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A MARKOVIAN MODEL FOR ASSESSMENT OF PERSONNEL HIRING PLANS

by Lawrence G. Katz

Goddard Space Flight Center
Greenbelt, Md. 20771

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16. Abstract

As a result of the current economic environment, many organizations are having to operate with fewer resources. In the manpower area, these constraints have forced organizations to operate within well-defined hiring plans. Exceeding personnel ceilings is in most cases an intolerable situation. A mathematical model, based on the theory of Markov processes, is presented which can be used to assess the chances of success of personnel hiring plans. The model considers a plan to be successful if the final population size, at the end of the planning period, lies within a range specified by management. Although this model was developed to assess personnel hiring plans at the Goddard Space Flight Center, it is directly applicable wherever personnel hiring plans are used.

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Lawrence G. Katz
Goddard Space Flight Center

INTRODUCTION

The environment in which the National Aeronautics and Space Administration is operating today is markedly different from the one in which it has operated in the past. A particular change is in the area of personnel employment ceilings. After more than a decade of operation, NASA underwent its first major work force adjustment in August 1971. A second major adjustment was completed during April 1973. NASA today finds itself in the position of having to operate with continually fewer people.

To complicate matters, the onset of the low-cost era brings with it major changes in the manner in which NASA can accomplish its goals. During this era, NASA will rely more heavily upon contractor personnel, in keeping with the President's desire to reduce the size of the Federal bureaucracy. A second major result of entry of NASA into the low-cost era concerns the use of standard subsystems for spacecraft in lieu of specially-designed units. NASA management has determined that operation in this era will make the use of standard subsystems a way of life. Consequently, the optimal skill mix for the low-cost era will undoubtedly be different from that for the present era. Thus, the skill mix problem will begin to take on added importance.

At Goddard Space Flight Center, directors of administrative and technical Directorates are submitting personnel plans on a fiscal year basis. In its broadest sense, a personnel plan is a scheduling of all personnel actions (hires, promotions, within-grade increases, quality increases, cash awards, noncash awards, training, and reassignments). GSFC management must combine the Directorates' personnel plans to obtain a personnel plan for this Center.

At present, two major personnel constraints are affecting NASA–population ceiling and average grade. Both of these are considered when a personnel plan is devised. The model presented in this report considers only the personnel ceiling constraint. However, the model possibly could be extended to incorporate the average grade constraint.

How can GSFC management rationally assess a particular personnel plan's chances of success as the planning period progresses? The model considers a plan to be successful if the final population size, at the end of the planning period, lies within a range specified by management.

At the Directorate level, this model can be used to assess the probability of success of plans concerned with:

- Skill codes
- Combinations of skill codes

At the GSFC level, this model can be used to estimate the probability of success of plans concerned with:

- Skill codes within combinations of Directorates
- Combinations of skill codes within combinations of Directorates

This model can also be used to keep track of plans of various durations, thus allowing management to modify a plan during the active planning period. In the midst of the planning period, it is possible to determine the effect of plan changes on the probability of success.

In an iterative manner this method can be used to devise plans with specified probabilities of success. Prior to the beginning of the planning period, plans can be assessed, modified, and reassessed until a suitable plan is devised.

The personnel plan assessment model described in the following section can be used to track personnel plans probabilistically. Briefly, the model is based on the theory of Markov processes. Probabilities of the population's size changing from one value to another, in one month's time, are estimated. These probabilities are used to determine the probabilities associated with future population sizes.

THE MODEL

This model utilizes the theory of Markov chains. In terms of a system where the various states of the system are the GSFC population sizes at specified points in time, a Markov chain can be defined in the following manner. A Markov chain is a stochastic process where the probability that the GSFC population size, one time period hence, depends only on the present size and not on past sizes. In this respect, a Markov chain is a memoryless process. It is not the author's intent to develop the theory of Markov chains in this document. A short, but excellent discussion of the theory of Markov chains may be found in Hillier and Lieberman (Reference 1). Various properties of Markov chains will be used, without formal proof, throughout the description of this model. Proofs of the properties may be found in various texts such as Karlin or Feller (References 2 and 3).

The size of the GSFC population is basically a function of two processes—a personnel departure (separation) process and a personnel arrival (hiring) process. For the purposes of this model, the arrival process is considered to be completely deterministic. That is, management completely controls the numbers and scheduling of personnel hiring. Thus, in a probabilistic sense, the GSFC population is a function of only one stochastic process—the departure process.

