AN EXPERIMENTAL AND NUMERICAL STUDY
OF WAVE MOTION AND UPSTREAM INFLUENCE
IN A STRATIFIED FLUID

by David A. Hurdis

Goddard Space Flight Center
Greenbelt, Md. 20771
A system consisting of two superimposed layers of liquid of different densities, with a thin transition layer at the interface, provides a good laboratory model of an ocean thermocline or of an atmospheric inversion layer. The object of this research was to gain knowledge about the propagation of disturbances within these two geophysical systems. The technique used was to observe the propagation of internal waves and of upstream influence within the density-gradient region between the two layers of liquid. The disturbances created by the motion of a vertical flat plate, which was moved longitudinally through this region, were examined both experimentally and numerically. An upstream influence, which resulted from a balance of inertial and gravitational forces, was observed, and it was possible to predict the behavior of this influence with the numerical model. The prediction included a description of the propagation of the upstream influence to steadily increasing distances from the flat plate and the shapes and magnitudes of the velocity profiles.
PREFACE

The work presented here was completed while the author, now an assistant professor of mechanical engineering at the University of Maryland, was an employee at Goddard Space Flight Center. The material was submitted in May, 1973, to the faculty of the School of Engineering and Architecture, at the Catholic University of America, as a doctoral dissertation.

The author wishes to express his gratitude to Professor H. P. Pao of Catholic University for his guidance throughout the course of this work. In addition thanks are due to Professors Timothy W. Kao and H. Bulent Atabek, also of Catholic University, for their helpful comments. The author's supervisors at GSFC, William W. Auer and Henry Maurer, Jr., showed continuing interest and helpfulness throughout his doctoral studies.
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>i</td>
</tr>
<tr>
<td>PREFACE</td>
<td>iii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>THEORETICAL STUDY</td>
<td>7</td>
</tr>
<tr>
<td>EXPERIMENTAL STUDY</td>
<td>15</td>
</tr>
<tr>
<td>DISCUSSION OF RESULTS</td>
<td>23</td>
</tr>
<tr>
<td>CONCLUSIONS</td>
<td>43</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>45</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>49</td>
</tr>
<tr>
<td>APPENDIX 1—TRANSFORMED GOVERNING EQUATIONS</td>
<td>A-1</td>
</tr>
<tr>
<td>APPENDIX 2—FINITE-DIFFERENCE FORM OF TRANSFORMED GOVERNING EQUATIONS</td>
<td>B-1</td>
</tr>
<tr>
<td>APPENDIX 3—COMPUTER PROGRAM</td>
<td>C-1</td>
</tr>
</tbody>
</table>
INTRODUCTION

Wave Propagation in Nonhomogeneous Fluids

The earliest significant study of waves in a nonhomogeneous fluid was that of Stokes (1847), who developed the theory of the propagation of infinitesimal waves in a two-layer fluid system. The excitation of waves in two fluid layers is discussed also by Lamb (1932, Section 231). In essence, it has been shown that for a two-layer fluid, two possible wave systems can exist for any given period of oscillation. One propagates at the free surface of the upper layer, while the other propagates at the interface of the two-layer fluid. Similarly, it can be shown for a three-layer fluid that three wave systems can exist, one for each surface of density discontinuity. In this context, a fluid with a continuous density stratification can be considered to be the limit of an n-layer fluid system as n → ∞; internal waves can be propagated at all levels in such a fluid. A semantic distinction is usually made between waves propagating at the interfaces of a multilayer system and those propagating through a continuously stratified fluid. The former are called interfacial waves, while the latter are referred to as internal waves.

Internal wave propagation in a continuously stratified fluid is characterized by the local Brunt-Vaisala frequency \( N = \left(-\frac{g}{\rho} \frac{d\rho}{dz}\right)^{1/2} \). (For stably stratified fluids, \( d\rho/dz \leq 0 \).) No waves can propagate with frequencies higher than the local value of \( N \), and the direction of propagation of a wave of frequency \( \sigma \) is \( \cos^{-1} (\sigma/N) \) from the vertical.

The earliest important experimental work on wave propagation in a nonhomogeneous fluid was published by Ekman (1904), who sought to explain the abnormally high resistance experienced by ships in the mouths of some Norwegian fjords where there is a layer of fresh water over salt water. Ekman demonstrated that the phenomenon is due to increased wave drag resulting from the ship’s exciting of waves not only at the free surface but also at the interface of the two layers.

Upstream Influence

Density stratification affects the flow of an inviscid, incompressible fluid in two ways. First, a change in density causes a change in the inertia per unit volume of the fluid. Second, in a gravitational field, a density change causes a change in the body force per
unit volume of the fluid. Since, in general, the body force is not the only force acting on the fluid, these two effects will not nullify each other, and the flow pattern of a nonhomogeneous fluid may differ considerably from that of a homogeneous fluid even though the boundary conditions are the same.

One flow phenomenon exhibited by stratified fluids is upstream influence. This phenomenon occurs during the flow of a stratified fluid past an obstacle, and consists of the propagation of a disturbance upstream from the obstacle. It is especially strong when the obstacle is so wide, in comparison with its height, that the flow past the obstacle is essentially two-dimensional. Upstream influence is also called forward influence or upstream wake in the literature. The limiting case and strongest manifestation of upstream influence occurs in the slow flow of a continuously stratified fluid past an obstacle, and is called blocking. Blocking is the stagnation with respect to an obstacle of the fluid within a horizontal slab containing the obstacle and having the same height as the obstacle. Closed streamline recirculation may occur within the slug of blocked fluid being pushed ahead of the obstacle (Janowitz 1971; Browand and Winant 1972). For very low flow speeds, blocking occurs both upstream and downstream of the obstacle. At higher speeds, the downstream blocking is replaced by a wake and lee waves, and as the speed increases still more, the upstream blocking becomes incomplete and is therefore more properly referred to as upstream influence.

Qualitative explanations for upstream influence may be given from two viewpoints, the buoyancy viewpoint and the internal wave viewpoint.

Buoyancy Viewpoint

As a two-dimensional body moves through a stably stratified fluid, it is necessary for the fluid to be displaced vertically to get by the body. Because of the stable stratification, however, the fluid is reluctant to be displaced vertically, and tends to pile up in front of the body, thus accounting for the upstream influence.

Internal Wave Viewpoint

The upstream influence may also be regarded as the propagation of an internal wave of zero frequency. From the above discussion of internal waves, it is clear that such a wave can propagate from the disturbance source only in a horizontal direction, and that as the wave does so, it causes the presence of the disturbance to be felt far upstream.

Investigations of Upstream Influence

Experimental

The fundamental experiment which established the existence of the phenomenon of upstream blocking in stratified flows was conducted by Long (1955). He studied the flow of a linearly stratified salt solution over a two-dimensional barrier which was towed along
the bottom of a channel. He observed that for values of the internal Froude number 
\( F_i = U/(g \Delta \rho/\rho) \) less than \( 1/\pi \), a disturbance was propagated upstream from a barrier 
moving through a fluid of depth \( H \). He noted that for this case his analytical flow predic-
tions became invalid, since they were dependent upon the assumption of undisturbed 
upstream flow.

More recent experiments have been performed by Pao (1967, 1968), Martin and Long 
(1968), Debler and Daily (1971), Browand and Winant (1972), and Laws and Stevenson 
(1972). The common features of these experiments are the use of a linearly stratified salt 
solution for the working fluid, and use of very slow obstacle speeds. All these investigators 
have reported upstream influence with a wavy velocity profile, thus confirming the early 
observations of Long (1955), who reported an upstream velocity profile consisting of 
alternate jets and stagnation regions.

**Theoretical**

The problem of stratified flow about an obstacle has also been studied analytically by 
several investigators. These studies generally have taken one of three theoretical approaches.

The first approach assumes inviscid steady flow everywhere and a uniform velocity profile 
upstream, includes nonlinear effects, and is generally based on the use of Long's equation; 
for example, Long (1953). For internal Froude numbers greater than \( 1/\pi \), good agreement 
is obtained between theoretical and experimental results. For lower internal Froude num-
bbers, the experiment shows upstream blocking, a condition which violates the theoretical 
assumption of undisturbed upstream flow. It is therefore not surprising that this approach 
fails to predict the upstream flow conditions for \( F_i < 1/\pi \).

By using the formulation of an inviscid, linearized initial value problem, the second approach 
permits study of the formation of the blocking column. However, introduction of viscosity 
into this formulation requires the assumption of a Schmidt number \( (\text{Sc} = v/D) \) equal to 
unity, which is far from accurate for the usual experimental conditions involving salt solu-
tions for which \( \text{Sc} \approx 800 \). This approach is the one used by Bretherton (1967).

The third approach is a steady viscous formulation for very slow, low Reynolds number 
flows. The fluid is assumed to be nondiffusive, and the low velocity assumption allows 
the nonlinear inertial terms to be neglected in the flow equations. This formulation is thus 
a balance of the gravitational and viscous forces. This approach was used by Pao (1967, 
1968), who used two-parameter perturbation expansions to solve for the flow field in front 
of and above a horizontal flat plate. He obtained good agreement with his experiment. 
Martin and Long (1968) and Brown (1968) have also arrived at solutions that predict 
upstream disturbance for slow stratified flow past a horizontal flat plate. Graebel (1969) 
solved for the flow field about a vertical flat plate by means of matched-asymptotic-expansion 
theory. His solution also predicts the existence of an upstream influence. Janowitz 
(1971) used an integral transform approach to examine the low Reynolds's number flow of 
a linearly stratified fluid about a vertical flat plate. His solution predicts the existence of
an upstream blocking column within which the fluid recirculates in closed streamlines. The experimental work of Browand and Winant (1972) appears to confirm this result.

In a separate class of theoretical papers, Benjamin (1970), Keady (1971), and McIntyre (1972) have examined the phenomenon of upstream influence from a more abstract viewpoint to determine whether Long's hypothesis of no disturbance far upstream is ever strictly correct. Benjamin concludes that an upstream disturbance exists in the form of a uniform, long wave extending to steadily increasing distances ahead of the body whenever lee waves are present downstream. Keady extended Benjamin's work to the case of interfacial waves in a two-fluid system. McIntyre, whose paper is still the subject of considerable controversy, discusses a kind of upstream influence different from that considered here. His theory predicts an upstream influence related to the formation of lee waves and manifested by an increase in upstream velocity with respect to the body rather than a decrease.

Geophysical Implications

The flow of a stratified ocean or atmosphere over a barrier such as a mountain ridge usually will produce lee waves downstream and blocking upstream. The prevailing west wind flowing over the San Gabriel Mountains east of Los Angeles, for example, produces blocking which makes it more difficult for the region's pollutants to be dispersed.

Background of This Study

Previous studies have dealt with the slow flow about a two-dimensional body of either a linearly stratified fluid or of a two-layer fluid. The linear-density model is an idealization which has proven to be very useful in studies of atmospheric and oceanic flows. In the real atmosphere, however, the near-linear density gradient may be interrupted by one or more inversions; and in the real ocean, a thermocline of one or more sheets is usually present. At these locations, the density undergoes a sudden but nevertheless continuous change, for which neither the linear nor the two-layer model is a valid representation. The real condition is somewhere between these two extremes.

