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FOREWORD

This compilation of papers and reports is the final report for Contract ME-(NGR-01-003-008), "Computer Techniques for Multivariant Function Model Generation Emphasizing Programs Applicable to Space Vehicle Guidance". The work was performed by those listed as authors of the papers and reports, for the National Aeronautics and Space Administration, Electronics Research Center, Cambridge, Massachusetts.
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A METHOD FOR DETERMINING OPTIMUM RE-ENTRY TRAJECTORIES

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SUMMARY

The Pontryagin Maximum Principle is used to formulate the problem of finding optimum atmospheric vehicular re-entry trajectories. The optimization problem is that of minimizing an integral which is a function of the state and control variables. The vehicle's motion is assumed to be influenced only by a gravitational force and an aerodynamic force. The problem is formulated and the necessary equations are developed simultaneously for three sets of Euler angles. Computational procedures are suggested so that numerical trajectories may be generated.
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<td>Center of gravity</td>
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<td>C.P.</td>
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<td>Ca</td>
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<td>CP</td>
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<td>m</td>
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$M$ Mass of the earth

$\overline{N}_{am}$ Aerodynamic moment (missile system)

$\overline{N}_{rm}$ Roll couple (missile system)

$q$ Dynamic pressure

$R_o$ Earth's radius

$|\overline{R}|$ Absolute value of the plumbline position vector

$S_\alpha$ Sine $\alpha$

$S_\alpha y$ Sine $\alpha_y$

$SP$ Sine $\phi_p$

$SR$ Sine $\phi_r$

$SY$ Sine $\phi_y$

$t$ Time

$T$ Kinetic energy of the vehicle

$\overline{V_r}$ Relative velocity vector (Aerodynamic System)

$\overline{V}_{rm}$ Relative velocity vector (Missile System)

$\overline{V}_R$ Relative velocity vector (Plumbline System)

$\overline{W}$ Velocity vector for abnormal air movement in plumbline system
\( \bar{x} \) Plumbline position vector

\( \bar{x}_a \) Aerodynamic system position vector

\( \bar{x}_{cp} \) Position of the center of pressure in the missile system

\( \bar{x}_m \) Missile system position vector

\( z_r \) Roll jet positions in the missile system

\( \alpha^* \) Angle of attack

\( \alpha_y \) Yaw angle of attack

\( \phi_p \) Pitch angle

\( \phi_r \) Roll angle

\( \phi_y \) Yaw angle

\( \rho \) Density of the Atmosphere

\( \bar{\omega} \) Angular velocity vector of the vehicle in the missile system

\( \bar{\omega}_e \) Angular velocity vector of the attracting body in the plumbline system
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I. INTRODUCTION

This paper is an extension of previous work done by Grady Harmon and W. A. Shaw, presented in NASA TM X-53024, March 14, 1964.

The objectives of this paper are (1) to present a method for treating optimum re-entry problems in a simplified manner and (2) to generalize the computational scheme outlined in the aforementioned paper. The computational scheme given allows for the optimization of any functional subject to the specified constraints. Atmospheric data, vehicle configuration and aerodynamic coefficients are incorporated in the computational scheme in tabular form. Thus, different vehicles and/or atmospheres may be considered by changing the appropriate tables.

The governing equations are developed for three different gimbal sets. A computational scheme is outlined for each case.

No numerical results are available at present, but development of the computer deck is underway at the Electronics Research Center.

This work is sponsored, in part, by a grant, NGR-01-003-008, from the Electronics Research Center.
II. STATEMENT OF THE PROBLEM

The problem is that of finding the optimum control process, $a_y(t)$, that will transfer a vehicle from an initial state, at time $t_0$, in an atmosphere to a terminal state, at time $t_1$, in the same atmosphere so that the value of the functional

$$J = \int_{t_0}^{t_1} f(\dot{x}, \ddot{x}, \dot{\phi}, \phi, \dot{\psi}, \psi, \alpha, a_\alpha, F) \, dt$$

is a minimum. The trajectory associated with this optimum control process is the optimum trajectory.

The rotational motion of the vehicle is treated in a simplified manner. The equations governing the vehicle's rotational motion are considered as a steady-state problem with only one component of the angular velocity vector present for any given gimbal set. A gimbal set is used to measure the Euler angles, $\phi_x$, $\phi_y$, and $\phi_p$. The equations of motion are developed simultaneously for three different gimbal sets.

The problem is formulated as a Pontryagin initial value problem. The relative velocity equations appear as algebraic constraints. The yaw angle of attack, $a_y(t)$, is the control variable.

Additional assumptions are made as follows:

1. The motion of the vehicle is influenced by an aerodynamic force that acts through the vehicle's center of pressure.
2. The attracting body is a rotating sphere with homogeneous mass.
3. The vehicle's centroid of mass and centroid of volume are not coincident.

4. The vehicle's center of mass is invariant with respect to the vehicle.

5. The center of pressure of the vehicle is invariant with respect to the vehicle.

6. A system of roll control jets is available on the vehicle that produce a pure roll couple as required by the optimum control process.
III. COORDINATE SYSTEMS

Three rectangular coordinate systems will be used in this paper. They are:

1. The plumpline space fixed coordinate system,
2. The vehicle fixed missile coordinate system,
3. The aerodynamic coordinate system.

A. Plumpline System

The plumpline system, Figure 1, has its origin at the earth's center with the Y-axis parallel to the gravity gradient at the launch point. The X-axis is parallel to the earth fixed launch azimuth and the Z-axis is chosen to form a right-handed system.

B. Missile System

The missile system, Figure 1, is located with its origin at the center of mass of the vehicle and its \( y_m \) axis parallel to the longitudinal axis of the vehicle. The \( x_m \) and \( z_m \) axes are chosen to form a right-handed system which is parallel to the plumpline system at the launch point.

As the vehicle moves along its trajectory, the missile system undergoes a displacement with respect to the plumpline system. This displacement is given by three Euler angles as measured by a gimbal set. The Euler angles uniquely specify the orientation of the vehicle at any time. Any particular orientation of the vehicle may be described by
Fig. 1. Plumpline and missile coordinate systems
different sets of Euler angles depending solely on the sequence in which the angles are measured. Therefore, it is mandatory that a specific sequence be followed in measuring the Euler angles. The three Euler angles are referred to as the yaw angle, $\theta_y$, the roll angle, $\theta_r$, and the pitch angle, $\theta_p$. The yaw angle is measured with respect to an X axis. The roll angle is measured with respect to a Y axis, and the pitch angle is measured with respect to a Z axis. An angle is considered positive counterclockwise when viewed from the positive end of the axis about which the rotation is taken. The angles are measured by a set of gimbals on the vehicle. A gimbal set measures the Euler angles in a specific sequence such as pitch, yaw, and roll. In this paper, equations that involve the angles yaw, roll, or pitch are developed simultaneously for three different sets of Euler angles. The angles are obtained from three gimbal sets. They will be referred to as follows:

1. A gimbal set which measures in the order of pitch, yaw, roll.
2. A gimbal set which measures in the order of pitch, roll, yaw.
3. A gimbal set which measures in the order of roll, yaw, pitch.

The Euler angles are shown in Figures 2, 3, and 4.

A position vector in the missile coordinate system may be written in terms of a position vector in the plumbline coordinate system.
The equations of transformation are given by the orthogonal rotation matrices

\[
[\varphi_y] = \begin{bmatrix}
1 & 0 & 0 \\
0 & CY & SY \\
0 & -SY & CY
\end{bmatrix}
\]

\[
[-\varphi_x] = \begin{bmatrix}
CR & 0 & SR \\
0 & 1 & 0 \\
-SR & 0 & CR
\end{bmatrix}
\]

\[
[\varphi_p] = \begin{bmatrix}
CP & SP & 0 \\
-SP & CP & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The particular combination of the above rotation matrices that relate a vector in the two coordinate systems is dependent on the gimbal set used. The relationship for gimbal set 1 is

\[
X_m = [-\varphi_x] [\varphi_y] [\varphi_p] X
\]  

(1a)

or

\[
X_m = [A_d]_1 X
\]  

(1b)
Fig. 2. Eulerian angles for gimbal set 1
where

\[
[A_d]_1 = \begin{bmatrix}
    \text{CRCP} + \text{SPSR} & \text{CRSP} - \text{SRSY} & \text{SRCY} \\
    -\text{CYSP} & \text{CYCP} & \text{SY} \\
    -\text{SRCP} + \text{CYSRSP} & -\text{SPSR} - \text{CRCPY} & \text{CRCY}
\end{bmatrix}
\]  

(1c)

is the combined product of the rotation matrices in equation (1a).

When \(\varphi_y = 90^\circ\) gimbal set 1 is oriented so that \(\varphi_r\) and \(\varphi_p\) are measured in the same direction, refer to Figure 2. This condition is referred to as gimbal lock.

The relationship for gimbal set (2) is

\[
\bar{x}_m = [\varphi_y] [-\varphi_r] [\varphi_p] \bar{x}
\]  

(2a)

or

\[
\bar{x}_m = [A_d]_2 \bar{x}
\]  

(2b)

where

\[
[A_d]_2 = \begin{bmatrix}
    \text{CRCP} & \text{CRSP} & \text{SR} \\
    -\text{CYSP} - \text{SYSRCP} & \text{CYCP} - \text{YSRSP} & \text{SYCR} \\
    \text{SYSP} - \text{CYSRCP} & -\text{SYCP} - \text{CYSRSP} & \text{CYCR}
\end{bmatrix}
\]  

(2c)

is the combined product of the rotation matrices in equation (2a).

Gimbal set 2 is locked when \(\varphi_r = 90^\circ\). At this orientation, refer to Figure 3, \(\varphi_y\) and \(\varphi_p\) are measured in the same direction.
Fig. 3. Eulerian angles for gimbal set 2
The relationship for gimbal set (3) is

$$\overline{X}_m = [\overline{\theta}_x] \overline{\theta}_y [\overline{\theta}_z] X$$  \hspace{1cm} (3a)

or

$$\overline{X}_m = [A_d]_3 \overline{X}$$  \hspace{1cm} (3b)

where

$$[A_d]_3 = \begin{bmatrix}
\text{CPCR-SPRSY} & \text{SPCY} & \text{CPSR+SPSYCR} \\
-\text{SPCR-CPSRSY} & \text{CPCY} & -\text{SPSR+CPSYCR} \\
-\text{SRCY} & -\text{SY} & \text{CYCR}
\end{bmatrix}$$  \hspace{1cm} (3c)

is the combined product of the rotation matrices in equation (3a).

Gimbal set 3 is locked when \( \overline{\theta}_y = 90^\circ \). At this orientation, refer to Figure 4, \( \overline{\theta}_p \) and \( \overline{\theta}_r \) are measured in the same direction.

The transformation matrices (1c), (2c), and (3c) will be referred to as

$$[A_d]_3 \text{ where } i = 1, 2, 3.$$

Equations (1b), (2b), and (3b) are restated as

$$\overline{X}_m = [A_d]_i \overline{X}.$$  \hspace{1cm} (4)
Fig. 4. Eulerian angles for gimbal set 3
C. Aerodynamic Coordinate System

The aerodynamic coordinate system is located as shown in Figure 5 with its origin at the center of pressure of the vehicle. The $Y_a$ axis lies in the plane containing the vehicle longitudinal axis of symmetry and the relative velocity vector. The relative velocity vector, $\vec{V}_R$, is defined as the velocity of the air with respect to the vehicle as measured from the inertial reference. The $X_a$ and $Z_a$ axes are chosen to form a right-handed system. As the vehicle moves along its trajectory, there will be a relative displacement between the missile fixed coordinate system and the aerodynamic coordinate system. The direction of the $Y_a$ axis is defined by the following rotations as shown in Figure 5:

1. Rotate the vehicle fixed reference frame about the $Y_m$ axis so that the $X_m$ axis lies in a plane parallel to the plane formed by the vehicle's longitudinal axis of symmetry and the relative velocity vector. The angle traversed is referred to as the yaw angle of attack, $\alpha_y$.

2. Rotate about the new $Z$ axis by the true angle of attack, $\alpha^*$. This specifies the orientation of the aerodynamic coordinate system.

The true angle of attack, $\alpha^*$, will be expressed in terms of the aerodynamic force in the next section.
Fig. 5. Aerodynamic and missile coordinate systems
A position vector in the aerodynamic coordinate system may be written in terms of a position vector in the missile fixed coordinate system. The orthogonal transformation matrices are

$$[-\alpha^*] = \begin{bmatrix} Ca^* & -Sa^* & 0 \\ Sa^* & Ca^* & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[\alpha_y] = \begin{bmatrix} Ca_y & 0 & -Sa_y \\ 0 & 1 & 0 \\ Sa_y & 0 & Ca_y \end{bmatrix}$$

A positive vector in the aerodynamic coordinate system is expressed in terms of a position vector in the missile fixed reference as

$$\bar{X}_a = [-\alpha^*] [\alpha_y] \bar{X}_m \quad (5a)$$

or

$$\bar{X}_a = [A_a] \bar{X}_m \quad (5b)$$

where

$$[A_a] = \begin{bmatrix} Ca^*Ca_y & -Sa^* & -Ca^*Sa_y \\ Sa^*Sa_y & Ca^* & -Sa^*Sa_y \\ Sa_y & 0 & Ca_y \end{bmatrix} \quad (5c)$$
is the combined product of the rotation matrices in equation (5a).
The aerodynamic coordinate system transformation matrix (5c) is independent of the sequence used in measuring the angles yaw, roll, and pitch.
IV. MECHANICS

A. Forces

Two forces are assumed to act on the vehicle as it moves along its trajectory. It was assumed that the attracting body is a homogeneous sphere. Thus, an inverse square gravitational force is written in terms of the plumbline coordinates as

$$F_g = -\frac{C M m}{|R|^3}$$  \hspace{1cm} (6)

The vehicle's motion is also influenced by an aerodynamic force. The force lies in the plane formed by the vehicle longitudinal axis of symmetry and the relative velocity vector and passes through the center of pressure of the vehicle, as shown in Figure 6.

The components of the aerodynamic force are defined by the equations

$$F_x = A q C_x (a^*)$$  \hspace{1cm} (7a)

and

$$F_z = A q C_z (a^*)$$  \hspace{1cm} (7b)
Fig. 6. Aerodynamic force components $F_x$ and $F_z$. 
A is the projected cross-section area of the vehicle and \( q \) is the dynamic pressure. \( C_x \) and \( C_z \) are experimentally determined factors that are dependent on the vehicle's shape and the angle of attack. It is assumed that \( C_z \) and \( C_x \) are known. The aerodynamic force is expressed in the aerodynamic system as

\[
\begin{bmatrix}
-F_x C_a^* + F_x S_a^* \\
-F_y C_a^* - F_z S_a^* \\
0
\end{bmatrix}
\] (8)

The aerodynamic force is expressed in terms of the missile fixed reference as

\[
\mathbf{F}_{am} = [A_a]^T \mathbf{F}_a
\] (9a)

(Note: The symbol \([A]^T\) is used to denote the transpose of matrix \( A \).)

Equation (9a) can be written in component form as

\[
\begin{bmatrix}
F_{amx} \\
F_{amy} \\
F_{amz}
\end{bmatrix}
= \begin{bmatrix}
C_a^* C_a y & S_a^* C_a y & S_a y \\
-S_a y & C_a^* & 0 \\
-C_a^* S_a y & -S_a^* S_a y & C_a y
\end{bmatrix}
\begin{bmatrix}
-F_a S_a C_a^* + F_a C_a S_a^* \\
-F_a C_a^* C_a y - F_a S_a S_a y \\
0
\end{bmatrix}
\] (9b)

When simplified, equation (9b) becomes

\[
\begin{bmatrix}
F_{amy} \\
F_{amz}
\end{bmatrix}
= \mathbf{F}_a
\begin{bmatrix}
-S_a C_a y \\
-C_a \\
S_a S_a y
\end{bmatrix}
\] (9c)
where the magnitude of the aerodynamic force is

\[ F_a = \sqrt{F_x^2 + F_z^2} \quad , \quad (10) \]

and \( c \) is expressed in terms of the components of the aerodynamic force through the equations

\[ S_a = \frac{F_z}{|F_a|} \quad , \quad (11a) \]

\[ C_a = \frac{F_x}{|F_a|} \quad , \quad (11b) \]

\[ \tan \alpha = \frac{S_a}{C_a} = \frac{F_z}{F_x} = \frac{C_a(c^*)}{C_x(c^*)} \quad . \quad (11c) \]

The magnitude of the aerodynamic force is related to the relative velocity through the dynamic pressure by the equation

\[ q = \frac{1}{2} \rho V_R^2 \quad . \quad (12) \]
It is assumed that the atmosphere normally moves with the attracting body (6). Hence, at all times there is an air mass movement with respect to the plumbline coordinate system. \( \vec{W} \) is a vector that represents any abnormal air movement. An equation expressing the velocity of the wind may be written as

\[
\vec{V}_{\text{wind}} = \vec{\omega}_e \times \vec{X} + \vec{W} \quad . (13)
\]

The relative velocity equation is

\[
\dot{\vec{X}} = \vec{V}_{\text{wind}} + \vec{V}_R \quad . (14)
\]

When equation (13) is substituted into equation (14), the result is

\[
\vec{V}_R = \dot{\vec{X}} + \vec{X} \times \vec{\omega}_e - \vec{W} \quad , (15a)
\]

or, in component form,

\[
\begin{bmatrix}
V_{RX} \\
V_{RY} \\
V_{RZ}
\end{bmatrix} = \begin{bmatrix}
\dot{X} \\
\dot{Y} \\
\dot{Z}
\end{bmatrix} + \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} \times \begin{bmatrix}
\omega_{ex} \\
\omega_{ey} \\
\omega_{ez}
\end{bmatrix} - \begin{bmatrix}
W_x \\
W_y \\
W_z
\end{bmatrix} \quad . (15b)
\]

The relative velocity may be expressed in terms of the aerodynamic, missile, or plumbline coordinate system variables. The relative velocity
vector is written in the missile coordinate system as

\[ \vec{V}_{\text{rm}} = [A_d]_i \vec{V}_R = [A_a]^T \vec{V}_r \]  \hspace{1cm} (16)

where

\[ \vec{V}_{\text{rm}} = \begin{bmatrix} V_{\text{rm}x} \\ V_{\text{rm}y} \\ V_{\text{rm}z} \end{bmatrix} \]

is the relative velocity vector in the missile system and

\[ \vec{V}_r = \begin{bmatrix} 0 \\ V_r \\ 0 \end{bmatrix} \]

is the relative velocity vector in the aerodynamic system. (Note that equation (16) represents three possible equations depending on i.)

The resultant force acting on the vehicle written in the plumpline coordinates is

\[ \vec{F}_R = \vec{F}_g + [A_d]^T [A_d]_i \vec{F}_{\text{am}} \]  \hspace{1cm} (17)
B. Couples and Moments

The motion of the vehicle is influenced by a moment and a couple. It is assumed that the center of pressure and the center of mass are invariant with respect to the vehicle. Thus, the center of pressure is located by a constant position vector, $\vec{x}_{cp}$, in the missile fixed reference. The aerodynamic moment is given by the vector product of the position vector, $\vec{x}_{cp}$, and the aerodynamic force, $\vec{F}_{am}$. The aerodynamic moment is written in the missile fixed reference as

$$\vec{M}_{am} = \vec{x}_{cp} \times \vec{F}_{am}$$

or

$$\begin{bmatrix} M_{ax} \\ M_{ay} \\ M_{az} \end{bmatrix} = \begin{bmatrix} F_a y_{cp} S_o s_y + F_a z_{cp} C_o \\ -F_a z_{cp} C_o S_o s_y - F_a x_{cp} S_o s_y \\ -F_a x_{cp} C_o + F_a y_{cp} S_o C_o s_y \end{bmatrix}$$

A system of roll jets is used to produce a pure roll control couple about the $Y_m$ axis. The jets are located with respect to the missile fixed coordinate system so that

$$\begin{bmatrix} F_{r1} \\ 0 \\ 0 \end{bmatrix} \text{ located at } \begin{bmatrix} 0 \\ 0 \\ Z_r \end{bmatrix}$$
and
\[
\mathbf{\bar{F}}_{r2} = \begin{bmatrix} -\bar{F}_r \\ 0 \\ 0 \end{bmatrix}
\]
located at \(\mathbf{\bar{Z}}_r = \begin{bmatrix} 0 \\ 0 \\ -\bar{Z}_r \end{bmatrix}\)

yield a roll couple

\[
\mathbf{\bar{M}}_{Tm} = 2 (\mathbf{\bar{Z}}_r \times \mathbf{\bar{F}}_r)
\]

which may be expressed as

\[
\mathbf{\bar{M}}_{Tm} = \begin{bmatrix} 0 \\ 2z_r \mathbf{\bar{F}}_r \\ 0 \end{bmatrix}
\]

The resultant moment about the center of mass of the vehicle in the missile fixed coordinate system is the sum of the roll couple and aerodynamic moment

\[
\mathbf{\bar{M}}_{Tm} = \mathbf{\bar{M}}_{am} + \mathbf{\bar{M}}_{Tm}
\]

When equations (18b) and (19b) are substituted into equation (20), the result is

\[
\mathbf{\bar{M}}_{Tm} = \begin{bmatrix} F_a y^2 \text{cp} Ca \text{Sa}_y + F_a z^2 \text{cp} Ca \\ -F_a z \text{cp} SaCa_y - F_a x \text{cp} SaCa_y + 2z_r \mathbf{\bar{F}}_r \\ -F_a x \text{cp} Ca + F_a y \text{cp} SaCa_y \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\mathbf{\bar{M}}_{Tm} = \begin{bmatrix} 0 \\ 2z_r \mathbf{\bar{F}}_r \\ 0 \end{bmatrix}
\]
which can be reduced to

\[
\begin{bmatrix}
F_a y_{cp} \Sigma S a y + F_a z_{cp} \Sigma c a \\
22 \Sigma f_x - F_a z_{cp} \Sigma c a y - F_a x_{cp} \Sigma S a y \\
-F_a x_{cp} \Sigma c a + F_a y_{cp} \Sigma S a y
\end{bmatrix}
\]

(21b)

C. Equations of Motion

It is possible to interpret the motion of a rigid body as the sum of two independent effects—the motion of the center of mass of the vehicle with respect to an inertial coordinate system and the rotational motion of the vehicle about its center of mass. The motion of a rigid body in general requires six independent coordinates to specify its orientation at any time. The six independent coordinates used in this problem are the three plumeline coordinates and three Eulerian angles.

The translational equations of motion are written for the center of mass in the inertial reference as

\[
\dddot{\bar{X}}_m = m \ddot{\bar{X}}
\]

(22a)

or

\[
\dddot{\bar{F}}_g + [A_d]^T \dddot{\bar{F}}_{am} = m \dddot{\bar{X}}
\]

(22b)

where
When expression (6) is substituted into equation (22b), the second order translational equations of motion become

$$\ddot{X} = - \frac{G \times \dot{X}}{|R|^3} + [A_d]_i^T \frac{F_{am}}{m}. \quad (22c)$$

The three second order differential equations, (22c), may be reduced to six first order differential equations by a change of variables. Let

$$\ddot{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \begin{bmatrix} \dot{X} \\ Y \\ Z \end{bmatrix} = \ddot{X}. \quad (23)$$

When the above transformation is used, the second order differential equations of motion, (22c), reduce to

$$\ddot{u} = - \frac{G \times \ddot{X}}{|R|^3} + [A_d]_i^T \frac{F_{am}}{m}. \quad (24)$$

For convenience, the following definitions are made:

$$g = - \frac{G M}{|R|^3}. \quad (25)$$

$$[A_d]_i^T \frac{F_{am}}{m} = \frac{F_a}{m} N_i \quad (26)$$
where
\[ F_a^* = \frac{F_a}{m} \]

and
\[
\begin{bmatrix}
N \\
\bar{N} \\
Q
\end{bmatrix}
= \begin{bmatrix}
-(S\alpha_C \gamma)(CRCP+SRSYSP) + C\alpha CYSP + (S\alpha S\alpha_S) (-SRCP+CRSYSP) \\
-(S\alpha C\alpha_S)(CRSP-SRSYCP) - C\alpha CYCP - (S\alpha S\alpha_S) (SPS\alpha + CR\alpha PS\gamma) \\
-(S\alpha C\alpha_S)(SRCY) - C\alpha SY + (CRCYS\alpha S\alpha_S) \\
\end{bmatrix}
\] 

(27a)

and
\[
\begin{bmatrix}
N \\
\bar{N} \\
Q
\end{bmatrix}
= \begin{bmatrix}
-(S\alpha C\alpha_S)(CRCP) + C\alpha (CYSP+SYSRCP) + S\alpha S\alpha_S (SYSP-CYS\alpha CP) \\
-(S\alpha C\alpha_S)(CRSP) - C\alpha (CYCP-SYSRSP) - (S\alpha S\alpha_S) (SYCP+SYSRSP) \\
-(S\alpha C\alpha_S)(SR) - C\alpha (SYCR) + (S\alpha S\alpha_S) (CRCY) \\
\end{bmatrix}
\] 

(27b)

and

\[
\begin{bmatrix}
N \\
\bar{N} \\
Q
\end{bmatrix}
\]
where

\[
\begin{bmatrix}
N \\
Q \\
P
\end{bmatrix} = \begin{bmatrix}
-(SaCa_y)(CPCR-SPSRSY) + Ca (SPCR+CPSRSY) - SaSa_y (CYSR) \\
-(SaCa_y)(SPCY) - Ca (CPCY) - SaSa_y (SY) \\
-(SaCa_y)(CPSR+SPSYCR) - Ca (-SPSR+CPSYCR) + (SaSa_y)(CYCR)
\end{bmatrix}
\] (27c)

When these definitions are used in equation (24), it may be written as

\[
\dot{u} = g_x \vec{X} + F^*_a \vec{N}_i
\] (28)

It is convenient to write the rotational equations of motion in the Lagrangian form. When the Eulerian angles (pitch, roll, yaw) are generalized coordinates, the rotational equations of motion take the form

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \theta_j} \right) - \frac{\partial T}{\partial \dot{\theta}_j} = M_{\theta_j} \quad j = p, y, r
\] (29)

\(T\) is the rotational kinetic energy of the vehicle and \(M_{\theta_j}\) is the moment associated with the \(\theta_j\) rotation. Based on the assumption of an offset center of mass, all components of the inertia matrix are assumed to be non-zero. The inertia matrix is

\[
[u] = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}
\] (30)
The rotational kinetic energy may be expressed with respect to the missile fixed coordinate system as

\[ T = \frac{1}{2} \mathbf{\bar{\omega}}^T [\mu] \mathbf{\bar{\omega}} \quad , \quad (31) \]

where \( \mathbf{\bar{\omega}} \) is the angular velocity of the vehicle in the missile fixed coordinate system.

When expressions (30) and (31) are substituted into equation (29), the result is

\[ \frac{d}{dt} \left( \frac{\partial \mathbf{\bar{\omega}}^T}{\partial \phi_j} \right) [\mu] \mathbf{\bar{\omega}} + \frac{\partial \mathbf{\bar{\omega}}^T}{\partial \phi_j} [\mu] \frac{d}{dt} (\mathbf{\bar{\omega}}) \]

\[ - \frac{\partial \mathbf{\bar{\omega}}^T}{\partial \phi_j} [\mu] \mathbf{\bar{\omega}} = \mathbf{M}_{\phi_j} \quad , \quad (32) \]

The angular velocity vector, \( \mathbf{\bar{\omega}} \), is obtained from a coordinate transformation of the angular velocity components \( \dot{\phi}_y, \dot{\phi}_r, \) and \( \dot{\phi}_p \) into the missile fixed reference. The transformation is dependent on the gimbal set used. The transformation matrix is developed for gimbal set 1. (Gimbal set 1 measures the Euler angles in the order pitch, yaw, roll.) A coordinate transformation is not required for \( \dot{\phi}_r \) since it is measured with respect to the missile coordinate system. The angular velocity
component $\hat{\varphi}_y$ is expressed in the missile fixed reference by use of the rotation matrix $[-\varphi_x]^T$. The transformation is:

$$
\begin{bmatrix}
\hat{\varphi}_y \\
0 \\
0
\end{bmatrix} = [-\varphi_x]^T \begin{bmatrix}
\varphi_y \\
0
\end{bmatrix}
$$

The angular velocity component $\hat{\varphi}_p$ is expressed in the missile fixed reference by use of two rotation matrices as follows:

$$
\begin{bmatrix}
\hat{\varphi}_p \\
0 \\
\varphi_p
\end{bmatrix} = [-\varphi_x]^T \begin{bmatrix}
0 \\
0 \\
\varphi_p
\end{bmatrix}
$$

Thus, the angular velocity vector

$$
\omega = \begin{bmatrix}
\hat{\varphi}_x \\
\hat{\varphi}_y \\
\hat{\varphi}_p
\end{bmatrix}_{\text{missile}}
$$

or, in component form,

$$
\begin{bmatrix}
\omega_{xm} \\
\omega_{ym} \\
\omega_{zm}
\end{bmatrix} = \begin{bmatrix}
0 \\
\hat{\varphi}_x \\
0
\end{bmatrix} + [-\varphi_x]^T \begin{bmatrix}
\varphi_y \\
0
\end{bmatrix} + [-\varphi_x]^T \begin{bmatrix}
\varphi_y \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
\varphi_p
\end{bmatrix}
$$
which may be expressed as

$$\ddot{\omega} = [\ddot{\phi}]_1 \ddot{\theta}$$

(33c)

where

$$\ddot{\theta} = \begin{bmatrix} \ddot{\phi}_\theta \\ \ddot{\phi}_r \\ \ddot{\phi}_p \end{bmatrix}$$

and the transformation matrix

$$[A_{\omega}]_1 = \begin{bmatrix} CR & 0 & SR \\ 0 & -1 & SY \\ -SR & 0 & CRCY \end{bmatrix}$$

(34a)

A similar argument is used to develop transformation matrices for gimbal sets 2 and 3:

$$[A_{\omega}]_2 = \begin{bmatrix} 1 & 0 & SR \\ 0 & -CY & CRSY \\ 0 & SY & CRCY \end{bmatrix}$$

(34b)

$$[A_{\omega}]_3 = \begin{bmatrix} CP & -SPCY & 0 \\ -SP & -CPCY & 0 \\ 0 & SY & 1 \end{bmatrix}$$

(34c)
The angular velocity vector, $\vec{\omega}$, is restated for the three gimbal sets as

$$\vec{\omega} = [A_{\omega}]_i \vec{\omega} \quad i = 1, 2, 3 \quad (35)$$

It should be noted that the transformation matrices $[A_{\omega}]_i$ are not orthogonal.

By use of the expressions obtained above, the rotational equations of motion become

$$\ddot{\phi}_i = [C]_i \left\{ \mu \phi_i - \left( \frac{d}{dt} [A_{\omega}]_i^T \right) \left[ \mu \right] [A_{\omega}]_i + [A_{\omega}]_i^T \left[ \mu \frac{d}{dt} [A_{\omega}]_i \right] \dot{\phi}_i \right\} + \vec{B}_i \quad (36)$$

where

$$[C]_i = \left( [A_{\omega}]_i^T \left[ \mu \right] [A_{\omega}]_i \right)^{-1} \quad (37a)$$

$$B_j = \frac{\partial}{\partial \phi_j} \left( \frac{d}{dt} [A_{\omega}]_i \right)^T \left[ \mu \right] [A_{\omega}] \ddot{\phi}_i \quad (37b)$$
\[ \bar{M}_{p} = [A]^T \bar{M}_{m} \quad , \quad (37c) \]

\[ \bar{B}_{p} = \begin{bmatrix} 3 \\ 4 \\ \vdots \\ 6 \end{bmatrix} , \quad \text{and} \quad \bar{\theta}_{p} = \begin{bmatrix} \theta \\ \phi \\ \gamma \end{bmatrix} \quad . \quad (37d) \]

\[ i = 1,2,3 \quad \quad j = p,y,r \]

Definitions for \( \bar{\theta} \) and \( \bar{\theta} \) are introduced that conform to the simplifications referred to in the problem statement. These definitions will be used throughout the remainder of the paper. For gimbal set 1:

\[ \bar{\theta}_{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \bar{\theta}_{1} = \begin{bmatrix} 0 \\ \phi \\ 0 \end{bmatrix} \quad . \]

For gimbal set 2:

\[ \bar{\theta}_{2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \bar{\theta}_{2} = \begin{bmatrix} \theta \\ 0 \\ 0 \end{bmatrix} \quad . \]
and for gimbal set 3:

\[
\begin{bmatrix}
\ddot{\varphi}_3 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
\dot{\varphi}_3 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
\varphi_i
\end{bmatrix}.
\]

It is noted that each of the matrices in the matrix product of equation (37a) is non-singular. Thus, the product is non-singular, and the rotational equations of motion, (36), can be reduced to the following form for each gimbal set.

\[
\mathbf{M}_T = [A\omega]_i^{-1} \left\{ \frac{d}{dt} [A\omega]_i^T [\mu] [A\omega]_i + [A\omega]_i^T [\mu] \frac{d}{dt} [A\omega]_i \right\} \begin{bmatrix}
\dot{\varphi}_i \\
- \varphi_i
\end{bmatrix}.
\]

Three rotational equations of motion are obtained for each gimbal set from equation (38). The three equations may be solved for three unknowns. Because a particular computational procedure is anticipated, the equations for each gimbal set are solved for the roll force, \( F_T \), the angle, \( \alpha \), and the angular velocity component that appears.
Gimbal Set 1

The three rotational equations of motion are:

\[ -\dot{\phi}_x I_{yz} = F_a y_{cp} s_a y + F_a z_{cp} c_a \]

\[ 0 = -F_a z_{cp} c_a y - F_a x_{cp} s_a y + 2F_x z_x \quad (39) \]

\[ \dot{\phi}_x I_{xy} = -F_a x_{cp} c_a + F_a y_{cp} s_a y \]

The first and third of equations (39) are solved for

\[ \alpha = \arctan \left( \frac{I_{zy} y_{cp} - I_{xy} x_{cp}}{y_{cp} (I_{zy} c_a y + I_{xy} s_a y)} \right) \quad (42a) \]

The second of equations (39) is solved for

\[ F_x = \frac{F_a s_a (x_{cp} s_a y + z_{cp} c_a y)}{2z_x} \quad (43a) \]

and the third is solved for

\[ \dot{\phi}_x = \pm \sqrt{\frac{F_a (y_{cp} s_a c_a y - x_{cp} c_a)}{I_{xy}}} \quad (44a) \]
Gimbal Set 2

The rotational equations of motion are:

\[ 0 = F_y \, S_a \, S_a \, \frac{\dot{Z}}{\dot{a}} + F_z \, C_a \, \frac{\dot{a}}{\dot{a}} \]

\[ \dot{\phi}_y \, I_{xz} = -F_a \, Z \, S_a \, C_a \, y - F_a \, X \, S_a \, S_a \, y + 2F \, Z \, x \]  

(40)

\[ \dot{\phi}_y \, I_{xy} = -F_a \, X \, C_a + F_a \, Y \, S_a \, C_a \, y \]

The first of equations (40) is solved for

\[ \alpha = \arctan \left( \frac{-Z}{C_p \, S_a \, y} \right) \]  

(42b)

The third of equations (40) is solved for

\[ \dot{\phi}_y = \sqrt{\frac{F \left( X \, C_a - Y \, S_a \, C_a \, y \right)}{I_{xy}}} \]  

(43b)

and the second is solved for

\[ F_x = \frac{\dot{\phi} \, 2I_{xz} + F \, S_a \, (Z \, C_a + X \, S_a \, y)}{2F \, Z \, x} \]  

(44b)
Gimbal Set 3

The three rotational equations of motion are:

\[ \dot{\varphi}_p^2 I_{yz} = F_Y SaSa + F_Z Ca \]
\[ -\dot{\varphi}_p^2 I_{xz} = -F_a z_{cp}SaC_y - F_a x_{cp}SaC_y + 2F_x Z \]
\[ 0 = -F_a x_{cp}C_a + F_a y_{cp}SaC_y \]

The third of equations (41) is solved for

\[ \alpha = \arctan \frac{X_{cp}}{Y_{cp} C_y} \]

The first of equations (41) is solved for

\[ \varphi_p = \pm \sqrt{\frac{F_a (Y_{SaSa} + Z_{Ca})}{F_a x_{cp} Y_{cp} I_{yz}}} \]

and the second is solved for

\[ F_r = \frac{F_a S (Z_{cp} C_y + X_{cp} C_y) - \dot{\varphi}_p^2 I_{xz}}{2Z_x} \]
V. THE RELATIVE VELOCITY CONSTRAINTS

The fact that the relative velocity vector may be written in terms of the three coordinate systems constitutes an algebraic constraint given by

\[ \vec{V}_{vm} = [A_d] \vec{V}_R = [A_a]^T \vec{V}_r \] (16)

where \( \vec{V}_{vm} \) is the relative velocity vector expressed in the missile coordinate system. Vector equation (16) yields three equations for each gimbal set. The three equations of each set are not independent; hence, they may not be solved for three unknowns. For each gimbal set, the three equations are solved for two angular displacements. The uniqueness of these angular displacements is discussed in Appendix B.

Gimbal Set 1

The constraint equations are:

\[
(CRCP+SRSYSP) \ V_{R_x} + (CRSP-SRSYCP) \ V_{R_y} + SRSYV_{R_z} = V_{rmx} \\
(-CYSP) \ V_{R_x} + CYCP \ V_{R_y} + SYV_{R_z} = V_{rmr} \] (45)

\[
(-SRCP+CRSYSP) \ V_{R_x} - (SRSP+CRCPY) \ V_{R_y} + CYV_{R_z} = V_{rmz} 
\]
The first and third of equations (45) are solved for

\[ \text{SP} = \frac{J V_{RY} - V_{RX} \sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \]  

(46a)

and

\[ \text{CP} = \frac{J V_{RX} + V_{RY} \sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(R_{RX}^2 + V_{RY}^2)} \]  

(46b)

where

\[ J = CR \frac{V_{RX}}{\text{rmx}} - SR \frac{V_{RX}}{\text{rmy}} \]

\[ \phi_p = \arctan \frac{\text{SP}}{\text{CP}} \]  

(46c)

As shown in Appendix B, equations (46) may be solved for a unique value of \( \phi_p \) only if

\[ -\pi \leq \phi_p \leq \pi \]

The second set of equations (45) is solved for

\[ \text{SY} = \frac{V_{RZ} - \kappa \sqrt{V_{RZ}^2 - V_{RZ}^2 + \kappa^2}}{(V_{RZ}^2 + \kappa^2)} \]  

(47a)
and

\[ C_Y = \frac{V_{RZ} \sqrt{V_{RZ}^2 - V_{RZ}^2 + k^2}}{(V_{RZ}^2 + k^2)} \]  \hspace{1cm} (47b)

where

\[ K = CP \cdot V_{RY} - SP \cdot V_{RX} \]

\[ \phi_y = \arctan \frac{SY}{CY} \]  \hspace{1cm} (47c)

As shown in Appendix B, equations (47) may be solved for a unique value of \( \phi_y \) only if

\[-\pi \leq \phi_y \leq \pi\]

Gimbal Set 2

The constraint equations are:

\[ V_{RX} \cdot CRCP + V_{RY} \cdot CRSP + V_{RZ} \cdot SR = V_{RMX} \]

\[ -V_{RX} \cdot CPSRSY - V_{RX} \cdot CYSR + V_{RY} \cdot (CYCP-SYRSRSP) + V_{RZ} \cdot SYCR = V_{RMY} \]  \hspace{1cm} (48)

\[ V_{RX} \cdot (SYSP-CYSRCP) - V_{RY} \cdot (SYCP+CYSRSP) + V_{RZ} \cdot CRCY = V_{RMZ} \]
The second and third of equations (48) are combined to give

\[ V_{RX} SP - V_{RX} CP = -V_{rmy} CY + V_{rmz} SY \]

which is solved for

\[ SP = \frac{F V_{RX} + V_{RY} \sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \]  \hspace{1cm} (49a)

and:

\[ CP = \frac{F V_{RY} + V_{RX} \sqrt{V_{RX}^2 - F^2 - V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \]  \hspace{1cm} (49b)

where

\[ F = -V_{rmy} CY + V_{rmz} SY \]

\[ \phi_p = \arctan \frac{SP}{CP} \]  \hspace{1cm} (49c)

As shown in Appendix B, equations (49) may be solved for a unique value of \( \phi_p \) only if

\[ -\pi \leq \phi_p \leq \pi \]
The first of equations (48) is solved for

\[
SR = \frac{V_{rZ} V_{rmx} - G \sqrt{V_{rZ}^2 - V_{rmx}^2 + G^2}}{(V_{rZ}^2 - G^2)}, \quad (50a)
\]

and

\[
SR = \frac{G V_{rmx} + V_{rZ} \sqrt{V_{rZ}^2 - V_{rmx}^2 + G^2}}{(V_{rZ}^2 + G^2)}, \quad (50b)
\]

where

\[
G = V_{rx} CP + V_{ry} SP
\]

\[
\phi_r = \arctan \frac{SR}{CR} \quad (50c)
\]

As shown in Appendix B, equations (50) may be solved for a unique value of \(\phi_r\) only if

\[-\pi \leq \phi_r \leq \pi\]

Gimbal Set 3

The constraint equations are:

\[
V_{rx}(CPCR-SPRSY) + V_{ry}SPCY + V_{rz}(CPSR+SPSYCR) = V_{rmx}
\]

\[-V_{rx}(SPCR+CPSRSY) + V_{ry}CPCY + V_{rz}(-SPSR+CPSYCR) = V_{rmy} \quad (51)
\]

\[-V_{rx}CYSR - V_{ry}SY + V_{rz}CYCR = V_{rmez}\]
The first and second of equations (51) are solved for

\[ SR = \frac{V_{RZ} A - V_{RX} \sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RZ}^2 + V_{RX}^2)} \]  

(52a)

and

\[ CR = \frac{V_{RX} A + V_{RZ} \sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RZ}^2 + V_{RX}^2)} \]  

(52b)

where

\[ A = CP V_{rmx} - SP V_{rmn} \]

\[ \phi_r = \arctan \frac{SR}{CR} \]  

(52c)

As shown in Appendix B, equations (52) may be solved for a unique value of \( \phi_r \) only if

\[ -\pi \leq \phi_r \leq \pi \]

The third of equations (51) is solved for

\[ SY = \frac{-V_{RY} V_{rmz} + B \sqrt{V_{R}^2 - V_{rmz}^2 + B^2}}{(V_{RY}^2 + B^2)} \]  

(53a)
and

\[
CY = \frac{BV_{Mz} + V_{RY} \sqrt{V_{RY}^2 - V_{Mz}^2 + B^2}}{(V_{RY}^2 + B^2)}
\]  

(53b)

where

\[
B = V_{RZCR} - V_{RXSR}
\]

\[
\theta_y = \arctan \frac{SY}{CY}
\]  

(53c)

As shown in Appendix B, equations (53) may be solved for a unique value of \( \theta_y \) only if

\[-\pi \leq \theta_y \leq \pi\]
VI. FORMULATION OF THE OPTIMIZATION PROBLEM

The optimization problem is that of finding the optimum control process, $\alpha_y(t)$, that will transfer a vehicle from an initial state to a terminal state in an atmosphere in a manner so that the functional

$$ J = \int_{t_0}^{t_1} f(X, \dot{X}, \theta, \phi, \psi, \alpha, \alpha_y, P_x) \, dt $$

is a minimum. Since the Pontryagin formulation is to be used, it is necessary to write the Pontryagin H function for each gimbal set (5).

Gimbal Set 1

The Pontryagin H function is

$$ H_1 = \lambda_1 \cdot \dot{X} + \lambda_{II} \cdot \dot{u} + \lambda_7 \theta_x + \lambda_8 J, \quad (55a) $$

which may be expressed as

$$ H_1 = \lambda_1 \cdot \dot{X} + \lambda_{II} \cdot (g \bar{X} + F_a \bar{N}_1) + \lambda_7 \sqrt{\frac{F_a (y_{cp} S_a C_{ay} - x_{cp} C_a)}{I_{xy}}} + \lambda_8 f(\bar{X}, \bar{\theta}, \bar{\psi}, \bar{\phi}, \alpha, \alpha_y, P_x) \quad (55b) $$
where

\[ \bar{\lambda}_I = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \text{ and } \bar{\lambda}_{II} = \begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} \]

The \( \lambda(t) \) are auxiliary variables used in a manner analogous to Lagrangian multipliers in the classical calculus of variations.

Gimbal Set 2

The Pontryagin \( H \) function is

\[ H_2 = \bar{\lambda}_{III} \cdot \dot{X} + \bar{\lambda}_{IV} \cdot \dot{u} + \lambda_{15} \dot{\phi}_y + \lambda_{16} \dot{J} \quad (56a) \]

which may be expressed as

\[ H_2 = \bar{\lambda}_{III} \cdot \dot{X} + \bar{\lambda}_{IV} \cdot (g \dot{X} + F_a \bar{N}_2) \]

\[ \pm \lambda_{15} \sqrt{\frac{F_a (x_{cp} Ca - y_{cp} Sa Ca_y)}{I_{xy}}} + \lambda_{16} f(\bar{X}, \dot{\bar{X}}, \dot{\phi}_2, \bar{\phi}, a, a_y, F_x) \quad (56b) \]

where

\[ \bar{\lambda}_{III} = \begin{bmatrix} \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \end{bmatrix}, \text{ and } \bar{\lambda}_{IV} = \begin{bmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \end{bmatrix} \]
Gimbal Set 3

The Pontryagin H function is

\[ H_3 = \lambda_V \cdot \ddot{X} + \lambda_{VI} \cdot \ddot{u} + \lambda_{23} \dot{\theta} + \lambda_{24} \dot{J} \]  

(57a)

which may be expressed as

\[ H_3 = \lambda_V \cdot \ddot{X} + \lambda_{VI} \cdot (g \ddot{X} + \frac{F^N}{a_3}) \]

\[ = \lambda_{23} \sqrt{F_a (\gamma_{cp} x + z_{cp} a)} \cdot \lambda_{24} f(\ddot{X}, \ddot{\theta}, \dot{\theta_3}, \theta, \alpha, \alpha_y, F_x) \]  

(57b)

where

\[ \lambda_y = \begin{bmatrix} \lambda_{17} \\ \lambda_{18} \\ \lambda_{19} \end{bmatrix} \]

and

\[ \lambda_{VI} = \begin{bmatrix} \lambda_{20} \\ \lambda_{21} \\ \lambda_{22} \end{bmatrix} \]
The expressions for the auxiliary variables are obtained from the H functions as follows:

**Gimbal Set 1**

\[ - \dot{\lambda}_I = \frac{\partial H_I}{\partial x} \]

\[ = \ell \dot{g} \frac{\partial (\bar{\lambda}_{II} \cdot \bar{N})}{\partial x} + (\bar{\lambda}_{II} \cdot \bar{N}) \frac{\partial \tilde{g}^*}{\partial x} - \bar{\lambda}_{II} \tilde{g} \]

\[ + (\bar{\lambda}_{II} \cdot \bar{x}) \frac{\partial \tilde{g}}{\partial x} \pm \lambda \sqrt{\frac{y_{cp} S_{cd} C_{dy} - x_{cp} C_{dx}}{I_{xy}}} \left( \frac{\partial (\ell \dot{g}^*)}{\partial x} \right)^{1/2} \]

\[ + \lambda \theta \frac{\partial \tilde{g}^*}{\partial x} \cdot \text{(58a)} \]

\[ - \dot{\lambda}_{II} = \frac{\partial H_{II}}{\partial u} \]

\[ = \bar{\lambda}_I + \ell \dot{g} \frac{\partial (\bar{\lambda}_{II} \cdot \bar{N})}{\partial u} + (\bar{\lambda}_{II} \cdot \bar{N}) \frac{\partial \tilde{g}^*}{\partial u} \]

\[ + \lambda \sqrt{\frac{y_{cp} S_{cd} C_{dy} - x_{cp} C_{dx}}{I_{xy}}} \left( \frac{\partial (\ell \dot{g}^*)}{\partial u} \right)^{1/2} \]

\[ + \lambda \theta \frac{\partial \tilde{g}^*}{\partial u} \cdot \text{(59a)} \]
\[ -\dot{\lambda}_7 = \frac{\partial H_i}{\partial \phi_r} = \mathcal{F}_o^* \left\{ \bar{\lambda}_{ix} \cdot \frac{\partial \bar{N}_r}{\partial \phi_r} \right\}, \quad (60a) \]

\[ -\dot{\lambda}_8 = \frac{\partial H_i}{\partial J} = 0. \quad (61a) \]

**Gimbal Set 2**

The expressions for the auxiliary variables are:

\[ -\dot{\lambda}_{ix} = \frac{\partial H_2}{\partial x} \]

\[ = \mathcal{F}_o^* \frac{\partial (\bar{\lambda}_{ix} \cdot \bar{N}_2)}{\partial x} + (\bar{\lambda}_{ix} \cdot \bar{N}_2) \frac{\partial \mathcal{F}_o^*}{\partial x} \]

\[ + \bar{\lambda}_{ix} g + (\bar{\lambda}_{ix} \cdot \bar{x}) \frac{\partial g}{\partial x} \]

\[ \pm \lambda_{15} \sqrt{\frac{x_p c_\omega - y_p s_\omega c_\omega}{I_{xy}}} \left\{ \frac{\partial (\mathcal{F}_*^*)}{\partial x} \right\}^{\frac{1}{2}} \]

\[ + \lambda_{16} \frac{\partial \mathcal{F}_o^*}{\partial x}. \quad (58b) \]
\[-\lambda_{14} = \frac{\partial H_2}{\partial u}\]
\[= \lambda_{14} + \frac{\partial \left( \lambda_{14} \cdot \overline{N}_2 \right)}{\partial u} + (\lambda_{14} \cdot \overline{N}_2) \frac{\partial \overline{F}_x^*}{\partial u}
\]
\[\pm \lambda_{15} \sqrt{\frac{y_{cp} S^u L_{y}^2}{\overline{J}_{x_y}} \left\{ \frac{\partial \left( \overline{F}_x^* \right)^{1/2}}{\partial u} \right\}}
\]
\[\pm \lambda_{16} \frac{\partial \overline{F}_x^*}{\partial u}.
\]

(59b)

\[-\lambda_{15} = \frac{\partial H_2}{\partial \phi_y} = \overline{F}_x^* (\lambda_{14} \cdot \overline{N}_2).
\]

(60b)

\[-\lambda_{16} = \frac{\partial H_3}{\partial J} = 0.
\]

(61b)

Gimbal Set 3

The expressions for the auxiliary variables are:

\[-\lambda_{14} = \frac{\partial H_2}{\partial \lambda_0} \]
\[= \frac{\partial \left( \lambda_{14} \cdot \overline{N}_2 \right)}{\partial \lambda_0} + (\lambda_{14} \cdot \overline{N}_2) \frac{\partial \overline{F}_x^*}{\partial \lambda_0} + \lambda_{14} g
\]
\[+(\lambda_{14} \cdot \overline{x}) \frac{\partial \theta}{\partial \lambda_0} \pm \lambda_{14} \sqrt{\frac{y_{cp} S^u S^y + z_{cp} L^u}{I_{yz}}} \left\{ \frac{\partial \left( \overline{F}_x^* \right)^{1/2}}{\partial \lambda_0} \right\}
\]
\[+ \lambda_{14} \frac{\partial \overline{F}_x^*}{\partial \lambda_0}.
\]

(58c)
\[
- \frac{\partial \lambda_{23}}{\partial \phi_p} = \frac{\partial H_3}{\partial \phi_p} = F^* \left\{ \lambda_{23} \cdot \frac{\partial \lambda_3}{\partial \phi_p} \right\},
\]

\[
- \frac{\partial \lambda_{24}}{\partial J} = \frac{\partial H_3}{\partial J} = 0.
\]

Equations (61a), (61b) and (61c) imply that \( \lambda_8 \), \( \lambda_{16} \), and \( \lambda_{24} \) are constant. The constant in each case is taken equal to plus c.e. This insures that a minimization of the \( H \) function is also a minimization of the payoff function.
The necessary condition for a critical value of $J$ is

$$\frac{\partial H_i}{\partial a_y} = 0 \quad (62)$$

where

$$i = 1, 2, 3$$

The inequality

$$\frac{\partial^2 H_i}{\partial a_y^2} \geq 0 \quad (63)$$

must also be satisfied to insure a minimum of the payoff function.

(Note: The criteria expressed in (62) and (63) are valid only if the $H$ function is differentiable at each point on the trajectory.) Partial differentiation of the $H$ functions as indicated in (62) and (63) produces the equations given on the following page.
\[ \frac{\partial H_1}{\partial \alpha_y} = \frac{5}{3} \xi \frac{\partial^3 \bar{W}}{\partial \alpha_y^3} + \lambda \frac{\partial}{\partial \alpha_y} \sqrt{\frac{f_0 (y_{cp} \Sigma_i x_i - x_{cp} \Sigma_i y_i)}{I_{xy}}} + \lambda \frac{2}{2 \alpha_y} f(\vec{x}, \vec{y}, \vec{\phi}, \vec{\theta}, \alpha_y, \alpha_y, F) = 0. \] (64a)

\[ \frac{\partial H_2}{\partial \alpha_y} = \frac{5}{3} \xi \frac{\partial^3 \bar{W}_2}{\partial \alpha_y^3} + \lambda \frac{\partial}{\partial \alpha_y} \sqrt{\frac{f_0 (x_{cp} \Sigma_i y_i - y_{cp} \Sigma_i x_i)}{I_{xy}}} + \lambda \frac{2}{2 \alpha_y} f(\vec{x}, \vec{y}, \vec{\phi}, \vec{\theta}, \alpha_y, \alpha_y, F) = 0. \] (64b)

\[ \frac{\partial H_3}{\partial \alpha_y} = \frac{5}{3} \xi \frac{\partial^3 \bar{W}_3}{\partial \alpha_y^3} + \lambda \frac{\partial}{\partial \alpha_y} \sqrt{\frac{f_0 (y_{cp} \Sigma_i y_i + z_{cp} \Sigma_i x_i)}{I_{yz}}} + \lambda \frac{2}{2 \alpha_y} f(\vec{x}, \vec{y}, \vec{\phi}, \vec{\theta}, \alpha_y, \alpha_y, F) = 0. \] (64c)
\[
\frac{\partial^2 H_1}{\partial y^2} = \bar{\lambda}_1 F \frac{\partial^2 N_2}{\partial y^2} + \lambda_2 \frac{\partial^2}{\partial y^2} \sqrt{\frac{F_0 (x_2 S_1 + x_1 C_2) - x_1 C_1}{I_{xy}}} \\
+ \lambda_8 \frac{\partial^2}{\partial y^2} f(x_2 \bar{x}_2 \bar{\phi}_1, \bar{\phi}_1, \alpha_2, \alpha_2, \Phi_1) > 0. \quad (65a)
\]

\[
\frac{\partial^2 H_2}{\partial y^2} = \bar{\lambda}_2 F \frac{\partial^2 N_2}{\partial y^2} + \lambda_3 \frac{\partial^2}{\partial y^2} \sqrt{\frac{F_0 (x_2 C_2 - y_2 S_2 C_2) + \frac{1}{\alpha_2}}{I_{xy}}} \\
+ \lambda_8 \frac{\partial^2}{\partial y^2} f(x_2 \bar{x}_2 \bar{\phi}_2, \bar{\phi}_2, \alpha_2, \alpha_2, \Phi_1) > 0. \quad (65b)
\]

\[
\frac{\partial^2 H_3}{\partial y^2} = \bar{\lambda}_3 F \frac{\partial^2 N_3}{\partial y^2} + \lambda_4 \frac{\partial^2}{\partial y^2} \sqrt{\frac{F_0 (y_2 S_2 S_2 + y_2 C_2)}{I_{yx}}} \\
+ \lambda_8 \frac{\partial^2}{\partial y^2} f(x_2 \bar{x}_2 \bar{\phi}_3, \bar{\phi}_3, \alpha_3, \alpha_3, \Phi_1) > 0. \quad (65c)
\]

The algebraic and differential constraint equations (28), (42), (43), (44), (46c), (47c), (49c), (50c), (52c), and (53c), and the characteristic equations (58), (59), (60), (61), and (64) form a complete set of equations for the problem. To insure that the payoff function has been minimized, the inequality (65) must also be satisfied.
VII. COMPUTATIONAL PROCEDURE

The problem formulated is of a general nature and the equations involved are quite complex. It is highly improbable that a closed form solution can be found. Therefore, no time has been spent in search of this type solution. A computational scheme is suggested in order that trajectories may be generated on a digital computer. For convenience in the discussion of the computational scheme, the principle equations are written in functional notation.

Gimbal Set 1

The important equations expressed in functional notation are:

\[ \alpha = \alpha(y) \] (66a)

\[ \dot{\phi}_r = \pm \dot{\phi}_r (\alpha, a_y, \bar{X}, \bar{\bar{X}}) \] (67a)

\[ \phi_p = \phi_p (\phi, \bar{X}, \bar{\bar{X}}, \alpha, a_y) \] (68a)

\[ \phi_y = \phi_y (\phi_p, \bar{X}, \bar{\bar{X}}, \alpha, a_y) \] (69a)

\[ \ddot{X} = \ddot{X}(\bar{X}, \bar{\bar{X}}, \bar{\bar{\phi}}, \alpha, a_y) \] (70a)

\[ F_r = F_r (\alpha, a_y) \] (71a)

\[ H_1 = H_1 (\bar{X}, \bar{\bar{X}}, \bar{\bar{\phi}}, \alpha, a_y, \lambda_i) \] (72a)
\[ \dot{\lambda}_i = \dot{\lambda}_i(\overline{X}, \dot{\overline{X}}, \overline{\varphi}, a, a_y, \lambda_i) \]  \hspace{1cm} (73a)

\[ \frac{\partial H_1}{\partial a_y} = \frac{\partial}{\partial a_y} H_1(\overline{X}, \dot{\overline{X}}, \overline{\varphi}, a, a_y, \lambda_i) = 0 \]  \hspace{1cm} (74a)

**Gimbal Set 2**

The important equations expressed in functional notation are:

\[ \alpha = \alpha(a_y) \]  \hspace{1cm} (66b)

\[ \dot{\varphi}_y = \dot{\varphi}_y(a, a_y, \overline{X}, \dot{\overline{X}}) \]  \hspace{1cm} (67b)

\[ \dot{\varphi}_r = \dot{\varphi}_r(\varphi_y, a, a_y, \overline{X}, \dot{\overline{X}}) \]  \hspace{1cm} (68b)

\[ \dot{\varphi}_p = \dot{\varphi}_p(\varphi_x, a, a_y, \overline{X}, \dot{\overline{X}}) \]  \hspace{1cm} (69b)

\[ \ddot{\overline{X}} = \ddot{\overline{X}}(X, \dot{\overline{X}}, \overline{\varphi}, a, a_y) \]  \hspace{1cm} (70b)

\[ F_r = F_r(\varphi_y, a, a_y) \]  \hspace{1cm} (71b)

\[ H_2 = H_2(\overline{X}, \dot{\overline{X}}, \overline{\varphi}, a, a_y, \lambda_i) \]  \hspace{1cm} (72b)

\[ \dot{\lambda}_i = \dot{\lambda}_i(\overline{X}, \dot{\overline{X}}, \overline{\varphi}, a, a_y, \lambda_i) \]  \hspace{1cm} (73b)

\[ \frac{\partial H_2}{\partial a_y} = \frac{\partial}{\partial a_y} H_2(\overline{X}, \dot{\overline{X}}, \overline{\varphi}, a, a_y, \lambda_i) = 0 \]  \hspace{1cm} (74b)
Gimbal Set 3

The important equations expressed in functional notation are:

\[ \alpha = \alpha(y) \quad (66c) \]

\[ \dot{\varphi}_p = \dot{\varphi}_p(\alpha, a_y, \bar{x}, \hat{x}) \quad (67c) \]

\[ \varphi_r = \varphi_r(\varphi_p, a, a_y, \bar{x}, \hat{x}) \quad (68c) \]

\[ \varphi_y = \varphi_y(\varphi_r, a, a_y, \bar{x}, \hat{x}) \quad (69c) \]

\[ \ddot{x} = \ddot{x}(\bar{x}, \hat{x}, \bar{\varphi}, a, a_y) \quad (70c) \]

\[ \ddot{F}_r = \ddot{F}_r(\varphi_y, a, a_y) \quad (71c) \]

\[ H_3 = H_3(\bar{x}, \hat{x}, \bar{\varphi}, a, a_y, \lambda_i) \quad (72c) \]

\[ \lambda_i = \lambda_i(\bar{x}, \hat{x}, \bar{\varphi}, a, a_y, \lambda_i) \quad (73c) \]

\[ \frac{\delta H_3}{\delta a_y} = \frac{\partial}{\partial a_y} H_3(\bar{x}, \hat{x}, \bar{\varphi}, a, a_y, \lambda_i) = 0 \quad (74c) \]

A complete set of equations has been developed for each gimbal set.