First, one may consider only the stochastic departure process. This process will describe, in a probabilistic sense, how the GSFC population size changes from one time period to the next. Any suitable length of time period may be used within the model. For the GSFC population size system, a time period length of 1 month was chosen. This was the finest time resolution available in the historical personnel data base. It was also felt that any finer time resolution would not be more useful from a planning point of view.

To provide information of the detail described in the Introduction, monthly data on personnel departures were collected by skill code (clerical, professional administrative, scientist and engineer, technician, and wage grade) within Directorate. These 29 subpopulations, shown in Table 1, can be combined to yield probabilistic descriptions of the larger populations, including the entire GSFC population.

Table 1

GSFC Skill Code and Directorate Categories.

Skill Code	Directorate Code						
	100/200	300	400	500	600	700	800
Clerical	X	X	X	X	X	X	X
Professional Administrative	X	X	X	X	X	X	X
Scientist, Engineer	X	X	X	X	X	X	X
Technician	X	X	X	X	X	X	X
Wage Grade	X						

Departure Process

Basically, the departure process can be represented by a square matrix such as that presented in Figure 1 for professional administrative personnel in Directorate code 300.

The entries, P_{ij}, for the 1-month transition matrix in Figure 1 are defined as follows:

P_{ij} = prob [Number of professional administrative personnel in code 300 1 month hence equals j, given the present number of professional administrative personnel in code 300 equals i]

In this example, both i and j take on the values 0, 1, 2, 3, 4, 5, and 6.

The method used to obtain the various 1-month transition probabilities will be described in the last section. At this point though, it will be beneficial to discuss some of the properties of the 1-month transition matrix:

1. The 1-month transition matrix is square (the number of rows equals the number of columns).

$$
\begin{array}{cc}
 & \overbrace{\hphantom{P_{66}\quad P_{65}\quad P_{64}\quad P_{63}\quad P_{62}\quad P_{61}\quad P_{60}}}^{\text{Number of professional administrative personnel 1 month hence}} \\
 & \begin{array}{ccccccc} 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array} \\
\text{Present number of professional administrative personnel}
\left\{\begin{array}{c} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array}\right. &
\begin{pmatrix}
P_{66} & P_{65} & P_{64} & P_{63} & P_{62} & P_{61} & P_{60} \\
P_{56} & P_{55} & P_{54} & P_{53} & P_{52} & P_{51} & P_{50} \\
P_{46} & P_{45} & P_{44} & P_{43} & P_{42} & P_{41} & P_{40} \\
P_{36} & P_{35} & P_{34} & P_{33} & P_{32} & P_{31} & P_{30} \\
P_{26} & P_{25} & P_{24} & P_{23} & P_{22} & P_{21} & P_{20} \\
P_{16} & P_{15} & P_{14} & P_{13} & P_{12} & P_{11} & P_{10} \\
P_{06} & P_{05} & P_{04} & P_{03} & P_{02} & P_{01} & P_{00}
\end{pmatrix}
\end{array}
$$

Figure 1. One-month transition matrix for code 300.

2. Since each P_{ij} is a probability, it must necessarily lie in the range $0 \leqslant P_{ij} \leqslant 1$.
3. The 1-month transition matrix enumerates all possible 1-month transitions.
4. Since a present population size of i must become some population size j in 1 month hence, all row sums must be unity

$$\sum_{\text{j columns}} P_{ij} = 1$$

5. Since this transition matrix probabilistically describes a departure process $P_{ij} \equiv 0$ for all $j > i$.

Considering these properties and with the assumption of no hires, the general 1-month transition matrix of Figure 1 may be written as in Figure 2.

Except for the entry 1 in the lower right corner of Figure 2, explicit use of property 4 has not been made. At this point, assuming no hiring takes place, it is possible to answer questions about the size of the code 300 professional administrative staff after a 1-month time period.

Examples of Departure Process

As an example, suppose the present size of the code 300 professional administrative staff is 4. What is the probability that the staff will be 3 one month from now? This probability can be found by entering the matrix at row 3 (present size = 4), moving across to column 4 (size 1 month hence = 3), and reading P_{43}.