Long referred to this situation in Part III of his well-known trilogy on stratified flows (1955). He suggested that for strong inversions the shock-wave type of phenomena (hydraulic jumps and drops), such as exhibited in the flow of a two-layer fluid over a barrier (Long 1954), would dominate. For weak inversions, he proposed that the flow would behave like the linear model. He concluded, however, that the intermediate case would be highly nonlinear and therefore difficult to solve analytically.

Davis (1967), while doing research for his doctoral dissertation on the stability of oscillatory internal waves, used an experimental model consisting of a layer of fresh water on top of a layer of salt water, with a thin transition region at the interface. He recognized, within the transition layer, the propagation of a new type of solitary wave which had not been reported in the literature. This type of solitary wave was new, in that it could be propagated in a fluid of infinite depth, whereas internal solitary waves were known previously
only as shallow-water phenomena. Davis added an investigation of this new wave to his dissertation (see also Davis and Acrivos 1967). Independently and at about the same time, Benjamin completed a theoretical study in which he predicted the existence of such waves (1967).

Davis's experience suggests that the flow behavior of a stratified fluid in which the density changes suddenly over a short distance cannot necessarily be inferred from previous work on linearly stratified or two-layer fluids. In view of the importance of sudden density changes in geophysical fluids (atmospheric inversions and ocean thermoclines), it was decided to undertake a study of flow about an obstacle by a fluid with a sudden but continuous density change.

Outline of This Study

Experimental

A laboratory flow model consisting of a layer of fresh water on top of a layer of salt water with a thin transition region at the interface was used for this study. A vertical flat plate of 2.54-cm height was towed through the interfacial layer at various speeds and for various density profiles. Upstream and downstream velocity profiles were measured by a flow visualization technique. A conductivity probe was used to determine the density profile before and after each run. In addition to the shape of the density profile, the present work differs from that of most previous experiments in another respect. Those workers whose theoretical work depended upon neglecting the inertial terms in the momentum equation considered only very slow flow speeds in their experiments. The top speed considered by Laws and Stevenson (1972), for example, was only 0.07 mm/s, corresponding to a Reynolds number of O(1) and a Richardson number (Ri = -g/ρ dρ/dz/(dU/dz)^2) of O(10^4). For the present study, flow speeds as high as 4 cm/s were examined. The corresponding dimensionless parameters were Re = O(10^3) and Ri = O(1).

Theoretical

The flow conditions described above were formulated as a nonlinear initial-value problem in a viscous diffusive medium. The Boussinesq approximation, which neglects the effect of density variation on the nonlinear inertial terms, was not used. The initial density profile was expressed mathematically by fitting experimental measurements to the equation ρ = ρ̂ (1 - ω tanh αz), where ω̂ is a dimensionless constant and α^-1 is a measure of the thickness of the density-gradient region. Formulation as an initial-value problem permitted theoretical study of the development of the upstream influence. The formulation is analytically intractable, and the problem was therefore solved numerically. Comparison of theoretical upstream velocity profiles with those measured experimentally shows good agreement, but the theory fails to predict the existence of the reverse lee jet which was observed downstream in the density-gradient layer.
THEORETICAL STUDY

Formulation

Dimensionless-flow equations will now be developed for the case of a vertical flat plate of height $2b$ moving to the right with velocity $U$ through the density-gradient layer of a viscous, diffusive, stably stratified fluid whose initial density profile is of the form $\rho = \bar{\rho} (1 - \tilde{\omega} \tanh \alpha z)$. The flow is viewed from a Cartesian coordinate system attached to the plate (see figure 1), and the plate is of infinite extent in the $y$ direction.

Figure 1. Geometry of the Flow Field

The continuity equation for a diffusive medium is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (u\rho) + \frac{\partial}{\partial z} (w\rho) = D \nabla^2 \rho.$$  \hspace{1cm} (1)

The incompressibility equation is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$ \hspace{1cm} (2)

Since this equation is not quite correct for the case of a diffusive fluid such as salt water (see Phillips 1969, p. 15), it introduces a slight error to the formulation. The momentum equations are

$$\rho \frac{Du}{Dt} = -\frac{\partial \rho}{\partial x} + \mu \nabla^2 u,$$ \hspace{1cm} (3a)

$$\rho \frac{Dw}{Dt} = -\frac{\partial \rho}{\partial z} - \rho g + \mu \nabla^2 w,$$ \hspace{1cm} (3b)

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \quad \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$
The stream function $\psi$ is defined by

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}$$

(4)

and the y component of the vorticity $\omega = \nabla \times \vec{q} = (\xi, \eta, \zeta)$ is

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

(5a)

so that by combining (4) and (5a)

$$\nabla^2 \psi = \eta.$$ 

(5b)

Subtracting the x derivative of (3b) from the z derivative of (3a) yields

$$\frac{\partial \rho}{\partial z} \frac{Du}{Dt} - \frac{\partial \rho}{\partial x} \frac{Dw}{Dt} + \rho \left[ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} (\mu \eta) + \frac{\partial}{\partial z} (\omega \eta) \right] = g \frac{\partial \rho}{\partial x} + \mu \nabla^2 \eta.$$ 

(6)

The Boussinesq approximation, which will not be made here, consists of neglecting the first two terms on the left side of (6), and replacing the factor $\rho$ in the third term by the average density $\bar{\rho}$.

The boundary conditions for the system (1), (4), (5b), and (6) for $t > 0$ are

$$\text{as } |x|, |z| \to \infty, \rho \to \bar{\rho} (1 - \omega \tanh \alpha z), \psi \to -Uz$$

(7a)

and at $x = 0, |z| \leq b, \frac{\partial \rho}{\partial x} = 0$$

(7b)

$$\frac{\partial \psi}{\partial x} = 0, \frac{\partial \psi}{\partial z} = 0.$$ 

(7c)

Since $u$, $w$, and $\eta$ are defined in terms of $\psi$ by (4) and (5b), no independent boundary conditions are possible for these quantities. Although (4), (5b), and (6) can be easily combined into one unwieldy equation, this is not done here, to retain clarity and manipulative ease.

The initial conditions are

$$\rho = \bar{\rho} (1 - \omega \tanh \alpha z) \quad \text{everywhere at } t = 0$$

(8a)

and

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z} = 0 \quad \text{everywhere at } t = 0.$$ 

(8b)

With the use of the density scale-length $\alpha^{-1}$ as a characteristic length, $U$ as a characteristic velocity, and $\bar{\rho}$ as a characteristic density, these equations can be put into dimensionless form. Thus
\((\tilde{x}, \tilde{z}) = \alpha (x, z), \quad (\tilde{u}, \tilde{w}) = \frac{1}{U} (u, w), \quad \tilde{\rho} = \frac{\rho}{\bar{\rho}}\)

\[ \tilde{t} = U \alpha t, \quad \tilde{\eta} = \frac{1}{U \alpha} \eta, \quad \tilde{\psi} = \frac{\alpha}{U} \psi. \]  \hspace{1cm} (9)

Equation (1) then becomes

\[ \frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} (\tilde{u} \tilde{\rho}) + \frac{\partial}{\partial \tilde{z}} (\tilde{w} \tilde{\rho}) = \frac{1}{\text{Sc} \cdot \text{Re}} \nabla^2 \tilde{\rho} \]  \hspace{1cm} (10)

where

\begin{align*}
\text{Sc}, \text{Schmidt number} &= \frac{\nu}{D} \\
\text{Re}, \text{Reynolds number} &= \frac{(U/c)}{\alpha}.
\end{align*}

Equation (6) becomes

\[ \frac{\partial \tilde{\rho}}{\partial \tilde{t}} \frac{\partial \tilde{u}}{\partial \tilde{t}} - \frac{\partial \tilde{\rho}}{\partial \tilde{x}} \frac{\partial \tilde{w}}{\partial \tilde{x}} + \tilde{\rho} \left[ \frac{\partial \tilde{\eta}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} (\tilde{u} \tilde{\eta}) + \frac{\partial}{\partial \tilde{z}} (\tilde{w} \tilde{\eta}) \right] = \frac{1}{F^2} \frac{\partial \tilde{\rho}}{\partial \tilde{x}} + \frac{1}{\text{Re}} \nabla^2 \tilde{\eta}, \]  \hspace{1cm} (11)

where

\[ F, \text{Froude number} = \frac{U}{(g/\alpha)^{1/2}}. \]

Similarly, (4) and (5b) become

\[ \tilde{u} = \frac{\partial \tilde{\psi}}{\partial \tilde{z}}, \quad \tilde{w} = -\frac{\partial \tilde{\psi}}{\partial \tilde{x}} \]  \hspace{1cm} (12)

and

\[ \nabla^2 \tilde{\psi} = \tilde{\eta}. \]  \hspace{1cm} (13)

Note that the Froude number appearing in (11) is of the overall form, and is not the internal Froude number, \(F_{\text{i}} = U/(g \Delta \rho/\rho \ H)^{1/2}\), that commonly appears in the formulation of stratified flow problems. This is a result of treating density in the present time-dependent problem as a separate unknown with a separate governing equation rather than combining the density profile into the momentum equation. To facilitate comparison of previous work with the results of this study, internal Froude numbers were computed for the conditions of the experimental runs.

The dimensionless form of the boundary conditions for \(\tilde{t} > 0\) is

\[ \text{as } |\tilde{x}|, |\tilde{z}| \to \infty, \quad \tilde{\rho} \to (1 - \omega \tanh \tilde{z}), \quad \tilde{\psi} \to -\tilde{z} \]  \hspace{1cm} (14a)
and at \( \tilde{x} = 0 \), \( \mid \tilde{z} \mid \leq ab \), \( \frac{\partial \tilde{\rho}}{\partial \tilde{x}} = 0 \) \( \frac{\partial \tilde{\psi}}{\partial \tilde{z}} = 0 \) \( \frac{\partial \tilde{\psi}}{\partial \tilde{z}} = 0 \) \( \frac{\partial \tilde{\psi}}{\partial \tilde{x}} = 0 \). \hspace{1cm} (14b)

The initial conditions become

\[ \tilde{\rho} = (1 - \bar{\omega} \tanh \tilde{z}) \hspace{1cm} \text{everywhere at } \tilde{t} = 0 \] \hspace{1cm} (15a)

and

\[ \frac{\partial \tilde{\psi}}{\partial \tilde{x}} = \frac{\partial \tilde{\psi}}{\partial \tilde{z}} = 0 \hspace{1cm} \text{everywhere at } \tilde{t} = 0 . \] \hspace{1cm} (15b)

Henceforth, the tildes (\( \sim \)) will be omitted and all variables will be understood to be dimensionless.

**Numerical Solution**

**Algorithm for Solution**

The solution of a modified form of (10) through (15) was accomplished by writing them as finite difference equations and solving these on a digital computer according to the following algorithm:

(a) At \( t = 0 \), the initial conditions (15a) and (15b) exist.

(b) At \( t = 0^+ \), the flow is irrotational, with \( \eta = 0 \). The solution for the \( \psi \) field is accomplished by making an initial guess of \( \psi = -z \) everywhere and then converging to the correct result by overrelaxation of (13), subject to the boundary conditions (14a) and (14c).

(c) The \( u \) and \( w \) velocity fields at \( t = 0^+ \) are computed from (12).

(d) Equation (10), subject to the boundary conditions (14a) and (14b), is used to make a time step in the density field.