Therefore, three independent, but similar, computational procedures are
written. All three computational procedures require the following initial data:

Atmospheric tables for \( \rho \) as a function of position
Atmospheric tables for \( \mathbf{W} \) as a function of position
Aerodynamic tables for \( C_x(a^*) \) and \( C_y(a^*) \) as a function of \( a^* \)

Values for:

\[
\begin{array}{ll}
A & \bar{R}_0 \\
G & \bar{X}_{cp} \\
m & \bar{z}_x \\
N & [\mu]
\end{array}
\]

Plumbline position, \( \bar{X}_o \), and velocity, \( \dot{\bar{X}}_o \), vectors at the initial point on the optimum trajectory.
Computational procedure for Gimbal Set 1

Initial values for the auxiliary variables, $\lambda_i$, and the roll angle, $\varphi_x$, are required. It is assumed that these values are known. These initial data are referred to as:

$$\lambda_{I_0} = \begin{bmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \end{bmatrix}, \quad \lambda_{II_0} = \begin{bmatrix} \lambda_{40} \\ \lambda_{50} \\ \lambda_{60} \end{bmatrix}, \quad \lambda_{70}, \quad \lambda_{80} = 1, \quad \varphi_{x_0}.$$

Preload Computation I

Use the initial data given to compute the following quantities in the order indicated.

1. Choose $\alpha_y = -180^\circ$
2. Choose the positive sign in equation (67a) and compute:
   a. $a$ from (66a); iterate (11c) for $a^*$
   b. $\varphi_x$ from (67a)
   c. $\varphi$ from (68a)
d. \( \gamma \) from (68a)

e. \( \tilde{\alpha} \) from (70a)

f. \( H_1 \) from (72a)

g. \( \frac{\partial H_1}{\partial \alpha_y} \) from (74a)

3. Choose \( \alpha_y = \alpha_y + 5^\circ \) and repeat step 2. Continue until \( \alpha_y = +180^\circ \).

4. Repeat steps 1 through 3 using the negative sign in equation (67a).

The results of Preload Computation I should be tabulated as follows:

<table>
<thead>
<tr>
<th>Eqn. + (67a)</th>
<th>( \alpha_y )</th>
<th>( H_1 )</th>
<th>( \frac{\partial H_1}{\partial \alpha_y} )</th>
<th>Eqn. - (67a)</th>
<th>( \alpha_y )</th>
<th>( H_1 )</th>
<th>( \frac{\partial H_1}{\partial \alpha_y} )</th>
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<tbody>
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</tbody>
</table>

A plot of \( H_1 \) vs \( \alpha_y \) should yield insight as to the number of solutions that exist. In addition, this plot should yield a starting value of \( \alpha_y \) for the iteration of equation (74a).
5. Use the positive sign in equation (67a) and the results of
   Preload Computation I to iterate equation (74a) for $a_y$.

6. Use the $a_y$ computed in step 5 to compute

   \[
   \frac{\partial^2 H_1}{\partial a^2_y}.
   \]

7. If the inequality

   \[
   \frac{\partial^2 H_1}{\partial a^2_y} > 0
   \]

   is satisfied, a minimum exists. Proceed to step 12. Use
   the positive sign in equation (67a) in all remaining cal-
   culations. If the inequality is not satisfied, proceed
   to step 8.

8. Use the negative sign in equation (67a) and the results from
   Preload Computation I to iterate equation (74a) for $a_y$.

9. Use the $a_y$ found in step 8 to compute

   \[
   \frac{\partial^2 H_1}{\partial a^2_y}.
   \]
10. Check to assure that

\[ \frac{\partial^2 l}{\partial a^2} > 0. \]

11. Proceed to step 12. Use the negative sign in equation (67a)
in all remaining calculations.

"W" line computation

12. Use the initial data and the correct sign (as determined in
  Preload Computation II) in equation (67a) to iterate (74a)
  for \( a_y \).

13. Use the \( a_y \) computed in step 12 and the initial data to compute:

  a. \( a \) from equation (66a); iterate (11c) for \( a^* \)
  b. \( \phi_x \) from equation (67a)
  c. \( \phi_y \) from equation (68a)
  d. \( \phi_y \) from equation (69a)
  e. \( \dot{X} \) from equation (70a)
  f. \( F_r \) from equation (71a)
  g. \( H_1 \) from equation (72a)
  h. \( \dot{X}_i \) from equation (73a)
14. Use a numerical integration technique to integrate

\[ \ddot{X} \text{ for } \ddot{X} \text{ for } X, \]

\[ \dot{\phi}_r \text{ for } \dot{\phi}_r, \]

\[ \dot{\xi} \text{ for } \dot{\xi}, \]

\[ \dot{\lambda}_I \text{ for } \dot{\lambda}_I, \]

\[ \dot{\lambda}_{II} \text{ for } \dot{\lambda}_{II}, \]

\[ \dot{\lambda}_7 \text{ for } \dot{\lambda}_7. \]

15. Use the integrated values from step 14 for the new initial values in the "N + 1" line computation.

**Computational procedure for Gimbal Set 2**

Initial values for the auxiliary variables and the yaw angle are required. It is assumed that these values are known. These initial data are referred to as:

\[ \lambda_{II0} = \begin{bmatrix} \lambda_9 \\ \lambda_{10} \\ \lambda_{110} \end{bmatrix}, \quad \lambda_{IV0} = \begin{bmatrix} \lambda_{120} \\ \lambda_{130} \\ \lambda_{140} \end{bmatrix} \]
4. Repeat steps 1 through 3 but use the negative sign in equation (67b).

The results of Preload Computation I should be tabulated as follows:

<table>
<thead>
<tr>
<th>Eqn. + (67b)</th>
<th>$\alpha_y$</th>
<th>$H_2$</th>
<th>$\frac{\partial H_2}{\partial \alpha_y}$</th>
<th>Eqn. - (67b)</th>
<th>$\alpha_y$</th>
<th>$H_2$</th>
<th>$\frac{\partial H_2}{\partial \alpha_y}$</th>
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A plot of $H_2$ vs $\alpha_y$ should yield insight as to the number of solutions that exist. In addition, this plot should aid in selecting an initial value for $\alpha_y$ to be used in the iteration of equation (74b).

Preload Computation II

5. Use the positive sign in equation (67b) and the results of Preload Computation I to iterate equation (74b) for $\alpha_y$.

6. Use the value of $\alpha_y$ found in step 5 to compute

$$\frac{\partial^2 H_2}{\partial \alpha^2_y}$$
7. If the inequality

\[ \frac{\partial^2 H_2}{\partial \alpha^2} > 0 \]

is satisfied a minimum exists. Proceed to step 12. Use the positive sign in equation (67b) in all remaining calculations. If the inequality is not satisfied, proceed to step 8.

8. Use the negative sign in equation (67b) and the results from Preload Computation I to iterate equation (74b) for \( \alpha \).

9. Use the value of \( \alpha \) found in step 8 to compute

\[ \frac{\partial^2 H_2}{\partial \alpha^2} \]

10. Check to assure that

\[ \frac{\partial^2 H_2}{\partial \alpha^2} > 0 \]

11. Proceed to step 12. Use the negative sign in equation (67b) in all remaining calculations.

"N" line computation

12. Use the initial data and the correct sign (as determined in Preload Computation II) in equation (67b) to iterate equation (74b) for \( \alpha \).
13. Use the value of $a_y$ computed in step 12 and the initial data to compute:

a. $a$ from equation (66b); iterate (11c) for $a^*$

b. $\phi_y$ from equation (67b)

c. $\phi_r$ from equation (68b)

d. $\phi_p$ from equation (69b)

e. $\bar{x}$ from equation (70b)

f. $\bar{F}_r$ from equation (71b)

g. $H_2$ from equation (72b)

h. $\dot{\lambda}_{III}$ from equation (73b)

i. $\dot{\lambda}_{IV}$ from equation (73b)

j. $\lambda_{15}$ from equation (73b)

14. Use a numerical integration technique to integrate

$\ddot{x}$ for $\bar{x}$ for $\bar{x}$,

$\dot{\phi}_y$ for $\phi_y$

$\ddot{\lambda}_{III}$ for $\dot{\lambda}_{III}$

$\ddot{\lambda}_{IV}$ for $\dot{\lambda}_{IV}$

$\lambda_{15}$ for $\lambda_{15}$
15. Use the integrated values computed in step 14 for the new initial values in the "N + 1" line computation.

Computational procedure for Gimbal Set 3

Initial values for the auxiliary variables and the pitch angle are required. It is assumed that these values are known. These initial data are referred to as:

\[
\begin{bmatrix}
\lambda_{170} \\
\lambda_{180} \\
\lambda_{190}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\lambda_{200} \\
\lambda_{210} \\
\lambda_{220}
\end{bmatrix}
\]

\[
\lambda_{230} = 1
\]

\[
\phi_{r_0}
\]
Preload Computation I

Use the initial data given to compute the following quantities in the order indicated.

1. Choose \( \alpha_y = -180^\circ \)

2. Choose the positive sign in equation (67c) and compute:
   a. \( \alpha \) from (66c); iterate (1lc) for \( \alpha^* \)
   b. \( \Theta_p \) from (67c)
   c. \( \Theta_r \) from (68c)
   d. \( \Theta_y \) from (69c)
   e. \( \bar{X} \) from (70c)
   f. \( \frac{H}{3} \) from (72c)
   g. \( \frac{\partial H}{\partial \alpha_y} \) from (74c)

3. Choose \( \alpha_y = \alpha_y + 5^\circ \) and repeat step 2. Continue until \( \alpha_y = +180^\circ \).

4. Repeat steps 1 through 3, but use the negative sign in equation (67c).
The results of Preload Computation I should be tabulated as follows:

<table>
<thead>
<tr>
<th>Eqn. + (67c)</th>
<th>$a_y$</th>
<th>$H_3$</th>
<th>$\frac{\partial H_3}{\partial a_y}$</th>
<th>Eqn. - (67c)</th>
<th>$a_y$</th>
<th>$H_3$</th>
<th>$\frac{\partial H_3}{\partial a_y}$</th>
</tr>
</thead>
</table>

A plot of $H_3$ vs $a_y$ should give insight as to the number of solutions that exist. In addition, this plot should aid in selecting an initial value for $a_y$ to be used in the iteration of equation (74c).

**Preload Computation II**

5. Use the positive sign in equation (67c) and the results of Preload Computation I to iterate equation (74c) for $a_y$.

6. Use the value of $a_y$ found in step 5 to compute

$$\frac{\partial^2 H_3}{\partial a_y^2}$$

7. If the inequality

$$\frac{\partial^2 H_3}{\partial a_y^2} > 0$$
is satisfied, a minimum exists. Proceed to step 12. Use the positive sign in equation (67c) in all remaining calculations. If the inequality is not satisfied proceed to step 8.

8. Use the negative sign in equation (67c) and the results from Preload Computation I to iterate equation (74c) for $a_y$.

9. Use the value of $a_y$ found in step 8 to compute

$$\frac{\delta^2 H_3}{\delta a_y^2}$$

10. Check to assure that

$$\frac{\delta^2 H_3}{\delta a_y^2} > 0$$

11. Proceed to step 12. Use the negative sign in equation (74c) in all remaining calculations.

"N" line computation

12. Use the initial data and the correct sign (as determined in Preload Computation II) in equation (67c) to iterate equation (74c) for $a_y$.

13. Use the value of $a_y$ computed in step 12 and the initial data to compute:

   a. $a$ from equation (66c); iterate (11c) for $a^*$

   b. $\phi_p$ from equation (67c)
c. \( \phi_r \) from equation (68c)
d. \( \phi_y \) from equation (69c)
e. \( \bar{X} \) from equation (70c)
f. \( F_x \) from equation (71c)
g. \( H_3 \) from equation (72c)
h. \( \dot{\lambda}_V \) from equation (73c)
i. \( \dot{\lambda}_{VI} \) from equation (73c)
j. \( \lambda_{23} \) from equation (73c)

14. Use a numerical integration technique to integrate

\[ \ddot{X} \text{ for } \dot{X} \text{ for } \bar{X} \text{ for } X \]

\[ \dot{\phi}_p \text{ for } \phi_p \]

\[ \dot{\lambda}_V \text{ for } \bar{\lambda}_V \]

\[ \dot{\lambda}_{VI} \text{ for } \bar{\lambda}_{VI} \]

\[ \dot{\lambda}_{23} \text{ for } \lambda_{23} \]

15. Use the integrated values computed in step 14 for the new initial values in the "N + 1" line computation.
VIII. DISCUSSION

The problem studied has application to cases involving the flight of any "unpowered" vehicle through any atmosphere—subject to the assumptions given in the problem statement. For example, any space vehicle returning to the earth's surface must pass through the earth's atmosphere. This paper provides a method for determining an optimum trajectory for the transfer of the vehicle through the atmosphere. The pay-off function to be minimized over the atmospheric trajectory is a function of the state and control variables. For example, it may be desirable to minimize quantities such as the accumulative aerodynamic drag or the aerodynamic heating.

In order to solve the rotational equations of motion for three unknowns, it was necessary to introduce particular definitions for the angular acceleration, \( \ddot{\theta} \), and the angular velocity, \( \dot{\theta} \), of the vehicle. The definitions essentially eliminate all angular acceleration and two of the three components of the angular velocity for any given gimbal set. Thus, response of equipment and/or crew on the vehicle to a particular angular velocity may dictate choice of gimbal sets.

In the numerical generation of a trajectory, it is possible that an Euler angle will be computed that produces gimbal lock. A trajectory that produces gimbal lock is not admissible since gimbals will not function when in the gimbal lock orientation. Should the situation of gimbal lock arise, a new set of initial values for the auxiliary variables may
be selected and a new trajectory generated. A particular set of auxiliary variables will yield an optimum trajectory for each gimbal set. The trajectory generated will not be the same for each gimbal set even though the same initial values of the auxiliary variables are chosen. No attempt has been made in this paper to determine the initial values of the auxiliary variables for any of the gimbal sets.
BIBLIOGRAPHY


## APPENDIX A

### Experimentally Determined Values of $C_x$ and $C_z$ for a Typical Space Vehicle

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APPENDIX B

Uniqueness of Solution for the Euler Angles

The relative velocity constraint equations are solved for two Euler angles in each gimbal set. The identity

\[ \sin^2 \theta + \cos^2 \theta = 1 \]

is used. Thus, the question arises as to which sign should be used with the radical that appears. This question is answered for each gimbal set by considering the way in which the coordinate systems are defined.

Gimbal Set 1

A first algebraic solution of equations (45) for \( \varphi_p \) and \( \varphi_y \) yields

\[
SP = \frac{J V_{RY} \pm V_{RX} \sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \quad (B1)
\]

and

\[
CP = \frac{J V_{RX} \pm V_{RY} \sqrt{V_{RX}^2 - J^2 + V_{RY}^2}}{(V_{RY}^2 + V_{RY}^2)} \quad (B2)
\]
where

$$J = CR \ V_{xmx} - SR \ V_{zmz}$$

(78)

$$SY = \frac{V_{xmy} \ V_{RZ} + K \sqrt{V_{RZ}^2 - V_{xmy}^2 + K^2}}{(V_{RZ}^2 + K^2)}$$

(78)

and

$$CY = \frac{V_{xmy} \ K + V_{RZ} \sqrt{V_{RZ}^2 - V_{xmy}^2 + K^2}}{(V_{RZ}^2 + K^2)}$$

(78)

where

$$K = CP \ V_{RY} - SP \ V_{RX}$$

The identity

$$SP^2 + CP^2 = 1$$

is satisfied only if opposite signs appear with the radical in (B1) and (B2). Let $\phi_T = \alpha = 0$. Then equations (B1) and (B2) reduce to

$$SP = \frac{\pm V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}}$$

(B5)

and

$$CP = \frac{\pm V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}}$$

(B6)
Consider the positive pitch angle, $\phi$, shown in Appendix Figure 1. Now restrict $\phi_p$, $-\pi \leq \phi_p \leq \pi$.

[Diagram showing coordinate system with positive pitch angle.

Coordinate System Showing A Positive Pitch Angle
Appendix Figure 1

Thus, the correct signs for the sine and cosine are

$$SP = -\frac{V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}}$$ \hspace{1cm} (B7)

and

$$CP = +\frac{V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}}$$ \hspace{1cm} (B8)
The identity

\[ SY^2 + CY^2 + 1 \]

is satisfied only if opposite signs appear with the radical in (B3) and (B4). Let \( \alpha_y = 90 \) and \( \phi_p = 0 \). Then equations (B3) and (B4) reduce to

\[ SY = \pm \frac{V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} \tag{B9} \]

and

\[ CY = \pm \frac{V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} \tag{10} \]

Consider the positive yaw angle, \( \phi_y \), shown in Appendix Figure 2. Now restrict \( \phi_y \), \(-\pi \leq \phi_y \leq \pi\).

Coordinate System Showing A Positive Yaw Angle \( \phi_y \)

Appendix Figure 2
Thus, the correct signs for the sine and cosine are

\[ SY = \frac{-V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} \quad , \quad (B11) \]

and

\[ CY = \frac{+V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} \quad . \quad (B12) \]

Gimbal Set 2

A first algebraic solution of equations (48) for \( \theta_p \) and \( \theta_r \) yields

\[ SP = \frac{F V_{RX} \pm V_{RY} \sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \quad , \quad (3) \]

and

\[ CP = \frac{-F V_{RY} \pm V_{RX} \sqrt{V_{RX}^2 - F^2 + V_{RY}^2}}{(V_{RX}^2 + V_{RY}^2)} \quad (B14) \]

where

\[ F = -V_{TM y} CY + V_{TM z} SY \]
\[
SR = \frac{V_{RZ} V_{TMX} + G \sqrt{V_{RZ}^2 - V_{TMX}^2 + G^2}}{(V_{RZ}^2 + G^2)}
\]

and

\[
CR = \frac{G V_{TMX} + V_{RZ} \sqrt{V_{RZ}^2 - V_{TMX}^2 + G^2}}{(V_{RZ}^2 + G^2)}
\]

where

\[
G = V_{RX} CP + V_{RY} SP
\]

The identity

\[
SP^2 + CP^2 = 1
\]

is satisfied only if the same sign appears with the radical in (B13) and (B14). Let \( \alpha = 0 \) and \( \varphi_y = 90^\circ \). Then equations (B13) and (B14) reduce to

\[
SP = \pm \frac{V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}}
\]

and

\[
CP = \pm \frac{V_{RX}}{\sqrt{V_{RX}^2 + V_{RY}^2}}
\]
Consider the positive pitch angle, $\phi_p$, shown in Appendix Figure 3.

Now restrict $\phi_p$, $-\pi \leq \phi_p \leq \pi$.

Thus, the positive sign is chosen for both the sine and cosine.

\[
SP = \frac{+ V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}}, \quad (B19)
\]

and

\[
CP = \frac{+ V_{RY}}{\sqrt{V_{RX}^2 + V_{RY}^2}}. \quad (B20)
\]
The identity

$$SR^2 + CR^2 = 1$$

is satisfied only if opposite signs appear with the radical in (B15) and (B16). Let $\alpha = \phi_p = 0$. Then equations (B15) and (B16) reduce to

$$SR = \frac{\pm V_{RX}}{\sqrt{V_{RZ}^2 + V_{RX}^2}}$$  \hspace{1cm} (B21)

and

$$CR = \frac{\pm V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RX}^2}}$$  \hspace{1cm} (B22)

Consider the positive roll angle, $\phi_r$, shown in Appendix Figure 4. Now restrict $\phi_r$, $-\pi \leq \phi_r \leq \pi$.

Coordinate System Showing A Positive Roll Angle

Appendix Figure 4
Thus, the correct signs for the sine and cosine are

\[ SR = \frac{-V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} \]  \hspace{1cm} \text{(B23)}

and

\[ CR = \frac{+V_{RX}}{\sqrt{V_{RZ}^2 + V_{RX}^2}} \]  \hspace{1cm} \text{(B24)}

**Gimbal Set 3**

A first algebraic solution of equations (51) for \( \theta_y \) and \( \theta_x \) yields

\[ SR = \frac{V_{RZ} A + V_{RX} \sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RX}^2 + V_{RZ}^2)} \]  \hspace{1cm} \text{(B25)}

where

\[ CR = \frac{V_{RX} A + V_{RZ} \sqrt{V_{RZ}^2 - A^2 + V_{RX}^2}}{(V_{RX}^2 + V_{RZ}^2)} \]  \hspace{1cm} \text{(B26)}

where

\[ A = CP \ V_{\text{max}} - SP \ V_{\text{my}} \]
\[ SY = -V_{RY} V_{xinz} + 3 \sqrt{V_{RY}^2 - V_{xinz}^2 + B^2} \frac{\sqrt{V_{RY}^2 - V_{xinz}^2 + B^2}}{(V_{RY}^2 + B^2)} \quad (B.7) \]

and

\[ CY = \frac{B V_{xinz} + V_{RY} \sqrt{V_{RY}^2 - V_{xinz}^2 + B^2}}{(V_{RY}^2 + B^2)} \quad (B.28) \]

where

\[ B = V_{RZ} CR - V_{RX} SR \]

The identity

\[ SR^2 + CR^2 = 1 \]

is satisfied only if opposite signs appear with the radical in (B25) and (B26). Let \( \alpha = \beta_p = 0 \). Then equations (B25) and (B26) reduce to

\[ SR = \pm \frac{V_{RX}}{\sqrt{V_{AX}^2 + V_{RZ}^2}} \quad (B.29) \]

and

\[ CR = \pm \frac{V_{RZ}}{\sqrt{V_{AX}^2 + V_{RZ}^2}} \quad (B.30) \]

Note, equations (B29) and (B30) are the same as (B21) and (B22). The identity

\[ SR^2 + CR^2 = 1 \]
is satisfied in the same way in each case. Hence, the signs for the sine and cosine are chosen the same as in equations (B27) and (B24).

The identity

$$S_Y^2 + C_Y^2 = 1$$

is satisfied only if the same sign appears with the radical in (B27) and (B28). Let $\alpha = \theta_2 = 0$. Then equations (B27) and (B28) reduce to

$$S_Y = \pm \frac{V_{RZ}}{\sqrt{V_{RY}^2 + V_{RZ}^2}}$$

and

$$C_Y = \pm \frac{V_{RY}}{\sqrt{V_{RY}^2 + V_{RZ}^2}}$$

Consider the positive yaw angle, $\phi_y$, shown in Appendix Figure 5. Now restrict $\phi_y$, $-\pi \leq \phi_y \leq \pi$.

![Coordinate System Showing A Positive Yaw Angle](image-url)

Appendix Figure 5
Thus, the positive sign is chosen for both the sine and cosine.

\[ SY = \frac{V_{RZ}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} \] \hspace{1cm} (B33)

and

\[ CY = \frac{V_{RY}}{\sqrt{V_{RZ}^2 + V_{RY}^2}} \] \hspace{1cm} (B34)
Semi-Annual Report on NASA Grant NGR-01-003-008

RESEARCH ON

DEVELOPMENT OF EQUATIONS FOR PERFORMANCE TRAJECTORY COMPUTATIONS

During the period November 1, 1967, to May 1, 1968, at the suggestion of Mr. W. E. Miner of NASA, ERC, Cambridge, Massachusetts, major emphasis was placed on investigating the analytical foundation of the Hamilton-Jacobi theory from the standpoint of its possible applications of space flight. Several references were obtained, as listed in the back of this report, and a study of previous work by several authors was undertaken.

As of May 1, 1968, a specific problem area had been defined as follows.

To attempt to utilize the first order perturbation theory, which has been developed for the motion of a uniaxial satellite in a gravitational field (reference 8) in studying the motion of a triaxial satellite in a gravity field. Also to expand the theory for the uniaxial case to higher order.
LIST OF REFERENCES BEING STUDIED


AUBURN UNIVERSITY

VEHICLE CONTROL FOR FUEL OPTIMIZATION

By

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Department of Mechanical Engineering

NASA Grant NGR-01-003-008-S-1

Prepared For

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ELECTRONICS RESEARCH CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
CAMBRIDGE, MASSACHUSETTS

November 1967

AUBURN, ALABAMA
VEHICLE CONTROL FOR FUEL OPTIMIZATION

Klaus D. Dannenberg and Grady R. Harmon

Department of Mechanical Engineering
Auburn University

ABSTRACT

The problem considered in this report is that of predicting a minimum fuel trajectory for a six degree of freedom vehicle which has a motion characterized by the first order differential equations of translational and rotational dynamics. The thrust direction and center of gravity of the vehicle are assumed to be fixed with respect to the vehicle. Thrust magnitude and the control moment are used as control variables and appear linearly in the equations of motion.

Pontryagin's Maximum Principle is used to solve the variational problem. With this formulation, the extremal controls are bang-bang with the exception of the singular case. A unique feature of this problem is a combination of nonlinear state and linear control will allow the computation of the initial values of the Lagrange multipliers by an appropriate choice of some of the initial states. Initial values of the multipliers are always necessary for the complete solution, but no process is generally available for their determination.
### NOMENCLATURE

**$A_D$** Matrix transformation from plumbline system to vehicle system

**$A_j$** Exit area of vehicle engines

**$A_w$** Matrix transformation or $\Psi$ vector into vehicle system

**$c$** Abbreviation for cosine

**$\mathcal{C}$** Control variable vector

**$C_D$** Coefficient of drag

**$C_L$** Coefficient of lift

**$\mathcal{F}$** Force vector

**$G$** Gravitational constant

**$H$** Hamiltonian

**$m$** Total mass of vehicle

**$\dot{m}_a$** Mass flow rate of air through vehicle engines

**$m_f$** Mass of vehicle's fuel

**$M$** Mass of earth

**$\mathcal{M}$** Moment vector

**$P_j$** Exit pressure of vehicle engines

**$P_0$** Freestream pressure

**$\mathcal{F}$** Arbitrary vector

**$s$** Abbreviation for sine

**$t$** Time

**$T$** Rotational kinetic energy of vehicle
\( \bar{u} \)  
Translational velocity of vehicle

\( v_j \)  
Exit velocity of air and fuel of vehicle engines

\( v_o \)  
Freestream velocity

\( \bar{x} \)  
Translational position of vehicle

\( \bar{x}_{cp} \)  
Position vector of center of pressure in vehicle system

\( \bar{x} \)  
State variable vector

\( \alpha \)  
Angle between y-axis and relative velocity vector

\( \alpha_y \)  
Angle measured in xz-plane from x-axis, locating plane containing relative velocity vector and y-axis

\( \bar{\lambda} \)  
Lagrange multiplier vector

\( \mu \)  
Inertia tensor of vehicle in vehicle coordinate system

\( \rho \)  
Freestream air density

\( \bar{\Phi} \)  
Eulerian angular position of vehicle

\( \bar{\psi} \)  
Time rate of change of \( \bar{\Phi} \)

\( \bar{\omega} \)  
Angular velocity vector in vehicle coordinate system

**Subscripts**

\( a \)  
Relating to aerodynamic force

\( p \)  
Relating to pitching motion about vehicle's z-axis

\( r \)  
Relating to rolling motion about vehicle's y-axis

\( t \)  
Relating to thrust force

\( v \)  
Relating to vehicle coordinate system

\( y \)  
Relating to yawing motion about vehicle's x-axis
INTRODUCTION

The Maximum Principle is a mathematical optimization process, yielding a continuous set of controls, as contrasted with the computer search technique of optimization. One of the primary drawbacks of the Maximum Principle is the necessity for determining the initial values of the Lagrange multipliers. Since no physical significance is attached to the Lagrange multipliers, a system of assumed initial values is commonly used with the hope that a maximum can be found.

In the problem formulated in this paper, a unique situation arises: the Hamiltonian is linear in the control variables and nonlinear in the state variables. If these nonlinearities are used with appropriate nonrestrictive initial values for some of the states, a set of equations is produced which can be solved for the initial Lagrange multipliers. Thus, a complete extremal solution can be found for the optimization problem presented in this report.
COORDINATE SYSTEMS

Two coordinate systems are used to describe the motion of the vehicle. One of these, the plumbline system, is fixed to the earth's center and is assumed to be a primary inertial system. The other is fixed to the vehicle at the center of gravity and moves with the vehicle. The directions of the vehicle axes are shown in Figure 1. The position of the center of gravity of the vehicle is given by its Cartesian coordinates relative to the plumbline system. The angular orientation is given by a series of three consecutive rotations, which are illustrated in Figure 2. From an initial position in which all axes of the vehicle and plumbline systems are parallel, the following rotations are made about the vehicle's center of gravity:

1) Yawing rotation $\phi_y$ about the x axis
2) Pitching rotation $\phi_p$ about the z axis
3) Rolling rotation $-\phi_r$ about the -y axis

Consequently,

$$\bar{r}_V = [-\phi_r][\phi_p][\phi_y]\bar{r} = [A_D]\bar{r}$$

or

$$\bar{r}_V = \begin{bmatrix} CRCP & CRSPCY - SRSY & CRSPSY + SRCY \\ -SP & CPCY & CPSY \\ -CPSR & -SPSRCY - SYCR & -SRSPSY + CYCR \end{bmatrix} \bar{r}$$
PROBLEM FORMULATION

The minimization of the performance index

\[ \int_0^t \dot{m}_f d\tau \]

will be accomplished through utilization of the Maximum Principle. Thus, for a minimum of

\[ \int_0^t \dot{m}_f dt, \]

a maximum of the Hamiltonian \( H \) is desired, where \( H \) is defined as

\[ H \equiv \bar{\lambda} \cdot \dot{X} \]

where \( \dot{X} \) is the state variable vector and \( \bar{\lambda} \) is the Lagrange multiplier vector.

The state variables chosen for this problem are the translational and rotational position and velocity \( \bar{X}, \bar{u}, \bar{\phi}, \) and \( \bar{\psi} \), respectively. From a knowledge of mechanics, the state equations are as follows:

\[ \begin{align*}
\dot{\bar{X}} & = \bar{u} \\
\dot{\bar{u}} & = \bar{F}/m - \dot{\bar{m}}\bar{u}/m \\
\dot{\bar{\phi}} & = \bar{\psi} \\
\dot{\bar{\psi}} & = [B]\dot{\bar{\Phi}} + [C]\bar{\psi} + [F]\bar{D}
\end{align*} \]
Thus, the Hamiltonian becomes

\[ H = \lambda_0 \dot{\lambda}_f + \lambda_I \cdot \ddot{u} + \lambda_{II} \cdot \left( \frac{\dot{F}}{m} - \frac{\dot{u}}{m} \right) + \lambda_{III} \]

\[ \cdot \bar{\psi} + \lambda_{IV} \cdot \{ [B] \bar{M} + [C] \bar{\psi} + [F] \bar{D} \} \]

After substitution of the forces and moments discussed in the appendix, the Hamiltonian takes the following form:

\[ H = \lambda_0 \left[ \frac{\dot{F}}{m} \frac{\dot{m}a(v_j - v_o)}{v_j} - A_j (p_j - p_o) \right] + \lambda_I \cdot \ddot{u} + \lambda_{II} \]

\[ \left\{ \frac{E}{m^2} + [A] T \frac{\dot{F}}{m} - \frac{GM}{|x|^3} \bar{x} \right. \]

\[ + \frac{m a(v_j - v_o) + A_j (p_j - p_o) - F_t}{v_j m} \left. \right\} \]

\[ + \lambda_{III} \cdot \bar{\psi} + \lambda_{IV} \cdot \{ [B] \bar{M} + [C] \bar{\psi} + [F] \bar{D} \} \]

From the Hamiltonian, the necessary conditions can be obtained as

\[ \bar{\lambda} = - \frac{\partial H}{\partial \bar{x}} \]

Expanding into scalar form, these equations become:

\[ \dot{\lambda}_0 = \lambda_{II} \left\{ \frac{E}{m^2} \frac{\dot{F}_a}{m^2} + [A] T \frac{\dot{F}_t}{m} \right. \]

\[ \left. + \frac{m a(v_j - v_o) + A_j (p_j - p_o) - F_t}{v_j m^2} \right\} \]

\[ \dot{\lambda}_1 = - \frac{1}{m} (\lambda_{II} \cdot E) \frac{\partial F_a}{\partial x} + \frac{GM}{|x|^3} \left( \frac{1}{|x|^3} - \frac{3x^2}{|x|^5} \right) \]

\[ \dot{\lambda}_2 = - \frac{1}{m} (\lambda_{II} \cdot E) \frac{\partial F_a}{\partial y} + \frac{GM}{|x|^3} \left( \frac{1}{|x|^3} - \frac{3y^2}{|x|^5} \right) \]
\[ \dot{\lambda}_3 = -\frac{1}{m} (\vec{\lambda}_{II} \cdot \vec{E}) \frac{\partial F_a}{\partial z} + GM \left( \frac{1}{|\vec{x}|^3} - \frac{3z^2}{|\vec{x}|^5} \right) \]

\[ \dot{\lambda}_4 = -\frac{\lambda_0}{v_j} \frac{\partial v_0}{\partial u} - \lambda_1 - \frac{\vec{\lambda}_{II} \cdot \vec{E}}{m} \frac{\partial F_a}{\partial u} + \frac{\lambda_4}{v_j} \left( v_c + u \frac{\partial v_0}{\partial u} \right) \]

\[ \dot{\lambda}_5 = -\frac{\lambda_0}{v_j} \frac{\partial v_0}{\partial v} - \lambda_2 - \frac{\vec{\lambda}_{II} \cdot \vec{E}}{m} \frac{\partial F_a}{\partial v} + \frac{\lambda_4}{v_j} \left( v_o + v \frac{\partial v_0}{\partial v} \right) \]

\[ \dot{\lambda}_6 = -\frac{\lambda_0}{v_j} \frac{\partial v_0}{\partial w} - \lambda_3 - \frac{\vec{\lambda}_{II} \cdot \vec{E}}{m} \frac{\partial F_a}{\partial w} + \frac{\lambda_4}{v_j} \left( v_o + w \frac{\partial v_0}{\partial w} \right) \]

\[ \dot{\lambda}_7 = -\vec{\lambda}_{II} \cdot \left\{ \frac{\partial E}{\partial \phi_y} \frac{F_a}{m} + \frac{\partial [A_n]^T}{\partial \phi_y} \frac{F_t}{m} \right\} - \vec{\lambda}_{IV} \]

\[ \cdot \left\{ \frac{\partial}{\partial \phi_y} [B] \bar{M} + \frac{\partial}{\partial \phi_y} [C] \bar{\psi} + \frac{\partial}{\partial \phi_y} ([F] \bar{D}) \right\} \]

\[ \dot{\lambda}_8 = -\vec{\lambda}_{II} \cdot \left\{ \frac{\partial E}{\partial \phi_r} \frac{F_a}{m} + \frac{\partial [A_n]^T}{\partial \phi_r} \frac{F_t}{m} \right\} - \vec{\lambda}_{IV} \]

\[ \cdot \left\{ \frac{\partial}{\partial \phi_r} [B] \bar{M} + \frac{\partial}{\partial \phi_r} [C] \bar{\psi} + \frac{\partial}{\partial \phi_r} ([F] \bar{D}) \right\} \]

\[ \dot{\lambda}_9 = -\vec{\lambda}_{II} \cdot \left\{ \frac{\partial E}{\partial \phi_p} \frac{F_a}{m} + \frac{\partial [A_n]^T}{\partial \phi_p} \frac{F_t}{m} \right\} - \vec{\lambda}_{IV} \]

\[ \cdot \left\{ \frac{\partial}{\partial \phi_p} [B] \bar{M} + \frac{\partial}{\partial \phi_p} [C] \bar{\psi} + \frac{\partial}{\partial \phi_p} ([F] \bar{D}) \right\} \]

\[ \dot{\lambda}_{10} = -\lambda - \bar{\lambda}_{IV} \cdot \left( \sum_{i=1}^{3} C_{11} + \frac{\partial}{\partial \phi_y} ([F] \bar{D}) \right) \]

\[ \dot{\lambda}_{11} = -\lambda - \bar{\lambda}_{IV} \cdot \left( \sum_{i=1}^{3} C_{12} + \frac{\partial}{\partial \phi_r} ([F] \bar{D}) \right) \]
\[ \dot{\lambda}_{12} = -\lambda_9 - \lambda_9^{(1)}, \quad \left( \sum_{i=1}^{3} c_{i3} + \frac{3}{\delta \psi_p} \left[ [F] \overline{p} \right] \right) \]

The solution of these equations for \( \overline{\lambda} \) depends on the initial values of \( \overline{\lambda} \). Since no physical significance can be given to the Lagrange multipliers, some method must be developed to determine their initial values. When one realizes that the Hamiltonian is of the form

\[ H = f(X, \overline{\lambda}) + \frac{\partial H}{\partial \overline{F}_t} \overline{F}_t + \frac{\partial H}{\partial \overline{M}} \overline{M}, \]

the possibility arises that the nonlinear function of state can be made zero at the initial time by an appropriate choice of initial state without the necessity of all states being zero. Consequently, since on an optimal path \( H = 0 \), the remainder of the Hamiltonian must be zero; i.e.,

\[ \frac{\partial H}{\partial \overline{F}_t} \overline{F}_t + \frac{\partial H}{\partial \overline{M}} \overline{M} = 0 \]

Since \( \overline{F}_t \) and \( \overline{M} \), in general, are not zero,

\[ \frac{\partial H}{\partial \overline{F}_t} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \overline{M}} = 0 \]

This is the normal necessary condition used for the case of nonlinear controls.

If one chooses the initial state to be a position of rest, i.e., \( \overline{\psi} = 0 \) and \( \overline{u} = 0 \), and if one selects an initial thrust which satisfies the equation

\[ - (\lambda_4 x + \lambda_5 y + \lambda_6 z) \frac{GM}{|x|^3} + \lambda_0 \left( m_a \left( \frac{v_j}{v_i} - v_j \right) + A_i \left( \frac{v_j - \overline{v}_j}{v_j} \right) \right) = 0 \]

the coefficients of the controls are zero at the initial time step, allowing an analytic solution for the unknown initial values of the twelve variable Lagrange multipliers. If one uses these initial values, the given differential equations can be solved for the time history of \( \overline{\lambda} \). Similarly, the state equations can be solved for a time history of the state variables.
Extremal control is determined by the coefficients of the control variables. Since the Hamiltonian is linear in all controls, the extremal control is bang-bang unless the control coefficient is zero; i.e., if

\[ \frac{\partial H}{\partial C_i} > 0, \quad C_i = C_{i\text{MAX}} \quad i = F_t, M \]
\[ \frac{\partial H}{\partial C_i} < 0, \quad C_i = C_{i\text{MIN}} \quad i = F_t, M \]

For the singular control case of a zero coefficient over a non-zero time interval, the equation(s)

\[ \frac{\partial H}{\partial C_i} = 0 \quad i = F_t \text{ or } M_x \text{ or } M_y \text{ or } M_z \]

can be added to the differential multiplier equations over the appropriate time period to solve for the extremal control.
CONCLUSIONS

A set of initial values of the Lagrange multipliers for the state problem can be found analytically through a choice of appropriate initial velocities. This is by no means a unique solution to the problem, but it is a method of making a feasible choice of initial multipliers for a certain realizable initial state. The actual numerical solution of the equations should present no major difficulties if the initial values are no longer a problem.

This method of solving for the initial Lagrange multipliers will not be applicable to most problems. With the selection of an appropriate number of initial states, the problem becomes too restrictive to be of any great general value.
BIBLIOGRAPHY


APPENDIX: MATHEMATICAL MODEL

A mathematical model for the basic mechanics of the problem will be deduced using the separability of the rotational and translational motions of a rigid body. The forces and moments will be discussed first.

A. Forces

An aerodynamic force \( \bar{F}_a \) is assumed to act at the vehicle’s center of pressure. The orientation of the aerodynamic force is determined by two rotations from the vehicle system to a new coordinate system denoted by \( \bar{F}_a \). The rotations align the aerodynamic force with the \(-y_a\) axis. The maneuvers necessary for this alignment (Appendix Figure 1) are:

1) Roll \( \alpha_y \) about the \( y \) axis.
2) Pitch \( \alpha \) about the \( z \) axis to align the \( y \) axis with the relative velocity vector.

Thus, \( \bar{F}_a = [-\alpha][\alpha_y] \bar{F} \).

Appendix Figure 1. Aerodynamic Force System
The magnitude of $\bar{F}_a$ is given by

$$|\bar{F}_a| = \frac{1}{2} v_o^2 A (C_D^2 + C_L^2)^{1/2}$$

A *thrust force* $\bar{F}_T$ is assumed to act along the longitudinal axis of the aircraft. The magnitude of this force is given by

$$|\bar{F}_T| = \dot{m}_a (v_j - v_o) + \dot{m}_f v_j + A_f (p_j - p_o)$$

where $\dot{m}_a$, $v_j$, and $p_j$ are known functions of $|\bar{F}_T|$ for a given engine.

The *gravitational force* of a spherical earth acting at the center of gravity of the vehicle is

$$\bar{F} = -\frac{GMm}{|x|^3} x$$

### B. Moments

An aerodynamic moment and a thrust moment are present as a result of the nonconcurrency of the center of pressure and the center of gravity. Collectively, the moments are

$$\bar{x}_{cp} \times \left\{ \begin{bmatrix} 0 \\ F_a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ F_T \\ 0 \end{bmatrix} \right\}$$

where $\bar{x}_{cp}$ is the position vector of the center of pressure in the vehicle system.

The control surface moment $\bar{M}_F$ is a control of the optimization problem. These are the collective moments resulting from the flaps, ailerons, and all other vehicle control surfaces.

Clasle's theorem for rigid body motion states that the motion may be divided into a pure translation of the center of gravity and a pure rotation about the center of gravity. Therefore, for the translational motion, the following equation results from Newton's law:

$$\dot{\bar{u}} = \frac{\bar{F}}{m} - \frac{\bar{\dot{m}} \bar{u}}{m}$$
or

\[ \dot{\mathbf{u}} = E \frac{F_a}{m} + [A_D]^T \frac{F_T}{m} - \frac{GM}{|x|^3} \mathbf{x} \]
\[ + m_a(v_j - v_o) + A_j(p_j - p_o) - \frac{F_T}{u} \]

where

\[ \mathbf{E} = [\mathbf{\phi}_y]^T [\mathbf{\phi}_p]^T [-\mathbf{\phi}_r]^T [\cdot \mathbf{\alpha}][\cdot \mathbf{a}] \frac{\mathbf{F}_a}{|\mathbf{F}_a|} \]

and

\[ [A_D] = [-\mathbf{\phi}_r][\mathbf{\phi}_p][\mathbf{\phi}_y] \]

The rotational motion equation is obtained from energy considerations. The rotational kinetic energy in matrix form is

\[ \mathbf{T} = \frac{1}{2} \mathbf{\omega}^T [\mu] \mathbf{\omega} \]

where \( \mathbf{\omega} \) is the vehicle-fixed angular velocity vector and \([\mu]\) is the inertia tensor for motion about the vehicle axes. The Lagrangian form for generalized coordinates of angular character is

\[ \frac{d}{dt} \left( \frac{\partial \mathbf{T}}{\partial \dot{\phi}_i} \right) - \frac{\partial \mathbf{T}}{\partial \phi_i} = \mathbf{M}_{\phi_i} \]

When one carries out the indicated operations, the Lagrangian equations become:

\[ \frac{d}{dt} \left( \frac{\partial \mathbf{\omega}^T}{\partial \dot{\phi}_i} \right) [\mu] \mathbf{\omega} + \frac{\partial \mathbf{\omega}^T}{\partial \dot{\phi}_i} [\mu] \frac{d}{dt} \frac{\partial \mathbf{\omega}^T}{\partial \phi_i} [\mu] \mathbf{\omega} = \mathbf{M}_{\phi_i} \]

After substitution of the angular velocity components of \( \dot{\phi}_p \), \( \dot{\phi}_y \), and \( \dot{\phi}_r \) for \( \mathbf{\omega} \) in the vehicle system and simplification, the resulting equation is

\[ \dot{\mathbf{\psi}} = [\mathbf{B}] \mathbf{\omega} + [\mathbf{C}] \mathbf{\psi} + [\mathbf{F}] \mathbf{D} \]
Semi-Annual Report on NASA Grant NGR-01-003-008

RESEARCH ON

DEVELOPMENT OF EQUATIONS FOR PERFORMANCE TRAJECTORY COMPUTATION

SUMMARY

During the second six months of the original one-year period of the grant work has progressed on two projects:

1. Development of a computer program for the study formulated earlier, as discussed in the last report, and
2. An analytical study of a minimum fuel flight for high speed aircraft.

Included in this report are a listing of the program to compute a minimum time re-entry into the atmosphere for an Apollo-type capsule, and a technical summary of the minimum fuel problem. A detailed report on item two is to be presented to the Guidance Laboratory of Electronics Research Center in Cambridge, Massachusetts, on April 19 and 20. A full report will be forwarded to you after this presentation.
A Maximum Principle Re-Entry Study by Grady R. Harmon
As Edited by Frank J. Walker, Jr. - February 1967

Input Card 5: JUMP = 1 PARTS 1 AND 2
Col No 35: = 2 PART 1 ONLY
IS JUMP: = 3 PART 2 ONLY

Symbols Used in Program

Planet Data

GM = Gravitational Constant of Planet
P0 = Radius of Planet
DHo = Density of Planet Atmosphere
DhoD = Partial of Ho W.R.T. Altitude
O'VEGA = Angular Velocity of Planet About Rotation Axis
(WFX, WFY, WfZ) = Angular Velocity Components of the Planet in the
Inertial Frame

Vehicle Data

CX = Aerodynamic Coefficient (Longitudinal Axis)
CXfHO = Partial of CX W.R.T. Alpha
CZ = Aerodynamic Coefficient (Perpendicular to Longitudinal
Axis)
CZfHO = Partial of CZ W.R.T. Alpha
A = Cross-Section of Vehicle
WM = Mass of Vehicle

General Data

(X, Y, Z) = Cartesian Coordinates (Inertial Frame)
(U, V, W) = Velocity Components (Inertial Frame)
R = Magnitude of Radius Vector to Vehicle
HGT = Altitude
(VRX, VRY, VRZ) = Relative Wind Velocity Components (Inertial Frame)
(VRX, VRfY, VRfZ) = Relative Wind Velocity Components (Missile-
Fixed Frame)
VR = Magnitude of Vehicle Velocity Relative to Air
FPA = Aerodynamic Acceleration
GGG = Gravitational Acceleration
H = Pontryagin H Function
PHA = Partial of H W.R.T. Alpha
PHAY = Partial of H W.R.T. Alpha Y

XLM(1) = Lagrange Multiplier (1)
XLM(2) = Lagrange Multiplier (2)
XLM(3) = Lagrange Multiplier (3)
XLM(4) = Lagrange Multiplier (4)
XLM(5) = Lagrange Multiplier (5)
XLM(6) = LAPRANGE MULTIPLIER (6)
XLM(7) = CONSTANT = +1

PREFIX OF R INDICATES ANGLE IS IN RADIANS. OTHERWISE IT IS ASSUMED TO BE IN DEGREES.
PHI0  = INERTIAL FRAME ORIENTATION ANGLE
AO    = INERTIAL FRAME ORIENTATION ANGLE
AND   = (AO - AO)
PHI1  = ROLL ANGLE
PHIY  = YAW ANGLE
PHIP  = PITCH ANGLE
ALFY  = ROLL ANGLE OF VEHICLE (AERODYNAMIC FRAME)
ALF   = ANGLE OF ATTACK OF THE VEHICLE
CRAFL = COS(RAFL)
SRAFL = SIN(RAFL)
CRAFLY = COS(RAFLY)
SRAFLY = SIN(RAFLY)

CPH10 = COS(PHI10)
SPH10 = SIN(PHI10)
CRA09 = COS(RA09)
CPh1R = COS(Ph1R)
SPh1R = SIN(Ph1R)
CPh1P = COS(Ph1P)
SPh1P = SIN(Ph1P)
CPhiY = COS(PhiY)
SPhiY = SIN(PhiY)

EQUIVALENCE (MASCOM(1),Odds(1)) (MASCOM(101) TARS(1)) (MASCOM(669)
1, VFX(1))
EQUIVALENCE (ODDS(1),ALF)(ODDS(2),RALF)(ODDS(3),CRALF)(ODDS(4),
1 SRAFL)(ODDS(5),ALFY)(ODDS(6),RALFY)(ODDS(7),CRAFLY)(ODDS(8),
2 SRAFLY)(ODDS(10),PHI10)(ODDS(11),SPH10)(ODDS(12)
3),(40)(ODDS(13),CR09)(ODDS(14),SRA09)(ODDS(15),PHIP)(ODDS(16)
4 CPh10)(ODDS(17),SP)(ODDS(18),PH1Y)(ODDS(19),CY)(ODDS(20),SY)
5 (ODDS(21),PHIR)(ODDS(22),CR)(ODDS(23),SR)(ODDS(24),UFEA)
6 (ODDS(25),VFX)(ODDS(26),VRY)(ODDS(27),WEZ)(ODDS(28),VR)(ODDS
7 (29),VFX)(ODDS(30),VRY)(ODDS(31),VRZ)(ODDS(32),VRIX)(ODDS(33)
8 VRY)(ODDS(34),VRM)(ODDS(35),VRMPD)(ODDS(36),VRMYD)(ODDS
9 (37),VRYP))
EQUIVALENCE (ODDS(38),CX)(ODDS(39),CXM)(ODDS(40),(ODDS(41),CZ)
1 (ODDS(42),CR)(ODDS(43),CRO)(ODDS(44),RH)(ODDS(45),RHON)
2 (ODDS(46),R)(ODDS(47),R0)(ODDS(47),H)(ODDS(48),A)(ODDS(49)
3 (VW)(ODDS(50),C)(ODDS(51),GG)(ODDS(52),FPA)(ODDS(53),XMDOT)
4 (ODDS(54),H)(ODDS(55),PH)(ODDS(56),PHAY)
EQUIVALENCE (ODDS(57),FA)
EQUIVALENCE (TARS(1),ALT(1))(TARS(89),PRESS(1))
1(TARS(25),ALPHA(1)) (TARS(303),TCZ(1)) (TARS(341),TCZP(1))
2(TARS(37),TCZP(1))(TARS(417),TCX(1))(TARS(457),TCXP(1))
3TARS(495),TCXP(1))
EQUIVALENCE (VFX(1),XM(1))(VFX(4),URDVT(1))(VFX(7),XBAR(1))
1(VFX(10),XLM(1))(VFX(19),XLM(1)(1))(VFX(19),XLM(1)(1))(VFX(19)
2XLM(1)(1))(VFX(22),UR(1))(VFX(2),XLM(7)
EQUIVALENCE (XN(1),XX)(XN(2),XNY)(XN(3),XNZ)
EQUIVALENCE (URDVT(1),UP)(URDVT(2),VD)(URDVT(3),WD)
EQUIVALENCE (XBAR(1),X)(XBAR(2),Y)(XBAR(3),Z)
EQUIVALENCE (XLM(1),XLM(1)(1))(XLM(2),XLM(2))(XLM(3),XLM(3)
EQUIVALENCE (XLM(1)(1),XLM(4))(XLM(1)(1),XLM(5))(XLM(1)(1),XLM(6)

FOUINTERNCE (XLMID(1),XLMID(2),XLMID(3),XLMID(4),XLMID(5),XLMID(6))
FOUINTERNCE (XLMID(1),XLMID(2),XLMID(3),XLMID(4),XLMID(5),XLMID(6))
COMMON MACRO
DIMENSION MACRO(64)
DIMENSION DORS(100),TAB(56),VEX(25)
DIMENSION ALT(44),PRESS(44),X(88)
DIMENSION ALPHAT(38),TCZ(38),TCSZ(38),TCPP(38),TCSZ(38),TCZ(38),TCPX(38)
TCSZ(38),TCPX(38)
DIMENSION UNDOT(3),XN(3),XRAR(3),XLAM1(3),XLMII(3),XLMID(3)
1 XLMID(3),UP(3)
DIMENSION OUT(4*100)
DOUBLE PRECISION FA,AST
DOUBLE PRECISION MACROM,ORDS,TABS,VEX,ALF,RAFl,CRAl,SKAl,ALF,
1 RALF,CRAFl,SPAFl,PHl0,CPHl0,SPHI0,ALF,PHIP,CP,SP,PHIY
2,CR,SY,PHIP,CR,SR,OMEGA,VEX,WFX,WFE,VEZ,VR,VPX,VPX,VRZ,VRX,VRMY,VRMZ
3,VRM,VP,YX,YX,YN,XY,YNX,UN,VV,NN,XY,XZ,XLM1,XLM2,XLM3,XLM4,
7,XLM5,XLM6,XLM11D,XLM21D,XLM3D,XLM40,XLM50,XLM60,
3ZD
DOUBLE PRECISION SRA09,CRAO9,PHIO,RA
DIMENSION STX(2),STY(3),STAY(3),RN(7)
DOUBLE PRECISION STX,STY,STAY,LP,STALF
DOUBLE PRECISION CONA,CONB,CONC,COND,Y0,Y2,Y22,Y22,DFL2
DOUBLE PRECISION HH
DIMENSION HH(2,4)
DIMENSION OF(115),OUTA(45),OUTC(205)
DATA OUTC(2),OUT(3),OUT(4),OUT(5),OUT(7),OUT(8),
1 OUTC(9),OUTC(10),OUTC(11),OUTC(13),OUTC(14),OUTC(15),OUTC(17),
2 OUTC(18),OUTC(19),OUTC(20),OUTC(22),OUTC(23),OUTC(24),OUTC(25),
3 OUTC(27),OUTC(28),OUTC(29),OUTC(30),OUTC(32),OUTC(33),OUTC(34),
4 OUTC(35),OUTC(37),OUTC(38),OUTC(39),OUTC(40),OUTC(42),OUTC(43),
5 OUTC(44),OUTC(45),OUTC(46),OUTC(47),OUTC(48),OUTC(49),OUTC(50),OUTC(51),
6 OUTC(53),OUTC(54),OUTC(55),OUTC(56),OUTC(57),OUTC(58),OUTC(59),OUTC(60),
7 OUTC(62),OUTC(63),OUTC(64),OUTC(65),OUTC(66),OUTC(67),OUTC(68),OUTC(69),
8 OUTC(70),OUTC(71),OUTC(72),OUTC(73),OUTC(74),OUTC(75),OUTC(77),OUTC(78),
9 OUTC(79),OUTC(80),OUTC(80),OUTC(83),OUTC(84),OUT(84),/67*6HBLANKS/;
DATA OUT(85),OUT(87),OUT(88),OUT(89),OUTC(90),
1 OUTC(91),OUTC(92),OUTC(93),OUTC(94),OUTC(95),OUTC(97),OUTC(98),
2 OUTC(99),OUTC(100),OUTC(102),OUTC(103),OUTC(104),OUTC(105),
3 OUTC(107),OUTC(108),OUTC(109),OUTC(110),OUTC(112),OUTC(113),
4 OUTC(114),OUTC(115),OUTC(117),OUTC(118),OUTC(119),OUTC(120),
5 OUTC(121),OUTC(123),OUTC(124),OUTC(125),OUTC(127),OUTC(128),
6 OUTC(129),OUTC(130),OUTC(132),OUTC(133),OUTC(134),OUTC(135),
7 OUTC(137),OUTC(138),OUTC(139),OUTC(140),OUTC(142),OUTC(143),
8 OUTC(144),OUTC(145),OUTC(147),OUTC(148),OUTC(149),OUTC(150),
9 OUTC(157),OUTC(153),OUTC(154),OUTC(155),/57*6HBLANKS/;
DATA OUT(157),OUT(158),OUT(159),OUT(160),OUTC(162),
1 OUTC(163),OUTC(164),OUTC(165),OUTC(167),OUTC(168),OUTC(169),
2 OUTC(170),OUTC(172),OUTC(174),OUTC(175),OUTC(177),
3 OUTC(178),OUTC(180),OUTC(182),OUTC(183),OUTC(184),
4 OUTC(185),OUTC(187),OUTC(188),OUTC(189),OUTC(190),OUTC(192),
5 OUTC(193),OUTC(194),OUTC(195),OUTC(197),OUTC(198),OUTC(199),
READ IN DATA
READ IN HYPERSONIC DATA TABLE
DO 8000 I=1,89
READ(5,1000) RD(1),RD(2),RD(3),RD(4),RD(5),RD(6),RD(7),J(I)
ALPHAT(I) = DBLE(RD(I))
TC7(I) = DBLE(RD(2))
TCZP(I) = DBLE(RD(3))
TCZPP(I) = DBLE(RD(4))
TCX(I) = DBLE(RD(5))
TCXP(I) = DBLE(RD(6))
DO 10 FORMAT(F10.3,F10.3,F10.2,F10.2,F10.2,F10.2)
DO 120 I=1,89
IF(J(I)=1) 101,120,101
CONTINUE
GO TO 100
10 WRITE(*,100)
20 FORMAT(1H1,1X,F8.2) DATA CARDS OUT OF ORDER
GO TO 888
READ IN ALTITUDE VS DENSITY TABLE
DO 1000 PROP = 1,88
READ(5,1000) RD(1),RD(2),K(I)
ALPHA(I) = DBLE(RD(I))
10 PRESS(I) = DBLE(RD(2))
REPRODUCIBILITY OF THE ORIGINAL PAGE IS POOR.

