{s(k_1-1)} \cdot \frac{(I+G_pG_c)_{12}}{\det(I+G_pG_c)} = 0 \tag{3.4}
\]

and

\[
\lim_{s \to 0} \frac{1}{s(k_2-1)} \cdot \frac{(I+G_pG_c)_{21}}{\det(I+G_pG_c)} = 0 \tag{3.5}
\]

where \( k_1 \) and \( k_2 \) are defined as in (2.8).

Let the inputs \( r_1(t) \) and \( r_2(t) \) be two step functions with arbitrary amplitudes, then by (2.8), \( k_1 = k_2 = 1 \) and \( r_1, r_2 \) are two arbitrary constants.

Then, by substituting (3.1), (3.2), and (3.3) into (3.4) and (3.5), we have

\[
\lim_{s \to 0} \frac{g_{p21}g_{c11}}{(1+g_{p11}g_{c11}+g_{p22}g_{c22}+g_{p11}g_{p22}g_{c11}g_{c22} - g_{p12}g_{p21}g_{c11}g_{c22})} = 0 \tag{3.6}
\]
and \[ \lim_{s \to 0} \frac{g_{p12} g_{c22}}{(1 + g_{p11} g_{c11} + g_{p22} g_{c22} + g_{p11} g_{p22} g_{c11} g_{c22})} = 0 \] 

Note that in both (3.6) and (3.7), the \( s \) factor from the Final Value Theorem was cancelled by the \( \frac{1}{s} \) factor in the input transforms, hence no explicit powers of \( s \) appear in (3.6) and (3.7).

Thus, for systems as in Figure 3.1 with arbitrary amplitude step inputs, the necessary and sufficient conditions for steady-state decoupling are (3.6) and (3.7).

For a given plant, all the \( g_{pij} \) are known, hence the design for steady-state decoupling is simply the determination of \( g_{c11} \) and \( g_{c22} \), such that (3.6) and (3.7) are satisfied.

For example, consider

\[
G_p(s) = \begin{bmatrix}
\frac{1}{s+1} & \frac{1}{s+2} \\
\frac{1}{s+3} & \frac{1}{s+4}
\end{bmatrix}
\]

By (3.6) and (3.7), if
\[ g_{c11}(s) = \frac{1}{s} \cdot g'_{c11}(s) \]  
\[ g_{c22}(s) = \frac{1}{s} \cdot g'_{c22}(s) \]

where \(g'_{c11}(s)\) and \(g'_{c22}(s)\) do not contain any pole or zero at the origin, or alternatively, \(\lim_{s \to 0} g'_{c11} = \text{finite constant}\) and \(\lim_{s \to 0} g'_{c22} = \text{finite constant}\), then

\[ g_{c11}(s) \cdot g_{c22}(s) = \frac{1}{s^2} \cdot g'_{c11}(s) \cdot g'_{c22}(s) \]

Since,

\[ \lim_{s \to 0} \left( g_{p11}g_{p22} - g_{p12}g_{p21} \right) = \frac{1}{12} \neq 0 \]  
(3.9)

the following term in the denominator of both (3.6) and (3.7),

\[ (g_{p11}g_{p22}g_{c11}g_{c22} - g_{p12}g_{p21}g_{c11}g_{c22}) \]

which contains a \(1/s^2\) factor, will go to infinity faster than both of the numerators in (3.6) and (3.7) as \(s\) approaches zero. Thus, (3.6) and (3.7) are satisfied and the system is steady-state decoupled. It is seen that the pure integrators in \(g_{c11}\) and \(g_{c22}\) and the constraint (3.9) are important. These constitute the highlights of the analysis that follows.
Let

\[ g_{pij} = s^{-t_{pij}} g_{pij} \]

\[ g_{cij} = s^{-t_{cij}} g_{cij} \]  \hspace{1cm} (3.10)

where \( t_{pij}, t_{cij} \) are integers that will be referred to as the type numbers of the corresponding transfer functions, and \( g_{pij}, g_{cij} \) are such that: \( \lim_{s \to 0} g_{pij} \) and \( \lim_{s \to 0} g_{cij} \) are non-zero constants (i.e. the numerators and denominators of \( g_{pij} \) and \( g_{cij} \) do not contain powers of \( s \) as their factors). Whenever \( g_{pij} = 0 \) or \( g_{cij} = 0 \), the corresponding \( g'_{pij} \) and \( g'_{cij} \) are defined to be identically zero, however \( t_{pij} \) and \( t_{cij} \) become indefinite in this case, and we will use the symbol \( x \) to identify them for reasons that will be clear in Chapter 4.

The matrices \( T_p = (t_{pij})_{n \times m} \) and \( T_c = (t_{cij})_{m \times n} \) will be referred to as the TYPE NUMBER MATRICES of the plant and compensator respectively.

For example, given

\[
G_P = \begin{bmatrix}
5 & \frac{2}{s(s+3)} \\
\frac{s}{s+2} & 0
\end{bmatrix}
\]
the type number matrix is

\[ T_p = \begin{bmatrix} 0 & 1 \\ -1 & x \end{bmatrix} \]

By separating the powers of \( s \) in each of the transfer functions as in (3.10), (3.6) and (3.7) can be expressed as:

\[
\lim_{s \to 0} \frac{-(t_{p11} + t_{c11})}{\Delta} \cdot g_{p21} g_{c11} = 0 \quad (3.11)
\]

\[
\lim_{s \to 0} \frac{-(t_{p12} + t_{c22})}{\Delta} \cdot g_{p12} g_{c22} = 0 \quad (3.12)
\]

where

\[
\Delta = \frac{1 + s}{1 + s} \cdot g_{p11} g_{c11} + s \cdot g_{p22} g_{c22} + s \cdot g_{p11} g_{p22} g_{c11} g_{c22} - s \cdot g_{p12} g_{p21} g_{c11} g_{c22} \quad (3.13)
\]

Thus, the necessary and sufficient conditions (3.6) and (3.7) assume different forms in (3.11) and (3.12). Again, once a plant
is given, \( g_{pij} \), and hence \( t_{pij} \), \( g'_{pij} \) are known. Therefore, only \( t_{ciij} \) and \( g'_{ciij} \) in (3.11) and (3.12) are left adjustable. It was shown in a previous example that pure integrators in \( g_{c11} \) and \( g_{c22} \) (see (3.8)) are important in steady-state decoupling. In terms of the expressions given in (3.10), this is the same as saying that the values of \( t_{c11} \) and \( t_{c22} \) are the key factors in the attainment of a steady-state decoupling scheme.

In order to find out the constraints on \( t_{c11} \) and \( t_{c22} \), such that (3.11) and (3.12) are satisfied, the following theorem is developed:

**THEOREM 3.1**

Let

\[
C_0(s) = \sum_{i=1}^{k} \frac{-n_i}{s^i} P_i(s) \frac{1+m_j}{1+\Sigma s^j q_j(s)}
\]

be a rational function in \( s \), where \( P_i(s) \), \( q_j(s) \) are themselves rational functions such that \( \lim_{s \to 0} P_i(s) \) and \( \lim_{s \to 0} q_j(s) \) are non-zero finite constants and \( n_i \), \( m_j \) are integers for all \( i = 1, \ldots, k \), \( j = 1, \ldots, \ell \).
b) \( \text{NAMax} \{n_1, \ldots, n_k\} \)

\( \text{MAMax} \{m_1, \ldots, m_k\} \)

c) \( \lim_{s \to 0} \sum_{i=1}^{k} P_i(s) \neq 0 \) \hspace{1cm} (3.15)

\( \lim_{s \to 0} \sum_{j=1}^{\ell} q_j(s) \neq 0 \) \hspace{1cm} (3.16)

\( \lim_{s \to 0} \sum_{j=1}^{\ell} q_j(s) \neq -1 \) \hspace{1cm} (3.17)

where \( \sum_{i=1}^{k} P_i(s) \) means that the summation is only over those \( P_i \) in \( n_i = N \) (3.14) that have \( s^{-N} \) as their multiplication factor. The other two summations in (3.16) and (3.17) are defined similarly.

Then \( \lim_{s \to 0} C_0(s) = 0 \) if and only if

either \( M > N \)

or \( M < 0, \ N < 0 \). \hspace{1cm} (3.18)
PROOF:

\( C_{0}(s) \) can be expressed as

\[
C_{0}(s) = \sum_{i=1}^{n_i \geq 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i = 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i < 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i > 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i \leq 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i \geq 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i < 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i > 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i \leq 0}
\]

Since

\[
\lim_{s \to 0} \sum_{i=1}^{n_i < 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i > 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i \leq 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i \geq 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i < 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i > 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i \leq 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i \geq 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i < 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i > 0} k_{-n_i} \prod_{i=1} \sum_{i=1}^{n_i \leq 0} = 0 \text{ by (a)}
\]

The limit value can be determined for each of the following nine possible cases:
1) \( N>0, \ M>0 \)

\[
\lim_{s \to 0} \xi s^{n_i} P_i(s) + \lim_{s \to 0} \xi s^{m_i} P_i(s)
\]

\[
\lim_{s \to 0} C_0(s) = \frac{\sum_{i=1}^{k} M - n_i}{\sum_{i=1}^{k} M - m_i}
\]

\[
\lim_{s \to 0} s^{M-n_i} P_i(s) = \frac{\sum_{i=1}^{k} M - n_i}{\sum_{i=1}^{k} M - m_i}
\]

\[
\lim_{s \to 0} \xi s^{j_q_j(s)} + \lim_{s \to 0} \xi s^{m_q_j(s)}
\]

\[
= 0 \text{ if and only if } M>N, \text{ by (3.16)}
\]

\& \text{(3.15)}

2) \( N>0, \ M=0 \)

\[
\lim_{s \to 0} s^{-N} \xi P_i(s)
\]

\[
\lim_{s \to 0} C_0(s) = \frac{1}{\sum_{j=1}^{k} j_q_j(s)}
\]

\[
= \infty, \text{ by (3.15)}
\]

3) \( N>0, \ M<0 \)

\[
\lim_{s \to 0} s^{-N} \xi P_i(s)
\]

\[
\lim_{s \to 0} C_0(s) = \frac{1}{\sum_{i=1}^{k} n_i=N}
\]

\[
= \infty, \text{ by (3.15)}
\]
4) \( N=0, M>0 \)

\[
\lim_{s \to 0} \frac{\sum_{i=1}^{k} P_i(s)}{s^{M} \sum_{j=1}^{L} q_j(s)} = 0, \\
\lim_{s \to 0} C(s) = 0, \text{ by (3.16)}
\]

5) \( N=0, M=0 \)

\[
\lim_{s \to 0} \frac{\sum_{i=1}^{k} P_i(s)}{s^{M} \sum_{j=1}^{L} q_j(s)} \neq 0, \\
\lim_{s \to 0} C(s) = 0, \text{ by (3.15)}
\]

6) \( N=0, M<0 \)

\[
\lim_{s \to 0} \frac{\sum_{i=1}^{k} P_i(s)}{s^{M} \sum_{j=1}^{L} q_j(s)} \neq 0, \\
\lim_{s \to 0} C(s) = 0, \text{ by (3.15)}
\]

7) \( N<0, M>0 \)

\[
\lim_{s \to 0} C(s) = 0, \\
\lim_{s \to 0} \frac{\sum_{i=1}^{k} P_i(s)}{s^{M} \sum_{j=1}^{L} q_j(s)} = 0, \text{ by (3.16)}
\]
8) \(N<0, M=0\)

\[
\lim_{s \to 0} C_o(s) = \frac{0}{1} = 0, \quad \text{by (3.17)}
\]

9) \(N<0, M<0\)

\[
\lim_{s \to 0} C_o(s) = \frac{0}{1} = 0
\]

Thus, \(\lim_{s \to 0} C(s) = 0\) if and only if one of (1), (4), (7), (8), (9) is true. Since the conditions in (1) to (8) are equivalent to \(\lim_{s \to 0} C_o(s) = 0\) if and only if \(M>N\), and (9) gives \(M<0, N<0\), the theorem is proved.

Q.E.D.

Note that (3.18) contains only the powers of \(s\), neither \(p_i\) nor \(q_j\) appears in this expression. Also, note that \(C_o(s)\) is of exactly the same form as the rational functions in (3.11) and (3.12), therefore, the theorem can be applied directly.
Compare (3.11) and (3.12) with (3.14), and by the definition of M and N in (b) of Theorem 3.1, we have:

\[ M = \max \{ t_{p11} + t_{c11}, t_{p22} + t_{c22}, t_{p11} + t_{p22} + t_{c11} + t_{c22}, t_{p12} + t_{p21} + t_{c11} + t_{c22} \} \]

\[ N_{12} = \max \{ t_{p21} + t_{c11}, t_{p12} + t_{c22} \} = t_{p21} + t_{c11} \tag{3.19} \]

\[ N_{21} = \max \{ t_{p12} + t_{c22} \} = t_{p12} + t_{c22} \]

Where the notation \( \max \{ \cdots \} \) denotes the maximum value among all the elements in the brackets and the subscripts on N are used in accordance with (3.2) and (3.3) to distinguish them from each other.

Since \( t_{p_{ij}} \) are known for a given plant, the only unknowns in (3.19) are \( t_{c11} \) and \( t_{c22} \), which can be chosen to satisfy (3.18). Once \( t_{c11}, t_{c22} \) are chosen, \( M, N_{12}, N_{21} \) are known, and (3.15), (3.16), (3.17) can then be written down explicitly.

In general, these expressions contain both \( g_{p_{ij}} \) and \( g_{c_{ij}} \). Since the pole and zero locations and the gains of each \( g_{c_{ij}} \) are free parameters, they can hopefully be adjusted to satisfy (3.15), (3.16), and (3.17).
These free parameters should also be designed for stability and transient response of the system, therefore, they cannot be adjusted with complete freedom. However, as was mentioned before, (3.18) does not depend on $g'_{cij}$, therefore, the design of stability will not destroy the steady-state decoupling as long as (3.15), (3.16) and (3.17) are not violated. Hence, once $t_{c11}$ and $t_{c22}$ are determined, stability can be considered.

After all $g'_{cij}$ are designed, however, (3.15), (3.16) and (3.17) must be checked. If satisfied, the design is completed, if not, slight adjustments of the free parameters, under the allowance of stability, can be made in order to satisfy these constraints and hence guarantees that the steady states are decoupled.

It might happen that in some cases, no adjustment in $g'_{cij}$ is possible to satisfy these constraints, e.g.,

$$\lim_{s \to 0} (g'_{p11}g'_{p22} - g'_{p12}g'_{p21}) g'_{c11}g'_{c22} \neq 0$$
is not possible if

$$\lim_{s \to 0} (g'_{p11}g'_{p22} - g'_{p12}g'_{p21}) = 0$$

happens to be true for the given plant.

In cases like this, another choice of $t_{c11}$ and $t_{c22}$ is necessary.
Following this procedure, we can, at present, assume that (3.15), (3.16) and (3.17) are satisfied. Then, by Theorem 3.1, if the constraints (3.15), (3.16) and (3.17) for the rational function in (3.11) are satisfied, then (3.11) is true if and only if

either \( M > N_{12} \) \hspace{1cm} (3.20)

or \( M < 0, N_{12} < 0 \)

Similarly, if the constraints (3.15), (3.16) and (3.17) for the function in (3.12) are satisfied, then (3.12) is true if and only if

either \( M > N_{21} \) \hspace{1cm} (3.21)

or \( M < 0, N_{21} < 0 \)

For steady-state decoupling, both (3.11) and (3.12) must be true, therefore, combining (3.20) and (3.21), each of the following four sets of criteria can be used:

\[
\begin{align*}
M &> N_{12} \\
M &> N_{21}
\end{align*}
\] \hspace{1cm} (3.22)
\[ M > N_{12} \]  
\[ M < 0, \quad N_{21} < 0 \]  
\[ (3.23) \]

\[ M > N_{21} \]  
\[ M < 0, \quad N_{12} < 0 \]  
\[ (3.24) \]

\[ M < 0 \]  
\[ N_{12} < 0 \]  
\[ N_{21} < 0 \]  
\[ (3.25) \]

It should be noticed that (3.23) and (3.24) are redundant since they are contained in (3.25).

The best choice among these four sets will depend on the type number matrix of the given plant.

Consider the following example: Given the 2x2 plant\(^3\)

\[
G_p(s) = \begin{bmatrix}
\frac{-s + 1}{(s+1)^2} & \frac{-s + 2}{(s+1)^2} \\
\frac{-3s + 1}{3(s+1)^2} & \frac{-s + 1}{(s+1)^2}
\end{bmatrix}
\]  
\[ (3.26) \]

\(^3\) The plant is taken from an example in (5).
Find a steady-state decoupling scheme using diagonal $G_c$ and the configuration of Figure 2.1 assuming that the inputs are arbitrary steps.

By inspection, $t_{pij} = 0$ for all $i, j=1, 2$, then by (3.19),

$$M = \text{Max} \left\{ t_{c11}, t_{c22}, t_{c11} + t_{c22}, t_{c11} t_{c22} \right\}$$

$$N_{12} = t_{c11}$$

$$N_{21} = t_{c22}$$

If (3.22) is used, $t_{c11}=t_{c22}=1$ is the simplest solution (note that the solution is not unique). For this particular choice, $g_{c11} = \frac{1}{s} g_{c11}, g_{c22} = \frac{1}{s} g_{c22}$, hence the introduction of pure integrators in the loops will cause the steady states to be decoupled.

(3.23), (3.24) and (3.25) can also be used, however, in this case, the solutions for both $t_{c11}$ and $t_{c22}$ will turn out to be negative, which corresponds to the introduction of differentiators in $G_c$, and is physically undesirable.
Since only one term appears in the numerators of (3.11) and (3.12), (3.15) is satisfied automatically by definition of \( g'_{pij} \) and \( g'_{cij} \) (see (3.10)).

By inspection of (3.13) and by noting that both of the last two terms contain \( s^{-M} \) for the above choice of \( t_{c11} \) and \( t_{c22} \), we have for (3.16)

\[
\lim_{s \to 0} \left( g'_{p11}g'_{p22}g'_{c11}g'_{c22} - g'_{p12}g'_{p21}g'_{c11}g'_{c22} \right) \neq 0
\]

Since

\[
\lim_{s \to 0} \left( g'_{p11}g'_{p22} - g'_{p12}g'_{p21} \right) = \frac{1}{3} \neq 0
\]

We have

\[
\lim_{s \to 0} g'_{c11}g'_{c22} \neq 0
\]

which is again satisfied automatically.