(e) Equation (11) is used to make a time step in the vorticity field. [The boundary conditions on \( \psi \) result in the following conditions on \( \eta \): as \( \mid x \mid, \mid z \mid \to \infty, \eta \to 0 \); and at \( x = 0, \mid z \mid \leq ab, (u = \partial \psi / \partial z = 0) \Rightarrow \eta = \partial^2 \psi / \partial x^2 \).]

(f) With the new result for the vorticity field, a new value for the \( \psi \) field is computed by overrelaxating (13), subject to (14a) and (14c).

(g) The \( u \) and \( w \) velocity fields are computed from (12). Steps d, e, f, and g are repeated until a specified time \( t = t_f \) has been reached. The computer can be
requested to print out the $\rho$, $\psi$, $\eta$, $u$, and $w$ fields at $t = 0^+$, $t = t_f$, and at intermediate times, so that the time development of the flow field can be studied.

As mentioned above, the numerical solution is accomplished with a modified form of (10) through (15). The nature of these modifications will now be discussed.

**Modification of Nonlinear Boussinesq Terms**

Numerical evaluation of nonlinear terms was accomplished by a technique called the special three-point noncentral differences method (Torrance and Rockett 1969). To use this method, it is necessary that nonlinear terms be of the form $\partial (fg)/\partial x$ rather than of the form $g \partial f/\partial x$. This is accomplished in the case of the first two terms on the left side of (11) by recalling (2) and adding $u (\partial u/\partial x + \partial w/\partial z)$ to the factor $Du/Dt$ and $w (\partial u/\partial x + \partial w/\partial z)$ to the factor $Dw/ Dt$. Equation (11) then becomes

$$
\frac{\partial \rho}{\partial t} \left[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial z} (uw) \right] = \frac{\partial \rho}{\partial x} \left[ \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} (uw) + \frac{\partial}{\partial z} (w^2) \right]
$$

(16)

$$
+ \rho \left[ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} (u\eta) + \frac{\partial}{\partial z} (w\eta) \right] = \frac{1}{\eta^2} \frac{\partial \rho}{\partial x} + \frac{1}{Re^2} \nabla^2 \eta.
$$

**Stretching the Coordinate System**

Previous attempts at numerical solution of similar problems have had difficulty with the application of boundary conditions at infinity. A dilemma is posed by the conflicting requirements of keeping the finite-difference grid mesh fine in the vicinity of the body where the gradients are largest, and at the same time having the last grid point be far enough away from the body that it is an accurate approximation of infinity. One is forced to choose among an overly coarse mesh near the body, or prohibitively large computer core requirements, or an infinity so close to the body that errors are introduced. This problem was circumvented by writing (10) and (12) through (16) with respect to a coordinate system such that its mapping to real-world coordinates has a fine-grid mesh near the origin, a coarser mesh farther away, and a last grid point as close to infinity as desired. This mapping, which involves the hyperbolic tangent function, will now be developed.

The coordinate system sought $(\bar{x}, \bar{z})$ is one whose mapping to ordinary Cartesian coordinates $(x, z)$ is such that as $\bar{x}$ and $\bar{z}$ vary from $-1$ to $+1$, $x$ and $z$ vary from $-\infty$ to $+\infty$. The transformation for such a system is given by

$$
\bar{x} = \tanh ax, \quad \bar{z} = \tanh cz
$$

(17)

where

$$
\frac{d\bar{x}}{dx} = a \cdot \text{sech}^2 ax = a(1 - \tanh^2 ax) = a(1 - \bar{x}^2)
$$

(18a)
and
\[ \frac{d\bar{z}}{dz} = c (1 - \bar{z}^2). \] (18b)

Therefore, for any arbitrary function \( f \),
\[ \frac{\partial f}{\partial x} = \frac{d\bar{x}}{dx} \frac{\partial f}{\partial \bar{x}} = a (1 - \bar{x}^2) \frac{\partial f}{\partial \bar{x}} \] (19a)

and
\[ \frac{\partial f}{\partial z} = c (1 - \bar{z}^2) \frac{\partial f}{\partial \bar{z}}, \] (19b)

so that from (12)
\[ u = \frac{\partial \psi}{\partial z} = c (1 - \bar{z}^2) \frac{\partial \psi}{\partial \bar{z}} = c (1 - \bar{z}^2) \bar{u} \] (19c)

and
\[ w = -\frac{\partial \psi}{\partial x} = -a (1 - \bar{x}^2) \frac{\partial \psi}{\partial \bar{x}} = a (1 - \bar{x}^2) \bar{w}. \] (19d)

Finally
\[ \frac{\partial^2 f}{\partial x^2} = a^2 \frac{\partial}{\partial x} \left[ (1 - \bar{x}^2) \frac{\partial f}{\partial \bar{x}} \right] = a^2 (1 - \bar{x}^2)^2 \frac{\partial^2 f}{\partial \bar{x}^2} - 2a^2 \bar{x}(1 - \bar{x}^2) \frac{\partial f}{\partial \bar{x}} \] (19e)

and
\[ \frac{\partial^2 f}{\partial z^2} = c^2 (1 - \bar{z}^2)^2 \frac{\partial^2 f}{\partial \bar{z}^2} - 2c^2 \bar{z}(1 - \bar{z}^2) \frac{\partial f}{\partial \bar{z}}. \] (19f)

At this point, it is useful to restate the untransformed dimensionless governing equations.
\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial z} (\rho w) = \frac{1}{Sc \cdot Re} \nabla^2 \rho \] (20a)

\[ \frac{\partial \rho}{\partial z} \left[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial z} (uw) \right] - \frac{\partial \rho}{\partial x} \left[ \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} (uw) + \frac{\partial}{\partial z} (w^2) \right] \\
+ \rho \left[ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} (\eta u) + \frac{\partial}{\partial z} (\eta w) \right] = \frac{1}{Pr^2} \frac{\partial \rho}{\partial x} + \frac{1}{Re} \nabla^2 \eta \] (20b)

\[ u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x} \] (20c)

\[ \nabla^2 \psi = \eta. \] (20d)
Boundary conditions for \( t > 0 \) are

\[
\text{as } |x|, |z| \to \infty, \quad \rho \to (1 - \infty \tanh z), \quad \psi \to -z \quad (20e)
\]

\[
\text{and at } x = 0, \quad |z| \leq \alpha b, \quad \frac{\partial \rho}{\partial x} = 0 \quad (20f)
\]

\[
\frac{\partial \psi}{\partial x} = 0, \quad \frac{\partial \psi}{\partial z} = 0. \quad (20g)
\]

Initial conditions are

\[
\rho = (1 - \infty \tanh z) \quad \text{everywhere at } t = 0 \quad (20h)
\]

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z} = 0 \quad \text{everywhere at } t = 0. \quad (20i)
\]

Substitution of the transformation (19) into the governing equations (20) results in a system of equations which can be put into finite-difference form and solved numerically by the algorithm discussed above. The transformed equations are so lengthy that their presentation has been deferred to appendix 1. The finite-difference form of these equations is given in appendix 2, and the computer program is presented in appendix 3.

At this point, it is appropriate to mention another approximation necessitated by numerical modeling. Although the vertical flat plate used in the experimental study was very thin in the \( x \)-direction compared to its height, it was necessary that the numerical model of the plate be at least two \( x \)-grid widths thick to avoid instability in the solution caused by singularities at \((0, b)\) and \((0, -b)\). This approximation is not believed to have significantly affected theoretical results upstream, but was found to be responsible for the discrepancy between theoretical and experimental results downstream.

**Numerical Results**

Figure 2 presents a typical theoretical result for the velocity profiles at four upstream locations at some time, \( t = t_1 \). Figure 3 shows a typical theoretical prediction for the variation of center line velocity with upstream distance at \( t = t_1 \). Further information on numerical results will be presented in the section, Discussion of Results, where they are interpreted in the light of experimental findings.
Figure 2. Typical Theoretical Result for Upstream Velocity Profiles

Figure 3. Typical Theoretical Prediction for the Variation of Centerline Velocity with Upstream Position
EXPERIMENTAL STUDY

Description of Experiments

Filling the Channel

The experiments were carried out in a Plexiglass channel 10 m long, 60 cm deep, and 35.5 cm wide. On the day before a set of experimental runs, the channel would be filled to a depth of 23 cm with a uniform salt water solution lightly tinted with nigrosine. To ensure its uniformity, the solution in the channel was thoroughly agitated and then allowed to sit for at least 12 hours. Two salt solution densities were used during this study, 1.013 g/cm$^3$ and 1.026 g/cm$^3$. On the day of the experimental runs, a layer of fresh water was slowly floated onto the salt solution until the total fluid depth was 46 cm. This was accomplished by letting the fresh water from the filling hose flow into a sponge rubber basin which was floating on a Styrofoam raft. The water seeped through the sponge, thereby dissipated most of its momentum, and floated with little mixing onto the salt solution. Gradient-layer thicknesses of the order of 4 cm were achieved by this method.

By making certain that the temperature of the inflowing fresh water differed by no more than 2° C from that of the salt water onto which it was floated, convective fluid motions were reduced. These small secondary motions could not be entirely eliminated, however.

Measurement of Density Profiles

Density profiles were measured before and after each run with a conductivity probe. The probe was calibrated with standard salt solutions whose densities had been measured by a hydrometer to an accuracy of ±0.0005 g/cm$^3$. Density measurements made with the probe are estimated to be accurate to ±0.0005 g/cm$^3$, while the z elevations for these density measurements have an estimated accuracy of ±0.2 cm. The resulting density profiles were fitted to Benjamin's expression (1967), $\rho = \tilde{\rho} (1 - \tilde{\omega} \tanh \alpha z)$, where $\tilde{\rho} = 1/2 (\rho_u + \rho_q)$ = average density of upper and lower layers; $\tilde{\omega} = (\rho_q - \rho_u)/(\rho_e + \rho_u)$; $\alpha$ is a number determined by least-squares fit whose reciprocal is a measure of the gradient thickness, and $z$ is the height above the level at which $\rho = \tilde{\rho}$. Figure 4 shows a typical experimentally-measured density profile and the function to which it was fitted. Interestingly enough, it was found that the increase in the gradient layer thickness $\alpha^{-1}$ was very nearly a linear function of the number of transits made through the layer by the vertical flat plate. This relationship is illustrated in figure 5. Since the diffusivity of salt in water is very low, the thickening is almost entirely due to mixing.

Typically, about four experiment runs and four subsequent retrievals of the plate could be made during the course of a day. Approximately an hour was allowed between runs for residual fluid motions to dampen out.
Figure 4. Typical Density Profile—Theoretical and Experimental

\[ \rho = \bar{\rho} (1 - \tilde{\omega} \tanh \alpha z) \]
\[ \bar{\rho} = 1.0118 \]
\[ \tilde{\omega} = 0.01408 \]
\[ \alpha = 0.320 \]

Figure 5. Thickening of the Gradient Layer Caused by Transits of the Vertical Flat Plate
Plate and Towing System

The vertical flat plate was a Plexiglass strip 2.54 cm high, 35.0 cm wide, and 2 mm thick. This was suspended below a carriage by two Plexiglass struts which had been machined to an easy shape to minimize their effect on the flow. The carriage rolled on four double-race ball-bearing roller skate wheels along rails made from aluminum angle. The towing mechanism consisted of a variable-speed motor at one end of the channel and a system of counterweights at the other end. The motor and the counterweights were connected to the carriage over pulleys by size 20 Hygrade flexible fiberglass cord.* Of several types of cords and cables tried, this material proved to be the most satisfactory for minimizing jerkiness in the towing motion. Considerable time was spent in trial and error with the towing system in an attempt to achieve smooth towing motion. Although towing motion was made adequately smooth for the range of plate speeds considered in this study, it was less than adequate at very slow speeds. During an experimental run, the motor was reversed so that the cord unwound from the motor pulley and the counterweight pulled the carriage along the rails. After each run, the plate was retrieved to its original starting position in preparation for the next run. Figure 6 is a schematic drawing of the plate and towing system.