01 FORMAT(F10.0,E11.4,4B13)
02 DD IZ2 I=1,98
03 IF(K(I)-1) 101*122,101
04 CONTINUE
05 READ INPUT PARAMETERS
06 CONTINUE
07 READ(5,1002) RD(1),RD(2),RD(3),RD(4),RD(5),L1
08 GY = DABLE(RD(1))
09 GO = DABLE(RD(2))
10 OVEGA = DABLE(RD(3))
11 AO = DABLE(RD(4))
12 PHI0 = DABLE(RD(5))
13 FORMAT(3F20.8,2F4.0,2X,12)
14 IF(L1-1)101,124,101
15 CONTINUE
16 GO TO 102
17 READ(5,1003) RD(1),RD(2),RD(3),RD(4),L2
18 ALE = DABLE(PD(1))
19 ALEY = DABLE(PD(2))
20 V = DABLE(RD(3))
21 A = DABLE(PD(4))
22 FORMAT(4F10.4,2CX,12)
23 IF(L2-2)101,126,101
24 CONTINUE
25 GO TO 103
26 READ(5,1004) RD(1),RD(2),RD(3),RD(4),RD(5),RD(6),L3
27 XO = DABLE(RD(1))
28 YO = DABLE(RD(2))
29 ZO = DABLE(RD(3))
30 UO = DABLE(RD(4))
31 VO = DABLE(RD(5))
32 AO = DABLE(RD(6))
33 FORMAT(3F10.0,2F10.3,10X,12)
34 IF(L3-3)101,128,101
35 CONTINUE
36 READ(5,1005) RD(1),RD(2),RD(3),RD(4),RD(5),RD(6),RD(7),L4
37 XLM10 = DABLE(RD(1))
38 XLM20 = DABLE(RD(2))
39 XLM30 = DABLE(RD(3))
40 XLM40 = DABLE(RD(4))
41 XLM50 = DABLE(RD(5))
42 XLM60 = DABLE(RD(6))
43 XLM70 = DABLE(RD(7))
44 FORMAT(7F10.3,12)
45 IF(L4-4)101,130,101
46 CONTINUE
47 GO TO 135
48 READ(5,1010) RD(1),RD(2),RD(3),JUMP,IFF,L5
49 TPRINT = DABLE(PD(1))
50 TPRINT = DABLE(PD(2))
51 TSTEP = DABLE(PD(3))
52 TPRINT MUST BE GREATER THAN OR EQUAL TO TSTEP
53 FORMAT(3F10.0,2F5.3,10X,12)
54 IF(L5-5)101,137,101
55 CONTINUE
56 IF((JUMP.EQ.2).OR.(JUMP.EQ.3)) GO TO 138
57 JUMP=1
58 CONTINUE
59 WRITE(6,1493)
60 FORMAT(111)
PRINT INPUT

OUTA(1) = SNGL(VM)
OUTA(2) = SNGL(A)
OUTA(3) = SNGL(GM)
OUTA(4) = SNGL(P0)
OUTA(5) = SNGL(XO)
OUTA(6) = SNGL(YO)
OUTA(7) = SNGL(ZO)
OUTA(8) = SNGL(II0)
OUTA(9) = SNGL(VO)
OUTA(10) = SNGL(WO)
OUTA(11) = SNGL(XLAM10)
OUTA(12) = SNGL(XLAM20)
OUTA(13) = SNGL(XLAM30)
OUTA(14) = SNGL(XLAM40)
OUTA(15) = SNGL(XLAM50)
OUTA(16) = SNGL(XLAM60)
OUTA(17) = SNGL(XLAM7)
OUTA(18) = SNGL(TSTEP)
OUTA(19) = SNGL(TPRINT)
OUTA(20) = SNGL(TLIMIT)
OUTA(21) = SNGL(ALF)
OUTA(22) = SNGL(ALFY)
OUTA(23) = SNGL(Q*FGA)
CALL CONV(OF,OUTA,23)
WRITE(6,6954)

FORMAT(1X,27HINPUT VALUES ARE AS FOLLOWS)
WRITE(6,6955) (OE(1L),LL=1,115)

INITIALIZE PROGRAM

CONTINUE
CALL TRAP
U=1.0
V=VO
W=WO
X=XO
Y=YO
Z=ZO
XLAM1=XLAM10
XLAM2=XLAM20
XLAM3=XLAM30
XLAM4=XLAM40
XLAM5=XLAM50
XLAM6=XLAM60
CALL JACOR(HH,-5.0,-5.0)
CALL INVPS(HH,INDX,100,2,4,KFRR)
DO 200 I=1,2
DO 200 N=1,2
DO HH(1,N)=-HH(1,N)
WRITE(6,299)
FORMAT(1HO,31HTHE FOLLOWING ARE VALUES FOR HH)
WRITE(6,31) ((HH(N+1),I=1,2),N=1,2)
FORMAT(2F20,5)
IF(JUMP,F0,3) GO TO 4002

}
THE FOLLOWING IS PART I AS CALLED FOR BY JUMP

AST = ALF
AST = ALFY
ALFY = -180.0
DO 4577 NN = 1, 71
ALFY = ALFY + 5.0
ALF = -180.0
DO 4500 JX = 1, 71
ALF = ALF + 5.
CALL HCALC
OUTD(1, JX) = SNGL(ALF)
OUTD(2, JX) = SNGL(H)
OUTD(3, JX) = SNGL(PHAY)
OUTD(4, JX) = SNGL(PHA)
IF (IFF) 4522 END
WRITE (6, 4511) ALFY
1 FORMAT (79H1 ALF ALFY = F6.1//)
WRITE (6, 4510) (OUTD(KKK, LLL), KKK = 1, 4), LLL = 1, 71
10 FORMAT (1HO, F10.2, 3E20.8)
CONTINUE
WRITE (6, 1493)
IF (JUMP.EQ.2) GO TO 888
JUMP = 3
ALF = AST
ALFY = AST1
GO TO 4001
CONTINUE

THE FOLLOWING IS PART II AS CALLED FOR BY JUMP

WRITE (6, 4534)
34 FORMAT (1HO, 16HITERATIONS BEGIN/)
30 TIPRC = 0.
TY = TPRINT
1 JZ = 0
ITERATE FOR ALPHAY
30 CONTINUE
IF (TIPRC.GE.TLIMIT) GO TO 888
COLAT = ACOS(DARS(Z)/DSORT(X*X + Y*Y + Z*Z))
VLAT = DSIGN((1.570796 - COLAT)*Z)*57.2958
VLONG = (DATAN2(Y, X) - OMEGA*TIPRC)*57.2958
CALL SLVMN(ALF, ALFY, HH, PHA, PHAY, 1, F-14, 28, TIPRC)
88 CONTINUE
CALL PPHY
IF (TIPRC, EQ.0.0) GO TO 8008
TY = TY + TSFP
IF (TY.LT.TPRINT) GO TO 6448
5 CONTINUE
TY = C.

PRINT OUTPUT
DOUBLE PRECISION RA09
DOUBLE PRECISION FA
DOUBLE PRECISION STALF,ALPHA1,C,CXMD1,CZMD1,DALF,B
DOUBLE PRECISION ARCS,COLAT,CRIT,DEL,TESP,TIREST,TLIMIT,TPRINT,
1TSTEP,XY,U0,
2 V0,VLAT,VLONG,WD,WD,WD,WD,WD,
320
DOUBLE PRECISION SRA09,CRA09,RPHIO,RAD
DOUBLE PRECISION CONA,CONB,CONC,YO,Y2,Y2,Y2,DEL2
RAD = 3.1415926535897932 / 180.0 ARCS (X) = DATAN2(DSORT(1. - X*X),X)
RPHIO = PHERI * RAD
RA09 = (YO - AN) * RAD
RALF = ALF * RAD
RALFY = ALFY * RAD
CRA09 = ARCS(RA09)
SRA09 = DSIN(RA09)
CRA09 = DSIN(RA09)
SRA09 = DSIN(RA09)
CALCULATE C=0,FA-F-bar
FX = CPHI0*SRA09*OMEGA
FY = SPHI0*OMEGA
FZ = -CPHI0*CRA09*OMEGA
CALCULATE VR
VRX = Y*WEZ-Z*WFY+U
VRY = Z*WEZ-X*VFZ+V
VRZ = X*WFY-Y*VFZ+W
VR = DSORT(VRX*VRX + VRY*VRY + VRZ*VRZ)
VR = VR
P = DSORT(X*X + Y*Y + Z*Z)
C = DSORT(VRX*VRX + VRZ*VRZ)
CALCULATE ALTITUDE
HGT= Ritch-RO
GGG=-G*V/R**3
CALCULATE VR=W-bar
VRX=VR*SRA09*CRA09R
VRY= VR*CRA09R
VRZ=-VR*SRA09*CRA09R
STALF= ALF
IF(ALF.LT.0.0.) ALF=-ALF
IF(ALF.LT.180.0.) GO TO 1390
ALF=ALF-360.0.
GO TO 1380
0 CONTINUE
DO 140 I=1,96
J=I+2
IF(ALPHAT(J).GE.ALF) GO TO 141
0 CONTINUE
WRITE(6,143)
3 FORMAT(11H1,15X,42HPROGRAM DUMPED BECAUSE ALPHAT IS LESS THAN/16X,
17HALF AS COMPUTED BY SUBROUTINE PRELOR.)
STOP
REPRODUCIBILITY OF THE ORIGINAL

DEL=(ALF-ALPHAT(J-1))/5
DEL2=DEL*DEL
CX=TCX(J-1)*5*(TCX(J)-TCX(J-2))*DEL+.5*(TCX(J)-2*TCX(J-1)+TCX(J-12))*DEL2
CZ=TCZ(J-1)*5*(TCZ(J)-TCZ(J-2))*DEL+.5*(TCZ(J)-2*TCZ(J-1)+TCZ(J-12))*DEL2
CX3=TCXP(J-1)*5*(TCXP(J)-TCXP(J-2))*DEL+.5*(TCXP(J)-2*TCXP(J-1)
1+TCZP(J-2)))*DEL2
CX7=TCZP(J-1)*5*(TCZP(J)-TCZP(J-2))*DEL+.5*(TCZP(J)-2*TCZP(J-1)
1+TCXP(J-2)))*DEL2
DO 202 I=1,N6
J=I+2
IF(ALT(J)*GF.HGT) GO TO 203
CONTINUE
WRITE(6,204)
FORMAT(11H1,15X,39HPROGRAM DUMPED BECAUSE ALT IS LESS THAN/16X,
137HHT AS COMPUTED BY SURROUND PRELOAD.)
STOP
3 YO=ALT(J-1)-ALT(J-1)
Y2=ALT(J)-ALT(J-1)
CONA=YO*Y2*(Y2-Y0)
Y02=YO*Y2
Y22=Y2*Y2
CONC=Y22*PRESS(J-1)+(Y02-Y22)*PRESS(J-1)-Y02*PRESS(J)
CONC=-Y22*PRESS(J-1)+(Y02-Y22)*PRESS(J-1)+Y02*PRESS(J)
CONB=CONA/CONA
CONC=CONC/CONA
DFL=HCT-ALT(J-1)
RHO=PRESS(J-1)+CONB*DEL+CONC*DEL*DEL
RHO=CONA+2.*CONC*DEL
ALF=STALF
CC=DSORT(CX,CX+CZ*CZ)
FPA=(A/(2.0*VM))RHO*VR*VR*CC
FA=FPA*VM
X=DOM=FPA**2
SPALFP=CZ/CC
CPALFP=CX/CC
CALCULATE PHI-P
VRMPO1=DSQRT(VRMX*VRMY+VRMY*VRMY)
SP=VRMX/VRMPO1
CP=VRMY/VRMPO1
PHIP=DATA0(SP,CP)
CALCULATE PHI-Y
VRMPO1=DATA01(SY,Y)
CALCULATE PHI-R
VRMPO1=DSQRT(VRX+VRZ+VRZ*VRZ)
SR=VRX/VRMPO1
CP=VRZ/VRMPO1
PHIP=DATA0(VR**2,SR,CP)
XNAX=- (CP*SR+SP*SY*SR)*SRALFP*CPALFP+(SP*CR-CP*SY*SR)*CPALFP+CY*SR*
1SRALFP*SRALFY
XNY=- (SP*CR-SP*SY*CR)*SRALFP*CPALFP-(SY*SRALFP*SRALFY)
XNZ=(CP*SR-SP*SY*CR)*SRALFY*SRALFP-(SP*SR+CP*SY*CR)*SRALFP
1+CY*CR*SRALFP*SRALFY
RETURN
END

XTC GET
END

SUBROUTINE HCALC

EQUIVALENCE (MASC0M(1),ODDS(1)),(MASC0M(101),TAB0(1)),(MASC0M(169)
1,VFX(1))

EQUIVALENCE (ODDS(1),ALF),((ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4)
1,RALF),(ODDS(5),RALFY),(ODDS(6),RALFY),(ODDS(7),RALFY),(ODDS(8)
2,SRALFY),(ODDS(10),PHI0),(ODDS(10),PHI0),(ODDS(11),PHI0),(ODDS(12)
3).APRO),(ODDS(14),SRA09),(ODDS(15),PHI0),(ODDS(16),PHI0),(ODDS(17)
4),(ODDS(17),SP),(ODDS(18),PHI0),(ODDS(19),SY)

EQUIVALENCE (ODDS(21),PHI0),(ODDS(22),CR),(ODDS(23),SR),(ODDS(24),OMEGA)
6),(ODDS(25),MEX),(ODDS(27),MEX),(ODDS(27),MEX),(ODDS(28),VR),(ODDS
7),(ODDS(29),VR),(ODDS(30),VR),(ODDS(31),VR),(ODDS(32),VR),(ODDS(33)
8),(ODDS(34),VR),(ODDS(35),VR),(ODDS(36),VE),(ODDS(37),VR),(ODDS
9),(ODDS(37),VR),(ODDS(38),CX),(ODDS(39),CX),(ODDS(40),CZ)
1),(ODDS(41),CZ),(ODDS(42),CC),(ODDS(43),RO),(ODDS(44),RO),(ODDS
2),(ODDS(45),RO),(ODDS(46),RO),(ODDS(47),HGT),(ODDS(48),A),(ODDS(49)
3),VR),(ODDS(50),CZ),(ODDS(51),CZ),(ODDS(52),CP),(ODDS(53),XNP04)
4),(ODDS(54),R),(ODDS(55),PHI),(ODDS(56),PHAY)

EQUIVALENCE (ODDS(57),FA)

EQUIVALENCE (TASS(11),ALT111),(TASS(89),PRFSS(1))

EQUIVALENCE (TASS(265),ALPHAT(11)),(TASS(304),TCSZ(11)),(TASS(141),TCZP(11)
1),(TASS(179),TCSZP(11)),(TASS(417),TCZ(11)),(TASS(457),TCXP(11))

EQUIVALENCE (VFX(1),XN(1)),(VFX(4),URB0T(1)),(VFX(7),XBAR(1))
1),(VFX(10),XLAM(1)),(VFX(13),XLAM(1)),(VFX(16),XLMID(1)),(VFX(19)
2),(VFX(22),XN(2)),(VFX(25),XLM7)

EQUIVALENCE (XN(1),XN(1)),(XN(1),XN(2)),(XN(1),XN(3)),(XN(1),XN(3))

EQUIVALENCE (URB0T(1),UNJ),(URB0T(2),UNJ),(URB0T(3),UNJ)

EQUIVALENCE (XBAR(1),Z),(XBAR(2),Z),(XBAR(3),Z)

EQUIVALENCE (XLAM(1),XLAM(1)),(XLAM(2),XLAM(2)),(XLAM(3),XLAM(3))

EQUIVALENCE (XLAM(1),XLAM(1)),(XLAM(2),XLAM(2)),(XLAM(3),XLAM(3))

EQUIVALENCE (XLAM(1),XLAM(1)),(XLAM(2),XLAM(2)),(XLAM(3),XLAM(3))

EQUIVALENCE (XLAM(1),XLAM(1)),(XLAM(2),XLAM(2)),(XLAM(3),XLAM(3))

10)

COMMON MASC0M

DIMENSION MASC0M(693)

DIMENSION ODDS(100),TAB0(568),VFX(25)

DIMENSION ALPHT(88),PRESS(47)

DIMENSION ALPHAT(38),TCSZ(38),TCZP(38),TCZP(38),TCX(38),TCXP(38)

1)

DIMENSION URB0T(3),XN(3),XBAR(3),XLAM(3),XLAM(3),XLAM(3)

XLAM(3),XBAR(3)

DOUBLE PRECISION DEL1, DELA2, H2

DOUBLE PRECISION DEL1, DEL2, DEL3, DEL4, DEL5, DEL6

DOUBLE PRECISION MASC0M, ODDS, TAB0, VFX, ALF, RALF, CRALF, SRALF, ALFY,
1, RALFY, CRA0F, CRALFY, SRLFY, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
2, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
3, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
4, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
5, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
6, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
7, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
8, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
9, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0
10, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0, PHI0

DOUBLE PRECISION SRA09, CPR09, RPHI0, RA0

DEL = 1

STORE = ALF
ALF = STORE + DEL
CALL PRELOD
CALL GETH
DA=H
ALF = ALF + DEL
CALL PRELOD
CALL GETH
DA=H
ALF = STORE - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF-DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = STORE + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = STORE - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = STORE - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = STORE + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = STORE - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = STORE + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = STORE - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF - DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = STORE + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF + DEL
CALL PRELOD
CALL GETH
DA=DA-H
ALF = ALF - DEL
CALL PRELOD
CALL GETH
DA=DA-H

PHAY = (-75*DA - .15*DB + DC/60.) / DEL
SDA = SNGL(DA)
SBR = SNGL(DB)
SDC = SNGL(DC)
WRITE(6,1000) SDA, SBR, SDC
FORMAT(*6H VALUES DA, DB, DC USED TO COMPUTE PHA/10X,3(E14.8,5X))
EQUIVALENCE (XN(1),XMX(1),XN(2),XNY(1),XN(3),XNZ)
EQUIVALENCE (URDST(1),UN), (URDST(2),VD), (URDST(3),WD)
EQUIVALENCE (XRAR(1),X), (XRAR(2),Y), (XRAR(3),Z)
EQUIVALENCE (XLM1(1),XLM1(2),XLM2(1),XLM2(2),XLM3(1),XLM3(2)
EQUIVALENCE (XLM1(1),XLM4(2),XLM1(2),XLM5(2),XLM2(1),XLM3(2)
EQUIVALENCE (XLM1(1),XLM2(2),XLM3(2),XLM4(2),XLM5(2),XLM6(2)
EQUIVALENCE (XLM1(1),XLM4(2),XLM5(2),XLM6(2)
COMMON MASC
DIMENSION MASCOW(693)
DIMENSION ODDS(100), TARS(568), VFX(25)
DIMENSION ALT(88), PRESS(88)
DIMENSION ALPHAT(38), TCZ(38), TCZP(38), TCZPP(38), TCX(38), TCXP(38)
1TCZP(38)
DIMENSION URRDST(3), XN(3), XRAR(3), XLM1(3), XLM2(3), XLM3(3)
1XLMD1(3), XR(3)
DIMENSION DERIV(14)
DOUBLE PRECISION DERIV
DOUBLE PRECISION MASCOW, ODDS, TARS, VFX, ALF, RLF, CRALF, SRLALF, ALFY
1 RALF, CRLFY, SRLFY, PHIO, PHIP, PHIO, A0, A09, CA09, PHI, CP, SP, PHLY
2 CY, SY, PHIR, CP, SR, OMEGA, WFX, WEY, WZ, VRX, VRY, VZ, VRXW, VRYW, VRHZ
3 VRMDP1, VR, VRMD21, CK, XKM2, CZ, C2M2, CC, RHD, RHOD, R, RG, HGT, A
4 VR, GM, GGG, FPA, XDOT, HPHI, PHAY, ALF, PRESS, ALPHAT, TCZ
5 TCZP, TCZPP, TCX, TCXP, TCZP, UNURDST, UAR, XLM1, XLM2, XLM3, XLM4
6 VR, XLM5, XLM6, XLM7, XLM8, XLM9, U, V, W
7 XLM5, XLM6, XLM7, XLM8, XLM9, XLM5, XLM5, XLM5, XLM5, U, V, W
8 SRLAFP, CRALP
DOUBLE PRECISION FA
DOUBLE PRECISION SRA09, CRA09, RPHIO, RAD
DO 1000 I=1,3
DERIV(I)=UR(I)
DERIV(4)=URDST(1)
DERIV(6)=XLM1(1)
0 DERIV(9)=XLM1D1
RETURN
END
TC STF1
SUBROUTINE STF1(XVAL)
EQUIVALENCE (MAJCOM(1), ODDS(1)), (MAJCOM(10), TARS(1)), (MAJCOM(669)
1), VFX(1))
EQUIVALENCE (ODDS(1), ALF), (ODDS(2), RLF), (ODDS(3), CRALF), (ODDS(4)
1), SRLALF, (ODDS(5), ALFY), (ODDS(6), RALF), (ODDS(7), CRLFY), (ODDS(8)
1), 2 SRLFY, (ODDS(9), PHIO), (ODDS(10), PHIP), (ODDS(11), PHIO), (ODDS(12)
3), A0, (ODDS(13), CRA09), (ODDS(14), SRA09), (ODDS(15), PHI), (ODDS(16)
4), C1, (ODDS(17), SP), (ODDS(18), PHIT), (ODDS(19), CY), (ODDS(20), SY)
5 (ODDS(21), PHR), (ODDS(22), CR), (ODDS(23), SR), (ODDS(24), OMEGA)
6 (ODDS(25), WFX), (ODDS(26), WEY), (ODDS(27), WZ), (ODDS(28), VR), (ODDS
7), VRX), (ODDS(29)), (ODDS(30), VRZ), (ODDS(31), VRZ), (ODDS(32), VRX), (ODDS(33)
8), VRY), (ODDS(34), VRIZ), (ODDS(35), VRMPD1), (ODDS(36), VRWYD1), (ODDS
9), VRMDP1)
EQUIVALENCE (ODDS(38), CX), (ODDS(39), CXMD), (ODDS(40), CZ)
1 (ODDS(41), CZMD), (ODDS(42), CC), (ODDS(43), RH), (ODDS(44), RHON)
2 (ODDS(45), RW), (ODDS(46), RO), (ODDS(47), HGT), (ODDS(48), A), (ODDS(49)
3), 2 VMM, (ODDS(40), GM), (ODDS(41), GGG), (ODDS(52), FPA), (ODDS(53), XMDOT)
4 (ODDS(54), H), (ODDS(55), PHA), (ODDS(56), PHAY)
EQUIVALENCE (ODDS(57), FA)
EQUIVALENCE (TARS(1), ALT(1)), (TARS(89), PRESS(1))
1 (TARS(265), ALPHAT(1)), (TARS(303), TCZ(1)), (TARS(341), TCZP(1)),
COMMON

DIMENSION MASCOM(693)
DIMENSION ODDS(100),TARS(568),VFX(25)
DIMENSION ALT(88),PRESS(88)

DO 1000 1=1,3
XVAL(1)=XRAR(1)
XVAL(1+3)=UR(1)
XVAL(1+6)=XVAL(I)
10 XVAL(1+9)=XVALI(1)
RETURN

END

SURROUNDF STF3(XVAL)

EQUIVALENCE (MASCOM(1),ODDS(1)),(MASCOM(10),TABS(1)),(MASCOM(669)
1,VFX(1))

EQUIVALENCE (ODDS(1),ALF),(ODDS(2),RALF),(ODDS(3),CRALF),(ODDS(4)
1,SLALF),(ODDS(5),ALFY),(ODDS(6),SLFY),(ODDS(7),CRALFY),(ODDS(8)
2,SLALFY),(ODDS(9),PHI0),(ODDS(10),PHIO),(ODDS(11),PHIO),(ODDS(12
3),RA09),(ODDS(13),CRALF),(ODDS(14),SRALF),(ODDS(15),SRALF)

Equivalences continued...
4 (ODDS(54), H) * (ODDS(55), PHA) * (ODDS(56), PHAY)
EQUIVALENCE (ODDS(57), FA)
EQUIVALENCE (TABS(1), ALT(1)) * (TABS(89), PRESS(1))
1 (TABS(25), ALPHAT(1)) * (TABS(30), TCZ(1)) * (TABS(34), TCZP(1))
2 (TABS(370), TCZP(1)) * (TABS(47), TCX(1)) * (TABS(457), TCX(1))
3 (TABS(495), TCXPP(1))
EQUIVALENCE (VFEX(1), XN(1)) * (VFEX(4), UMD(1)) * (VFEX(7), XBAR(1))
1 (VFEX(10), XLAM1(I)) * (VFEX(12), XLAM2(I)) * (VFEX(16), XMID(1)) * (VFEX(19))
2 (XVAL(1), VEX(22)) * (VEX(24), XLAH(1))
EQUIVALENCE (XN(1), XNX) * (XN(2), XNY) * (XN(3), XNZ)
EQUIVALENCE (UM(1), U) * (UM(2), U) * (UM(3), WD)
EQUIVALENCE (XBAR(1), X) * (XBAR(2), Y) * (XBAR(3), Z)
EQUIVALENCE (XLAM1(I), XLAM1(I)) * (XLAM2(I), XLAM2(I)) * (XLAM3(I), XLAM3(I))
EQUIVALENCE (XLAM4(I), XLAM4(I)) * (XLAM5(I), XLAM5(I))
EQUIVALENCE (XLAM1(I), XLAM1(I)) * (XLAM2(I), XLAM2(I)) * (XLAM3(I), XLAM3(I))
EQUIVALENCE (XLAM4(I), XLAM4(I)) * (XLAM5(I), XLAM5(I))
COMMON MASCOW
DIMENSION MASCOW(693)
DIMENSION ORDS(100), TABS(568), VEX(25)
DIMENSION ALT(88), PRES(88)
DIMENSION ALPHAT(38), TCZ(38), TCZP(38), TCZPP(38), TCX(38), TCXP(38)
1 TCXPP(38)
DIMENSION UMD(3), XN(3), XBAR(3), XLAM1(3), XLAM2(3), XLAM3(3), XLAM4(3),
1 XLAM5(3), U(3)
DIMENSION XVAL(14)
DOUBLE PRECISION XVAL
DOUBLE PRECISION MASCOW, ORDS, TABS, VEX, ALF, RALF, CRALF, SRALF, ALFY
1 RALF, CRALF, SRALF, PHI0, CPPHI0, CPPHI0, PH0, A0, CA09, CA09, PHIP, CP, SP, PHIY
2 CY, SY, PH1, CR, SN, OMEGA, FX, FY, FY, VRX, VRX, VRX, VRX, VRX, VRX, VRX, VRX,
3 VRXP, VRXP, VRXP, VRXP, VRXP, VRXP, VRXP, VRXP, VRXP, VRXP,
4 VRXP, VRXP, VRXP, VRXP, VRXP, VRXP, VRXP, VRXP, VRXP, VRXP,
5 TCZP, TCZPP, TCX, TCXP, TCXPP, XN, UMD, XBAR, XLAM1, XLAM2, XLAM3, XLAM4,
6 U, XLAM7, XNX, XNY, XNZ, U, VD, XD, X, Y, XLAM1, XLAM2, XLAM3, XLAM4,
7 XLAM5, XLAM6, XLAM7, XLAM8, XLAM9, XLAM10, XLAM11, XLAM12, XLAM13, XLAM14,
8 XLAM15, XLAM16, XLAM17, XLAM18, XLAM19, XLAM20, XLAM21, XLAM22, XLAM23, XLAM24,
XRALFP, CRALFD
DOUBLE PRECISION FA
DOUBLE PRECISION SRA9, CRA9, CPPHI0, RAD
DO 1000 I = 1, 9
XR1(I) = XVAL(I)
1000 XLAM1(I) = XVAL(I + 6)
0 XLAM1(I) = XVAL(I + 9)
RETURN
END
TC SHIFT
SUBROUTINE SHIFT (A, R, K)
DIMENSION A(14), R(14)
DOUBLE PRECISION A, R
DO 1000 I = 1, K
A(I) = A(I)
1000 RETURN
END
TC TGRATE
SUBROUTINE TGRATF(N, DT)
DIMENSION XVAL(14), STDVR(14), STORX(14), PCN(14)
DOUBLE PRECISION STDVR, STORX, PCN, XVAL
IF (N GT 3) GO TO 1000
CALL PDHY
CALL GETH
CALL RVINT(DT)
CALL STF3(STDRV(1,N))
RETURN
DO CONTINUE
CALL STF1(XVAL)
CALL PDHY
CALL GETH
CALL STF2(STDRV(1,4))
CALL SHIFT(STORX,XVAL,14)

PREDICTOR - ADAMS BASHFORTH
DO 1100 J=1,14
XVAL(I)=STORX(I)+DT*(-9.*STDRV(I,1)+37.*STDRV(I,2)-59.*STDRV(I,3)
+56.*STDRV(I,4))/24.
DO CONTINUE
DO 1200 J=1,13
DO CALL SHIFT(STDRV(1,J),STDRV(1,J+1),14)

CORRECTOR - ADAMS BASHFORTH
CALL STF3(XVAL)
CALL PDHY
CALL GETH
CALL STF2(STDRV(1,4))
DO 1300 I=1,14
STORX(I)=STORX(I)+DT*(STDRV(I,1)-5.*STDRV(I,2)+9.*STDRV(I,3)
+9.*STDRV(I,4))/24.
DO PCV(I)=STORX(I)-XVAL(I)
CALL STF3(STORX)
RETURN
END

SUBROUTINE CONV(R,A,K)
DIMENSION A(I),R(I)
REAL MINUS
DATA PLUS/1H+/,MINUS/1H-/EQUIVALENCE(IX,IX)
DO 1000 I=1,K
IX=0
JJA = 5*J-I-3
X=A(I)
IF(X) 1005,1001,1010
1 R(JJA)=PLUS
R(JJA+2) = PLUS
R(JJA+1) = 0,0
GO TO 1000
R(JJA)=MINUS
X=-X
GO TO 1020
10 R(JJA) = PLUS
DO CONTINUE
R(JJA+2) = MINUS
IF(X,LT,1.) GO TO 1035
R(JJA+2) = PLUS
15 IF(X,LT,1.) GO TO 1040
IX=IX+1
X=X/10.
GO TO 1025
SOLVE A SYSTEM OF N NONLINEAR EQUATIONS

DESCRIPTION OF INPUT PARAMETERS
X - INITIAL VALUE OF VECTOR X
H - APPROXIMATION OF THE INVERSE JACOBIAN MATRIX, H=1/A
N - NUMBER OF VARIABLES AND EQUATIONS
EVALF - FORTRAN SUBROUTINE TO COMPUTE VECTOR F
CRIT - PRESCRIBED ACCURACY LIMIT OF NORM(F)

DESCRIPTION OF OUTPUT PARAMETERS
X - FINAL VALUE OF VECTOR X
H - APPROXIMATION OF THE INVERSE JACOBIAN MATRIX, H=1/A
F - VECTOR OF N FUNCTIONS
ITFR - NUMBER OF ITERATIONS

SUBROUTINES REQUIRED
MATMPY
LINCOM
FNORM
GFTT
SHIFT
EVALF

DIMENSION X(2),H(2),F(2)
DIMENSION P(10),Y(10),FNN(10),XN(10)
DOUBLE PRECISION P,Y,F,Y,XN,H,F,VALO,VALO,X,X1,X2,FF,FFY,SCALF
N=2
NST=N+1
ITFR=0
X(1)=X1
X(2)=X2
F(1)=FF
F(2)=FFY
00 CALL EVALF(X,F,N)

START NEW ITERATION
EVALUATE VECTOR P=H*F

00 CALL MATMPY(H,F,P,N,N+1)
ITFR=ITFR+1
CALL FNORM(F,N,VALO)
CALL GETT(T,VALO)

FIND A VALUE OF T SUCH THAT THE NORM OF F(X+T*P) IS LESS
THAN THE NORM OF F(X)
XN IS THE NEW TRIAL VALUE OF X, OBTAINED AS XN=X+T*P
-VAL- AND -VALO- ARE THE NORM OF F(XN) AND F(X) RESPECTIVELY

25
DO 1075 I=1,10
CALL LINCOM(I,X,T,P,XN,N,1)
CALL FVALF(XN,FN,N)
CALL FNORM(FN,N,VAL)
IF(VAL.LT.VAL0) GO TO 1080
CALL GFTT(I,T,VAL)
CONTINUE
GO TO 2000

ONCE A SATISFACTORY T WAS FOUND, X IS REPLACED BY XN,
F IS REPLACED BY FN
IF REQUIRED ACCURACY IS OBTAINED OR ALLOWED NUMBER OF INTERATIONS
EXHAUSTED RETURN TO CALLING PROGRAM.
A NEW APPROXIMATION OF MATRIX H IS COMPUTED.
NEW H IS OBTAINED AS H=H-(H*Y+T*P)*(P*H/SCALE)

CONTINUE
IF(ITER.GT.NTOP) GO TO 2000
CALL SHIFT1(X,XN,N)

COMPUTE Y=FN-F

CALL LINCOM(1.,FN,-1.,F,Y,N,1)

REPLACE F BY FN

CALL SHIFT1(F,FN,N)

COMPUTE H*Y

CALL MATMPY(H*Y,FN,N,N,1)

COMPUTE H*Y+T*P

CALL LINCOM(1.,FN,T,P,FN,N,1)

COMPUTE P*H

CALL MATMPY(P*H,XN,1,N,N)

COMPUTE SCALE=(P*H)*Y

CALL MATMPY(XN,Y,SCALE,1,N,1)

COMPUTE P*H/SCALE

DO 1100 I=1,N
  10 XN(I)=XN(I)/SCALE
DO 1200 I=1,N
DO 1200 J=1,N
  10 H(I,J)=H(I,J)-FN(I)*XN(J)
IF(VAL.LT.CRIT) GO TO 2000
GO TO 1050

WRITE(6,3300) ITER,TIFEC
3300 FORMAT(12H ITERATIONS=,I6,5X,2HT=E14.8)
  XX1=X(1)
  XX2=X(2)
  FF=F(1)
  FFY=F(2)
RETURN
FND

SUBROUTINE MATPY
DIMENSION A(N1,N2),B(N2,N3),C(N1,N3)
DOUBLE PRECISION A,B,C,TEMP
DO 3 I=1,N1
DO 2 K=1,N3
TMP=0.
DO 1 J=1,N2
TEMP=TEMP+A(I,J)*B(J,K)
C(I,K)=TEMP
CONTINUE
3 CONTINUE
RETURN
END

SUBROUTINE LINCOM
DIMENSION A(M,N),B(M,N),C(M,N)
DOUBLE PRECISION A,B,C
DO 2 I=1,N
DO 1 J=1,N
C(I,J)=S*A(I,J)+T*B(I,J)
CONTINUE
2 CONTINUE
RETURN
END

SUBROUTINE FNORM
DIMENSION F(N)
DOUBLE PRECISION F
VAL=0.
DO 1 I=1,N
VAL=VAL+F(I)*F(I)
SVAL = SNGL(VAL)
WRITE(6,1000) SVAL
FORMAT(*H VAL=,E14.8)
RETURN
END

SUBROUTINE GETT(IT,T,F)
IF(IT.EQ.0) GO TO 1
T=1.
F0=F
RETURN
IF(IT.LE.1) GO TO 2
F1=F
TH=F1/F0
T=(SORT(1.+6.*TH)-1.)/3./TH
RETURN
T=-T/2.
RETURN
END

SUBROUTINE SHIFT1(A,B,K)
DIMENSION A(1), B(1)
DOUBLE PRECISION A,B
DO 1000 I=1, K
A(I)=B(I)
RETURN
END
DIMENSION STX(3),STY(3),STY(3),STAY(3)
DOUBLE PRECISION STX,STY,STAY,SLOPE
DOUBLE PRECISION STLAF
DOUBLE PRECISION CONA,CONR,CONC,Y0,Y2,Y02,Y22,DEL2
DOUBLE PRECISION XX,F
DIMENSION Rd(7)
DIMENSION OF(110),OUTA(40),OUTC(200)
DIMENSION XX(N1),F(N1)
"ALF=XX(1)
ALFY=XX(2)
CALL HCALC
F(1)=PHA
F(2)=PHAY
RETURN
END.

TC INVERS
SUBROUTINE INVERS(A,INDX,ORD,N,NN,KERR)
DOUBLE PRECISION A,R
DIMENSION A(N,NN)
DIMENSION INDX(N),ORD(N)
IF(RPR=0
J1=N+1
J2=2*N
DO 23 I=1,N
J3=I+N
DO 24 J=J1,J2
A(I,J)=0.0
A(I,J3)=1.0
DO 10 M=1,N
INDX(M)=0
NO=N-]
DO 11 M=1,NO
DO 12 M=1,NO
IF(INDX(M),CF,0.0) GO TO 13
CONTINUE
P=-1.0+36
K=I
IL=1
IT=1
DO 14 M=K,N
IF(INDX(M),CF,0) GO TO 14
IL=IL+1
IF(A(M,J),CF,0) GO TO 17
IT=IT+1
GO TO 14
IF((2-NARS(A(M,J)))*GT,0.0) GO TO 14
P=A(M,J)
MM=M
CONTINUE
IF((IT-IL),CF,0) GO TO 19
WRITE(6,21)
FORMAT(///3DH MATRIX INVERSION NOT POSSIBLE///)
IFRPR=1
RETURN
INDX(MM)=1
ORD(J)=MM
IP=J+1
DO 25 J=IP,J2
A(MM,JJ)=A(MM,JJ)/A(MM,J)
DO 26 K=1,N
IF((MV-K).EQ.0) GO TO 26
DO 28 JJ=IP+J2
A(K+JJ)=A(K+JJ)-A(K,J)*A(NM,JJ)
CONTINUE
CONTINUE
DO 29 I=1,N
IF=IORD(I)
DO 29 J=1,N
L=J+N
A(I,J)=A(IF,L)
RETURN
END

AP TRAP DECK
ENTRY TRAP
AXT **+4
TRA **
SX A TRAP-1,4
CLA /8
STA RESET+1
CLA FIX
TSX S.5CCR+4
STO /8
TRA TRAP-1
CLA 0
TMJ **
APS 20
LAT
TRA **+2
TR A* RESET+1
SX A OUT+4
TSX S.5PIT+4
PZF **+5
AXT **+4
ZAC
LRS 35
TRA* 0
TRA RESET
PCI 3. **** UNDFRFLOW
END

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1812 -00122 -000084 2
1772 -00040 -00248 3
1710 -002432 -000996 4
1626 -002786 -000476 5
1520 -002880 -000328 6
1438 -003114 -000252 7
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The procedure for solving the problem of flying a minimum fuel point-to-point transfer with a high speed aircraft is as follows.

Minimize the integral

\[ I = \int_{t_0}^{t} \dot{m}_f(t) dt \]

where \( \dot{m}_f(t) \) is the time variable fuel burning rate, subject to the differential constraints

\[ \dot{x} = \bar{u} \]  
(1)

\[ \dot{u} = \frac{[T]F_T}{m} + \frac{[T][\alpha, \alpha_o]}{m} F_a + \frac{F_g}{m} - \frac{F_T \bar{u}}{V_{e m}} \]  
(2)

\[ \dot{\phi} = \bar{\psi} \]  
(3)

and \[ \dot{\bar{\psi}} = [B_2]^{-1} (\bar{M}_\phi + (A) \bar{\psi} + B_3) \]  
(4)

which are the equations of motion, and the algebraic constraints

\[ \bar{V}_R = [\alpha, \alpha_o] \bar{V}_x. \]  
(5)

The Hamiltonian is formed as

\[ H = \bar{\lambda}_I \cdot \dot{x} + \bar{\lambda}_{II} \cdot \dot{u} + \bar{\lambda}_{III} \cdot \dot{\phi} + \bar{\lambda}_{IV} \cdot \dot{\bar{\psi}} + \lambda_1 \dot{m}_f \]  
(6)

The control variables will be \( F_T \) and \( \bar{M}_p \) where \( F_T \) is the thrust magnitude and \( \bar{M}_p \) is the moment generated about the C.P. by the aircraft flaps.
The canonical and control equations become

\[
\begin{align*}
\dot{x}_I &= -\frac{\partial H}{\partial \alpha} \\
\dot{x}_{II} &= -\frac{\partial H}{\partial u} \\
\dot{x}_{III} &= -\frac{\partial H}{\partial \phi} \\
\dot{x}_{IV} &= -\frac{\partial H}{\partial \nu} \\
\lambda_{13} &= -\frac{\partial H}{\partial m_f} \\
\frac{\partial H}{\partial F_T} &= 0 \\
\frac{\partial H}{\partial \bar{F}_T} &= 0
\end{align*}
\]  

(7) (8) (9) (10) (11) (12) (13)

There are four control variables \( F_T, \bar{M}_p \), fourteen state variables, \( \bar{x}, \dot{x}, \bar{\phi}, \dot{\phi}, \alpha, \alpha_y \), and thirteen multipliers. Equations (1-13) provide thirty-one scalar equations from which to determine thirty-one unknowns.

From Equation (13)

\[
\frac{\partial H}{\partial \bar{M}_p} = 0 + \bar{\lambda}_{IV} = 0
\]  

(14)

\[
\ddot{\lambda}_{IV} = 0 = \bar{\lambda}_{III}
\]  

(15)

\[
\ddot{\lambda}_{III} = -\frac{\partial H}{\partial \phi} = f_2(\bar{\lambda}_{II}, F_T, \bar{\phi}, \alpha, \alpha_y, \bar{x}, \dot{x})
\]  

(16)

\[
\frac{\partial H}{\partial F_T} = 0 = f_3(\bar{\phi}, \bar{\lambda}_{II})
\]  

(17)

Solve Equations (16) and (17) simultaneously for \( \bar{\lambda}_{II} \) and \( F_T \).
\[
\lambda_{II} = \lambda_{II}(\dot{\phi}, \alpha, \alpha_y, \ddot{x}, \dot{x})
\]

\[
P_T = P_T(\phi, \alpha, \alpha_y, \ddot{x}, \dot{x})
\]

Compute \(\dot{\lambda}_{II} = f_5(\dot{\phi}, \dot{\alpha}, \ddot{x}, \ddot{u}, \ddot{u})\)

From (8) \(\dot{\lambda}_{II} = \frac{\partial H}{\partial u} = -\dot{\lambda}_{II} - \frac{\partial (\lambda_{II} \cdot \ddot{u})}{\partial u}\)

Solve for \(\dot{\lambda}_{I} = -\dot{\lambda}_{II} - \frac{\partial (\lambda_{II} \cdot \ddot{u})}{\partial u}\)

Compute \(\dot{\lambda}_{I} = f_7(\dot{\phi}, \ddot{\alpha}, \ddot{x}, \ddot{u}, \ddot{u}, \ddot{u})\)

From (7) \(\dot{\lambda}_{I} = \frac{\partial H}{\partial x} = f_8(\dot{\phi}, \ddot{\alpha}, \ddot{x}, \ddot{u}, \ddot{u}, \ddot{u})\)

Solve for \(\ddot{\alpha}\).

Plug \(\ddot{\alpha}\) into Equation (4) and solve for \(\mathcal{H}_p\).
Technical Report

SOME SUGGESTED APPROACHES
TO SOLVING THE HAMILTON-JACOBI EQUATION
ASSOCIATED WITH CONSTRAINED RIGID BODY MOTION

Prepared by
Philip M. Fitzpatrick, Grady R. Harmon, John E. Cochran
and W. A. Shaw

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Computer Research Laboratory
Electronics Research Center
National Aeronautics and Space Administration

On
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(May 1 – November 1, 1968)

January 1969

ENGINEERING EXPERIMENT STATION
AUBURN UNIVERSITY
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ABSTRACT

Some methods of approaching a solution to the Hamilton-Jacobi equation are outlined and examples are given to illustrate particular methods. These methods may be used for cases where the Hamilton-Jacobi equation is not separable and have been particularly useful in solving the rigid body motion of an earth satellite subjected to gravity torques. It is felt that these general methods may also have applications in studying the motion of satellites with aerodynamic torque and in studying space vehicle motion where thrusting is involved.
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INTRODUCTION

During the six months included in this reporting period (May to November 1968), work has continued on an investigation of the analytical foundation of the Hamilton-Jacobi theory and its application to space flight problems.

In studying the literature, many questions arose. An attempt was made to formulate these questions and then find satisfactory answers to them. The first work during this reporting period was directed toward comparing the different methods available for solving the Hamilton-Jacobi partial differential equation. Five different methods for obtaining a generator \( S \) were studied:

1. \( S = \int -L dt \), Where \( L \) Is the Lagrangian
2. Liouville's Theorem for Obtaining \( S \)
3. Jacobi's Method of Integration of Partial Differential Equations
4. Separation of Variables
5. Method of Characteristics

The following questions arose during the discussions of the different methods available for solving the Hamilton-Jacobi equation.

1. Can a solution be obtained by Jacobi's method; i.e., by obtaining half the integrals for \( p_i \) and then building \( S \) from

\[
dS = p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n
\]

that cannot be obtained by quadratures

\[
\frac{dt}{I} = \frac{dq_1}{\partial H/\partial p_1} = \cdots = \frac{dq_n}{\partial H/\partial p_n} \\
= -\frac{dp_1}{\partial H/\partial q_1} = \cdots = -\frac{dp_n}{\partial H/\partial q_n}
\]

which result from Hamilton's equations?
2. Same question as one except separation of variables versus Jacobi's method?

3. Same question except quadratures versus separation of variables?

4. All three of the above questions with the Hamiltonian given as an explicit function of time?

In discussing Jacobi's method, the following question and answer was developed. Given one complete integral, is there any technique for constructing another distinct complete integral? Yes, an infinite number of other distinct complete integrals can be constructed. Given a complete integral containing two arbitrary constants \( \alpha \) and \( \beta \), another complete integral can be constructed by replacing \( \alpha \) and \( \beta \) as arbitrary functions of two other arbitrary constants \( A \) and \( B \). Thus, the integration constants associated with each distinct complete integral of the Hamilton-Jacobi equation can be functionally related. There is a question as to whether any of these constants are canonical. Also, if the same problem were solved by integrating Hamilton's equations by quadratures, then there would be other constants of integration. One would want to know how these constants are related to those obtained from the Hamilton-Jacobi equation. Also, are they canonical?

Some of these questions are answered in subsequent sections of this report. One paper (see Appendix) has grown out of this work and has been submitted to the American Journal of Physics for possible publication.
DEFINITIONS OF ANGLES

The angles $\theta$, $\phi$, $\theta'$, $\phi'$, $\theta^*$, and $\psi^*$ are defined by their geometry in the spherical triangle (see Figure 1):

$$\cos \theta = \frac{a_2a_3}{h^2} - \frac{\sqrt{(h^2 - a_2^2)(h^2 - a_3^2)}}{h^2} \cos \frac{h}{A} (t - \beta_1)$$

$$= \cos \theta' \cos \theta^* - \sin \theta' \sin \theta^* \cos \phi^*$$

$$\cos(\phi - \phi') = \frac{a_3 - a_2 \cos \theta}{\sqrt{h^2 - a_2^2 \sin \theta}} = \frac{\cos \theta^* - \cos \theta' \cos \theta}{\sin \theta' \sin \theta}$$

$$\sin(\phi - \phi') = \frac{\sin \phi^* \sin \theta^*}{\sin \theta}$$

$$\cos \theta' = \frac{a_2}{h}$$

$$\sin \theta' = \frac{\sqrt{h^2 - a_2^2}}{h}$$

Figure 1
\[
\cos \phi^* = \frac{a_2 a_3 - h^2 \cos \theta}{\sqrt{(h^2 - a_2^2)(h^2 - a_3^2)}} = \cos \frac{h}{A}(t - \beta_1)
\]

\[
\cos \theta^* = \frac{a_3}{h}
\]

\[
\sin \theta^* = \frac{\sqrt{h^2 - a_3^2}}{h}
\]

\[
\cos(\psi - \psi^*) = \frac{a_2 - a_3 \cos \theta}{\sqrt{h^2 - a_3^2} \sin \theta} = \frac{\cos \theta' - \cos \theta^* \cos \theta}{\sin \theta^* \sin \theta}
\]

\[
\sin(\psi - \psi^*) = \frac{\sin \phi^* \sin \theta}{\sin \theta}
\]

The angles \(\theta_H\), \(\phi_H\), and \(\psi_H\) are defined by their geometry.

\[
\cos \theta_H = \cos i \cos \theta^* + \sin i \sin \theta^* \cos(\psi^* - \Omega)
\]

\[
cot(\psi^* - \phi_H) = \frac{\cos i \sin \theta^* - \sin i \cos \theta^* \cos(\psi^* - \Omega)}{\sin i \sin(\psi^* - \Omega)}
\]

\[
cot \psi_H = \frac{\cos i \sin \theta^* \cos(\psi^* - \Omega) - \sin i \cos \theta^*}{\sin \theta^* \sin(\psi^* - \Omega)}
\]
Functional Relations

\[ h = h(a_1, a_2) \]

\[ \psi = \psi(a_1, a_2, a_3, \beta_3, \theta) = \psi(a_1, a_2, a_3, \beta_1, \beta_3, t) \]

\[ \psi^* = \psi^*(\beta_3) \]

\[ \psi_H = \psi_H(\psi^*, \theta^*; i, \Omega) = \psi_H(a_1, a_2, a_3, \beta_3; i, \Omega) \]

\[ \theta = \theta(a_1, a_2, a_3, \beta_1, t) \]

\[ \theta' = \theta'(a_1, a_2) \]

\[ \theta^* = \theta^*(a_1, a_2, a_3) \]

\[ \theta_H = \theta_H(\psi^*, \theta^*; i, \Omega) = \theta_H(a_1, a_2, a_3, \beta_3; i, \Omega) \]

\[ \phi = \phi(a_1, a_2, a_3, \beta_2, \theta) = \phi(a_1, a_2, a_3, \beta_1, \beta_2, t) \]

\[ \phi' = \phi'(a_1, a_2, a_3, \beta_2, \theta) = \phi'(a_1, a_2, a_3, \beta_1, \beta_2, t) \]

\[ \phi^* = \phi^*(a_1, a_2, a_3, \theta) = \phi^*(a_1, a_2, a_3, \beta_1, t) \]

\[ \phi_H = \phi_H(\psi^*, \theta^*, \phi^*; i, \Omega) = \phi_H(a_1, a_2, a_3, \beta_1, \beta_3, t; i, \Omega) \]
 Canonical Transformations

The motion of the body is such that $\theta$ oscillates between $\theta_0$ and $\theta_1$, where $\theta_0 = \theta' + \theta^*$ and $\theta_1 = |\theta' - \theta^*|$. Let $t_0$ denote an instant at which $\theta = \theta_0$. Let $\theta_{01}$ refer to $\theta$ at the instant $t_0$, $\theta_{11}$ refer to $\theta$ at the first instant after $t_0$ that $\theta = \theta_1$, $\theta_{02}$ refer to $\theta$ at the first instant after $\theta_{11}$ that $\theta = \theta_0$, $\theta_{12}$ refer to $\theta$ at the first instant after $\theta_{02}$ that $\theta = \theta_1$, and so forth.

A generator, $S$, of a canonical transformation is given by

$$S = -a_1 t + a_2 \phi + a_3 \psi - \int_{\theta_{01}}^{\theta} Q(\theta) d\theta,$$

where

$$Q(\theta) = \begin{cases} \sqrt{f(\theta)}, & \theta_{0n} + \theta = \theta_{1n}, \\ -\sqrt{f(\theta)}, & \theta_{1n} + \theta = \theta_{0(n+1)}. \end{cases}$$

and

$$f(\theta) = 2Aa_1 - \frac{A}{C} a_0^2 - \csc^2 \theta (a_3 - a_2 \cos \theta)^2.$$

The symbol $\theta_{0n} + \theta = \theta_{1n}$ means that $\theta$ has passed through $\theta_{0n}$ and is going toward $\theta_{1n}$.

In terms of the variables $\phi'$, $\phi^*$, and $\psi^*$,

$$S' = -a_1 t + a_2 \phi' + a_3 \psi^* + h \phi^*$$

$$Q(\theta) = h \sin \theta^* \sin(\psi - \psi^*)$$

When $\theta = \theta_0$, it can be shown that $\phi^*$, $\phi' - \phi$, and $\psi - \psi^*$ must be multiples of $2\pi$. To avoid ambiguity, $\phi^* = \phi - \phi' = \psi - \psi^* = 0$ is defined when $\theta = \theta_{01}$. 
\[ P_\psi = a_3 \]
\[ P_\phi = a_2 \]
\[ P_\theta = -Q(\theta) = -h \sin \theta \sin (\psi - \psi^*) \]
\[ \beta_1 = t - \frac{A}{h} \phi^* \]
\[ \beta_2 = -\phi' - \frac{a_2}{h} \left( \frac{C - A}{C} \right) \phi^* \]
\[ \beta_3 = -\psi^* \]
\[ \phi^* = \frac{h}{A} (t - \beta_1) \]
\[ \phi' = -\beta_2 - \frac{a_2}{A} \left( \frac{C - A}{C} \right) (t - \beta_1) \]
\[ \psi^* = -\beta_3 \]

Alternatively,

\[ \beta_1 = t + A \int_{\theta_0}^{\theta} \frac{d\theta}{Q(\theta)} \]
\[ = t - \frac{A}{h} \cos^{-1} \frac{a_2 a_3 - h^2 \cos \theta}{\sqrt{(h^2 - a_2^2)(h^2 - a_3^2)}} \bigg|_{\theta_0}^{\theta} \]
\[ = t - \frac{A}{h} \cos^{-1} \frac{a_2 a_3 - h^2 \cos \theta}{\sqrt{(h - a_2^2)(h - a_3^2)}} \]

\[ \beta_2 = -\phi + \int_{\theta_0}^{\theta} \left\{ \left( \frac{a_3 - a_2 \cos \theta}{\sin^2 \theta} \right) \cos \theta - \frac{A a_2}{Q(\theta)} \right\} \frac{d\theta}{Q(\theta)} = \]
\[ \beta_3 = \int_{\theta_0}^{\theta} \left( \frac{a_3 - a_2 \cos \theta}{\sin^2 \theta} \right) d\theta \]

The multi-valued \( \cos^{-1} \) functions appearing above are to be interpreted as follows:

\[
\cos^{-1} g(\theta) \equiv \begin{cases} 
2(n - 1)\pi + \cos^{-1} g(\theta), & \theta_0 n + \theta \to \theta_1 n \\
2n\pi - \cos^{-1} g(\theta), & \theta_1 n + \theta \to \theta_0 (n+1)
\end{cases}
\]

where \( \cos^{-1} \) denotes the principal value (that is, the value between 0 and \( \pi \)) of the \( \cos^{-1} \) function.
Miscellaneous

\[ h^2 = 2A_{\alpha_1} + \left( \frac{C - A}{C} \right) a_2 \]

\[ \cos \theta = \frac{\alpha_2 a_3 - \sqrt{(h^2 - a_2^2)(h^2 - a_3^2)} \cos \frac{h}{A}(t - \beta_1)}{h^2} \]

= \cos \theta' \cos \theta^* - \sin \theta' \sin \theta^* \cos \phi^*

\[ \phi = -\beta_2 + \cos^{-1} \frac{\alpha_3 - \alpha_2 \cos \theta}{\sin \theta \sqrt{h^2 - a_2^2}} \]

\[ \psi = -\beta_3 + \cos^{-1} \frac{\alpha_2 - \alpha_3 \cos \theta}{\sin \theta \sqrt{h^2 - a_3^2}} \]
A NOTE ON DISTINCT COMPLETE INTEGRALS

Problem: Show that the differential equation

\[ 4XZQ^2 + P = 0, \quad P = \frac{\partial Z}{\partial X}, \quad Q = \frac{\partial Z}{\partial Y} \]

possesses the distinct complete integrals

\[ Z^2 = aY - \alpha^2X^2 + \beta \]

and

\[ Z^2(4X^2 + a) = (Y + b)^2 \]

Find a functional relation between \( \alpha, \beta, a, \) and \( b; \) hence, find the second solution as a particular case of the general integral obtained from the first.