Number of professional administrative personnel 1 month hence

Present number of professional administrative personnel

$$\begin{array}{cc}
 & \begin{array}{ccccccc} 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array} \\
\left\{\begin{array}{c} 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array}\right. &
\begin{pmatrix}
P_{66} & P_{65} & P_{64} & P_{63} & P_{62} & P_{61} & P_{60} \\
0 & P_{55} & P_{54} & P_{53} & P_{52} & P_{51} & P_{50} \\
0 & 0 & P_{44} & P_{43} & P_{42} & P_{41} & P_{40} \\
0 & 0 & 0 & P_{33} & P_{32} & P_{31} & P_{30} \\
0 & 0 & 0 & 0 & P_{22} & P_{21} & P_{20} \\
0 & 0 & 0 & 0 & 0 & P_{11} & P_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}$$

Figure 2. One-month separation transition matrix for code 300.

In a similar fashion, one may also answer questions of the following type: Assume the present size is 4; What is the probability that the staff size 1 month hence will be between 3 and 1, inclusive? That is, what is the probability that the staff size 1 month from now will be 3, 2, or 1? All possible staff sizes 1 month hence are mutually exclusive (only one can occur at a time). Thus, one really has three separate questions:

1. What is the probability that the staff size will be 3?

2. What is the probability that the staff size will be 2?

3. What is the probability that the staff size will be 1?

The procedure for answering each of the three questions is identical to that used in answering the question in the first part of this example. For question 1 the result is P_{43}. For questions 2 and 3 it is P_{42} and P_{41}, respectively. Since these staff sizes are mutually exclusive, one adds the three probabilities. Thus, the probability of a staff size between 3 and 1, inclusive, is $P_{43} + P_{42} + P_{41}$. The same result could have been obtained by making use of property 4. Property 4 states that $P_{44} + P_{43} + P_{42} + P_{41} + P_{40} = 1$. Using this information, the answer is $1 - P_{44} - P_{40}$. It is immediately evident that $P_{43} + P_{42} + P_{41} = 1 - P_{44} - P_{40}$.

Example of Arrival-Departure Process

A deterministic hiring plan is assumed to be in effect. Again, suppose the present staff size is 4; but now, during the month, one additional staff member is scheduled to be hired. We now assume that this new member will not leave during the remaining portion of the month. What is the probability that the staff size 1 month from now will remain 4? This question is probabilistically equivalent to: What is the probability that the staff size will decrease from 4 to 3 in one month? There is no staff size, other than 3, that will, with the hiring of one additional member result in a staff size of four, 1 month hence. Thus, as in the previous example, the solution to this problem is P_{43}.

Although, in these examples, the 1-month transition matrix was easily displayed, in most cases this is not feasible. In some cases, the number of rows and columns necessary to probabilistically describe the possible 1-month transitions exceeds 100. For this reason a FORTRAN program for the 360/91 was written. This program is used to obtain solutions to problems similar to the ones presented in the previous three examples.

Tracking Periods in Excess of 1 Month

The model as described to this point can be used to track personnel plans for each skill code/Directorate category for periods of 1 month. One can increase the tracking period from 1 month up to K months where K = 2, 3, . . ., 12. Thus, answers can be obtained to questions about staff sizes for time periods up to 12 months from the present. It is assumed that transitions from one month to the next are independent. This is nothing more than the memoryless property of Markov chains referred to above. Increased time periods are easily accommodated, at least in principle. To obtain a matrix which probabilistically describes transitions over a K-month period, the 1-month transition matrix is raised to the Kth power. That this accomplishes what is needed is not immediately evident. Karlin and Feller (References 2 and 3) may be consulted for proof of this assertion. The following illustration shows how this matrix multiplication accounts for all possible transitions.

Consider the simple 1-month transition matrix:

$$
\begin{array}{cc}
 & \overbrace{\begin{array}{ccc} 2 & 1 & 0 \end{array}}^{\text{Size 1 Month Hence}} \\
\text{Present Size} \left\{ \begin{array}{c} 2 \\ 1 \\ 0 \end{array} \right. & \begin{pmatrix} P_{22} & P_{21} & P_{20} \\ P_{12} & P_{11} & P_{10} \\ P_{02} & P_{01} & P_{00} \end{pmatrix}
\end{array}
$$

Starting with present size = 2, Table 2 enumerates all possible ways that the system size may be 0 after 2 months.