Similarly, by inspection of (3.13), (3.17) is also satisfied automatically, since by the above choice of \( t_{c11} \) and \( t_{c22} \), none of the terms in (3.13) has 0 as the power of the associated \( s \) factor.
Therefore, we are guaranteed to have a steady-state decoupled system by introducing one pure integrator in each of $g_{c11}$ and $g_{c22}$.

Actually, in this case we don't need (3.15) and (3.17), since $M=2>0$, $N_{12}=N_{21}=1>0$, and by (1) in the proof of Theorem 3.1, only (3.16) is sufficient.

It should also be noticed from the proof of Theorem 3.1 that the constraint (3.15) was used only to make (3.18) also a necessary condition. If (3.15) is not true, the sufficient part of the theorem is still guaranteed by (3.16) and (3.17). Therefore, it is usually only necessary to check (3.16) and (3.17) in practical design.

For inputs other than steps, Theorem 3.1 must be generalized as follows:

**THEOREM 3.2**

Let

$$C_t(s) = \frac{1}{s^t} C_0(s) \quad (3.28)$$

where $t = 0, 1, 2, \ldots$, $C_0(s)$ is as defined in (3.14) and (b), (c) are the same as in Theorem 3.1.
Then

$$\lim_{s \to 0} C_t(s) = 0 \quad \text{if and only if}$$

either \( M < 0, \quad N + t < 0 \)  \hspace{1cm} (3.29)

or \( M > N + t \)

PROOF: By writing:

$$C_t(s) = \frac{k - (n_i + t) \sum_{i=1}^{k_2}(s)}{1 + \sum_{j=1}^{\infty} \frac{m \cdot q_j(s)}{s}} \sum_{i=1}^{k_2} p_i(s)$$

the result follows immediately by Theorem 3.1.

Q.E.D.

Now let input 2 be a ramp, while input 1 is still a step (i.e., \( k_1 = 1, \quad k_2 = 2 \)). Then by (3.4), (3.6) and (3.11) are still the same. However, for input 2, since \( k_2 = 2 \) in (3.5), there will be an additional \( s \) factor in the denominators of (3.7) and (3.12). Since the only difference is this additional \( s \), (3.19) remains unchanged, and the application of Theorem 3.2 gives \( M > N_{21} + 1 \). Therefore, the conditions corresponding to (3.22) becomes:
For the plant (3.26), by (3.27) we have $t_{c11} = 2$, $t_{c22} = 1$ as the simplest solution.

Thus, we need one more integrator in $g_{c11}$ in order to decouple the steady states, if input 2 is a ramp instead of a step.

3.2 3x3 CASE

For a given 3-input, 3-output plant,

$$G_p = \begin{bmatrix} g_{p11} & g_{p12} & g_{p13} \\ g_{p21} & g_{p22} & g_{p23} \\ g_{p31} & g_{p32} & g_{p33} \end{bmatrix}$$

if the diagonal compensator matrix

$$G_c = \begin{bmatrix} g_{c11} & 0 & 0 \\ 0 & g_{c22} & 0 \\ 0 & 0 & g_{c33} \end{bmatrix}$$

is used, the system configuration is as shown in Figure 3.2.
Expressions for $\det(I + G G_{ij})$ and $(I + G G_{ij})_p$ can be obtained either by direct expansion as was done in Section 3.1 (see (3.1), (3.2) and (3.3)), or by using the formulae (4.5) and (4.6), then by (2.10) of Theorem 2.2 and assuming step inputs (i.e. $k_1 = k_2 = k_3 = 1$), a set of 6 limit expressions similar to (3.11) and (3.12) can be obtained. Compare these with (3.14), we have

$$M = \max \left\{ t_{p11} + t_{c11}, t_{p22} + t_{c22}, t_{p33} + t_{c33}, t_{p11} + t_{p22} + t_{c11} + t_{c22}, t_{p12} + t_{p21} + t_{c11} + t_{c22}, t_{p11} + t_{p33} + t_{c11} + t_{c33}, t_{p13} + t_{p31} + t_{c11} + t_{c33}, t_{p22} + t_{p32} + t_{c22} + t_{c33}, t_{p23} + t_{p32} + t_{c22} + t_{c33}, t_{p31} + t_{p32} + t_{c33}, t_{p11} + t_{p22} + t_{p33} + t_{c11} + t_{c22} + t_{c33}, t_{p12} + t_{p23} + t_{p31} + t_{c11} + t_{c22} + t_{c33}, t_{p13} + t_{p22} + t_{p32} + t_{c11} + t_{c22} + t_{c33}, t_{p15} + t_{p21} + t_{p32} + t_{c11} + t_{c22} + t_{c33}, t_{p13} + t_{p22} + t_{p31} + t_{c11} + t_{c22} + t_{c33}, t_{p12} + t_{p21} + t_{p33} + t_{c11} + t_{c22} + t_{c33}, t_{p11} + t_{p23} + t_{p32} + t_{c11} + t_{c22} + t_{c33} \right\}$$

(3.30)
\[ N_{12} = \max \left\{ t_{p21} + t_{c11}, t_{p21} + t_{p33} + t_{c11} + t_{c33}, t_{p23} + t_{p31} + t_{c11} + t_{c33} \right\} \]

\[ N_{13} = \max \left\{ t_{p31} + t_{c11}, t_{p31} + t_{p22} + t_{c11} + t_{c22}, t_{p32} + t_{p21} + t_{c11} + t_{c22} \right\} \]

\[ N_{21} = \max \left\{ t_{p12} + t_{c22}, t_{p12} + t_{p33} + t_{c22} + t_{c33}, t_{p13} + t_{p32} + t_{c22} + t_{c33} \right\} \]

\[ N_{23} = \max \left\{ t_{p32} + t_{c22}, t_{p32} + t_{p11} + t_{c11} + t_{c22}, t_{p31} + t_{p12} + t_{c11} + t_{c22} \right\} \]

\[ N_{31} = \max \left\{ t_{p13} + t_{c33}, t_{p13} + t_{p22} + t_{c22} + t_{c33}, t_{p23} + t_{p12} + t_{c22} + t_{c33} \right\} \]

\[ N_{32} = \max \left\{ t_{p23} + t_{c33}, t_{p23} + t_{p11} + t_{c11} + t_{c33}, t_{p21} + t_{p13} + t_{c11} + t_{c33} \right\} \]

(3.30)

Note that if \( g_{pij} = 0 \) for some \( i, j \) in a given plant, any term in (3.30) that contains the corresponding \( t_{pij} \) has to be dropped. In Chapter 4, an analytical scheme will be designed to take care of this.
Similar to (3.22) through (3.25), we have 64 \((=2^n(n-1))\) possible sets of criteria here to choose from. However, similar to the previous case, only the two corresponding to (3.22) and (3.25) are not redundant, these are:

\[
\begin{align*}
M &> N_{12} \\
M &> N_{13} \\
M &> N_{21} \\
M &> N_{23} \\
M &> N_{31} \\
M &> N_{32}
\end{align*}
\]

(3.31)

and

\[
\begin{align*}
M &< 0 \\
N_{12} &< 0 \\
N_{13} &< 0 \\
N_{21} &< 0 \\
N_{23} &< 0 \\
N_{31} &< 0 \\
N_{32} &< 0
\end{align*}
\]

(3.32)

Consider the following example:
EXAMPLE:

Given a 3x3 plant with

\[
\begin{align*}
\mathcal{g}_{p11}(s) &= 0.081(s-0.205)(s+0.967+j1.379) \\
&\quad (s+0.967-j1.379)/D^T_{DH}
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{p12}(s) &= -6.12(s+0.837)(s+0.947+j1.144) \\
&\quad (s+0.947-j1.144)/D^T_{DF}
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{p13}(s) &= -202(s+1.885)(s-13.037)/D^T_{DE}
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{p21}(s) &= -0.00163(s+2.881)(s+0.032+j0.313) \\
&\quad (s+0.032-j0.313)/D^T_{DH}
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{p22}(s) &= -0.153(s+0.824)(s-0.047+j0.205) \\
&\quad (s-0.047-j0.205)/D^T_{DF}
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{p23}(s) &= -9.07(s+26.339)(s+0.03+j0.361) \\
&\quad (s+0.03-j0.361)/D^T_{DE}
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{p31}(s) &= 0.00209(s-1.049)(s+0.268)/D^T_{TH}
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{p32}(s) &= 0.0995(s-0.12)(s+3.485)/D^T_{DF}
\end{align*}
\]

\[
\begin{align*}
\mathcal{g}_{p33}(s) &= -235.5(s+0.361+j0.076) \\
&\quad (s+0.361-j0.076)/D^T_{DE}
\end{align*}
\]

\[\text{4NASA STOL C-8A aircraft, with thrust, flap angle and elevator angle as the inputs and velocity, angle of attack, pitch angle as the outputs.}\]
where

\[ D(s) = (s+0.018+j0.336)(s+0.018-j0.336) \\
        (s+1.103+j1.277)(s+1.103-j1.277) \]

\[ D_{TH}(s) = (s+0.99+j0.479)(s+0.99-j0.479) \]

\[ D_E(s) = (s+3.3+j10.49)(s+3.3-j10.49) \]

\[ D_F(s) = s + 1 \]

Find a steady-state decoupling scheme using diagonal \( G_c \), and the configuration of Figure 3.2, assuming that the inputs are arbitrary steps.

By inspection of (3.33), \( t_{ij} = 0 \) for all \( i,j = 1,2,3 \). Thus (3.30) assumes the following simple form:

\[ M = \text{Max} \{ t_{c11}, t_{c22}, t_{c33}, t_{c11} + t_{c22}, t_{c11} + t_{c33}, t_{c22} + t_{c33}, t_{c11} + t_{c22} + t_{c33} \} \]

\[ N_{12} = \text{Max} \{ t_{c11}, t_{c11} + t_{c33} \} \]

\[ N_{13} = \text{Max} \{ t_{c11}, t_{c11} + t_{c22} \} \]

\[ N_{21} = \text{Max} \{ t_{c22}, t_{c22} + t_{c33} \} \]

\[ N_{23} = \text{Max} \{ t_{c22}, t_{c22} + t_{c11} \} \]

\[ N_{31} = \text{Max} \{ t_{c33}, t_{c33} + t_{c22} \} \]

\[ N_{32} = \text{Max} \{ t_{c33}, t_{c33} + t_{c11} \} \]
By (3.31) and (3.34), it is clear that $M$ must be $t_{c11} + t_{c22} + t_{c33}$, and $t_{c11}$, $t_{c22}$, $t_{c33}$ must be positive (otherwise $t_{c11} + t_{c22} + t_{c33}$ cannot be a maximum). Hence the simplest solution is $t_{c11} = t_{c22} = t_{c33} = 1$. This means that the introduction of one pure integrator in each of the compensators $g_{c11}$, $g_{c22}$, $g_{c33}$ will cause the steady states of the system to be decoupled.

Since $M > 0$, $N_{ij} > 0$, only (3.16) has to be checked. It can readily be found that this is satisfied, therefore we are guaranteed to have a steady-state decoupled system.

(3.32) can also be used, however, as in the 2x2 example of Section 3.1, the result requires pure differentiators in $g_{c11}$, $g_{c22}$ and $g_{c33}$, hence also not desirable for this particular plant.
4. GENERAL nxm CASE

Results of Chapter 3 show that, for the two special cases considered, the steady-state decoupling criterion can be written as a set of \( n(n-1) \) inequalities, where \( n \) is the number of outputs of the plant (or number of inputs or outputs of the system).

These results will be generalized in this chapter to systems consisting of \( m \)-input, \( n \)-output plant \( G(nxm) \), \( n \)-input, \( m \)-output compensator \( G_c(mxn) \), and unity feedbacks are employed as shown in Figure 2.1. Exactly the same approach as in Chapter 3 is presumed and it will be seen that both Theorem 3.1 and 3.2 are applicable.

As was shown in Chapter 3, the first step is to obtain expressions for \( \det(I+G_pG_c) \) and \( (I+G_pG_c)_{ji} \) as in (3.1), (3.2) and (3.3). For the general case, this can be accomplished by using the following formulae which are proved in Appendix A.

For any \( nxn \) square matrix \( G \)

\[
\det (I+G) = 1 + \sum_{\ell=1}^{n} \sum_{1\leq k_1<\cdots<k_{\ell}\leq n} G(k_1,\cdots,k_{\ell}) \tag{4.1}
\]

\[
(I+G)_{ji} = -\sum_{\ell=0}^{n-2} \sum_{1\leq k_1<\cdots<k_{\ell}\leq n} G(i,k_1,\cdots,k_{\ell}) \tag{4.2}
\]
where \( G(i_1, \ldots, i_\ell) \) denotes the minor formed from rows \( i_1, \ldots, i_\ell \) and columns \( j_1, \ldots, j_\ell \) of the matrix \( G \) with \( 1 \leq \ell \leq n \).

Let \( G = G_p G_c \) where \( G_p \) is the nxm plant matrix and \( G_c \) is the mxn compensator matrix. Then by Binet-Cauchy formula (6), we have

\[
G(k_1, \ldots, k_\ell)
\]

\[
= \begin{cases} 
\sum_{1 \leq \sigma_1 < \cdots < \sigma_\ell \leq m} G_p(\sigma_1, \ldots, \sigma_\ell) G_c(\sigma_1, \ldots, \sigma_\ell) \quad \ell \leq m \\
0 \quad \ell > m
\end{cases}
\]

and

\[
G(i, k_1, \ldots, k_\ell)
\]

\[
= \begin{cases} 
\sum_{1 \leq \rho_0 < \cdots < \rho_\ell \leq m} G_p(i, k_1, \ldots, k_\ell) G_c(\rho_0, \rho_1, \ldots, \rho_\ell) \quad \ell \leq m-1 \\
0 \quad \ell > m-1
\end{cases}
\]
Combining (4.1) and (4.3), (4.2) and (4.4), respectively, we have

\[ \det(I+G_pG_c) = 1 + \sum_{\ell=1}^{\min(n,m)} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq n} 1 \leq \sigma_1 < \cdots < \sigma_{\ell} \leq m \]

\[ \sum_{k_1, \cdots, k_{\ell} \neq i, j} \]  

(4.5)

\[ (I+G_pG_c)_{ji} = - \sum_{\ell=0}^{\min(n-2,m-1)} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq n} 1 \leq \rho_0 < \cdots < \rho_{\ell} \leq m \]

(4.6)

Then, by expanding the associated minors, (4.5) can be written as

\[ \det(I+G_pG_c) = 1 + \sum_{\ell=1}^{\min(n,m)} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq n} 1 \leq \sigma_1 < \cdots < \sigma_{\ell} \leq m \]

\[ \sum_{\delta \sigma_1', \cdots, \sigma_{\ell}'} G_p (\sigma_1') \cdots G_p (\sigma_{\ell}') \]

\[ \sum_{\delta \sigma_1'', \cdots, \sigma_{\ell}''} G_c (\sigma_1'') \cdots G_c (\sigma_{\ell}'') \]
\[
\min(n,m) = 1 + \sum_{k=1}^{\min(n,m)} \sum_{1 \leq k_1 < \cdots < k_\ell \leq n} \sum_{1 \leq \sigma_1 < \cdots < \sigma_\ell \leq m} \Sigma \delta_{\sigma_1', \ldots, \sigma_\ell'} \prod_{j=1}^{\ell} G_p(\sigma_j', k_j) \prod_{j=1}^{\ell} G_c(\sigma_j'', k_j) \] 

\tag{4.7}

where

\[
\delta_{\sigma_1', \ldots, \sigma_\ell'} = \begin{cases} 
1 & \text{if } \sigma_1', \ldots, \sigma_\ell' \text{ is an even permutation of } \sigma_1'', \ldots, \sigma_\ell'' \\
-1 & \text{if } \sigma_1', \ldots, \sigma_\ell' \text{ is an odd permutation of } \sigma_1'', \ldots, \sigma_\ell'' 
\end{cases}
\]

and \(\Sigma\), \(\Xi\) represent summations over all possible permutations of \(\sigma_1', \ldots, \sigma_\ell', \) and \(\sigma_1'', \ldots, \sigma_\ell''\), respectively.

The identity \(\delta_{\sigma_1', \ldots, \sigma_\ell'} = \delta_{\sigma_1, \ldots, \sigma_\ell} \delta_{\sigma_1'', \ldots, \sigma_\ell''} \delta_{\sigma_1', \ldots, \sigma_\ell'}\) is used, which can be proved by first rearranging \(\sigma_1', \ldots, \sigma_\ell'\) into \(\sigma_1, \ldots, \sigma_\ell\), then \(\sigma_1'', \ldots, \sigma_\ell''\) and by using simple reasoning.