Solution: First, transform to new variables according to the scheme

\[ X + x_1, \quad Y + x_2, \quad Z + x_3 \]

\[ P = -\frac{P_1}{P_3}, \quad Q = -\frac{P_2}{P_3}, \quad P_3 = \frac{\partial u}{\partial x_3} \]

See Frederic H. Miller, Partial Differential Equations (New York: John Wiley & Sons, 1949), Chapter V, for details on transformation. The differential equation

\[ F(X, Z, Q, P) = 4XZQ^2 + P = 0 \]  \hspace{1cm} (1)

now becomes

\[ F(x_1, x_3, P_1, P_2, P_3) = 4x_1x_3P_2^2 - P_1P_3 = 0 \]  \hspace{1cm} (1')

Jacobi's method will be used to solve Eq (1'). First, write

\[
\begin{align*}
\frac{dp_1}{df} & = \frac{dp_2}{df} = \frac{dp_3}{df} = -\frac{dx_1}{dp_1} = -\frac{dx_2}{dp_1} = -\frac{dx_3}{dp_3}
\end{align*}
\]  \hspace{1cm} (2)
Explicitly,
\[ \frac{dp_1}{4x_3p_2} = \frac{dp_2}{0} = \frac{dp_3}{4x_1p_2} = \frac{dx_1}{x_2} \]
\[ = -\frac{dx_2}{8x_1x_3p_2} = \frac{dx_3}{p_1} \]  \hspace{1cm} (2')

Using the second ratio,
\[ F_1 = p_2 = a_1 = \text{constant} \]  \hspace{1cm} (3)

Using the first and sixth ratios,
\[ p_1 dp_1 = 4a_1^2 x_3 dx_3 \]
and
\[ F_2 = p_1^2 - 4a_1^2 x_3^2 = a_2 = \text{constant} \]  \hspace{1cm} (4)

Using the third and fourth ratios,
\[ p_3 dp_3 = 4a_1^2 x_1 dx_1 \]
and
\[ F_2^* = p_3^2 - 4a_1^2 x_1^2 = a_2^* = \text{constant} \]  \hspace{1cm} (5)

\((F_1,F_2) = 0; \text{ also, } (F_1,F_2^*) = 0, \text{ as is readily verified. Using } \)
\[ F_1 = p_2 = a_1, \text{ and } F_2 = p_1^2 - 4a_1^2 x_3^2 = a_2, \text{ take} \]
\[ P_1 = \sqrt{a_2 + 4a_1^2 x_3^2} \]

Substitute into Eq (1') and solve for \(p_3\)
\[ 4x_1 x_3 a_1^2 - \sqrt{a_2 + 4a_1^2 x_3^2} p_3 = 0 \]
\[ p_3 = \frac{4x_1 x_3 a_1^2}{\sqrt{a_2 + 4a_1^2 x_3^2}} \]

Use \(F_1\) and \(F_2\) in conjunction with \(F\) to obtain a complete integral.
One has \((F_1,F_2) = 0\)
\[ \frac{du}{dx} = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \]

\[ \frac{du}{dx_1} = p_1 \]

\[ \frac{du}{dx_2} = p_2 \]

\[ \frac{du}{dx_3} = p_3 \]

\[ \frac{du}{dx_2} = \frac{3f}{\partial x_1} = p_1 + f = p_1x_1 + g(x_3) \]

\[ \frac{du}{dx_3} = \frac{3f}{\partial x_3} = x_1 \frac{3p_1}{\partial x_3} + g'(x_3) \]

\[ = p_3 + g'(x_3) = p_3 \cdot x_1 \frac{\partial^2}{\partial x_3^2} \]

But,

\[ p_3 - x_1 \frac{\partial p_1}{\partial x_3} = p_3 - x_1 \left[ \frac{1}{2} (a_2 + 4a_1^2x_3^2)^{-1/2} (8a_1^2x_3) \right] \]

\[ = p_3 - p_3 = 0 + g(x_3) = a_3 = \text{constant} \]

\[ u = a_1x_2 + x_1p_1 + a_3 = 0 \]

\[ p_1 = \sqrt{a_2 + 4a_1^2x_3^2} \]

\[ x_1p_1 = -a_3 - a_1x_2 \]

\[ p_1^2 = a_2 + 4a_1^2x_3^2 = \frac{(a_3 + a_1x_2)^2}{x_1^2} \]

\[ 4a_1^2x_3^2 + a_2 = \frac{(a_3 + a_1y)^2}{x^2} = \frac{a_2^2}{x^2} \left( \frac{a_3}{a_1} + y \right)^2 \]

\[ 4\lambda^2x_3^2 + \frac{a_2}{a_1^2} x^2 = \left( \frac{a_3}{a_1} + y \right)^2 \]
Set \( A \) equal to \( a_2/a_1^2 \) and \( B \) equal to \( a_3/a_1 \). Then,

\[
X^2(42^2 + A) = (Y + B)^2
\]

and Eq (6) is a complete integral of Eq (1). If Eqs (3) and (5) are used in conjunction with Eq (1'), observing that Eq (1') is unchanged if \( p_1 \rightarrow p_3 \) and \( x_1 \rightarrow x_3 \) are interchanged, one has

\[
u = a_1x_2 + x_3p_3 + a_3^* = 0
\]

\[
p_3^2 = \frac{(a_3 + a_1x_2)^2}{x_3^2}
\]

\[
x_3^2(4a_1^2x_1^2 + a_2^*) = (a_3 + a_1x_2)^2 = a_1^2\left(\frac{a_3}{a_1} + x_2\right)^2
\]

\[
z^2\left(4x^2 + \frac{a_3^*}{a_1^2}\right) = \left(\frac{a_3}{a_1} + Y\right)^2
\]

Set \( a \) equal to \( a_2^*/a_1^2 \) and \( b \) equal to \( a_3/a_1 \)

\[
z^2(4x^2 + a) = (Y + b)^2
\]

Eq (7) is a complete integral of Eq (1).

Still another distinct complete integral of Eq (1) can be obtained by separating the variables in Eq (1'). Since \( 4x_1x_3p_2^2 - p_1p_3 = 0 \) is free of \( x_2 \), \( p_2 = \partial u/\partial x_2 = a_1 \), a constant, and

\[
4x_1x_3a_1^2 - p_1p_3 = 0 = 4x_1x_3a_1^2 - \frac{\partial u}{\partial x_1} \cdot \frac{\partial u}{\partial x_3}
\]

Assume a solution of Eq (8) of the form

\[
u = f_1(x_1) + f_2(x_2) + f_3(x_3)
\]

\[
= a_1x_2 + f_1(x_1) + f_3(x_3)
\]

Substitute Eq (9) into Eq (8) to obtain
\[
\frac{4x_1a_1^2}{df_1/dx_1} = \frac{1}{x_3} \frac{df_3}{dx_3} = C_1
\]

\[
f_1 = \frac{2a_1^2 x_3^2}{C_1} + C_2
\]

\[
f_3 = \frac{C_1}{2} x_3^2 + C_3
\]

\[
u = \frac{2a_1^2}{C_1} x_1^2 + \frac{C_1}{2} x_3^2 + C_2 + C_3 + a_1x_2 = 0
\]

where \(x_3 = Z, x_1 = X, \) and \(x_2 = Y.\)

\[
\frac{C_1}{2} Z^2 = -(C_2 + C_3) - a_1 Y - \frac{2a_1^2}{C_1} X^2
\]

\[
Z^2 = -\frac{2}{C_1} (C_2 + C_3) - \frac{2a_1}{C_1} Y - \frac{4a_1^2}{C_1} X^2
\]

Set \(\alpha\) equal to \(-2a_1/C_1\) and \(\beta\) equal to \(-2(C_2 + C_3)/C_1\)

\[
Z^2 = \alpha Y - \alpha^2 X^2 + \beta
\]

and Eq (11) is a complete integral of Eq (1).

Consider the distinct integrals Eqs (7) and (11), and renumber them I and II

\[
Z^2(4X^2 + a) = (Y + b)^2
\]

\[
Z^2 = \alpha Y - \alpha^2 X^2 + \beta
\]

\[
\frac{\partial I}{\partial X}; \quad ZP(4X^2 + a) + 4XZ^2 = 0
\]

\[
\frac{\partial II}{\partial Y}; \quad ZQ(4X^2 + a) - (Y + b) = 0
\]

\[
\frac{\partial II}{\partial X}; \quad ZP + \alpha^2 X = 0
\]

\[
\frac{\partial II}{\partial Y}; \quad 2ZQ - \alpha = 0
\]
X, Y, Z, P, and Q must be eliminated from the six equations above. From Eq (III) and (V), eliminate P:

\[ 4Z^2 = a^2(4X^2 + a) \]

Use Eq (II):

\[ 4aY - 4a^2X^2 + 4\beta = a^2(4X^2 + a) \]

Solve for Y to obtain

\[ Y = 2ax^2 + \frac{aa}{4} - \frac{\beta}{a} \quad (VII) \]

From Eqs (IV) and (VI), eliminate ZQ:

\[ Y + b = \frac{a}{2}(4X^2 + a) \]

Solve for Y to obtain

\[ Y = 2ax^2 + \frac{aa}{2} - b \quad (VIII) \]

Equating Eqs (VII) and (VIII), one obtains

\[ \beta = ab - \frac{aa^2}{4} \quad (IX) \]

Substitute Eq (IX) into Eq (II) to obtain

\[ Z^2 = aY - a^2x^2 + ab - \frac{aa^2}{4} \quad (X) \]

\[ Y - 2ax^2 + b - \frac{2aa}{4} = 0 \]

\[ a\left(\frac{a}{2} + 2X^2\right) = b + Y \]

\[ a = \frac{2(b + Y)}{a + 4X^2} \quad (XI) \]
Substitute Eq (XI) into Eq (X):

\[ Z^2 = \alpha(b + Y) - \alpha^2 \left( \frac{a}{4} + x^2 \right) \]

\[ = \frac{2(b + Y)^2}{a + 4x^2} - \frac{4(b + Y)^2}{(a + 4x^2)^2} - \frac{(a + 4x^2)}{4} \]

\[ = \frac{(b + Y)^2}{a + 4x^2} \]

\[ Z^2(4x^2 + a) = (b + Y)^2 \]

*A Further Note:

\[ F = Z^2 - \alpha Y + \alpha^2 x^2 - \beta = 0 \quad \left\{ \begin{array}{c} \beta = f(\alpha), \quad \frac{\partial F}{\partial \alpha} = 0 \end{array} \right\} \]

\[ G = Z^2(4x^2 + a) - (Y + b)^2 = 0 \quad \text{General Integral} \]

F and G are two distinct complete integrals. Let

\[ \beta_1 = \alpha B - \frac{Aa^2}{4} \]

Note that \( \beta_1 \) is one possible functional form of \( \beta = f(\alpha) \). For all possible choices A and B in \( \beta_1 \), only a subset of the elements for the arbitrary choice \( \beta = f(\alpha) \) is obtained. Better said: Let \( H \) be the set of functions of \( \alpha \)

\[ \beta = Ba - \frac{Aa^2}{4} \]

for fixed A and B. \( H \) is a proper subset of the set Q of all possible functions \( \beta = f(\alpha) \). By inserting \( \beta = Ba - Aa^2/4 \) into F and forming \( \partial F/\partial \alpha \) for fixed A and B, \( \alpha \) can be eliminated, and the two-parameter family of surfaces G can be obtained. Thus, the surfaces G are part of the totality of envelopes which go to make up the general integral.
A NOTE ON OBTAINING A COMPLETE INTEGRAL
OF THE HAMILTON-JACOBI EQUATION

On page 324 of A Treatise on the Analytical Dynamics of Particles and Rigid Bodies (Cambridge: The University Press, 1937), E. T. Whittaker states--without proof--the following lemma.

Lemma: If \( dW \) is the perfect differential of some function \( W(q_i, a_i, t) \), then the first Pfaff's system of the differential form

\[
dW = \sum_{i=1}^{n} \frac{\partial W}{\partial a_i} \, da_i
\]

is

\[
d\left( \frac{\partial W}{\partial a_i} \right) = 0, \quad da_i = 0 \quad (i=1,2,\ldots,n)
\]

Let \( W(q_i, a_i, t) \) be in \( C_1 \) but otherwise arbitrary, and consider the differential form

\[
\sum_{i=1}^{n} p_i dq_i - H(q_i, p_i, t) \, dt
\]  

and the transformation

\[
p_i = f_i(q_i, a_i, t), \quad q_i = q_n \quad (i=1,2,\ldots,n)
\]  

The following theorem is established.

Theorem 1: If the transformation Eq (2) transforms Eq (1) into the differential form

\[
dW = \sum_{i=1}^{n} \frac{\partial W}{\partial a_i} \, da_i
\]
where
\[ dW = \sum_{i=1}^{n} f_i dq_i - H_i(q_i, f_i, t) dt + \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} d\alpha_i \]

\[ = \sum_{i=1}^{n} \frac{\partial W}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial W}{\partial \alpha_i} d\alpha_i + \frac{\partial W}{\partial t} dt \]

is a perfect differential of some function \( W(q_i, \alpha_i, t) \) of the variables \((q_i, \alpha_i, t)\), which contains \( n \) independent constants \( \alpha_i \), then \( W \) is a complete integral of the Hamilton-Jacobi equation.

**Proof:** By equating coefficients, the necessary conditions can be obtained

\[ f_i = \frac{\partial W}{\partial q_i} \]

\[ H_i(q_i, f_i, t) + \frac{\partial W}{\partial t} (q_i, \alpha_i, t) = 0 \]

Thus,

\[ H_i(q_i, \frac{\partial W}{\partial q_i}, t) + \frac{\partial W}{\partial t} (q_i, \alpha_i, t) = 0 \]

which establishes the theorem.

**Note:** This result agrees with a statement in Pars, p. 450, if it is assumed that a typographical error has been made there and that he means equation 16.5-4 rather than 16.5-6. This would be consistent with his earlier reference to 16.5-4 as "the modified partial differential equation."

**Example--Central Orbit, Polar Coordinates:**

(a) \( H = \frac{1}{2} p_r^2 + \frac{1}{2} \frac{p_{\theta}^2}{r^2} \) \( + V(r) = h \)

(b) \( p_{\theta} = \alpha = \text{constant} \)

Solve (a) for \( p_r \). One has*

*"See next page."
where

\[ f(r) = 2h - 2V - \frac{a^2}{r^2}. \]

\[ dW = p_r \, dr + p_\theta \, d\theta - h \, dt \]

(c) \[ W = -ht + \alpha \theta + \int_{r_1}^{r} f(r) \, dr \]

Either yields a complete integral of the Hamilton-Jacobi equation. The Hamilton-Jacobi equation is:

(d) \[ \frac{\partial W}{\partial t} + \frac{1}{2} \left( \frac{\partial W}{\partial r} \right)^2 + \frac{1}{2r^2} \left( \frac{\partial W}{\partial \theta} \right)^2 + V = 0 \]

\[ \frac{\partial W}{\partial \theta} = \alpha \]

\[ \frac{\partial W}{\partial t} = -h \]

\[ \frac{\partial W}{\partial r} = \pm f(r) \]

Substituting into (d), one finds

\[ -h + \frac{1}{2} f(r) \, \alpha^2 + V = -h + \frac{1}{2} \left( 2h - 2V - \frac{a^2}{r^2} \right) \]

\[ + \frac{1}{2r^2} \, \alpha^2 + V = 0 \]

*A theorem on page 323 of Whittaker's A Treatise on the Analytical Dynamics of Particles and Rigid Bodies assures the reader that the transformation
\[
\begin{align*}
p_\theta &= \alpha \\
p_r &= \pm \sqrt{f(r)}
\end{align*}
\]
transforms
\[
\sum_{i=1}^{n} p_i dq_i - H(q_i, p_i, t) dt
\]
into the differential form
\[
dW - \sum_{i=1}^{n} \frac{\partial W}{\partial q_i} dq_i
\]
It is a simple matter to show that the functions
\[
\begin{cases}
(a) \quad \frac{1}{2} p_r^2 + \frac{1}{r^2} p_\theta^2 + V(r) = \phi_1 \\
(b) \quad p_\theta = \alpha = \phi_2
\end{cases}
\]
are in involution; i.e., \([\phi_1, \phi_2] = 0\). Poisson brackets are zero, so that the theorem just cited may be applied.

It may be that there are \(n\) distinct integrals (in involution)
\[
\phi_i(q_i, p_i, t) = \alpha_i \quad (i=1,2,\ldots,n) \quad (3)
\]
where \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) are arbitrary constants, for the dynamical system
\[
\begin{align*}
\dot{q_i} &= \frac{\partial H}{\partial p_i}(q_i, p_i, t) \\
\dot{p_i} &= -\frac{\partial H}{\partial q_i}(q_i, p_i, t)
\end{align*} \quad (i=1,2,\ldots,n)
\]
It may be that all of these integrals cannot be solved for \((p_1, p_2, \ldots, p_n)\) so that they can be obtained in the form
\[ p_i = f_i(q_i, \alpha_i, t) \quad (i=1,2,\ldots,n) \quad (5) \]

However, it may be possible to solve \( I(\ell < n) \) of these integrals for the \( p \)'s and express them in the form of Eq (5). Suppose further that the remaining can be solved in integrals \( (m < n, \ell + m = n) \) for the \( q \)'s and express them in terms of \( p_i, \alpha_i, \) and \( t \). Relabel the coordinates, setting \( P_j, (j=1,2,\ldots,\ell), \) in one-to-one correspondence with the \( p \)'s which have been solved for explicitly, taken in any order, and \( Q_k, (k=1,2,\ldots,m), \) in one-to-one correspondence with the \( q \)'s which have been solved for explicitly, taken in any order. Thus, the \( n \) integrals may be written in the form

\[
\begin{align*}
  P_j &= F_j(q_i, \alpha_i, t) \quad (j=1,2,\ldots,\ell) \\
  Q_k &= H_k(p_i, \alpha_i, t) \quad (k=1,2,\ldots,m) 
\end{align*}
\]

(6)

Suppose now that in Eqs (6) none of the \( P_j \) appear in the right-hand sides of the expressions for \( Q_k \) and that none of the \( Q_k \) appear in the right-hand sides of the expressions for \( P_j \).

Introduce the following canonical transformation of coordinates

\[
\begin{align*}
  p_j^* &= P_j, \quad (j=1,2,\ldots,\ell) \\
  p_{\ell+k}^* &= -Q_k, \quad (k=1,2,\ldots,m) \\
  q_k^* &= P_{\ell+k}, \quad (k=1,2,\ldots,m) \\
  q_{m+j}^* &= Q_{m+j}, \quad (j=1,2,\ldots,\ell)
\end{align*}
\]

(7)

Equations (6) may now be written in the form

\[ p_i^* = f_i(q_i^*, \alpha_i, t) \quad (i=1,2,\ldots,n) \quad (8) \]

Since Eq (8) is in the form of Eq (2), Theorem 1 may be applied, in conjunction with Whittaker's theorem, to obtain a complete integral of the Hamilton-Jacobi equation expressed in the \textit{starred} coordinates.
A NOTE ON DISTINCT INTEGRALS FOR A PARTICLE
IN A UNIFORM GRAVITY FIELD IN A PLANE

\[ H = \frac{1}{2}(p_x^2 + p_h^2) + gh \]
\[ \dot{x} = \frac{\partial H}{\partial p_x} = p_x \]
\[ \dot{h} = \frac{\partial H}{\partial p_h} = p_h \]
\[ H_0 = \frac{1}{2}(p_x^2 + p_h^2) \]
\[ \dot{p}_x = 0 + p_x = a \]
\[ \dot{p}_h = -\frac{\partial H}{\partial h} = 0 + p_h = b \]

Direct integration of canonical equations:
\[
\begin{align*}
\dot{x} &= a, & x &= at + c \\
\dot{h} &= b, & h &= bt + d
\end{align*}
\]

(1)

Unperturbed problem (Hamilton-Jacobi Equation):
\[
\frac{dS}{dt} + \frac{1}{2}\left(\left(\frac{dS}{dx}\right)^2 + \left(\frac{dS}{dh}\right)^2\right) = 0
\]

Assume
\[
S = -\alpha t + S_1(x) + S_2(h)\]
\[
\left(\frac{dS_1}{dx}\right)^2 + \left(\frac{dS_2}{dh}\right)^2 = 2\alpha
\]

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\[
\left(\frac{dS_1}{dx}\right)^2 = 2a_1 - \left(\frac{dS_2}{dh}\right)^2 = a_2^2
\]
\[
S_1 = a_2x
\]
\[
\left(\frac{dS_2}{dh}\right)^2 = 2a_1 - a_2^2
\]
\[
S_2 = \sqrt{2a_1 - a_2^2}h
\]
\[
S = -a_1t + a_2x + \sqrt{2a_1 - a_2^2}h
\]
\[
px = \frac{\partial S}{\partial x} = a_2
\]
\[
ph = \frac{\partial S}{\partial h} = \sqrt{2a_1 - a_2^2}h
\]
\[
\beta_1 = -\frac{\partial S}{\partial a_1} = t - \frac{h}{\sqrt{2a_1 - a_2^2}}
\]
\[
\beta_2 = -\frac{\partial S}{\partial a_2} = \frac{h\beta_2}{\sqrt{2a_1 - a_2^2}} - x
\]

A comparison of solution (3) with Eq (1) yields

\[
\begin{align*}
a_2 &= a, & \beta_1 &= t - \frac{h}{b} = -\frac{d}{b} \\
a_1 &= \frac{a^2 + b^2}{2}, & \beta_2 &= c + \frac{ad}{b}
\end{align*}
\]

Substitute into Eq (2) to get

\[
\begin{align*}
S^\ast &= -\frac{(a^2 + b^2)}{2}t + ax + bh \\
px^\ast &= \frac{\partial S^\ast}{\partial x} = a, & \beta_1^\ast &= -\frac{\partial S^\ast}{\partial a} = at - x \\
ph^\ast &= \frac{\partial S^\ast}{\partial h} = b, & \beta_2^\ast &= -\frac{\partial S^\ast}{\partial b} = bt - h
\end{align*}
\]

It appears that \(\beta_1^\ast = -c\) and \(\beta_2^\ast = -d\).
Perturbation:

\[
\begin{align*}
\dot{\alpha}_1 &= \frac{\partial H_1}{\partial \beta_1}, & \dot{\beta}_1 &= -\frac{\partial H_1}{\partial \alpha_1} \\
\dot{\alpha}_2 &= \frac{\partial H_1}{\partial \beta_2}, & \dot{\beta}_2 &= -\frac{\partial H_1}{\partial \alpha_2}
\end{align*}
\]

(7)

where, since \( h = \sqrt{2\alpha_1 - \alpha_2^2} \) \((t - \beta_1)\), \( H_1 = H_1(\alpha_1, \alpha_2, \beta_1, t) \).

\[
\begin{align*}
\dot{a} &= \frac{\partial H_1}{\partial \beta_1*}, & \dot{\beta}_1* &= -\frac{\partial H_1}{\partial a} \\
\dot{b} &= \frac{\partial H_1}{\partial \beta_2*}, & \dot{\beta}_2* &= -\frac{\partial H_1}{\partial b}
\end{align*}
\]

(8)

where, since \( h = bt - \beta_2* \), \( H_1 = H_1(b, \beta_2*, t) \).

Variation of Parameters:

Assume

\[
\begin{align*}
x &= at + c, & p_x &= a \\
h &= bt + d, & p_h &= b
\end{align*}
\]

is a transformation of variables from the canonical set of equations

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p_x}, & \dot{p}_x &= -\frac{\partial H}{\partial x} \\
\dot{h} &= \frac{\partial H}{\partial p_h}, & \dot{p}_h &= -\frac{\partial H}{\partial h}
\end{align*}
\]

to new coordinates \( a, b, c, \) and \( d \). Thus,

\[
\begin{align*}
\dot{x} &= a + \dot{a}t + \dot{c} & (1) \quad \dot{c} + \dot{a}t = 0 \quad c = c_1 \\
\dot{h} &= b + \dot{b}t + \dot{d} & (2) \quad \dot{d} + \dot{b}t = 0 \quad a = a_1 \\
\dot{p}_x &= \dot{a} & (3) \quad \dot{a} = 0 \quad b = -gt + b_1 \\
\dot{p}_h &= \dot{b} & (4) \quad \dot{b} = g \quad d = \frac{gt^2}{2} + d_1
\end{align*}
\]
\[ x = a_1 t + c_1 \]

\[ h = -\frac{gt^2}{2} + v_1 t + d_1 \]

Return to Eq (8) \[ H_1 = g(b_1 - \beta_2^* \) \]

1. \[ \dot{a} = \frac{\partial H_1}{\partial \beta_1^*} = 0 \], \[ \beta_1^* = -\frac{\partial H_1}{\partial a} = 0 \]

2. \[ \dot{b} = \frac{\partial H_1}{\partial \beta_2^*} = -g \], \[ \beta_2^* = -\frac{\partial H_1}{\partial b} = -gt \]

3. \[ \dot{a} = a^* \], \[ \beta_1^* = \beta_1^{**} \]

4. \[ \dot{b} = -gt + b^* \], \[ \beta_2^* = -\frac{gt^2}{2} + \beta_2^{**} \]

\[ x = a'^t - \beta_1^{**} \]

\[ h = -\frac{gt^2}{2} + b'^t - \beta_2^{**} \]

Return to Eq (7) \[ H_1 = g/2a_1 - \alpha_1 (t - \beta_1) \]

1. \[ \dot{a}_1 = \frac{\partial H_1}{\partial \beta_1} = -g/2a_1 - \alpha_1 \], \[ \beta_1 = -\frac{\partial H_1}{\partial a_1} = \frac{g(t - \beta_1)}{\sqrt{2a_1 - \alpha_2}} \]

2. \[ \dot{a}_2 = 0 \], \[ \beta_2 = -\frac{\partial H_1}{\partial a_2} = \frac{ga_2(t - \beta_2)}{\sqrt{2a_1 - \alpha_2}} \]

Conclusion: The constants which appear in the solution \( a' \) Hamilton's equations obtained by quadratures are not in general canonical even though in some problems it appears so.
HAMilton Function for Triaxial Body (No Forces)

Let

\[ f(\phi) = \left( \frac{\sin^2 \phi}{2A} + \frac{\cos^2 \phi}{2B} \right) \]

\[ g(\phi) = \left( \frac{1}{A} - \frac{1}{B} \right) \sin \phi \cos \phi \]

\[ H = \frac{1}{\sin^2 \phi} \left( p_\psi - P_\theta \cos \phi \right)^2 + \frac{1}{2C} F_\phi^2 \]

\[ + \frac{g}{\sin \phi} p_\psi p_\theta - \frac{g}{\sin \phi} p_\theta p_\theta \cos \phi \]

\[ - \frac{f}{\sin^2 \phi} 2p_\psi p_\theta \cos \phi \]

If \( A = B \), \( f(\phi) = 1/2A \) and \( g(\phi) = 0 \). and one has

\[ H = \frac{1}{2A \sin^2 \phi} (P_\psi - P_\phi \cos \theta)^2 + \frac{1}{2C} P_\phi^2 + \frac{1}{2A} P_\theta^2 \]

Let

\[ q(\phi) = \frac{\cos^2 \phi}{2A} + \frac{\sin^2 \phi}{2B} \]

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and rewrite the general form of $H$. Note that $q(\phi) = A + B/2AB - f(\phi)$.

$$H = \frac{f(\phi)}{\sin^2 \theta} \left[ p_\psi - p_\phi \cos \theta \right]^2 + \frac{1}{2C} p_\phi^2$$

$$+ \frac{g(\phi)}{\sin \theta} p_\theta (p_\psi - p_\phi \cos \theta) + q(\phi)p_\theta^2$$

The Hamilton-Jacobi equation may be written: $\partial S/\partial t + H = 0$, where $H = a_1$, a constant.

$$\frac{\partial S}{\partial t} = -a_1$$

$$a_1 = \frac{f}{\sin^2 \theta} z^2 + \frac{1}{2C} \left( \frac{\partial S}{\partial \phi} \right)^2 + \frac{g}{\sin \theta} \frac{\partial S}{\partial \theta} z + q(\phi) \left( \frac{\partial S}{\partial \theta} \right)^2$$

where

$$z = p_\psi - p_\phi \cos \theta = \frac{\partial S}{\partial \psi} - \frac{\partial S}{\partial \phi} \cos \theta$$

Assume

$$S = S_1(t) + S_2(\phi) + S_3(\psi) + S_4(\theta)$$

Then

$$\frac{\partial S}{\partial t} = \frac{\partial S_1}{\partial t} = -a_1$$

and

$$S_1 = -a_1 t$$

$$a_1 = \frac{f}{\sin^2 \theta} z^2 + \frac{1}{2C} \left( \frac{\partial S_2}{\partial \phi} \right)^2 + \frac{g}{\sin \theta} \frac{\partial S_4}{\partial \theta} z + q(\phi) \left( \frac{\partial S_4}{\partial \theta} \right)^2$$

where

$$z = \frac{\partial S_3}{\partial \psi} - \frac{\partial S_2}{\partial \phi} \cos \theta$$

Using the quadratic formula, one may write:

$$z^2 + \frac{\sin^2 \theta \left( \frac{\partial S_2}{\partial \phi} \right)^2}{2Cf \left( \frac{\partial \phi}{\partial \phi} \right)^2} + \frac{\sin \theta g(\phi) \left( \frac{\partial S_4}{\partial \theta} \right) z + q \sin^2 \theta \left( \frac{\partial S_4}{\partial \theta} \right)^2}{f \left( \frac{\partial \theta}{\partial \theta} \right)^2} = \frac{\sin^2 \theta a_1}{f}$$
\[ z^2 + \frac{g}{f} \sin \theta \left( \frac{dS_2}{d\theta} \right)^2 + \frac{\sin^2 \theta}{f} \left[ \frac{1}{2C} \left( \frac{dS_2}{d\phi} \right)^2 + q \left( \frac{dS_2}{d\theta} \right)^2 - \alpha_1 \right] = 0 \]

\[ z = -\frac{g}{2f} \sin \theta \left( \frac{dS_2}{d\theta} \right) = \frac{1}{2} \left( \frac{g^2}{f} \sin^2 \theta \left( \frac{dS_2}{d\theta} \right)^2 \right) - 4 \sin^2 \theta \left[ \frac{1}{2C} \left( \frac{dS_2}{d\phi} \right)^2 + q \left( \frac{dS_2}{d\theta} \right)^2 - \alpha_1 \right]^{1/2} \]

\[ \frac{dS_3}{d\phi} = \frac{dS_2}{d\phi} \cos \theta + \sin \theta \left[ -\frac{g}{f} \left( \frac{dS_2}{d\theta} \right) \right] + \sqrt{\left( \frac{g^2}{f} \sin^2 \theta \right)^2 + \frac{4\sin^2 \theta}{f} \left[ \alpha_1 - \frac{1}{2C} \left( \frac{dS_2}{d\phi} \right)^2 + q \left( \frac{dS_2}{d\theta} \right)^2 \right]} \]

\[ = \alpha_3 \]

\[ S_3 = \alpha_3 \psi \]

\[ \frac{g^2}{f^2} \left( \frac{dS_2}{d\theta} \right)^2 - \frac{4q}{f} \left( \frac{dS_2}{d\theta} \right)^2 = \left( \frac{dS_2}{d\theta} \right)^2 \left( \frac{1}{2C} - \frac{g^2}{f^2} \right) \]

\[ \alpha_3 = \frac{dS_3}{d\psi} = \frac{dS_2}{d\phi} \cos \theta + \sin \theta \left[ -\frac{g}{f} \left( \frac{dS_2}{d\theta} \right) \right] + \sqrt{\left( \frac{g^2}{f^2} - \frac{4q}{f} \right) \left( \frac{dS_2}{d\theta} \right)^2 + \frac{4\alpha_1}{f} - \frac{4}{2Cf} \left( \frac{dS_2}{d\phi} \right)^2} \]

\[ g^2 + \frac{1}{A^2} - \frac{2}{AB} + \frac{1}{B^2} \sin^2 \phi \cos^2 \phi \]

\[ 4qf = 4 \left( \sin^2 \phi + \cos^2 \phi \right) \left( \frac{\sin^2 \phi}{2A} + \frac{\cos^2 \phi}{2B} \right) \]

\[ = \frac{\sin^2 \phi \cos^2 \phi}{A^2} + \frac{\sin^2 \phi \cos^2 \phi}{B^2} + \sin^4 \phi + \cos^4 \phi \]
\[ g^2 - 4qf = - \left( \frac{\sin^2 \phi + 2 \sin^2 \theta \cos^2 \phi - \cos^4 \phi}{AB} \right) \]

\[ = - \left( \frac{\sin^2 \phi + \cos^2 \phi}{AB} \right)^2 = - \frac{1}{AB} \]

\[ a_3 = \frac{dS_2}{d\phi} \cos \theta + \frac{\sin \theta}{2} \left[ - \frac{g}{f} \left( \frac{dS_1}{d\theta} \right) \right. \]

\[ + \left. \sqrt{\frac{4a_1}{f} - \frac{1}{ABf^2} \left( \frac{dS_1}{d\theta} \right)^2 - \frac{2}{CF} \left( \frac{dS_2}{d\phi} \right)^2} \right] \]

There does not appear to be any way to separate the right-hand side of the preceding equation.

\[ 2 \csc \theta (a_3 - p_\phi \cos \theta) = - \frac{g}{f} p_\theta + \sqrt{\frac{4a_1}{f} - \frac{1}{ABf^2} p_\theta^2 - \frac{2}{CF} p_\phi^2} \]

The explicit dependence on \( \theta \) can be eliminated by using the relationship

\[ p_\theta^2 = h^2 - p_\phi^2 - \csc^2 \theta (a_3 - p_\phi \cos \theta)^2 \]

which is valid for the triaxial problem with no forces if \( h \) is constant.

\[ 4 \csc^2 \theta (a_3 - p_\phi \cos \theta)^2 = \frac{4a_1}{f} - \frac{1}{ABf^2} p_\theta^2 - \frac{2}{CF} p_\phi^2 + \frac{g^2}{f^2} p_\theta^2 \]

\[ - \frac{2g}{f} p_\theta \sqrt{\frac{4a_1}{f} - \frac{1}{ABf^2} p_\theta^2 - \frac{2}{CF} p_\phi^2} \]

\[ 4(h^2 - p_\phi^2 - p_\theta^2) = \frac{4a_1}{f} + \frac{1}{f^2} \left( g^2 - \frac{1}{AB} \right) p_\theta^2 - \frac{2}{CF} p_\phi^2 \]

\[ - \frac{2g}{f} p_\theta \sqrt{\frac{4a_1}{f} - \frac{1}{ABf^2} p_\theta^2 - \frac{2}{CF} p_\phi^2} \]
\[
\frac{2}{\text{Cf}} P_\phi^2 + 4 \left( h^2 - P_\phi^2 - \frac{a_1}{f} \right) = \left[ 4 + \frac{1}{\text{f}^2} \left( g^2 - \frac{1}{AB} \right) \right] P_\epsilon^2
\]

\[
- \frac{2g}{f} P_\mu \sqrt{\frac{4a_1}{f} - \frac{1}{AB}} + \frac{2}{\text{Cf}} P_\phi^2
\]

Let

\[
\tau \equiv \left[ \frac{2}{\text{Cf}} P_\phi^2 + 4 \left( h^2 - P_\phi^2 - \frac{a_1}{f} \right) \right]
\]

\[
\nu \equiv 4 + \frac{1}{\text{f}^2} \left( g^2 - \frac{1}{AB} \right)
\]

\[
\xi \equiv \frac{\tau}{\nu}
\]

\[
n = - \frac{2g}{f \nu}
\]

\[
n \sqrt{\frac{4a_1}{f} - \frac{1}{AB} \text{f}^2} P_\theta^2 - \frac{2}{\text{Cf}} P_\phi^2 P_\theta = \xi - P_\theta^2
\]

\[
n^2 \left( \frac{4a_1}{f} - \frac{1}{AB} \text{f}^2 \right) P_\theta^2 - \frac{2}{\text{Cf}} P_\phi^2 \right) = \xi^2 - 2\xi P_\theta^2 + P_\theta^4
\]

\[
p_\theta^4 + \left( \frac{n^2}{AB} - 2\xi \right) P_\theta^2 = \xi^2 - n^2 \left( \frac{4a_1}{f} - \frac{2}{\text{Cf}} P_\phi^2 \right)
\]
RELATIONSHIP BETWEEN CONJUGATE MOMENTA AND ANGULAR MOMENTUM

Let

\[
\vec{p} = \begin{pmatrix} p_\psi \\ p_\theta \\ p_\phi \end{pmatrix}, \quad \vec{h} = \begin{pmatrix} h_x^* \\ h_y^* \\ h_z^* \end{pmatrix}, \quad \vec{h}' = \begin{pmatrix} h_x' \\ h_y' \\ h_z' \end{pmatrix}
\]

where the Euler angles \( \psi, \theta, \) and \( \phi \) are shown below, relating the body-fixed axes \( 0x'y'z' \) to the space-fixed axes \( 0x^*y^*z^* \).

The matrix

\[
\vec{\xi} = \begin{pmatrix} \psi \\ \theta \\ \phi \end{pmatrix}
\]

and \( \vec{p} \) represents the conjugate momenta matrix while \( \vec{h} \) and \( \vec{h}' \) represent the angular momentum vector referenced to space-fixed and body-fixed axes, respectively.
\[ \ddot{P} = \dot{x} \xi \]  

(1)

Explicitly,

\[
\begin{pmatrix}
\dot{p}_\psi \\
\dot{p}_\theta \\
\dot{p}_\phi \\
\end{pmatrix} = \\
\begin{pmatrix}
(A \sin^2 \phi + B \cos^2 \phi) \sin^2 \theta + C \cos^2 \theta, (A - B) \sin \phi \cos \phi \sin \theta, C \cos \theta \\
(A - B) \sin \phi \cos \phi \sin \theta, A \cos^2 \phi + B \sin^2 \phi, 0 \\
C \cos \theta, 0, C \\
\end{pmatrix}
\]

The kinetic energy for a force free triaxial rigid body is given by

\[ T = \frac{1}{2A} h_x^2 + \frac{1}{2B} h_y^2 + \frac{1}{2C} h_z^2, \]  

(2)

where

\[
\begin{align*}
\dot{h}_x &= A \dot{\omega}_x = A \dot{\theta} \cos \phi + A \dot{\psi} \sin \theta \sin \phi \\
\dot{h}_y &= B \dot{\omega}_y = -B \dot{\theta} \sin \phi + B \dot{\psi} \sin \theta \cos \phi \\
\dot{h}_z &= C \dot{\omega}_z = C \dot{\phi} + C \dot{\psi} \cos \theta \\
\end{align*}
\]

(3)

Also,

\[ H = \tilde{T} H' \]  

(4)

Explicitly,

\[
\begin{pmatrix}
\dot{h}_x' \\
\dot{h}_y' \\
\dot{h}_z' \\
\end{pmatrix} = \\
\begin{pmatrix}
\cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta, -\cos \psi \sin \phi - \sin \psi \cos \phi \cos \theta, \sin \psi \sin \theta \\
\sin \psi \cos \phi + \cos \psi \sin \phi \cos \theta, -\sin \psi \sin \phi + \cos \psi \cos \phi \cos \theta, -\cos \psi \sin \theta \\
\sin \phi \sin \theta, \cos \phi \sin \theta, \cos \theta \\
\end{pmatrix}
\]
\[ \vec{p} = \vec{N}T_{H'} \]  

Explicitly,
\[
\begin{pmatrix}
 p_\psi \\
p_\theta \\
p_\phi
\end{pmatrix} =
\begin{pmatrix}
 \sin \phi \sin \theta & \cos \phi \sin \theta & \cos \theta \\
 \cos \phi & -\sin \phi & 0 \\
 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
 h_{x'} \\
h_{y'} \\
h_{z'}
\end{pmatrix}
\]

\[ \vec{p} = \vec{N}T_{H'} \]  

Explicitly,
\[
\begin{pmatrix}
 p_\psi \\
p_\theta \\
p_\phi
\end{pmatrix} =
\begin{pmatrix}
 0 & 0 & 1 \\
 \cos \psi & \sin \psi & 0 \\
 \sin \psi \sin \theta & -\cos \psi \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
 h_{x''} \\
h_{y''} \\
h_{z''}
\end{pmatrix}
\]

Also,
\[
\begin{align*}
h_{x'} &= p_\theta \cos \phi + \frac{\sin \phi}{\sin \theta} (p_\psi - p_\phi \cos \theta) \\
h_{y'} &= -p_\theta \sin \phi + \frac{\cos \phi}{\sin \theta} (p_\psi - p_\theta \cos \theta) \\
h_{z'} &= p_\phi = h \cos \theta'
\end{align*}
\]  

Consider the case in which the direction of the angular momentum vector is fixed in space. Choose this direction as an axis and redesignate it by the letter \( \zeta \). Let the line of nodes of the angular momentum plane (a plane through the center of mass of the body perpendicular to the \( \zeta \) axis) with the space-fixed plane \( x'y'' \) be designated by \( \xi \). Consider the figure below.
$H$ may be represented in the form

$$
H = \begin{pmatrix}
0 \\
0 \\
h
\end{pmatrix}
$$

(8)

If Eq (6) is used with $\psi$ and $\theta$ replaced by $\phi^*$ and $\theta'$, respectively,

$$
\begin{pmatrix}
\mathbf{p}_{\phi^*} \\
\mathbf{p}_{\theta'} \\
\mathbf{p}_{\phi'}
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 1 \\
\cos \phi^* & \sin \phi^* & 0 \\
\sin \phi^* \sin \theta' & -\cos \phi^* \sin \theta' & \cos \theta'
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
h
\end{pmatrix}
$$

(9)

$$
\mathbf{p}_{\phi^*} = h \\
\mathbf{p}_{\theta'} = 0 \\
\mathbf{p}_{\phi'} = h \cos \theta'
$$

Since $\mathbf{p}_{\theta'} = 0$,

$$
\begin{align*}
\mathbf{h}_{x'} &= \frac{\sin \theta'}{\sin \theta'} (\mathbf{p}_{\phi^*} - \mathbf{p}_{\phi'} \cos \theta') \\
\mathbf{h}_{y'} &= \frac{\cos \theta'}{\sin \theta'} (\mathbf{p}_{\phi^*} - \mathbf{p}_{\phi'} \cos \theta') \\
\mathbf{h}_{z'} &= \mathbf{p}_{\phi'}
\end{align*}
$$

(10)

By utilizing the fact that $\mathbf{p}_{\phi'} = h \cos \theta'$, one can write $\mathbf{T}$ in the form

$$
\mathbf{T} = \frac{1}{2A} \mathbf{h}_{x'}^2 + \frac{1}{2B} \mathbf{h}_{y'}^2 + \frac{1}{2C} \mathbf{h}_{z'}^2
$$

$$
\mathbf{T} = \left( \frac{\sin^2 \phi'}{2A} + \frac{\cos^2 \phi'}{2B} \right) (\mathbf{p}_{\phi^*}^2 - \mathbf{p}_{\phi'}^2) + \frac{1}{2C} \mathbf{p}_{\phi'}^2
$$

= Hamilton Function

(11)
Designate $H = \text{Hamilton function} = \alpha_1$, a constant of motion.

\[
\begin{align*}
\phi' &= \frac{\partial H}{\partial p_\phi}, \\
\phi^* &= \frac{\partial H}{\partial p_{\phi^*}}, \\
p_{\phi^*} &= -\frac{\partial H}{\partial \phi^*} = 0 \Rightarrow h = \text{constant} = p_{\phi^*} \\
p_{\phi'} &= -\frac{\partial H}{\partial \phi'} 
\end{align*}
\]

(12)

Also,

\[
\cos \theta' = \frac{p_{\phi'}}{h}
\]

and

\[
p_{\theta'} = 0
\]

Suppose $\phi'$, $\phi^*$, $p_{\phi^*}$, $p_{\phi'}$, and $p_{\theta'}$ are known. Are $\theta$, $\psi$, $\phi$, $p_{\theta}$, $p_{\psi}$, and $p_\phi$ referenced to some space-fixed system $0x*y*z^*$ known?

\[
\begin{align*}
\theta &= \theta(\theta', \theta^*, \phi^*) \\
\psi &= \psi(\theta', \theta^*, \phi^*, \psi^*) \\
\phi &= \phi(\theta', \theta^*, \phi^*, \phi') \\
p_{\phi} &= h \cos \theta' = p_{\phi'} \\
p_{\psi} &= h \cos \theta^* = p_{\phi^*} \cos \theta^* \\
p_{\theta} &= \cos \psi h_{x^*} + \sin \psi h_{y^*}
\end{align*}
\]

(13)
Using Eq (4) with $\psi$, $\theta$, and $\phi$ replaced by $\phi^*$, $\theta^*$, and $0$,

\[
\begin{pmatrix}
  h_{x^*} \\
  h_{y^*} \\
  h_{z^*}
\end{pmatrix} =
\begin{pmatrix}
  \cos \psi^* & -\sin \psi^* \cos \theta^* & \sin \psi^* \sin \theta^* \\
  \sin \psi^* & \cos \psi^* \cos \theta^* & -\cos \psi^* \sin \theta^* \\
  0 & \sin \theta^* & \cos \theta^*
\end{pmatrix}
\begin{pmatrix}
  0 \\
  h \\
  0
\end{pmatrix}
\]

\[
\begin{aligned}
  h_{x^*} &= h \sin \psi^* \sin \theta^* \\
  h_{y^*} &= h \cos \psi^* \sin \theta^* \\
  h_{z^*} &= h \cos \theta^*
\end{aligned}
\]

(14)

\[
\begin{align*}
  P_\theta &= \cos \psi (h \sin \psi^* \sin \theta^*) + \sin \psi (-\cos \psi^* \sin \theta^*) \\
  &= -h \sin \theta^* \sin (\psi - \psi^*)
\end{align*}
\]

(15)

If $p_{\phi'}$ is known, $\theta'$ is also known. Also, $\theta^*$ and $\psi^*$ are prescribed constants, independent of each other and independent of $\phi'$, $\phi^*$, $p_{\phi'}$, and $p_{\phi^*}$. Hence, $\psi$, $\theta$, $\phi$, $p_{\psi}$, $P_\theta$, and $p_{\phi^*}$ are known. Thus, the independent variables $\phi'$, $\phi^*$, $p_{\phi'}$, $p_{\phi^*}$, $\theta^*$, and $\psi^*$ serve to describe the motion of the triaxial body with respect to the space-fixed system $Ox'y'z'$.

If Eqs (6) and (14) are used, $p_{\psi} = h \cos \theta^*$ is obtained. Since $p_{\psi}$ does not depend on $\psi$, $p_{\psi}^* = h \cos \theta^*$ may be written where $\psi$ and $\psi^*$ lie in the same plane. If $p_{\phi^*}$ is known, $\phi^*$ is known; hence, the independent variables ($\phi'$, $\phi^*$, $\psi^*$, $p_{\phi'}$, $p_{\phi^*}$, and $p_{\psi^*}$) will serve to describe the triaxial motion with respect to the space-fixed system $Ox'y'z'$. Kinetic energy $T = H$ (for this extended problem) is expressed in terms of ($\phi'$, $\phi^*$, $\psi^*$, $p_{\phi'}$, $p_{\phi^*}$, and $p_{\psi^*}$) and still given by Eq (11). The canonical equations may be extended to include

\[
\begin{aligned}
  \dot{\psi}^* &= \frac{\partial H}{\partial \psi^*} = 0 \\
  \dot{\phi}^* &= \frac{\partial H}{\partial \phi^*} = 0
\end{aligned}
\]

(16)

since Eqs (16) are consistent with the facts that $p_{\psi^*}$ and $\psi^*$ are constants of motion.
Thus, $H$ can be interpreted, as given by Eq (11), as the Hamilton function for the motion of a triaxial body with respect to the space-fixed system under no forces. The corresponding canonical equations are

$$
\begin{align*}
\dot{p}_\phi &= -\frac{\partial H}{\partial \phi}, & \dot{\phi} &= \frac{\partial H}{\partial p_\phi}, \\
\dot{p}_\psi &= -\frac{\partial H}{\partial \psi}, & \dot{\psi} &= \frac{\partial H}{\partial p_\psi} \\
\end{align*}
$$

(17)

The differential equations are explicitly:

$$
\begin{align*}
\dot{p}_\phi &= \left(p_{\phi*}^2 - p_{\phi'}^2\right)\frac{(A - B)}{AB} \sin \phi' \cos \phi' \\
\dot{p}_\phi &= 0 \\
\dot{p}_\psi &= 0 \\
\dot{\phi'} &= \left[\frac{1}{C} - \left(\frac{\sin^2 \phi'}{A} + \frac{\cos^2 \phi'}{B}\right)\right] p_{\phi'} \\
\dot{\phi*} &= \left(\frac{\sin^2 \phi'}{A} + \frac{\cos^2 \phi'}{B}\right) p_{\phi*} \\
\dot{\psi*} &= 0 \\
\end{align*}
$$

(18)

Then,

$$
\begin{align*}
p_{\phi*} &= \alpha_2, \text{ a constant} \\
p_{\psi*} &= \alpha_3, \text{ a constant} \\
\psi* &= -\beta_3, \text{ a constant} \\
\end{align*}
$$

(19a, 19b, 19c)
The differential equations for \( p_\phi' \), \( \phi' \), and \( \phi^* \) may now be written:

\[
\begin{align*}
\dot{p}_\phi' &= \left(a_2^2 - p_\phi'^2\right) \left(\frac{A - B}{AB}\right) \sin \phi' \cos \phi' \quad (20a) \\
\dot{\phi}' &= \left[\frac{1}{C} - \left(\frac{\sin^2 \phi' + \cos^2 \phi'}{A^2}\right)^\prime\right] p_\phi' \quad (20b) \\
\dot{\phi}^* &= \left(\frac{\sin^2 \phi' + \cos^2 \phi'}{A^2}\right)^\prime \alpha_2 \quad (20c)
\end{align*}
\]

From Eqs (20a) and (20b),

\[
- \frac{\dot{w}}{w} = \frac{\dot{u}}{u} \quad (21)
\]

where

\[
\begin{align*}
w &= a_2^2 - p_\phi'^2. \\
u &= \frac{1}{C} - \left(\frac{\sin^2 \phi' + \cos^2 \phi'}{A^2}\right)^\prime
\end{align*}
\]

Integration of Eq (21) yields

\[
w u = \kappa, \text{ a constant} \quad (22)
\]

To evaluate \( \kappa \), it is noted from Eq (11) that

\[
\alpha_1 = \frac{\mu_\phi'^2}{2} - \frac{a_2^2 \sin^2 \phi' + \cos^2 \phi'}{2}
\]

and from Eq (22), it is found that

\[
w u = \kappa = \frac{a_2^2}{C} - 2\alpha_1 \quad (23)
\]

Equation (23) permits the expression of \( p_\phi' \), in terms of \( \phi' \). One first writes

\[
\begin{align*}
u &= \frac{c' + d' \sin^2 \phi'}{ABC} \\
c' &= A(B - C) \\
d' &= C(A - B)
\end{align*}
\]
Then,

\[ p_{\phi^t}^2 = \alpha_2^2 - \frac{(\alpha_2^2 - aC_1)}{Cu} \]

This last equation reduces to

\[ p_{\phi^t}^2 = \frac{C[a' + b' \sin^2 \phi']}{[c' + d' \sin^2 \phi']} \]

\[
\begin{align*}
a' &= A(2B_1 - \alpha_2^2) \\
b' &= \alpha_2^2(A - B) \\
c' &= A(B - C) \\
d' &= C(A - B)
\end{align*}
\]

whence,

\[ p_{\phi^t} = \sqrt{\frac{C(a' + b' \sin^2 \phi')}{c' + d' \sin^2 \phi'}} \]
APPENDIX

ON A METHOD OF OBTAINING A COMPLETE INTEGRAL
O: THE HAMILTON-JACOBI EQUATION ASSOCIATED WITH A DYNAMICAL SYSTEM

Philip M. Fitzpatrick and John E. Cochran

Consider a dynamical system whose equations of motion are

\[
\begin{align*}
\dot{q}_i &= \frac{\partial H(q_j; p_j; t)}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H(q_j; p_j; t)}{\partial q_i}
\end{align*}
\]

where the Hamiltonian, \( H(q_j; p_j; t) \), is understood to be a function of the generalized coordinates, \( q_j \), and their conjugate momenta, \( p_j \), \( j=1,2,...,n \), and possibly the time, \( t \). If one-half of the integrals of Eq (1) have been obtained in a suitable form, there is a well-known theorem, due to Liouville,\(^1\) which may be used to find the remaining integrals. The purpose of this note is to point up the related, but perhaps not so well-known fact that a method of obtaining a complete integral of the Hamilton-Jacobi partial differential equation associated with (1) is implicitly contained in the theorem. Since a complete integral of (1) will permit us to express the solution of (1) in terms of canonical constants of integration, recognition of this fact is of importance in studying perturbations of the original system. The method will be discussed and applied in what follows.

Suppose that \( n \) integrals of a dynamical system with \( 2n \) degrees of freedom are known in the form

\[
\Phi_i(q_j; p_j; t) = a_i, \quad i=1,2,...,n; \quad j=1,2,...,n
\]

where the $\alpha_j$ form a set of $n$ independent constants of integration. If the Poisson bracket expression, $[\phi_i, \phi_j]$, vanishes for each $i$ and $j$ and if the $\phi_i$ are solvable for the $p_i$ in the form

$$ p_i = f_i(q_j; \alpha_j; t), \quad i=1,2,\ldots,n; \quad j=1,2,\ldots,n $$

(3)

the Liouville theorem states that the difference between

$$ \sum_{i=1}^{n} f_i dq_i $$

and $H(q_i; \alpha_i; t) dt$ is the perfect differential of a function $W(u_j; \alpha_j; t)$ and that the remaining $n$ integrals of the system are given by

$$ \frac{dW}{dq_i} = \beta_i, \quad i=1,2,\ldots,n $$

(4)

where the $\beta_i$ form a set of $n$ constants of integration which are independent of each other and of the set formed by the $\alpha_i$.

To say that

$$ \sum_{i=1}^{n} f_i dq_i - H(q_j; \alpha_j; t) dt, \quad j=1,2,\ldots,n $$

(5)

is the perfect differential of a function $W(q_j; \alpha_j; t)$ means that

$$ \frac{dW}{dq_i} = f_i = p_i, \quad i=1,2,\ldots,n $$

(6)

$$ \frac{dW}{dt} = -H $$

(7)

Thus, implicit in the Liouville theorem is the fact that the function $W$ is a complete integral of (7) which is the Hamilton-Jacobi partial differential equation associated with the system.

When the $n$ integrals of (2) can be solved for the $q_i$ instead of the $p_i$, $i=1,2,\ldots,n$, the theorem may also be applied, if the canonical transformation
to new variables \((Q_i, P_i)\) is first introduced. Even if we are not able to solve the \(n\) integrals (2) explicitly for the \(P_i\), or for the \(q_i\), a complete integral may still be obtained in certain important cases now to be discussed.

Suppose we are able to solve the integrals (2) explicitly for \(P_i\) of \(q_i\), or for the \(\xi\) momenta and \(\eta\) coordinates. Suppose further that, after reordering the subscripts, the expressions for the \(\xi\) momenta and \(\eta\) coordinates can be written in the restricted form

\[
\begin{align*}
P_i &= f_i(q_k; p_m; a_j; t), & i &= 1, 2, \ldots, k \leq \xi; \\
& & m &= 1, 2, \ldots, n; \\
q_i &= h_i(q_m; p_k; a_j; t), & i &= \xi + 1, \xi + 2, \ldots, n; k > \xi; \\
& & m &= \leq \xi; j &= 1, 2, \ldots, n
\end{align*}
\] (9)

By introducing the canonical transformation

\[
\begin{align*}
P_i^* &= P_i, & q_i^* &= q_i, & i &= 1, 2, \ldots, \xi \\
P_i^* &= -q_i, & q_i^* &= P_i, & i &= \xi + 1, \xi + 2, \ldots, n
\end{align*}
\] (10)

Eqs (9) may be written in the form

\[
P_i^* = f_i^*(q_j^*; a_j; t), & i &= 1, 2, \ldots, n; j = 1, 2, \ldots, n
\] (11)

Equations (11) are in the form (3) and the theorem may be applied.

**Example 1: Central Orbit in the Plane, Polar Coordinates**

For a particle moving in a plane under a central force derivable from the potential \(V(r)\), the Hamiltonian function is a constant \(a_1\). If we designate by \((p_r, p_\theta)\), the momenta conjugate to the polar coordinates \((r, \theta)\), respectively, see Figure 1, the system has the well-known integrals

\[
p_\theta = a_2, \text{ a constant}
\] (12)
Figure 1

\[ p_r = \pm \sqrt{2[a_1 + V(r)] - \frac{a_2^2}{r^2}} \]  

(13)

From (5), we write

\[ dW = p_r dr + p_\theta d\theta - a_1 dt \]  

(14)

If \( r_0 \) is chosen so that no new independent constant is introduced, the function

\[ W = \int_{r_0}^{r} p_r dr + a_2 \theta - a_1 t \]  

(15)

obtained by integrating (14), satisfies (7). Also, \( W \) is a complete integral of (7) since it contains two non-additive independent constants \( a_1 \) and \( a_2 \).

**Example 2: Free Motion of a Triaxial Rigid Body**

For the free rotation of a triaxial, rigid body about a fixed point \( 0 \), the Hamiltonian function, which is a constant of the motion, \( a_1 \), may be written in terms of the Euler angles \( (\theta, \phi, \psi) \), which specify the position of principal axes at \( 0 \) relative to space-fixed axes \( 0\xi\eta\zeta \) and their conjugate momenta \( (p_\theta, p_\phi, p_\psi) \). See Figure 2.
Three known integrals for this dynamical system are\(^2\)

\[ p_\psi = \alpha_3, \text{ a constant} \quad (16) \]

\[ \theta = \tan^{-1} \left\{ \frac{r^2 - \alpha_2^2 - p_\theta^2}{\alpha_3} \right\} \]

\[ -\tan^{-1} \left\{ \frac{r^2 - p_\theta^2 - p_\phi^2}{p_\phi} \right\} \quad (17) \]

\[ \phi = \tan^{-1} \left\{ \frac{p_\theta}{\sqrt{r^2 - p_\phi^2 - p_\theta^2}} \right\} \]

\[ + \tan^{-1} \left\{ \frac{(A)(2B\alpha_1 - \alpha_2^2)C + (C - B)p_{\phi}^2}{B(2A\alpha_1 - \alpha_2^2)C + (C - A)p_{\phi}^2} \right\}^{1/2} \quad (18) \]

where \(A, B,\) and \(C\) are the principal moments of inertia at \(0\) and \(\alpha_2\) is the constant magnitude of the angular momentum about \(0\).

\(^2\)See Whittaker, p. 325.
Although it is not possible to solve (17) and (18) so that \( p_\phi \) and \( p_\theta \) are expressed in the form (3), the set of equations (16), (17), and (18) is of the form (9); hence, the canonical transformation

\[
\begin{align*}
  p_1 &= -\phi \\
  p_2 &= -\theta \\
  p_3 &= p_\psi \\
  q_1 &= p_\phi \\
  q_2 &= p_\theta \\
  q_3 &= \psi
\end{align*}
\] (19)

allows us to write (16), (17), and (18) in the form (11). Then, from (5), we write

\[
dW = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 - \alpha_1 dt
\] (20)

If \( q_{10} \) and \( q_{20} \) are chosen in a manner so that no new independent constants are introduced, the function

\[
W = -\alpha_1 t + \alpha_3 q_3 + \int_{q_{20}}^{q_2} \tan^{-1}\left\{ \frac{\sqrt{\alpha_2^2 - \alpha_3^2 - x^2}}{\alpha_3} \right\} dx \\
- \int_{q_{20}}^{q_2} \tan^{-1}\left\{ \frac{\sqrt{\alpha_2^2 - \alpha_1^2 - x^2}}{\alpha_1} \right\} dx \\
+ \int_{q_{10}}^{q_1} \tan^{-1}\left\{ \frac{\sqrt{\alpha_2^2 - \alpha_1^2 - x^2}}{\alpha_1} \right\} dx \\
+ \int_{q_{10}}^{q_1} \tan^{-1}\left\{ \frac{(2B - \alpha_1^2)C + (C - B)x^2}{(2B - \alpha_1^2)C + (C - A)x^2} \right\} dx
\] (21)

obtained by integrating (20), is a complete integral of (7).
HAMILTON/JACOBI PERTURBATION METHODS
APPLIED TO THE ROTATIONAL MOTION OF A RIGID BOI
IN A GRAVITATIONAL FIELD

by

Philip M. Fitzpatrick, Grady R. Harmon, Joseph J. F. Li;
and John E. Cochran

Six Months Report to
National Aeronautics and Space Administration
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Computational Theory and Techniques Branch
Cambridge, Massachusetts 02139

on

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(November 1, 1968 – May 1, 1969)

October 1969

ENGINEERING EXPERIMENT STATION

AUBURN UNIVERSITY
Auburn, Alabama
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ABSTRACT

The formalism for studying perturbations of a triaxial rigid body within the Hamilton-Jacobi framework is set up. In particular, the motion of a triaxial artificial earth satellite about its center of mass is studied. Variables are found which permit separation, and the Euler angles and associated conjugate momenta are obtained as functions of canonical constants and time.
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INTRODUCTION

This report summarizes the results obtained on Grant NGR-01-003-008-S-2(NE) between November 1968 and May 1969.* These studies are aimed at applying the perturbation methods of celestial mechanics to the rigid body problem with particular emphasis on the problem of motion of an artificial earth satellite about its center of mass. During this reporting period, the investigators were able to express the Hamiltonian for the triaxial rigid body in terms of variables in which it is readily separable. This, in turn, permits introduction of a canonical transformation to new parameters which are constants in the torque-free motion. The equations of transformation are then inverted to allow the investigators to express the original Euler angles and associated conjugate momenta in terms of the canonical constants and the time. Thus, they are able to set up the formalism for studying perturbations of a triaxial rigid body within the Hamilton-Jacobi framework.