Table 2

Two-Month System Size Changes.

Case	Present Size	Size 1 Month Hence	Size 2 Months Hence
1	2	2	0
2	2	1	0
3	2	0	0

Because of the independence of the monthly transitions, the probabilities associated with these three exclusive and exhaustive possibilities are easily found:

Case	Probability
1	$P_{22} \cdot P_{20}$
2	$P_{21} \cdot P_{10}$
3	$P_{20} \cdot P_{00}$

Thus, the probability of transition from a present size of 2 to a size of 0 in exactly 2 months is given by

$$P_{22} \cdot P_{20} + P_{21} \cdot P_{10} + P_{20} \cdot P_{00}$$

Now, the square of the 1-month transition matrix is

$$\begin{pmatrix} P_{22} & P_{21} & P_{20} \\ P_{12} & P_{11} & P_{10} \\ P_{02} & P_{01} & P_{00} \end{pmatrix}^2 = \begin{pmatrix} P_{22} & P_{21} & P_{20} \\ P_{12} & P_{11} & P_{10} \\ P_{02} & P_{01} & P_{00} \end{pmatrix} \times \begin{pmatrix} P_{22} & P_{21} & P_{20} \\ P_{12} & P_{11} & P_{10} \\ P_{02} & P_{01} & P_{00} \end{pmatrix}$$

Performing the indicated matrix multiplication, under rules found in many texts such as Sokolnikoff and Redheffer (Reference 4), the following 2-month transition matrix is obtained

$$\begin{array}{ll}
 & \overbrace{\begin{array}{ccc} \qquad\qquad 2 \qquad\qquad & \qquad\qquad 1 \qquad\qquad & \qquad\qquad 0 \qquad\qquad \end{array}}^{\text{Size 2 Months Hence}} \\
\text{Present Size} \left\{ \begin{array}{c} 2 \\ 1 \\ 0 \end{array} \right. & \begin{pmatrix}
P_{22} \cdot P_{22} + P_{21} \cdot P_{12} + P_{20} \cdot P_{02} & P_{22} \cdot P_{21} + P_{21} \cdot P_{11} + P_{20} \cdot P_{01} & P_{22} \cdot P_{20} + P_{21} \cdot P_{10} + P_{20} \cdot P_{00} \\
P_{12} \cdot P_{22} + P_{11} \cdot P_{12} + P_{10} \cdot P_{02} & P_{12} \cdot P_{21} + P_{11} \cdot P_{11} + P_{10} \cdot P_{01} & P_{12} \cdot P_{20} + P_{11} \cdot P_{10} + P_{10} \cdot P_{00} \\
P_{02} \cdot P_{22} + P_{01} \cdot P_{12} + P_{00} \cdot P_{02} & P_{02} \cdot P_{21} + P_{01} \cdot P_{11} + P_{00} \cdot P_{01} & P_{02} \cdot P_{20} + P_{01} \cdot P_{10} + P_{00} \cdot P_{00}
\end{pmatrix}
\end{array}$$

The upper right-hand element of this matrix is seen to be the same expression as for the 2-month transition probability after the previous enumeration. Again, this is not intended as proof, but merely as an illustration. In similar fashion, the K-month transition matrices may be developed from 1-month transition matrices.

Problems of the type encountered in the previous examples for time periods in excess of 1 month, can be handled with the K-month transition matrices. The same methods of attack may be used, since the only difference in interpretation would be the time periods over which the transitions occur. It should be noted that the ranges of present sizes and 1-month future sizes are preserved under the matrix multiplication.

Matrix multiplication, especially on large matrices, can be very cumbersome. Raising even a 3-by-3 matrix to the 12th power is a fairly time consuming undertaking when done by hand. A second FORTRAN program for the 360/91 was written to perform all necessary matrix multiplications. Thus, by executing the two FORTRAN programs sequentially, solutions to questions about possible future staff sizes can be obtained.

One additional technique is still required before solutions can be obtained for problems of the type referred to in the Introduction. Some method must be found to combine various skill code/Directorate groups in a probabilistically consistent way.

Examination of the first three properties of the transition matrices reveals that each row of the matrices is itself a probability density function. A complete definition and description of probability density functions can be found in Feller (Reference 3). Given the present population size, a complete probability description of future population sizes is available within each row. Thus, a combination of skill code/Directorate categories amounts to a logical summation of probability density functions.