![Figure 6. Plate and Towing System](image)

Early in the study, some qualitative experiments were run with two other body shapes: a 2.54 cm-diameter circular cylinder, and a 2.54 cm-thick lenticular cylinder. The vertical flat plate was chosen for all quantitative experiments, because it produced the strongest upstream influence.

* The cord is manufactured by Markel & Sons of Norristown, Pa.
Flow Visualization

Visualization of fluid motion in the gradient layer was aided by injecting droplets of a mixture of carbon tetrachloride, mineral oil, and oil red dye into the upper layer with a syringe and hypodermic needle. The density of the mixture was such that the droplets were neutrally buoyant at the middle of the gradient layer. The syringe and hypodermic needle were required both to control droplet size and to prevent the mixture from being held at the free surface by surface tension. Droplets of a mixture containing toluene instead of mineral oil were tried earlier in the study. That mixture was rejected because of the toxicity of toluene fumes, high volatility of the toluene which made it difficult to control the mixture density, and the mixture's tendency to dissolve and permanently discolor the Plexiglass channel.

In addition to the droplets in the gradient layer, upstream flow visualization was accomplished by the use of vertical dye lines. These dye lines were created by dropping nigrosine dye crystals upstream of the approaching body. Arrival of the upstream disturbance was made visible by the horizontal displacement of the dye lines. When the first set of dye lines became so distorted that accurate measurement of their displacement was no longer possible, a second group of dye crystals was dropped. The second drop created dye lines which made it possible to evaluate the flow in the near upstream vicinity of the plate. Figure 7 is a photograph showing the displacement of dye lines due to the arrival of an upstream disturbance.

Measurement of Upstream Velocity Profiles

Upstream velocity profiles were evaluated by photographing the droplets and dye lines at known time intervals and measuring the displacement which they experienced during those intervals. As figure 7 shows, the displacements were measured with respect to a rectangular grid fixed to the rear wall of the channel. Also visible in the figure is one of the three stopclocks used. These clocks were started simultaneously with the start of the plate towing motor so that they measured the total duration of the run as well as the time intervals between photographs. The clock shown, an Aristo stopclock, measures time in minutes and seconds. Also used, but not shown in the figure, was a larger stopclock (manufactured by the A. W. Haydon Co.) which measures time in seconds and hundredths of seconds. The time intervals between photographs were measured with an accuracy of ±0.01 seconds.

As shown schematically in figure 8, two cameras with overlapping fields of view were used so that detailed photographs were made of a section of the channel about 125 cm long. Nikon-F cameras were used with 135 mm Nikkor telephoto lenses. Each camera had a motorized shutter release and film advance which made possible short time intervals between successive exposures. Camera motors were connected in parallel to a single switch so that the cameras made simultaneous exposures. The cameras were focused on the mid-plane of the channel, so that both the rectangular grid and the clocks were in focus.
Figure 7. Displacement of Vertical Dye Lines Caused by the Arrival of an Upstream Disturbance
Typical shutter and aperture settings were 1/250 second at \( f/4.0 \), using Hi-Speed Ektachrome color film developed for an effective film speed of ASA 400.

**Measurement of Downstream Velocity Profiles**

The measurement of downstream velocity profiles was considerably less accurate, not only because no dye crystals were dropped there, but also because the mixing in the wake combined with the sudden change of index of refraction in the gradient region to cause such optical distortion that accurate displacement measurements were impossible. By observing the motion of the oil droplets, however, qualitative assessments of the downstream flow could be made. The most striking feature of the downstream flow was the presence of a reverse jet in the gradient layer, which carried oil droplets about 75 cm downstream with respect to the fixed cameras before dissipating. This reverse jet is discussed in the section, **Discussion of Results**.

**Measurement of Body Speed**

The speed of the vertical flat plate was determined by two methods. By stopping the towing motor and the clocks simultaneously at the completion of a run, the body speed could be computed directly from the length and duration of the run. Secondly, the body speed could be computed as the slope of a straight line determined from a least-squares fit of photographically recorded body positions and elapsed times. Excellent agreement was obtained between the two methods.
**Measurement of Speed of Disturbance Front**

The least-squares method was also used to determine the propagation speed of the front of the upstream disturbance.

**Parameters for Experimental Runs**

Table 1 lists values of the density and flow parameters for the ten quantitative experimental runs. Data from the last six runs are considered most reliable.
<table>
<thead>
<tr>
<th>Run</th>
<th>$\rho$</th>
<th>$\alpha$</th>
<th>$\alpha^{-1}$</th>
<th>$U$</th>
<th>$U$</th>
<th>$F$</th>
<th>$F_i$</th>
<th>$\frac{Ri}{(Ri)^2}$</th>
<th>$N^2$</th>
<th>$N_2$</th>
<th>$N$</th>
<th>$\frac{\omega}{(g/\alpha)^{1/2}}$</th>
<th>$\frac{\rho U - \rho_{\mu}}{\rho_{\mu} + \rho_{U}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>1.0053</td>
<td>0.00771</td>
<td>0.50</td>
<td>2.00</td>
<td>2.07</td>
<td>1.94</td>
<td>1.94</td>
<td>0.0718</td>
<td>1.94</td>
<td>1.49</td>
<td>1.94</td>
<td>0.818</td>
<td>0.818</td>
</tr>
<tr>
<td>14</td>
<td>1.0053</td>
<td>0.00771</td>
<td>0.48</td>
<td>2.07</td>
<td>1.95</td>
<td>1.90</td>
<td>1.90</td>
<td>0.0432</td>
<td>1.90</td>
<td>4.14</td>
<td>1.90</td>
<td>0.492</td>
<td>0.492</td>
</tr>
<tr>
<td>16</td>
<td>1.0127</td>
<td>0.01417</td>
<td>2.44</td>
<td>0.41</td>
<td>1.94</td>
<td>0.94</td>
<td>0.94</td>
<td>0.0968</td>
<td>0.94</td>
<td>1.51</td>
<td>0.94</td>
<td>0.813</td>
<td>0.813</td>
</tr>
<tr>
<td>17</td>
<td>1.0127</td>
<td>0.01417</td>
<td>1.09</td>
<td>0.92</td>
<td>3.80</td>
<td>3.68</td>
<td>3.68</td>
<td>0.1268</td>
<td>3.68</td>
<td>0.88</td>
<td>3.68</td>
<td>1.065</td>
<td>1.065</td>
</tr>
<tr>
<td>18</td>
<td>1.0127</td>
<td>0.01417</td>
<td>0.73</td>
<td>1.37</td>
<td>2.60</td>
<td>3.76</td>
<td>3.76</td>
<td>0.0710</td>
<td>3.76</td>
<td>3.18</td>
<td>3.76</td>
<td>0.596</td>
<td>0.596</td>
</tr>
<tr>
<td>19</td>
<td>1.0127</td>
<td>0.01417</td>
<td>0.65</td>
<td>1.54</td>
<td>2.60</td>
<td>4.22</td>
<td>4.22</td>
<td>0.0670</td>
<td>4.22</td>
<td>3.16</td>
<td>4.22</td>
<td>0.563</td>
<td>0.563</td>
</tr>
<tr>
<td>20</td>
<td>1.0127</td>
<td>0.01417</td>
<td>0.65</td>
<td>1.54</td>
<td>2.60</td>
<td>4.22</td>
<td>4.22</td>
<td>0.0670</td>
<td>4.22</td>
<td>3.16</td>
<td>4.22</td>
<td>0.563</td>
<td>0.563</td>
</tr>
<tr>
<td>21</td>
<td>1.0118</td>
<td>0.01408</td>
<td>0.32</td>
<td>3.13</td>
<td>3.16</td>
<td>1094.</td>
<td>1094.</td>
<td>0.0598</td>
<td>1094.</td>
<td>2.10</td>
<td>1094.</td>
<td>0.504</td>
<td>0.504</td>
</tr>
<tr>
<td>22</td>
<td>1.0118</td>
<td>0.01408</td>
<td>0.31</td>
<td>3.28</td>
<td>3.16</td>
<td>1094.</td>
<td>1094.</td>
<td>0.0558</td>
<td>1094.</td>
<td>2.05</td>
<td>1094.</td>
<td>0.470</td>
<td>0.470</td>
</tr>
<tr>
<td>23</td>
<td>1.0118</td>
<td>0.01408</td>
<td>0.30</td>
<td>3.38</td>
<td>3.16</td>
<td>1094.</td>
<td>1094.</td>
<td>0.0709</td>
<td>1094.</td>
<td>2.02</td>
<td>1094.</td>
<td>0.598</td>
<td>0.598</td>
</tr>
<tr>
<td>24</td>
<td>1.0118</td>
<td>0.01408</td>
<td>0.29</td>
<td>3.51</td>
<td>3.16</td>
<td>1094.</td>
<td>1094.</td>
<td>0.0708</td>
<td>1094.</td>
<td>1.98</td>
<td>1094.</td>
<td>0.597</td>
<td>0.597</td>
</tr>
</tbody>
</table>

Table 1: Flow Parameters for Experimental Runs
DISCUSSION OF RESULTS

Qualitative Description of Observed Flow

Figure 9 is a schematic drawing which shows the type of flow observed to be caused by the motion of a vertical flat plate in a region of sudden density variation.

![Schematic Drawing](image)

At the front of the upstream disturbance an internal solitary wave of the type predicted by Benjamin (1967) and observed by Davis (1967) was always present. Behind the solitary wave was a region of upstream influence. This upstream influence did not take the form of complete blocking such as has been observed in the very slow motion of bodies through stratified fluids. Nevertheless, a strong upstream wake was observed in the gradient layer. (The solitary wave and the upstream influence when discussed together are called here the upstream disturbance.) Both the solitary wave and the upstream influence propagated upstream to steadily increasing distances in front of the plate. This is consistent with the internal wave concept of upstream influence, discussed in the Introduction. The connection, if any, between the upstream influence and the solitary wave is not entirely clear. Present indications are, however, that the solitary wave is a distinct phenomenon generated by the impulsive start of plate motion.

As shown also in figure 9, the upstream flow in the fresh- and salt-water layers was opposite in direction to that in the gradient layer.

The downstream flow observed in the gradient layer was somewhat surprising. Nearby the body, the flow looked like ordinary wake flow. About 10 cm downstream, however, a reverse jet was observed to carry oil droplets downstream not only with respect to the moving plate, but also with respect to the fixed cameras. The following explanation is suggested: The observed upstream influence did not take the form of complete blocking, and some upstream fluid was sufficiently displaced vertically that it cleared the plate. After passing the plate, the stratified fluid rushed back to its position of neutral buoyancy.
In doing so, it accumulated a residual rearward momentum manifested in the form of the observed lee jet.

The reverse lee jet dissipated about a meter downstream where it was replaced by a system of lee waves in which the oil particles, as seen by the fixed cameras, simply orbited in closed paths.