RECTANGULAR COORDINATE SYSTEM AND EULER ANGLES

Let O represent the center of mass of the rigid body. Choose a space-fixed rectangular system Oξηζ such that the positive ζ axis lies along the angular momentum vector H and in the sense of H. Consider a plane through the center of mass and perpendicular to the ζ axis. This plane intersects the fundamental plane of the space-fixed, but otherwise arbitrary, rectangular frame Ox'y'z' in a line of nodes ON, shown in the figure. The ζ axis is chosen to lie along the line of nodes, its positive sense being arbitrarily chosen. Then, the η axis is chosen to form a right-handed system.

Let Ox'y'z' be a body-fixed (principal axes) rectangular frame and let φ*, θ*, and ψ* represent the Euler angles relating the Ox'y'z' and Oξηζ systems. The x'y' plane will be called the body-fixed plane. The angle ψ* is the angle between the x* and the ξ axes, measured in the x*y' plane while the angle θ* is the angle between the positive z* and ζ axes.

*Work co-sponsored by Contract NAS8-20175 with the George C. Marshall Space Flight Center.
Angular-Momentum Plane

Body-Fixed Plane

Space-Fixed Axes

Body-Fixed Axes

Angular-Momentum Axes
SOLUTION OF THE HAMILTON/JACOBI EQUATION

ASSOCIATED WITH A TRIAXIAL BODY PROBLEM WITH NO EXTERNAL FORCES

Hamilton Function and Canonical Equations

Although the eventual goal is to give a complete description of the motion in the Ox*y*z* system, the description of the motion will first be given in the Oξηζ system. In this manner, a straightforward, coherent approach to the problem and its solution can be presented.

Let

\[
\begin{pmatrix}
    \frac{\partial}{\partial \psi} \\
    \frac{\partial}{\partial \theta} \\
    \frac{\partial}{\partial \phi}
\end{pmatrix}
\]

\[\mathbf{p} = \begin{pmatrix}
    p_{\psi} \\
    p_{\theta} \\
    p_{\phi}
\end{pmatrix}\] \quad (1a)

\[
\begin{pmatrix}
    \dot{\psi} \\
    \dot{\theta} \\
    \dot{\phi}
\end{pmatrix}
\]

\[\mathbf{\xi} = \begin{pmatrix}
    \dot{\psi} \\
    \dot{\theta} \\
    \dot{\phi}
\end{pmatrix}\] \quad (1b)

\[
\begin{pmatrix}
    h_x^* \\
    h_y^* \\
    h_z^*
\end{pmatrix}
\]

\[\mathbf{H}^* = \begin{pmatrix}
    h_x^* \\
    h_y^* \\
    h_z^*
\end{pmatrix}\] \quad (1c)

\[
\begin{pmatrix}
    h_x^- \\
    h_y^- \\
    h_z^-
\end{pmatrix}
\]

\[\mathbf{H}^- = \begin{pmatrix}
    h_x^- \\
    h_y^- \\
    h_z^-
\end{pmatrix}\] \quad (1d)

where \(\mathbf{p}\) represents the conjugate momenta matrix and \(\mathbf{H}\) and \(\mathbf{H}^\prime\) represent the angular momentum w.r.t. space-fixed and body-fixed axes, respectively. A recapitulation of some of the formulas from an
earlier report (Some Suggested Approaches to Solving the Hamilton-Jacobi Equation Associated with Constrained Rigid Body Motion, January 1969, pp. 31-35) is given below to help the reader follow the subsequent discussion. It should be pointed out for anyone who has a copy of the referenced report that $H$ should read $H^*$ through Eq (6); the other notation is correct.

One has

$$ \mathbf{P} = \lambda \mathbf{F} \quad (2) $$

$$ \begin{pmatrix} \mathbf{P}_\phi \\ \mathbf{P}_\theta \\ \mathbf{P}_\psi \end{pmatrix} = \begin{pmatrix} (A \sin^2 \phi + B \cos^2 \phi) \sin^2 \theta + C \cos^2 \theta & (A - B) \sin \phi \cos \phi \sin \theta & C \cos \phi \\ (A - B) \sin \phi \cos \phi \sin \theta & A \cos^2 \phi + B \sin^2 \phi & 0 \\ C \cos \theta & 0 & C \end{pmatrix} \begin{pmatrix} \phi \\ \theta \\ \phi \end{pmatrix} $$

$$ H^* = \mathbf{TH}^\dagger \quad (3) $$

or

$$ \begin{pmatrix} \mathbf{h}_x^* \\ \mathbf{h}_y^* \\ \mathbf{h}_z^* \end{pmatrix} = \begin{pmatrix} \cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta & -\cos \psi \sin \phi - \sin \psi \cos \phi \cos \theta & \sin \psi \sin \theta \\ \sin \psi \cos \phi + \cos \psi \sin \phi \cos \theta & -\sin \psi \sin \phi + \cos \psi \cos \phi \cos \theta & -\cos \psi \sin \theta \\ \sin \phi \sin \theta & \cos \phi \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \mathbf{h}_x' \\ \mathbf{h}_y' \\ \mathbf{h}_z' \end{pmatrix} $$

$$ \mathbf{P} = \mathbf{N}^\dagger \mathbf{H}^* \quad (4) $$
or

\[
\begin{pmatrix}
  p_{\psi} \\
p_{\theta} \\
p_{\phi}
\end{pmatrix} =
\begin{pmatrix}
  \sin \phi \sin \theta & \cos \phi \sin \theta & \cos \theta \\
  \cos \phi & -\sin \phi & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
h_x^- \\
h_y^- \\
h_z^-
\end{pmatrix}
\]

or

\[P = M^T H^*\] (5)

and from Eq (3),

\[H^- = T^I H^*\] (6)

or explicitly,

\[
\begin{pmatrix}
h_x^- \\
h_y^- \\
h_z^-
\end{pmatrix} =
\begin{pmatrix}
p_{\theta} \cos \phi + \frac{\sin \phi}{\sin \theta}(p_{\psi} - p_{\phi} \cos \theta) \\
-p_{\theta} \sin \phi + \frac{\cos \phi}{\sin \theta}(p_{\psi} - p_{\phi} \cos \theta) \\
p_{\phi} = h \cos \theta^-
\end{pmatrix}
\]

In the Oξηζ system the angular momentum can be written as

\[
H = \begin{pmatrix}
0 \\
0 \\
h
\end{pmatrix}
\] (7)

If Eq (5) is used and \(\psi, \theta, \) and \(\phi\) are replaced by \(\phi^*, \theta^*, \) and \(\phi^*\), respectively, then
\[
\begin{pmatrix}
\dot{p}_\phi^* \\
\dot{p}_\theta^- \\
\dot{p}_\phi^-
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 1 \\
\cos \phi^* & \sin \phi^* & 0 \\
\sin \phi^* \sin \theta^- & -\cos \phi^* \sin \theta^- & \cos \phi^-
\end{pmatrix}\begin{pmatrix}
0 \\
h
\end{pmatrix}
\] (8)

or

\[
\begin{align*}
p_{\phi^*} &= h \\
p_{\theta^-} &= 0 \\
p_{\phi^-} &= h \cos \theta^-
\end{align*}
\]

Similarly, after Eq (8) is used and with \( \psi, \theta, \) and \( \phi \) replaced by \( \phi^*, \theta^-\), and \( \phi^- \), respectively, from Eq (6) one has

\[
\begin{pmatrix}
h_x^- \\
h_y^- \\
h_z^-
\end{pmatrix}
= \begin{pmatrix}
\frac{\sin \phi^-}{\cos \theta^-}(p_{\phi^*} - p_{\phi^-} \cos \theta^-) \\
\frac{\cos \phi^-}{\sin \theta^-}(p_{\phi^*} - p_{\phi^-} \cos \theta^-) \\
p_{\phi^-}
\end{pmatrix}
\] (9)

Using Eq (9), coupled with (8), the kinetic energy (the Hamiltonian function) of the rigid body can be written in the form

\[
T = \frac{1}{2A} h_x^2 + \frac{1}{2B} h_y^2 + \frac{1}{2C} h_z^2 = H
\]
or

\[
H = T = \left(\frac{\sin^2 \phi^-}{2A} + \frac{\cos^2 \phi^-}{2B}\right)(p_{\phi^*}^2 - p_{\phi^-}^2) + \frac{p_{\phi^-}^2}{2C}
\] (10)

where \( A, B, \) and \( C \) are the moments of inertia of the rigid body referenced to the principal axes \( Ox^-y^-z^- \). It is assumed that

\[
A > B > C
\]

The associated canonical equations are

\[
\dot{\phi}^* = \frac{\partial H}{\partial p_{\phi^*}} = h\left(\frac{\sin^2 \phi^-}{A} + \frac{\cos^2 \phi^-}{B}\right)
\] (11a)
\[
\dot{\phi} = \frac{3H}{\partial \phi} = -h \cos \theta \left( \frac{\sin^2 \phi - \cos^2 \psi}{A} \right) + \frac{p_{\phi}'}{C} \tag{11b}
\]

\[
\dot{\psi} = -\frac{3H}{\partial \psi} = 0 \tag{11c}
\]

\[
\dot{\psi} = -\frac{3H}{\partial \psi} = h^2 \left( \frac{1}{B} - \frac{1}{A} \right) \sin \phi \cos \phi \sin^2 \theta' = -h \sin \theta \dot{\phi}' \tag{11d}
\]

\[
p_{\theta}' = 0 \tag{11e}
\]

\[
\cos \theta' = \frac{p_{\phi}'}{h} \tag{11f}
\]

**Description of the Motion in the Ox'y'z' System**

A set of relationships is given which allows the description of the motion in the space-fixed system (\(\psi, \theta, \phi, p_\psi, p_\theta, \) and \(p_\phi\)) to be obtained completely from the description of the motion in the body-fixed system (\(\phi', \theta', \phi', p_\psi', p_\theta', \) and \(p_\phi'\)).

From elementary trigonometry,

\[
\cos \theta = \cos \theta' \cos \theta' - \sin \theta' \sin \theta' \cos \phi' \tag{12a}
\]

\[
\sin \theta = \sqrt{1 - \cos^2 \theta} \tag{12b}
\]

\[
\sin \theta' \sin \theta \cos(\psi - \psi') = \cos \theta - \cos \theta' \cos \theta \tag{12c}
\]

\[
\sin \theta \sin(\psi - \psi') = \sin \phi' \sin \theta' \tag{12d}
\]

\[
\sin \theta' \cos \theta' \cos(\phi - \phi') = \cos \theta' - \cos \theta' \cos \theta \tag{12e}
\]

\[
\sin \theta \sin(\phi - \phi') = \sin \phi' \sin \theta' \tag{12f}
\]

With Eqs (3), (4), and (5), the variables \(p_\psi, p_\theta, \) and \(p_\phi\) can be related to the variables \(p_{\phi'}, p_{\phi'}, \theta', \psi', \) and \(\phi'.\) Explicitly, these relationships can be written as

\[
p_{\theta} = -p_{\phi'} \sin \theta' \sin(\phi - \phi') \tag{13a}
\]

\[
p_{\psi} = h \cos \theta' \tag{13b}
\]

\[
P_{\phi} = p_{\phi'} \tag{13c}
\]
Since $\theta^*$ and $\psi^*$ are prescribed constants, independent of each other and independent of $\phi^*$, $\psi^*$, $P_\phi^*$, and $P_\psi^*$, the independent quantities $(\phi^*, \psi^*, \theta^*, \psi^*, P_\phi^*,$ and $P_\psi^*)$ serve to describe the motion of the triaxial body in the $Ox^*y^*z^*$ system.

**Generator and Equations of Transformation**

The Hamilton-Jacobi equation associated with Eq (10) is

\[
\frac{1}{2}\left(\frac{\sin^2\phi^*}{A} + \frac{\cos^2\phi^*}{B}\right)\left(\frac{3S}{\partial \phi^*}\right)^2 - \left(\frac{3S}{\partial \phi^*}\right)^2 + \frac{1}{2C} \left(\frac{3S}{\partial \psi^*}\right)^2 + \frac{3S}{\partial t} = 0
\]  

(14)

from which the generator $S$ of a canonical transformation is to be determined. A complete integral $S$ of Eq (14) can be obtained by separation of variables. It is found that

\[
S = -\alpha_1t + h\phi^* + \alpha_3\psi^* + S_1(\phi^*)
\]  

(15)

where

\[
\alpha_1 = H
\]

\[
h = p_{\phi^*} = \frac{3S}{\partial \phi^*}
\]  

(16)

\[
\alpha_3 = p_{\psi^*} = \frac{3S}{\partial \psi^*}
\]

are independent canonical variables. The function $S_1(\phi^*)$ is related to $\alpha_1$ and $h$ through the expression

\[
S_1(\phi^*) = \int_{\phi^0}^{\phi^*} p_{\phi^*} d\phi^*
\]  

(17)

where

\[
p_{\phi^*} = \pm \sqrt{\frac{a^2 + b^2 \sin^2 \phi^*}{c^2 + d^2 \sin^2 \phi^*}}
\]

*The variables $(\phi^*, \psi^*, \theta^*, p_{\phi^*}, P_\phi^*, P_\psi^*)$ in which $\theta^*$ is replaced by $\cos^{-1}(p_{\psi^*}/h)$ are introduced here (see "Perturbation of the Force Free Motion of the Triaxial Rigid Body, page 20, for justification").
and

\[ a' = A(2B\alpha_1 - h^2) \]
\[ b' = h^2(A - B) \]
\[ c' = A(B - C) \]
\[ d' = C(A - B) \]

(19)

The complete set of transformation equations from \((\psi^*, \phi^*, \phi', p_{\psi^*}, p_{\phi^*}, p_{\phi'})\) to \((\alpha_1, h, \alpha_3, \beta_1, \beta_2, \beta_3)\) is obtained from Eq (15). The equations are:

\[ \beta_1 = -\frac{\partial S}{\partial \alpha_1} = t - L(\phi') \] (20a)
\[ \beta_2 = -\frac{\partial S}{\partial h} = M(\phi') - \phi^* \] (20b)
\[ \beta_3 = -\frac{\partial S}{\partial \alpha_3} = -\psi^* \] (20c)
\[ p_{\psi^*} = \frac{\partial S}{\partial \psi^*} = \alpha_3 \] (20d)
\[ p_{\phi^*} = \frac{\partial S}{\partial \phi^*} = h \] (20e)
\[ p_{\phi'} = \frac{\partial S}{\partial \phi'} = \pm \sqrt{C\left( \frac{a' + b' \sin^2 \phi'}{c' + d' \sin^2 \phi'} \right)} \] (20f)

where

\[ L(\phi') = \pm AB\sqrt{C} I_2(\phi') \] (21)
\[ M(\phi') = \sqrt{C} \alpha_2 I_3(\phi') \]

and

\[ I_2(\phi') = \int_{\phi'}^{\phi^*} \frac{d\phi'}{\sqrt{(a' + b' \sin^2 \phi')(c' + d' \sin^2 \phi')}} \] (22a)
\[ I_3(\phi') = \int_{\phi'}^{\phi^*} \frac{[(A - B)\sin^2 \phi' - A]d\phi'}{\sqrt{(a' + b' \sin^2 \phi')(c' + d' \sin^2 \phi')}} \] (22b)
In three of the six Eqs (20), the right-hand sides are preceded by * symbols. The choice of the sign in these equations is determined by the choice of sign for $p_\phi^-$. Also,

$$P = MT_H$$

or

$$\begin{pmatrix} p_{\phi}^- \\ p_{\theta}^- \\ p_{\phi}^+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \cos \phi^- & \sin \phi^- & 0 \\ \sin \phi^- \sin \theta^- & -\cos \phi^- \sin \theta^- & \cos \theta^- \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

(23)

hence,

$$p_{\phi}^- = h \cos \theta^-$$

Thus, the sign of $p_{\phi}^-$ depends upon whether $\cos \theta^-$ is positive or negative. It is assumed that $0 < \theta^- < \pi/2$. Therefore, Eqs (20) and (21) become

$$t - \beta_1 = L(\phi^-)$$

(24a)

$$\phi^- + \beta_2 = M(\phi^-)$$

(24b)

$$\beta_3 = -\psi^*$$

(24c)

$$p_{\phi}^- = \sqrt{C \left( \frac{a^- b^- \sin^2 \phi^-}{c^- d^- \sin^2 \phi^-} \right)}$$

(24d)

$$p_{\phi}^* = h$$

(24e)

$$p_{\psi}^* = a_3$$

(24f)

where

$$L(\phi^-) = AB \sqrt{C} I_2(\phi^-)$$

(25a)

$$M(\phi^-) = -h \sqrt{C} I_3(\phi^-)$$

(25b)
INVERSION OF THE SOLUTION FOR THE TRIAXIAL RIGID BODY PROBLEM
WITH NO EXTERNAL FORCES

Equation (24) must be inverted to express the variables \((\phi^*, \theta^*, \psi^*, P_\phi^*, P_\theta^*, P_\psi^*)\) in terms of the canonical constants \((a_1, h, a_3, \beta_1, \beta_2, \beta_3)\) and time \(t\). The inversion is shown below.

Inversion of the Equation \(t - \beta_1 = L(\phi^*)\)

Since it is assumed that \(A > B > C\), the quantities \(b^*, c^*, \text{ and } d^*,\) given in Eqs (19), are positive. The quantity \(a^*\) may be either positive or negative. In what follows, it is assumed that \(a^* > 0\).

From Eq (8), it is noted that

\[
a^* \frac{b^*}{c^* + b^*} < \frac{c^*}{d^*} \tag{26}
\]

since

\[
\frac{2Ca_1}{h^2} = \frac{AC\omega_x^2 + BC\omega_y^2 + C^2\omega_z^2}{A^2\omega_x^2 + B^2\omega_y^2 + C^2\omega_z^2}
\]

where \(\omega_x^*, \omega_y^*, \text{ and } \omega_z^*\) are components of the angular velocity w.r.t. the primed system.

For convenience the following parameters are defined:

\[
n_1^2 = \frac{b^*}{a^* + b^*} \tag{27a}
\]

\[
n_2^2 = \frac{d^*}{c^* + d^*} \tag{27b}
\]

\[
\xi = \frac{1}{\sqrt{(a^* + b^*)(c^* + d^*)}} = \frac{1}{B\sqrt{(A - C)(2Aa_1 - h^2)}} \tag{27c}
\]

\[
k = \sqrt{n_1^2 - n_2^2} = \sqrt{(A - B)(h^2 - 2Ca_1)} \tag{27d}
\]

\[
g = \frac{1}{\sqrt{1 - n_2^2}} = \sqrt{\frac{B(A - C)}{A(B - C)}} \tag{27e}
\]

\[
k^* = \sqrt{1 - k^*} = \sqrt{\frac{(A - C)(2B a_1 - h^2)}{(B - C)(2A a_1 - h^2)}} \tag{27f}
\]
Clearly, \( 1 > \frac{n_1^2}{n_2^2} > 0 \); thus, \( 0 < k < 1 \), and \( k' \) is real since \( a'/b' < c'/d' \).

To cast Eq (22a) into a more convenient form, a new variable is introduced by the equation

\[
\alpha = \phi' + \pi/2
\]  

(28)

It follows immediately, by substituting \( \alpha \) and the parameters in Eq (16) into (22a), that

\[
I_2(\phi') = \xi \int_{\phi_0}^{\alpha} r(\alpha) d\alpha
\]  

(29)

where

\[
\alpha_0 = \phi_0' + \pi/2
\]

(30)

\[
r(\alpha) = \frac{1}{\sqrt{(1 - n_1^2 \sin^2 \alpha)(1 - n_2^2 \sin^2 \alpha)}}
\]

Since the lower limit of integration of Eq (29) may be taken to be an absolute constant, \( \phi_0' = -\pi/2 \) is chosen; hence, \( \alpha_0 = 0 \). Therefore,

\[
I_2(\phi') = \xi \int_{0}^{\alpha} r(\alpha) d\alpha
\]  

(31)

In what follows, the formulas which appear in Byrd and Friedman [1] will be referenced. Such formula numbers will be indicated by prefixing the numbers with the designation B-F.

Using B-F (284.00) and Eq (24a),

\[
\int_{0}^{\alpha} r(\alpha) d\alpha = gu = \frac{1}{\xi \sqrt{AB_1}} (t - \beta_1)
\]  

(32)

or

\[
u = \lambda t + \epsilon
\]  

(33)
where

\[ \lambda \equiv \frac{1}{g \xi ABvC} = \sqrt[ABC]{(2A\alpha_1 - h^2)(B - C)} \]  

(34a)

\[ \xi \equiv -\lambda \beta_1 \]  

(34b)

Also from Byrd-Friedman,

\[ \sin^2 u \equiv [\sin(\mu)]^2 = \frac{(1 - n_2^2)\sin^2 \alpha}{1 - n_2^2 \sin^2 \alpha} \]  

(35)

Solving the above equation for \( \sin \alpha \), one writes

\[ \sin \alpha = \frac{\sin u}{\sqrt{1 - n_2^2 \cos^2 u}} \]  

(36a)

and

\[ \cos \alpha = \frac{\sqrt{1 - n_2^2 \cos u}}{\sqrt{1 - n_2^2 \cos^2 u}} \]  

(36b)

where

\[ \cos \mu = \cos(\mu) \]

and

\[ \sin^2 u + \cos^2 u = 1 \]

Since \( \alpha = \phi' + \pi/2 \),

\[ \sin \phi' = \frac{\sqrt{1 - n_2^2 \cos \mu}}{\sqrt{1 - n_2^2 \cos^2 \mu}} \]  

(37a)

\[ \cos \phi' = \frac{\sin u}{\sqrt{1 - n_2^2 \cos^2 u}} \]  

(37b)

\[ \tan \phi' = -\frac{n_2 \cos \mu}{\sin \mu} \]  

(37c)

The quadrant of \( \phi' \) is uniquely determined by studying the signs of \( \cos \mu \) and \( \sin u \).

Equation (37) is not in a convenient form for calculation since powers of \( u \) appear in the expressions for \( \cos \mu \) and \( \sin u \). This difficulty can be avoided by introducing theta functions. From B-F (907.01), (907.02), (907.03), (900.04), and (901.01), for \( |u| < K' \),

\[ \sin u = u - (1 + k^2) \frac{u^3}{3!} + (1 + 14k^2 + k^4) \frac{u^5}{5!} \]

\[ - (1 + 135k^2 + 135k^2 + k^6) \frac{u^7}{7!} + \cdots \]  

(38a)
\[ cnu = 1 - \frac{u^2}{2l} + (1 + 4k^2) \frac{u^4}{4l} - (1 + 44k^2 + 16k^4) \frac{u^6}{6l} \]
\[ + (1 + 408k^2 + 912k^4 + 64k^6) \frac{u^8}{8l} - \cdots \]  
(38b)

\[ dnu = 1 - k^2 \frac{u^2}{2l} + (4 + k^2)k^2 \frac{u^4}{4l} \]
\[ - (16 + 44k^2 + k^4)k^2 \frac{u^6}{6l} + \cdots \]  
(38c)

where

\[ K' = K(k') \]  
(39a)

\[ K = \frac{\pi}{2} \left(1 + 4 \sum_{m=1}^{\infty} \frac{q^m}{1 + q^{2m}}\right) = \int_0^{\pi/2} \frac{dv}{\sqrt{1 - k^2 \sin^2 \nu}} \]  
(39b)

\[ q = k' \left[1 + 2 \left(\frac{k_1}{2}\right)^4 + 15 \left(\frac{k_1}{2}\right)^8 + 150 \left(\frac{k_1}{2}\right)^{12} \right. \]
\[ \left. + 1707 \left(\frac{k_1}{2}\right)^{16} + \cdots \right] \]  
(39c)

\[ k' = \frac{1 - \sqrt{k^2}}{1 + \sqrt{k^2}}(k_1^2 < k^2 < 1) \]  
(39d)

If B-F (1050.01), (1052.01), and (1052.02) are used, then

\[ \tan \phi' = -\frac{\sqrt{k^2} \cos \nu + q^2 \cos 3\nu + q^6 \cos 5\nu + \cdots}{g \sin \nu - q^2 \sin 3\nu + q^6 \sin 5\nu - \cdots} \]  
(40)

where

\[ \nu = \frac{\pi}{2K} u \]

The series (c) of (39) for computing \(q\) converges rapidly. Hence, the angle \(\phi'\) can be expressed in terms of canonical parameters and time through Eq (37) and can be computed by using expression (40).

**Inversion of the Equation \(\phi' + \beta_1 = M(\phi')\)**

By using Eqs (27), (28), and (30) and recalling that \(a_0 = 0\), Eq (24b) can be rewritten as
\[ \phi^* + \beta_2 = -h \sqrt{\xi} \int_0^a [(A - B) \cos^2 \alpha - A] r(\alpha) \, d\alpha \] (41)

From Eq (32),

\[ \phi^* + \beta_2 = \frac{h}{B} (t - \beta_1) - h \sqrt{\xi} (A - B) \int_0^a \cos^2 \alpha \, r(\alpha) \, d\alpha \] (42)

Using B-F (284.08) and (432.03), Eq (42) becomes

\[ \phi^* + \beta_2 = \frac{h}{B} (t - \beta_1) - \left( \frac{\pi}{2K} \right)[\Omega_5 - u \Lambda_0(\beta, k)] \] (43)

where

\[ \beta = \sin^{-1} \frac{1}{\sqrt{1 - \gamma^2}} \] (44a)

\[ \gamma^2 = -\frac{n_2^2}{1 - n_2^2}, \quad (1 < \gamma^2 < \infty) \] (44b)

and the functions \( \Omega_5 \) and \( \Lambda_0 \) are defined in B-F, Sections 430 and 150, respectively. Since \( u = \lambda(t - \beta_1) \), it can be written

\[ \phi^* + \beta_2 = M^*(t - \beta_1) - \frac{\pi}{2K} (\Omega_5 - u) \] (45)

where

\[ M^* = \frac{h}{B} - \frac{\pi}{2K} [1 - \Lambda_0(\beta, k)] \lambda \] (46)

Expressions for \( p_{\phi^*} \) and \( \phi^* \)

By applying Eqs (27) and (37), Eq (24d) can be written as

\[ p_{\phi^*} = \sqrt{C(a^\prime + b^\prime)(1 - n_2^2)} (k^{-2} + k^2 \text{cn}^2 u)^{1/2} \] (47)

From B-F (121.00), one has

\[ \text{dn}^2 u = k^{-2} + k^2 \text{cn}^2 u \] (48)
Hence, Eq (47) takes the form

\[ p_\phi^* = \frac{\sqrt{C(a^* + b^*)(1 - n^*)}}{\sqrt{C^*}} \, \text{dnu} \]

\[ = \sqrt{\frac{C(2Aa_1 - h^2)}{A - C}} \, \text{dnu} \]  

(49)

and since \( p_\phi^* = h \cos \theta^* \),

\[ \cos \theta^* = \frac{p_\phi^*}{h} = \sqrt{\frac{C(2Aa_1 - h^2)}{h^2(A - C)}} \, \text{dnu} \]  

(50)

**Inverted Solution for the Triaxial Rigid Body Problem with No External Forces**

The general solution for the triaxial rigid body problem with no external forces can then be summarized as follows:

\[ \tan \phi^* = -\frac{cnu}{g \, snu} \]

\[ = -\frac{\sqrt{k^*}}{g} \cos \nu + q^2 \cos 3\nu + q^6 \cos 5\nu + \cdots \]  

(51a)

\[ \phi^* + \beta_2 = M^* (t - \beta_1) - \frac{\pi}{2k} (\Omega_5 - t) \]  

(51b)

\[ \psi^* = \beta_3 \]  

(51c)

\[ p_\phi^* = \sqrt{\frac{C(2Aa_1 - h^2)}{A - C}} \, \text{dnu} \]  

(51d)

\[ p_{\phi^*} = h \]  

(51e)

\[ p_{\psi^*} = a_3 \]  

(51f)

This solution coupled with Eqs (12) and (13) gives a complete description of the motion of the triaxial body in the space-fixed system \( Ox^*y^*z^* \) in terms of the canonical constants and time.
UNIAXIAL SOLUTION

By letting A equal B, the triaxial solution (51) can be reduced to the corresponding uniaxial solution. To distinguish between the canonical parameters which appear in the triaxial solution and the reduced solution, the latter will be labeled with the subscript u; that is, \(\alpha_{1u}, \alpha_{3u}, \beta_{1u}, \beta_{2u}, \) and \(\beta_{3u}.

For the case \(A = B,\) one has

\[
\begin{align*}
n_2^2 &= n_2^2 = 0, \\
k &= 0, \\
k' &= 1, \\
s_{nu} &= \sin u, \\
c_{nu} &= \cos u,
\end{align*}
\]

and

\[
\lambda = \sqrt{\frac{(2\alpha_{1u} - h_u^2)(A - C)}{A^2 C}}
\]

Thus, from Eq (37e), one obtains

\[
\phi^* - \phi_0 = \sqrt{\frac{(2\alpha_{1u} - h_u^2)(A - C)}{A^2 C}} (t - \beta_{1u})
\]

(52)

When \(A = B,\)

\[
\begin{align*}
\beta &= \frac{\pi}{2}, \\
\Lambda_0\left(\frac{\pi}{2}, 0\right) &= 1, \\
M* &= \frac{h_u}{A}, \\
\Omega_5 &= u;
\end{align*}
\]

therefore, Eq (45) reduces to

\[
\phi^* = -\beta_{2u} + \frac{h_u}{A} (t - \beta_{1u})
\]

(53)

Furthermore, for \(A = \lambda, \) \(dn(u, 0) = 1,\) and Eq (49) reduces to

\[
\rho^* = \sqrt{\frac{C(2\alpha_{1u} - h_u^2)}{A - C}}
\]

(54)
In summary, the uniaxial solution is given as follows:

$$\phi^* - \phi_0^* = \sqrt{\frac{(2A_{11} - h_{11}^2)(A - C)}{A^2C}} (t - \beta_{1u})$$  \hspace{1cm} (55a)$$

$$\phi^* = -\beta_{2u} \cdot \frac{h_u}{A}(t - \beta_{1u})$$ \hspace{1cm} (55b)$$

$$\psi^* = -\beta_{3u}$$ \hspace{1cm} (55c)$$

$$p_{\phi^*} = \sqrt{\frac{C(2A_{11} - h_{11}^2)}{A - C}}$$ \hspace{1cm} (55d)$$

$$p_{\phi^*} = h_u$$ \hspace{1cm} (55e)$$

$$p_{\psi^*} = \alpha_{3u}$$ \hspace{1cm} (55f)$$

and the corresponding generator is

$$S_u = -a_{11}^u t + h_u \phi^* + \alpha_{3u} \psi^* + \sqrt{\frac{C(2A_{11} - h_{11}^2)}{A - C}} (\phi^* - \phi_0^*)$$ \hspace{1cm} (56)$$

Through the use of Eqs (12) and (13), the complete solution of the force-free uniaxial motion can be obtained in the space-fixed system $Ox^*y^*z^*$.

The parameters which appear in the treatment of the force-free uniaxial problem, given in [2], will be labeled with superscript asterisks; that is, $\alpha_{1u}^*$, $\alpha_{2u}^*$, $\alpha_{3u}^*$, $\beta_{1u}^*$, $\beta_{2u}^*$, and $\beta_{3u}^*$. It has been shown that

$$h_u^2 = 2A_a_{11}^* + \left(\frac{C - A}{C}\right)\alpha_{2u}^*$$ \hspace{1cm} (57)$$

The corresponding generator, in which $h_u$ is interpreted as a function of $\alpha_{1u}^*$ and $\alpha_{2u}^*$ through Eq (57), takes the form

$$S_u^* = -a_{11}^* t + \sqrt{2A_{a11}^* + \left(\frac{C - A}{C}\right)\alpha_{2u}^*} \phi^* + \alpha_{3u}^* \psi^*$$

$$+ \alpha_{2u}^*(\phi^* - \phi_0^*)$$ \hspace{1cm} (58)$$
After inversion, the associated equations of transformation are

\[
\phi^* - \phi_0^* = -\beta_2^* + \frac{a_2^*}{A} \left( \frac{A - C}{C} \right) (t - \beta_1^*)
\quad (59a)
\]

\[
\phi^* = \sqrt{2Aa_1^* + \left( \frac{A - C}{C} \right) a_2^*^2} (t - \beta_1^*)
\quad (59b)
\]

\[
\psi^* = -\beta_3^*
\quad (59c)
\]

\[
P_{\phi^*} = a_2^* 
\quad (59d)
\]

\[
P_{\phi^*} = \sqrt{2Aa_1^* + \left( \frac{C - A}{C} \right) a_2^*} 
\quad (59e)
\]

\[
P_{\psi^*} = a_3^* 
\quad (59f)
\]

If Eqs (59) and (60) are compared, the parameters \((a_{1u}^*, a_{2u}^*, a_{3u}^*, b_{1u}^*, b_{2u}^*, \text{and } b_{3u}^*)\) and \((a_{1u}, h_u, a_{3u}, b_{1u}, b_{2u}, \text{and } b_{3u})\) are related as follows:

\[
a_{1u}^* = a_{1u} 
\quad (60a)
\]

\[
a_{2u}^* = \sqrt{C(2Aa_{1u} - h_u^2)} 
\quad (60b)
\]

\[
a_{3u}^* = a_{3u}
\quad (60c)
\]

\[
b_{1u}^* = b_{1u} + \frac{A}{h_u} b_{2u}
\quad (60d)
\]

\[
b_{2u}^* = -\frac{1}{h_u} \sqrt{(2Aa_{1u} - h_u^2)(A - C)} b_{2u}
\quad (60e)
\]

\[
b_{3u}^* = b_{3u}
\quad (60f)
PERTURBATION OF THE FORCE FREE MOTION OF THE TRIAXIAL RIGID BODY

Recalling the section entitled "Generator and Equations of Transformation," page eight, \( \Theta^* \) must be replaced with an equivalent parameter \( \Psi^* \), the momentum conjugate to \( \Psi^* \), to use the canonical perturbation equations of Hamilton-Jacobi theory in studying the perturbations of the force free motion of the triaxial rigid body. It follows from Eq (13b) that either \( \Theta^* \) or \( \Psi^* \) will give an equivalent description of the motion. It is clear from this equation that the momentum conjugate to any angle \( \Psi \) which lies in the \( x^*y^*z^* \) plane is independent of the angle \( \Psi \) and depends only on \( h \) and \( \Theta^* \). Therefore,

\[
P_{\Psi} = P_{\Psi^*} = h \cos \Theta^* = p_{\Phi^*} \cos \Theta^*
\]

Thus, the six independent quantities \( (\Phi^*, \Phi^*, \Psi^*, P_{\Phi^*}, P_{\Phi^*}, P_{\Psi^*}) \) will completely describe the motion of the triaxial rigid body with respect to the \( Ox^*y^*z^* \) system. The Hamilton function from which \( \Phi^*, \Phi^*, \Psi^*, P_{\Phi^*}, P_{\Phi^*}, P_{\Psi^*} \) are to be obtained is, of course, still given by Eq (10). Furthermore, \( H \), as given in Eq (10), can be considered to be the Hamilton function of an extended system of variables \( (\Phi^*, \Phi^*, \Psi^*, P_{\Phi^*}, P_{\Phi^*}, P_{\Psi^*}) \), which satisfy the canonical equations of motion.

\[
\dot{\Phi}^* = \frac{\partial H}{\partial P_{\Phi^*}} \tag{62a}
\]

\[
\dot{\Phi}^* = \frac{\partial H}{\partial P_{\Phi^*}} \tag{62b}
\]

\[
\dot{\Phi}^* = \frac{\partial H}{\partial P_{\Phi^*}} \tag{62c}
\]

\[
\dot{P}_{\Phi^*} = -\frac{\partial H}{\partial \Phi^*} \tag{62d}
\]

\[
\dot{P}_{\Phi^*} = -\frac{\partial H}{\partial \Phi^*} \tag{62e}
\]

\[
\dot{P}_{\Psi^*} = -\frac{\partial H}{\partial \Psi^*} \tag{62f}
\]

subject to the constraints

\[
\Psi^* = \text{constant} \tag{63a}
\]

\[
P_{\Psi^*} = h \cos \Theta^* = \text{constant} \tag{63b}
\]

This follows from the fact that the two differential equations (62c) and (62f), which have been added to the system, are entirely consistent with (63), the equations of constraint.
GRAVITY GRADIENT POTENTIAL FOR THE TRIAXIAL BODY

The gravity gradient potential \( V \) for the triaxial body is given by

\[
V = -\frac{3}{2} \kappa [(A - C) \cos^2 \chi + (A - B) \cos^2 \beta] \tag{64}
\]

where \( \kappa = n^{-2} \) and \( n \) is the mean motion of the Earth about the triaxial body. A circular orbit will be considered for which \( \kappa \) is a constant. The angles \( \alpha, \beta, \) and \( \chi \) are the direction angles of the line segment from the center of mass of the body to the center of mass of the Earth with respect to \( \text{O}x'y'z' \), the principal axes of the body. Since \( \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \), Eq (64) can be rewritten as

\[
V = \frac{3}{2} \kappa (B - A) + W \tag{65}
\]

where

\[
W = \left( \frac{B - C}{A - C} \right) W_1 + W_2 \tag{66}
\]

and

\[
W_1 = -\frac{3}{2} \kappa (B - C) \cos^2 \chi \tag{67}
\]

\[
W_2 = \frac{3}{2} \kappa (A - B) \cos^2 \alpha \tag{68}
\]

The expression for \( \cos \chi \) in terms of canonical constants and time \( t \) is deduced in [1] and can be written in the form

\[
\cos \chi = D_1 \cos \theta^* + D_2 \sin \theta^* \sin \phi^* + D_3 \sin \theta^* \cos \phi^* \tag{69}
\]

where

\[
D_1 = \sin i \sin \ell \cos \theta^* - [\cos i \sin \ell \cos (\alpha + \beta_3) + \cos \ell \sin (\alpha + \beta_3)] \sin \theta^* \tag{70}
\]

\[
D_2 = -\cos i \sin \ell \sin (\alpha + \beta_3) + \cos \ell \cos (\alpha + \beta_3) \tag{71}
\]

\[
D_3 = -\sin i \sin \ell \sin \theta^* - [\cos i \sin \ell \cos (\alpha + \beta_3) + \cos \ell \sin (\alpha + \beta_3)] \cos \theta^* \tag{72}
\]

Note that \( D_1, D_2, \) and \( D_3 \) are functions of three canonical constants only, namely, \( \alpha_2 = h, \alpha_3, \) and \( \beta_3 \) and contain \( t \) explicitly only through \( \ell \) and \( \Omega \), which are both linear in \( t \).
A suitable expression for \( \cos \alpha \) can be derived similarly. From spherical trigonometry,

\[
\cos \alpha = \cos \phi \cos \phi_H - \sin \phi \sin \phi_H \cos \theta^*
\]  \hspace{1cm} (70)

and

\[
\cos \theta_H = \cos \theta \cos \phi - \sin \theta \sin \phi \cos (\Omega + \beta_3)
\]  \hspace{1cm} (71)

Introducing

\[
E_1 = \cos(\phi^* - \phi_H) = \frac{\cos i - \cos \theta \cos \theta^*}{\sin \theta \sin \theta^*}
\]  \hspace{1cm} (72)

\[
E_2 = \sin(\phi^* - \phi_H) = \frac{\sin i \sin (\Omega + \beta_3)}{\sin \theta_H}
\]  \hspace{1cm} (73)

Equation (7) can be written in the form

\[
\cos \alpha = E_1(\cos \phi^* \cos \phi^* - \cos \theta^* \sin \phi^* \sin \phi^*)
\]

\[
- E_2(\cos \phi^* \sin \phi^*)
\]

\[
+ \cos \theta^* \sin \phi^* \cos \phi^*)
\]  \hspace{1cm} (74)

Note that \( E_1 \) and \( E_2 \) are functions of only three canonical constants, namely, \( \alpha_2, \alpha_3, \) and \( \beta_3 \) and contain \( \theta \) explicitly only through \( \phi \) and \( \Omega \). It is important to note that \( D_1, D_2, D_3, \) etc., and \( E_1 \) and \( E_2 \) do not contain the moments of inertia \( A, B, \) and \( C. \) Thus, these coefficients can be treated as constants when \( \cos \chi \) and \( \cos \alpha \) are expanded in Taylor's series about their values at \( B = A. \) The reason for the expansion is the angles \( \phi^*, \phi^*, \) and \( \theta^* \) for the unperturbed triaxial body are no longer either constant or simple linear functions of time (as was the case in the uniaxial problem). Thus, since difficulties are anticipated in the integration of the perturbation equations, attempts are made to linearize the arguments of the trigonometric functions which will appear in the integration.

Introducing the notation

\[
f(\chi) \equiv \cos \chi
\]  \hspace{1cm} (75)

\[
g(\alpha) \equiv \cos \alpha
\]

\( f(\chi) \) and \( g(\alpha) \) are treated as functions of \( B \) and are expanded about the value \( B = A. \) Using prime notation to indicate derivatives with respect to \( B, \) one has

\[
f(\chi) = f(B) - f'(B)(A - B) + \frac{1}{2}f''(B)(A - B)^2 + O(A - B)^3
\]  \hspace{1cm} (76)
where

\[ f(B) = D_1 [\cos \theta']_{B=A} + D_2 [\sin \theta' \sin \phi^*]_{B=A} + D_3 [\sin \theta' \cos \phi^*]_{B=A} \]

\[ f'(B) = D_1 \left[ \frac{\partial}{\partial B} \cos \theta' \right]_{B=A} + D_2 \left[ \frac{\partial^2}{\partial B^2} (\sin \theta' \sin \phi^*) \right]_{B=A} + D_3 \left[ \frac{\partial}{\partial B} (\sin \theta' \cos \phi^*) \right]_{B=A} \] (77)

\[ f''(B) = D_1 \left[ \frac{\partial^2}{\partial B^2} \cos \theta' \right]_{B=A} + D_2 \left[ \frac{\partial^2}{\partial B^2} (\sin \theta' \sin \phi^*) \right]_{B=A} + D_3 \left[ \frac{\partial}{\partial B} (\sin \theta' \cos \phi^*) \right]_{B=A} \]

and

\[ g(a) = g(B) - g'(B) (A - B) + O((A - B)^2) \] (78)

where

\[ g(B) = E_1 [\cos \theta' \cos \phi^* - \cos \theta' \sin \phi' \sin \phi^*]_{B=A} + E_2 [\cos \phi' \sin \phi^* + \cos \theta' \sin \phi' \cos \phi^*]_{B=A} \]

\[ g''(B) = E_1 \left[ \frac{\partial}{\partial B} (\cos \phi' \cos \phi^* - \cos \phi' \sin \phi' \sin \phi^*) \right]_{B=A} + E_2 \left[ \frac{\partial}{\partial B} (\cos \phi' \sin \phi^* + \cos \theta' \sin \phi' \cos \phi^*) \right]_{B=A} \] (79)

In Eq (78), only two terms are carried since \( g(a) \) is multiplied by the fact \( r(A - B) \) in \( W \).

Equations (66), (67), (76), and (78) yield

\[ W = \left( \frac{B - C}{A - C} \right) W_{1u} + W_{2t} + O((A - B)^3) \] (80)
where

\[ W_{1u} = - \frac{3\epsilon}{2} (A - C) [f(B)]^2 \]

\[ W_{2t} = \frac{3\epsilon}{2} (A - B) \{ 2(B - C)f(B)f'(B) + [g(B)^2] \}
- \frac{3\epsilon}{2} (A - B)^2 \{ (B - C)[f'(B)]^2
+ f(B)f''(B) \} + 2g(B)g'(B) \}

These expressions for \( W_{1u} \) and \( W_{2t} \) can be used to study the perturbations of the variables \( (\alpha_1, h, \alpha_3, \beta_1, \beta_2, \text{ and } \beta_3) \) which are given by the following relations

\[ \dot{\alpha}_i = \left( \frac{B - C}{A - C} \right) \frac{\partial W_{1u}}{\partial \beta_i} + \frac{\partial W_{2t}}{\partial \beta_i} \quad (i=1,2,3), \quad (\alpha_2 = h) \quad (81) \]

\[ \dot{\beta}_i = - \left( \frac{B - C}{A - C} \right) \frac{\partial W_{1u}}{\partial \alpha_i} - \frac{\partial W_{2t}}{\partial \alpha_i} \]
REFERENCES


STRESSES IN DOME-SHAPED SHELLS OF REVOLUTION WITH DISCONTINUITIES AT THE APEX

C. H. Chen, J. C. M. Yu, W. A. Shaw

ABSTRACT

Asymptotic solutions to Novozhilov's equations of shells of revolution are derived for axisymmetric and first harmonic loadings. The solutions obtained are valid throughout the shallow and nonsallow regions.

Stresses in dome-shaped shells of revolution with a discontinuity in the form of a circular hole; or a circular rigid insert; or a nozzle, at the apex have been investigated. Numerical results are obtained for spheres, ellipsoids, and paraboloids, containing a discontinuity under an internal pressure and a moment. Good correlation between theoretical and experimental stresses is obtained for the spherical shell. Curves depicting stress distributions are given. The influence of three types of discontinuity on the stresses of the shells is also investigated.
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**Upper-case Letters**

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<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$A_1$, $A_2$</td>
<td>Lame' parameters.</td>
</tr>
<tr>
<td>$D$</td>
<td>Extensional stiffness, $Eh/(1 - \mu^2)$</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's modulus.</td>
</tr>
<tr>
<td>$G(\ )$</td>
<td>Differential operator defined by Eq. (2-26).</td>
</tr>
<tr>
<td>$G_1(\ )$</td>
<td>Differential operator defined by Eq. (2-29).</td>
</tr>
<tr>
<td>$H_1$</td>
<td>$= A_1(1 + z/R_1)$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>$= A_2(1 + z/R_2)$</td>
</tr>
<tr>
<td>$I_0$, $I_1$</td>
<td>Modified Bessel functions of the first kind of order zero and one, respectively.</td>
</tr>
<tr>
<td>$J_0$, $J_1$</td>
<td>Bessel functions of the first kind of order zero and one, respectively.</td>
</tr>
<tr>
<td>$K$</td>
<td>Bending rigidity, $Eh^2/(12(1 - \mu^2))$.</td>
</tr>
<tr>
<td>$K_0$, $K_1$</td>
<td>Modified Bessel functions of the second kind of order zero and one, respectively.</td>
</tr>
<tr>
<td>$M_1$, $M_2$, $M_{12}$, $M_{21}$</td>
<td>Bending moments and twisting moments defined by Eqs. (A-13).</td>
</tr>
<tr>
<td>$M$</td>
<td>$= M_1 + M_2$, or applied external moment.</td>
</tr>
<tr>
<td>$N_1$, $N_2$</td>
<td>Transverse shears defined by Eqs. (A-13).</td>
</tr>
<tr>
<td>$R_1$, $R_2$</td>
<td>Principal radii of curvature.</td>
</tr>
<tr>
<td>$R_o$</td>
<td>Minimum radius of curvature of a shell of revolution.</td>
</tr>
<tr>
<td>$R^*$</td>
<td>Radius of curvature at the apex of a shell of revolution.</td>
</tr>
<tr>
<td>$S$</td>
<td>$= T_{12} - M_{21}/R_2 = T_{21} - M_{12}/R_1$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>A complex force</td>
</tr>
</tbody>
</table>
\(T_1, T_2, T_{12}, T_{21}\) Normal forces and in plane shearing forces defined by Eqs. (A-13).

\(T_1, T_2, \bar{T}\) Complex forces defined by Eqs. (A-26).

\(T = T_1 + T_2\) Membrane direct forces.

\(T_1, \bar{T}_2\) Complex auxiliary functions defined by Eqs. (2-24).

\(U, V\) Complex function defined by Eq. (2-30).

\(x_2\) Rotation about the line \(a_2 = \text{constant}\).

\(Y_0, Y_1\) Bessel functions of the second kind of order zero and one, respectively.

**Lower-case Letters**

- \(a, l\) Major and minor semi-axes of an ellipsoid.
- \(b^2\) \(= \sqrt{3(1-u^2)} x_0/h^*\)
- \(b_1\) \(= b(1+1/(2b^2))\)
- \(b_2\) \(= b(1-1/(2b^2))\)
- \(c\) \(= h/\sqrt{12(1-u^2)}\)
- \(c^*\) \(= h^*/\sqrt{12(1-u^2)}\)
- \(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_n\) Unit vectors in the directions of \(a_1, a_2, a_3\) and along the normal to the middle surface.
- \(e_{11}, e_{22}, e_{12}\) Strain components.
- \(h, h^*\) Thicknesses of a shell of revolution and a cylindrical shell, respectively.
- \(h_1\) \(= (R_0/R_2)^{3/4}(\eta/\lambda \sin \theta)^{1/2}\)
- \(k_{e_1}, k_{e_2}, k_{e_1}, k_{e_1}\) Thompson functions of order zero and one.
- \(p\) Internal pressure.
- \(p(\theta)\) A function of \(\theta\).
\( q_1, q_2, q_n \)  
Components of a surface loading in the directions of \( \hat{e}_1, \hat{e}_2, \hat{e}_n \), respectively.

\( q(\theta) \)  
A function of \( \theta \).

\( r_0 \)  
Radius of a circular cylindrical shell.

\( u, v, w \)  
Displacements of the middle surface in the directions of \( \hat{e}_1, \hat{e}_2, \hat{e}_n \), respectively.

\( u_z, v_z, w_z \)  
Displacements of a point, at a distance \( z \) from the middle surface, in the directions of \( \hat{e}_1, \hat{e}_2, \hat{e}_n \), respectively.

\( \tilde{u}, \tilde{v}, \tilde{w} \)  
Complex displacements.

\( z \)  
Distance along the normal from the middle surface.

**Greek Letters**

\( \alpha, \beta \)  
Coordinates of a middle surface.

\( \theta, \phi \)  
Coordinates of the middle surface of a shell of revolution, see Eq. (2-2).

\( \alpha, \beta \)  
Coordinates of the middle surface of a cylindrical shell, see Eq. (2-41).

\( \sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}, \sigma_{33}, \sigma_{23}, \sigma_{32} \)  
Stress components.

\( \mu \)  
Poisson ratio.

\( \lambda^2 = 12(1 - \mu^2) \frac{R_0}{h} \)

\( \eta = \int_0^\theta \frac{R_2}{\sqrt{R_2 R_0}} \, d\theta \)

\( \phi, \psi \)  
Angles of rotation defined by Eqs. (A-3d).

\( \Delta(\_\_) \)  
Differential operator defined by Eqs. (A-3a), (2-5), (2-7) and (2-43).
I. INTRODUCTION

Background

(A) General Theory

The basic equations of the linear theory of thin elastic shells have been well developed (1-5)*. These equations involve the deformation-displacement relations, the equations of equilibrium and relations between forces, moments and the deformation parameters. The derivation of these equations and methods for effecting solutions, especially the deduction by means of complex transformations to a fourth order system of differential equations, are briefly outlined in the Appendix.

(B) Shells of Revolution

The basic equations for shells of revolution can be deduced from those of the general theory of thin elastic shells by proper choice of a coordinate system and Lamé' parameters. There are several methods of reduction of the basic equations to a system of equations from which solutions can be readily derived.

Axisymmetric Deformation

Reduction of the basic equations to a set of two equations which relate the rotation $\Theta$ and the transverse shear $N_1$ was first obtained by Reissner(6) for spherical shells and generalized by Meissner(7) for shells of revolution. The procedure of this derivation is also given in reference (3). By similar procedure, Naghdi and DeSilva(8) extended the work of

* Numbers inside the brackets refer to the references.
Meissner and obtained two equations which, for uniform thickness and for some classes of variable thickness, can be combined to give a single complex differential equation. Novozhilov[2], from the approach of the complex transformation, obtained a single differential equation, which is valid only for shells of uniform thickness.

Exact solutions to these equations have been derived for some particular classes of shells of revolution, such as circular cylindrical shells, conical shells and spherical shells[2,3,5], of which the curvatures of the generators of the middle surfaces are either zero or a constant. When the curvatures of the generators are functions of position, as is the case in ellipsoidal, paraboloidal and other shells of revolution, the exact solution becomes prohibitively difficult. Development of approximate solutions such as asymptotic solutions is indispensable to the analysis.

The method of asymptotic integration has been widely applied to obtaining approximate solutions for shell equations, which for shells of revolution may be transformed into the form

$$\frac{d^2 w}{d \theta^2} = \left( \lambda p(\theta) + q(\theta) \right) w$$

(1-1)

where \( \lambda \) is a large parameter. The asymptotic character of the solutions of Eq.(1-1) as \( \lambda \) approaches infinity can take many different forms depending on the properties of \( p(\theta) \) and \( q(\theta) \). However, three cases are usually encountered in the equations of shells of revolution. The first case, the so-called classical type, is an asymptotic solution of Eq.(1-1) in which, on some interval \( \theta_1 < \theta < \theta_2 \), \( p(\theta) \) and \( q(\theta) \) are both bounded and \( p(\theta) \) is also bounded from zero. The second case is an asymptotic solution of Eq.(1-1) containing a turning point. In this case, \( p(\theta) \) vanishes at a
point \( \theta_0 \) within the interval \( \theta_1 \leq \theta \leq \theta_2 \), such a point is called a turning point. The third case is an asymptotic solution of Eq. (1-1) containing a singular point. In such a case, there exists a point \( \theta_0 \) in the interval \( \theta_1 \leq \theta \leq \theta_2 \) at which \( q(\theta) \) may have a pole of first or second order and \( p(\theta) \) contains as a factor \((\theta - \theta_0)^a\) where \( a \) is a real nonnegative constant, and \( p(\theta) \) and \( q(\theta) \) are both bounded in the rest of the interval. The solutions of these three cases have been investigated extensively by Langer [9] and Olver [10, 11]. The first case occurs in the differential equation for shells of revolution with two open edges where the region of interest lies in the nonshallow region (large values of \( \theta \)). The second case occurs in the differential equation for toroidal shells, and the third case is encountered in the differential equation for dome-shaped shells of revolution where the region of interest lies in the shallow region (small values of \( \theta \)) including a singular point at the apex.

Asymptotic integration of the third case has been applied to the investigation of ellipsoidal, paraboloidal and other dome-shaped shells of revolution. Naghibi and DeSilva [12] applied this method to the study of deformations of ellipsoidal shells of revolution of uniform thickness under axisymmetric loading. Solutions valid in the shallow region were obtained in terms of Kelvin functions. Clark and Reissner [13] obtained the solution based on the bending theory for complete ellipsoidal shells of revolution subjected to internal pressure by the use of small-parameter expansion. Deformation of paraboloidal shells of uniform thickness subjected to a load uniformly distributed over a small region about the apex and clamped at the open edge was studied by DeSilva and Arbor [14]. Study of dome-shaped shells of revolution subjected to axisymmetric loading.
was made by Baker and Cline [15], and Steele and Hartung [16].

Application of the first case of asymptotic solution which is valid only in the nonshallow region was made by Novozhilov in the study of nonshallow shells of revolution under axisymmetric loads.

**Nonsymmetric Deformation**

There are three basic procedures in reducing the basic equations of shells of revolution subjected to arbitrary loads. In the first of these, the basic equations are reduced to three differential equations which relate the displacements $u$, $v$ and $w$. This procedure was employed by Vlasov [1] and Donnell [17] in deriving the governing equations for circular cylindrical shells. Stelle [18] also used the same procedure for reduction of the basic equations of shells of revolution under nonsymmetric edge loads, and obtained, by neglecting the transverse shear terms in the first two equations of equilibrium, three differential equations which relate the displacements $u$, $v$ and $w$. The membrane and bending solutions that are valid throughout the shallow and nonshallow regions were obtained by means of asymptotic integration. In the second, a stress function is introduced and the governing equations are reduced to two differential equations which relate the stress function $F$ and the normal displacement $w$. Reissner [19] employed this method and obtained a set of two equations for small deformation of shallow spherical shells. In the third, the basic equations are reduced by means of complex transformation developed by Novozhilov [2] to two differential equation which relate to two complex functions. The procedure of derivation is given in the Appendix and in Chapter II.

Asymptotic solutions to Novozhilov's equations valid in in the nonshallow region were derived by Schile [20] for external loads including (a) sinusoi-
dal loading and (b) higher harmonic load distribution. No literature on
solutions to Novozhilov's equations that are valid in the shallow region
is known to the author.

(C) Application

Numerous investigations have been made on the application of the
solutions mentioned previously to engineering structures. Attention here
will be limited to dome-shaped shells of revolution having a discontinuity
of the types: (a) a hole; (b) a rigid insert; (c) a nozzle attachment.

The problem of the stress distribution around holes in shells has
been investigated by a number of workers. Hemispherical shells with a
circular opening at the vertex subjected to axisymmetric self-equilibrating
forces were studied by Galletly [21]. An elliptical opening in a spherical
shell under internal pressure was investigated by Leckie and Payne [22] who
expressed the equation in elliptical coordinates and obtained the solution
in terms of Mathieu functions. For a more general case, Savin [23] inves-
tigated the stress distribution around an arbitrary hole with smooth con-
tour in thin shells and obtained solutions to the shell equations which
had been transformed by the use of conformal mapping into a coordinate
system such that along the contour of the hole one of the coordinates is
constant. The general method was described and applied to a cylindrical
shell with a circular hole and to a spherical shell with either a circular
or an elliptical hole. Further studies [24] were made of a spherical shell
under internal pressure weakened by an elliptical hole, square and tri-
angular holes with rounded corners.

Spherical shells with a circular rigid insert have been considered
by Bijlaard [25] and, with an elliptical rigid insert, by Leckie and Payne.
[22], and Foster[26].

The effect of local loading on spherical shells in which external loads are transmitted from a nozzle radially attached to the shells has been investigated extensively by Bijlaard[27] and Leckie and Payne[28]. Studies of the case in which the nozzle is obliquely attached to a spherical shell were made by Johnson[29] and Yu, Chen and Shaw[30].

All the investigations mentioned here are restricted to shallow shells with a hole or a rigid insert, the size of which is small compared to the radii of curvatures of the shells so that shallow shell equations hold for the problems under consideration. The case of a nozzle attachment has been also limited to nozzle-to-spherical shells.

As far as the author knows, little attention has been given to systematic studies of stress distribution in nonshallow shells of dome shape around a discontinuity of a size which is not necessarily small compared to the radius of curvature of the shells.