To answer questions regarding combinations of various skill code/Directorate categories a number of probability density functions must be combined. To keep the problem tractable, only certain probability density functions were feasible for fitting to the matrix rows. Combinations of the probability density function chosen should possess an easily determinable form. Transitions within the various skill code/Directorate categories were assumed to be mutually independent. Each row of the various transition matrices was fitted by a unique Poisson probability density function. This function was chosen because of the binomial nature of the transition probabilities. It can be shown (Reference 3) that sums of independent, identical, or nonidentical Poisson probability density functions result in another Poisson probability density function. This probability density function was then subjected to further analysis to yield the needed information about transitions within the combination of skill code/Directorate categories.

At GSFC, since many of the skill code/Directorate transition matrices are rather large, problems regarding probabilities of various transitions are handled within the Resource Planning Office of the Financial Management Division on a one-by-one basis. It is not feasible to publish the totality of results since these would involve an astronomical number of combinations of possibilities.

In an operational mode, this model can provide information on a monthly basis. Figure 3 is an example of the manner in which the information could be provided to management. Information pertaining to each Directorate would appear on a separate sheet.

Directorate 100/200
Planning Period 7/73 - 7/74
Months Left in Planning Period 10

Original Hiring Plan		Current Hiring Plan	
48	Clerical	39	Clerical
39	Professional Administrative	34	Professional Administrative
4	Scientist, Engineer	4	Scientist, Engineer
6	Technician	4	Technician
12	Wage Grade	8	Wage Grade

Current Plan Success Probability 0.85

History of Plan Success Probabilities

Months to Go	Plan Success Probability
12	0.96
11	0.89
10	0.85
9	X
8	X
7	X
6	X
5	X
4	X
3	X
2	X
1	X

Note: X indicates future time periods.

Figure 3. Example of code 100/200 hiring plan assessment.

DERIVATION OF 1-MONTH TRANSITION PROBABILITIES

The historical personnel data base for the 3-year period FY 1970 through FY 1972 was analyzed to obtain the number of separations per month in each of the 29 skill code/Directorate categories listed in Table 1. Also obtained from this data base were average fiscal year population sizes for these categories. The probability that any individual departs from his particular skill code/Directorate category during any one month was estimated in the following manner:

Let D_i^h = number of departures from skill code/Directorate category h during month i

i = 1, 2, . . ., 36 (months of FY 1970-1972)

S_k^h = average skill code/Directorate category h population size during FY k

k = FY 1970, 1971, 1972

P^h = prob [any individual departs from skill code/Directorate category h in any one month]

Then the 1-month departure probabilities are

$$P^h = \sum_{i=1}^{36} D_i^h \Bigg/ 12 \sum_{K=1970}^{1972} S_k^h \tag{1}$$

Analysis of the 3-year historical personnel data base yielded 1-month departure probabilities, P^h, for each of the 29 ($h = 1, \ldots, 29$) skill code/Directorate categories. These values are presented in Table 3.

At the beginning of each month, each individual must decide whether or not to remain in the same skill code/Directorate category. Probabilistically, this amounts to an individual tossing a biased coin to make the decision. A toss resulting in a "head" will indicate the decision that the individual will remain in the same skill code/Directorate category during the month, while the outcome "tail" will indicate the decision that the individual will leave his particular skill code/Directorate category during the month.

Toss Outcome	Prob [Toss Outcome]	Action
Tail	P^h	leave
Head	$1 - P^h$	remain

The result is a series of Bernoulli trials, each individual within the same skill code/Directorate category having to make the same decision. If there are N individuals in skill code/Directorate category h, then the probability that exactly M ($M \leq N$) will leave during the