It can easily be shown by letting \(n=m=2\) in (4.7) that (3.1) can be obtained through this expression. Similarly, general expressions for the cofactors \((I + G_p G_c)_{ji}\) can be obtained by (4.6). Thus, by using (4.1), (4.2), (4.3), and (4.4), the problem of expressing \(\det(I + G_p G_c)\) and \((I + G_p G_c)_{ji}\) in terms of the transfer functions \(g_{pij}\) and \(g_{cij}\) explicitly is solved.
Then, following the approach of Chapter 3, limit expressions similar to (3.6) and (3.7) can be obtained. In order to evaluate the values of these limits, it was found convenient to express each transfer function as in (3.10). By doing so, (4.7) can be written as:

$$\det(I+G G G) = 1 + \sum_{\substack{l=1
1\leq k_1 < \cdots < k_{\ell} \leq n
1\leq \sigma_1 < \cdots < \sigma_{\ell} \leq m}} \sum_{\substack{\delta = 1
\delta = 1, \ldots, \delta_{\ell} \in \sigma_1, \ldots, \sigma_{\ell}}} \sum_{\substack{\epsilon = 1
\epsilon = 1, \ldots, \epsilon_{\ell} \in \sigma_1, \ldots, \sigma_{\ell}}} \left[ T_P (\sigma_{\ell}) + \cdots + T_P (\sigma_1) + T_C (k_{\ell}) + \cdots + T_C (k_1) \right]$$

$$\cdot G^I (\sigma_{\ell}) \cdots G^I (\sigma_1) \cdot G^C (k_{\ell}) \cdots G^C (k_1)$$

(4.8)

Note that there are

$$\min(n, m) \sum_{\ell=1}^{\min(n, m)} \prod_{\ell=1}^{\min(n, m)} n_{P_{\ell}} \cdot m_{P_{\ell}} \cdot \ell! \cdot \ell! = \sum_{\ell=1}^{\min(n, m)} \prod_{\ell=1}^{\min(n, m)} n_{P_{\ell}} \cdot m_{P_{\ell}} \cdot \ell! \cdot \ell!$$

(4.9)

terms in (4.8). Similarly, (4.6) can be manipulated into the following form:
\[(I + G G_c)_{ji} = \min(n-2,m-1) \sum_{z=0}^{\infty} \prod_{1 \leq k_1 < \cdots < k_z \leq n} 1 \leq \rho_0 < \cdots < \rho_z \leq m k_1, \ldots, k_z \neq i, j \]

\[
\sum_{z=0}^{\infty} \prod_{1 \leq k_1 < \cdots < k_z \leq n} 1 \leq \rho_0 < \cdots < \rho_z \leq m k_1, \ldots, k_z \neq i, j
\]

\[
- \left[ T_p(\rho_0) + T_p(\rho_i) + \cdots + T_p(\rho_k) + T_c(\rho_0') + T_c(\rho_i') + \cdots + T_c(\rho_k') \right]
\]

\[
G_p(\rho_0') G_p(\rho_i') G_p(\rho_k') G_c(\rho_0') G_c(\rho_i') G_c(\rho_k')
\]

(4.10)

Again, note that there are

\[
\min(n-2,m-1) \sum_{z=0}^{\infty} \binom{n-2}{z} \cdot \binom{m}{z+1} \cdot (z+1)! \cdot (z+1)! \]

(4.11)

terms in (4.10).

By (4.8), (4.10) and Theorem 2.2 (assuming arbitrary \( r_j \)), \( n(n-1) \)

limit expressions similar to (3.11) and (3.12) can be obtained.

These limits are, according to Theorem 2.2, necessary and sufficient
conditions for steady-state decoupling. In order to satisfy these
conditions, Theorem 3.1 and 3.2 were developed to find the
constraints on \( t_{ij} \).
By comparing (4.8) and (4.10) to the denominator and the numerator of (3.14), it can be seen that they are of exactly the same form, only that $n_i,m_j,p_i,q_j$ assume more complicated forms here. Therefore, similar to what was done in Chapter 3, constraints on $t_{cij}$ for steady-state decoupling can be obtained by applying Theorem 3.1 (or Theorem 3.2, if the inputs contain ramps or parabolas besides steps) to each of these $n(n-1)$ limits.

To be more precise, let's go through these step by step as follows:

1. Assume step inputs with arbitrary amplitudes, (i.e., $k_j = 1$, $r_j$ arbitrary for all $j = 1, \ldots, n$), and consider the configuration Figure 2.1. By Theorem 2.2, the system is steady-state decoupled if and only if

$$\lim_{s \to 0} \frac{(I+G_pG_c)_{ji}}{\det(I+G_pG_c)} = 0$$

(4.12)

for all $i, j = 1, \ldots, n$ and $i \neq j$.

Note that $1/(s^{k_j} - 1) \equiv 1$ since $k_j = 1$ for step inputs.
2. By comparing the denominator and the numerator of (3.14) to those of (4.12) which are given respectively in (4.8) and (4.10), we have J m's, J q's, L n's and L p's (see (4.9) and (4.11)) as follows:

\[ m_j = T \left( \frac{k_1}{\sigma_1} \right) + \cdots + T \left( \frac{k_\ell}{\sigma_\ell} \right) + T \left( \frac{\sigma_1'}{k_1} \right) + \cdots + T \left( \frac{\sigma_\ell'}{k_\ell} \right) \quad (4.13) \]

\[ n_i = T \left( \frac{i}{\rho_0'} \right) + T \left( \frac{k_1}{\rho_1'} \right) + \cdots + T \left( \frac{k_\ell}{\rho_\ell'} \right) + T \left( \frac{\rho_0''}{j} \right) + T \left( \frac{\rho_1''}{k_1} \right) + \cdots + T \left( \frac{\rho_\ell''}{k_\ell} \right) \quad (4.14) \]

\[ q_j = G \left( \frac{k_1}{\sigma_1'} \right) \cdots G \left( \frac{k_\ell}{\sigma_\ell'} \right) G \left( \frac{\sigma_1'}{k_1} \right) \cdots G \left( \frac{\sigma_\ell'}{k_\ell} \right) \quad (4.15) \]

\[ p_i = G \left( \frac{i}{\rho_0'} \right) G \left( \frac{k_1}{\rho_1'} \right) \cdots G \left( \frac{k_\ell}{\rho_\ell'} \right) G \left( \frac{\rho_0''}{j} \right) G \left( \frac{\rho_1''}{k_1} \right) \cdots G \left( \frac{\rho_\ell''}{k_\ell} \right) \quad (4.16) \]

where each possible combination of k's, \( \sigma \)'s, and \( \rho \)'s under the restrictions in (4.8) and (4.10) contributes to one of the above.

3. Then, for each of the \( n(n-1) \) limits (4.12), Theorem 3.1 can be applied and a set of inequalities consisting of

\[ \text{Max} \{ m_j, \mid j' = 1, \cdots, J \} \quad (4.17) \]

and \[ \text{Max} \{ n_i, \mid i' = 1, \cdots, L \} \quad (4.18) \]
can be obtained as in Chapter 3. Again, since all the $T_i's$ in (4.13) and (4.14) are known for any given plant, the only unknowns are the $T_c's$ which can be chosen to satisfy the inequalities and hence achieve a steady-state decoupling scheme.

4. Whenever any transfer function in $G_p$ or $G_c$ is identically zero, those terms in the summations of (4.8) and (4.10) that contain such a factor will also be identically zero, hence the number of non-trivial terms in (4.8) and (4.10) will be less than $J$ and $L$, respectively. The number of $m_j$ and $n_i$ will also be reduced. Thus, those $m_j$ and $n_i$ in (4.17) and (4.18) associated with the identical zero term should be dropped, since they don't even appear in (3.14). In order to express this analytically, the identification symbol "x" introduced in Chapter 3 (see the discussion following (3.10)) will be used. Also, the following definition of annihilation sum is needed:

The annihilation sum is defined to be a summation, which will sum up to be an empty set whenever there exists at least one identification symbol $x$ in the summands, otherwise it is the same as algebraic sum. The symbol (+) will be used for such kind of summation, e.g.

$$1(+)2(+)3 = 1+2+3 = 6$$
$$1(+)x(+)3 = \emptyset,$$ an empty set.
By using these concepts, (4.13), (4.14) become

\[
m_j' = T_p\left(\begin{array}{c} k_1 \\ \sigma^1_1 \\
\end{array}\right) (+) \cdots (+) T_p\left(\begin{array}{c} k_L \\ \sigma^L_1 \\
\end{array}\right) + T_c\left(\begin{array}{c} \sigma^1_1 \\ k_1 \\
\end{array}\right) (+) \cdots (+) T_c\left(\begin{array}{c} \sigma^L_1 \\ k_L \\
\end{array}\right) \tag{4.19}
\]

\[
n_i' = T_p\left(\begin{array}{c} i \\ \rho^i_0 \\
\end{array}\right) (+) T_p\left(\begin{array}{c} k_1 \\ \rho_1^i \\
\end{array}\right) (+) \cdots (+) T_p\left(\begin{array}{c} k_L \\ \rho_L^i \\
\end{array}\right) (+) T_c\left(\begin{array}{c} \rho^o_0 \\ j \\
\end{array}\right) (+) T_c\left(\begin{array}{c} \rho^o_1 \\ k_1 \\
\end{array}\right) (+) \cdots (+) T_c\left(\begin{array}{c} \rho^o_L \\ k_L \\
\end{array}\right) \tag{4.20}
\]

Now, starting from the type number matrices \( T_p \) and \( T_c \) (see Chapter 3), we know immediately from (4.19) and (4.20) which \( m_j' \) and \( n_i' \) are to be included and which should be discarded.

5. Define \( M, N_{ji} \) to be the \( 1 \times J \) and \( 1 \times L \) vectors with their elements corresponding to the \( J \) and \( L \) annihilation sums given in (4.19) and (4.20). Note that some of their elements can be an empty set, whenever "x" is contained in those particular elements.

In this way, whether a term should be dropped or not is expressed analytically. Then, by Theorem 2.2 and Theorem 3.2, the following general theorem for steady-state decoupling can be given:

**Theorem 4:** Let the given \( n \times m \) plant \( G_p \) be compensated by an \( m \times n \) compensator \( G_c \) as in Figure 2.1. \( T_p \) (\( n \times m \)), \( T_c \) (\( m \times n \)) are the type number matrices of the plant and compensator, respectively. \( M, N_{ji} \)
are defined to be the maximum among the elements of $M$ and $N_{ji}$, respectively. Where $M$ and $N_{ji}$ are the $1xJ$, and $1xL$ vectors defined above, then under the following constraints:

\begin{align*}
\text{(i)} & \quad \lim_{s \to 0} \sum_{\ell=0}^{\min(n-2,m-1)} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq n} \sum_{1 \leq \rho_0 < \cdots < \rho_{\ell} \leq m} \sum_{\rho_1^2} \sum_{\rho_2^2} \cdots \sum_{\rho_{\ell}^2} G_i(\rho_1^2) G_i^{(k_1)} \cdots G_i^{(k_{\ell})} \\
& \qquad \quad G_j(\rho_0^2) G_j^{(k_1)} \cdots G_j^{(k_{\ell})} \neq 0 \tag{4.21}
\end{align*}

\begin{align*}
\text{(ii)} & \quad \lim_{s \to 0} \sum_{\ell=1}^{\min(n,m)} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq n} \sum_{1 \leq \sigma_1 < \cdots < \sigma_{\ell} \leq m} \sum_{\sigma_1^2} \sum_{\sigma_2^2} \cdots \sum_{\sigma_{\ell}^2} G_i(\sigma_1^2) G_i^{(k_1)} \cdots G_i^{(k_{\ell})} \\
& \qquad \quad G_j(\sigma_0^2) G_j^{(k_1)} \cdots G_j^{(k_{\ell})} \neq 0 \tag{4.22}
\end{align*}

\begin{align*}
\text{(iii)} & \quad \lim_{s \to 0} \sum_{\ell=1}^{\min(n,m)} \sum_{1 \leq k_1 < \cdots < k_{\ell} \leq n} \sum_{1 \leq \sigma_1 < \cdots < \sigma_{\ell} \leq m} \sum_{\sigma_1^2} \sum_{\sigma_2^2} \cdots \sum_{\sigma_{\ell}^2} G_i(\sigma_1^2) G_i^{(k_1)} \cdots G_i^{(k_{\ell})} \\
& \qquad \quad G_j(\sigma_0^2) G_j^{(k_1)} \cdots G_j^{(k_{\ell})} \neq -1 \tag{4.23}
\end{align*}
Where \( <N> \) under the summation sign of (4.21) denotes that the sum is only over all those terms with \( s^{-N} \) as their multiplication factor in (4.10). The \( <M> \) and \( <0> \) in (4.22) and (4.23) are defined similarly.

The system is steady-state decoupled if and only if either

\[
M > N_{ji} + k_j - 1 \quad \text{for all } i = j
\]

or

\[
M < 0 \quad N_{ji} < 1 - k_j \quad i, j = 1, \ldots, n
\]

Theorem 4 looks formidable, however, for systems with less than 4 inputs and 4 outputs, long-hand calculation is still feasible, especially when \( G_c \) assumes some simple forms like being diagonal, as were shown in Chapter 3, which is often of practical importance.

Besides, due to its analytical nature, Theorem 4 can be programmed into computers, thus making the design easier.
PART II

STABILITY AND DESIGN
5. INTRODUCTION TO PART II

Part I gives the scheme for decoupling the steady states of a system, however, it should be noticed that:

1. The result does not guarantee stability.

2. The whole discussion is meaningful only when the closed-loop system is stable.

Therefore, stability must be considered after the steady-state decoupling scheme is achieved.

The problem of stability and design of multivariable systems has been widely investigated (e.g. (5), (7), (8), (9), (10), (11)), and a survey of the existing methods was given by Anderson (12). In general, efforts have been made to utilize the beauty of the existing single-loop techniques such as Nyquist-Bode-Nichol's methods and root locus design.

In this part of the thesis, a new connection between single-loop and multivariable systems is seen by properly factorizing the closed-loop characteristic equation. This makes the design of multivariable systems possible by using any suitable classical single-loop method. An extended root locus method is developed and diagonal $G_c$ will be considered primarily.
6. DESIGN OF CLOSED-LOOP SYSTEMS WITH 2x2 PLANT AND DIAGONAL $G_c$

The characteristic equation for single-loop systems with cascade compensation and unity feedback is

$$1 + g_p(s) \cdot g_c(s) = 0 \quad (6.1)$$

where $g_p(s)$ is the given plant and $g_c(s)$ is the cascade compensator function to be designed.

Two major techniques for the design of $g_c(s)$ are Bode's method and the root locus method (see (14)). However, only one unknown function can be handled in each of these methods. Therefore, they cannot be applied directly to multivariable cases, since in general, there exists $n \cdot m$ unknown compensator functions to be designed.

A simple factorization which shows the connection between single-loop and multivariable cases will be given in this chapter. It can then be seen that the above-mentioned single-loop methods can still be applied for multivariable systems.

The design philosophy will be illustrated through an example in Section 6.4. Before that, however, several important steps must be
established. These are given in Section 6.1 to 6.3 as follows:

6.1 CHARACTERISTIC EQUATION

It is proved in (7) and (8) that the stability of a multivariable system as shown in Figure 2.1 is determined by the zeros of $N_1(s)$ and $N(s)$, which are defined as follows:

\[
\frac{N_1(s)}{D_1(s)} \triangleq \text{det}(I+G_p(s) \cdot G_c(s))
\]

\[
\hat{N}(s) \triangleq \frac{\Delta_c(s) \cdot \Delta_p(s)}{D_1(s)}
\]

Where the rational function $N_1(s)/D_1(s)$ is in irreducible form, i.e., no common factor between $N_1(s)$ and $D_1(s)$ is left uncancelled. And $\Delta_c(s)$ $\Delta_p(s)$ represent the characteristic polynomials of the rational transfer function matrices $G_c(s)$ and $G_p(s)$, respectively.

The characteristic polynomial of a proper rational transfer function matrix $G(s)$ is defined to be the least common denominator of all the minors (in irreducible rational form) of $G(s)$ (see e.g. (13)).

Therefore, by (4.5), if all the minors of $G$ and $G_c$ are non-zero, $G_p$
and no common pole exists between $G_p$ and $G_c$, the denominator of $\det(I+G_{p}G_{c})$ is $\Delta_c(s)\Delta_p(s)$ before any pole-zero cancellation is performed. Since $N_1(s)/D_1(s)$ is in irreducible form, it is clear by (6.3) that $\hat{N}(s)$ simply consists of all those factors that were cancelled in getting the irreducible $N_1(s)/D_1(s)$ form. Therefore, in this case, if all the common factors in (6.2) are left uncancelled, the zeros of (6.2) alone determine the stability of the system. Unfortunately, the same conclusion is not true in general if zero minor(s) of $G_p$ and (or) $G_c$ exists. Furthermore, if "all" the common factors are left uncancelled, erroneous results can still be obtained as was shown by Chen in (7).

However, it is found (see Appendix B) that if

1. cancellations are selected systematically by using (4.5),

2. poles of $G_c$ are carefully selected,

then (6.2) alone determines the stability of the system.

Since (1) above can always be done and (2) can be taken care of fairly easily in the process of design, the mathematical possibilities in which zeros of (6.3) must be considered can be bypassed.
Thus, \( \det(I + G_p G_c) = 0 \) \( (6.4) \)

will be referred to as the characteristic equation for multi-variable systems as shown in Figure 2.1.

Details are given in Appendix B.

6.2 CONNECTION BETWEEN SINGLE-LOOP AND MULTIVARIABLE CASES

For 2x2 plant \( G_p \), and 2x2 diagonal \( G_c \),

\[
\det(I + G_p G_c) = 1 + gp_{11}gc_{11} + gp_{22}gc_{22} + (\det G_p)gc_{11}gc_{22} \tag{6.5}
\]

which can be factored as

\[
\det(I + G_p G_c) = (1 + gp_{11}gc_{11}) \left( 1 + \frac{gp_{22} + (\det G_p)gc_{11}}{1 + gp_{11}gc_{11}} \right) gc_{22} \tag{6.6}
\]

\[
= (1 + gp_{11}gc_{11}) (1 + G_{eq}gc_{22}) \tag{6.7}
\]

where \( G_{eq} \triangleq \frac{gp_{22} + (\det G_p)gc_{11}}{1 + gp_{11}gc_{11}} = \frac{gp_{22}}{1 + gp_{11}gc_{11}} \left( 1 + \frac{\det G_p}{gp_{22} gc_{11}} \right) \tag{6.8} \)

By (6.7), the roots of the characteristic equation are simply the zeros of the rational function \( (1 + gp_{11}gc_{11}) (1 + G_{eq}gc_{22}) \).
It is also clear by (6.6) that for non-trivial cases \( G_{eq}(s) \neq 0 \), all the zeros of the first factor will be cancelled exactly by some of the poles of the second factor. Therefore, the roots of

\[
1 + G_{eq} \cdot g_{c22} = 0 \quad (6.9)
\]

will determine the stability of the system.