Statement of the Problems

Investigation of the following problems is suggested upon the review made in the preceding sections:

(1) Derivation of solutions to Novozhilov's equations for shells of revolution which are valid in the shallow and nonshallow regions under axisymmetric and first harmonic loads. This extends the work of Novozhilov who derived the equations and obtained solutions valid only in the non-shallow region. The development here also differs from that of Steele in that it does not neglect the transverse shear terms in the first two equations of equilibrium.
(2) Application of the solution derived in (1) to the study of the stresses in the vicinity of a discontinuity at the apex of dome-shaped shells of revolution under external loads including (a) internal pressure and (b) a couple applied to the discontinuity. The discontinuity is in the form of a circular hole, or a circular rigid insert, or a nozzle. No restriction is placed on the size of the discontinuity in relation to the radius of the shells. Application to discontinuities is embedded in a uniform treatment and includes discontinuities in geometries on which little information is available.

(3) Analysis of the influence of the different types of discontinuity on the stresses of the shells.

For systematic study of these problems, the procedures for the reduction of the basic shell equations to a fourth order system of three equations are briefly outlined in the Appendix. Further reductions to a second order differential equation in terms of a complex force are derived in Chapter II. Solutions to this equation valid in the shallow region are derived in Chapter III using the method of asymptotic integration.

Applications of these solutions to the study of problem (2) are investigated in Chapter IV in which the boundary conditions for each of the appropriate cases are derived. The study of problem (3) is given in Chapter V.
II. GOVERNIG DIFFERENTIAL EQUATIONS

A second order differential equation governing the deformation of dome-shaped shells of revolution and of circular cylindrical shells will be deduced from the system of differential equations (A-27) for both axisymmetric and first harmonic loads.

Shells of Revolution

The coordinate system chosen for shells of revolution will be \( \theta \) and \( \phi \), which determine the position of a point on the middle surface (Fig. 2-la). Let \( R_1 \) be the radius of curvature of the meridian \( (\phi = \text{constant}) \) and \( R_2 \) be the length along the normal to the middle surface between the axis of revolution and the middle surface. \( R_2 \) is sometimes referred to as the second radius of curvature. Thus, the first fundamental form of the surface is (Fig. 2-1b)

\[
(ds)^2 = (R_1 d\theta)^2 + (R_2 \sin \theta d\phi)^2
\]

(2-1)

By comparison of Eq. (2-1) with Eq. (A-1) for shells of arbitrary shape one sees that

\[
\begin{align*}
C_1 &= \theta, & C_2 &= \phi \\
A_1 &= R_1, & A_2 &= R_2 \sin \theta
\end{align*}
\]

(2-2)

The last two of the conditions of Gauss-Codazzi, Eq. (A-2), are identically satisfied, since \( R_1 \) and \( R_2 \) are functions of \( \theta \) only. The first condition reduces to

\[
(R_2 \sin \theta)' = R_1 \cos \theta
\]

(2-3)
\( R_1 \): radius of curvature of the line \( \phi = \text{constant} \).

\( R_2 \): length between the axis of revolution and the middle surface.

(a)

Fig. 2-1: Coordinate system of a shell of revolution
where the prime indicates differentiation with respect to \( \theta \). By use of these relations, Eqs. (A-27) and (A-29) as given in the Appendix are expressed by

\[
\frac{1}{R_1} \frac{\partial^2 \tilde{\tau}_{1}}{\partial \theta^2} - \cot \theta \left( \frac{\tilde{\tau}_1}{r_1} - \frac{\tilde{\tau}_2}{r_2} \right) + \frac{1}{R_2 \sin \theta} \frac{\partial \tilde{\tau}_3}{\partial \phi} + \frac{i}{R_1} \frac{\partial \tilde{\tau}_1}{\partial \theta} = - \varphi_1
\]

\[
\frac{1}{R_1} \frac{\partial \tilde{\tau}_3}{\partial \theta} + 2 \cot \theta \tilde{\tau}_3 + \frac{1}{R_2 \sin \theta} \frac{\partial \tilde{\tau}_3}{\partial \phi} + \frac{c}{R_1} \frac{\partial \tilde{\tau}_{1}}{\partial \theta} \frac{\partial \varphi}{\partial \phi} = - \varphi_2
\]  

\[
\tilde{\tau}_1 \frac{1}{R_1} + \tilde{\tau}_2 \frac{1}{R_2} - i c \Delta(\tilde{\tau}) = \varphi_n
\]

and

\[
R_1 \tilde{a}_1 = \frac{\partial \tilde{a}}{\partial \theta} + \tilde{w} = \frac{R_1}{E_h} \left( \tilde{\tau}_1 - \mu \tilde{\tau}_2 \right)
\]

\[
R_2 \tilde{a}_2 = \frac{1}{\sin \theta} \frac{\partial \tilde{\tau}_3}{\partial \phi} + \tilde{u} \cot \theta + \tilde{w} = \frac{R_2}{E_h} \left( \tilde{\tau}_3 - \mu \tilde{\tau}_4 \right)
\]

\[
\tilde{a}_1 = - \frac{i}{R_1} \frac{\partial}{\partial \theta} \left[ \frac{1}{R_1} \left( \frac{\partial \tilde{\tau}_3}{\partial \phi} - \tilde{u} \right) - \frac{\cot \theta}{R_1} \frac{\partial \tilde{\tau}_3}{\partial \theta} \right] = \frac{i}{cE_h} \left( \tilde{\tau}_2 - \tilde{\tau}_4 \right)
\]

\[
\tilde{a}_2 = \frac{1}{R_2} \frac{\partial \tilde{\tau}_3}{\partial \phi} \left( \frac{1}{\sin \theta} \frac{\partial \tilde{\tau}_3}{\partial \phi} - \tilde{v} \right) \cot \theta + \tilde{v} \tilde{\tau}_3 + \frac{i}{cE_h} \left( \tilde{\tau}_2 - \tilde{\tau}_4 \right)
\]

\[
R_1 \tilde{w} = \frac{R_1}{R_2} \tilde{a}_1 \left( \frac{\partial \tilde{w}}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial \tilde{w}}{\partial \phi} \right) = \frac{2(1+\mu)R_2}{E_h} \tilde{w}
\]

\[
R_2 \tilde{w} = \frac{1}{R_1} \frac{\partial \tilde{w}}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \tilde{w}}{\partial \phi} + \tilde{v} \right) + \frac{1}{R_1} \frac{\partial \tilde{w}}{\partial \theta} \left( \frac{\partial \tilde{w}}{\partial \theta} - \tilde{v} \right) = - \frac{i}{cE_h} (\tilde{\tau}_2 - \tilde{\tau}_4)
\]

where \( \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \) and \( \tilde{\tau}_4 \) are complex forces defined by Eqs. (A-26); \( \varphi_1, \varphi_2, \) and \( \varphi_n \) are components of surface loading in the directions of \( \tilde{\tau}_1, \tilde{\tau}_2, \) and \( \tilde{\tau}_4 \), respectively; \( \tilde{a}_1, \tilde{a}_2, \tilde{w}, \tilde{v}, \tilde{\tau}_1, \tilde{\tau}_2, \) and \( \tilde{\tau}_3 \) are the complex deformation parameters of the middle surface and \( a, v, \) and \( \tau \) are the complex displacements; and

\[
\Delta(\cdot) = \frac{1}{R_1^2} \frac{\partial^2 \tilde{a}_1}{\partial \theta^2} + \left( \frac{\cot \theta}{R_1 R_2} - \frac{1}{R_1} \frac{dR_2}{d\theta} \right) \frac{\partial \tilde{a}_2}{\partial \theta} + \frac{1}{R_2^2 \sin^2 \theta} \frac{\partial \varphi}{\partial \phi^2}
\]

(A) Axisymmetric Deformation
Because of the assumed symmetry all quantities are independent of \( \phi \). If, in addition,

\[
q_1 = q_2 = 0 \quad \text{and} \quad j_n = p = \text{constant}
\]

then, Eqs. (2-4a) reduce to the form

\[
\begin{align*}
\frac{1}{R_1} \tilde{T}' + \frac{\cot \theta}{R_2} (\tilde{T}_1 - \tilde{T}_2) + i \frac{c}{R_1^2} \tilde{T}' &= 0 \\
\frac{1}{R_1} \tilde{S}' + 2 \frac{\cot \theta}{R_2} \tilde{S} &= 0 \\
\frac{\tilde{T}_1}{R_1} + \frac{\tilde{T}_2}{R_2} - i c \Delta(t) &= p
\end{align*}
\]

(2-6)

where

\[
\Delta(\theta) = \frac{1}{R_1} \left( \frac{1''}{R_1 R_2} + \frac{\cot \theta}{R_1 R_2} - \frac{1}{R_1^2} \right) R_1' (\tilde{T}')
\]

(2-7)

and the prime indicates differentiation with respect to \( \theta \). By use of the first Gauss-Codazzi condition, Eq. (2-3), the second of Eqs. (2-6) may be written in the form

\[
\frac{d \tilde{S}}{\tilde{S}} + 2 \frac{d R_2 \sin \theta}{R_2 \sin \theta} = 0
\]

which, upon integration, has the solution

\[
\tilde{S} = \frac{c_1}{R_2 \sin \theta}
\]

(2-8)

where \( c_1 \) is a complex constant of integration. Since, due to symmetry, \( \tilde{S} \) vanishes on an edge of \( \theta = \text{constant} \), \( c_1 \) must be set to zero.

Next, the solution for the auxiliary functions \( \tilde{T}_1 \) and \( \tilde{T}_2 \) (see Eqs. (A-26) in the Appendix) will be obtained from the first and the third of Eqs. (2-6). By use of Eq. (2-3) the first of Eqs. (2-6) may be written in the form

\[
\frac{1}{R_1 R_2 \sin \theta} (R_2 \sin \theta \tilde{T}_1)' - \frac{\cot \theta}{R_2} \tilde{T}_2 + i \frac{c}{R_1^2} \tilde{T}' = 0
\]

(2-9)
Eliminating $\bar{T}_2$ from Eq. (2-9) by multiplying the third of Eqs. (2-6) by $\cot \theta$ and adding it to Eq. (2-9), and multiplying the result by $R_1 R_2 \sin \theta$, there results

$$
(R_2 \sin \theta \bar{T}_1)' + R_2 \cos \theta \bar{T}_1 + i c R_1 R_2 \sin \theta (-\frac{1}{R_1} \bar{T}' - \cot \theta \Delta (\bar{T})) = R_1 R_2 \cos \theta R
$$

(2-10)

The first two terms of Eq. (2-10) may be combined to give

$$
(R_2 \sin \theta \bar{T}_1)' + R_2 \cos \theta \bar{T}_1 = \frac{1}{\sin \theta} (R_2 \sin^2 \theta \bar{T}_1)'
$$

and the third term of Eq. (2-10) can be shown to be equal to

$$
-\frac{j c}{\sin \theta} (\frac{R_2}{R_1} \sin \theta \cos \theta \bar{T}')'
$$

Thus, Eq. (2-10) reduces to

$$
(R_2 \sin^2 \theta \bar{T}_1)' - ic (\frac{R_2}{R_1} \sin \theta \cos \theta \bar{T}') = p R_1 R_2 \sin \theta \cos \theta
$$

(2-11)

Now, introduce a function $\bar{U}$ defined by

$$
\bar{U} = R_2 \sin^2 \theta \bar{T}_1 - ic \frac{R_2}{R_1} \sin \theta \cos \theta \bar{T}'
$$

(2-12)

Eq. (2-11) becomes

$$
\bar{U}' = p R_1 R_2 \sin \theta \cos \theta
$$

It follows upon integration that

$$
\bar{U} = \bar{c}_2 + \frac{p}{2} R_2 \sin^2 \theta
$$

(2-13)

The fourth of Eqs. (A-26), i.e.,

$$
\bar{T} = \bar{T}_1 + \bar{T}_2
$$

(2-14)

can be substituted into the third of Eqs. (2-6) to eliminate $\bar{T}_2$. Also, $\bar{T}_1$ can be eliminated by using Eq. (2-12). The final result of this manipulation is a second order differential equation on $\bar{T}$ which can be written as
\[ \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \left( \frac{\bar{U}}{R_2 \sin^2 \theta} + i e \frac{1}{R_1} \cot \theta \bar{T}' \right) + \frac{\bar{T}}{R_2} - i e \alpha(\bar{T}) = p \]

or after rearrangement

\[ \bar{T}'' + \left[ \left( \frac{2 - R_2}{R_2} - 1 \right) \cot \theta - \frac{R_1'}{R_1} \right] \bar{T}' + i \frac{R_2^2}{R_2 e} \bar{T} = i \frac{R_1^2}{R_2 e} F(\theta) \quad (2-15) \]

where

\[ F(\theta) = p R_2 - \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{\bar{U}}{\sin^2 \theta} \quad (2-16) \]

Thus, the axisymmetric deformation of shells of revolution under internal pressure reduces to the solution of the second order differential equation (2-15). Eqs. (2-12) and (2-14) can be written as

\begin{align*}
\bar{T}_1 &= i \frac{e}{R_1} \cot \theta \bar{T}' + \frac{\bar{U}}{R_2 \sin^2 \theta} \\
\bar{T} &= \bar{T} - \bar{T} + \dots + \bar{T} - \bar{T} = \dots \bar{U} \quad \quad (2-17)
\end{align*}
\[
\frac{1}{R_1} (\tilde{u}' + \tilde{w}) = \frac{1}{E_h} \left( \tilde{T}_1 - \mu \tilde{T}_2 \right)
\]
\[
\frac{1}{R_2} (\tilde{u} \cot \theta + \tilde{w}) = \frac{1}{E_h} \left( \tilde{T}_2 - \mu \tilde{T}_1 \right)
\]
\[
\frac{1}{R_1} \left[ \frac{1}{R_1} (\tilde{w}' - \tilde{u}^*') \right]' = \frac{i}{c \epsilon h} (T_2 - T_2^*)
\]
\[
-\frac{\cot \theta}{R_1 R_2} (\tilde{w}' - \tilde{u}) = \frac{i}{c \epsilon h} (T_1 - T_1^*)
\]

The last one of Eqs. (2-19) may be written in the form
\[
\frac{1}{R_1} (\tilde{w}' - \tilde{u}) = \frac{i R_1}{c \epsilon h} \tan \theta (T_1 - T_1^*)
\]

Comparing this equation with the third of Eqs. (2-19), one observes that these two equations are compatible only if
\[
\frac{1}{R_1} \left[ R_2 \tan \theta (T_1 - T_1^*) \right]' = T_2 - T_2^*
\]

is identically satisfied. Eq. (2-21), upon substitution for \( \tilde{T}_1 \) and \( \tilde{T}_2 \) by their expressions from Eqs. (2-17) and with the consideration of Eq. (2-15), becomes
\[
\tilde{T} - \tilde{T} - \frac{ic}{R_1} \cot \theta \tilde{T}' + \frac{1}{R_1} \left[ R_2 \tan \theta (\tilde{u} \frac{\tilde{u}}{R_2 \sin \theta} - T_1^*) \right]' = \tilde{T} - \tilde{T} - \frac{ic}{R_1} \cot \theta \tilde{T}'
\]

This equation is satisfied if
\[
T_1^* = \frac{\tilde{u}}{R_2 \sin \theta}
\]
and
\[
F = T_2^* + \frac{\tilde{u}}{R_2 \sin \theta} = \frac{T_1^*}{R_2 \sin \theta} + \frac{\tilde{u}}{R_1}
\]

Comparison of the first of Eqs. (2-22) with the third of Eqs. (2-18) yields
\[
\tilde{u} = U
\]
from which it follows that
\[
\tilde{C}_2 = C_2 = \text{real constant}
\]
(B) Non-symmetric Deformation – Edge Loads only

In that which follows, equations will be developed for the non-symmetric deformation of shells of revolution due to edge effects only. In addition, deduction to a single second-order differential equation will be obtained for the special case where the resultant edge loads consist only of moment.

Since the surface loads \( q_1, q_2, \) and \( q_n \) are zero, Eqs. (2-4) become

\[
\frac{1}{R_1} \frac{\partial^2}{\partial \theta^2} \left( R \right) + \frac{\cot \theta}{R_2} \left( \frac{R}{R_2} \right) + \frac{1}{R_2 \sin \theta} \frac{\partial}{\partial \phi} + i \frac{\xi}{R_2} \frac{\partial}{\partial \phi} = 0
\]

\[
\frac{1}{R_1} \frac{\partial^2}{\partial \theta^2} \left( R \right) + 2 \frac{\cot \theta}{R_2} \frac{\partial}{\partial \theta} \left( R \right) + \frac{1}{R_2 \sin \theta} \frac{\partial}{\partial \phi} + i \frac{\xi}{R_2 \sin \theta} \frac{\partial}{\partial \phi} = 0 \quad (2-23)
\]

\[
\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \phi} - i \zeta A(t) = 0
\]

Following the procedure of reduction to a single second-order differential equation for symmetric deformation, one may introduce, on the basis of Eqs. (2-8) and (2-12), two auxiliary functions

\[
\bar{U} = R \sin^2 \theta \bar{T} - i \frac{R_e \sin \theta \cos \theta}{R_1} 
\]

\[
\bar{V} = R \sin^2 \theta \bar{S} \quad (2-24)
\]

Eqs. (2-23), through certain manipulations with the help of Eqs. (2-3) and (2-24), may be reduced to the following system of three partial differential equations [2] of which the first two involve two unknowns \( \bar{U} \) and \( \bar{T} \).

\[
G(\bar{U}) - \left[ 1 - i \zeta \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{1}{\sin^2 \theta} \right] \frac{\partial^2 \bar{T}}{\partial \phi^2} = 0
\]

\[- i \zeta G(\bar{T}) + \bar{T} + \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{\sin \theta}{R_2} \bar{U} = 0 \quad (2-25)
\]

\[
\frac{\partial \bar{V}}{\partial \phi} + \frac{R \sin \theta}{R_1} \frac{\sin \theta}{\phi} - i \zeta R \cos \theta \frac{\sin \theta}{R_1} \frac{\partial^2 \bar{T}}{\partial \phi^2} = 0
\]
where
\[
G_1(\gamma) = \frac{1}{R_1 R_2 \sin \theta} \left[ \frac{R_1^2 \sin \theta}{R_1} \gamma + \frac{1}{R_2 \sin^2 \theta} \phi^2 \right]
\]  
(2-26)

Thus, the analysis of shells of revolution subject to any type of edge loading has been reduced to the solution of the system of Eqs. (2-25).

However, the following will be restricted to the case where the resultant edge load at \( \theta = \theta_0 \) (near the apex) of a shell of revolution is equivalent to a moment. For this particular case the auxiliary functions \( \tilde{T}, \tilde{U} \) and \( \tilde{V} \) may be expressed as

\[
\begin{align*}
\tilde{T} &= \tilde{T}^*(\theta) \cos \phi \\
\tilde{U} &= \tilde{U}^*(\theta) \cos \phi \\
\tilde{V} &= \tilde{V}^*(\theta) \sin \phi
\end{align*}
\]  
(2-27)

Substitution of Eqs. (2-27) into Eqs. (2-25) yields

\[
\begin{align*}
G_1(\tilde{U}^*) + \left[ \frac{1}{R_1} - \frac{1}{R_2} \right] \frac{\partial}{\partial \theta} \tilde{T}^* &= 0 \\
-i \xi G_1(\tilde{T}^*) + \tilde{T}^* + \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{1}{R_2 \sin^2 \theta} \tilde{U}^* &= 0 \\
\tilde{V}^* + \frac{R_2^2 \sin \theta}{R_1} \tilde{U}^* + i \xi R_2 \cos \theta \tilde{T}^* &= 0
\end{align*}
\]  
(2-28)

where

\[
G_1(\gamma) = \frac{1}{R_1 R_2 \sin \theta} \left[ \frac{R_1^2 \sin \theta}{R_1} \phi' - \frac{1}{R_2 \sin^2 \theta} \phi \right]
\]  
(2-29)

The first two of Eqs. (2-28) may be uncoupled by subtracting the second equation from the first and then introducing the new function

\[
\tilde{W} = \tilde{U}^* + i \xi \tilde{T}^*
\]  
(2-30)

into the result. In this way there results

\[
G_1(\tilde{W}) - \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{1}{R_2 \sin^2 \theta} \tilde{W} = 0
\]  
(2-31)
Expanding this equation with the help of Eq. (2-29) one arrives at

\[
\left( \frac{R^2_t \sin \theta}{R_t} \bar{W}' \right)' - \frac{R_t}{\sin \theta} \bar{W} = 0
\]  

(2-32)

It may be verified that one of the solutions of Eq. (2-32) is

\[
\bar{W} = \frac{1}{R^2_t \sin \theta}
\]

The second solution may be obtained by assuming

\[
\bar{W} = \frac{A}{R^2_t \sin \theta}
\]  

(2-33)

where \( A \) is a function of \( \theta \). Eq. (2-32) upon substitution for \( \bar{W} \) from Eq. (2-33) reduces to the form

\[
\left( \frac{1}{R_t \sin \theta} A' \right)' = 0
\]

from which it follows

\[
A = B_1 + B_2 \int R_t \sin \theta \, d\theta
\]

Thus

\[
\bar{W} = \frac{B_1}{R^2_t \sin \theta} + \frac{B_2}{R^2_t \sin \theta} \int R_t \sin \theta \, d\theta
\]  

(2-34)

Eliminating \( \bar{U}^0 \) in the second of Eqs. (2-28) by its expression from Eq. (2-30), one arrives at the following differential equation in a single unknown \( T^0 \)

\[
G_1(\tilde{T}^0) + \frac{i}{c} \tilde{T}^0 + \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{1}{\sin^2 \theta} \tilde{T}^0 = -\frac{i}{c} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{i}{\sin^2 \theta} \bar{W}
\]

which may be written in expanded form

\[
\tilde{T}^0 + \left( 2 - \frac{R_2}{R_1} \right) \cos \theta - \frac{R_2'}{R_1} \tilde{T}' + \frac{R_1}{R_2} \left( 1 - 2 \frac{R_2}{R_1} \right) \frac{1}{\sin^2 \theta} \tilde{T}^0
\]

\[
+ i \frac{R_1^4}{R_2^4} = i \frac{R_1^4}{R_2^4} F_1(\theta)
\]  

(2-35)

where

\[
F_1(\theta) = -\left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{i}{\sin^2 \theta} \bar{W}
\]  

(2-36)
Once $\bar{U}^0$ and $\bar{T}^0$ have been found the auxiliary function $\bar{V}^0$ can be found from the third of Eqs. (2-28). It is noted that, by letting $c = 0$ in the third of Eqs. (2-23), the first of Eqs. (2-24) and Eq. (2-30), one arrives at the membrane theory solution.

$$\bar{U}^0 = W, \quad \bar{T}_{z^o} = \frac{W}{R_2 \sin^2 \phi}$$

$$\bar{T}_{e^o} = -\frac{W}{R_1 \sin^2 \phi}, \quad \bar{S}^o = -\frac{W'}{R_1 \sin \phi}$$

(2-37)

And Eq. (2-36) becomes

$$F_z(0) = \bar{T}_{z^o} + \bar{T}_{e^o}$$

There remains the evaluation of the displacements which for this particular problem are assumed to be

$$\bar{u} = \bar{u}_1(\phi) \cos \phi$$

$$\bar{v} = \bar{v}_1(\phi) \sin \phi$$

$$\bar{w} = \bar{w}_1(\phi) \cos \phi$$

(2-38)

On substituting these expressions into Eqs. (2-4), a system of equations relating the complex displacements $\bar{U}_1$, $\bar{V}_1$, $\bar{W}_1$ to the complex forces is obtained

$$\bar{U}_1' + \bar{W}_1 = \frac{R_1}{E h} \left( \bar{T}_{z^o} - \mu \bar{T}_{e^o} \right)$$

$$\frac{1}{\sin \phi} \bar{V}_1 + \bar{U}_1 \cot \theta + \bar{W}_1 = \frac{R_1}{E h} \left( \frac{\bar{T}_{z^o}}{R_2} - \mu \bar{T}_{e^o} \right)$$

$$\frac{R_2}{R_1} \bar{V}_1' - \bar{V}_1 \cot \theta - \frac{1}{\sin \phi} \bar{U}_1 = \frac{2(1+\mu)R_2}{E h} \bar{S}^o$$

$$-\frac{1}{R_1} \left( \frac{\bar{V}_1' - \bar{V}_1}{R_1} \right)' = \frac{i}{c E h} \left( \frac{\bar{T}_{z^o}}{R_1} - \frac{\bar{T}_{e^o}}{R_2} \right)$$

$$\frac{1}{R_2 \sin \phi} \left( \frac{1}{\sin \phi} \bar{W}_1 + \bar{V}_1 \right) - \frac{c \cot \theta}{R_1 R_2} \left( \bar{V}_1' - \bar{V}_1 \right) = \frac{i}{c E h} \left( \frac{\bar{T}_{z^o}}{R_1} - \bar{T}_{e^o} \right)$$

$$\frac{1}{R_1 \sin \phi} \left( \frac{\bar{W}_1' + \bar{V}_1}{R_1} \right)' - \frac{1}{R_1 R_2 \sin \phi} \left( \bar{U}_1 + \bar{V}_1 \cos \theta \right) = -\frac{i}{c E h} \left( \frac{\bar{S}^o}{R_2} - \bar{S}^o \right)$$
This completes the reduction of the basic equations of the general theory to the governing equations for shells of revolution pertaining to investigation stated in Chapter I.

**Circular Cylindrical Shells**

The coordinates identifying the position of points on the middle surface are $\alpha$ and $\beta$ (Fig. 2-2) and $r_0$ is the radius of a circular cross section. Thus, the first fundamental form of the surface is

$$(ds)^2 = (r_0 \, d \alpha)^2 + (r_0 \, d \beta)^2$$  \hspace{1cm} (2-40)

from this one may verify that

$$\alpha_1 = \alpha, \quad \alpha_2 = \beta$$

$$A_1 = A_2 = r_0$$

Hence, the differential equations (A-27) for complex forces may be written in the form

$$\frac{\partial \overline{\tau}}{\partial \alpha} + \frac{\partial \overline{\sigma}}{\partial \beta} = - \bar{\gamma} r_0$$

$$\frac{\partial \overline{\tau}}{\partial \alpha} + \frac{\partial \overline{\sigma}}{\partial \beta} + \frac{i}{2b^2} \frac{\partial \overline{\tau}}{\partial \beta} = - \bar{\gamma} r_0$$

$$\bar{T}_2 - \frac{j}{2b^4} \Delta(\bar{T}) = \bar{\gamma}_n r_0$$

where

$$4b^4 = 12(1 - \mu^2)(\frac{L}{b})^2$$

$$\Delta(\bar{T}) = \frac{\partial \overline{\tau}}{\partial \alpha} + \frac{\partial \overline{\tau}}{\partial \beta}$$

$$\bar{T} = \bar{T}_1 + \bar{T}_2$$

The complex forces in these equations are related to the forces and
\[
\alpha \approx s/r_0
\]

Fig. 2-2: Cylindrical Coordinate system
moments by the following expressions

\[ \bar{T}_1 = T_1 - i \frac{2b^2}{l_o} \frac{M_2 - \mu M_1}{1 - \mu^2} \]
\[ \bar{T}_2 = T_2 - i \frac{2b^2}{l_o} \frac{M_1 - \mu M_2}{1 - \mu^2} \]
\[ \bar{S} = S + i \frac{2b^2}{l_o} \frac{H}{1 - \mu} \]

(2.44)

Also \( q_1, q_2 \) and \( q_n \) are the components of the surface loading in the directions of \( \delta_1, \delta_2 \) and \( \delta_n \), respectively.

The relations (A-29) between complex displacements and complex forces for circular cylindrical shells (taking into account Eq. (2.41) and \( P_\perp = \infty, R_2 = r_o \)) may be expressed in the form

\[ \bar{\varepsilon}_1 = \frac{1}{r_o} \left( \frac{\partial \bar{u}}{\partial a} \right) = \frac{l}{Eh} \left( \bar{T}_1 - \mu \bar{T}_2 \right) \]
\[ \bar{\varepsilon}_2 = \frac{1}{r_o} \left( \frac{\partial \bar{v}}{\partial a} + \bar{w} \right) = \frac{j}{Eh} \left( \bar{T}_2 - \mu \bar{T}_1 \right) \]
\[ \bar{\omega} = \frac{1}{r_o} \left( \frac{\partial \bar{v}}{\partial a} + \frac{\partial \bar{u}}{\partial a} \right) = \frac{2(1 + \mu)}{Eh} \bar{S} \]
\[ \bar{k}_1 = -\frac{l}{r_o} \left( \frac{\partial \bar{w}}{\partial a} \right) = i \frac{2b^2}{Eh r_o} \left( \bar{T}_2 - \bar{T}_1^{*} \right) \]
\[ \bar{k}_2 = -\frac{l}{r_o} \left( \frac{\partial \bar{w}}{\partial a} \right) = i \frac{2b^2}{Eh r_o} \left( \bar{T}_1 - \bar{T}_2^{*} \right) \]
\[ \bar{\eta} = -\frac{l}{r_o} \left( \frac{\partial \bar{w}}{\partial a} \right) = i \frac{2b^2}{Eh r_o} \left( \bar{S} - \bar{S}^{*} \right) \]

(2.45)

(A) Axisymmetric Deformation

On account of the assumed symmetry all quantities are independent of \( \beta \), and \( q_2 = 0 \). Thus, Eqs. (2.42) reduce to

\[ \bar{T}_1' = -\bar{\eta} r_o \]
\[ \bar{S}' = 0 \]
\[ \bar{T}_2 - i \frac{2b^2}{l_o} \bar{\eta} = \bar{\eta} r_o \]

(2.46)

The prime indicates the derivative with respect to \( \alpha \). From the second of
Eqs. (2-46) one obtains, in view of symmetry
\[ \bar{S} = \bar{C}_1 = 0 \]  
(2-47)
The first of Eqs. (2-46) gives
\[ \bar{T}_1 = \bar{C}_2 - r_c \int g_1 \, da \]  
(2-48)
Eliminating \( \bar{T} \) in the third of Eqs. (2-46) by its expression in terms of \( \bar{T}_1 \) and taking into consideration Eq. (2-48) one arrives at a second order differential equation for a single unknown \( \bar{T}_2 \)
\[ \bar{T}_2''' + i2b^2 \bar{T}_1 = i2b^2 g_2 \bar{r}_c + r_c g_2' \]  
(2-49)
The displacements can be obtained from Eqs. (2-45) which, for this case, reduce to the form
\[ \frac{f}{h} \bar{u}' = \frac{f}{Eh} (\bar{T}_1 - \mu \bar{T}_2) \]  
(2-50)
\[ \frac{f}{h} \bar{w} = \frac{f}{Eh} (\bar{T}_2 - \mu \bar{T}_1) \]  
(2-50)

(B) Non-symmetric Deformation

Eliminating \( \bar{S} \) from the first two of Eqs. (2-42), there results
\[ \frac{2b^2}{\omega a^2} \Delta (\bar{T}_2) - \frac{i}{2b^2} \frac{\partial^2 \bar{T}}{\partial \beta^2} = r_c (\frac{\partial^2 g_1}{\partial \beta} - \frac{\partial^2 g_1'}{\partial \alpha}) \]  
(2-51)
Substitution in Eq. (2-51) for \( \bar{T}_2 \) by its expression from the third of Eqs. (2-42) yields a fourth order partial differential equation in a single unknown \( \bar{T} \)
\[ \Delta \bar{T} + \frac{\partial^2 \bar{T}}{\partial \beta^2} + i2b^2 \frac{\partial^2 \bar{T}}{\partial \alpha^2} = i2b^2 r_c [\frac{\partial^2 g_1}{\partial \beta} - \frac{\partial^2 g_1'}{\partial \alpha} + \Delta (g_1)] \]  
(2-52)
Thus, the analysis of the non-symmetric deformation of a circular cylindrical shell has been reduced to the solution of this equation. Once \( \bar{T} \) has been obtained the complex forces may be found from the following equations:
\[ \bar{T}_2 = \bar{r}_\beta + \frac{i}{2b^2} \Delta (\bar{T}) \]
\[ \bar{T}_1 = \bar{T} - \bar{T}_2 = \bar{T} - \frac{i}{2b^2} \Delta (\bar{T}) - \bar{r}_\beta \]
\[ \frac{\partial \bar{T}}{\partial \alpha} = -\frac{i}{2b^2} \left[ \frac{\partial}{\partial \beta} \Delta (\bar{T}) + \frac{\partial}{\partial \beta} \right] - \bar{r}_\beta \left[ \frac{\partial}{\partial \beta} + \bar{q}_2 \right] \tag{2-53} \]
\[ \frac{\partial \bar{T}}{\partial \beta} = -\bar{c} \frac{\partial \bar{T}}{\partial \alpha} + \frac{i}{2b^2} \frac{\partial}{\partial \alpha} \Delta (\bar{T}) + \bar{r}_\beta \bar{q}_2 - \bar{q}_1 \]

As was done in the non-symmetric deformation for shells of revolution the problem will be restricted to that of pure bending. For such a case
\[ q_1 = q_2 = q_n = 0 \]
and the complex forces can be assumed to be
\[ \bar{T} = \bar{T}_0 (\alpha) \cos \beta, \quad \bar{T}_1 = \bar{T}_0 (\alpha) \cos \beta \]
\[ \bar{T}_2 = \bar{T}_2 (\alpha) \cos \beta, \quad \bar{S} = \bar{S}_0 (\alpha) \sin \beta \tag{2-54} \]

On substitution in Eq. (2-52) for \( \bar{T} \) by its expression from the first of Eqs. (2-54), there results an ordinary differential equation for \( \bar{T}^0 \)
\[ \bar{T}^{0''} + (i2b^2 - 2) \bar{T}^0'' = 0 \tag{2-55} \]
where the prime denotes differentiation with respect to \( \alpha \).

The complex displacements for the given case are assumed to be of the form
\[ \bar{u} = \bar{u}_0 (\alpha) \cos \beta, \quad \bar{v} = \bar{v}_0 (\alpha) \sin \beta, \quad \bar{w} = \bar{v}_0 (\alpha) \cos \beta \tag{2-56} \]
On substituting these expressions into the first three of Eqs. (2-45) the following equations are obtained for the determination of the complex displacements \( \bar{u}_1, \bar{v}_1, \) and \( \bar{w}_1 \).
\[ \bar{u}_1 = \frac{\bar{r}_\beta}{\bar{E}h} \left( \bar{T}_0^2 - \mu \bar{T}_2^2 \right) \]
\[ \bar{v}_1 + \bar{w}_1 = \frac{\bar{r}_\beta}{\bar{E}h} \left( \bar{T}_0^2 - \mu \bar{T}_2^2 \right) \]
\[ \bar{v}_1 - \bar{v}_1 = \frac{2(1+\mu)h}{\bar{E}h} \bar{S} \tag{2-57} \]
III. SOLUTIONS OF THE GOVERNING DIFFERENTIAL EQUATIONS

In this Chapter solutions are obtained to the governing differential equations derived in Chapter II. In addition, formulas for forces, moments and displacements are listed in tables.

Shells of Revolution

(A) Axisymmetric Deformation - Internal Pressure

The analysis of shells of revolution under internal pressure has been reduced to the solution of the second order differential equation

\[ \bar{T}'' + \left[ (2 - \frac{R_2}{R_1} - 1) \cot \theta - \frac{R_1'}{R_1} \right] \bar{T} + i \frac{R_2^2}{R_1^5} \bar{T} = i \frac{R_2^2}{R_1^5} F(\theta) \]  

(2-15)

where

\[ F(\theta) = R_2 p - \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{\bar{U}}{\sin^2 \theta} = T_1'' + T_2'' \]

\[ \bar{U} = C_2 + \frac{P}{2} R_2^2 \sin^2 \theta \]  

(2-16) (2-13)

It is noticed that the coefficient \( iR_1^2/\bar{R}_2 \bar{C} \) of \( \bar{T} \) is a magnitude of order \( R/h \). For convenience of analysis this coefficient will be expressed in terms of a parameter \( \lambda \)

\[ i \frac{R_2}{R_1} \frac{\bar{R}_2}{\bar{C}} = i \lambda^2 \frac{R_1^2}{R_2} \]  

(3-1)

where

\[ \lambda^2 = \frac{R_2}{\bar{C}} = \sqrt{1 - \bar{U}^2} \frac{R_2}{R} \]  

(3-2)

and \( R_0 \) is the minimum radius of curvature of a shell of revolution. For thin shells \( \lambda^2 \) is a large parameter.
(a) **Homogeneous Solution**

It is well known that a second order differential equation of the type

\[ y'' + p(x)y' + q(x) y = 0 \]

may be reduced to the form

\[ \xi'' + m(x) \xi = 0 \]  \hspace{1cm} (3-3)

by the transformation

\[ y = \xi(x) \eta(x) \]

where

\[ \xi = y e^{\int \frac{P}{2} \, dx} \]

\[ m(x) = \frac{q}{2} - \frac{P'}{2} - \frac{P^2}{4} \] \hspace{1cm} (3-4)

Now, let \( \tilde{\xi} \) be the solution of the homogeneous equation

\[ \tilde{\xi}'' + \left[ (2 \frac{R_i}{R_i} - 1) \cot \theta - \frac{R'}{R_i} \right] \tilde{\xi} + i \lambda^2 \frac{R_i^2}{R_0 R_1} \tilde{\xi} = 0 \] \hspace{1cm} (3-5)

Applying the results shown above to Eq. (3-5) one obtains

\[ \xi'' + m(\theta) \xi = 0 \] \hspace{1cm} (3-6)

where

\[ \xi = \tilde{\xi} \exp \left\{ \frac{1}{2} \int \left[ (2 \frac{R_i}{R_i} - 1) \cot \theta - \frac{R'}{R_i} \right] d\theta \right\} = \tilde{\xi} \tilde{R}_\theta \left( \frac{\sin \theta}{R_i} \right)^{1/2} \]

\[ m(\theta) = i \lambda^2 \frac{R_i^2}{R_0 R_1 \tilde{R}_\theta} - \frac{2 + \cos^2 \theta}{4 \sin^2 \theta} + \frac{R_i}{R_0 \sin^2 \theta} - \frac{R'}{2 R_i \sin \theta} \cot \theta + \frac{R_i''}{2 R_i} - \frac{3 R_i^2}{4 R_i^2} \] \hspace{1cm} (3-7)

The condition of Codazzi has been used in the above transformation.

It is noted that the coefficient \( m(\theta) \) contains a singular point at \( \theta = 0 \), which characterizes solutions of Eq. (3-6) as two completely different types. The first type is an asymptotic solution of classic type which is valid only in the nonshallow region, i.e., large values of \( \theta \).
The other type is an asymptotic solution valid in all regions including the singular point \( \theta = 0 \). Attention here will be directed more to the second than the first, since the problem of interest is that of the stress distribution near the apex which belongs to the second type.

As regards the first type of solution, comparison of the magnitude of each term in \( m(\theta) \) shows that, if the region of interest lies in the non-shallow region, the first term is \( O(\lambda^4) \) and the remaining terms are \( O(1) \), provided the shell is sufficiently smooth so that the derivatives of \( R_1 \) and \( R_2 \) have the same order of magnitudes as \( R_1 \) and \( R_2 \). Thus, Eq. (3-6) may be written in the form

\[
\xi'' + \left[ i \lambda^2 \frac{R_1}{R_2 R_0} \left( 1 + O(\lambda^2) \right) \right] \xi = 0 \tag{3-8}
\]

Through the use of the transformations

\[
\xi_1 = (i \lambda^2 \frac{R_1}{R_2 R_0})^{\frac{1}{2}} \xi, \quad \xi_2 = (i \lambda^2 \frac{R_2}{R_2 R_0})^{\frac{1}{2}} d\theta
\]

and by neglecting terms of order \( \lambda^2 \) in comparison with unity, Eq. (3-8) may be reduced to a familiar form

\[
\frac{d^2 \xi_1}{d\theta^2} + \xi_1 = 0
\]

which has the solution

\[
\xi_1 = e^{\pm i\lambda \theta}
\]

Expressing this solution in terms of the original variables \( \xi \) and \( \theta \), one obtains the solution of Eq. (3-8)

\[
\xi = (\frac{R_2 R_1}{R_2^2})^{\frac{1}{4}} e^{i \frac{\lambda}{2} \eta}
\tag{3-9}
\]

where

\[
\eta(\theta) = \lambda \int_0^\theta \frac{R_1}{\sqrt{R_2 R_0}} d\theta
\]
Hence

\[ \tilde{f} = \frac{1}{R^2 \sin^2 \theta} \left[ \bar{c}_1 e^{\frac{i}{2} (1-i) \eta} + \bar{c}_2 e^{-\frac{i}{2} (1-i) \eta} \right] \quad (3-10) \]

in which \( \bar{c}_1 \) and \( \bar{c}_2 \) are complex constants of integration.

As regards the second type of solution which is valid in the entire region including a singular point \( \theta = 0 \), it is necessary to rewrite Eq. (3-6) in the form

\[ \xi'' + [i \lambda^2 \psi^2(\theta) + \Lambda \psi(\theta)] \xi = 0 \quad (3-11) \]

where

\[ \psi^2 = \frac{R^2_t}{R_s R_x} \]

\[ \Lambda = \frac{i}{\sin^2 \theta} \left( \frac{R^2_t}{R^2_x} - \frac{2 \cos^2 \theta}{4} - \frac{R^2_t}{2 R_s} \cot \theta + \frac{R^2_x}{2 R_s} - \frac{3}{4} \left( \frac{R^2_t}{R_s} \right)^2 \right) \quad (3-12) \]

It was shown by Langer [9] that there exists, corresponding to Eq. (3-11), a related differential equation whose solution is asymptotic with respect to the solution of Eq. (3-11). The domain of validity of this asymptotic solution depends on the function in the coefficient of \( \xi' \), i.e., \( \psi^2(\theta) \) and \( \psi^4(\theta) \), which meet the following requirements:

(i) Within the interval \( I_0 \) which includes a singular point \( \theta_s \), \( \psi^2(\theta) \) is of the form

\[ \psi^2(\theta) = (\theta - \theta_s)^{d-2} \psi^2(\theta) = (\theta - \theta_s)^{d-2} \left[ 1 + a_1(\theta - \theta_s) + a_2(\theta - \theta_s)^2 + \ldots \right] \]

with \( d \) being any real positive constant.

(ii) Within \( I_0 \), \( \Lambda(\theta) \) is of the form

\[ \Lambda(\theta) = \frac{A_1}{(\theta - \theta_s)^2} + \frac{B_1}{(\theta - \theta_s)} + C_1(\theta) \]

with \( A_1 \) and \( B_1 \) any constants and \( C_1(\theta) \) is analytic and bounded uniformly with respect to \( \lambda \) in \( I_0 \).
If the constants $a_1$ and $B_1$ are both zero, the differential equation will be defined to be normal. Thus the normal form of the differential equation which reflects the foregoing requirements can be represented by

$$
\xi^* + [i \lambda^2 (\theta - \theta_0)^{d/2} \psi^2 + \frac{A_1}{(\theta - \theta_0)^2} + C_1(\theta)] \xi = 0 \quad (3-13)
$$

If $a_1$ and $B_1$ are not zero, the differential equation may always be normalized by substitution

$$
\theta - \theta_0 = z^2/4, \quad \xi = z^{1/2} u
$$

Then, according to Langer the functions

$$
\begin{bmatrix}
\bar{z}_1 \\
\bar{z}_2
\end{bmatrix} = \psi^{-1/2}(\theta) \sigma^{1/2} \begin{bmatrix}
J_p(\sigma) \\
Y_p(\sigma)
\end{bmatrix}
$$

are the solutions of the related differential equation

$$
\chi^{*} + [i \lambda^2 \psi^2 + \frac{A_1}{(\theta - \theta_0)^2} + \Omega(\theta)] \chi = 0 \quad (3-15)
$$

where $\lambda(\theta)$ is analytic and bounded with respect to $\lambda$ in $I_0$, $J_p(\sigma)$ and $Y_p(\sigma)$ are Bessel functions of the first and second kinds and

$$
\begin{align*}
\rho &= c/d, \\
\sigma &= \int_{\theta_0}^{\theta} (i \lambda^2)^{1/2} \psi(\theta) d\theta
\end{align*} \quad (3-16)
$$

It will be shown that the functions in the coefficient of $\xi$ in Eq. (3-11) satisfy the requirements stipulated above, provided the shells are smooth at the apex, i.e., if

$$
R_1, R_2 \rightarrow R^* \quad \text{as} \quad \theta \rightarrow 0,
$$

or more specifically, if

$$
\frac{R_2}{R_1} = 1 + f(\theta) \sin^2 \theta \quad (3-17)
$$

where $f(\theta)$ is analytic and bounded in $I_0$. For such a shell
\[ R_2' = R_1(1 + f \sin^2 \theta) + R_1' f \sin^2 \theta ]' \]

By use of the condition of Codazzi and Eq. (3-17) the preceding equation may be written in the form

\[ R_1' = -\frac{R_1}{1 + f \sin^2 \theta} \left[ f \sin \theta \cos \theta + (f \sin^2 \theta)' \right] \quad (3-18) \]

As an example, shells of revolution generated by rotation of the second order curves

\[ R_1 = \frac{R^n}{(1 + f \sin^2 \theta)^{3/2}} \]
\[ R_2 = \frac{R^n}{(1 + f \sin^2 \theta)^{1/2}} \quad (3-19) \]

satisfy the condition given by Eq. (3-17). In fact, these curves generate the classes of surfaces including (i) sphere for \( r = 0 \); (ii) paraboloids for \( r = -1 \); (iii) ellipsoids for \( r > 1 \); and (iv) hyperboloids for \( r < -1 \).

By use of Eqs. (3-17) and (3-18) \( \Lambda(\theta) \) in the second of Eqs. (3-12) reduces to

\[ \Lambda(\theta) = \frac{1}{4} \left( 1 - \sin^2 \theta \right) + \Lambda_1(\theta) \quad (3-20) \]

where

\[ \Lambda_1(\theta) = \frac{1}{2} + \frac{R_1^n}{2R_1^2} - \frac{3}{4} \left( \frac{R_1'}{R_1} \right)^2 + \frac{1}{2(1 + f \sin^2 \theta)} \left( 3 f \cos^2 \theta + 2 \sin \theta \cos \theta \right) \]

and is bounded in \( I_2 \), and Eq. (3-21) becomes

\[ \xi'' + \left( i \lambda^2 + \frac{1}{4} \left( 1 - \sin^2 \theta \right) + \Lambda_1(\theta) \right) \xi = 0 \quad (3-21) \]

To make this equation fit the form of Eq. (3-13), a new independent variable \( x \) will be introduced

\[ x = \sin \theta/2, \quad dx = \frac{1}{2}(1 - x^2)^{1/2} \]

Thus, Eq. (3-21) becomes
\frac{d^2\bar{\xi}}{dx^2} - \frac{x}{1-x^2} \frac{d\bar{\xi}}{dx} + \left[4i\lambda^2 - \frac{4\psi^2}{1-x^2} + \frac{1}{4x^2} + \frac{1}{1-x^2}\left(\frac{3}{4} + \frac{1}{3\csc(\frac{1}{2}+1)}\right)\right] \bar{\xi} = 0 \tag{3-22}

Now, by means of the transformation

\bar{\xi} = \xi (1-x^2)^{\nu/4} \tag{3-23}

Eq. (3-22) reduces to the desired form

\frac{d^2\bar{\xi}}{dx^2} + \left[4i\lambda^2 - \frac{4\psi^2}{1-x^2} + \frac{1}{4x^2} + \Lambda_2\right] \bar{\xi} = 0 \tag{3-24}

where

\Lambda_2 = \frac{1}{1-x^2}\left(1\Lambda_1 - \frac{3}{4} + \frac{1}{4\csc(\frac{1}{2}+1)}\right) \tag{3-25}

is bounded in \(|x| < 1\), i.e., \(0 < \theta < \pi\). From this one finds

\begin{align*}
C &= (1-4A_1)^{\nu/2} = 0, & R = c/d = 0 \\
\sigma &= i^{\frac{1}{2}} \lambda \int_0^x \frac{2\psi}{(1-x^2)^{\nu/2}} dx = i^{\frac{1}{2}} \lambda \int_0^\theta \frac{R_1}{\sqrt{R_0 R_2}} d\theta = i^{\frac{1}{2}} \eta
\end{align*}

where

\eta(\theta) = \lambda \int_0^\theta \frac{R_1}{\sqrt{R_0 R_2}} d\theta \tag{3-26}

Thus the asymptotic solutions of Eq. (3-24) are given by

\begin{align*}
\begin{bmatrix}
\bar{\xi}_1 \\
\bar{\xi}_2
\end{bmatrix} &= \left(\frac{4\psi^2}{1-x^2}\right)^{-\frac{1}{4}} \eta^{\frac{1}{4}} \begin{bmatrix} J_0(i^{\frac{1}{2}} \eta) \\ Y_0(i^{\frac{1}{2}} \eta) \end{bmatrix} \tag{3-27}
\end{align*}

which, in terms of \(\xi\), becomes

\begin{align*}
\begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix} &= \left(\frac{R_0 R_2}{R_1^2}\right)^{\frac{1}{4}} \eta^{\frac{1}{4}} \begin{bmatrix} J_0(i^{\frac{1}{2}} \eta) \\ Y_0(i^{\frac{1}{2}} \eta) \end{bmatrix} \tag{3-28}
\end{align*}

\(J_0(i^{\frac{1}{2}} \eta)\) and \(Y_0(i^{\frac{1}{2}} \eta)\) are Bessel functions of the first and second kinds which are the solutions of the differential equation

\begin{align*}
y'' + \frac{1}{\eta} y' + i y = 0
\end{align*} \tag{3-29}
Since these solutions are not tabulated for complex arguments, they will be transformed to modified Bessel functions which are well tabulated in terms of Thompson functions. To do this, let
\[ \eta = i^{\nu} x \]
Equation (3-29) is thus transformed to
\[ \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - y = 0 \]
which has solution
\[ y = \tilde{A} I_\nu(x) + \tilde{B} K_\nu(x) = \tilde{A} I_\nu(i^{\nu} \eta) + \tilde{B} K_\nu(i^{\nu} \eta) \]
where \( I_\nu \) and \( K_\nu \) are modified Bessel functions of the first and second kinds and are related to Thompson functions by
\[ I_\nu(i^{\nu} \eta) = \text{Ber} \eta - i \text{Bei} \eta \]
\[ K_\nu(i^{\nu} \eta) = \text{Ker} \eta - i \text{Kei} \eta \]
(3-30)
Using the relation between \( \xi \) and \( \tilde{F} \) given by Eq. (3-7) one finally obtains the solution of Eq. (3-5) to be
\[ \tilde{F} = \left( \frac{\tilde{A}}{\tilde{B}} \right) \left( \frac{R_0}{R_\varepsilon} \right)^{\frac{3}{24}} \left( \frac{\eta}{\lambda \sin \theta} \right)^{\nu/2} \left[ I_\nu(i^{\nu} \eta) \right] \]
(3-31)
It was shown in [9] that for \( \lambda \gg 1 \), Eq. (3-31) furnishes asymptotic solution of Eq. (3-5) to within terms of relative order \( 1/\lambda \) uniformly on an interval \( 0 \leq \theta < \pi \) provided the function \( \psi(\theta) = 0(1) \) on the interval.

(b) Remarks on the Characteristics of the Solution

The following observations on the characteristics of the function are of importance:

(1) The coefficient outside the bracket of Eq. (3-31) is a non-zero slowly varying function of \( \theta \) while the terms in the bracket vary rapidly with
respect to 9. In view of this fact, this coefficient may be regarded as a constant in performing differentiation, admitting the same order of error as the asymptotic solution. This consideration results in a great algebraic simplification.

(ii) The order of magnitude between \( \bar{f} \) and its derivatives obeys the relation

\[
\bar{f}'' = a 0(\bar{f}') = a^2 0(\bar{f})
\]

Thus, the differential equation, Eq.(3-5), is essentially equivalent to the following in the non-shallow region.

\[
\bar{f}'' + i a^2 \psi^2 \bar{f} = 0
\]

(iii) Let

\[
h = \left( -\frac{R_o}{R_z} \right)^{3/4} \left( \frac{\eta}{\lambda \sin \theta} \right)^{1/2}
\]

By regarding \( h \) as constant in performing differentiation, it may be shown from the property of Bessel function that the solution \( \bar{f} \) given by Eq.(3-31) satisfies the differential equation

\[
\bar{f}'' + \frac{1}{\theta} \bar{f}' + i a^2 \psi^2 \bar{f} = 0
\]

Transition to this equation from Eq.(3-5), i.e.,

\[
\bar{f}'' + \left( \frac{2R_o}{R_z} - 1 \right) \cot \theta \bar{f}' + i a^2 \psi^2 \bar{f} = 0
\]

is made possible by the assumption that the shell is smooth near the apex. Thus, in the \( \bar{f}' \) term, one may approximate \( R_1/R_2 \) by unity and neglect the terms of \( 0(\theta) \) in comparison with \( 1/\theta \), since the \( \bar{f}' \) term is significant only in the shallow region. However, it should be noted that one can not make the same approximation on the last term, which is of the order \( a^2 \psi^2 0(\bar{f}) \).
Since in the expression for $\psi^2$, i.e.,

$$\psi^2 = \frac{R_2}{R_0} \frac{R_0}{R_2}$$

$R_1/R_2$ may be far removed from unity in the non-shallow region.

(c) **Reduction to the Solution of Spherical Shells**

The solution for the spherical shell is obtained from Eq. (3-31) by letting

$$R_1 = R_2 = R_0$$

and

$$\eta = \lambda \theta$$

Thus, Eq. (3-31) reduces to

$$\mathcal{F} = \left( \frac{A}{B} \right) \left( \frac{\theta}{\sin \theta} \right)^{3/2} \begin{bmatrix} I_0(i^{3/2} \eta) \\ K_0(i^{3/2} \eta) \end{bmatrix}$$

(3-32)

If attention is restricted to shallow spherical shells, then, one may write

$$\sin \theta = \theta \left( 1 + \frac{i}{3!} \theta^3 + \cdots \right) = \theta \left( 1 + O(\lambda') \right)$$

which may be approximate by $\theta$ within an error of $O(\lambda')$ if $\theta$ is restricted to the interval $0 \leq \theta \leq \theta_1 = 0(1/R)$. Thus, the standard solution for shallow spherical shell is obtained

$$\mathcal{F} = \begin{bmatrix} I_0(i^{3/2} \eta) \\ K_0(i^{3/2} \eta) \end{bmatrix}$$

(3-33)

(d) **Complex Forces**

With the solution for $\mathcal{F}$, the complex forces are ready to compute. In the following the manipulation will be performed only for the solution associated with $B$. The other solution may be simply obtained from that associated with $B$ by replacing $K_0$ with $I_0$ and $K'_0$ with $I'_0$.

$$\mathcal{F}_1 = \frac{i e}{R_i} \cot \theta \mathcal{F} = B i \sqrt{\frac{e}{R_i}} \cot \theta \mathcal{F}$$

(3-34)
Upon separating the real and imaginary part of Eqs. (3-34) and (3-35) and applying the definition of complex forces and also Eq. (3-30), the forces and moments are obtained which are listed in Table 3-1.

(e) **Particular Solution**

Let \( \bar{T} \) be the particular solution of the equation

\[
\bar{T}'' + \left( z \frac{R_1}{R_2} - \frac{1}{R_1} \right) \frac{d}{d\theta} \left( \frac{1}{R_1} \right) \bar{T}' + \frac{1}{R_1} \psi^2 \bar{T} = i \lambda^2 \psi^2 F(\theta) \quad (2-15)
\]

where

\[
F(\theta) = R_2 p - \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{U}{\sin^2 \theta} = T_1^* + T_2^* \quad (2-16)
\]

\[
U = C_2 + \frac{P}{R_2} \frac{R_2}{\sin^2 \theta} \quad (2-13)
\]

\[
T_1^* = \frac{U}{R_2 \sin^2 \theta} \quad T_2^* = R_2 p - \frac{U}{R_1 \sin^2 \theta} \quad (3-36)
\]

The constant \( C_2 \) will be determined prior to finding the solution \( \bar{T} \). The equilibrium of the forces (Fig. 3-1) in the vertical direction requires that

\[
T_1 \sin \theta - N_1 \cos \theta = \frac{P}{2} R_2 \sin \theta \quad (3-37)
\]

It may be shown that the left hand side of this equation is the real part of the complex force \( \bar{V}_z \)

\[
\bar{V}_z = T_1 \sin \theta - i \frac{\xi}{R_1} T_1 \cos \theta \quad (3-38)
\]

The second term on the right hand side is deduced from

\[
\tilde{T}_z = \tilde{T}_1 - \tilde{T}_1 = \frac{\xi}{R_1} \tilde{T}_1 \cos \theta \quad (3-39)
\]
\[ N_1 = \frac{1}{1+\mu} \frac{1}{R_1} \frac{dM}{d\theta} = R_0 \left( i \frac{c}{R_1} \frac{d\phi}{d\theta} \right) \]

which is the first of Eqs. (A-24). Substituting for \( \bar{\pi}_1 \) in Eq. (3-38) by its expression from the first of Eqs. (2-17) one obtains

\[ \bar{V}_2 = \frac{U}{R_2 \sin \theta} = \frac{C_2}{R_2 \sin \theta} + \frac{P}{2} R_2 \sin \theta \]

It follows from Eq. (3-37) that

\[ C_2 = 0 \]

\[ F(\theta) = R_2 p - \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{P}{2} R_2^2 \]

If there were a vertical force \( V \) applied at the apex, then, \( C_2 = V/2\pi \).