Table 3

GSFC Skill Code/Directorate 1-Month Separation Probabilities

Skill Code	Directorate	P^h
Clerical	100/200	0.018
Professional Administrative	100/200	0.008
Scientist, Engineer	100/200	0.011
Technician	100/200	0.006
Wage Grade	100/200	0.006
Clerical	300	0.012
Professional Administrative	300	0.010
Scientist, Engineer	300	0.003
Technician	300	0.002
Clerical	400	0.019
Professional Administrative	400	0.0001*
Scientist, Engineer	400	0.002
Technician	400	0.0001*
Clerical	500	0.012
Professional Administrative	500	0.0001*
Scientist, Engineer	500	0.001
Technician	500	0.002
Clerical	600	0.014
Professional Administrative	600	0.007
Scientist, Engineer	600	0.004
Technician	600	0.003
Clerical	700	0.013
Professional Administrative	700	0.004
Scientist, Engineer	700	0.003
Technician	700	0.003
Clerical	800	0.011
Professional Administrative	800	0.001
Scientist, Engineer	800	0.005
Technician	800	0.003

*Indicates 0 separations during 3-year period FY 1970-1972; P^h assumed equal to 0.0001.

month can be shown (Reference 3) to follow the binomial probability density function $b(M;N, P^h)$:

$$b\,(M;\,N,\,P^h) \equiv \binom{N}{M} \left[P^h\right]^M \left[1 - P^h\right]^{N-M} \tag{2}$$

where $M = 0, 1, \ldots, N$

$$\binom{N}{M} = \frac{N!}{M!\ (N-M)!}$$

$$N! = N(N-1)\ (N-2)\ \ldots\ (N-(N-1))$$

$b\,(M;\,N,\,P^h)$, being the probability of M departures from a population size of N, is identical to the transition probability $P^h_{N,N-M}$.

From this formula it can be seen that the probability that M individuals will leave skill code/Directorate category h during any one month depends upon more than just P^h and M. It is also a function of the population size, N, of category h. This dependence upon the population size of the category has a marked influence on the values of the 1-month transition probabilities. As an illustration, consider the matrix in Figure 1. Although P_{64} and P_{53} each refer to a situation where two professional administrative personnel depart code 300 during any one month, the values of these probabilities are in general different if computed by using Equation (2):

P_{64}:

$N = 6$

$M = 6 - 4 = 2$

$$P_{64} = \binom{6}{2} \left[P^h\right]^2 \left[1 - P^h\right]^{6-2}$$

P_{53}:

$N = 5$

$M = 5 - 3 = 2$

$$P_{53} = \binom{5}{2} \left[P^h\right]^2 \left[1 - P^h\right]^{5-2}$$

$$\frac{P_{64}}{P_{53}} = \frac{\left[6!/2!4!\right] \left[P^h\right]^2 \left[1 - P^h\right]^4}{\left[5!/2!3!\right] \left[P^h\right]^2 \left[1 - P^h\right]^3}$$

$$= \left[\frac{6}{4}\right] \left[1 - P^h\right]$$

Thus $P_{64} = P_{53}$ only if $[1 - P^h] = \frac{4}{6}$ or equivalently, only if $P^h = \frac{1}{3}$.

For the purposes of this model, it is assumed that skill code/Directorate category h has a lowest allowable population size L^h. This assumption was made to facilitate the necessary matrix computations. The error introduced will be directly proportional to the probability of occurrence of a transition to a population size below L^h. An estimate of L^h was obtained by the use of Tchebycheff's Inequality (Reference 3). One version of this inequality states that the probability of occurrence of a value of a random variable more than n standard deviations above or below its mean is less than $1/n^2$. Since this model is ultimately concerned with tracking personnel plans for periods of time of up to 1 year, Tchebycheff's Inequality was used with mean annual departure and annual standard deviation. (The mean annual departures are 12 times the mean monthly departures.) The relation between the monthly departures standard deviation and the annual departures standard deviation is:

$$\text{Annual departures standard deviation} = (12)^{1/2} \times (\text{Monthly departures standard deviation}).$$

Table 4 lists the mean annual departures and annual standard deviations for the 29 skill code/Directorate categories as obtained from an analysis of the historical personnel data base for the 3 years FY 1970 through 1972.

The current (May 1973) population sizes for the various skill code/Directorate categories are listed in Table 5.

Utilizing the data presented in Tables 4 and 5 along with Tchebycheff's Inequality, a 99 percent lower bound (L^h_{99}) was computed for each of the 29 skill code/Directorate categories. This 99 percent lower bound is to be interpreted as follows: In the long run, only 1 time in 100 will the final subpopulation (skill code/Directorate) size be less than L^h_{99}. Values of this bound are listed in Table 6.