**Examination of the Internal Solitary Wave**

Figure 10 is a photograph showing the propagation of an internal solitary wave from left to right ahead of the upstream influence which is in turn ahead of the vertical flat plate.

**Comparison With Theory**

The solitary waves which were observed in this study to head the upstream disturbance have been examined in the light of the work of Benjamin (1967) and Davis and Acrivos (1967).

Benjamin predicted that for a region whose density profile is described by \( \rho = \bar{\rho} (1 - \omega \tanh \alpha z) \), an infinitesimal solitary wave of the first mode would propagate with the speed \( c_{sw}^0 = (g\omega / 2\alpha)^{\frac{1}{2}} \). A wave with finite amplitude \( a \) is predicted to propagate with the speed

\[
 c_{sw} = c_{sw}^0 (1 + \frac{3}{5} \alpha a)^{\frac{1}{2}}. \tag{21}
\]

In the present study, the speed \( c_{sw} \) could be determined accurately from photographs by the least-squares method (see the section, Experimental Study). Thus, it was possible to compare Benjamin’s theory with experimental results by measuring the observed solitary wave amplitudes, using (21) to predict wave speeds, and comparing these predictions with experiment. For the purpose of these measurements, the amplitude was defined as the maximum vertical displacement of the streamline at \( z = \alpha^{-1} \). The results of the comparison are presented in table 2.

Considering the crudeness of the experimental amplitude measurements, the agreement between the last two columns of table 2 is quite good.

A correlation between the dimensionless amplitude, \( \alpha \alpha \), and the dimensionless group \( \lambda = g/2\alpha c_{sw}^2 \ln (\rho_s / \rho_u) \) is predicted by Davis’s theory. Moreover, for small values of \( \alpha \alpha \), the relation \( \lambda = 2.0 - 1.2 \alpha \alpha \) is predicted to be a good approximation to this correlation. Figure 11 is a plot of \( \lambda \) vs \( \alpha \alpha \) for the solitary waves observed in the present study. The theoretical predictions of both Davis and Benjamin are plotted in the same figure. The agreement between these theories and the results of the present experiment is quite good. Since the theories have nothing to do with upstream influence, this agreement indicates that the observed solitary waves are generated by the impulsive start of plate motion and are independent of the existence of upstream influence.
<table>
<thead>
<tr>
<th>Run</th>
<th>( \bar{\omega} ) (cm(^{-1}))</th>
<th>( \alpha )</th>
<th>( \alpha^0 = \left( \frac{\bar{g} \bar{\omega}}{2 \alpha} \right)^{1/2} ) (cm/s)</th>
<th>Measured Amplitude ( a ) (cm)</th>
<th>Measured Wave Speed ( c_{sw} ) (cm/s)</th>
<th>Calculated Wave Speed ( c_{sw} ) [Equation (21)] (cm/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>0.00771</td>
<td>0.50</td>
<td>2.75</td>
<td>2.5</td>
<td>3.73</td>
<td>3.64</td>
</tr>
<tr>
<td>14</td>
<td>0.00771</td>
<td>0.48</td>
<td>2.81</td>
<td>0.6</td>
<td>2.95</td>
<td>3.04</td>
</tr>
<tr>
<td>16</td>
<td>0.01417</td>
<td>2.44</td>
<td>1.69</td>
<td>1.5</td>
<td>3.18</td>
<td>3.02</td>
</tr>
<tr>
<td>17</td>
<td>0.01417</td>
<td>1.09</td>
<td>2.52</td>
<td>2.8</td>
<td>4.75</td>
<td>4.24</td>
</tr>
<tr>
<td>19</td>
<td>0.01417</td>
<td>0.73</td>
<td>3.08</td>
<td>1.8</td>
<td>4.11</td>
<td>4.12</td>
</tr>
<tr>
<td>20</td>
<td>0.01417</td>
<td>0.65</td>
<td>3.27</td>
<td>1.3</td>
<td>4.06</td>
<td>4.01</td>
</tr>
<tr>
<td>21</td>
<td>0.01408</td>
<td>0.32</td>
<td>4.64</td>
<td>0.0</td>
<td>4.62</td>
<td>4.64</td>
</tr>
<tr>
<td>22</td>
<td>0.01408</td>
<td>0.31</td>
<td>4.72</td>
<td>0.0</td>
<td>4.65</td>
<td>4.72</td>
</tr>
<tr>
<td>23</td>
<td>0.01408</td>
<td>0.30</td>
<td>4.80</td>
<td>0.6</td>
<td>5.02</td>
<td>5.05</td>
</tr>
<tr>
<td>24</td>
<td>0.01408</td>
<td>0.29</td>
<td>4.88</td>
<td>0.6</td>
<td>5.05</td>
<td>5.13</td>
</tr>
</tbody>
</table>
Figure 11. Correlation of the Parameter $\lambda$ with the Dimensionless Amplitude $\alpha$ for Observed Solitary Waves and Theoretical Predictions of Benjamin and Davis.

Figure 12. Correlation of Solitary Wave Speed $c_{sw}$ with Plate Speed $U$. $c_{sw} = U + 1.0$.
Correlation of Plate Velocity and Wave Speed

A plot of solitary wave speed $c_{sw}$ versus plate speed $U$ appears in figure 12. The correlation between these two quantities is close enough to suggest a contradiction with the above conclusion that the solitary wave is independent of the existence of upstream influence. This writer suggests, however, that the correlation shown in figure 12 is quite indirect: that wave speed $c_{sw}$ depends on the wave amplitude $a$, which depends on the strength of the starting impulse of the plate and on the speed $U$ for which the towing motor is set when it is switched on.

The results are therefore not as clear as one might wish, but the bulk of the evidence seems to indicate that the initially generated solitary wave and the upstream influence do not interact.

Examination of Upstream Influence

Brunt-Vaisala Frequency

Now, recall from the Introduction the significance to the propagation of internal waves of the Brunt-Vaisala frequency $N = (-g/\rho \, dp/dz)^{1/2}$. It was noted that:

(a) For stably stratified fluids, $dp/dz \leq 0$, so that $N > 0$.

(b) No internal wave can be propagated with frequency higher than the local value of $N$.

(c) The direction of propagation of a wave of frequency $\sigma$ is $\cos^{-1}(\sigma/N)$ from the vertical.

Finally, it was pointed out that one may consider upstream influence to be the propagation of an internal wave having zero frequency, and that the direction of propagation of such a wave would be horizontal.

Figure 13 is a plot of $N$ versus elevation $\alpha z$ for a typical density profile of the type considered in this study. Note that $N$ reaches its maximum at the center line $\alpha z = 0$ and drops off rapidly to zero at short distances above and below the centerline. The significance of this is that any wave having a frequency $\sigma$ such that $0 < \sigma < N$ would propagate in an increasingly vertical direction as it left the centerline, but could not propagate beyond the elevation at which $\sigma = N$. Thus, such waves would be trapped within the gradient layer and after multiple reflections would eventually die out. The waves with higher frequencies would be reflected more often and would tend to die out faster.

Phase Velocity of the Upstream Influence

Experimental measurement of the propagation speed $c_{ui}$ of upstream influence was not possible in this study due to the presence of the solitary wave. It was not clear where the disturbance caused by the solitary wave ended and where the front of the upstream
Figure 13. Brunt-Vaisala Frequency $N$ as a Function of Elevation above Centerline in a Region of Sudden Density Variation

Figure 14. Correlation of Predicted Upstream-influence Propagation Speed $c_{ui}$ with Froude Number $F$
influence began. Moreover, the velocity profiles were not observed over a time interval long enough to permit experimental evaluation of their variation with time. A phase velocity for the upstream influence could be determined, however, from the results of the numerical study. Since, as will be shown later, the numerical solutions successfully predicted upstream velocity profiles as a function of time and position, the determination of $c_u$ from these solutions is thought to be quite reliable. Table 3 presents these results. In figure 14, $c_u/U$ is plotted as a function of the Froude number $F = U/(g/\alpha)^{1/2}$.

**Suggested Relationship Between $c_{ui}$ and Group Velocity of Infinitesimal Internal Waves**

The $i^{th}$ component of the group velocity of an infinitesimal internal wave of frequency $\sigma$ and wave number vector $k$ is $(c_{ui}) = \delta\sigma/\delta k_i$, $i = 1, 2, 3$. The relationship $\sigma = \sigma (k_i)$ between frequency and wave number is called the dispersion relation, and is obtained from the analytic solution of an eigenvalue problem formulated from the equations of motion for a stratified fluid with appropriate boundary conditions (see Phillips 1969, pp. 161-165). Phillips points out that for the general case in which the Brunt-Vaisala frequency is an arbitrary function of $z$, simple analytic solutions to the eigenvalue problem are not possible. For the present study, $N(z) = (g \omega \alpha)^{1/2} \text{sech} \alpha z$ (see figure 13), and the solution of the eigenvalue problem is such a formidable task that it has not been attempted here. Nevertheless, it is tempting to suggest that the speed $c_{ui}$ is related to the group velocity of infinitesimal internal waves. The existence of such a relationship is suggested by the results of Kao, Pao, and Wei (1972), who found that when a linearly stratified fluid in a channel suddenly begins to discharge into a line sink, a disturbance propagates upstream from the sink with velocity equal to the group velocity of internal gravity waves of zero horizontal wave number. In view of the analogy between rotating and density stratified flows, a relationship between $c_{ui}$ and group velocity is also suggested by the study of Benjamin and Barnard (1964). They found that the motion of a cavity in a rotating fluid induced an upstream disturbance which propagated at the maximum group velocity for infinitesimal traveling waves. Confirmation of the existence of such a relationship for the conditions of this study will await the solution for the dispersion relation.

**Comparison of Observed Upstream Influence With Predictions of Janowitz and Experiments of Browand and Winant**

The Introduction mentioned the theoretical study by Janowitz (1971) of the slow motion of a vertical flat plate of height $2b$ in a linearly stratified fluid. This study, whose results are valid when $1/Ri \ll Re \ll 1$, predicts a steady-state, upstream blocking column of length $10^{-2} b (Re \cdot Ri)$ composed of cells of closed streamlines. The experiments of Browand and Winant (1972) appear to confirm the prediction of a region of recirculating flow upstream.