Now, return to the particular solution of Eq. (2-15). In view of the assumption that \( R_1 \) and \( R_2 \) are sufficiently smooth so that \( F(\theta) \) is a slowly varying function, the particular solution may be assumed to be

\[ \bar{\tau} = \bar{\tau}^{(m)} + \frac{i}{\lambda^4} \bar{\tau}^{(n)} + \frac{i}{\lambda^4} \bar{\tau}^{(s)} + \ldots \]  

(3-39)

On substituting this expression into Eq. (2-15) and equating to zero the coefficients of each power of \( \lambda \), there results the following equations for determination of \( \bar{\tau}^{(m)}, \bar{\tau}^{(n)}, \ldots \)

\[ \bar{\tau}^{(n)} = F(\theta) \]

(3-40)

\[ \bar{\tau}^{(n)} = i \frac{R_0 R_2}{R_1^2} \left[ \left( \frac{R_2}{R_1} - 1 \right) \cos \theta - \frac{R_1^2}{R_1} \right] \bar{\tau}^{(n-1)} \]

\[ n = 1, 2, 3, \ldots \]

Having determined \( \bar{\tau}^{(n)} \) from the first equation, \( \bar{\tau}^{(n)}, \bar{\tau}^{(s)}, \ldots \) may be successively found from the second of Eqs. (3-40). However, for consistency in the degree of accuracy with the homogeneous solution only the first term of Eq. (3-39) will be retained as the particular solution of Eq. (2-15). In this way one simply identifies the particular integral as the solution of
TABLE 3-1
BENDING SOLUTIONS OF A SHELL OF REVOLUTION UNDER INTERNAL PRESSURE

\[ u = -\frac{R_2}{E h} \left( \frac{R_2}{R_1} + \mu \right) \frac{c}{R_2} \hat{R}_1 \left[ B_1 \text{kei}' \eta - B_2 \text{ker}' \eta \right] \]

\[ w = \frac{R_2}{E h} \hat{R}_1 \left[ B_1 \text{ker} \eta + B_2 \text{kei} \eta \right] \]

\[ \chi_2 = -\frac{1}{E h} \int \frac{R_2}{c} \hat{R}_1 \left[ B_1 \text{ker}' \eta + B_2 \text{kei} \eta \right] \]

\[ J_1 = \sqrt{\frac{c}{R_2}} \cot \theta \hat{R}_1 \left[ B_1 \text{kei}' \eta - B_2 \text{ker}' \eta \right] \]

\[ J_2 = \hat{R}_1 \left[ B_1 \left( \text{ker} \eta - \sqrt{\frac{c}{R_2}} \cot \theta \text{kei}' \eta \right) + B_2 \left( \text{kei} \eta + \sqrt{\frac{c}{R_2}} \cot \theta \text{ker}' \eta \right) \right] \]

\[ M_1 = c \hat{R}_1 \left[ B_1 \left( \text{kei} \eta + \left( 1 - \mu \right) \sqrt{\frac{c}{R_2}} \cot \theta \text{ker}' \eta \right) + B_2 \left( \text{ker} \eta - \left( 1 - \mu \right) \sqrt{\frac{c}{R_2}} \cot \theta \text{kei}' \eta \right) \right] \]

\[ M_2 = c \hat{R}_1 \left[ B_1 \left( \mu \text{ker} \eta - \left( 1 - \mu \right) \sqrt{\frac{c}{R_2}} \cot \theta \text{kei}' \eta \right) + B_2 \left( -\mu \text{kei} \eta - \left( 1 - \mu \right) \sqrt{\frac{c}{R_2}} \cot \theta \text{ker}' \eta \right) \right] \]

\[ N_1 = \sqrt{\frac{c}{R_2}} \hat{R}_1 \left[ B_1 \text{kei}' \eta - B_2 \text{ker}' \eta \right] \]

TABLE 3-2
MEMBRANE SOLUTIONS OF A SHELL OF REVOLUTION UNDER INTERNAL PRESSURE

\[ u = -\frac{p}{E h} \sin \theta \int R_2^2 \frac{I + R_1 \mu}{\sin \theta} \left[ 1 - \frac{1}{2} \left( \frac{R_2}{R_1} - 1 \right) \right] d\theta \]

\[ w = \frac{p}{E h} R_2^2 \left( \frac{1}{2} - \mu \left( 1 - \frac{1}{2} \frac{R_2}{R_1} \right) \right) + \frac{p}{E h} \cos \theta \int R_2^2 \frac{1 + R_1 \mu}{\sin \theta} \left[ 1 - \frac{1}{2} \left( \frac{R_2}{R_1} - 1 \right) \right] d\theta \]

\[ T_1^* = \frac{1}{2} PR_2 \]

\[ T_2^* = PR_2 - \frac{P}{2} \frac{R_2}{R_1} \]
membrane theory, and the homogeneous solution as the solution of bending theory. Thus, the complex forces $\bar{t}_1$ and $\bar{t}_2$ are found from the expressions

$$\bar{t}_1 = \frac{U}{R_z \sin^2 \theta} + i \frac{c}{R_i} \cot \theta \bar{t}' \approx t_1$$

$$\bar{t}_2 = \bar{t} - \bar{t}_1 \approx t_2$$

They are also listed in Table 3-2.

(f) Displacements

The displacements for symmetric deformation may be found from the first two of Eqs. (2-19)

$$\bar{u}' + \bar{w} = \frac{R_L}{Eh} (\bar{t}_r - \mu \bar{t}_i) \quad (3-42)$$

$$\bar{u} \cot \theta + \bar{w} = \frac{R_1}{Eh} (\bar{t}_2 - \mu \bar{t}_1)$$

Eliminating $\bar{w}$ by subtracting the second from the first equation and taking into consideration the relations between complex forces, one obtains

$$\sin \theta \left(\frac{\bar{u}}{\sin \theta}\right)' = \frac{ic(1+\mu)}{Eh} [\left(1 + \frac{R_2}{R_1}\right) \cot \theta \bar{t}' + \frac{i}{c} \frac{R_2 + \mu R_i}{1 + \mu} \bar{t}] \quad (3-43)$$

in which

$$\bar{t} = \bar{t}' + \bar{t}$$

is the general solution of the governing equation (2-15). Within the admissible error, it has been concluded that this solution is the sum of the solution of membrane theory and bending theory.

(i) Membrane solution

Let $c = 0$, Eq. (3-43) reduces to

$$\sin \theta \left(\frac{\bar{u}}{\sin \theta}\right)' = \frac{i}{Eh} \left(\frac{R_2}{R_1} + \mu R_i\right) \bar{t}$$

$$\bar{u} = -\frac{\sin \theta}{Eh} \int \frac{R_2 + \mu R_i}{\sin \theta} \left[ R_2 \left(\frac{1}{R_i} - \frac{1}{R_2}\right) \frac{PR_1^2}{2} \right] d\theta \quad (3-44)$$

and
\[ w = \frac{P}{Eh} \left[ \frac{1}{2} \mu \left( 1 - \frac{1}{E} \frac{R_t}{R_i} \right) \right] \cot \Theta \tag{3-45} \]

(ii) Bending solution

Upon substitution of \( \bar{T} \) by \( \bar{T} \) Eq. (3-43) becomes

\[ \sin \theta \left( \frac{\bar{u}}{\sin \Theta} \right)' = \frac{i \epsilon (1+\mu)}{Eh} \left[ (1+\frac{R_t}{R_i}) \cot \Theta \bar{T}' + \frac{i}{\epsilon} \frac{R_t + \mu R_i}{1+\mu} \bar{T} \right] \tag{3-46} \]

Exact integration of this equation is difficult, however, it is possible to determine an approximate solution within the admissible error.

Observing the characteristics of the solution mentioned in the previous section one may write this equation in the form

\[ \sin \theta \left( \frac{\bar{u}}{\sin \Theta} \right)' = \frac{i \epsilon (1+\mu)}{Eh} \left[ 2 \cot \Theta \bar{T}' + \frac{i}{\epsilon} \frac{R_t}{R_i} (-\bar{T}'' - \cot \Theta \bar{T}') \right] \]

\[ = - \frac{R_t}{Eh} \frac{i \epsilon}{R_i} (\frac{R_t}{R_i} + \mu) (\frac{\bar{T}'}{\sin \Theta} \sin \Theta) \]

It follows that

\[ \bar{u} = - \frac{R_t}{Eh} \frac{i \epsilon}{R_i} (\frac{R_t}{R_i} + \mu) \bar{T}' \]

and

\[ \bar{w} = \frac{R_t}{Eh} (\frac{\bar{T}'}{R_i} - \mu \bar{T}) - \bar{u} \cot \Theta = \frac{R_t}{Eh} \bar{T} \tag{3-47} \]

The real parts of Eqs. (3-47) are also listed in Table 3-1.

(B) Non-symmetric Deformation - under a Moment

The analysis of shells of revolution subject to a moment has been reduced to the integration of the second order differential equation (2-35)

\[ \bar{T}'' + \left[ \left( 2 \frac{R_t}{R_i} - 1 \right) \cot \Theta - \frac{R_t'}{R_i} \right] \bar{T}' + \frac{R_t}{R_i} \left( 1 - 2 \frac{R_i}{R_t} \right) \frac{1}{\sin^2 \Theta} \bar{T}'' + \frac{i \lambda^2}{R_t} \frac{R_t^2}{R_e R_s} \bar{T} \]

where

\[ = i \lambda^2 \frac{R_t^2}{R_e R_s} \mathcal{F}_i(\Theta) \tag{2-35} \]
\[ F_1(\omega) = -\left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{\tilde{W}}{\sin^2 \theta} = -T_1^{\omega} + T_2^{\omega} \]

\[ \tilde{W} = \frac{1}{R_2 \sin \theta} \left[ \tilde{B}_1 + \tilde{B}_2 \int R_1 \sin \theta \, d\theta \right] \quad (3-40) \]

\[ T_1^{\omega} = \frac{\tilde{W}}{R_2 \sin \theta}, \quad T_2^{\omega} = -\frac{\tilde{W}}{R_1 \sin^2 \theta} \]

(a) Homogeneous Solution

Let \( \tilde{\xi}^* \) be the homogeneous solution of Eq. (2-35). By use of the transformation given in Eq. (3-4) the homogeneous part of Eq. (2-35) becomes

\[ \ddot{\xi} + m(\theta) \dot{\xi} = 0 \quad (3-49) \]

where

\[ \xi = \tilde{\xi}^* R_2 \left( \frac{\sin \theta}{R_2} \right)^{\frac{1}{2}} \quad (3-50) \]

\[ m(\theta) = i \lambda^2 \frac{R_2}{R_0 R_2} + \frac{3 - 3 \sin^2 \theta}{\sin^2 \theta} \left[ \frac{R_1}{R_2} \left( 1 - \frac{R_1}{R_2} \right) \frac{1}{4} \right] \]

\[ + \cot \theta \left( \frac{R_1}{2 R_1} + \frac{R_1}{R_1^{'}} \cdot \frac{R_1}{R_2} \right) + \frac{R_1^{'}}{2 R_1} - \frac{3}{4} \left( \frac{R_1}{R_1^{'}} \right)^2 \quad (3-51) \]

The assumption that shells are smooth near the apex gives the relation (3-17), from which \( m(\theta) \) may be reduced to the form

\[ m(\theta) = i \lambda^2 \frac{R_2^2}{R_0 R_2} - \frac{3}{4 \sin^2 \theta} + \lambda(\theta) \quad (3-52) \]

and Eq. (3-49) becomes

\[ \ddot{\xi} + \left[ i \lambda^2 \psi^2 - \frac{3}{4 \sin^2 \theta} + \lambda(\theta) \right] \xi = 0 \quad (3-53) \]

where

\[ \psi^2 = \frac{R_1^2}{R_0 R_2} \quad (3-54) \]

and \( \lambda(\theta) \) is analytic and small with respect to \( \lambda \) in \( 0 \leq \theta < \pi \). Then, with the procedure established in the previous section, it is found that
\[ c = (1 - 4A_1)^{1/2} = ? \quad \eta = \frac{c}{d} = 1 \] 

(3-55)

where

\[ \eta = \int_0^\theta \frac{R_1}{\sqrt{R_0 R_2}} \, d\theta \] 

(3-56)

Thus, the asymptotic solutions of Eq. (3-53) are given by

\[
\begin{bmatrix}
\hat{t}_1 \\
\hat{t}_2
\end{bmatrix} = \left( \frac{R_0 R_2}{R_1 R_2} \right)^{1/4} \eta^{1/2} \begin{bmatrix}
I_1(i^{1/2} \eta) \\
K_1(i^{1/2} \eta)
\end{bmatrix}
\] 

(3-57)

where \( I_1, K_1 \) are first and second kinds of modified Bessel function of order one. Using the relation between \( t \) and \( \hat{t} \) given by Eq. (3-50) one finally finds the homogeneous solution of Eq. (2-35)

\[ \bar{\mathcal{T}}^* = \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{E}_1(\theta) \begin{bmatrix}
I_1(i^{1/2} \eta) \\
K_1(i^{1/2} \eta)
\end{bmatrix} \] 

(3-58)

For \( \lambda \gg 0 \), Eq. (3-58) furnishes asymptotic solution to within \( O(\frac{1}{\lambda^2}) \) on the interval \( 0 \leq \theta < \pi \). The foregoing statements on the characteristics of the solution in Section (A-1) also apply to this solution, which in this case may be regarded as the solution of the differential equation

\[ \mathcal{T}^* + \cot \theta \mathcal{T}' + (i \lambda^2 \psi^2 - \frac{i}{\sin^2 \theta}) \mathcal{T} = 0 \] 

(3-59)

The terms with coefficients \( \cot \theta \) and \( \frac{i}{\sin^2 \theta} \) are significant only in the shallow region.

The complex forces \( \bar{\mathcal{T}}^* \) is computed from Eq. (2-24) and \( \bar{\mathcal{T}}^*_s \) from Eq. (2-14)

\[
\bar{\mathcal{T}}^*_1 = i B \sqrt{\frac{c}{R_0}} \mathcal{H}_1 \left[ \cot \theta K_1 - \frac{1}{\sqrt{R_2}} \cot \theta K_2 \right] \\
\bar{\mathcal{T}}^*_2 = \mathcal{H}_1 \left[ \mathcal{K}_1 - i \left( \sqrt{\frac{c}{R_0}} \cot \theta K_1' - \frac{c}{R_2} \frac{i}{\sin^2 \theta} K_1 \right) \right]
\] 

(3-60)

From the third of Eq. (2-28) and the second of Eqs. (2-24) one obtains
\[ \tilde{S}^* = \frac{ie}{R_1 \sin \theta} \left( -\frac{R_i}{R_2} \cot \theta \tilde{T}^* + \tilde{T}' \right) \]
\[ = \frac{e}{R_2} \frac{1}{\sin \theta} \tilde{E} \left[ K_1' - \frac{e}{R_2} \cot \theta K_1 \right] \]  
(3-61)

Separation of the real and imaginary parts of Eqs. (3-60) and (3-61) yields the expressions for the forces and moments which are listed in Table 3-3.

(b) **Particular Solution**

Let \( t \) be the particular solution of Eq. (2-35). From the assumption given by Eq. (3-17) it may be shown that
\[ \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \frac{1}{\sin \theta} = \frac{\dot{t}}{R_2} \]  
(3-62)

Hence, the expression for \( F_1(\theta) \) becomes
\[ F_1(\theta) = -\frac{\dot{t}}{R_2} \tilde{W} \]  
(3-63)

Thus, \( t \) will be assumed in the form
\[ t = t^{(o)} + \frac{1}{\lambda} t^{(i)} + \frac{1}{\lambda^2} t^{(ii)} + \ldots \]  
(3-64)

Substituting this expression into Eq. (2-35) and equating to zero the coefficients of each power of \( \lambda \) one obtains for the determination of \( t^{(o)} \), \( t^{(i)} \ldots \) the system of equations
\[ t^{(o)} = F_1(\theta) \]
\[ t^{(n)} = \frac{\dot{t}}{\lambda^n} \left[ t^{(n-1)} + \left( -2 \frac{R_i}{R_2} - \frac{R_i'}{R_1} \right) t^{(n-1)} + \ldots \right] \]  
(3-65)

\[ n = 1, 2, 3, \ldots \]

Notice that \( \tilde{W} \) satisfies Eq. (2-32), which, in the expanded form, is
\[ \tilde{W}'' + \left( -2 \frac{R_i}{R_2} \cot \theta - \frac{R_i'}{R_1} \right) \tilde{W}' - \frac{\dot{t}}{R_2} \frac{1}{\sin^2 \theta} \tilde{W} = 0 \]
### Table 3-3

**Bending Solutions of a Shell of Revolution Under a Moment**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>(- \frac{R_2}{EA} (\frac{R_2^3}{R_1} + \mu) \frac{c}{R_2} ) ( h_1 \left( D_1 \cos \eta - D_2 \sin \eta \right) \cos \phi )</td>
</tr>
<tr>
<td>( v )</td>
<td>( \frac{R_2}{EA} \left( 2 + \mu - \frac{R_2^3}{R_1} \right) \frac{c}{R_2} ) ( h_1 \left( D_1 \cos \eta - D_2 \sin \eta \right) \sin \phi )</td>
</tr>
<tr>
<td>( w )</td>
<td>( \frac{R_2}{EA} ) ( h_1 \left( D_1 \cos \eta + D_2 \sin \eta \right) \cos \phi )</td>
</tr>
<tr>
<td>( \chi_x )</td>
<td>(- \frac{1}{Et} ) ( \frac{R_2}{c} ) ( h_1 \left( D_1 \cos \eta + D_2 \sin \eta \right) \cos \phi )</td>
</tr>
<tr>
<td>( J_1 )</td>
<td>( \frac{c}{R_2} ) ( h_1 \left( D_1 \cos \eta - D_2 \sin \eta \right) \left( \cot \theta \cos \eta \cos \phi \right) )</td>
</tr>
<tr>
<td>( J_2 )</td>
<td>( \frac{c}{R_2} ) ( h_1 \left( D_1 \cos \eta - D_2 \sin \eta \right) \left( \cot \theta \cos \eta \cos \phi \right) )</td>
</tr>
<tr>
<td>( J_3 )</td>
<td>( \frac{c}{R_2} ) ( h_1 \left( D_1 \cos \eta - D_2 \sin \eta \right) \left( \cot \theta \cos \eta \cos \phi \right) )</td>
</tr>
<tr>
<td>( M_1 )</td>
<td>( c_k \left( D_1 \left( \cos \eta - D_2 \sin \eta \right) \right) \left( \cot \theta \cos \eta \cos \phi \right) )</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>( c_k \left( D_1 \left( \cos \eta - D_2 \sin \eta \right) \right) \left( \cot \theta \cos \eta \cos \phi \right) )</td>
</tr>
<tr>
<td>( M_3 )</td>
<td>( c_k \left( D_1 \left( \cos \eta - D_2 \sin \eta \right) \right) \left( \cot \theta \cos \eta \cos \phi \right) )</td>
</tr>
<tr>
<td>( N_1 )</td>
<td>( h_1 \sqrt{\frac{c}{R_2}} ) ( \left( D_1 \cos \eta - D_2 \sin \eta \right) \cos \phi )</td>
</tr>
</tbody>
</table>
Through the use of this equation the expression for computing $t$ is obtained in the form

$$
t''' = \frac{i}{\rho^2} \{-2(\frac{R}{R_2})' \tilde{W}' - \left[(\frac{R}{R_2})'' + ((z \frac{R}{R_2} - 1) \cot \theta - \frac{R_1'}{R_1})(\frac{R}{R_2})' + 2 \frac{R_1}{R_1} (\frac{R}{R_2})^2 \right] \tilde{W} \}
$$

It is noted that the terms containing $\tilde{u}'$ and $\cot \theta \tilde{W}$ in the above equation involve a singularity ($\theta = 0$) of one order higher than $\tilde{u}$. For the solution given by Eq. (3-65) to be applicable in the shallow region, a restriction must be imposed on the function $f$ such that the order of magnitude of these terms is at most the order of $\tilde{W}$. The condition which is sufficient for this purpose is

$$f/R_2 = k = \text{constant}$$

Thus, the expression for $t'''$ reduces to

$$t''' = -2 i k^2 R_0 \tilde{W} \quad (3-66)$$

However, this restriction is not necessary if the solution sought is in the non-shallow region.

For consistency in the degree of accuracy with the homogeneous solution, only the first term of Eq. (3-64) will be retained. In doing this, one essentially identifies the particular integral of Eq. (2-35) with the solution of membrane theory. Accordingly, one may write

$$t_1 = T_1^w = \frac{W}{R_0 \sin \theta}, \quad t_2 = T_2^w = -\frac{W}{R_1 \sin \theta} \quad (3-67)$$

which are listed in Table 3-4.

(c) **Displacements**

With the solution for $T$ and complex forces, the displacements may
### Table 3-4

**Membrane Solutions of a Shell of Revolution Under a Moment**

<table>
<thead>
<tr>
<th>( W )</th>
<th>( \frac{1}{R_2 \sin \theta} (D_3 + D_4) (R_1 \sin \theta d\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon_1^* )</td>
<td>( \frac{1}{E_h} \frac{W}{\sin^2 \theta} (\frac{1}{R_2} + \frac{\mu}{R_1}) )</td>
</tr>
<tr>
<td>( \varepsilon_2^* )</td>
<td>( -\frac{1}{E_h} \frac{W}{\sin^2 \theta} (\frac{1}{R_1} + \frac{\mu}{R_2}) )</td>
</tr>
<tr>
<td>( \omega^* )</td>
<td>( -\frac{2(1+\mu)}{E_h} \frac{W'}{R_1 \sin \theta} )</td>
</tr>
<tr>
<td>( \phi^* )</td>
<td>( \frac{1}{E_h} { \frac{W}{\sin^2 \theta} \left[ \frac{1}{R_1} \left(1 + \frac{R_1^2}{R_2^2}\right) + \frac{\mu}{R_2} \right] } - 2(1+\mu) \left{ \frac{1}{R_1} W'' + \frac{\sin \theta}{R_2} \left(\frac{R_2}{R_1 \sin \theta}\right) W' \right} )</td>
</tr>
<tr>
<td>( u )</td>
<td>( R_2 \sin \theta \int \frac{\phi}{\sin^2 \theta} d\theta - \cos \theta \int R_1 \sin \theta \left( \int \frac{\phi}{\sin^2 \theta} d\theta \right) d\theta )</td>
</tr>
<tr>
<td>( v )</td>
<td>( \int R_1 \sin \theta \left( \int \frac{\phi}{\sin^2 \theta} d\theta \right) d\theta )</td>
</tr>
<tr>
<td>( w )</td>
<td>( R_2 \left( \frac{\varepsilon_2^<em>}{\cos \theta} + \cos \theta \frac{\omega^</em>}{\sin \phi} \right) - R_2 \sin \theta \cos \theta \int \frac{\phi}{\sin^2 \theta} d\theta )</td>
</tr>
<tr>
<td>( T_1^* )</td>
<td>( \frac{W}{(R_2 \sin^2 \theta)} )</td>
</tr>
<tr>
<td>( T_2^* )</td>
<td>( -\frac{W}{(R_1 \sin^2 \theta)} )</td>
</tr>
<tr>
<td>( S^* )</td>
<td>( -\frac{W'}{(R_1 \sin \theta)} )</td>
</tr>
</tbody>
</table>

| \( \varepsilon_1 \) | \( \cos \phi \) |
| \( \varepsilon_2 \) | \( \cos \phi \) |
| \( \omega \) | \( \sin \phi \) |
| \( \phi \) | \( 1 \) |
| \( u \) | \( \cos \phi \) |
| \( v \) | \( \sin \phi \) |
| \( w \) | \( \cos \phi \) |
| \( T_1^* \) | \( \cos \phi \) |
| \( T_2^* \) | \( \cos \phi \) |
| \( S^* \) | \( \sin \phi \) |
be found from the system of Eqs. (2-39). The first three are

\[ \ddot{u}_i + \ddot{w}_i = R_i \varepsilon_i^* \]

\[ \frac{1}{\sin \theta} \ddot{v}_i + \dot{w}_i \cot \theta + \dot{w}_i = R_i \varepsilon_i^* \]

\[ \frac{R_i}{R_i} \ddot{v}_i - \ddot{v}_i \cot \theta - \frac{1}{\sin \theta} \ddot{u}_i = R_i \ddot{\omega}_i \]

where \( \varepsilon_i^*, \varepsilon_i^*, \) and \( \ddot{\omega}_i \) are related to the complex strain components by the relation

\[ \begin{bmatrix} \varepsilon_i, \varepsilon_i^*, \ddot{\omega}_i \end{bmatrix} = \begin{bmatrix} \varepsilon_i^* \cos \phi, \varepsilon_i^* \cos \phi, \ddot{\omega}_i \sin \phi \end{bmatrix} \]

Elimination of \( \ddot{u}_i \) from the first two of Eqs. (3-68) by subtracting the second from the first gives

\[ \sin \theta \left( \frac{\ddot{v}_i}{\sin \theta} - \ddot{v}_i \right) = R_i \varepsilon_i^* - R_i \varepsilon_i^* \]

(3-69)

The third of Eqs. (3-68), upon using the relation of Codazzi, may be written in the form

\[ R_i \sin \theta \left( \frac{\ddot{v}_i}{R_i} \right) - \frac{R_i}{R_i} \ddot{u}_i = R_i \ddot{\omega}_i \]

(3-70)

Elimination of \( \ddot{u}_i / \sin \theta \) from Eqs. (3-69) and (3-70) yields after some rearrangement

\[ \frac{\sin \theta}{R_i} \left( \frac{R_i^2 \sin \theta}{R_i} \right) \left( \ddot{v}_i \right) - \ddot{v}_i = \frac{R_i}{R_i} \varepsilon_i^* - \varepsilon_i^* + \frac{\sin \theta}{R_i} (R_i \ddot{\omega}_i) \]

(3-71)

Now, letting

\[ z = \frac{\ddot{v}_i}{R_i \sin \theta} \]

\[ \phi = \frac{R_i}{R_i} \varepsilon_i^* - \varepsilon_i^* + \frac{\sin \theta}{R_i} (R_i \ddot{\omega}_i) \]

equation (3-71) reduces to

\[ \frac{\sin \theta}{R_i} \left( \frac{R_i^2 \sin \theta}{R_i} \right) \left( \ddot{v}_i \right) - z = \phi \]

(3-72)

which takes essentially the same form as Eq. (2-32). Hence, the transfor-
mation

\[ z = \frac{z_1}{R_2 \sin \Theta} \]

reduces Eq. (3-72) to the form

\[ \sin^2 \Theta \left( \frac{z_1'}{R_1 \sin \Theta} \right)' = \phi \]

from which it follows that

\[ \bar{v}_i = \dot{z}_i = c_i + c_2 \int R_1 \sin \Theta d \Theta + \int R_2 \sin \Theta \left( \int \frac{\phi}{\sin^2 \Theta} d \Theta \right) d \Theta \quad (3-73) \]

The solution associated with \( c_1 \) and \( c_2 \) are the solutions of the homogeneous system of Eqs. (3-68), i.e., solutions of Eqs. (3-68) with \( \ddot{\varphi} = \ddot{\psi} = \ddot{\omega} = 0. \)

Hence, these two solutions are rigid body displacements and will be discarded in the following computation.

The displacement \( \bar{v}_1 \), which may be obtained from Eq. (3-70), is

\[ \bar{v}_1 = R_2 \sin^2 \Theta \left[ \frac{\dot{\phi}}{\sin^2 \Theta} - \cos \Theta \right] \int R_1 \sin \Theta \left( \int \frac{\phi}{\sin^2 \Theta} d \Theta \right) - R_2 \sin \Theta \bar{w} \quad (3-74) \]

and \( \bar{w}_1 \), which is found from the first of Eqs. (3-68), takes the form

\[ \bar{w}_1 = R_2 \left( \ddot{\varphi} + \cos \Theta \ddot{\psi} - \sin \Theta \right) \int R_1 \sin \Theta \left( \int \frac{\phi}{\sin^2 \Theta} d \Theta \right) - R_2 \sin \Theta \bar{w} \quad (3-75) \]

(1) Membrane solution

The strain components are related to the solution \( \ddot{\bar{w}} \) by the expressions

\[ \varepsilon_i' = \frac{I}{E h} \frac{W}{\sin^2 \Theta} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \]

\[ \varepsilon_2' = \frac{I}{E h} \frac{W}{\sin^2 \Theta} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad (3-76) \]

\[ \omega = - \frac{2(1+\mu)}{E h} \frac{W'}{R_2 \sin \Theta} \]

where

\[ W = \frac{I}{R_2 \sin \Theta} \left[ B_1 + B_2 \int R_1 \sin \Theta d \Theta \right] \]

Substitution of these relations into the expression for \( \phi \) yields
\[
\Phi = \frac{I}{\sin^2 \theta} \left[ \frac{2(1 + \mu)W}{R^2} + \frac{2\mu}{R^2} \left( 1 + \frac{R_i}{R_z} \right) \right]
\]

which are then substituted into Eqs. (3-73) to Eqs. (3-75) to obtain the displacements due to the particular solution \( \Phi \). These displacements are also listed in Table 3-4.

(ii) Bending solution

Approximate integration of Eq. (3-71) is possible, however, it involves considerable algebraic manipulation. Only some intermediate steps are shown below. Observing the properties of the function \( \tilde{\theta}^o \), and the smoothness of the shell near the apex, one may write the deformation parameters in the following form

\[
\tilde{e}_1^o = \frac{I}{E_h} \left[ \frac{2(1 + \mu)W}{R^2} \left( 1 + \frac{R_i}{R_z} \right) \right]
\]

\[
\tilde{e}_2^o = \frac{I}{E_h} \left[ \frac{2c}{R_i} \frac{2}{\sin^2 \theta} - \frac{2}{\sin^2 \theta} \tilde{\theta}^o \right]
\]

Substitution of these equations into the expression for \( \phi \) yields

\[
\phi = \frac{I}{E_h} \left[ \frac{2c}{R_i} \left( 1 + \mu \right) \tilde{\theta}^o \right]
\]

With the observation that \( \tilde{\theta}^o \) satisfies the differential equation

\[
\tilde{\theta}'' + \cot \theta \tilde{\theta}'' + \frac{2}{\sin^2 \theta} \tilde{\theta} = 0
\]

the expression for \( \phi \) reduces to the form

\[
\phi = \frac{I}{E_h} \left[ \frac{2c}{R_i} \left( 1 + \mu \right) \tilde{\theta}^o \right]
\]

from which it follows that

\[
\int \frac{\phi}{\sin^2 \theta} d\theta = \frac{I}{E_h} \left[ \frac{2c}{R_i} \left( 1 + \mu \right) \tilde{\theta}^o \right]
\]
\[
\int R \sin \theta \left( \int \frac{d\phi}{\sin \phi} \right) d\theta = \frac{i}{Eh} \frac{ic}{\left( 2 + \mu - \frac{R_2}{R_1} \right)} \frac{\bar{J}^*}{\sin \theta} \tag{3-80b}
\]

Thus, the displacement \( \bar{v}_1 \) is obtained
\[
\bar{v}_1 = \frac{R_2}{Eh} \frac{ic}{\bar{R}_1} \left( 2 + \mu - \frac{R_2}{R_1} \right) \frac{\bar{J}^*}{\sin \theta} \tag{3-81}
\]

and \( \bar{u}_1 \) and \( \bar{w}_1 \) are found from Eqs. (3-74) and (3-75), respectively.
\[
\bar{u}_1 = -\frac{R_2}{Eh} \frac{ic}{\bar{R}_1} \left( \frac{R_2}{R_1} + \mu \right) \bar{J}^* \tag{3-82}
\]
\[
\bar{w}_1 = \frac{R_2}{Eh} \bar{J}^* \tag{3-83}
\]

It is noticed from Eqs. (3-81) to (3-83) that the magnitudes of displacements obey the following order of magnitude relationships
\[
\bar{v}_1 = \frac{1}{\lambda^2} O(\bar{w}_1), \quad \bar{u}_1 = \frac{1}{\lambda} O(\bar{w}_1)
\]

and
\[
\frac{d^2 q}{d\theta^2} = \lambda^2 O(q)
\]

where \( q \) denotes one of \( \bar{u}_1, \bar{v}_1, \) and \( \bar{w}_1 \). With these relations at the outset, the displacement \( \bar{u}_1 \) could have been easily obtained from the fourth of Eqs. (2-39), which is
\[
-\frac{1}{\bar{R}_1} \left( \frac{\bar{w}_1'}{\bar{R}_1} \right)' = \frac{i}{Eh} \left( \frac{\bar{J}^*}{\bar{R}_1} - \frac{\bar{J}^*}{\bar{R}_1} \right)
\]

Neglecting \( \bar{u}_1 \) in comparison with \( \bar{u}_1' \) from the above equation, there results
\[
\left( \frac{\bar{w}_1'}{\bar{R}_1} \right)' = -\frac{1}{Eh} \frac{iR_1 \bar{J}^*}{\bar{R}_1} = -\frac{1}{Eh} \frac{iR_1}{\bar{R}_1} \left( \bar{J}^* \frac{ic}{\bar{R}_1} \left( \cot \theta \bar{J}^* - \frac{R_2}{R_2} \sin \theta \bar{J}^* \right) \right)
\]

By virtue of Eq. (3-59) the preceding equation reduces to
\[
\left( \frac{\bar{w}_1'}{\bar{R}_1} \right)' = \frac{i}{Eh} \frac{R_2}{R_1} \bar{J}^*
\]

It follows that
\[
\bar{w}_1 = \frac{R_2}{Eh} \bar{J}^*
\]
Circular Cylindrical Shells

(A) Axisymmetric Deformation - Internal Pressure

The analysis of cylindrical shells under internal pressure has been reduced to the solution of Eqs. (2-48) and (2-49), which after dropping the terms containing \( q_1 \) give

\[
\begin{align*}
T_1 &= C \quad (3-84) \\
T_2^* + 12b^2 T_2 &= 12b^2 \pi c
\end{align*}
\]

The last equation has the solution

\[
\bar{T}_2 = \bar{A} e^{(i-\delta)_b^*} + \bar{B} e^{-(i-\delta)_b^*} + \pi c \quad (3-85)
\]

The displacements may be obtained from Eqs. (2-50), which, upon substitution for \( T_1 \) and \( T_2 \) by their expressions from the first of Eq. (3-84) and (3-85), yield

\[
\bar{v} = \frac{f_o}{g} \left[ -\mu \frac{L+1}{2b} (\bar{A} e^{(i-\delta)_b^*} - \bar{B} e^{-(i-\delta)_b^*}) - \mu \pi c q + \bar{B}_e^* \right] \quad (3-86)
\]

\[
\bar{w} = \frac{f_o}{g} \left[ \bar{A} e^{(i-\delta)_b^*} + \bar{B} e^{-(i-\delta)_b^*} + \pi c - \bar{B}_e^* \mu \right] \quad (3-87)
\]

It is noted that the fifth of Eqs. (2-45) is compatible only if

\[
\bar{T}_1 = \bar{T}_1^* \quad (3-88)
\]

from which it follows that

\[
\bar{T}_1 = B \quad \text{real constant}
\]

Letting

\[
\bar{B} = B_2 + i B_4
\]

and separating the real and imaginary parts of Eqs. (3-85) to (3-87), then, using the definition of the complex forces given by Eqs. (2-44) one obtains the forces, moments, and displacements as shown in Table 3-5, in which the
solution associated with \( \bar{\alpha} \) has been dropped by virtue of the property that it becomes unbounded when \( \alpha \) increases.

(B) **Non-symmetric Deformation - under a Moment**

The analysis of circular cylindrical shells due to a moment loading has been reduced to the solution of the differential equation (2-55)

\[
\tau_0'''' + (i 2b^2 - 2) \tau_0'' = 0
\]  

(2-55)

It follows that upon integration

\[
\tau_0'' + (i 2b^2 - 2) \tau_0 = (i 2b^2 - 2) (\bar{B}_Y + \bar{B}_\alpha \alpha)
\]  

(3-88)

which has solution

\[
\tau_0 = \bar{b} e^{-\alpha w} + E e^{\alpha w} + \bar{B}_Y + \bar{B}_\alpha \alpha
\]  

(3-89)

where

\[
\alpha = i (i 2b^2 - 2) = -b [(1 + \frac{i}{2b^2}) - i (1 - \frac{i}{2b^2})]
\]  

(3-90)

The complex forces obtained from Eqs. (2-53) take the form

\[
\bar{T}_x^* = (1 + \frac{i}{2b^2}) \tau_0'' - (1 + \frac{i}{2b^2}) (\bar{B}_Y + \bar{B}_\alpha \alpha)
\]  

(3-91)

\[
\bar{T}_y^* = -\frac{i}{2b^2} \tau_0'' + (1 + \frac{i}{2b^2}) (\bar{B}_Y + \bar{B}_\alpha \alpha)
\]  

(3-92)

\[
\bar{S}^* = \frac{i}{2b^2} \tau_0'' - (1 + \frac{i}{2b^2}) \bar{B}_\alpha
\]  

(3-93)

With the complex forces expressed in terms of \( \bar{T}_0 \) and its derivatives, the displacements are obtainable from Eqs. (2-58)

\[
\bar{u}_i' = \frac{\bar{F}_x}{Eh} ( \bar{T}_i^* - \mu \bar{T}_2^*)
\]

\[
\bar{v}_i + \bar{w}_i = \frac{\bar{F}_x}{Eh} ( \bar{T}_i^* - \mu \bar{T}_2^*)
\]  

(2-58)

\[
- \bar{u}_i' + \bar{v}_i' = \frac{\bar{F}_x}{Eh} 2(1+\mu) \bar{S}^*
\]
TABLE 3-5
SOLUTIONS OF A CYLINDRICAL SHELL UNDER INTERNAL PRESSURE

\[ u = \frac{r_o}{E_h} \left[ \frac{M}{2b} e^{-b\alpha} \left\{ B_3 (\cos \beta a - \sin \beta a) - B_4 (\sin \beta a + \cos \beta a) \right\} + (B_5 - \mu p r_o) a \right] \]

\[ w = \frac{r_o}{E_h} \left[ e^{-b\alpha} \left\{ B_3 \cos \beta a - B_4 \sin \beta a \right\} + p r_o - \mu B_5 \right] \]

\[ T_1 = B_5 \]

\[ T_2 = e^{-b\alpha} [B_3 \cos \beta a - B_4 \sin \beta a] + p r_o \]

\[ M_1 = -ze^{-b\alpha} [B_3 \sin \beta a + B_4 \cos \beta a] \]

\[ M_2 = -c^* \mu e^{-b\alpha} [B_3 \sin \beta a + B_4 \cos \beta a] \]

\[ N_1 = \frac{c^*}{l_0} b e^{-b\alpha} [B_3 (\sin \beta a - \cos \beta a) + B_4 (\cos \beta a + \sin \beta a)] \]

\[ c^* = \frac{h^*}{\sqrt{12(1-\mu^2)}} \]
Substituting in the first of Eqs. (2-58) for $\bar{T}_1$, $\bar{T}_2$ by their expressions from Eqs. (3-91) and (3-92) and taking into consideration that $\bar{T}^0$ satisfies Eq. (3-88), one has, after neglecting terms of order $1/b^2$ compared with 1

$$\bar{U}' = \frac{K}{E_h} \left[ -\mu \frac{I}{2b^3} \bar{T}'' + (D_7 + D_8\alpha) \right]$$

Integration of this equation yields

$$\bar{U} = \frac{K}{E_h} \left[ -\mu \frac{I}{2b^2} \bar{T}' + D_7\alpha + \frac{1}{2} D_8\alpha^2 \right] \quad (3-94)$$

The third of Eqs. (2-58) gives

$$\bar{V}' = \frac{K}{E_h} \left[ (2 + \mu) \frac{I}{2b^3} \bar{T}' + D_7\alpha + D_8 \left( -2(1 + \mu) + \frac{1}{2} \alpha^2 \right) \right]$$

which yields the solution for $\bar{V}_1$ upon integration

$$\bar{V} = \frac{K}{E_h} \left[ (2 + \mu) \frac{I}{2b^2} \bar{T} + \frac{D_7\alpha^2}{2} + D_8 \left( -4(1 + \mu) + \frac{1}{2} \alpha^3 \right) \right] \quad (3-95)$$

Finally, $\bar{V}_1$ is obtained from the second of Eqs. (2-58)

$$\bar{W} = \frac{K}{E_h} \left[ \bar{T} - D_7 (1 + \mu + \frac{1}{2} \alpha^2) + D_8 \left((1 + \mu) - \frac{1}{6} \alpha^3 \right) \right] \quad (3-96)$$

It may be shown that the constants $D_7$ and $D_8$ are real. This follows from the first of Eqs. (2-45) that it is compatible only if

$$\bar{T}^0 = D_7 + D_8\alpha = \text{real value}$$

The forces, moments and displacements are obtained upon substitution for $\bar{T}^0$ into Eq. (3-91) through Eq. (3-96) by its expression from Eq. (3-89) and then separation of the real and imaginary parts. The results of these manipulations are shown in Table 3-6.
TABLE 3-6

SOLUTIONS OF A CYLINDRICAL SHELL UNDER A MOMENT

\[ u : \quad \frac{r_0}{Eh^2} \left\{ -\frac{\mu}{2c^2} e^{ia} \left[ D_6 (b_2 \sin b_2 \alpha - b_1 \cos b_2 \alpha) + D_6 (b_2 \cos b_2 \alpha + b_1 \sin b_2 \alpha) \right] + D_7 a + \frac{D_9 \alpha^2}{2} \right\} \cos \beta \]

\[ v : \quad \frac{r_0}{Eh^2} \left\{ \frac{2+\mu}{2b^2} e^{-ia} \left( -D_6 \sin b_2 \alpha - D_6 \cos b_2 \alpha + \frac{D_9 \alpha^2}{2} \right) + D_8 (-2(1+\mu) + \frac{1}{6} \alpha^3) \right\} \sin \beta \]

\[ w : \quad \frac{r_0}{Eh^2} \left\{ e^{ia} \left( D_6 \cos b_2 \alpha - D_6 \sin b_2 \alpha \right) - D_7 (\mu + \frac{1}{6} \alpha^2) + D_8 \left( (2+\mu) - \frac{1}{6} \alpha^3 \right) \right\} \cos \beta \]

\[ T_1 : \quad \frac{1}{2b^2} e^{-ia} \left( D_7 \sin b_2 \alpha + D_7 \cos b_2 \alpha \right) + D_7 + D_8 \alpha \cos \beta \]

\[ T_2 : \quad e^{-ia} \left( D_7 \cos b_2 \alpha - D_7 \sin b_2 \alpha \right) \cos \beta \]

\[ M_1 : \quad -c^* \left( e^{-ia} \left( D_7 \sin b_2 \alpha + D_7 \cos b_2 \alpha \right) - \frac{1-M}{2b^2} (D_7 + D_8 \alpha) \right) \cos \beta \]

\[ M_2 : \quad -c^* \left( e^{-ia} \left( D_7 \sin b_2 \alpha + D_7 \cos b_2 \alpha \right) + \frac{1-M}{2b^2} (D_7 + D_8 \alpha) \right) \cos \beta \]

\[ T_{12} : \quad \frac{1}{2b^2} e^{-ia} \left[ D_6 (b_1 \sin b_2 \alpha - b_2 \cos b_2 \alpha) + D_6 (b_1 \cos b_2 \alpha + b_2 \sin b_2 \alpha) \right] - D_8 \sin \beta \]

\[ M_{12} : \quad -\frac{c^*(1-M)}{2b^2} e^{-ia} \left[ D_6 (b_2 \cos b_2 \alpha + b_1 \sin b_2 \alpha) + D_6 (-b_1 \sin b_2 \alpha - b_2 \cos b_2 \alpha) \right] - \frac{c^*(1-M)}{2b^2} \pi \sin \beta \]

\[ N_1 : \quad -\frac{c^*}{r_0} e^{-ia} \left[ D_6 (b_2 \cos b_2 \alpha - b_1 \sin b_2 \alpha) + D_6 (-b_1 \sin b_2 \alpha - b_2 \cos b_2 \alpha) \right] \cos \beta \]

\[ b_1 = b \left( 1 + \frac{l_1}{2b^2} \right), \quad l_2 = b \left( 1 - \frac{l_1}{2b^2} \right), \quad c^* = \frac{\rho}{\sqrt{3(1-\mu^2)}}\]
IV. BOUNDARY CONDITIONS AND DETERMINATION OF CONSTANTS

As an application of the solutions derived in the previous Chapters, the stresses of a shell of revolution due to the presence of a discontinuity in terms of either a circular hole, a circular rigid insert, or a nozzle will be studied. The external loading is an internal pressure or a moment.

Axisymmetric Deformation - Internal Pressure

Case a: a circular hole at the apex

The discontinuity presented in this case is a small circular hole described by \( \theta = \theta_0 \). The boundary of the hole is free from stresses. However, the internal pressure must be equilibrated with a vertical shear uniformly distributed along \( \theta = \theta_0 \). The boundary conditions are (Fig. 4-1)

\[
M_1 = 0 \quad \text{at} \quad \theta = \theta_0 \quad (4-1)
\]

\( Q_x = 0 \)

in which \( Q_x \) is the component of force in the direction perpendicular to the axis of the shell, i.e.,

\[
Q_x = T_1 \cos \theta + N_1 \sin \theta
\]

Substitution in Eqs. (4-1) for \( M_1 \) and \( Q_x \) by their expressions from Table 3-1 yields, for the determination of the

Fig. 4-1: Internal pressure equilibrated with vertical shear acting along the hole.
two constants $B_1$ and $B_2$, the following equations

$$A_{11} B_1 + A_{12} B_2 = 0$$
$$A_{21} B_1 + A_{22} B_2 = H_2$$

where

$$A_{11} = h \left[ K_0 \eta + \frac{F}{R^2} \cot \theta \, \text{Ker} \eta \right] e_0$$
$$A_{12} = h \left[ -K_0 \eta + \frac{F}{R^2} \cot \theta \, \text{Ker} \eta \right] e_0$$
$$A_{21} = h \sqrt{\frac{F}{R^2}} \frac{1}{\sin \theta} \, \text{Ker} \eta \big|_{e_0}$$
$$A_{22} = -h \sqrt{\frac{F}{R^2}} \frac{1}{\sin \theta} \, \text{Ker} \eta \big|_{e_0}$$
$$H_e = -T_e \cos \theta \big|_{e_0}$$

Having determined $B_1$ and $B_2$, the direct stress $\sigma_p$ and the bending stress $\sigma_B$ are obtained by the formulas

$$\sigma_{jp} = \frac{T_j}{h} \quad j = 1, 2$$
$$\sigma_{jB} = \frac{6M_j}{h^2}$$

Case b: a circular rigid insert at the apex

Since the rigid insert, by its definition, does not deform during the deformation of the shell, the rotation $X_2$ of the shell about the line $\theta = \theta_0$ and the strain $\varepsilon_2$ of the shell along the insert $\theta = \theta_0$ should be zero. Thus

$$\varepsilon_2 = 0 \quad \text{at} \ \theta = \theta_0$$
$$X_2 = 0$$

where

$$\varepsilon_2 = \left( T_2 - \mu T_1 \right) / Eh$$
and \( X_2 \) is the real part of Eq.(2-20)

\[
X_2 = -\frac{j}{R_0} \left( \frac{R_0}{2} \tan \Theta \right) = -\frac{R_0^2}{2E_h} J''
\]

Substitution for \( X_2 \) and \( \varepsilon_2 \) in Eqs.(4-5) by their expressions from Tables 3-1 and 3-2 results in a system of two equations for the determination of the constants \( B_1 \) and \( B_2 \)

\[
A_31 B_1 + A_32 B_2 = H_3
\]

\[
A_41 B_1 + A_42 B_2 = 0
\]

where

\[
A_{31} = h_1 \left[ \text{Kein} - \frac{R_0}{c} \cot \Theta \text{Kein} (\tau + \mu) \right] \theta_0
\]

\[
A_{32} = h_1 \left[ \text{Kein} + \frac{R_0}{c} (\tau + \mu) \cot \Theta \text{Kein} \right] \theta_0
\]

\[
A_{41} = h_1 \frac{R_0}{c} \text{Kein} \theta_0
\]

\[
A_{42} = h_1 \frac{R_0}{c} \text{Kein} \theta_0
\]

\[
H_3 = - \tau_2'' + \mu \tau_1'' \theta_0
\]

Case c: a nozzle at the apex

The discontinuity in this case is a nozzle attached to the apex of a shell of revolution (Fig.4-2). The conditions of equilibrium and continuity across the junction of the nozzle with the shell of revolution at \( \phi = 0 \) and \( \Theta = \theta_0 \) require that the following conditions be satisfied

\[
M_i = M_i^e, \quad \varepsilon_i = \varepsilon_i^e
\]

\[
Q_i = -Q_i^e, \quad X_i = X_i^e
\]

The quantities on the left of the equal signs of Eqs.(4-9) represent the
moment, force, strain, and rotation of the shell of revolution, while those on the right hand side with superscript c denote the corresponding quantities for nozzle (or cylinder), in which \( \epsilon_c \) and \( X_2^c \) are given by

\[ \epsilon_c = \frac{M_2 - M_1}{E h^*} \]

\[ X_2^c = \frac{1}{H} \frac{dM_c}{da} \]

Substitution for those quantities in Eqs. (4-9) by their expressions from Tables 3-1, 3-2 and 3-5, with \( B_5 \) set equal to \( \frac{1}{2} p r_0 \), yields a system of four equations for determination of the four constants \( B_1, B_2, B_3 \) and \( B_4 \):

\[ A_{11} B_1 + A_{12} B_2 + A_{14} B_4 = 0 \]
\[ A_{21} B_1 + A_{22} B_2 + A_{23} B_3 + A_{24} B_4 = K_2 \]
\[ A_{31} B_1 + A_{32} B_2 + A_{33} B_3 = C_3 \]
\[ A_{41} B_1 + A_{42} B_2 + A_{43} B_3 + A_{44} B_4 = 0 \]

in which \( A_{jk} \) (\( k = 1, 2, 3, 4 \)) and \( K_2 \) have been given by Eqs. (4-8) and (4-3). The rest are defined by

\[ A_{14} = \alpha^* / \alpha \]
\[ A_{23} = -A_{24} = \alpha^* b / r_0 \]
\[ A_{33} = h / h^* \]
A_{43} = A_{44} = -bh/h^* \quad (4-11d,e) \\
G_3 = H_3 - h/h^* \pi r_0 (1 - \mu/2)

**Non-symmetric Deformation - under a Moment**

A couple is applied in the plane $\phi = 0$ either at the apex of a shell of revolution or at the far end of a nozzle when it is attached to the shell. The constants $J, D_4, D_7, D_8$ which associate with the membrane solutions of the shell of revolution and the cylindrical shell, respectively shown in Tables 3-4 and 3-6 will be first determined from the condition of equilibrium. Notice the properties of the functions ker, kei, which diminish rapidly when their argument becomes large. Hence, the bending solutions are insignificant in the range of large values of $\theta$. The state of stress in this region is, in fact, of the membrane type. The equilibrium of moment about the plane $\phi = \frac{\pi}{4}$ (Fig.4-3) gives

$$
\int_0^{2\pi} T_i \sin \theta (R \sin \theta \cos \phi) R \sin \theta d \phi = M \quad (4-12)
$$

in which

$$
T_i = T_i^* + \tau_i \approx T_i^* \quad \text{for large } \theta
$$

Equation (4-12) upon introducing the expression for $T_1$ and performing integration reduces to

$$
\pi (D_4 + D_8) \int_0^{2\pi} R \sin \theta d \theta = M \quad (4-13)
$$

The equilibrium of the forces in the direction of $\phi = 0$ gives

$$
\int_0^{2\pi} (T_i \cos \theta \cos \phi - S \sin \phi) R \sin \theta d \phi = 0 \quad (4-14)
$$

**Fig. 4-3:** Free body diagram of a shell of revolution under a moment
Substitution for $T_1$ and $S$ in Eq. (4-14) by their expressions from Table 3-5 and then integration give the result that

$$D_4 = 0$$

from which Eq. (4-15) yields

$$D_3 = M/\pi$$

Similarly, when the moment is applied at the far end of the nozzle (Fig. 4-4), the equilibrium of moment about the plane $\beta = \pi/2$ and sum of the forces in the direction $\beta = 0$ gives

$$\int_0^{2\pi} T_1 r_0 \cos \beta (r_0 \, d\beta) = M \quad (4-15)$$

$$\int_0^{2\pi} (S \sin \beta - M, \cos \beta) r_0 \, d\beta = 0$$

Equations (4-15) upon substitution for $T_1$ and $S$ from Table 3-6 and then integration reduce to

$$\pi r_0^2 (D_2 + D_3 \, \sigma) = M$$

$$D_3 = 0$$

Hence,

$$D_2 = M/(\pi r_0^2)$$

Case a: a circular hole at the apex

A couple $M$ is applied by means of a vertical force distribution along the hole $\theta = \theta_0$ with the magnitude of $M \cos \theta / (R_2^2 \sin^2 \theta)$. The boundary conditions are

$$M_1 = 0, \quad Q_x = 0 \quad \text{at} \ \theta = \theta_0 \quad (4-16)$$
in which $Q_\phi$ is the component of force in the direction perpendicular to the axis of the shell, i.e.,

$$Q_\phi = T_1 \cos \theta + \left( N_1 + \frac{1}{R_2 \sin \theta} \frac{\partial M_2}{\partial \phi} \right) \sin \theta$$

Substitution for $M_1$ and $Q_\phi$ from Tables 3-3 and 3-4 into Eqs. (4-16) results in a system of two equations for the two constants $D_1$ and $D_2$

$$E_{11} D_1 + E_{12} D_2 = 0$$

$$E_{21} D_1 + E_{22} D_2 = F_2$$

where

$$E_{11} = h_1 \left[ K_1 \eta + (1-\mu) \frac{F_2}{R_2} \left\{ \cot \theta K_1 \eta - \frac{1}{R_2} \frac{\partial}{\partial \eta} \cot \theta K_1 \eta \right\} \right]$$

$$E_{12} = h_1 \left[ -K_1 \eta + (1-\mu) \frac{F_2}{R_2} \left\{ \cot \theta K_1 \eta - \frac{1}{R_2} \frac{\partial}{\partial \eta} \cot \theta K_1 \eta \right\} \right]$$

$$E_{21} = h_1 \frac{F_2}{R_2} \left( \cot \eta \eta - \frac{F_2}{R_2} \cot \theta K_1 \eta \right)$$

$$E_{22} = h_1 \frac{F_2}{R_2} \left( -K_1 \eta + \frac{F_2}{R_2} \cot \theta K_1 \eta \right)$$

$$F_2 = -T_1 \cos \theta / \cos \phi$$

Case b: a circular rigid insert at the apex

As shown in Fig. 4-5 the rigid insert does not deform but rotates through an angle when the moment $M$ is applied. The shell has to rotate through the same angle to keep its original angle between the insert and the shell. The boundary conditions are
\[ \epsilon_z = 0, \quad x_z = -w/r_0 \text{ at } \theta = \theta_0 \quad (4-19) \]

Substitution for \( \epsilon_z, x_2 \) and \( w \) from Tables 3-3 and 3-4 into Eqs. (4-19) yields a system of two equations for the determination of the two constants \( D_1 \) and \( D_2 \)

\[ E_{31} D_1 + E_{32} D_2 = F_3 \]
\[ E_{41} D_1 + E_{42} D_2 = 0 \quad (4-20) \]

where

\[ E_{31} = h_1 \left[ \text{Kei}, \eta - (1 + \mu) \frac{E}{R_2} \left\{ \cot \theta \text{Kei}, \eta - \frac{1}{R_2} \frac{1}{\sin \theta} \text{Kei}, \eta \right\} \right] \theta_0 \]
\[ E_{32} = h_1 \left[ \text{Kei}, \eta + (1 + \mu) \frac{E}{R_2} \left\{ \cot \theta \text{Kei}, \eta - \frac{1}{R_2} \frac{1}{\sin \theta} \text{Kei}, \eta \right\} \right] \theta_0 \]
\[ E_{41} = h_1 \left[ -\frac{E}{R_2} \text{Kei}, \eta + \frac{1}{\sin \theta} \text{Kei}, \eta \right] \theta_0 \quad (4-21) \]
\[ E_{42} = h_1 \left[ -\frac{E}{R_2} \text{Kei}, \eta + \frac{1}{\sin \theta} \text{Kei}, \eta \right] \theta_0 \]
\[ F_3 = \frac{T_3^p - \mu T_{3\nu}}{\cos \phi} \]

Case c: A nozzle at the apex

The boundary conditions are the same as those in the case c for the axisymmetric deformation, except the rotation which, for this case, is shown in Fig. 4-6.

\[ M_1 = M_1^0, \quad \epsilon_z = \epsilon_z^0 \]
\[ Q_x = -Q_x^0, \quad x_2 + x_2^0 = -w/r_0 \]

at \( s = 0 \) and \( \theta = \theta_0 \).
These conditions upon substitution for the quantities \( M_1 \), \( Q_x \), \( \xi \) and \( X_2 \) by their expressions from Tables 3-3 and 3-6 result in a system of four equations for determination of the four constants \( D_1 \), \( D_2 \), \( D_5 \) and \( D_6 \):

\[
\begin{align*}
E_{11} D_1 + E_{12} D_2 + E_{16} D_6 &= F_1 \\
E_{21} D_1 + E_{22} D_2 + E_{25} D_5 + E_{26} D_6 &= F_2 \\
E_{31} D_1 + E_{32} D_2 + E_{35} D_5 + E_{36} D_6 &= G_3 \\
E_{41} D_1 + E_{42} D_2 + E_{45} D_5 + E_{46} D_6 &= 0
\end{align*}
\] (4-43)

in which \( E_{jk} \) for \( j = 1, 2, 3, 4 \) and \( k = 1, 2 \) and \( F_2 \) have been defined in Eqs. (4-16) and (4-21). The remainder are given by:

\[
\begin{align*}
E_{16} &= c^* / c \\
E_{25} &= -E_{26} = -b_2 c^* / r_0 \\
E_{35} &= h / h^* \\
E_{45} &= b_1 h / h^* \\
E_{36} &= (h / h^*) \mu / (2b^2) \\
E_{46} &= b_2 h / h^* \\
F_1 &= -(1 - \mu) / (2b^2) M / (\pi r_0^2) \\
G_3 &= F_3 - (h / h^*) \mu M / (\pi r_0^2)
\end{align*}
\] (4-24)

and

\[
r_0 = R_2 \sin \theta \bigg|_{\theta_0}
\]
V. ANALYSIS OF NUMERICAL RESULTS

Numerical results are obtained for spherical shells, ellipsoids, and paraboloids, which are of common interest in engineering structures, of which the generating curves (Fig. 5-1) are defined by the equations

\[ R_1 = R^*/(1 + r \sin^2 \theta)^{3/2} \]
\[ R_2 = R^*/(1 + r \sin^2 \theta)^{3/2} \]  

(5-1)

The results are compared with the limited experimental data which are available only for the spherical shell attached to a cylindrical nozzle. For each class of shells stresses are computed for three different types of discontinuity. Physical interpretation as to the effects on the stresses due to the presence of a discontinuity is given with the spherical shell under internal pressure. A study of the optimum ratio \( r_0/h^* \) of the nozzle which makes the stresses of a given spherical shell a minimum has been determined. Determination of a favorable ratio \( a/l \) among ellipsoids with a nozzle attachment, which contain the same volume and use the same amount of material, is also studied. A computer program feasible for all these studies has been written in Fortran IV language to accomplish all the necessary computation.

Comparison of Theoretical and Experimental Stresses

Let \( r = 0 \) and \( R^* = R \) in Eqs. (5-1) from which one obtains the equations for the spherical shell.
Fig. 5-1: Generating curves of shells of revolution

\[ R_1 = R^*/(1 + r sin^2 \theta)^{3/2} \]

\[ R_2 = R^*/(1 + r sin^2 \theta)^{1/2} \]
\[ R_1 = R_2 = R = \text{constant} \]

The dimensions of the experimental model tested by Maxwell and Holland [31] and the external loads are as follows:

\[
\begin{align*}
R &= 15.255 \text{ in.} & h &= 0.38 \text{ in.} \\
r_0 &= 1.281 \text{ in.} & h^* &= 0.0625 \text{ in.} \\
p &= 200 \text{ psi} & M &= 2,400 \text{ in-lbs.}
\end{align*}
\]

In all cases Poisson ratio \( \mu \) is set equal to 0.3. Comparisons of theoretical and experimental stresses are shown in Fig.5-2 for the pressure loading and in Fig.5-3 for the moment loading. In general, good agreement is obtained except for \( \sigma_z \) of the outer surface of the sphere (Fig. 5-2) which shows a different trend between theoretical and experimental stress near the junction. However, this discrepancy is rather insignificant because of its smallness in magnitude in comparison with the magnitude of \( \sigma_z \). It is seen that better agreement is obtained in the moment loading (Fig. 5-3).