The similarity of the form of (6.9) to that of (6.1) suggests immediately the possibility of applying the single-loop methods mentioned above to multivariable cases. But, unlike the \( g_p \) in (6.1), which is known for a given plant, \( G_{eq} \) of (6.9) is not a known function. Hence, neither Bode plot nor root locus for \( G_{eq} \) can be drawn at this stage. By inspection of (6.8), the only unknown function contained in \( G_{eq} \) is \( g_{c11} \). We can, of course, choose an arbitrary function for \( g_{c11} \), then the only unknown function left to be designed is \( g_{c22} \), and the design is reduced to that of the single-loop case. For example, let \( g_{c11} \) be an arbitrary constant, say 3, then for any given plant, \( G_{eq} \) can be obtained through (6.8), and the design of \( g_{c22} \) can be carried through by using (6.9).

However, in doing so, \( G_{eq} \) may turn out to be very unstable, which will make the design of \( g_{c22} \) extremely difficult. Therefore, instead of choosing it arbitrarily, a guide in designing \( g_{c11} \) is preferable.
By inspection of (6.8), it can be seen that the roots of

\[ 1 + \frac{\text{det} G_p}{g_{p22}} \cdot g_{c11} = 0 \quad (6.10) \]

and

\[ 1 + g_{p11} g_{c11} = 0 \quad (6.11) \]

consistute part of the zeros and poles of $G_{eq}$, respectively.

Again, both (6.10) and (6.11) are of the standard form (6.1). Furthermore, $\text{det} G_p/g_{p22}$ and $g_{p11}$ are now known functions.

Therefore, any single-loop method be used in designing $g_{c11}$, to place the roots of (6.10) and (6.11) at desirable locations. Since these roots will be part of the poles and zeros of $G_{eq}$, what is meant by placing them at desirable locations is that $g_{c11}$ should be designed such that these roots, together with the other known poles and zeros (see (6.8)) form a reasonably good pole-zero pattern for $G_{eq}$ (i.e., $G_{eq}$ is not badly unstable). Once $g_{c11}$ is designed, all the poles and zeros of $G_{eq}$ are known, and the problem of designing $g_{c22}$ is, by (6.9), reduced to that of the single-loop case, and can be done by either Bode's or root locus method.
In summary, what has been accomplished so far is that the effect of \( g_{c11} \) on the pole-zero pattern of \( G_{eq} \) and hence on the design of \( g_{c22} \) can be seen through (6.10) and (6.11). Therefore, (6.10) and (6.11) serve as a guide in designing \( g_{c11} \) in order to make easy the design of \( g_{c22} \).

Because of the standard forms involved, both Bode's and root locus design techniques can be applied. For better insight of the problem, the root locus method will be considered primarily.

6.3 IDENTIFICATION OF POLES, AND ZEROS

Since the design will be concerned with (6.9), the poles and zeros of \( G_{eq}(s) \) must be well identified, and the problem of pole-zero cancellation must be considered carefully. This can be done generally by considering the sum and ratio of the following two rational functions \( G_1(s) \) and \( G_2(s) \):

\[
G_1(s) = \frac{N_1(s)}{D_1(s)}
\]

(6.12)

\[
G_2(s) = \frac{N_2(s)}{D_2(s)}
\]

(6.13)
Where \( N_i(s), D_i(s) \) are the numerator and the denominator polynomials of \( G_i(s) \) for \( i=1,2 \), also note that \( N_1(s), D_1(s) \) here are different from those in (6.2).

For simplicity, the argument \( s \) will be omitted in the following discussion.

Let \( D_{12} \) be the greatest common factor between \( D_1 \) and \( D_2 \), and \( D_{11}, D_{22} \) are the remaining factors as shown below:

\[
D_1 = D_{11} D_{12}
\]  

(6.14)

\[
D_2 = D_{22} D_{12}
\]

Then, the sum of \( G_1 \) and \( G_2 \) is

\[
G_1 + G_2 = \frac{N_1}{D_1} + \frac{N_2}{D_2}
\]

\[
= \frac{N_1 D_{22} + N_2 D_{11}}{D_{11} D_{12} D_{22}}
\]

(6.15)

Note that \( D_{11} D_{12} D_{22} \) is the least common multiplier of \( D_1 \) and \( D_2 \).
Now, consider the sum $1 + \frac{G_2}{G_1}$, by (6.12), (6.13) and (6.14)

$$1 + \frac{G_2}{G_1} = 1 + \frac{N_2}{D_2} \cdot \frac{D_1}{N_1}$$

$$= 1 + \frac{N_2}{D_{22} D_{12}} \cdot \frac{D_{12} D_{11}}{N_1}$$  \hspace{1cm} (6.16)$$

If the common factor $D_{12}$ between $D_1$ and $D_2$ is cancelled, and no other cancellation is performed, (6.16) becomes

$$1 + \frac{G_2}{G_1} = \frac{D_{22} N_1 + D_{11} N_2}{D_{22} N_1}$$ \hspace{1cm} (6.17)$$

The numerator of (6.17) is exactly that of (6.15). Therefore, the zeros of $G_1 + G_2$, before any possible cancellation by the poles, would be the same as those of $1 + \frac{G_2}{G_1}$, if $D_{12}$ (and only $D_{12}$) is cancelled. No other cancellation in $G_2/G_1$ is allowed, even if it can be done. Otherwise, some zero of (6.15) would not appear in (6.17).

Therefore, if only $D_{12}$ is cancelled in forming $G_2/G_1$, the root locus for $G_2/G_1$ would give all the zeros of $G_1 + G_2$ if no cancellation is done between the numerator and the denominator of (6.15).
Let $P_G = \{ \ldots \}$ and $Z_G = \{ \ldots \}$ denote the set of poles and zeros of the rational function $G$, where the multiplicity of each pole or zero is counted, e.g.,

let

$G(s) = \frac{(s+3)^3(s+5)^2}{s(s+1)^2(s+2)}$

$H(s) = \frac{(s+3)^2(s+5)^2}{s^2(s+1)^3(s+2)}$

then

$P_G = \{0,-1,-1,-2\}$
$Z_G = \{-3,-3,-3,-5,-5\}$
$P_H = \{0,0,-1,-1,-1,-2\}$
$Z_H = \{-3,-3,-5,-5\}$

(6.18)

Also, let $s_1 \cap s_2$ denote the intersection of the two sets $s_1$ and $s_2$, defined as in set theory only that multiplicity is taken into account here, e.g., in (6.18)

$P_G \cap P_H = \{0,-1,-1,-2\}$
$Z_G \cap Z_H = \{-3,-3,-5,-5\}$
Furthermore, \( s_1 + s_2 \) is defined to be the set consisting of all the elements of \( s_1 \) and those of \( s_2 \), (counting multiplicity), e.g., in (6.18)

\[
P_G + P_H = \{0, -1, -1, -2, 0, 0, -1, -1, -1, -2\}
\]

Similarly, \( s_1 - s_2 \) is defined to be the set formed by taking away all the elements (counting multiplicity) of \( s_2 \) from \( s_1 \), and is defined only when \( s_2 \) is a subset of \( s_1 \), e.g., in (6.18)

\[
P_H - P_G = \{0, -1\}
\]
\[
Z_G - Z_H = \{-3\}
\]

By using these notations and by inspection of the denominator of (6.15), we have

\[
P_{G_1 + G_2} = P_{G_1} + P_{G_2} - (P_{G_1} \cap P_{G_2}) \quad (6.19)
\]

Similarly, by comparing the numerators of (6.15) and (6.17), we have

\[
Z_{G_1 + G_2} = Z_1 \cdot \frac{G_2}{G_1} \quad (6.20)
\]
The poles and zeros for the ratio $G_2/G_1$ can also be obtained in a similar way by inspection of the second term in (6.16)

$$P_{G_2/G_1} = Z_{G_1} + P_{G_2} - (P_{G_1} \cap P_{G_2})$$ \hspace{1cm} (6.21)

$$Z_{G_2/G_1} = Z_{G_2} + P_{G_1} - (P_{G_1} \cap P_{G_2})$$ \hspace{1cm} (6.22)

Note that in (6.19), (6.20), (6.21) and (6.22), $D_{12}$ is cancelled, and no other cancellation is performed.

The application of (6.19), (6.20), (6.21) and (6.22) will be illustrated through a design example in the next section. At present, however, (6.20) will be used to justify one statement pointed out in Section 6.2, i.e., the roots of (6.9) will determine the stability of the system.

Let

$$G_1 = 1 + g_{p11}g_{c11}$$

$$G_2 = g_{p22} + (\det G_p)g_{c11}$$

then (6.5) becomes

$$\det(I + G_{pc}) = G_1 + G_2g_{c22}$$ \hspace{1cm} (6.23)
By (6.20),

\[ 2G_1 + G_2 G_{c22} = Z \frac{G_2}{1 + G_1 G_{c22}} \]

if the common factors between the denominators of \( G_1 \) and \( G_2 G_{c22} \) (and only these common factors) are cancelled.

Since \( G_2 / G_1 = G_{eq} \) by (6.8),

\[ Z_1 \frac{G_2}{G_1 G_{c22}} = Z_1 + G_{eq} G_{c22} \]

Hence, the zeros of (6.23) are exactly the same as those of \( 1 + G_{eq} G_{c22} \) i.e., the roots of (6.9) are the same as those of the characteristic equation (6.4). Therefore, they do determine the stability of the system (under the restrictions given in Appendix B). No pole would be lost on account of the factorization and the using of (6.19), (6.20), (6.21) and (6.22).

### 6.4 DESIGN EXAMPLE

Consider the 2x2 plant

\[
P = \begin{bmatrix}
\frac{s+3}{s(s+1)} & 4 \\
\frac{3}{s+2} & -\frac{2}{s}
\end{bmatrix}
\]  \hspace{1cm} \text{(6.24)}
For diagonal $G_c$, the characteristic equation (6.4) is, by (6.5)

$$1 + g_{p11}g_{c11} + g_{p22}g_{c22} + (\text{det } G_p)g_{c11}g_{c22} = 0 \quad (6.25)$$

By the factorization (6.7) and the discussions in Section 6.2 and Section 6.3, the design philosophy follows:

1. Design $g_{c11}$ according to (6.10) and (6.11), to achieve a reasonably good pole-zero pattern for $G_{eq}$.

2. Design $g_{c22}$ according to (6.9) to meet system specifications.

where (6.9), (6.10) and (6.11) are repeated below:

$$1 + G_{eq}g_{c22} = 0 \quad (6.9)$$

$$1 + \frac{\text{det } G}{g_{p22}} g_{c11} = 0 \quad (6.10)$$

$$1 + g_{p11}g_{c11} = 0 \quad (6.11)$$

By (6.24),

$$g_{p11} = \frac{s+3}{s(s+1)} \quad (6.26)$$
\[ g_{p22} = -\frac{2}{s} \quad (6.27) \]

\[ \text{det} G_p = -\frac{14s^2 + 10s + 12}{s^2(s+1)(s+2)} \quad (6.28) \]

\[ \text{det} G_p \]
\[ g_{p22} = \left[ -\frac{14s^2 + 10s + 12}{s^2(s+1)(s+2)} \right] \cdot \left( -\frac{s}{2} \right) \]
\[ = \frac{7s^2 + 5s + 6}{s(s+1)(s+2)} \quad (6.29) \]

Note that the common factor \( s \) between the denominators of \( \text{det} G_p \) and \( g_{p22} \) is cancelled. Also note that the use of diagonal \( G_c \) is allowed in this example, since multiplicity of each plant pole is not reduced in the minor \( \text{det} G_p \) (see Appendix B).

Let \[ G_1 = 1 + g_{p11} g_{c11} \quad (6.30) \]
\[ G_2 = g_{p22} + (\text{det} G_p) g_{c11} \quad (6.31) \]

then, by (6.8),
\[ G_{eq} = \frac{G_2}{G_1} \quad (6.32) \]
The poles and zeros of $G_{eq}$ are by (6.21) and (6.22),

$$P_{G_{eq}} = Z_{G_1} + P_{G_2} - (P_{G_2} \cap P_{G_1})$$
$$Z_{G_{eq}} = Z_{G_2} + P_{G_1} - (P_{G_2} \cap P_{G_1})$$  \hspace{1cm} (6.33)

The poles and zeros of $G_1$ and $G_2$ are by (6.19) and (6.20),

$$P_{G_1} = P_{g_{p_{11}g_{c_{11}}} + P_{g_{c_{11}}} = \{0,-1\} + P_{g_{c_{11}}}$$  \hspace{1cm} (6.34)

$$Z_{G_1} = Z_1 + g_{p_{11}g_{c_{11}}}$$  \hspace{1cm} (6.35)

$$P_{G_2} = P_{g_{p_{22} + P_{(det_{G_p})g_{c_{11}}} - [P_{g_{p_{22}}} \cap P_{(det_{G_p})g_{c_{11}}}]]}$$
$$= \{0\} + \{0,0,-1,-2\} + P_{g_{c_{11}}} - [\{0\} \cap ((0,0,-1,-2) + P_{g_{c_{11}}})]$$
$$= \{0,0,0,-1,-2\} + P_{g_{c_{11}}} - \{0\}$$
$$= \{0,0,-1,-2\} + P_{g_{c_{11}}}$$  \hspace{1cm} (6.36)

$$Z_{G_2} = Z_1 + \frac{det_{G_p}}{g_{p_{22}}} g_{c_{11}}$$  \hspace{1cm} (6.37)

By (6.34) and (6.36),

$$P_{G_1} \cap P_{G_2} = \{0,-1\} + P_{g_{c_{11}}} = P_{G_1}$$  \hspace{1cm} (6.38)
Then by (6.33),

\[ P_{G_{eq}} = Z_{1 + g_{p11} g_{c11}} + \{0,0,-1,-2\} + \frac{g_{11}}{g_{c11}} \left[ \{0, -1\} + \frac{p_{22}}{g_{c11}} \right] \]

\[ = Z_{1 + g_{p11} g_{c11}} + \{0, -2\} \]

\[ Z_{G_{eq}} = Z_{G_{2}} = Z_{1 + \frac{\text{det}G_{P}}{g_{p22}} g_{c11}} \]

Thus, \( G_{eq} \) has two known poles at \( s=0 \) and \( s=-2 \), the other poles are the roots of \( 1 + g_{p11} g_{c11} = 0 \), which is (6.11). The zeros of \( G_{eq} \) are simply the roots of \( 1 + \frac{\text{det}G_{P}}{g_{p22}} g_{c11} = 0 \), which is (6.10).

Then, the design procedure follows:

1. Prepare the root loci for

\[ 1 + k_{1} g_{p11} = 0 \]

and \( 1 + k_{1} \frac{\text{det}G_{P}}{g_{p22}} = 0 \)

where \( k_{1} \) is a real parameter. The result is shown in Figure 6.1 and 6.2. The loci for positive and negative values of \( k_{1} \) are represented by solid and dashed curves, respectively.
2. The roots of Figure 6.1 will be poles of $G_{eq}$, therefore, for negative values of $k_1$, there will be one pole of $G_{eq}$ in the right half plane (on branch $\Theta$). Similarly, by Figure 6.2, a zero of $G_{eq}$ will be in the right half plane (on branch $\Theta$) if $k_1<0$. This is certainly undesirable. Thus, negative $k_1$ will not be considered.

3. Mark down the known poles $s=0$, and $s=-2$ of $G_{eq}$ on Figure 6.3 and superimpose the loci of Figure 6.1 and 6.2 corresponding to positive $k_1$ on top of it. Note that roots on branch $\Theta$ and $\Theta$ correspond to poles and zeros of $G_{eq}$, respectively.

4. Increase the value of $k_1$ from 0 to $\infty$, and observe the change of pole-zero pattern. It can be seen that

(i) $0<k_1<5$ is not desirable, since the poles will be clustered together near the origin.

(ii) $k_1>5$ is also not desirable, since the pole on branch $\Theta$ will be pushed into the negative real axis, hence, dominant roots of (6.9) will probably be determined by the two known poles $s=0$, $s=-2$ and the zeros on branch $\Theta$, which are too close to the origin.
5. When \( k_1 = 5 \), the roots on branch \( \circ \) which are the zeros of \( G_{eq} \) will be close to the two zeros on Figure 6.2 and the pole on branch \( \bullet \) is as shown. Another zero, by Figure 6.2 will be on the negative real axis at about \( s = -35 \). This pole-zero configuration looks to be the best, since it is possible to confine the roots of \( 1+k_2 G_{eq} = 0 \), corresponding to the two poles \( s=0 \) and \( s=-2 \) and the two conjugate zeros, to be on the negative real axis. And at the same time, the root of \( 1+k_2 G_{eq} = 0 \) on the branch starting from the pole at \( k_1=5 \) will be someplace to the upper left of the pole, which is a good location for dominant root. Therefore, try \( k_1=5 \).

6. Once \( k_1=5 \) is determined, all the poles and zeros of \( G_{eq} \) are known. The root locus gain (see Appendix C) of \( G_{eq} \), denoted by \( k_{eq} \) is found as follows:

\[
G_{eq} = \frac{g_p 22 + (\text{det} G_p) g_{c11}}{1+g_{p11} g_{c11}}
\]

\[
= \frac{-2 - 14s^2 + 10s + 12}{s s^2(s+1)(s+2)}
\]

\[
= 1 + \frac{s+3}{s(s+1)}
\]

\[
= \frac{-2s^3 + \cdots}{s^4 + \cdots}
\]

\[
= \frac{-2(s-z_1)(s-z_2)(s-z_3)}{(s-p_1)(s-p_2)(s-p_3)(s-p_4)}
\]

(6.41)
where \( z_i \), and \( p_i \) are the zeros and poles. By inspection of (6.41) and the definition of root locus gain, we have

\[
K_{eq} = -2
\]

7. The root locus for \( 1 + k_2 G_{eq} = 0 \) is then drawn as shown in Figure 6.4. The choice of \( k_2 \) is now strictly that of a single-loop problem. It is easily found that the two small real roots meet each other at about \( k_2 k_{eq} = 0.36 \), i.e. \( k_2 = \frac{0.36}{k_{eq}} = -0.18 \). For this value of \( k_2 \), the smallest real root on branch \( \Theta \) will be at approximately the breakaway point and will hopefully be the best among all the possible locations on this particular branch. The root on branch \( \Box \) corresponding to \( k_2 k_{eq} = 0.36 \) is also shown in Figure 6.4. It is seen that this is a pretty good dominant root.
FIGURE 6.1 Root locus for $1+\frac{b_1}{b_2^{10}}=0$

FIGURE 6.2 Root locus for $1+\frac{b_1}{b_2^{12}}=0$

FIGURE 6.3 SUPERPOSITION OF FIGURE 6.1 and FIGURE 6.2 FOR $b_2=0$

FIGURE 6.4 Root locus for $1+b_2^{20}=0$

When $b_2=0$, another zero at $s=-5$.