In the present study, the body speeds were much higher, with $Ri = O(1)$ and $Re \gg 1$. Upstream influence was definitely present, but not in the extreme form of complete blocking. Nevertheless, it is instructive to calculate the slug lengths $10^{-2} b (Re \cdot Ri)$ for the conditions observed. These are presented in table 4.
Table 3
Propagation Speed of Upstream Influence

<table>
<thead>
<tr>
<th>Run</th>
<th>Body Speed U (cm/s)</th>
<th>Froude Number F U/(g/a)^{1/2}</th>
<th>Solitary Wave Speed c_{sw} (cm/s)</th>
<th>Propagation Speed of Upstream Influence c_{ui} (cm/s)</th>
<th>(c_{ui}/U)</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>3.18</td>
<td>0.0718</td>
<td>3.73</td>
<td>0.225</td>
<td>0.071</td>
</tr>
<tr>
<td>14</td>
<td>1.95</td>
<td>0.0432</td>
<td>2.95</td>
<td>0.857</td>
<td>0.440</td>
</tr>
<tr>
<td>16</td>
<td>1.94</td>
<td>0.0968</td>
<td>3.18</td>
<td>0.058</td>
<td>0.030</td>
</tr>
<tr>
<td>17</td>
<td>3.80</td>
<td>0.1268</td>
<td>4.75</td>
<td>0.080</td>
<td>0.021</td>
</tr>
<tr>
<td>19</td>
<td>2.60</td>
<td>0.0710</td>
<td>4.11</td>
<td>0.216</td>
<td>0.083</td>
</tr>
<tr>
<td>20</td>
<td>2.60</td>
<td>0.0670</td>
<td>4.06</td>
<td>0.325</td>
<td>0.125</td>
</tr>
<tr>
<td>21</td>
<td>3.31</td>
<td>0.0598</td>
<td>4.62</td>
<td>0.724</td>
<td>0.219</td>
</tr>
<tr>
<td>22</td>
<td>3.16</td>
<td>0.0558</td>
<td>4.65</td>
<td>0.800</td>
<td>0.253</td>
</tr>
<tr>
<td>23</td>
<td>4.08</td>
<td>0.0709</td>
<td>5.02</td>
<td>0.354</td>
<td>0.087</td>
</tr>
<tr>
<td>24</td>
<td>4.15</td>
<td>0.0708</td>
<td>5.05</td>
<td>0.324</td>
<td>0.078</td>
</tr>
</tbody>
</table>
Table 4
Length of Upstream Blocking Column as Predicted by the Janowitz Formula for Present Experiments

<table>
<thead>
<tr>
<th>Run</th>
<th>13</th>
<th>14</th>
<th>16</th>
<th>17</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-2} b \cdot (Re \cdot Ri)$ (cm)</td>
<td>3.0</td>
<td>5.0</td>
<td>48.</td>
<td>11.</td>
<td>11.</td>
<td>10.</td>
<td>4.0</td>
<td>4.0</td>
<td>3.0</td>
<td>3.0</td>
</tr>
</tbody>
</table>

These results confirm that Janowitz's theory is indeed inapplicable to this study, for no recirculating flow of the type described was observed within the predicted distances of the plate. Had such closed streamline flow been present, the oil droplets and dye lines would surely have revealed it.

**Influence of the End Wall**

Another way in which this study differs from the experiment of Browand and Winant is that it was unnecessary to correct the data for the finite length of the channel. The front of the upstream disturbance was still several meters from the end wall when the velocity profile photographs were taken.

**Theoretical and Experimental Results for Upstream Flow Field**

The upstream velocity profiles predicted by the numerical study (see section, Theoretical Study) will now be compared with the experimental results. Figures 15 and 16 present the results for two typical experimental runs together with the corresponding theoretical results. Profiles of dimensionless velocity with respect to the plate, $u/U$, are shown at four upstream locations. Agreement between theory and experiment is excellent in the gradient layer. Theoretical flow velocities in the homogeneous layers are in good qualitative agreement with the experimental, although the predicted velocities are somewhat greater than those actually observed.

Figures 17 and 18 show theoretical and experimental results for the variation of centerline ($z = 0$) velocity with upstream position, $ax$. Here again, the agreement between theory and experiment is excellent. Shown on the same plots are the predictions for upstream-velocity variation for potential flow. These plots show that the upstream velocity variation is somewhat like potential flow in the immediate vicinity of the plate, but rapidly deviates from this idealization for $ax > 0.5$.

Finally, these experimental results are compared with the theoretical predictions of Janowitz in figure 19, where dimensionless centerline velocity with respect to the plate, $u/U$, is plotted versus $|x/L^3 f_4^1$. The quantity $L$, which Janowitz defines as $L = b/(U\nu/\beta g)^{1/3}$, can be expressed in terms of the parameters used in the present study as follows.
Figure 15. Theoretical and Experimental Results for Upstream Velocity Profiles, Run 21

Figure 16. Theoretical and Experimental Results for Upstream Velocity Profiles, Run 23
Janowitz considered the density profile

\[ \rho = \bar{\rho} (1 - \beta z) \quad (22a) \]

from which

\[ \frac{d\rho}{dz} = -\bar{\rho} \beta \quad (22b) \]

This study has considered

\[ \rho = \bar{\rho} (1 - \bar{\omega} \tanh \alpha z) \quad (23a) \]

for which

\[ \frac{d\rho}{dz} = -\bar{\rho} \bar{\omega} \alpha \text{sech}^2 \alpha z \quad (23b) \]

or, at \( z = 0 \)

\[ \frac{d\rho}{dz} = -\bar{\rho} \bar{\omega} \alpha \quad (23c) \]
Thus, by analogy between (22b) and (23c), one can define for a plate of height $2b$

$$L = b/(U_r/\omega ag)^{1/3}$$

(24)

Figure 19 again illustrates that Janowitz's theory, which is formulated for high Richardson number and very low Reynolds number, does not apply to the present experiment. Agreement is particularly poor near the body, where Janowitz's theory predicts the existence of a blocking column with recirculating flow.

**Approach to Steady State**

In figures 20 and 21 are plotted, for the two runs and at the four upstream stations previously considered, the variation of the theoretical centerline velocity $u/U$ with dimensionless time $U_at$. The times during which the experimental data were taken are indicated in the figures. These plots indicate that at the time of the experimental measurements the flow near the body had nearly reached steady state, but that farther upstream it was still quite unsteady.
Figure 19. Comparison of Experimental Results for Positional Variation of Centerline Velocity with Janowitz's Theory for Slow-flow Blocking

Figure 20. Approach to Steady State. Dimensionless Centerline Velocity as a Function of Dimensionless Time at Four Upstream Locations, Run 21
Relative Importance of Various Terms in the Theoretical Formulation and Conclusions about the Nature of the Observed Upstream Influence

It will be recalled that the formulation presented in the section, Theoretical Study, was made as general as possible. It treated the flow of a diffusive, viscous, non-Boussinesq, stratified fluid about a vertical flat plate. A series of numerical solutions were run on the computer to evaluate the relative importance of various terms in the governing equations and to determine thereby the nature of the observed upstream influence:

(a) When the diffusive term \(1/(Sc \cdot Re) \nabla^2 \rho\) was neglected from the formulation, the numerical results for the velocity field were absolutely unchanged. This is not surprising, since, for the conditions of these experiments, \(Sc = 800\), while \(Re\) varied from 84 to 1538. Thus, for the conditions of the present study, the flow was nondiffusive.

(b) The Boussinesq approximation resulted in a small but insignificant (1 percent) decrease in the predicted velocities. Evidently, the Boussinesq approximation is a valid assumption for the flow under study.

(c) Neglecting the viscous terms \(1/Re \nabla^2 \eta\) also resulted in a negligibly small (1 percent) decrease in the predicted velocities. Hence, one concludes that the plate speeds were fast enough that the upstream flow could be regarded as inviscid.
Variation in the Froude number $F = U/(g/\alpha)^{1/2}$, however, had an overwhelming impact on the numerical solution. An increase in $F$ resulted in a dramatic reduction in the length of the predicted region of upstream influence.

From these results one can conclude that the flow under study was governed almost entirely by a balance between inertial and gravitational forces. This is a significant result, for it means that the observed upstream influence is of an entirely different regime from the slow-flow blocking studied by Janowitz and others. In the case of these very slow flows, a balance exists between the viscous and gravitational forces.

The effect of the Froude number on the numerical solution is illustrated in figures 22, 23, and 24. Figure 22 shows the predicted variation of centerline velocity with upstream position for $F = 0.02, 0.06, 0.10, \text{ and } 0.20$ and for potential flow. This figure shows that as $F$ increases, the upstream influence becomes weaker and the variation of velocity with position approaches that for potential flow. For very small values of $F$, on the other hand, the upstream solution shows a closed-streamline recirculation region near the body, a result suggestive of the findings of Janowitz. This causes the solution to be somewhat unstable, as is shown clearly in figure 24, where dimensionless velocity at four upstream positions is plotted versus dimensionless time for $F = 0.02$. This result contrasts with the very stable solution for $F = 0.10$, as illustrated in figure 23. For intermediate values of $F$ corresponding to the conditions of our experiments, the numerical solutions were quite stable with no evidence of recirculating flow near the body.

Examination of Downstream Flow Field

The photograph presented as figure 25 shows a dye line created by a crystal that happened to drop downstream of the moving plate. The shape of the dye line in the gradient layer shows very clearly the existence of the reverse lee jet previously described.

As mentioned earlier, there were difficulties in obtaining downstream velocity profiles. The optical distortion in the downstream-gradient layer is evident in the figure. Nevertheless, an attempt was made to determine downstream-velocity profiles from the motion of the oil drops that were not swept upstream by the plate. This attempt is summarized in figures 26 and 27. Figure 26 presents profiles of dimensionless velocity with respect to the plate $u/U$ at four downstream locations. Figure 27 shows the variation of centerline velocity with downstream position $\alpha x$.

The numerical solution, which models a plate of finite thickness, failed to predict the existence of the reverse lee jet. When the numerical model of the plate was shrunk to zero thickness in the $x$-direction, however, the solution for $U\alpha t < 40$ did indeed show the beginning of the development of a reverse downstream wake. Unfortunately, the singularities at the tips of the infinitesimally thin plate created a disturbance which caused the numerical solution to become noticeably unstable both upstream and downstream for $U\alpha t = 40$ and to become increasingly degenerate with time. The problem of the numerical solution of the equations of motion near mathematically sharp edges has been studied by Lugt and
Ohring (1971). They found that they could avoid numerical instability only by manipulating their coordinate system so that the singularity did not coincide with a grid point. This is not a feasible alternative for the geometry of the present study.

Thus, although qualitative prediction of the nature of the downstream flow could be accomplished by numerical modeling of a plate of infinitesimal thickness, that procedure created more problems than it solved.

An Examination of the Numerical Study

In summary, the numerical model successfully predicted the observed flow, including:

(a) The wavelike propagation of the upstream influence to steadily increasing distances in front of the plate

(b) The correct shape and magnitude of upstream velocity profiles.

Moreover, the model permitted identification of the observed upstream influence as the product of a balance between inertial and gravitational forces, thus distinguishing it from the recirculating blocking column produced when viscous and gravitational forces predominate.
Figure 23. Stable Approach to Steady State for $F = 0.10$, Showing Dimensionless Centerline Velocity as a Function of Dimensionless Time at Two Upstream Locations

Figure 24. Instability of the Numerical Solution for $F = 0.02$, Showing Dimensionless Centerline Velocity as a Function of Dimensionless Time at Four Upstream Locations
Figure 26. Typical Experimental Results for Downstream Velocity Profiles

Figure 27. Typical Experimental Results for the Variation of Centerline Velocity with Downstream Position
The numerical model was unsuccessful, however, in predicting the propagation of an internal solitary wave ahead of the upstream influence. Evidently, to successfully predict the generation of such a wave, more work is needed on accurate numerical representation of the impulsive start of the plate.

The model also failed to describe the existence of the reverse lee jet. As mentioned above, this could be accomplished by making the plate infinitesimally thin, but the procedure introduced singularities which eventually destroyed the numerical solution both upstream and downstream.

CONCLUSIONS

Summary of Findings

The significant conclusions of this study are as follows:

(a) The increase during a series of experimental runs of the gradient-layer thickness $\alpha^{-1}$ was very nearly a linear function of the number of transits made through the layer by the vertical flat plate (see figure 5).

(b) The observed upstream influence propagated as an internal wave of zero frequency whose wave speed depended on the Froude number $F = U/(g/\alpha)^{1/2}$.