**Pressure Loading**

(A) **Spherical Shells**

**Effect of a Discontinuity on Stresses and Its Physical Interpretation**

To study the effect of the different types of discontinuity on the stresses, the numerical results were obtained for the following set of data

\[ R/h = 100, \quad r_0/h^* = 20 \]

and were shown in Fig.5-4 for pressure loading.

Study of Fig.5-4 reveals that the stress concentration in the case
Fig. 5-2: Comparison of theoretical and experimental stresses internal pressure, 200 psi.
Fig. 5-3: Comparison of theoretical and experimental stresses in the plane of loading - moment 2400 in-lbs.
of the hole is much higher than that in the case of the rigid insert. Presence of the hole causes large values of hoop stress $\sigma_h$, while presence of the rigid insert induces significant meridian stress $\sigma_t$.

These results can be deduced from the consideration of the deformation. Suppose that the shell does not have any discontinuity, then, due to the application of internal pressure, the shell is essentially in the state of membrane stresses for which $\tau_1 = \tau_2 = \frac{1}{2} \rho R$. Let $Q_1$, $Q_2$ be the horizontal and vertical components of $T_1$, respectively. The radius $r_0$ before deformation is stretched into $r^*_0$ after deformation (Fig. 5-5), and the strain $\varepsilon_2$ in the circumferential direction is equal to $(1-\mu)\rho R/2Eh$.

When a discontinuity in terms of a circular hole of radius $r_0$ is present the boundary conditions imply that

$$M_{1h} = 0, \quad Q_{xh} + T_{1h} = 0$$

along the hole (where subscript $h$ is associated with the hole). The hole of radius $r_0$ deforms into a hole of radius $r^*_0$ (Fig. 5-6), which, because of the zero value of $Q_{xh}$, will be larger than $r^*_0$. Consequently, the strain $\varepsilon_{z_h}$ will be also larger than $\varepsilon_2$. From this it follows that the hoop tension $T_{2h} = Eh \varepsilon_{z_h}$ is also larger than $T_2$.

To show there exists a moment $M_{2h}$ in the circumferential direction, it is noticed that

$$M_{2h} = \frac{Eh^3}{12(1-\mu^2)} (k_1 + \mu k_2) = 0$$

From this it follows that

$$k_1 = -u k_2$$

and

$$M_{2h} = \frac{Eh^3}{12(1-\mu^2)} (k_2 + \mu k_1) = -\frac{Eh^3}{12\mu} k_1 = -\frac{Eh^3}{12\mu} \frac{8x_3}{R \Theta}$$
Fig. 5-4: Comparison of stresses among different types of discontinuity for internal pressure. (Sphere, \( l/h = 100, r_o/h = 20, \theta_o = 5^\circ \))
Fig. 5-5: Deformation of a spherical shell

Fig. 5-6: Deformation of a spherical shell with a circular hole

Fig. 5-7: Deformation of a spherical shell with a rigid insert
in which \( X_2 \) is the rotation about the line \( \theta = \theta_0 \). It can be seen from Fig. 5-6 that \( \frac{\partial X_2}{\partial \theta} \) is a negative quantity, hence, \( M_{2h} \) is a positive value. This agrees with the stress shown in Fig. 5-4. The stress on the surface of the shell is computed using the formula

\[
\sigma_2 = \frac{T_2}{h} \pm \frac{6M_2}{h^2}
\]

Hence, \( \sigma_2 \) of the outer surface is a significant stress in the case of the circular hole discontinuity.

When a discontinuity in terms of a rigid insert is present in a shell, the strain \( \varepsilon_2 \) and rotation \( X_2 \) vanish along the rigid insert. The deformation of the shell is shown in Fig. 5-7 in two steps. Because of the zero strain, \( r_{0R}^* \) (the subscript \( R \) is associated with rigid insert) must be equal to its original length \( r_0 \). To fulfil this condition, the horizontal force \( Q_{xR} \) has to be larger than \( Q_x \) of the membrane state. As a consequence of this larger \( Q_{xR} \), a rotation is produced as shown in Fig. 5-7b. Since the shell has to retain zero rotation along the insert, a negative moment is required to compensate this rotation. The final configuration is shown in Fig. 5-7c. The zero value of strain along the insert implies that

\[
T_{2R} = \mu T_{1R}
\]

To show the relative magnitude between \( M_1 \) and \( M_2 \) it is necessary to evaluate the change of curvature \( k_2 \).

\[
k_2 = -\frac{\cot \theta}{R^2} (m-u) = \frac{\cot \theta}{R} \frac{\partial \theta}{\partial \theta} \quad \text{such that} \quad \theta = 0
\]

Thus,

\[
M_{2R} = \mu M_{1R}
\]
Notice that $M_{1R}$ is a negative value, hence, $M_{2R}$ is also a negative value. This agrees with the stress shown in Fig. 5-4. Both the ratios $T_{2R}/T_{1R}$ and $M_{2R}/M_{1R}$ equal $\mu$, which is less than $\frac{1}{2}$ for most of the materials. Hence, $\sigma_1$ of the inner surface is a significant stress in the case of rigid insert.

Next, when the shell is connected by a nozzle, with a rigidity between that of a rigid insert and that of a circular hole, one would anticipate that the stresses of the shell would fall in between these two extreme cases. The rigidity of a nozzle of $r_0/h^* = 20$ being used for computing the numerical results is rather close to the flexibility of a circular hole, in which case $\sigma_2$ is of significance. Consequently, the stress $\sigma_2$ of the shell should close to that in the case of a circular hole. This result again agrees with the stress $\sigma_2$ shown in Fig. 5-4. However, the stress $\sigma_1$ does not follow this conclusion at and near the junction. The physical interpretation of this behavior is possible, however, it is complicated by the fact that four conditions are required to be fulfilled across the junction. Besides, the magnitude of $\sigma_1$ is less important. No attempt is made to analyze this behavior.

Optimum ratio $r_0/h^*$ of a Nozzle

From the previous analysis it is understood that a discontinuity of a circular hole causes a higher stress concentration than that of a rigid insert. With a nozzle attached to a shell the stress variations of the shell between these two extreme cases can be studied by changing the ratio $r_0/h^*$ of the nozzle. It is believed that a proper choice of a nozzle could minimize the stress concentration in the shell. The stresses
of a shell with \( r/h = 1,000 \) have been computed for various values of \( r_0/h^* \) of a nozzle and the nondimensional stresses \( \sigma_1/p \) and \( \sigma_2/p \) at the junction (\( \theta = 5^\circ \)) are plotted in Fig. 5-8. The stress \( \sigma_1 \) on both the outer and the inner surface attains its maximum values at \( r_0/h^* \) around 80, and decreases as \( r_0/h^* \) increases and finally approaches zero as \( r_0/h^* \) goes to infinity (which is the case of the circular hole). \( \sigma_2 \) of the inner and the outer surface increases as the nozzle becomes thinner and thinner and finally approaches the values of the stresses for the case of a circular discontinuity as the ratio \( r_0/h^* \) reaches infinity. The stresses of the shell with a discontinuity of rigid insert are shown on the left hand side of the figure. The curves shown in solid lines are terminated at \( r_0/h^* = 20 \) since below this value the accuracy of thin shell theory is questionable. Nevertheless, the curves showing the stresses in the region between \( r_0/h^* = 20 \) and rigid insert are connected in a manner with stresses obtained from thin shell theory as a guide. It is quite interesting to see that all curves meet at a point where the stress \( \sigma/p \) is approximately equal to 500, which is the membrane stress. At this point \( \sigma_{1, \text{inner}} = \sigma_{1, \text{outer}} \) and the moment \( M_1 = 0 \). For this optimum value the ratio \( r_0/h^* \) is located around 8.

(8) Ellipsoids

When the value of \( r \) is great than \(-1\), Eqs. (5-1) represent generating curves of ellipsoids. \( r \) is related to the ratio of semiaxes by

\[
 r = \frac{a^2}{b^2} - 1
\]

Two ellipsoids with \( r = 0.2 \) and \(-0.2 \), which are equivalent to having the
Fig. 5-8: Stresses of a spherical shell at the junction.
square ratio of semiaxes $a^2/l^2$ equal to 1.2 and 0.8 (Fig. 5-1), respectively, are chosen for computing the stresses. Other parameters used are $l/h=100$, and $r_0/h^* = 20$. The length of semiaxis $l$ remains the same for the two ellipsoids and equals the radius $R$ of the sphere.

Comparison of the stresses due to the effects of three types of discontinuity are shown in Fig. 5-9 and Fig. 5-10. The stress variations along the meridian reveal a similar pattern to those of the spherical shell shown in Fig. 5-4. The ellipsoid with $a^2/l^2 = 1.2$ appears to have higher stresses and another one with $a^2/l^2 = 0.8$ has lower stresses than the spherical shell. The effects of the discontinuity on the stresses also show that a circular hole type of discontinuity gives higher stresses than a rigid type and that the stresses for the shell with a nozzle fall in between.

(c) Paraboloids

When $r = -1$, Eqs. (5-1) represent generating curves of paraboloids and are reduced to

$$R_1 = R^*/\cos^3 \theta$$

$$R_2 = R^*/\cos \theta$$

$R^*$ is chosen to be equal to $a^2/2l$ such that the generating curve passes through the end points of the major axis of the ellipsoid with $a^2/l^2 = 1.2$ as shown in Fig. 5-1. The stresses are shown in Fig. 5-11 for three types of discontinuity. Similar conclusions to the spherical shell are obtained except that the magnitudes are lower than those of the spherical shell.
Fig. 5-9: Comparison of stresses among different types of discontinuity, internal pressure, (Ellipsoid, $a^2/e^2 = 1.2$, $l/h = 100$, $r_0/h^* = 20$, $\theta_o = 5^\circ$)
Fig. 5-10: Comparison of stresses among different types of discontinuity, internal pressure. (Ellipsoid, $a^2/l^2 = 0.8$, $h/l = 100$, $r_0/h = 20$, $\theta_0 = 5^\circ$)
Fig. 5-11: Comparison of stresses among different types of discontinuity, internal pressure. (Paraboloid, $r^* = 1$, $l/h = 100$, $x_0/h = 20$, $\theta_0 = 90'$)
(D) **Optimum Ratio a/l of Ellipsoids with a Nozzle under Internal Pressure**

It is the attempt of this section to find, among ellipsoids which contain the same volume and use the same amount of material, the one which has minimum stress due to the effect of a nozzle attachment under internal pressure.

Let V and S be the volume and surface area, respectively. For a spherical shell having thickness h, its volume and surface area are given by

\[ V = 4 \pi R^3 / 3 \]
\[ S = 4 \pi R^2 \]

For an ellipsoid with its major axis as the axis of revolution, and its semi-major a, semi-minor b, thickness h, the volume and surface area are given by

\[ V = 4 \pi a^2 b / 3 \]
\[ S = 2 \pi a^2 + 2 \pi \frac{a l}{\varepsilon} \sin^{-1} \varepsilon \]

where \( \varepsilon \) is the eccentricity defined by

\[ \varepsilon^2 = 1 - \frac{a^2}{b^2} \]

The condition that all ellipsoids have the same volume as the spherical shell of radius R gives

\[ \frac{a}{R} = \sqrt[3]{1 - \left(\frac{a}{l}\right)^2}, \quad \frac{a}{R} = \left(\frac{a}{l}\right)^{1/3} \]

Another condition that they use the same amount of material as the spherical shell gives

\[ 4 \pi R^2 h = 2 \pi (a^2 + \frac{a l}{\varepsilon} \sin^{-1} \varepsilon) h_0 \]

from which one obtains, after certain manipulation
The data chosen for this study are

\[ \frac{I}{h_e} = \frac{1}{4} \frac{R}{\pi} \left( 1 + \frac{\pi n^2}{6} \frac{I}{h_e} \right) \]

The stresses are computed for various values of the ratio \( a/l \), and are shown in Fig. 5-12 for \( \sigma_z \) on the outer surface, which gives the maximum stress. For these ellipsoids the corresponding ratios \( I/h_e \) are

<table>
<thead>
<tr>
<th>( a/l )</th>
<th>( I/h_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>0.8</td>
<td>117.03</td>
</tr>
<tr>
<td>0.6</td>
<td>146.59</td>
</tr>
<tr>
<td>0.5</td>
<td>170.92</td>
</tr>
</tbody>
</table>

As shown in Fig. 5-12 when the value of \( a/l \) decreases the stress \( \sigma_z \) at the junction \( (\theta_o = 5^\circ) \) decreases, however, it increases at \( \theta = 90^\circ \) where the effect due to discontinuity disappears. The ellipsoid which acquires the minimum stress falls somewhere between \( a/l = 0.6 \) and 0.5.

**Moment Loading**

The stresses of spherical shells due to the effect of three types of discontinuity under moment loading are plotted in Fig. 5-13. It can be seen from this figure that high hoop tensile stress \( (\sigma_z) \) occurs in the discontinuity of a circular hole, while the meridian stress \( (\sigma_r) \) is significant in the discontinuity of a rigid insert, and that \( \sigma_z \) in the former case is higher than \( \sigma_r \) in the latter case. In other word, a circular hole causes a higher stress concentration than does a rigid insert in the same spherical shell. The stresses of the sphere with a nozzle
Fig. 5-12: Stress distributions of ellipsoids having the same volume and the same weight.
Fig. 5-13: Comparison of stresses among different types of discontinuity, moment loading. (Sphere, $R/h = 100, r_0/h^2 = 20, \theta_0 = 5^\circ$)
attachment fall in between these two extreme cases. The same conclusions were also true in the case of pressure loading.

The stress distribution along the meridian for ellipsoids under moment loading shows a pattern similar to that of a spherical shell except for a slight difference in the magnitude of the stresses. In the case of nozzle attachment, there is little difference in $\sigma_1$ among the ellipsoids of the ratio $a^2/l^2 = 0.8, 1, \text{ and } 1.2$ with $l$ remaining constant (Fig. 5-14). However, the difference in $\sigma_1$ is pronounced, which increases rapidly as the ratio $a^2/l^2$ decreases. This result is contrary to the one obtained in the pressure loading, where the stresses decrease as $a^2/l^2$ decreases.
VI. SUMMARY AND CONCLUSIONS

Governing differential equations for shells of revolution pertaining to axisymmetric and moment loadings have been reduced from the basic equations of the general theory of thin shells in terms of complex forces.

For the axisymmetric case, the analysis of shells of revolution has been reduced to the integration of a second order differential equation. Method of asymptotic integration is employed. The solution valid in the region $0 < \theta < \pi$ is obtained in terms of Thompson function of order zero, provided the shell is sufficiently smooth near the apex.

For moment loading applied at the apex the problem has been further reduced to the integration of a second order differential equation. Asymptotic solutions valid in the region $0 < \theta < \pi$ are also obtained in terms of Thompson function of order one.

Formulas for displacements, forces, and moments for both axisymmetric and moment loadings are also obtained and listed. Side by side with the shell of revolution the governing differential equations for circular cylindrical shells are also derived. Solutions in terms of exponential functions are obtained for both axisymmetric and moment loadings.

As an application of the solutions derived previously, three cases of discontinuity at the apex of shells of revolution have been studied: a circular hole, a circular rigid insert, and a nozzle. The boundary conditions and the determination of the constants for each of the
appropriate cases have been derived.

Numerical results in terms of dimensionless stresses are obtained for shells of revolution having the shapes of spheres, ellipsoids, and paraboloids in which each of the three types of discontinuity is present. Good agreement between theoretical and experimental stresses has been obtained for a spherical pressure vessel with a nozzle. Careful studies of these results reveal significant phenomena from which the following conclusions can be drawn:

(1) A circular hole present at the apex of a shell of revolution weakens the shell more than does a rigid insert on the same shell, that is, the stress concentration in the former is higher than that in the latter.

(2) For the case of a circular hole, the hoop stress $\sigma_1$ is higher than the meridian stress $\sigma_2$, and the maximum stress ($\sigma_3$) occurs on the outer surface of the hole. On the other hand, in the case of a rigid insert, $\sigma_1$ is larger than $\sigma_2$; the maximum stress ($\sigma_3$) also occurs on the outer surface of the insert.

(3) The stresses of a shell of revolution with a nozzle attached at the apex fall in between the stresses of the case of a circular hole and the case of a rigid insert. When the radius to thickness ratio $r_0/h$ of the nozzle becomes large the stress distribution of the shell tends toward the case of a circular hole.

(4) The stress concentration due to the attachment of a nozzle may be alleviated, to a certain extent, by proper choice of the value $r_0/h$ of the nozzle.

(5) By proper adjustment of the ratio of semiaxes of ellipsoid, it is
possible to obtain, among ellipsoidal pressure vessels containing the same volume and using the same amount of materials, the one which has the minimum stress concentration due to the effect of a nozzle attached at the apex.

The solutions obtained in this dissertation can be easily extended to include the study of the problems in which the external loads are one of the followings: (a) a vertical load; (b) a torsion; (c) a horizontal force, applied at the apex of a shell of revolution. The same computer program with a slight modification can be used in obtaining the stresses for these three cases of loadings.
REFERENCES


Small deformation of shells of revolution under axisymmetric loading was considered. The basic equations were reduced to a single complex equation which is valid for uniform thickness, as well as for a large class of variable thickness.


Asymptotic solutions of the differential equation

\[ \frac{d^2w}{ds^2} + \{ \lambda \psi(s) + \tau(\lambda, s) \} w = 0 \]

are investigated, where \( \lambda \) represents a large parameter, and the coefficients \( \psi(s) \) and \( \tau(\lambda, s) \) take the following forms:
Asymptotic solutions of the differential equations

\[ \frac{d^n w}{dx^n} = \left\{ u z^n \right\} + \left\{ \frac{r}{z^{n+2}} \right\} w \]

for large positive values of \( u \), have the formal expansions

\[ w = \sum_{n=0}^{\infty} \frac{A_n(z)}{u^n} + \sum_{n=0}^{\infty} \frac{B_n(z)}{u^n} \]

where \( P \) is an exponential or Airy function for \( n = 0 \) or 1, respectively. The coefficients \( A_n(z) \) and \( B_n(z) \) are given by recurrence relations. This paper proves that solutions of the differential equations exist whose asymptotic expansions in Poincare's sense are given by these series, and that the expansions are uniformly valid with respect to the complex variable \( z \).

The differential equation

\[ \frac{d^n w}{dx^n} = \left\{ u z^n + \frac{r}{z^{n+2}} + g(z) \right\} w \]

where \( n \) is an integer \( \geq 1 \), \( u \) a parameter and \( r \) a constant, has the formal solution

\[ w = P(z) \left\{ 1 + \sum_{n=0}^{\infty} \frac{A_n(z)}{u^n} \right\} + \sum_{n=0}^{\infty} \frac{B_n(z)}{u^n} \]

where \( P \) is a solution of the equation

\[ \frac{d^n P}{dx^n} = \left\{ u z^n + \frac{r}{z^{n+2}} \right\} P \]

The coefficients \( A_n(z) \) and \( B_n(z) \) are given by recurrence relations. It is shown that they are analytic at \( z = 0 \) if, and only if, the differential equation for \( w \) can be transformed into a similar equation with \( n = 0, r = 0 \), or \( n = 1, r = 0 \), or \( n = -1 \). The first two cases have been treated in (10) of this reference. The third case, for
which $P$ is a Bessel function of order $\frac{1}{2}$, is examined in
detail in the present paper.

12. Naghdi, P. M. and DeSilva, C. N., "Deformations of Elastic Ellipsoi-

Deformations of thin elastic ellipsoidal shells of revolution of
uniform thickness under axisymmetric loading are considered in
detail. By means of asymptotic integration due to Langer, a solution
is obtained which is valid at the apex of the shell and involves
Kelvin functions. The stress distribution is obtained for ellip-
soidal shells under both uniformly distributed surface and edge
loadings.

13. Clark, R. A. and Reissner, E., "On stresses and Deformations of
Solids, 1957, 63-70.

Complete ellipsoidal shells of revolution subjected to internal
pressure were considered. Stresses and deformations based on bending
theory were obtained and compared with those obtained from membrane
theory. The range of validity of the membrane solution was established
and quantitative corrections to this solution were also determined.

14. DeSilva, C. N. and Arbor, A., "Deformation of Elastic Paraboloidal

Asymptotic solution is obtained for paraboloidal shells of uniform
thickness subjected to a load uniformly distributed over a small
region about the apex and clamped at the open edge. Naghdi's equa-
tions for shells of revolution in which the effect of transverse shear
deformation on the bending is considered are employed.


The differential equations governing the deformation of shells of
revolution of uniform thickness subjected to axisymmetric self-
equilibrating edge loads are transformed into a form suitable for
asymptotic integration. Asymptotic solutions are obtained for all
sufficiently thin shells that possess a smooth meridian curve and
that are spherical in the neighborhood of the apex.

Shells of Revolution," Lockheed Missiles and Space Co., Technical
Report, March 1964.

Stresses and deformations in thin, homogeneous, orthotropic shells
of revolution under action of axisymmetric loads is reduced to
solution of a single inhomogeneous second-order differential equation with complex dependent variable; asymptotic solutions are obtained which are uniformly valid in both steep and shallow regions of dome-shaped shell; the solution is equivalent to well-known membrane solution in steep region of shell but in shallow region it gives significant bending stresses.


Bending solutions that are uniformly valid in both shallow and non-shallow regions of a dome with arbitrary meridian are determined for edge loads that vary sinusoidally in the circumferential direction. The membrane and inextensible deformation solutions are obtained in terms of a function which satisfies a simple integral equation. For specific application, curve and formulas are obtained for the stresses and deformations of a dome with rigid rings clamped to the edges under the action of axial forces, side force and tilting moment.


In the first paper, a set of two equations which relate stress function $F$ and normal displacement $w$ was obtained for small deformation of shallow spherical shells. The assumptions used in the derivation are: (a) shallow shell; (b) the transverse shear terms in the first two equations of equilibrium are neglected; (c) the tangential displacements $u$ and $v$ are neglected in the expressions of bending and twisting deformation parameters.

In the second paper, solutions to these two equations were obtained for the case of axisymmetry. Applications were given to obtain results for the following problems: (a) a shell with no edge restraint carrying a point load at the apex, (b) a shell with no edge restraint carrying a load uniformly distributed over a small area with center at the apex, (c) a shell with edge restraint carrying a point load at the apex.


Asymptotic solutions of Novoshilov's equations are derived for a constant thickness, nonshallow shell of revolution. Two cases of loadings are considered: (a) sinusoidal loading and (b) higher harmonic load distribution. The asymptotic solutions are obtained by the use of small-parameter expansions and by the use of a standard method for the singular perturbation problem and are valid for a nonshallow shell free of singularities.

Hemispherical shells of thin, constant thickness with a circular opening at the vertex subjected to axisymmetric self-equilibrating forces were considered. Three methods for obtaining influence coefficients were investigated and utilized in a numerical example.


In part 1, shallow-shell equations expressed in elliptical coordinates have been solved in terms of Mathieu functions. Boundary conditions for the rigid insert and for the unreinforced hole are discussed in some detail. Results for an unreinforced opening are compared with experiment and satisfactory agreement is obtained. In part 2, a parametric study has been made of factors affecting the stresses at rigid inclusions and at unreinforced holes of elliptical shape in spherical shells.


The method yields the perturbation of membrane state of stress caused by holes with smooth countours, the size of which is small compared to radii of curvatures of the shell. The perturbation, being confined to a narrow zone, can be established by theory of shallow shells. The isometric system of curvilinear coordinates pertaining to the shell is in this zone replaced by a system such that along the contour one of the coordinates is constant. The deflection and the stress function are combined into one unknown complex function; likewise, the two homogeneous equations for the shallow shell into one complex equation. The general method is described and applied to the cylindrical shell with a circular hole and to a spherical shell with either a circular or an elliptical hole; the solutions are given for the case of the sphere.


As a first approach to the determination of the effect of local loads acting upon attachments to spherical shells, the case of a rigid cylindrical insert, loaded by a radial load or an external moment, is considered. Direct solutions are obtained by using the theory of shallow spherical shells. The numerical results are presented in graphs.

Shallow spherical shells with a rigid elliptical insert at the apex subjected to radial and tangential forces, and external moment applied to the insert were investigated. Reissner's equations for shallow spherical shells were expressed in elliptical coordinate system. Solutions were obtained in terms of Mathieu functions.


In the first paper, a spherical vessel with a radially inserted tube is investigated for the case where the tube is subjected to a radial load. After deriving the solutions for cylindrical shells and shallow spherical shells the continuity conditions between tube and vessel are established in order to determine the constants involving in the solutions. Graphs showing deflections, forces and moments of the vessel for some values of geometric parameters are presented. In the second paper, the same structure is investigated for the case of external moment acting on an insert pipe. Solutions in the form of the first harmonic are shown to be suitable for this analysis. Curves showing deflections, forces and moments are also presented.


The methods of calculation presented apply to the penetration of a spherical pressure vessel by a radial, circular cylindrical nozzle. This nozzle may protrude into the vessel, and allowance is made for the case when local reinforcement in the form of pad is present. Solutions for various loadings given include pressure and nozzle thrust, moment and shear.


An analytical investigation is made of stresses due to external forces and moments acting on an elastic nonradial circular cylindrical nozzle attached to a spherical shell. Results are obtained by combining solutions from shell theory by a Galerkin-type method so as to satisfy boundary conditions at the intersection of the two shells. It is found that, as the nozzle inclination increases, the stresses change gradually from those previously given by Bijlaard for the radial nozzle.

APPENDIX

A BRIEF REVIEW OF THE GENERAL THEORY OF THIN ELASTIC SHELLS

The derivation of the basic equations for thin elastic shells has been well established and can be found in most of the books on thin shells, for example, in [1,2,3]. For completeness of the text and convenience of application, a general procedures as to the deduction of these basic equations to a system of differential equations which may be readily applied to the problems studied here, will be outlined. The basic assumptions and their consequences will be pointed out wherever they are introduced.

The fundamental assumptions in shell theory are:
(a) Straight fibers normal to the middle surface of a shell before deformation remain so after deformation and do not change their length.
(b) The normal stress acting on surfaces parallel to the middle surface may be neglected in comparison with the other stresses.
(c) The relative thickness of the shell is sufficiently small in comparison with unity.
(d) The displacements are small compared to the thickness of the shell.

In that which follows, the notation and procedures used are those introduced by Novozhilov[2].

Coordinate System and Conditions of Gauss-Codazzi
Let \( \sigma_1 = \text{constant}, \ \sigma_2 = \text{constant} \) be the coordinate lines of the principal curvature of the middle surface of a shell and \( R_1 \) and \( R_2 \) be the corresponding radii of curvature (Fig. A-1). Since the lines of principal curvature are orthogonal, the first fundamental form of a surface may be written in the form

\[
(ds)^2 = (A_1 \, d\sigma_1)^2 + (A_2 \, d\sigma_2)^2 \quad (A-1)
\]

where \( ds \) is the length of the differential segment of a line on the middle surface and \( A_1, A_2 \) are called Lamé' parameters.

The parameters \( A_1, A_2, R_1 \) and \( R_2 \) are related by the conditions of Gauss-Codazzi

\[
\frac{\partial}{\partial \sigma_1} \left( \frac{A_2}{R_2} \right) = \frac{1}{R_1} \frac{\partial A_1}{\partial \sigma_1},
\]

\[
\frac{\partial}{\partial \sigma_2} \left( \frac{A_1}{R_1} \right) = \frac{1}{R_2} \frac{\partial A_2}{\partial \sigma_2},
\]

\[
\frac{\partial}{\partial \sigma_1} \left( \frac{1}{A_1} \frac{\partial A_1}{\partial \sigma_1} \right) + \frac{\partial}{\partial \sigma_2} \left( \frac{1}{A_2} \frac{\partial A_2}{\partial \sigma_2} \right) = -\frac{A_1 A_2}{R_1 R_2}
\]

The first two conditions may be obtained from the identity

\[
\frac{\partial^2 \xi_n}{\partial \sigma_1 \partial \sigma_2} = \frac{\partial^2 \xi_n}{\partial \sigma_2 \partial \sigma_1}
\]

and the third one from

\[
\frac{\partial^2 \xi_n}{\partial \sigma_1 \partial \sigma_2} = \frac{\partial^2 \xi_n}{\partial \sigma_2 \partial \sigma_1}
\]

where \( \xi_1 \) is a unit vector tangent to the line \( \sigma_1 = \text{constant} \) and \( \xi_n \) is a
unit normal to the middle surface (Fig. A-1). A surface is uniquely defined if the parameters \( A_1, A_2, R_1 \) and \( R_2 \) satisfy the condition \((A-2)\).

Hence, these conditions are usually referred to as the compatibility conditions of a surface.

**Strain-Displacement Relations and Compatibility Equations**

Let \( u, v, w \) be the displacements of a point \( A \) on the middle surface in the directions of \( \xi_1, \xi_2, \xi_n \), respectively, and \( u_z, v_z, w_z \) be the displacements of a point \( B \) on the normal through \( A \), at a distance \( z \) from the middle surface (Fig. A-1). The assumption (a) implies that

\[
\epsilon_{31} = \epsilon_{32} = \epsilon_{33} = 0
\]

Expressing these relations in terms of the displacements one obtains

\[
H_1 \frac{\partial}{\partial z} \left( \frac{u_z}{H_1} \right) + \frac{1}{H_1} \frac{\partial w_z}{\partial a_1} = 0
\]

\[
H_2 \frac{\partial}{\partial z} \left( \frac{v_z}{H_2} \right) + \frac{1}{H_2} \frac{\partial w_z}{\partial a_2} = 0
\]

\[
\frac{\partial w_z}{\partial z} = 0
\]

in which

\[
H_1 = A_1 (1 + z/R_1)
\]

\[
H_2 = A_2 (1 + z/R_2)
\]

(Eq. A-3b)

Equations \((A-3a)\) upon integration with respect to \( z \) over \((0, z)\) and use of the relation \((u_z, v_z, w_z) = (u, v, w)\) at \( z = 0 \), yield

\[
u_z = u + z \phi
\]

\[
v_z = v + z \phi
\]

\[
w_z = w
\]

(Eq. A-3c)

where
\[
\begin{align*}
J &= -\frac{1}{A_1} \frac{\partial N}{\partial \alpha_1} + \frac{u}{R_1} \quad (A-3d) \\
\psi &= -\frac{1}{A_2} \frac{\partial \omega}{\partial \alpha_2} + \frac{v}{R_2}
\end{align*}
\]

Equations (A-3c) show that the variation of the displacements through the thickness is linear and \( u_z \) is independent of \( z \).

The remaining three strain components are related to the displacements by

\[
\begin{align*}
\epsilon_{11} &= \frac{1}{M_1} \left( \frac{\partial u_1}{\partial \alpha_1} + \frac{1}{M_2} \frac{\partial H_1}{\partial \alpha_2} v_z \right) \\
\epsilon_{22} &= \frac{1}{M_2} \left( \frac{\partial u_2}{\partial \alpha_2} + \frac{1}{M_1} \frac{\partial H_2}{\partial \alpha_1} u_z \right) \\
\epsilon_{12} &= \frac{M_2}{M_1} \frac{\partial}{\partial \alpha_1} \left( \frac{v_z}{M_2} \right) + \frac{M_1}{M_2} \frac{\partial}{\partial \alpha_2} \left( \frac{u_z}{M_1} \right) \quad (A-4)
\end{align*}
\]

Substitution of Eqs. (A-3b) and (A-3c) into Eqs. (A-4) and use of conditions of Codazzi yield, after certain manipulation, the following explicit expressions

\[
\begin{align*}
\epsilon_{11} &= \frac{1}{1 + z/R_1} \left( \epsilon_1 + z \kappa_1 \right) \\
\epsilon_{22} &= \frac{1}{1 + z/R_2} \left( \epsilon_2 + z \kappa_2 \right) \quad (A-5) \\
\epsilon_{12} &= \frac{1}{1 + z/R_1} \left( \omega_1 + z \tau_1 \right) + \frac{1}{1 + z/R_2} \left( \omega_2 + z \tau_2 \right)
\end{align*}
\]

where

\[
\begin{align*}
\epsilon_1 &= \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} v + \frac{w}{R_1} \\
\epsilon_2 &= \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} u + \frac{w}{R_2} \\
\kappa_1 &= \frac{1}{A_2} \frac{\partial y}{\partial \alpha_1} + \frac{\psi}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} \\
\kappa_2 &= \frac{1}{A_2} \frac{\partial y}{\partial \alpha_2} + \frac{\psi}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \quad (A-6)
\end{align*}
\]

and
\[
\begin{align*}
\omega_1 &= \frac{1}{A_1} \frac{\partial y}{\partial a_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial a_2} u \\
\omega_2 &= \frac{1}{A_2} \frac{\partial y}{\partial a_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} u \\
\tau_1 &= \frac{1}{A_1} \frac{\partial \phi}{\partial a_1} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial a_2} \phi \\
\tau_2 &= \frac{1}{A_2} \frac{\partial \phi}{\partial a_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \phi
\end{align*}
\]

(A-7)

It is possible to reduce the last of Eqs. (A-5) to a form involving only two parameters. In doing this, observing the identity

\[
\tau_1 + \frac{\omega_2}{R_1} = \tau_2 + \frac{\omega_1}{R_2}
\]

and introducing the new notations

\[
\omega = \omega_1 + \omega_2
\]

\[
\tau = \tau_1 + \frac{\omega_2}{R_1} = \tau_2 + \frac{\omega_1}{R_2}
\]

one reduces the last of Eqs. (A-5) to the following form

\[
\epsilon_{iz} = \frac{1}{(1 + z/R_1 x_1 + z/R_2)} \left \{ (1 - \frac{z^2}{R_1 R_2}) \omega + 2 \left \{ 1 + (\frac{1}{R_1} + \frac{1}{R_2}) \frac{z}{2} \right \} \tau \right \}
\]

(A-8)

where

\[
\omega = \omega_1 + \omega_2 = \frac{A_1}{A_2} \frac{\partial}{\partial a_1} \left ( \frac{u}{A_1} \right ) + \frac{A_2}{A_1} \frac{\partial}{\partial a_1} \left ( \frac{v}{A_2} \right )
\]

\[
\tau = -\frac{1}{A_1 A_2} \left ( \frac{\partial^2 w}{\partial a_1 \partial a_2} - \frac{1}{A_1} \frac{\partial w}{\partial a_1} - \frac{1}{A_2} \frac{\partial w}{\partial a_1} \right )
\]

(A-9)

\[
+ \frac{1}{R_1 A_2} \left ( \frac{\partial u}{\partial a_2} - \frac{1}{A_2} \frac{\partial u}{\partial a_1} \right ) + \frac{1}{R_2 A_1} \left ( \frac{\partial v}{\partial a_1} - \frac{1}{A_1} \frac{\partial v}{\partial a_1} \right )
\]

Thus, the deformation of the middle surface is completely described by the six parameters \( \epsilon_1, \epsilon_z, \omega, \kappa_1, \kappa_2 \) and \( \tau \), which are usually referred to as the deformation parameters of a middle surface.

Neglecting the terms \( z/R_1 \) and \( z/R_2 \) in Eqs. (A-5) in comparison with unity one obtains the expressions given in [3] which differ only in \( \tau \).
from Novozhilov's expressions, i.e.,

$$\tau' = \tau_1 + \tau_2 = \frac{A_1}{A_1} \frac{\partial}{\partial z} \left( \frac{A_1}{A_1} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial z} \left( \frac{A_1}{A_1} \right) \quad (A-10)$$

Expanding the factors \((1 + z/R_1)^{-1}\) and \((1 + z/R_2)^{-1}\) in Eqs. (A-5) in the form of a series in the variable \(z\) and collecting the terms in the coefficients of \(z^0\) and \(z^1\), one obtains Vlasov's expressions [1] which relate Novozhilov's expressions by

$$\epsilon_i'' = \epsilon_i, \quad \epsilon_{i'}'' = \epsilon_{i'}, \quad \omega'' = \omega$$

$$\kappa_i = \kappa_i - \epsilon_i/R_1, \quad \kappa_{i'} = \kappa_{i'} - \epsilon_{i'}/R_1 \quad (A-11)$$

$$\tau'' = \tau_1 + \tau_2 - \frac{\omega}{R_1} - \frac{\omega}{R_2} = 2\tau - \frac{1}{R_1} + \frac{1}{R_2} \omega$$

The six parameters relating to the displacements must satisfy the compatibility conditions of the strains, which are given below [2]

$$\frac{2}{3x_1} (A_i, \kappa_i) - \kappa_i \frac{2A_i}{\varrho \varrho_2} - \frac{2A_i}{\varrho \varrho_1} \frac{\partial}{\partial z} \left( \frac{2A_i}{\varrho \varrho_2} \right) - \frac{\tau}{R_1} \frac{2A_i}{\varrho \varrho_1} \frac{\partial}{\partial z} \left( \frac{2A_i}{\varrho \varrho_2} \right) - \frac{\omega}{R_1} \frac{2A_i}{\varrho \varrho_1} \frac{\partial}{\partial z} \left( \frac{2A_i}{\varrho \varrho_2} \right) = 0 \quad (A-12)$$

The relations (A-12) play in the theory of shells the same role as the compatibility equations in the theory of elasticity, the fulfilment of which ensures the possibility of determining displacements from the given deformation parameters of a shell.
Equations of Equilibrium

The equations of equilibrium of a shell element may be derived in a similar manner as those which are derived in the theory of elasticity, except in the theory of shell, the stresses are replaced by statically equivalent forces and moments (Fig. A-2), which are defined by the following expressions

\[
T_x = \int_{-h/2}^{h/2} \sigma_{11} (1 + z/R_1) \, dz
\]

\[
T_{12} = \int_{-h/2}^{h/2} \sigma_{12} (1 + z/R_2) \, dz
\]

\[
N_1 = \int_{-h/2}^{h/2} \sigma_{13} (1 + z/R_3) \, dz
\]

\[
T_2 = \int_{-h/2}^{h/2} \sigma_{22} (1 + z/R_2) \, dz
\]

\[
T_{21} = \int_{-h/2}^{h/2} \sigma_{21} (1 + z/R_1) \, dz
\]

\[
N_2 = \int_{-h/2}^{h/2} \sigma_{23} (1 + z/R_3) \, dz
\]

\[
M_1 = \int_{-h/2}^{h/2} \sigma_{11} z (1 + z/R_2) \, dz
\]

\[
M_{12} = \int_{-h/2}^{h/2} \sigma_{12} z (1 + z/R_2) \, dz
\]

\[
M_2 = \int_{-h/2}^{h/2} \sigma_{22} z (1 + z/R_2) \, dz
\]

\[
M_{21} = \int_{-h/2}^{h/2} \sigma_{21} z (1 + z/R_1) \, dz
\]

The condition that the equilibrium of a shell element requires that the resultant force and moment vanish yields the following equations.
The last of Eqs. (A-14) is identically satisfied. This can be verified upon substitution into the equation the forces and moments by their expressions from Eqs. (A-13).

Relations between the Forces, Moments and the Deformation Parameters

The relations between the forces, moments and the deformation parameters (from now on called constitutive equations) can be obtained from Eqs. (A-13). For this purpose, the stress components in these equations are replaced by the strain components through the use of Hooke's law (neglecting $\sigma_{zz}$ in comparison with $\sigma_{ii}$ and $\sigma_{zz}$)

$$\sigma_{ii} = \frac{E}{1-\mu^2} (\epsilon_{ii} + \mu \epsilon_{zz})$$

$$\sigma_{zz} = \frac{E}{1-\mu^2} (\epsilon_{zz} + \mu \epsilon_{ii})$$

$$\epsilon_{iz} = \frac{E}{2(1+\mu)} \sigma_{iz}$$
and then, the strain components are replaced by the deformation parameters from Eqs. (A-5) and (A-8). On carrying out integration on the result of these manipulations and then, neglecting terms of the order h/R in comparison with unity, Eqs. (A-13) finally yield the following relations:

\[
\begin{align*}
T_1 &= D (\varepsilon_1 + \mu \varepsilon_2) , \\
T_2 &= D (\varepsilon_2 + \mu \varepsilon_1) , \\
T_{12} &= T_{21} = \frac{D(1-\mu)}{2} \omega , \\
M_1 &= K (k_1 + \mu k_2) , \\
M_2 &= K (k_2 + \mu k_1) , \\
M_{12} &= M_{21} = K(1-\mu) \tau ,
\end{align*}
\]

where

\[
D = \frac{E h}{1-\mu^2} , \quad K = \frac{E h^3}{12(1-\mu^2)} .
\]

Adopting these relations one is essentially disregarding the differences between \( T_{12} \) and \( T_{21} \), and \( M_{12} \) and \( M_{21} \). On substituting these relations into the last of Eqs. (A-14) it may be verified that this equation is not satisfied identically. As mentioned previously, the fact that this equation is identically satisfied secures the symmetry of the stress tensor \( \sigma_{ij} = \sigma_{ji} \) from which it follows that Eqs. (A-16) contradict the symmetric properties of the stress tensor.

This contradiction can be avoided if the constitutive equations are developed from the variational principle of the potential energy by neglecting terms of order h/R in comparison with unity. This approach yields (2)

\[
\begin{align*}
T_1 &= D (\varepsilon_1 + \mu \varepsilon_2) , \\
T_2 &= D (\varepsilon_2 + \mu \varepsilon_1) , \\
T_{12} &= T_{21} = \frac{D(1-\mu)}{2} (\omega + \frac{h^2}{6R_1} \tau) , \\
T_{21} &= T_{21} = \frac{D(1-\mu)}{2} (\omega + \frac{h^2}{6R_1} \tau) , \\
M_1 &= K (k_1 + \mu k_2) , \\
M_2 &= K (k_2 + \mu k_1) , \\
M_{12} &= M_{21} = K(1-\mu) \tau
\end{align*}
\]
Introducing the new notations

\[ S = T_{12} - M_{31}/R_z = T_{22} - M_{12}/R_z, \]

\[ H = M_{12} = M_{21}. \]

and substituting from Eqs. (A-17) in Eqs. (A-18), one obtains

\[ S = \frac{D(1-\mu)}{2} \omega, \quad H = K(1-\mu) \tau \]  \hspace{1cm} (A-19)

For later use the inverse relation of Eqs. (A-17) is obtained as follows:

\[ e_1 = \frac{1}{Eh} (T_1 - \mu T_2), \quad e_2 = \frac{1}{Eh} (T_2 - \mu T_1) \]

\[ \omega = \frac{2(1+\mu)}{Eh} S \quad k_1 = \frac{12}{Eh^3} (M_1 - \mu M_2) \]  \hspace{1cm} (A-20)

\[ k_2 = \frac{12}{Eh^3} (M_2 - \mu M_1) \quad \tau = \frac{12(1+\mu)}{Eh^3} H \]

**Reduction of the Basic Equations to a Fourth Order System**

So far, a system of nineteen equations including six strain-displacement relations, five equations of equilibrium and eight constitutive equations, has been introduced. These equations involve the same number of unknowns, i.e., six forces, four moments, six deformation parameters and three displacements. One now faces the problem of solving these equations subject to appropriate boundary conditions. As in the theory of elasticity, there exist two methods of solving problems of thin elastic shells - in terms of the displacements of the middle surface or in terms of the forces and moments. Before proceeding to further discussion of these methods, the equations of equilibrium will be first simplified. To do this, the forces \( N_1 \) and \( N_2 \) in the first three of Eqs. (A-14) will be eliminated by substituting for them their expressions as given by the
fourth and fifth of Eqs. (A-14). Then, taking into consideration the notations given in Eqs. (A-18) and the conditions of Codazzi, the first three of Eqs. (A-14) may be written in the form

\[
\frac{\partial A_2 T_1}{\partial a_1} + \frac{\partial A_2 S}{\partial a_2} + \frac{\partial A_2}{\partial a_1} S = -2 \cdot \frac{\partial A_1}{\partial a_1} T_2
\]

\[
+ \frac{1}{R_1} \left[ \frac{\partial A_2 H}{\partial a_1} - \frac{\partial A_2 M_2}{\partial a_2} + 2 \frac{\partial A_1 H}{\partial a_2} - 2 \frac{R_1 \partial A_1}{\partial a_1} M_2 \right] = -A_1 A_2 g_1
\]

\[
\frac{\partial A_2 S}{\partial a_1} + \frac{\partial A_2 T_2}{\partial a_2} + \frac{\partial A_2}{\partial a_2} S = -2 \cdot \frac{\partial A_1}{\partial a_2} T_1
\]

\[
+ \frac{1}{R_2} \left[ \frac{\partial A_2 M_1}{\partial a_2} - \frac{\partial A_2 M_2}{\partial a_1} + 2 \frac{\partial A_1 H}{\partial a_1} + 2 \frac{R_2 \partial A_1}{\partial a_2} H \right] = -A_1 A_2 g_2
\]

\[
\frac{T_1}{R_1} + \frac{T_2}{R_2} - \frac{1}{A_1 A_2} \left\{ \frac{\partial A_2 M_1}{\partial a_1} + \frac{\partial A_2 H}{\partial a_2} + \frac{\partial A_2}{\partial a_2} H - \frac{2}{R_1} \frac{\partial A_1}{\partial a_1} M_1 \right\}
\]

\[
+ \frac{2}{R_2} \frac{1}{A_2} \left[ \frac{\partial A_1 H}{\partial a_1} + \frac{\partial A_2 H}{\partial a_2} + \frac{\partial A_1}{\partial a_2} H - \frac{2}{R_1} \frac{\partial A_2}{\partial a_1} M_1 \right] = g_s
\]

Now, return to methods of obtaining solutions. The first method involves replacement in Eqs. (A-21) the forces and moments by their expressions in terms of the strains of the middle surface. Then, one obtains, upon substitution for the strains by their expressions in terms of the displacements as given in Eqs. (A-6) and (A-9) a system of three partial differential equations in terms of the three displacements of the middle surface.

The second method consists in supplementing the equations of equilibrium (A-21) by the compatibility equations (A-12), which, for this purpose, must be expressed in terms of the forces and moments. Then, one obtains a system of six partial differential equations for the determination of unknowns \( T_1, T_2, S, M_1, M_2 \) and \( H \).

In that which follows, attention will be limited to the second method of solution. Substituting in Eqs. (A-12) the strains from Eqs. (A-20) one
obtains the compatibility equations in terms of the forces and moments

\[
\frac{\partial A_2}{\partial \sigma_1} \left( M_2 - \mu M_1 \right) - (1 + \mu) \left( \frac{\partial A_1}{\partial \sigma_1} \frac{M_1}{R_1} - \frac{\partial A_2}{\partial \sigma_2} \left( M_1 - M_2 \right) \right) - \frac{h^2}{12 R_1} \frac{\partial A_2}{\partial \sigma_1} \left( T_1 - \mu T_2 \right) - 2(1 + \mu) \frac{\partial A_2}{\partial \sigma_2} \left( T_1 - \mu T_2 \right) - 2(1 + \mu) \frac{R_1}{R_2} \frac{\partial A_1}{\partial \sigma_1} S = 0
\]

\[
\frac{\partial A_1}{\partial \sigma_2} \left( M_1 - \mu M_2 \right) - (1 + \mu) \left( \frac{\partial A_1}{\partial \sigma_2} \frac{M_2}{R_2} - \frac{\partial A_2}{\partial \sigma_1} \left( M_2 - M_1 \right) \right) - \frac{h^2}{12 R_2} \frac{\partial A_2}{\partial \sigma_1} \left( T_2 - \mu T_1 \right) - 2(1 + \mu) \frac{\partial A_2}{\partial \sigma_2} \left( T_2 - \mu T_1 \right) - 2(1 + \mu) \frac{R_2}{R_1} \frac{\partial A_1}{\partial \sigma_2} S = 0 \quad (A-22)
\]

\[
\frac{M_2 - \mu M_1}{R_1} + \frac{M_1 - \mu M_2}{R_2} + \frac{h^2}{12} \frac{1}{A_1 A_2} \frac{1}{\partial \sigma_1} \left[ \frac{\partial A_2}{\partial \sigma_1} \left( T_1 - \mu T_2 \right) - (1 + \mu) \left( \frac{\partial A_2}{\partial \sigma_2} \frac{M_2}{R_2} - \frac{\partial A_2}{\partial \sigma_1} \left( T_2 - \mu T_1 \right) \right) \right] = 0
\]

The fulfilment of Eqs. (A-22) ensures the possibility of determining the displacements from the given forces and moments. Eqs. (A-22), after transformation employing the equations of equilibrium and then neglecting a number of terms of the order \( h/R \) compared with unity, can be reduced to the form

\[
(1 + \mu) N_1 = \frac{1}{A_1} \frac{\partial M}{\partial \sigma_1} - \frac{h^2}{12} \frac{1}{R_1 A_1} \frac{\partial T}{\partial \sigma_1}
\]

\[
(1 + \mu) N_2 = \frac{1}{A_2} \frac{\partial M}{\partial \sigma_2} - \frac{h^2}{12} \frac{1}{R_2 A_2} \frac{\partial T}{\partial \sigma_2}
\]

\[
\frac{M_2 - \mu M_1}{R_1} + \frac{M_1 - \mu M_2}{R_2} + \frac{h^2}{12} \frac{1}{A_1 A_2} \frac{1}{\partial \sigma_1} \left[ \frac{\partial A_2}{\partial \sigma_1} \left( T_1 - \mu T_2 \right) - (1 + \mu) \left( \frac{\partial A_2}{\partial \sigma_2} \frac{M_2}{R_2} - \frac{\partial A_2}{\partial \sigma_1} \left( T_2 - \mu T_1 \right) \right) \right] = 0
\]

in which

\[
M = M_1 + M_2, \quad T = T_1 + T_2 \quad (A-23a)
\]

\[
\Delta(\tau) = \frac{1}{A_1 A_2} \left[ \frac{\partial}{\partial \sigma_1} (\frac{A_2}{A_1} \frac{\partial T}{\partial \sigma_1}) + \frac{\partial}{\partial \sigma_2} \left( \frac{A_1}{A_2} \frac{\partial T}{\partial \sigma_2} \right) \right]
\]

The second term on the right hand side of the first two of Eqs. (A-23) is likewise of negligible magnitude. Thus, the first two of the compatibi-
lity equations can be written in the following simple form

\[(1+\mu)N_1 = \frac{1}{A_1} \frac{\partial M}{\partial \alpha_1} \]

\[(1+\mu)N_2 = \frac{1}{A_2} \frac{\partial M}{\partial \alpha_2} \]  \hspace{1cm} (A-24)

The compatibility equations have been simplified in the form of Eqs. (A-23) or (A-24), which will be employed to eliminate \(N_1\) and \(N_2\) in the equations of equilibrium. Eliminating from the first three of Eqs. (A-14), letting \(T_{12} = T_{21}\), the normal shearing forces \(N_1\) and \(N_2\) by use of Eqs. (A-24), and from the fourth and fifth of Eqs. (A-14) \(N_1\) and \(N_2\) by use of Eqs. (A-23), one obtains a system of six equations with the last one coming from the third of Eqs. (A-23)

\[\frac{1}{A_1A_2} \left[ \frac{\partial A_2 T_1}{\partial \alpha_1} + \frac{\partial A_1 S}{\partial \alpha_2} + \frac{\partial A_1 S}{\partial \alpha_2} - \frac{\partial A_2 T_2}{\partial \alpha_1} \right] + \frac{1}{1+\mu} \frac{1}{R_1A_1} \frac{\partial M}{\partial \alpha_1} + g_1 = 0\]

\[\frac{1}{A_1A_2} (\frac{\partial A_2 (M_1-\mu M_2)}{\partial \alpha_1} + (1+\mu)(\frac{\partial A_1 H}{\partial \alpha_1} + \frac{\partial A_1 H}{\partial \alpha_2} H) + \frac{\partial A_2 (M_1-\mu M_2)}{\partial \alpha_1}) + \frac{h^2}{12 R_1 A_1} \frac{\partial T}{\partial \alpha_1} = 0\]

\[\frac{1}{A_1A_2} \left[ \frac{\partial A_2 S}{\partial \alpha_1} + \frac{\partial A_1 T_1}{\partial \alpha_1} + \frac{\partial A_1 S}{\partial \alpha_2} - \frac{\partial A_2 T_2}{\partial \alpha_1} \right] + \frac{1}{1+\mu} \frac{1}{R_2 A_2} \frac{\partial M}{\partial \alpha_2} + g_2 = 0\]  \hspace{1cm} (A-25)

\[\frac{1}{A_1A_2} (\frac{\partial A_2 (M_1-\mu M_2)}{\partial \alpha_2} + (1+\mu)(\frac{\partial A_1 H}{\partial \alpha_1} + \frac{\partial A_1 H}{\partial \alpha_2} H) + \frac{\partial A_2 (M_1-\mu M_2)}{\partial \alpha_2}) + \frac{h^2}{12 R_2 A_2} \frac{\partial T}{\partial \alpha_2} = 0\]

\[-\frac{T_1}{R_1} + \frac{T_2}{R_2} - \frac{1}{1+\mu} \Delta M - g_n = 0\]

\[\frac{M_1-\mu M_2}{R_1} + \frac{M_2-\mu M_1}{R_2} + \frac{h^2}{12} \Delta (T) = -\frac{h^2}{12 A_1A_2} \left[ \frac{\partial}{\partial \alpha_2} (A_2 g_1) + \frac{\partial}{\partial \alpha_1} (A_1 g_2) \right]\]

These six equations constitute an eighth order system and can be reduced to three equations of fourth order system by the use of complex transform-
mation. For this purpose, the auxiliary functions

\[
\begin{align*}
\bar{T}_1 &= T_1 - \frac{j}{c} \frac{M_3 - \mu M_4}{1 - \mu^2} \\
\bar{T}_2 &= T_2 - \frac{j}{c} \frac{M_1 - \mu M_2}{1 - \mu^2} \\
\bar{S} &= S + \frac{j}{c} \frac{H}{1 - \mu} \\
\bar{T} &= \bar{T}_1 + \bar{T}_2
\end{align*}
\]

will be introduced, where

\[
c = \frac{\hbar}{\sqrt{12(1 - \mu^2)}}
\]

Substituting in Eqs. (A-25) the forces \( T_1, T_2, S \) by their expressions in terms of \( \bar{T}_1, \bar{T}_2, \bar{S} \) and \( M_1, M_2, H \) as defined in Eqs. (A-26). In this way one obtains a system of six equations from which the quantities \( M_1, M_2, H \) may be eliminated. This process leads to the following system of three partial differential equations in terms of three complex forces \( \bar{T}_1, \bar{T}_2, \) and \( \bar{S} \).

\[
\begin{align*}
\frac{1}{A_1 A_2} \left[ \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_1}{\partial y} \bar{S} - \frac{\partial A_2}{\partial y} \bar{T}_1 \right] + \frac{j}{c} \frac{e}{R_1 A_1} \frac{\partial \bar{T}_1}{\partial x} + g_1 &= 0 \\
\frac{1}{A_1 A_2} \left[ \frac{\partial A_1}{\partial x_2} + \frac{\partial A_2}{\partial x_1} + \frac{\partial A_2}{\partial y} \bar{S} - \frac{\partial A_1}{\partial y} \bar{T}_1 \right] + \frac{j}{c} \frac{e}{R_2 A_2} \frac{\partial \bar{T}_1}{\partial x} + g_2 &= 0 \quad (A-27)
\end{align*}
\]

where

\[
\mathbf{\bar{g}}_n = \mathbf{g}_n + j c \frac{e}{A_1 A_2} \left[ \frac{\partial A_2}{\partial x_1} + \frac{\partial A_1}{\partial x_2} \right]
\]

Equations (A-27) include the equations of equilibrium of the shell element and the equations of compatibility for the strains of the middle surface.

It is a fourth order system with three unknowns, and is half the number of equations, order and unknowns of the system (A-25).
Letting \( c = 0 \) and identifying \( \tilde{T}_1, \tilde{T}_2, \tilde{S} \) by \( T_1^*, T_2^*, S^* \), respectively in Eqs. (A-27), this system reduces to the equations of the membrane theory.

\[
\frac{1}{A_1 A_2} \left[ \frac{\partial A_1 T_1^*}{\partial z} + \frac{\partial A_1 S^*}{\partial y} + \frac{\partial A_1}{\partial y} \frac{\partial S^*}{\partial y} - \frac{\partial A_1}{\partial y} \frac{\partial T_2^*}{\partial y} \right] + \tilde{g}_1 = 0
\]

\[
\frac{1}{A_1 A_2} \left[ \frac{\partial A_1 S^*}{\partial z} + \frac{\partial A_1 T_2^*}{\partial z} \right] + \tilde{g}_2 = 0 \quad (A-28)
\]

\[
\frac{\tilde{T}^*}{R_1} + \frac{\tilde{T}^*}{R_2} = 0
\]

To get a complete solution, the displacements of the middle surface have to be found. Define the complex displacements \( \tilde{u}, \tilde{v}, \tilde{w} \) which relate to the complex forces by six differential equations

\[
\tilde{e}_1 = \frac{1}{Eh} \left( \tilde{T}_1 - \mu \tilde{T}_2 \right), \quad \tilde{e}_2 = \frac{1}{Eh} \left( \tilde{T}_2 - \mu \tilde{T}_1 \right)
\]

\[
\tilde{\omega} = \frac{2(1+\mu)}{Eh} \tilde{S}, \quad \tilde{\zeta}_1 = \frac{i}{\mu} \frac{1}{Eh} \left( \tilde{T}_1 - \tilde{T}_2^* \right) \quad (A-29)
\]

\[
\tilde{\zeta}_2 = \frac{i}{\mu} \frac{1}{Eh} \left( \tilde{T}_2 - \tilde{T}_1^* \right), \quad \tilde{\zeta} = -\frac{i}{\mu} \frac{1}{Eh} \left( \tilde{S} - \tilde{S}^* \right)
\]

In these equations \( \tilde{e}_1, \tilde{e}_2, \tilde{\omega}, \tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta} \) are related to \( \tilde{u}, \tilde{v}, \tilde{w} \) in the same way as the strain-displacement relations given in Eqs. (A-6) and (A-9), and \( T_1^*, T_2^*, S^* \) are solutions of the membrane theory, i.e., of the system (A-28). The real parts of \( \tilde{u}, \tilde{v}, \tilde{w} \) are the displacements \( u, v, w \), respectively.

Thus, the solution of problems of a shell reduces to the determination of the complex forces \( \tilde{T}_1, \tilde{T}_2, \tilde{S} \) from Eqs. (A-27) and the complex displacements \( \tilde{u}, \tilde{v}, \tilde{w} \) from Eqs. (A-29) subject to appropriate boundary conditions.

In conclusion it is noted that the error introduced in the system (A-27) is of order \( h/R \) compared with unity. Hence, the system of Eqs. (A-29) are only approximately compatible with each other within an error of this order.