One zero at $s=-5$. $b_2=0.25$
The schematic for the designed system is as shown in Figure 6.5.

![Schematic Diagram]

**FIGURE 6.5**

The simulation result is shown in Figure 6.6, in which \( r_1 = 10 \) and \( r_2 = 5 \) were used as reference inputs and the time responses for the two outputs \( y_1 \) and \( y_2 \) are as shown.

It is clear that stability has been achieved. The transient response of \( y_2 \) is very good, however, that of \( y_1 \) is kind of slow. If this is not allowed by the specification, a redesign is necessary. However, as in the single-loop case, every trial, despite of its failure, provides some guide for the next trial.
FIGURE 6.6
TIME RESPONSE OF OUTPUTS $y_1$ & $y_2$ IN FIGURE 6.5 WITH

$r_1 = 10 \cdot u(t)$

$r_2 = 5 \cdot u(t)$

$u(t):$ UNIT STEP
In this example, it can be seen from Figure 6.4 that the roots on branch 1 are, compared with that on branch 2, very close to the origin. This is probably the cause of the slow response exhibited in $y_1$. What we can do is put pole(s) and zero(s) in $g_{c11}$ (instead of just using a pure gain) to push the roots of Figure 6.2 farther away from $j\omega$ axis. Then the two conjugate zeros in Figure 6.4 will also be farther away from the $j\omega$ axis and the resultant root locus for $G_{eq}$ will move toward the left, thus improving the transient response.

Incidentally, the steady states of the outputs are decoupled. This is because $g_{p11}$ and $g_{p22}$ are of type 1 and can be proved very easily by (3.19) and (3.22).
7. DESIGN OF CLOSED-LOOP SYSTEMS WITH 3x3 PLANT AND DIAGONAL \( G_c \)

7.1 GENERAL

It will be shown in this chapter that the same design philosophy given in Chapter 6 can be carried over to 3-input, 3-output plant. Again, the central idea is in the factorization of the characteristic equation \( \det(I+G_pG_c) = 0 \).

For 3x3 \( G \) and diagonal 3x3 \( G_c \), the system configuration is as shown in Figure 3.2 and the characteristic equation is by (4.5),

\[
\det(I+G_pG_c) = 1 + g_{p11}c_{11} + g_{p22}c_{22} + g_{p33}c_{33} + (G_p)_{33}g_{11}c_{11}c_{22} \\
+ (G_p)_{22}g_{11}c_{11}c_{33} + (G_p)_{11}g_{22}c_{22}c_{33} + (\det G_p)g_{c11}c_{22}g_{c33} \\
= 0 \quad (7.1)
\]

where \( (G_p)_{11} \), \( (G_p)_{22} \), \( (G_p)_{33} \) denote the cofactors of \( g_{p11}, g_{p22} \) and \( g_{p33} \), respectively.
(7.1) can be factored as

\[
(1+g_{p11}g_{c11}) \left[ \frac{1+g_{p22}+(G_p)_{33}g_{c11}}{1+g_{p11}g_{c11}} g_{c22} + \frac{g_{p33}+(G_p)_{22}g_{c11}}{1+g_{p11}g_{c11}} g_{c33} \right. \\
+ \left. \frac{(G_p)_{11}+(\text{det}G_p)g_{c11}}{1+g_{p11}g_{c11}} \cdot g_{c22}g_{c33} \right] = 0 \quad (7.2)
\]

By defining,

\[
G_1 \Delta 1 + g_{p11}g_{c11} \\
G_2 \Delta g_{p22}+(G_p)_{33}g_{c11} \\
G_3 \Delta g_{p33}+(G_p)_{22}g_{c11} \\
G_4 \Delta (G_p)_{11}+(\text{det}G_p)g_{c11}
\]

(7.1) and (7.2) become

\[
G_1 + G_2 g_{c22} + G_3 g_{c33} + G_4 g_{c22}g_{c33} = 0 \quad (7.4)
\]

\[
G_1 \left[ \frac{1+G_2}{G_1} g_{c22} + \frac{G_3}{G_1} g_{c33} + \frac{G_4}{G_1} g_{c22} g_{c33} \right] = 0 \quad (7.5)
\]
Similar to the 2x2 case considered in Chapter 6, the roots of (7.4) will be exactly the same as those of:

\[ 1 + \frac{G_2}{G_1} g_{c22} + \frac{G_3}{G_1} g_{c33} + \frac{G_4}{G_1} g_{c22} g_{c33} = 0 \]  

(7.6)

if the common factors between the denominators of \( G_2 \) and \( G_1 \), \( G_3 \) and \( G_1 \), \( G_4 \) and \( G_1 \) (and only these common factors) are cancelled in \( G_2/G_1 \), \( G_3/G_1 \) and \( G_4/G_1 \), respectively.

Thus, if the poles and zeros of \( G_2/G_1 \), \( G_3/G_1 \) and \( G_4/G_1 \) are obtained through (6.21) and (6.22), which were designed to meet the above cancellation restrictions, the roots of (7.6) are the same as those of the characteristic equation (7.1).

By defining

\[ F_1 \triangleq \frac{G_2}{G_1} = \frac{g_{p22} + (G_p)_{33} g_{c11}}{1 + g_{p11} g_{c11}} \]  

(7.7)

\[ F_2 \triangleq \frac{G_3}{G_1} = \frac{g_{p33} + (G_p)_{22} g_{c11}}{1 + g_{p11} g_{c11}} \]  

(7.8)

\[ F_3 \triangleq \frac{G_4}{G_1} = \frac{(G_p)_{11} + (\det G_p) g_{c11}}{1 + g_{p11} g_{c11}} \]  

(7.9)
(7.6) becomes

\[ 1 + F_1 g_{c22} + F_2 g_{c33} + F_3 g_{c22} g_{c33} = 0 \]  \hspace{1cm} (7.10)

(7.10) is of the same form as (6.5), therefore, the same factorization can be done on (7.10) to give

\[ (1 + F_1 g_{c22}) \cdot \left[ 1 + \frac{F_2 + F_3 g_{c22}}{1 + F_1 g_{c22}} g_{c33} \right] = 0 \]  \hspace{1cm} (7.11)

Again, by (6.20), if the common factor between the denominators of \( F_2 + F_3 g_{c22} \) and \( 1 + F_1 g_{c22} \) (and only this common factor) is cancelled in

\[ G_{eq} \triangleq \frac{F_2 + F_3 g_{c22}}{1 + F_1 g_{c22}} \]  \hspace{1cm} (7.12)

the roots of (7.10) will be the same as those of

\[ 1 + G_{eq} g_{c33} = 0 \]  \hspace{1cm} (7.13)

Since (7.10) is simply (7.6), and the roots of (7.6) are the same as those of the characteristic equation (7.1), if (6.21) and (6.22) are used in determining the poles and zeros of \( F_1 \), \( F_2 \) and \( F_3 \), the following conclusion can be made:
The roots of (7.13) are the same as those of the characteristic equation (7.1), if (6.19), (6.20), (6.21) and (6.22) are used in determining the poles and zeros at each stage.

Thus, stability design can be considered through (7.13). The form of (7.13) is exactly that of a single-loop characteristic equation (6.1). Therefore, similar to the design of $g_{c22}$ for the 2x2 case in Chapter 6, any single-loop design method can be applied in designing $g_{c33}$, once all the poles, zeros and the root locus gain of $G_{eq}$ are known.

The ingredients of the poles and zeros of $G_{eq}$ can be found by substituting (7.7), (7.8) and (7.9) into (7.12) to express $G_{eq}$ in terms of the elements of $G_p$ and $G_c$ explicitly. However, due to the fact that some cancellations must be done while some others are not allowed, this approach may sometimes lead to an erroneous result. Therefore, the analytical schemes (6.19), (6.20), (6.21) and (6.22) designed to take care of the pole-zero cancellations, are recommended.

By applying (6.21) and (6.22) on (7.12), we have

$$P_{G_{eq}} = Z_1 + F_1 g_{c22} + P_{F_2 + F_3} g_{c22} - (P_{F_2 + F_3} g_{c22} \cap P_{1 + F_1} g_{c22}) \quad (7.14)$$

$$Z_{G_{eq}} = Z_{F_2 + F_3} g_{c22} + P_{1 + F_1} g_{c22} - (P_{F_2 + F_3} g_{c22} \cap P_{1 + F_1} g_{c22}) \quad (7.15)$$
(7.14) tells us that the roots of

\[ 1 + F_1 g_{c22} = 0 \]  \hspace{1cm} (7.16)

constitute part of the poles of \( G_{eq} \).

Also, by (6.20),

\[ Z_{F_2+F_3 \cdot g_{c22}} = Z_{1+F_3 g_{c22}} \]

Hence, by (7.15), the roots of

\[ 1 + \frac{F_3}{F_2} g_{c22} = 0 \]  \hspace{1cm} (7.17)

constitute part of the zeros of \( G_{eq} \).

These justify what can be seen by inspection of (7.12). Actually, these were done by inspection in Section 6.2 (see (6.8), (6.9), (6.10) and (6.11)), before the development of (6.19), (6.20), (6.21) and (6.22).
Thus, root loci for (7.16) and (7.17) identify part of the poles and zeros of $G_{eq}$, and similar to the 2x2 case in Chapter 6, these root loci can be used in the design of $g_{c22}$ (note: this corresponds to $g_{c11}$ in Chapter 6, compare (6.10), (6.11) with (7.16) and (7.17)).

However, unlike the 2x2 case, these two root loci cannot be drawn directly, since $F_1$ and $F_3/F_2$ depend on $g_{c11}$, which is also to be designed (hence not known yet!). The dependences of $F_1$, $F_2$ and $F_3$ on $g_{c11}$ are given in (7.7), (7.8) and (7.9).

Again, by repeated application of (6.19), (6.20), (6.21) and (6.22) on (7.7), (7.8) and (7.9), or simply by inspection, it can be seen that the root loci for

$$1 + g_{p11} g_{c11} = 0 \quad (7.18)$$

and

$$1 + \frac{(G_p)_{33}}{g_{p22}} g_{c11} = 0 \quad (7.19)$$

give part of the poles and the zeros of $F_1$, respectively.
And the root loci for

\[ 1 + \frac{(G_p)_{22}}{g_{p33}} g_{c11} = 0 \]  

(7.20)

and

\[ 1 + \frac{\text{det}G_p}{(G_p)_{11}} g_{c11} = 0 \]  

(7.21)

give part of the poles and the zeros of $F_3/F_2$, respectively.

Now, let's look back and see what we've got:

1. We concluded that roots of (7.13) are the same as those of the characteristic equation (7.1). Therefore, the design of $g_{c33}$ can be done through (7.13) if $G_{eq}$ is known.

2. Some of the poles and zeros of $G_{eq}$ are adjustable through $g_{c22}$, and the relationships are given in (7.16) and (7.17). Thus, the effect of $g_{c22}$ on the pole-zero pattern of $G_{eq}$ and, hence, on the design of $g_{c33}$, can be seen through (7.16) and (7.17). Therefore, (7.16) and (7.17) serve as a guide in designing $g_{c22}$ to make the design of $g_{c33}$ not formidable. This is exactly what was obtained in Section 6.2 for the 2x2 case.
3. **Since both (7.16) and (7.17) are of the standard single-loop form (6.1),** $g_{c22}$ can be designed if $F_1$, and $F_3/F_2$ are known.

4. **Again, some poles and zeros of $F_1$ and $F_3/F_2$ are adjustable through $g_{c11}$, and the relationships are given in (7.18), (7.19), (7.20) and (7.21). Thus, the effect of $g_{c11}$ on the pole-zero pattern of $F_1$ and $F_3/F_2$, and hence on the design of $g_{c22}$, can be seen through (7.18), (7.19), (7.20) and (7.21). Therefore, these four equations can be used as a guide in designing $g_{c11}$.**

Thus, it is clear that (7.16), (7.17), (7.18), (7.19), (7.20) and (7.21) are important in stability design. By the standard forms they assume and by their similarity to those in Chapter 6, it can be concluded that root loci for these six equations can help us design $g_{c11}$ and $g_{c22}$ to get a stable enough $G_{eq}$ such that $g_{c33}$ can be designed according to (7.13).

Since $g_{p11}$, $g_{p22}$, $g_{p33}$, $(G_p)_{11}$, $(G_p)_{22}$, $(G_p)_{33}$ and $\det G_p$ are known for a given plant, the four uncompensated root loci (i.e., $g_{c11}=k$, a free parameter) for (7.18), (7.19), (7.20) and (7.21) can be constructed right after the plant is given. However, the other two root loci for (7.16) and (7.17) cannot be drawn until after $g_{c11}$ is designed, since both $F_1$ and $F_3/F_2$ depend on $g_{c11}$. As mentioned
before, $F_1$ and $F_3/F_2$ have some poles and zeros other than those given by (7.18), (7.19), (7.20) and (7.21). Therefore, the identification of these poles and zeros is necessary, both for constructing root loci for (7.16) and (7.17) and for guiding the design of $g_{c11}$. This constitutes the topic of the following section.

7.2 IDENTIFICATION OF POLES AND ZEROS

As mentioned in the previous section, direct algebraic manipulation may lead to an erroneous result, so let's apply the analytical schemes (6.19), (6.20), (6.21) and (6.22) to identify all the poles and zeros we are interested in.

The expressions for all the poles and zeros of $G_{eq}$ have already been given in (7.14) and (7.15). The poles and zeros in the sets $Z_1 + F_1 g_{c22}$ and $Z_2 + F_2 g_{c22}$ are the roots of (7.16) and (7.17), respectively, and can be taken care of by the corresponding root loci. The other poles and zeros left to be identified are respectively the elements of the following two sets:

$$P_{F_2 + F_3 g_{c22}} = (P_{F_2 + F_3 g_{c22}} \cap P_{1 + F_1 g_{c22}})$$

$$P_{1 + F_1 g_{c22}} = (P_{F_2 + F_3 g_{c22}} \cap P_{1 + F_1 g_{c22}})$$

(7.22)
In order to express these explicitly in terms of the poles and zeros in the plant \( G_p \) and the compensator \( G_c \), repeated applications of (6.19), (6.20), (6.21), and (6.22) on \( 1 + F_1 g_{c22}, F_2 + F_3 g_{c22}, F_1, F_2, F_3/F_2 \) are necessary. The results can be written down by inspection as follows:

\[
P_{F_2 + F_3 g_{c22}} = P_{F_2} + P_{F_3} + P_{g_{c22}} - (P_{F_2} \cap P_{F_3 g_{c22}}) \tag{7.23}
\]

\[
P_{1 + F_1 g_{c22}} = P_{F_1} + P_{g_{c22}}
\]

\[
P_{F_1} = Z_{1+g_{p11}g_{c11}} + P_{g_{p22}+(G_p)_{33}g_{c11}}
- (P_{1+g_{p11}g_{c11}} \cap P_{g_{p22}+(G_p)_{33}g_{c11}})
\]

\[
Z_{F_1} = Z_{g_{p22}+(G_p)_{33}g_{c11}} + P_{1+g_{p11}g_{c11}}
- (P_{1+g_{p11}g_{c11}} \cap P_{g_{p22}+(G_p)_{33}g_{c11}})
\tag{7.24}
\]

\[
P_{F_2} = Z_{1+g_{p11}g_{c11}} + P_{g_{p33}+(G_p)_{22}g_{c11}}
- (P_{1+g_{p11}g_{c11}} \cap P_{g_{p33}+(G_p)_{22}g_{c11}})
\]
\[ Z_{F_2} = z_{g_{p33}} + (G_p)_{22}^{g_{c11}} + P_{1+g_{p11}}^{g_{c11}} \]
\[ P_{F_3} = Z_{1+g_{p11}}^{g_{c11}} + P_{G_p}^{G_p} + (\det G_p)^{g_{c11}} \]
\[ Z_{F_3} = Z_{G_p}^{G_p} + (\det G_p)^{g_{c11}} + P_{1+g_{p11}}^{g_{c11}} \]

(7.24)

where

\[ P_{g_{p22}}^{G_p} = P_{g_{p22}}^{G_p} P_{G_p}^{G_p} + P_{g_{c11}}^{G_p} \]
\[ P_{g_{p33}}^{G_p} = P_{g_{p33}}^{G_p} P_{G_p}^{G_p} + P_{g_{c11}}^{G_p} \]

(7.25)
\[ p_{1} + g_{p11}g_{c11} = p_{g_{p11}} + p_{c11} \]  \hspace{1cm} (7.25)

Also

\[ P_{F_{3}/F_{2}} = Z_{F_{2}} + P_{F_{3}} - (P_{F_{2}} \cap P_{F_{3}}) \]  \hspace{1cm} (7.26)

\[ Z_{F_{3}/F_{2}} = Z_{F_{3}} + P_{F_{2}} - (P_{F_{2}} \cap P_{F_{3}}) \]

By proper substitutions of (7.25) into (7.24), then (7.26), (7.23) and (7.22), the poles and zeros of interest can be identified.

This will be illustrated through a numerical example in the following section.