Equation (2) will not yield correct transition probabilities for those matrices for which $L^h > 0$. In this case, M takes on all values between L^h and N, inclusive, while the general binomial variable takes on all values between 0 and N inclusive. With $L^h > 0$, the transition probability is conditional; that is, the probability that the future size is j given the present size i, is conditioned on the future size being equal to or greater than L^h. This is quantitatively expressed in Equation (3), where $M = L^h, \ldots, N$

$$b\,(M; N, P^h, L^h) = \frac{\binom{N}{M} \left[P^h\right]^M \left[1 - P^h\right]^{N-M}}{\sum_{M=L^h}^{N} \binom{N}{M} \left[P^h\right]^M \left[1 - P^h\right]^{N-M}} \tag{3}$$

If L^h is equal to 0, Equation (3) and Equation (2) are identical because the denominator of the right side of Equation (3) is then equal to 1. Equation (3) was used to generate the necessary 1-month transition probabilities since, as shown above for $b(M; N, P^h)$

$$b\,(M; N, P^h, L^h) = P^h_{N,N-M}$$

Table 4

GSFC Skill Code/Directorate Mean and Standard Deviation of Annual Departures

Skill Code	Directorate	Mean Annual Departures	Annual Standard Deviation
Clerical	100/200	65.9	8.0
Professional Administrative	100/200	44.3	6.6
Scientist, Engineer	100/200	8.4	2.9
Technician	100/200	10.0	3.2
Wage Grade	100/200	11.6	3.4
Clerical	300	3.5	1.8
Professional Administrative	300	0.7	0.8
Scientist, Engineer	300	6.6	2.6
Technician	300	1.6	1.2
Clerical	400	6.1	2.5
Professional Administrative	400	0.004	0.07
Scientist, Engineer	400	4.1	2.0
Technician	400	0.0004	0.07
Clerical	500	8.8	2.9
Professional Administrative	500	0.05	0.21
Scientist, Engineer	500	12.1	3.5
Technician	500	1.1	1.0
Clerical	600	11.0	3.3
Professional Administrative	600	2.3	1.5
Scientist, Engineer	600	16.8	4.1
Technician	600	4.9	2.2
Clerical	700	9.0	3.0
Professional Administrative	700	0.4	0.6
Scientist, Engineer	700	16.0	4.0
Technician	700	9.5	3.1
Clerical	800	8.9	2.9
Professional Administrative	800	1.0	1.0
Scientist, Engineer	800	18.2	4.3
Technician	800	4.4	2.1

Table 5

GSFC Skill Code/Directorate Population Sizes, May 1973

Skill Code	Directorate	Population Size
Clerical	100/200	305
Professional Administrative	100/200	461
Scientist, Engineer	100/200	64
Technician	100/200	138
Wage Grade	100/200	162
Clerical	300	24
Professional Administrative	300	6
Scientist, Engineer	300	182
Technician	300	63
Clerical	400	27
Professional Administrative	400	3
Scientist, Engineer	400	168
Technician	400	3
Clerical	500	61
Professional Administrative	500	38
Scientist, Engineer	500	335
Technician	500	45
Clerical	600	66
Professional Administrative	600	27
Scientist, Engineer	600	349
Technician	600	138
Clerical	700	58
Professional Administrative	700	7
Scientist, Engineer	700	442
Technician	700	263
Clerical	800	67
Professional Administrative	800	76
Scientist, Engineer	800	303
Technician	800	123

Table 6

GSFC Skill Code/Directorate Size Lower Bound (99 Percent)

Skill Code	Directorate	L^h_{99}
Clerical	100/200	159
Professional Administrative	100/200	350
Scientist, Engineer	100/200	26
Technician	100/200	96
Wage Grade	100/200	116
Clerical	300	2
Professional Administrative	300	0
Scientist, Engineer	300	149
Technician	300	49
Clerical	400	0
Professional Administrative	400	2
Scientist, Engineer	400	143
Technician	400	2
Clerical	500	23
Professional Administrative	500	35
Scientist, Engineer	500	288
Technician	500	33
Clerical	600	22
Professional Administrative	600	9
Scientist, Engineer	600	291
Technician	600	111
Clerical	700	19
Professional Administrative	700	0
Scientist, Engineer	700	386
Technician	700	242
Clerical	800	29
Professional Administrative	800	65
Scientist, Engineer	800	241
Technician	800	97

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