(c) The behavior of the observed upstream influence can be adequately predicted by a numerical solution of the equations formulated in the section, Theoretical Study. This includes a description of the propagation of the upstream influence to steadily increasing distances from the obstacle, and the prediction of the shape and magnitude of the upstream velocity profiles.

(d) The Boussinesq approximation can properly be used in the formulation of equations to describe the observed upstream influence; moreover, neither mass diffusion nor viscosity are important in the upstream flow.

(e) The upstream influence observed in the present experiments has been identified as the result of a balance between inertial and gravitational forces. Furthermore, no recirculating flow was observed upstream. The observed flow is thus of a different regime from the recirculating blocking column produced in very slow flows where viscous and gravitational forces predominate.

(f) The impulsive start of a vertical flat plate in a region of suddenly varying density will produce an internal solitary wave (Davis and Acrivos 1967; Benjamin 1967). The numerical flow model used in this study, however, fails to predict the generation of this wave.

(g) A reverse lee jet was observed downstream of the moving plate. The mechanism for the formation of the jet is thought to be a buoyancy effect. The present numerical flow model with its plate of finite thickness fails to predict the existence of the lee jet.
Suggestions for Future Research

The dilemma confronted in attempting to predict numerically the existence of the reverse lee jet, without destroying the solution with singularities, suggests that further study is needed in the area of numerical flow modeling in the vicinity of mathematically sharp edges.

Another area needing study is the internal structure of the upstream flow when \( \text{Re} = O(1) \) so that the inertial and viscous forces are of the same order. Of interest would be the nature of the transition between a recirculating viscous blocking column and a laminar, inertial upstream influence.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland      May 1973
502-21-28-01-51
NOMENCLATURE

A  overrelaxation constant
a, c constants used in the mapping from stretched to real coordinates
a  solitary-wave amplitude
b  half-height of the vertical flat plate
cg  internal-wave group velocity
cw  propagation speed of an infinitesimal, internal solitary wave
csw propagation speed of an internal solitary wave of finite amplitude
cui propagation speed of the upstream influence
D  mass diffusion coefficient
F  U/(g/α)^½, Froude number
Fi  U/(g Δρ/ρ H)^½, or (1/Ri)^½, internal Froude number
g  acceleration of gravity
H  symbol used by Long for total depth of a stratified fluid
i  x-grid index in the finite-difference solution
j  z-grid index in the finite-difference solution
k  wave number vector of an internal wave
L  b/(Uυ/βg)^1/3, dimensionless parameter used by Janowitz
N  (-g/ρ dρ/dz)^½, Brunt-Vaisala frequency
p  pressure
Re  U/αν, Reynolds number
Ri  (N/Uα)^2, Richardson number
s  number of iterations through overrelaxation routine
Sc ν/D, Schmidt number
t  time
τ  Uαt, dimensionless time
u  horizontal fluid velocity with respect to the vertical flat plate
ũ  u/U, dimensionless horizontal velocity
NOMENCLATURE (Continued)

\( \tilde{u} \)  dimensionless horizontal velocity with respect to stretched coordinates
\( U \) velocity of the vertical flat plate with respect to the experimenter
\( w \) vertical fluid velocity with respect to the vertical flat plate
\( \tilde{w} \) \( w/U \), dimensionless vertical velocity
\( \overline{w} \) dimensionless vertical velocity with respect to stretched coordinates
\( x, z \) Cartesian coordinates
\( \tilde{x}, \tilde{z} \) \( \alpha x, \alpha z \), dimensionless Cartesian coordinates
\( \tilde{x}, \tilde{z} \) stretched coordinates
\( \alpha \) a parameter of the density profile; \( \alpha^{-1} \) is a measure of the gradient-layer thickness
\( \beta \) \( -1/\tilde{\rho} \left( \frac{d\tilde{\rho}}{dz} \right) \), symbol for constant-density gradient used by those investigators who have considered linear density profiles
\( \Delta t \) finite-difference time increment
\( \Delta x \) finite-difference x-grid mesh (stretched coordinates)
\( \Delta z \) finite-difference z-grid mesh (stretched coordinates)
\( \zeta \) \( z \) component of vorticity
\( \eta \) \( y \) component of vorticity
\( \tilde{\eta} \) \( \eta/U\alpha \), dimensionless vorticity
\( \lambda \) \( \left( \frac{g}{2 \alpha c_{sw}^{2}} \right) \ln \left( \frac{\rho_{s}/\rho_{u}}{\rho_{s}} \right) \), dimensionless parameter used by Davis in his study of internal solitary waves
\( \mu \) viscosity
\( \nu \) \( \mu/\rho \), kinematic viscosity
\( \xi \) \( x \) component of vorticity
\( \rho \) fluid density
\( \tilde{\rho} \) fluid density at the horizontal centerline, \( z = 0 \)
\( \overline{\rho} \) \( \rho/\tilde{\rho} \), dimensionless density
\( \rho_{g} \) density of the homogeneous salt-water layer
\( \rho_{u} \) density of the homogeneous fresh-water layer
\( \omega \) frequency of an internal wave
NOMENCLATURE (Continued)

ψ  stream function defined by \( u = \partial \psi / \partial z, \ w = -\partial \psi / \partial x \)

\( \tilde{\psi} \)  \( \psi \alpha/U \), dimensionless stream function

\( \bar{a} \) a parameter of the density profile \( (\rho_\lambda - \rho_u)/(\rho_\lambda + \rho_u) \)

\( \bar{z} \) vorticity \( (\nabla \times \bar{\Omega} = (\xi, \eta, \zeta)) \)
REFERENCES


Ekman, V. W., 1904, "On Dead Water," *Scientific Results of the Norwegian North Polar Expedition, Pt. XV.*


APPENDIX 1

TRANSFORMED GOVERNING EQUATIONS

The coordinate transformation (19) applied to the governing equations (20) produces the following transformed system. The continuity equation becomes

$$\frac{\partial \rho}{\partial t} + a(1 - \bar{x}^2) \cdot c(1 - \bar{z}^2) \left[ \frac{\partial}{\partial \bar{x}} (\bar{u}\rho) + \frac{\partial}{\partial \bar{z}} (\bar{w}\rho) \right] = \frac{1}{Sc \cdot Re} \left\{ a^2 (1 - \bar{x}^2)^2 \frac{\partial^2 \rho}{\partial \bar{x}^2} - 2a^2 \bar{x} (1 - \bar{x}^2) \frac{\partial \rho}{\partial \bar{x}} + c^2 (1 - \bar{z}^2)^2 \frac{\partial^2 \rho}{\partial \bar{z}^2} - 2c^2 \bar{z} (1 - \bar{z}^2) \frac{\partial \rho}{\partial \bar{z}} \right\}$$

(A-1)

The transformed momentum equation is

$$c(1 - \bar{z}^2) \frac{\partial \rho}{\partial \bar{z}} \left[ c(1 - \bar{z}^2) \frac{\partial \bar{u}}{\partial t} + a(1 - \bar{x}^2) \cdot c(1 - \bar{z}^2)^2 \frac{\partial}{\partial \bar{x}} (\bar{u}^2) \right] + a(1 - \bar{x}^2) \cdot c^2(1 - \bar{z}^2) \left[ (1 - \bar{z}^2) \frac{\partial}{\partial \bar{z}} (\bar{u} \bar{w}) - 2 \bar{u} \bar{w} \bar{x} \right]$$

$$- a(1 - \bar{x}^2) \frac{\partial \rho}{\partial \bar{x}} \left\{ a(1 - \bar{x}^2) \frac{\partial \bar{w}}{\partial \bar{t}} + a^2(1 - \bar{x}^2) \cdot c(1 - \bar{z}^2) \left[ (1 - \bar{x}^2) \frac{\partial}{\partial \bar{x}} (\bar{u} \bar{w}) - 2 \bar{u} \bar{w} \bar{x} \right] \right\} + a^2(1 - \bar{x}^2)^2 \cdot c(1 - \bar{z}^2) \frac{\partial}{\partial \bar{x}} (\bar{w}^2) \right\} + \rho \left\{ \frac{\partial \bar{\eta}}{\partial \bar{t}} + a(1 - \bar{x}^2) \cdot c(1 - \bar{z}^2) \left[ \frac{\partial}{\partial \bar{x}} (\bar{u} \bar{\eta}) + \frac{\partial}{\partial \bar{z}} (\bar{w} \bar{\eta}) \right] \right\}$$

$$= a(1 - \bar{x}^2) \frac{1}{F^2} \frac{\partial \rho}{\partial \bar{x}} + \frac{1}{Re} \left\{ a^2(1 - \bar{x}^2)^2 \frac{\partial^2 \bar{\eta}}{\partial \bar{x}^2} - 2a^2 \bar{x} (1 - \bar{x}^2) \frac{\partial \bar{\eta}}{\partial \bar{x}} + c^2(1 - \bar{z}^2)^2 \frac{\partial^2 \bar{\eta}}{\partial \bar{z}^2} - 2c^2 \bar{z} (1 - \bar{z}^2) \frac{\partial \bar{\eta}}{\partial \bar{z}} \right\}$$

(A-2)

The velocities are

$$\bar{u} = \frac{\partial \psi}{\partial \bar{z}}, \quad \bar{w} = -\frac{\partial \psi}{\partial \bar{x}}$$

(A-3)

and the vorticity expressed as the Laplacian of the stream function is

$$\bar{\eta} = a^2(1 - \bar{x}^2)^2 \frac{\partial^2 \psi}{\partial \bar{x}^2} - 2a^2 \bar{x} (1 - \bar{x}^2) \frac{\partial \psi}{\partial \bar{x}} + c^2(1 - \bar{z}^2)^2 \frac{\partial^2 \psi}{\partial \bar{z}^2} - 2c^2 \bar{z} (1 - \bar{z}^2) \frac{\partial \psi}{\partial \bar{z}}$$

(A-4)
The transformed boundary conditions for $t > 0$ are

$$
as |\bar{x}|, |\bar{z}| \to 1, \rho \to \left[ 1 - \frac{1}{c} \arctanh \left( \frac{\bar{z}}{\bar{x}} \right) \right], \quad |\psi| \to \infty \quad (A5)$$

and at $\bar{x} = 0$, $|\bar{z}| \leq \tanh \coth$,

$$
\frac{\partial \rho}{\partial \bar{x}} = 0, \quad \frac{\partial \psi}{\partial \bar{x}} = 0. \quad (A6)
$$

The transformed initial conditions are

$$
\rho = \left[ 1 - \frac{1}{c} \arctanh \left( \frac{1}{c} \arctanh \frac{\bar{z}}{\bar{x}} \right) \right] \text{ everywhere at } t = 0 \quad (A8)
$$

$$
\frac{\partial \psi}{\partial \bar{x}} = \frac{\partial \psi}{\partial \bar{z}} = 0 \quad \text{ everywhere at } t = 0 \quad (A9)
$$
APPENDIX 2