At present, however, some simplifications on the above general expressions can be made. It is observed that if,

\[ p_{g_{p22}} \subset p_{(G_{p})_{33}} \]

\[ p_{g_{p33}} \subset p_{(G_{p})_{22}} \]  \hspace{1cm} (7.27)

\[ p_{(G_{p})_{11}} \subset p_{detG_{p}} \]
where \( A \subseteq B \) denotes that set \( A \) is a subset of set \( B \), again multiplicities of the elements in each set are counted. Then, we have

\[
P_{g_p^{22}} \cap P(G_p)^{33} g_{c11} = P_{g_p^{22}}
\]

\[
P_{(G_p)^{11}} \cap P(\det G_p) g_{c11} = P_{(G_p)^{11}}
\]

\[
P_{g_p^{33}} \cap P(G_p)^{22} g_{c11} = P_{g_p^{33}}
\]

then (7.25) becomes

\[
P_{g_p^{22}} \cap (G_p)^{33} g_{c11} = P(G_p)^{33} + P g_{c11}
\]

\[
P_{g_p^{33}} \cap (G_p)^{22} g_{c11} = P(G_p)^{22} + P g_{c11}
\]

\[
P(G_p)^{11} \cap (\det G_p) g_{c11} = P(\det G_p)^{+} + P g_{c11}
\]

\[
P_{1} + g_{p11} g_{c11} = P_{g_p^{11}} + P g_{c11}
\]
and

\[ P_{1+g_{p11}g_{c11}} \bigcap P_{g_{p22} + (G_p)_{33}g_{c11}} = P_{g_{c11}} + \{ P_{g_{p11} \bigcap P(G_p)_{33}} \} \]

\[ P_{1+g_{p11}g_{c11}} \bigcap P_{g_{p33} + (G_p)_{22}g_{c11}} = P_{g_{c11}} + \{ P_{g_{p11} \bigcap P(G_p)_{22}} \} \quad (7.30) \]

\[ P_{1+g_{p11}g_{c11}} \bigcap P(G_p)_{11} + (\det G_p)g_{c11} = P_{g_{c11}} + \{ P_{g_{p11} \bigcap P(\det G_p)} \} \]

By (7.29) and (7.30), (7.24), (7.26) become

\[ P_{F_1} = Z_{1+g_{p11}g_{c11} + P(G_p)_{33}} - \{ P_{g_{p11} \bigcap P(G_p)_{33}} \} \]

\[ Z_{F_1} = Z_{g_{p22} + (G_p)_{33}g_{c11} + P_{g_{p11} \bigcap P(G_p)_{33}}} \]

\[ P_{F_2} = Z_{1+g_{p11}g_{c11} + P(G_p)_{22}} - \{ P_{g_{p11} \bigcap P(G_p)_{22}} \} \quad (7.31) \]

\[ Z_{F_2} = Z_{g_{p33} + (G_p)_{22}g_{c11} + P_{g_{p11} \bigcap P(G_p)_{22}}} \]

\[ P_{F_3} = Z_{1+g_{p11}g_{c11} + P(\det G_p)} - \{ P_{g_{p11} \bigcap P(\det G_p)} \} \]

\[ Z_{F_3} = Z_{(G_p)_{11} + (\det G_p)g_{c11} + P_{g_{p11} \bigcap P(\det G_p)}} \]
and

\[ F_{3/F_2} = \frac{P_{c33} + (G_p)_{22} c_{11} + P_{c11} - \{P_{c11} \cap P(G_p)_{22}\} + P_1(detG_p)}{\{P(G_p)_{22} - (P_{c11} \cap P(G_p)_{22}\} \cap}

(7.32)

\[ Z_{F_3/F_2} = \frac{Z(G_p)_{11} + (detG_p) c_{11} + P_{c11} - \{P_{c11} \cap P(G_p)_{22}\} + P_2(detG_p)}{\{P(G_p)_{22} - (P_{c11} \cap P(G_p)_{22}\} \cap}

Once a plant is given, \( P(G_p)_{33} \cap P(G_p)_{33} \) is a known set. Therefore, by \( P_{F_1} \) of (7.31), all the poles of \( F_1 \) are well identified. They consist of all the roots of (7.18) which are adjustable through \( g_{c11} \), and some other fixed poles given by the above set which is known.

Similarly, by inspection of (7.31) and (7.32), all the zeros of \( F_1 \), all the poles and zeros of \( F_3/F_2 \) are well identified. Some of them are fixed, the others, which are the roots of (7.19), (7.20) and (7.21) respectively, are adjustable through \( g_{c11} \). Therefore, \( g_{c11} \) can be designed according to the four root loci for (7.18), (7.19), (7.20) and (7.21), to realize a presumably good pole-zero pattern for \( F_1 \) and \( F_3/F_2 \), such that the design of
$g_{c22}$ according to (7.16) and (7.17) will not be formidable.

As anyone who is familiar with root locus design knows, there is a certain amount of trial-and-error involved. This is more so in the multivariable case because of the successive dependence of the root loci described so far. However, a little experience can always lead to good judgements that would reduce the amount of the trial-and-error. For example, in the design of $g_{c11}$ described above, if the root loci for (7.18), (7.19), (7.20) and (7.21) extend well into the right-half-plane, more sections for $g_{c11}$ is, in general, recommended. Otherwise, it is very probable that the resulting $F_1$ or $F_3/F_2$ (or both) contain poles and zeros well in the right-half-plane, thus making $g_{c22}$ difficult to design. This is, of course, a trade-off between $g_{c11}$ and $g_{c22}$. If more sections of compensation are used in $g_{c11}$, $F_1$ and $F_3/F_2$ can be made more stable, hence, less sections are required in $g_{c22}$. Conversely, if $g_{c11}$ is chosen to be too simple, more sections should be needed in $g_{c22}$. Judicious choice can be made by investigating the four root loci (7.18), (7.19), (7.20) and (7.21), and $g_{c11}$ can be designed accordingly.

The poles and zeros of $G_e$ can also be identified in a similar way. This will be clearer after the consideration of the numerical example in the following section.
7.3 DESIGN EXAMPLE

As was found in Section 7.1 and 7.2, $g_{c11}$ can be designed according to the four root loci

\[ 1 + g_{p11}g_{c11} = 0 \quad (7.18) \]

\[ 1 + \frac{(G_p)_{33}g_{c11}}{g_{p22}} = 0 \quad (7.19) \]

\[ 1 + \frac{(G_p)_{22}g_{c11}}{g_{p33}} = 0 \quad (7.20) \]

\[ 1 + \frac{\det G_p}{(G_p)_{11}} g_{c11} = 0 \quad (7.21) \]

And $g_{c22}$ can be designed by the other two

\[ 1 + F_1 g_{c22} = 0 \quad (7.16) \]

\[ 1 + \frac{F_3}{F_2} g_{c22} = 0 \quad (7.17) \]

Finally, $g_{c33}$ is designed according to

\[ 1 + G_{eq} \cdot g_{c33} = 0 \quad (7.13) \]
where \( F_1, F_2, F_3 \) and \( G_{eq} \) are given in (7.7), (7.8), (7.9), and (7.12), respectively.

The general design procedure then follows:

1. Identify all the poles and zeros of \( F_1, F_3/F_2 \) and \( G_{eq} \).

2. Prepare root loci for (7.18), (7.19), (7.20) and (7.21) with \( g_{c11} = k_1 \), a free parameter.

3. By varying the value of \( k_1 \) from \(-\infty\) to \( \infty \), observe the accompanying changes in root locations.

4. Choose the value of \( k_1 \) that corresponds to the best pole-zero pattern for \( F_1 \) and \( F_3/F_2 \) in the sense that \( g_{c22} \) can be designed most easily to give good pole-zero pattern for \( G_{eq} \).

5. If no value of \( k_1 \) gives satisfactory \( F_1 \) and \( F_3/F_2 \), use pole-zero pair as necessary in \( g_{c11} \) to pull the loci toward the left and determine the gain value for best pole-zero pattern for \( F_1 \) and \( F_3/F_2 \).

6. Construct the root loci for (7.16) and (7.17), using \( g_{c22} = k_2 \), a free parameter.
7. Adjust $k_2$ as in 3 to find best pole-zero pattern for $G_{eq}$. Put in poles and zeros as necessary, as in 5.


9. Design $g_{c33}$ to meet specifications.

Consider the 3-input, 3-output plant (3.33) repeated below:

\[
g_{p11}(s) = 0.081(s-0.205)(s+0.967+j1.379) \\
\quad (s+0.967-j1.379)/D\cdot D_{TH}
\]

\[
g_{p12}(s) = -6.12(s+0.837)(s+0.947+j1.144) \\
\quad (s+0.947-j1.144)/D\cdot D_{P}
\]

\[
g_{p13}(s) = -202(s+1.885)(s-13.037)/D\cdot D_{E}
\]

\[
g_{p21}(s) = -0.00163(s+2.881)(s+0.032+j0.313) \\
\quad (s+0.032-j0.313)/D\cdot D_{TH}
\]

\[
g_{p22}(s) = -0.153(s+0.824)(s-0.047+j0.205) \\
\quad (s-0.047-j0.205)/D\cdot D_{P}
\]

\[
g_{p23}(s) = -9.07(s+26.339)(s+0.03+j0.361) \\
\quad (s+0.03-j0.361)/D\cdot D_{E}
\]

\[
g_{p31}(s) = -0.00209(s-1.049)(s+0.268)/D\cdot D_{TH}
\]

\[
g_{p32}(s) = 0.0995(s-0.12)(s+3.485)/D\cdot D_{P}
\]

\[
g_{p33}(s) = -235.5(s+0.361+j0.076) \\
\quad (s+0.361-j0.076)/D\cdot D_{E}
\]
where

\[ D(s) = (s + 0.018 + j 0.336)(s + 0.018 - j 0.336) \]
\[ \quad (s + 1.103 + j 1.277)(s + 1.103 - j 1.277) \]
\[ D_{TH}(s) = (s + 0.99 + j 0.479)(s + 0.99 - j 0.479) \]
\[ D_{E}(s) = (s + 3.3 + j 10.49)(s + 3.3 - j 10.49) \]
\[ D_{F}(s) = s + 1 \]

By manipulation,

\[ (G_p)_{11} = g_{p22}g_{p33} - g_{p23}g_{p32} = \frac{36.85(s - 0.096)}{D \cdot D_F \cdot D_E} \]

\[ (G_p)_{22} = g_{p11}g_{p33} - g_{p13}g_{p31} = \frac{-19.08(s + 0.229)}{D \cdot D_E \cdot D_{TH}} \]

\[ (G_p)_{33} = g_{p11}g_{p22} - g_{p12}g_{p21} = \frac{-0.0224(s^2 + 1.656s + 0.694)}{D \cdot D_F \cdot D_{TH}} \]

\[ \det G = \frac{5.232}{D \cdot D_E \cdot D_F \cdot D_{TH}} \]

By inspection of (7.33) and (7.35), it is clear that (7.27) is true. Therefore, by (7.31),

\[ P_{F1} = Z_1 + g_{p11}g_{c11} + Z_{DF} \]
\[ Z_{F1} = Z_{g_{p22}} + (G_p)_{33}g_{c11} \]
\[ P_{F_2} = Z_{1+g_{p11}g_{c11}} + Z_{D_E} \]  
\[ Z_{F_2} = Z_{g_{p33} + (G_p)_{22}g_{c11}} \]  
\[ P_{F_3} = Z_{1+g_{p11}g_{c11}} + Z_{D_E} + Z_{D_F} \]  
\[ Z_{F_3} = Z_{(G_p)_{11} + (\text{det}G_p)g_{c11}} \]  

(7.37)  
(7.38)  

where  
\[ Z_D = \{(-0.018\pm j0.336), (-1.103\pm j1.277)\} \]  
\[ Z_{D_{TH}} = \{-0.99\pm j0.479\} \]  
\[ Z_{D_E} = \{-3.3\pm j10.49\} \]  
\[ Z_{D_F} = \{-1\} \]  

are the sets of the zeros of \( D(s), D_{TH}(s), D_E(s), D_F(s) \) in (7.34), respectively,  

and note that  
\[ P(G_p)_{33} = Z_D + Z_{D_F} + Z_{D_{TH}}, \text{ etc.}, \]  

and by (7.32) or by (7.26), (7.37) and (7.38)  
\[ \frac{P_{F_3}}{P_{F_2}} = \frac{Z_{g_{p33} + (G_p)_{22}g_{c11}} + Z_{D_F}}{Z_{1+g_{p11}g_{c11}}} \]  
\[ \frac{Z_{F_3}}{Z_{F_2}} = \frac{Z_{(G_p)_{11} + (\text{det}G_p)g_{c11}}}{Z_{1+g_{p11}g_{c11}}} \]  

(7.39)
Then, by (7.23)

\[ P_{F_2} + F g_{c22} = Z_1 + g_{p11} g_{c11} + Z_{D_E} + Z_D + P g_{c22} \]

(7.40)

Finally, by (7.14), (7.15) and (7.40), we have

\[ P_{G_{eq}} = Z_1 + F g_{c22} + Z_{D_E} \]

(7.41)

\[ Z_{G_{eq}} = Z_{F_2} + F g_{c22} \]

The outline of the design then follows:

1. Prepare the four root loci for (7.18), (7.19), (7.20) and (7.21), using \( g_{c11} = k_1 \). These are shown in Figure 7.1 to Figure 7.4 (note that dashed curves are the loci corresponding to negative \( k_1 \)).

By inspection of these loci, the following can be observed:

(i) \( k_1 < 0 \) is undesirable, since for negative \( k_1 \), there will be one branch in each of these four plots that extends along positive real axis to \( + \infty \) and this will tend to
produce a pole-zero pair on positive real axis for $G_{eq}$, which is certainly undesirable.

(ii) When $k_1 \approx 50$, there will be a pole-zero pair of $G_{eq}$ close to the point $(0.5, 1.5)$. The reason is that the roots corresponding to $k_1 \approx 50$ on branch $\odot$ of Figure 7.1 and Figure 7.2 are close to each other. By (7.36), these two roots will be one pole and one zero of $F_1$ respectively. Since they are close to each other, the root for $1+F_1 g_{c22}=0$ corresponding to this pole-zero pair is very difficult to push far away from this region. Thus, by (7.41), a pole of $G_{eq}$ will be around $(0.5, 1.5)$. Similarly, by Figure 7.3 and Figure 7.4, there will be a zero of $G_{eq}$ close to the same point. Thus, a pole-zero pair of $G_{eq}$ exists near the point $(0.5, 1.5)$ in the right-half-plane. This will most probably cause the corresponding root of (7.13) close to the same point, hence, undesirable.

(iii) For $k_1 > 50$, the situation is obviously worse. Therefore, the range of $k_1$ that remains to be investigated is $0 < k_1 < 50$.

(iv) It can be seen that when $k_1$ is too close to 0, the root on branch $\odot$ of each plot will all be close to the origin, hence also not desirable.
2. When $k_1 = 10$, the pole-zero locations seem to be the best. Thus, try $k_1 = 10$.

3. Using $g_{c11} = k_1 = 10$, construct the root loci for (7.16) and (7.17) with $g_{c22} = k_2$, a free parameter. The results are shown in Figure 7.5 and Figure 7.6, respectively. By inspection of these two plots, the following can be observed.

(i) $k_2 > 0$ is undesirable for the same reason as that in 1(i) above.

(ii) $0 > k_2 >> -10$ looks better than the other range, since the root on branch 1 in both Figure 7.5 and Figure 7.6 will be farther away from the $j\omega$ axis, hence, more stable $G_{eq}$ can be expected (note that roots in Figure 7.5 and Figure 7.6 give poles and zeros of $G_{eq}$ respectively, see (7.41)).

(iii) Although the root on branch 2 of Figure 7.5 will be close to the origin for $0 > k_2 >> -10$, it can still be tolerated because the same thing does not happen in Figure 7.6. Thus, the root of (7.13) corresponding to this pole-zero pair of $G_{eq}$ can still be adjusted to be not too close to the origin.

Thus, try $g_{c22} = k_2 = -1$. 
4. The roots corresponding to $g_{c22} = -1$ can be read off from Figure 7.5 and Figure 7.6. By (7.14), these constitute part of the poles and zeros of $G_{eq}$. Together with the other known poles (roots of $D_E(s) = 0$, by (7.41)), the root locus for (7.13) can be constructed as shown in Figure 7.7, with $g_{c33} = k_3$, a free parameter. By inspection, $g_{c33} = k_3 = -2$ is a good value.

Thus, we have determined all three compensator functions with only gain adjustments. The resulting system schematic is as shown in Figure 7.8. For the three step inputs $r_1 = 126.7$ ft./sec., $r_2 = -0.25$ radian and $r_3 = -0.5$ radian, the simulation results for the three outputs $y_1(t)$, $y_2(t)$ and $y_3(t)$ are given in Figure 7.9, Figure 7.10, and Figure 7.11, respectively. It is clear that the resulting system is stable.
FIGURE 7.8
Figure 7.9

Time response of output $y_1$ in Figure 7.8 with

$r_1 = 126.7 \cdot u(t)$

$r_2 = -0.25 \cdot u(t)$

$r_3 = -0.5 \cdot u(t)$

$u(t)$: Unit step
FIGURE 7.10

TIME RESPONSE OF OUTPUT $y_2$ IN FIGURE 7.8 WITH

- $r_1 = 126.7 \cdot u(t)$
- $r_2 = -0.25 \cdot u(t)$
- $r_3 = -0.5 \cdot u(t)$
- $u(t)$: UNIT STEP
FIGURE 7.11

TIME RESPONSE OF OUTPUT $y_3$ IN FIGURE 7.8 WITH

\[ r_1 = 126.7 \cdot u(t) \]
\[ r_2 = -0.25 \cdot u(t) \]
\[ r_3 = -0.5 \cdot u(t) \]

$u(t)$: UNIT STEP
PART III

APPLICATION TO STOL AIRCRAFT
8. COMPENSATION DESIGN FOR STOL C-8A AIRCRAFT

WITH STEADY-STATE DECOUPLING

8.1 GENERAL

The simulation results, Figure 7.9, Figure 7.10 and Figure 7.11, for the system in Figure 7.8, justify that the root locus technique developed in Chapter 6 and Chapter 7, can be used for designing both the stability and the transient response of a multivariable system. Stability and transient response are certainly the most important factors to consider when designing a system, however, some other factors are also important. Among them (e.g., steady-state accuracy, integrity, sensitivity, etc.), steady-state accuracy is usually the most important. In single-loop theory, the restriction on steady-state accuracy usually makes it impossible to adjust some parameters with complete freedom. In root locus terminology, root relocation zones exist, which limits some of our abilities to relocate those roots of interest.

For the multivariable case, due to the existence of mutual coupling, the problem of steady-state accuracy is more complex. This can be seen by comparing the steady-state values in Figure 7.9, Figure 7.10 and Figure 7.11 to the input commands. For example, input $r_1$ is a step of magnitude 126.7 ft./sec., while the velocity output $y_1$ is only 93.8 ft./sec. when steady-state is reached.
One way to reduce or eliminate the steady-state errors in multi-variable systems is decoupling the steady states of the system. The concept of steady-state decoupling was developed in Part I of this thesis, and the schemes obtained can be used directly to determine what types of functions should be used in the compensator. Then, results of Part II are applied to stabilize the system.

8.2 DESIGN PROCEDURE

The plant under consideration is the longitudinal mode of the NASA STOL (Short Take Off and Landing) C-8A Buffalo Aircraft which is a 3-input, 3-output plant as shown below in block diagram form.

Thus, the plant can be represented by a 3x3 matrix $G_p$. The transfer functions $g_{pij}$ are given in (3.33) and (7.33).
By using the feedback configuration as that in Figure 3.2, a steady-state decoupling scheme was obtained in Section 3.2 to be 
\[ t_{c11} = t_{c22} = t_{c33} = 1 \], i.e.,

\[ g_{c11}(s) = \frac{1}{s} g'_{c11}(s) \]
\[ g_{c22}(s) = \frac{1}{s} g'_{c22}(s) \]
\[ g_{c33}(s) = \frac{1}{s} g'_{c33}(s) \] (8.1)

where, as defined in (3.10), numerators and denominators of \( g'_{c11}(s), g'_{c22}(s) \) and \( g'_{c33}(s) \) do not contain power(s) of \( s \) as their factors.

(8.1) tells us that the introduction of one pure integrator in each of the three compensator transfer functions will cause the steady states of the system to be decoupled. Thus, one pole (at the origin) is required in each of the three unknown functions \( g_{c11}(s), g_{c22}(s) \) and \( g_{c33}(s) \). The other poles and zeros and the gain values are unknown and have to be determined for stability and transient response. The design of these unknowns can be done in exactly the same manner as that in Chapter 7, except that the existence of the extra pole (\( s = 0 \)) in each of \( g_{c11}(s), g_{c22}(s), \) and \( g_{c33}(s) \) has to be taken into account. Also note that the pole-zero expressions (7.41) (7.36) (7.39) are still valid, since the same plant is considered. Then the design procedure follows:
1) Prepare the 4 root loci for (7.18), (7.19), (7.20) and (7.21) with $g_{c11}^5 = \frac{k_1}{s}$, where $k_1$ is a free parameter.

These are shown in Figure 8.2 to Figure 8.5 (note that these loci can be drawn by adding the additional pole at the origin in Figure 7.1 to Figure 7.4).

By inspection, it is observed that

(i) $k_1 < 0$ is not desirable, for the same reason as 1 (i) of the design in Section 7.3.

(ii) For $k_1 > 0$, branch on all four plots are quite unstable, which is the effect of the introduced pole at the origin. By the same argument as that in 1 (ii) of the design in Section 7.3, it can be concluded that no range of $k_1$ is desirable.

Therefore, only one pole (at the origin) and no zero in $g_{c11}$ is most probably not enough to give satisfactory system performance, some other pole(s) and (or) zero(s) are recommended.

---

The argument s will be dropped whenever no confusion exists.
2) By spirule check, or by noting that any introduced zero (that goes with the decoupling pole at the origin to form one section of filter) on the negative real axis cannot overcome the effect of the pole at the origin, it can be seen that if only one section of filter (pole at origin, zero on negative real axis) is used, the situations will always be worse than those in Figure 7.1 to Figure 7.4. Since that design was only marginally successful and the situations now are worse, several sections of filter are recommended.

3) By spirule, or by inspection (if experienced enough) it can be seen that two sections of filters are enough to pull branch \( \Theta \) in each of these four plots into the left-half-plane. Then no poles and zeros of \( F_1 \) and \( F_3/F_2 \) are in the right-half-plane. (Note that by (7.36), roots in Figure 8.3 are the zeros of \( F_1 \). Those in Figure 8.2, together with the known pole at \( s=-1 \), which is the root of \( D_F(s)=0 \), give all the poles of \( F_1 \). Similarly, by (7.39), roots in Figure 8.4, Figure 8.5 and the known pole at \( s=-1 \) give all the poles and zeros of \( F_3/F_2 \), and the design of \( g_{c22} \) and \( g_{c33} \) can be continued in the same manner as that in Section 7.3. However, by noting that each new root locus that has to be constructed out of these four has also an additional pole at the origin (comes from \( g_{c22} \) and \( g_{c33} \), see (8.1)). This will tend to destabilize the results and make the design of \( g_{c22} \) and \( g_{c33} \) more difficult. Thus, try a three section lead compensator.
4) By spirule (or by inspection), three zeros close to the origin and two poles far away from the origin can stabilize Figure 8.2 to Figure 8.5 to a great extent. For one specific choice,

$$g_{c11}(s) = \frac{k_1(s+1)(s+0.5)^2}{s(s+4)(s+10)}$$

where $k_1$ is a free parameter, the results are shown in Figure 8.6 to Figure 8.9. Note that the loci for $k_1 < 0$ are not shown, since $k_1 < 0$ is not desirable for the same reason as in 1 (i).

5) By inspection of these root loci and by considering the effect of $k_1$ on the pole-zero pattern of $F_1$ and $F_3/F_2$ (see (7.36) and (7.39)), $k_1 = 1000$ is chosen for the same reason as in the previous design of Section 7.3. For this value of $k_1$, the root loci for $F_1$ and $F_3/F_2$ can be drawn. Then, $g_{c22}$ can be designed according to these two root loci and their relationship with the poles and zeros of $G_{eq}$ (see (7.41)). By trial-and-error (or by inspection),

$$g_{c22} = \frac{k_2(s+0.5)}{s}$$

(k_2 is a free parameter) was found to be good. For this $g_{c22}$, the root loci for (7.16) and (7.17) are shown in Figure 8.10 and Figure 8.11. Loci for $k_2 > 0$ are not shown since for $k_2 > 0$, one branch in each plot extends along positive real axis to $+\infty$, hence undesirable.

6) By inspection of Figure 8.10 and Figure 8.11 and by (7.41), $k_2 = -400$ is a good value.
7) Using $k_2 = -400$, the root locus for (7.13) with $g_{c33}$

$$g_{c33} = \frac{k_3(s+0.3)(s+0.4)}{s(s+10)}$$

can be drawn (see (7.41)), and $g_{c33}$ can be designed accordingly. The result is shown in Figure 8.12, from which the best value for $k_3$ can be seen by inspection to be $k_3 \approx -20$.

Thus, a design with

$$g_{c11}(s) = \frac{1000(s+1)(s+0.5)^2}{s(s+4)(s+10)}$$

$$g_{c22}(s) = \frac{-400(s+0.5)}{s}$$

$$g_{c33}(s) = \frac{-20(s+0.3)(s+0.4)}{s(s+10)}$$

is completed.

The schematic diagram for the designed system is shown in Figure 8.13.

For step inputs of magnitudes 126.7 ft./sec., -0.25 radian, -0.5 radian in $r_1$, $r_2$, and $r_3$, the simulation results for $y_1$, $y_2$, and $y_3$ are shown in Figure 8.14, Figure 8.15 and Figure 8.16, respectively. It is seen that both stability and steady-state decoupling have been achieved. Furthermore, due to the introduced
pole at the origin in each of \( g_{c11} \), \( g_{c22} \), and \( g_{c33} \), the steady-state error in each of the outputs is zero, which is the most desirable situation. Thus, by decoupling the steady states, steady-state accuracy has also been achieved as a byproduct.
FIGURE 8.14
TIME RESPONSE OF OUTPUT $y_1$ IN FIGURE 8.13 WITH

$r_1 = 126.7 \cdot u(t)$
$r_2 = -0.25 \cdot u(t)$
$r_3 = -0.5 \cdot u(t)$

$u(t)$: UNIT STEP
FIGURE 8.15

TIME RESPONSE OF OUTPUT $y_2$ IN FIGURE 8.13 WITH

$r_1 = 126.7 \cdot u(t)$

$r_2 = -0.25 \cdot u(t)$

$r_3 = -0.5 \cdot u(t)$

$u(t)$: UNIT STEP
Figure 8.16

Time response of output $y_3$ in Figure 8.13 with

$r_1 = 126.7 \cdot u(t)$
$r_2 = -0.25 \cdot u(t)$
$r_3 = -0.5 \cdot u(t)$

$u(t)$: Unit Step
9. CONCLUSIONS

9.1 COMPARISON OF THE RESULT IN PART I TO THAT OF STATE VARIABLE FEEDBACK APPROACH

In part I, a constructive criterion for decoupling the steady states of a linear time-invariant multivariable system was developed. Transfer function matrix representation, unity feedbacks and cascade compensation were used as shown in Figure 2.1.

Another approach using linear state variable feedbacks was investigated by Wolovich (2). The result in terms of transfer function matrix representation is given in Chapter 1, and repeated here as follows:

A system characterized by an \((nxm)\) proper rational transfer function matrix, \(G_p(s)\), having no poles at the origin \((s=0)\) can be steady-state decoupled (via linear state variable feedback or perhaps some other less ambitious scheme) if and only if

\[
\rho(G_p(0)) = n
\]  

(9.1)

where \(\rho(G_p(0))\) denotes the rank of the matrix \(G_p(s)\) when \(s\) approaches zero.
Several advantages of the result given in Part I over that described above are in order:

1. Cascade compensators and output feedbacks are much easier to implement than measuring the states.

2. The rank condition (9.1) is not necessary, e.g., given the 2x2 plant (n=m=2):

\[
G_p(s) = \begin{bmatrix}
\frac{s}{s+1} & 1 \\
\frac{s}{s+3} & \frac{1}{s+4}
\end{bmatrix}
\]  

(9.2)

By (3.19), (3.10) and (3.18), steady states can be decoupled by introducing two and one pure integrators in \( g_{c11} \) and \( g_{c22} \) in Figure 3.1, respectively. However, by (9.1), this cannot be done through linear state variable feedbacks.

3. Poles at the origin in the given plant are allowed. Actually, such poles are very helpful for steady-state decoupling as was shown in Figure 6.6 for the plant (6.24).

9.2 POSSIBLE GENERALIZATION OF THE SYSTEM CONFIGURATION

The discussion of this thesis has been restricted to the configuration given in Figure 2.1. For the general feedback configuration in which the unity feedbacks are replaced by a transfer function
matrix $G_f(s)$ (nxn), the simple relation (2.5) is no longer valid. However, if $G_f(s)$ is diagonal and nonsingular (in the field of rational functions of $s$), which are of practical importance, similar result to (2.5) can still be obtained as follows:

$$H = (I+G_f G_c G_f)^{-1} G_p G_c$$

$$= (I+G_p G_c G_f)^{-1} G_p G_c G_f G_f^{-1}$$

$$= (I-(I+G_p G_c G_f)^{-1}) G_f^{-1}$$

Since $G_f$ is diagonal, $G_f^{-1}$ is also diagonal. Therefore, simple expressions for the off-diagonal elements of $H(s)$ can still be obtained easily. Then, with a slight modification, the results in Part I can still be applied.

9.3 STABILITY AND DESIGN

In Parts II and III, stability of a linear time-invariant multi-variable system was considered. A design technique, using an extended root locus method was also developed and applied successfully to 2x2 and 3x3 cases. The major achievement is the revelation of the simple connection between single-loop and multivariable cases. Such connections made the application of
single-loop design methods to multivariable systems possible, as was seen through the design examples in Chapter 6, Chapter 7 and Chapter 8. Some other advantages of the design techniques are:

1. Consideration of integrity problems is possible in the process of the design by forming pertinent root loci. This means that the system can be designed such that possible failure of any loop (or combination of loops) do not cause the system to be unstable (e.g., see (10)).

2. The problem of input output permutation, like "which output should be fed back to a particular input?" can be solved to some extent by inspection.

3. More insight to the problem is achieved through the root locus approach.

4. The problem of meeting system specifications can be done in the same manner as in any single-loop design method.

A few disadvantages, however, do exist. For example, the successive dependence of each root locus on the previous ones causes more design difficulty as the number of inputs and outputs of the plant increases. Also, like the single-loop frequency domain methods, trial-and-error is inherent in this technique.
However, with the help of computers, these problems can be minimized and the design can be done within a reasonable amount of time. Besides, with some experience in handling the root locus, the effort can be further reduced.
REFERENCES


APPENDIX A: FORMULAE (4.1) AND (4.2)

The proof of (4.1) can be done by applying Lemma A.1 in Appendix A of (15) directly to the two nxn square matrices I and G. For (4.2) however, direct application of this lemma has a little difficulty.

For better analyticity, an independent proof using mathematical induction has been developed. An outline of the proof is given below:

1. (4.1), (4.2) are satisfied for n=2 and n=3 by direct expansion.

2. Suppose (4.1) is true for n=N, (4.2) is true for n=N+1, then (4.1) is true for n=N+1.

3. Suppose (4.1) is true for both n=N-1, and n=N, then (4.2) is true for n=N+1.

Thus, starting from N=2, it can be induced that (4.1) and (4.2) are true for any positive integer.

Details of the proof in 2 and 3 above are omitted.
APPENDIX B: CHARACTERISTIC EQUATION

It was pointed out in Section 6.1 that the stability of a multi-variable system as shown in Figure 2.1 is determined by the zeros of both (6.2) and (6.3). It will be shown in this appendix that if pole-zero cancellations are done deliberately, and if the compensator poles are chosen carefully, zeros of (6.2) alone can determine the stability. The constraints under which this is true are very practical and can be fulfilled in a systematic manner. Thus, the mathematical possibilities in which zeros of (6.3) must be considered are bypassed.

By the definition of characteristic polynomials given in Section 6.1, \( \Delta_c(s) \) and \( \Delta_p(s) \) can be expressed analytically as:

\[
\Delta_c(s) = \text{LCD} \left\{ G_c \left( \begin{array}{c} i_1, \ldots, i_z \\ j_1, \ldots, j_z \end{array} \right) \right\}_{k=1, \ldots, \min(n,m), \ 1 \leq i_1 < \cdots < i_z \leq m, \ 1 \leq j_1 < \cdots < j_z \leq n} \tag{B.1}
\]

\[
\Delta_p(s) = \text{LCD} \left\{ G_p \left( \begin{array}{c} i_1, \ldots, i_z \\ j_1, \ldots, j_z \end{array} \right) \right\}_{k=1, \ldots, \min(n,m), \ 1 \leq i_1 < \cdots < i_z \leq n, \ 1 \leq j_1 < \cdots < j_z \leq m} \tag{B.2}
\]

Where, as in Chapter 4, \( G_c \left( \begin{array}{c} i_1, \ldots, i_z \\ j_1, \ldots, j_z \end{array} \right) \) and \( G_p \left( \begin{array}{c} i_1, \ldots, i_z \\ j_1, \ldots, j_z \end{array} \right) \) denote the
\&th-order minors of \( G_c \) and \( G_p \) formed from rows \( i_1, \ldots, i_L \) and columns \( j_1, \ldots, j_L \) of each matrix, respectively.

\( \text{LCD} \{ \ldots \} \) denotes the Least Common Denominator of all the rational functions described in the brackets and all the minors are assumed to be in irreducible rational forms.

Let \( N_p \left( \begin{array}{c} i_1, \ldots, i_L \\ j_1, \ldots, j_L \end{array} \right) \), \( N_c \left( \begin{array}{c} i_1, \ldots, i_L \\ j_1, \ldots, j_L \end{array} \right) \), \( D_p \left( \begin{array}{c} i_1, \ldots, i_L \\ j_1, \ldots, j_L \end{array} \right) \) and \( D_c \left( \begin{array}{c} i_1, \ldots, i_L \\ j_1, \ldots, j_L \end{array} \right) \) denote the numerators and denominators of the irreducible minors, \( G_p \left( \begin{array}{c} i_1, \ldots, i_L \\ j_1, \ldots, j_L \end{array} \right) \) and \( G_c \left( \begin{array}{c} i_1, \ldots, i_L \\ j_1, \ldots, j_L \end{array} \right) \) respectively. Then,

\[
\Delta_c(s) = \text{LCM}\{D_c \left( \begin{array}{c} i_1, \ldots, i_L \\ j_1, \ldots, j_L \end{array} \right) \mid L = 1, \ldots, \min(n,m) \ 1 \leq i_1 < \ldots < i_L \leq m, \\
1 \leq j_1 < \ldots < j_L \leq n \} 
\]

(B.3)

\[
\Delta_p(s) = \text{LCM}\{D_p \left( \begin{array}{c} i_1, \ldots, i_L \\ j_1, \ldots, j_L \end{array} \right) \mid L = 1, \ldots, \min(n,m) \ 1 \leq i_1 < \ldots < i_L \leq n, \\
1 \leq j_1 < \ldots < j_L \leq m \} 
\]

(B.4)

Where \( \text{LCM} \{ \ldots \} \) denotes the Least Common Multiplier of all the polynomials described in the brackets.
These analytical expressions for $\Delta_c(s)$ and $\Delta_p(c)$ will be useful later in this appendix.

For single-loop systems,

$$G_p(s) = \begin{bmatrix} g_p(s) \end{bmatrix}_{1x1}$$
$$G_c(s) = \begin{bmatrix} g_c(s) \end{bmatrix}_{1x1}$$

Let $N_p(s)$, $N_c(s)$, $D_p(s)$, $D_c(s)$ denote the numerators and the denominators of $g_p(s)$ and $g_c(s)$ (both in irreducible rational forms) respectively, we have for (6.2)

$$\frac{N_1(s)}{D_1(s)} = 1 + \frac{N_p(s)}{D_p(s)} \cdot \frac{N_c(s)}{D_c(s)}$$

(Note that $N_1(s)/D_1(s)$ is in irreducible form.)

Also, by definition of the characteristic polynomial,

$$\Delta_p(s) = D_p(s)$$
$$\Delta_c(s) = D_c(s)$$
Hence, (6.3) becomes

\[ N(s) = \frac{D_c(s) D_p(s)}{D_1(s)} \]  \hspace{1cm} (B.6)

The right hand side of (B.5) can be written as

\[ \frac{D_p(s) D_c(s) + N_p(s) N_c(s)}{D_p(s) D_c(s)} \]

Let

\[ D_o(s) \triangleq D_p(s) D_c(s) \]

\[ N_o(s) \triangleq D_p(s) D_c(s) + N_p(s) N_c(s) \]

which are the denominator and numerator of (6.2) before cancellation (if any). Also, let \( C(s) \) denote the greatest common factor between \( D_o(s) \) and \( N_o(s) \) (\( C(s) = 1 \), if no common factor exists), then

\[ D_o(s) = C(s) \cdot D_1(s) \]  \hspace{1cm} (B.7)

\[ N_o(s) = C(s) \cdot N_1(s) \]

By (B.6) and (B.7), we have:

\[ N_1(s) = \frac{N_o(s)}{C(s)} \]  \hspace{1cm} (B.8)

\[ \hat{N}(s) = \frac{D_o(s)}{D_1(s)} = C(s) \]
(B.8) tells us that the zeros of \( N_1(s) \), together with those of \( \hat{N}(s) \) are the zeros of \( N_0(s) \) alone.

Therefore, the following conclusion, which is well-known in single-loop theory, can be made.

The stability for systems shown in Figure B.1 is determined by the roots of

\[
1 + g_p(s)g_c(s) = 0
\]  \hspace{1cm} (B.9)

if and only if no pole-zero cancellation is allowed in (B.9), even if a common factor exists.

![Figure B.1](image)

As an example, consider the system shown in Figure B.1 with

\[
g_p(s) = \frac{s-1}{s+1}
\]

\[
g_c(s) = \frac{1}{s-1}
\]
By (6.1) and (6.3),

\[
\frac{N_1(s)}{D_1(s)} = \frac{s+2}{s+1}
\]

\[
\hat{N}(s) = \frac{(s-1)(s+1)}{s+1} = s-1
\]

Since the zero of \(\hat{N}(s)\) is in the right-half plane, the system is unstable.

By (B.9),

\[
1 + \frac{1}{s-1} \cdot \frac{s-1}{s+1} = 0
\]

If the common factor \((s-1)\) is cancelled, we have \(\frac{s+2}{s+1}=0\). Therefore, only the \(s=-2\) pole is retained and erroneous conclusion that the system is stable is reached.

However, if \((s-1)\) is not cancelled, we have \(\frac{(s-1)(s+2)}{(s-1)(s+1)}=0\). Hence, both the zero of \(N_1(s)\) and \(\hat{N}(s)\) are retained.

Therefore, for single-loop systems, if no cancellation is allowed, (6.2) alone gives all the zeros of \(N_1(s)\) and \(\hat{N}(s)\), hence determines the stability.
The following question then arises naturally: "Can we use (6.2) alone in determining the stability of a multivariable system by the same requirement that no cancellation is allowed?" The answer is "no" as was shown by Chen (7) through the following 2x2 example:

Consider

\[
G_p(s) = \begin{bmatrix}
\frac{-s^2+s+1}{(s+1)(s-1)} & 
\frac{1}{s-1} \\
\frac{1}{(s+1)(s-1)} & 
\frac{1}{s-1}
\end{bmatrix}
\]

\(G_c(s) = I\)  \hspace{1cm} (B.11)

where I denotes the 2x2 unity matrix.

Then,

\[
\det(I + G_p G_c) = \frac{s^2}{(s+1)(s-1)^2} - \frac{1}{(s+1)(s-1)^2}
\]

\[
= \frac{(s+1)(s-1)}{(s-1)^2(s+1)}
\]

There exists a right-half-plane zero in (B.12) at s=1.

However, by (6.2) and (6.3)

\[
N_1(s) = 1 \hspace{1cm} \text{ (B.13)}
\]

\[
\hat{N}(s) = s+1
\]
(Note: $A_c(s) = 1$, $A_p(s) = (s+1)(s-1)$ and $D_1(s) = s-1$) the system is clearly stable. Thus, leaving all the common factors uncancelled does not work.

However, if the general formula (4.5) is used and let all the minors of $G_p$ be in irreducible forms, we have

$$\det(I+G_p G_c) = \det(I+G_p)$$

$$= 1 + g_{p11} + g_{p22} + \det G_p$$

$$= 1 + \frac{-s^2 + s + 1}{(s+1)(s-1)} + \frac{1}{s-1} + \frac{-s}{(s+1)(s-1)}$$

$$= \frac{s+1}{(s+1)(s-1)}$$

(\text{B.14})

It is seen that if the common factor $(s+1)$ is not cancelled, the zero of (\text{B.14}) is exactly the same as those given by $N_1(s)$ and $N(s)$ as shown in (\text{B.13}).

Note that the misleading factor $(s-1)$ in (\text{B.12}) does not appear in (\text{B.14}). The reason is that we started out with irreducible minors and in forming the 2nd order irreducible minor $\det G_p$, the $(s-1)$ factor was cancelled.
Therefore, by selecting cancellations, stability of a multivariable system can still be determined by (6.2) alone. All common factors in the minors must be cancelled to get the irreducible forms, while the others are not allowed. This strict rule together with the application of formula (4.5) make the whole procedure completely systematic, no confusion will arise.

As another example, consider

\[ G_p(s) = \begin{bmatrix} \frac{-s}{s-1} & \frac{s}{s+1} \\ 1 & \frac{-2}{s+1} \end{bmatrix} \]

\[ G_c(s) = I \]

which is also an example in (7).

By direct manipulation, it can be found that \( \det (I+G_pG_c) = -1 \). However, by selecting cancellations as described above, we have

\[
\det(I+G_pG_c) = 1 + g_{p11}g_{p22} + \det G_p \\
= 1 + \frac{-s}{s-1} + \frac{-2}{s+1} + \frac{-s^2 + 3s}{(s+1)(s-1)} \\
= -\frac{(s+1)(s-1)}{(s+1)(s-1)}
\]

(B.15)
Once again, if the common factor \((s+1)(s-1)\) is not cancelled, the two zeros of \((B.15)\) are exactly what one would obtain as zeros for \(N_1(s)\) and \(\hat{N}(s)\) by \((6.2)\) and \((6.3)\). Thus, these two uncancelled zeros of \(\det(I+G_pG_c)\) determine the stability of the system. Since one of them is in the right-half-plane, the feedback system is unstable.

These two examples suggest that \((6.2)\) alone can be used as characteristic equation for a multivariable system if we select cancellations as described above. But, is this true in general? To answer this question, consider the general expression \((4.5)\) for feedback system as shown in Figure 2.1. For simplicity, consider 2x2 \(G_p\) and \(G_c\) first.

By \((4.5)\),

\[
\det(I+G_pG_c) = 1+G_p(1)G_c(1)+G_p(1)G_c(2)+G_p(2)G_c(1)+G_p(2)G_c(2)
\]

\[
+G_p(1,2)G_c(1,2)
\]

\[
= 1+\frac{N_p(1)}{D_p(1)}\cdot\frac{N_c(1)}{D_c(1)}+\frac{N_p(1)}{D_p(1)}\cdot\frac{N_c(2)}{D_c(2)}+\frac{N_p(2)}{D_p(2)}\cdot\frac{N_c(1)}{D_c(1)}
\]

\[
+\frac{N_p(2)}{D_p(2)}\cdot\frac{N_c(2)}{D_c(2)}+\frac{N_p(1,2)}{D_p(1,2)}\cdot\frac{N_c(1,2)}{D_c(1,2)}
\]

\[(B.16)\]

Where the notations for numerator and denominator of each irreducible minor used in \((B.3)\) and \((B.4)\) are employed, e.g., \(N_p(1), D_p(1)\)
denote the numerator and denominator of the first order minor 
\( G_p(1) = g_{p11} \), etc. Note that no common factor exists between 
\( N_p(1) \) and \( D_p(1) \), \( N_c(1) \) and \( D_c(1) \), etc.

Let \( N_o(s) \), \( D_o(s) \) be the numerator and denominator of (B.16), after
collecting all the terms at the right-hand-side under the 
restriction that no pole-zero cancellation is allowed (even if a 
common factor exists). Then,

\[
D_o(s) = \text{LCM}(D_p(1)D_c(1), D_p(1)D_c(2), D_p(2)D_c(1),
D_p(2)D_c(2), D_p(1,2)D_c(1,2))
\]

(B.17)

Also, by (B.3) and (B.4),

\[
\Delta_c(s) = \text{LCM}(D_c(1), D_c(1), D_c(2), D_c(2), D_c(1,2))
\]

(B.18)

\[
\Delta_p(s) = \text{LCM}(D_p(1), D_p(1), D_p(2), D_p(2), D_p(1,2))
\]

(B.19)

If no common factor exists between \( \Delta_c(s) \) and \( \Delta_p(s) \), which can be 
realized by not using any plant pole\(^6\) as a pole in the compensator,

\(^6\)G(s) has a pole at \( s = \lambda \), whenever at least one element of \( G(\lambda) \) is \( \infty \).
we have

\[ D_0(s) = \Delta_c(s) \cdot \Delta_p(s) \]  \hspace{1cm} (B.20)

This can be proved by the following arguments:

1. If no common factor exists between \( \Delta_c(s) \) and \( \Delta_p(s) \), then no common factor can exist between any element in (B.18) and any of those in (B.19). Otherwise, a common factor will exist between \( \Delta_c(s) \) and \( \Delta_p(s) \).

2. Any factor of \( D_0(s) \) must be a factor of either \( \Delta_c(s) \) or \( \Delta_p(s) \) and with the same multiplicity. The reason is that any factor of \( D_0(s) \) must exist in at least one of the five elements in (B.17). And the multiplicity of this factor must be the same as that of the element that has the maximum multiplicity of the same factor. By 1, this factor can be either in the \( D_p \)'s or in the \( D_c \)'s of (B.17), but not both. Therefore, by (B.19) or (B.18), the factor with the same multiplicity must appear in either \( \Delta_p(s) \) or \( \Delta_c(s) \).

3. Any factors of \( \Delta_c(s) \) \( \Delta_p(s) \) must also be a factor of \( D_0(s) \) with the same multiplicity. This can be seen by similar arguments as that in 2 above, but starting from (B.18) and (B.19) instead of (B.17).
Now, similar to what was done in the 1x1 case, let $C(s)$ be the common factor between $N_o(s)$ and $D_o(s)$. Then,

$$D_o(s) = C(s) \cdot D_1(s)$$

$$N_o(s) = C(s) \cdot N_1(s)$$

By (6.3) and (B.20),

$$\hat{N}(s) = \frac{\Delta_c(s) \cdot \Delta_p(s)}{D_1(s)}$$

$$= \frac{D_o(s)}{D_1(s)}$$

$$= C(s)$$

Therefore, the zeros of $N_o(s)$ are exactly those of $N_1(s)$ together with those of $\hat{N}(s)$. Thus, stability of a 2x2 multivariable system can be considered by (6.2) alone; If (1) irreducible minors are used in (4.5), (2) all the other cancellations are not allowed, (3) no plant pole is used in the compensator, and (4) $g_{cij}(s) \neq 0$ for all $i=1, \ldots , m$ $j=1, \ldots , n$.

The condition $g_{cij}(s) \neq 0$ in (4) was added because any zero $g_{cij}$ will cause one corresponding element of (B.17) to be missing which will impair the equality of (B.20) if the associated $D_p$ term happens
to be the only one among all the \( D_p \)'s that contain the highest multiplicity of any factor. This is shown by the following example:

Let

\[
G_p(s) = \begin{bmatrix}
\frac{1}{s+1} & \frac{1}{s-5} \\
\frac{s-5}{(s+3)(s+1)} & \frac{1}{s+2}
\end{bmatrix}
\]

\( (B.21) \)

Then

\[
g_{p11}(s) = \frac{1}{s+1}
\]

\[
g_{p12}(s) = \frac{1}{s-5}
\]

\[
g_{p21}(s) = \frac{s-5}{(s+3)(s+1)}
\]

\[
g_{p22}(s) = \frac{1}{s+2}
\]

\[
\text{det} G_p(s) = \frac{1}{(s+1)(s+2)(s+3)}
\]
By (B.19), we have

\[ \Delta_p(s) = \text{LCM}\{(s+1), (s-5), (s+3)(s+1), (s+2), (s+1)(s+2)(s+3)\} \]

\[ = (s+1)(s+2)(s+3)(s-5) \]

Note that the highest multiplicity for the factor \((s-5)\) comes from the second element in the bracket, which is \(D_p(\frac{1}{2})\), denominator of \(g_{p12}(s)\) (or \(G_p(\frac{1}{2})\)). Now if \(g_{c21}(s) = 0\) is used (e.g., diagonal \(G_c\)), the third term in (B.16) is zero, hence the element \(D_p(\frac{1}{2}) D_c(\frac{2}{1})\) will not appear in (B.17). Therefore, the factor \((s-5)\) will not appear in \(D_o(s)\). This makes (B.20) to be not true. The \((s-5)\) factor will show up in \(\hat{N}(s)\) of (6.3) which makes the system always unstable as long as \(g_{c21}(s) \neq 0\). Therefore, it is impossible to stabilize the system with diagonal compensator matrix \(G_c(s)\).

Therefore, whenever diagonal \(G_c(s)\) is employed, care must be taken to see if situation like this happens. This can be checked very easily by forming all the pertinent irreducible minors of \(G_p(s)\) and then check to see if the multiplicity of each plant pole is retained in these minors. If yes, zero element is allowed in \(G_c(s)\) and (6.2) alone can determine the stability. If not, (6.3) must also be considered as was shown in the above example. Incidentally, this is a trivial case in single-loop systems. Since there is only one element in \(G_c(s)_{1x1}\), which obviously cannot be identically zero.
The above analysis for 2x2 $G_p(s)$ can be carried over in exactly the same manner to the nxm case.

Therefore, for systems as shown in Figure 2.1, if

1. The general formula (4.5) for $\det(I+G_pG_c)$ is used.

2. All the common factors in the minors of $G_p$ are cancelled to get irreducible forms.

3. All other cancellations are not allowed.

4. Plant poles are not used as poles of $G_c$.

5. Appearance of zero elements in $G_c$ is carefully checked.

then the zeros of $\det(I+G_pG_c)$ alone determine the stability of the system. If all of them are in the open left-half-plane, the system is stable, otherwise it is not.

Therefore, as in the single-loop theory,

$$\det(I+G_pG_c)=0$$

(B.22)

is referred to as the characteristic equation for multivariable systems and stability can be considered through this equation.
APPENDIX C: ROOT LOCUS GAIN

When root locus approach is used, it is convenient to express transfer functions in terms of root locus forms (14) as shown below:

\[ G(s) = \frac{k_j(s+z_j)}{s^{N_k}(s+p_k)} \]  \hspace{1cm} (C.1)

where \( j, k \) are positive integers and \( s=-z_j, s=-p_k \) are the zeros and poles of \( G(s) \) respectively.

The gain constant \( k \) in (C.1) is referred to as the "root locus gain" of the function \( G(s) \) (see 14).

In step 6 of Section 6.4, some algebraic manipulation was performed to find the root locus gain \( k_{eq} \) of the function \( G_{eq}(s) \). In most cases, this step can be bypassed as shown below:

Let \( k_1, k_2 \) be the root locus gains of the two transfer functions \( G_1(s) \) and \( G_2(s) \) respectively. Let \( (G_s(s), G_r(s) \) be the sum and the ratio \( (G_2/G_1) \) of \( G_1(s) \) and \( G_2(s) \), in root locus forms and \( k_s, k_r \) denote the corresponding root locus gains. Also, let the order of the numerator and denominator of any rational function \( H(s) \) be denoted by \( ON(H) \) and \( OD(H) \) respectively.
Then, we have

**THEOREM C.1**

The root locus gain \( k_s \) of \( G(s) = G_1(s) + G_2(s) \) is

(i) \( k_s = k_1 \) if and only if \( \text{ON}(G_1) + \text{OD}(G_2) > \text{ON}(G_2) + \text{OD}(G_1) \)

(ii) \( k_s = k_2 \) if and only if \( \text{ON}(G_1) + \text{OD}(G_2) < \text{ON}(G_2) + \text{OD}(G_1) \)

(iii) \( k_s = k_1 + k_2 \) if and only if \( \text{ON}(G_1) + \text{OD}(G_2) = \text{ON}(G_2) + \text{OD}(G_1) \) and \( k_1 \neq -k_2 \)

**THEOREM C.2**

The root locus gain \( k_r \) of \( G(s) = \frac{G_2(s)}{G_1(s)} \) is \( k_r = \frac{k_2}{k_1} \).

Proofs for both Theorem C.1 and C.2 are straightforward, hence, omitted.

Both of these theorems are very simple in nature, however, they are very useful tools in evaluating the root locus gains, as illustrated below for the determination of \( k_{\text{eq}} \) in Section 6.4 (see (6.41)).
By (6.8),

\[ G_2 = g_{p22} + (\text{det} G_p) g_{c11} \]

\[ G_1 = 1 + g_{p11} g_{c11} \]

and \[ G_{eq} = \frac{G_2}{G_1} \]

For \( G_2 \),

\[ \text{ON}(g_{p22}) + \text{OD}(\text{det} G_p \cdot g_{c11}) = 0 + 4 = 4 \]

\[ \text{ON}(\text{det} G_p \cdot g_{c11}) + \text{OD}(g_{p22}) = 2 + 1 = 3 \]

Since \( 4 > 3 \), we have by Theorem C.1 (i)

\[ k_{G_2} = k_{g_{p22}} = -2 \]

where \( k_{G_2}, k_{g_{p22}} \) denote the root locus gains of \( G_2(s) \) and \( g_{p22}(s) \) respectively.

Similarly, for \( G_1 \)

\[ \text{ON}(1) + \text{OD}(g_{p11} g_{c11}) = 0 + 2 = 2 \]

\[ \text{ON}(g_{p11} g_{c11}) + \text{OD}(1) = 1 + 0 = 1 \]
Again, since $2 > 1$, by Theorem C.1 (i), we have

\[ k_{G_1} = 1 \]

Then, by Theorem C.2,

\[ k_{eq} = \frac{k_{G_2}}{k_{G_1}} = -2 \]

which agrees with what was obtained in Section 6.4 through algebraic manipulation.

Remark: Whenever $k_1 + k_2 = 0$ in case (iii) of Theorem C.1, no conclusion can be obtained through the theorem, since no obvious analytic expression exists for the coefficient of the second higher order terms. However, direct algebraic manipulation can always be used in such cases.