FINITE-DIFFERENCE FORM OF TRANSFORMED GOVERNING EQUATIONS

The stream function $\psi$ is determined by overrelaxation of (A4) subject to the boundary conditions (A5) and (A7). The overrelaxation scheme is given by

$$\psi_{i,j}^{(s+1)} = \psi_{i,j}^{(s)} + A \left( \psi_{i,j}^{(s)} - \psi_{i,j}^{(s+1)} \right) \quad (B1)$$

where $i$ and $j$ are the $x$ and $z$ grid indices, respectively, so that $\psi_{i,j}^{(s+1)}$ is the value computed for the stream function at the grid point $(i, j)$ on the $(s+1)$th iteration. The value of $\psi_{i,j}$ is obtained from the finite-difference form of (A4). If the overrelaxation converges, then as $s$ increases

$$\left( \psi_{i,j} \rightarrow \psi_{i,j}^{(s)} \right) \Rightarrow \left( \psi_{i,j}^{(s+1)} \rightarrow \psi_{i,j}^{(s)} \right).$$

$A$ is an overrelaxation constant whose value affects the speed of convergence but not the final result for $\psi_{i,j}^{(s+1)}$. For this study, $A$ was set equal to 1.75. The overrelaxation equation for the stream function is therefore given by

$$\psi_{i,j}^{(s+1)} = (1 - A) \psi_{i,j}^{(s)} + \left[ 0.5 A \left( a^2 (1 - \bar{x}_i^2)^2 + c^2 (1 - \bar{z}_j^2)^2 \left( \frac{\Delta x}{\Delta z} \right)^{2} \right) + (\Delta x)^2 \eta_{i,j} \right]$$

$$+ a^2 (1 - \bar{x}_i^2)^2 (\psi_{i+1,j} + \psi_{i-1,j}) - a^2 \bar{x}_i (1 - \bar{x}_i^2) \Delta x (\psi_{i+1,j} - \psi_{i-1,j})$$

$$+ c^2 (1 - \bar{z}_j^2)^2 \left( \frac{\Delta x}{\Delta z} \right)^2 (\psi_{i,j+1} + \psi_{i,j-1}) - c^2 \bar{z}_j (1 - \bar{z}_j^2) \left( \frac{\Delta x}{\Delta z} \right)^2 (\psi_{i,j+1} - \psi_{i,j-1}) \quad (B2)$$

The boundary conditions on $\psi$ at the solid boundary of the obstacle are derived as follows. The $\psi = 0$ streamline is coincident with the horizontal centerline of the flow field except at the obstacle where it bifurcates and follows the contour of the solid boundary. The forward-difference expression for the derivative of a function $f$ at a solid boundary is

$$f'(\bar{x}_i) \approx \frac{1}{2(\Delta x)} \left[ -3 f_i + 4 f_{i+1} - f_{i+2} \right] + O \left( (\Delta x)^2 \right) \quad (B3a)$$

and the backward-difference expression is

$$f'(\bar{x}_i) \approx \frac{1}{2(\Delta x)} \left[ 3 f_i - 4 f_{i-1} + f_{i-2} \right] + O \left( (\Delta x)^2 \right) \quad (B3b)$$

(Conte 1965, p. 110). The nonslip condition, together with the additional constraint that
\( \psi = 0 \) along the solid boundary, combines with (B3a) and (B3b) to give the following boundary conditions for \( \psi \):

\[
\begin{align*}
\psi_{M3-1,j} &= \frac{1}{4} \psi_{M3-2,j} & j &= N3, \ldots, N4 \\
\psi_{M4+1,j} &= \frac{1}{4} \psi_{M4+2,j} & j &= N3, \ldots, N4 \\
\psi_{i,N3-1} &= \frac{1}{4} \psi_{i,N3-2} & i &= M3, \ldots, M4 \\
\psi_{i,N4+1} &= \frac{1}{4} \psi_{i,N4+2} & i &= M3, \ldots, M4
\end{align*}
\]

(B4)

where the obstacle is represented as a rectangle contained within the grid lines \( i = M3 \) to \( i = M4 \) and \( j = N3 \) to \( j = N4 \). The \( u \) and \( w \) velocity fields, as given by (A3), become

\[
\bar{u}_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2(\Delta z)}, \quad \bar{w}_{i,j} = \frac{\psi_{i-1,j} - \psi_{i+1,j}}{2(\Delta x)}
\]

(B5)

The continuity equation (A1) can be written in finite-difference form as

\[
\rho'_{i,j} = \rho_{i,j} + \Delta t \left\{ \frac{1}{Sc \cdot Re} \left[ a^2 (1 - \bar{x}_i^2)^2 \frac{\rho_{i+1,j} - 2 \rho_{i,j} + \rho_{i-1,j}}{(\Delta x)^2} \\
- 2 a^2 \bar{x}_i (1 - \bar{x}_i^2) \frac{\rho_{i+1,j} - \rho_{i-1,j}}{2(\Delta x)} + c^2 (1 - \bar{z}_j^2)^2 \frac{\rho_{i,j+1} - 2 \rho_{i,j} + \rho_{i,j-1}}{(\Delta z)^2} \\
- 2 c^2 \bar{z}_j (1 - \bar{z}_j^2) \frac{\rho_{i,j+1} - \rho_{i,j-1}}{2(\Delta z)} \right] - a (1 - \bar{x}_i^2) \cdot c (1 - \bar{z}_j^2) \left[ \frac{\partial}{\partial \bar{x}} \left( \bar{u} \rho \right) \right]_{i,j} \\
+ \left( \frac{\partial}{\partial \bar{z}} \left( \bar{w} \rho \right) \right)_{i,j} \right\}
\]

(B6)

where the unprimed variables represent values at \( t = t \) and \( \rho'_{i,j} \) is the new value of the density at grid point \((i, j)\) and at \( t = t + \Delta t \). As mentioned in the section, Theoretical Study, the nonlinear terms \([\partial / \partial \bar{x} (\bar{u} \rho)]_{i,j}\) and \([\partial / \partial \bar{z} (\bar{w} \rho)]_{i,j}\) were evaluated by the "special three-point noncentral difference method" of Torrance and Rockett (1969).

A derivation analogous to that for the boundary conditions on \( \psi \) gives the following boundary conditions for \( \rho \).
The momentum equation, which is used to evaluate the change of free stream vorticity with time, can be written in finite-difference form as

\[
\begin{align*}
\eta_{i,j}' &= \eta_{i,j} + \Delta t \left\{ a (1 - x_i^2) \cdot c (1 - z_j^2) \left[ \frac{\partial}{\partial x} (\bar{u} \eta) \right]_{i,j} + \left[ \frac{\partial}{\partial z} (\bar{w} \eta) \right]_{i,j} \\
&+ \frac{1}{\rho} \left[ a (1 - x_i^2) \frac{\rho_{i+1,j} - \rho_{i-1,j}}{2(\Delta x)} + \frac{1}{Re} \left( a^2 (1 - x_i^2)^2 \frac{\eta_{i+1,j} - 2 \eta_{i+1} + \eta_{i-1,j}}{(\Delta x)^2} \right) \right. \\
&+ \left. 2 a^2 x_i (1 - x_i^2) \frac{\eta_{i+1,j} - \eta_{i-1,j}}{2(\Delta x)} + c^2 (1 - z_j^2)^2 \frac{\eta_{i,j+1} - 2 \eta_{i,j} + \eta_{i,j-1}}{(\Delta z)^2} \right] \\
&- \frac{a (1 - x_i^2) \cdot c^2 (1 - z_j^2)^2 \left[ \frac{\partial}{\partial x} (\bar{u} \bar{u}) \right]_{i,j} + a (1 - x_i^2) \cdot c^2 (1 - z_j^2) \left[ (1 - z_j^2) \left[ \frac{\partial}{\partial z} (\bar{w} \bar{w}) \right] \right]_{i,j} \\
&+ \frac{a^2 (1 - x_i^2) \cdot c (1 - z_j^2) \left[ (1 - x_i^2) \left[ \frac{\partial}{\partial x} (\bar{u} \bar{w}) \right] \right]_{i,j} - 2 \bar{u}_{i,j} \bar{w}_{i,j} \bar{x}_i}{\Delta t} + \frac{a^2 (1 - x_i^2)^2 \cdot c (1 - z_j^2) \left[ \frac{\partial}{\partial x} (\bar{w} \bar{w}) \right]_{i,j}}{\Delta t} \right\} 
\end{align*}
\]
Finally, the vorticity at the solid boundaries is evaluated from \( \eta = \nabla^2 \psi \) by taking note of the boundary conditions on \( \psi \) and using the forward- and backward-difference expressions for a second derivative at a boundary. These expressions are (Conte 1965, p. 110)

\[
f''(\bar{x}_i) \approx \frac{1}{(\Delta x)^2} \left[ -f_{i+3} + 4f_{i+2} - 5f_{i+1} + 2f_i \right] + O [(\Delta x)^2] \tag{B9a}
\]

\[
f''(\bar{x}_i) \approx \frac{1}{(\Delta x)^2} \left[ -f_{i-3} + 4f_{i-2} - 5f_{i-1} + 2f_i \right] + O [(\Delta x)^2] \tag{B9b}
\]

and the expressions for \( \eta \) at the boundary are

\[
\eta_{M3,j} = a^2 \left( 1 - \bar{x}_i^2 \right)^2 \left( \frac{-\psi_{M3-3,j} + 4\psi_{M3-2,j} - 5\psi_{M3-1,j} + 2\psi_{M3,j}}{(\Delta x)^2} \right), \quad j = N3, \ldots, N4
\]

\[
\eta_{M4,j} = a^2 \left( 1 - \bar{x}_i^2 \right)^2 \left( \frac{-\psi_{M4+3,j} + 4\psi_{M4+2,j} - 5\psi_{M4+1,j} + 2\psi_{M4,j}}{(\Delta x)^2} \right), \quad j = N3, \ldots, N4
\]

\[
\eta_{i,N3} = c^2 \left( 1 - \bar{z}_j^2 \right)^2 \left( \frac{-\psi_{i,N3-3} + 4\psi_{i,N3-2} - 5\psi_{i,N3-1} + 2\psi_{i,N3}}{(\Delta z)^2} \right), \quad i = M3, \ldots, M4
\]

\[
\eta_{i,N4} = c^2 \left( 1 - \bar{z}_j^2 \right)^2 \left( \frac{-\psi_{i,N4+3} + 4\psi_{i,N4+2} - 5\psi_{i,N4+1} + 2\psi_{i,N4}}{(\Delta z)^2} \right), \quad i = M3, \ldots, M4
\]
The computer program used to make the theoretical flow predictions for this study is based on the finite difference equations of Appendix 2. It is written in FORTRAN-IV for use on an IBM 360 computer. It uses double precision, and for the present 81 by 41 grid mesh, requires 620 K bytes of storage and 30 to 40 minutes of central processing unit (CPU) time to model a complete experimental run. Most of the CPU time is spent in the overrelaxation step of the computation. The core and CPU time requirements are considerably reduced for coarser grid meshes, but not without some sacrifice in accuracy.

**INPUT-DATA FORMAT**

Cards 1, 2, 3, and 4 are title cards. A descriptive title can be punched on these cards in columns 1 to 80.

Card 5 contains 12 floating-point data inputs in fields of six columns each:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DX = Δx = finite-difference x-grid mesh (&quot;stretched&quot; coordinates)</td>
<td>DZ = Δz = finite-difference z-grid mesh (&quot;stretched&quot; coordinates)</td>
<td>DT = Δt = dimensionless finite difference time increment</td>
<td>T = time at start of computation</td>
<td>F = Froude number = U/(g/α)^{1/2}</td>
<td>Re = Reynolds number = U/vα</td>
<td>EPS = (Schmidt number)^{-1} = D/ν</td>
<td>DAVG = p = fluid density at z = 0</td>
<td>OMEG = ω = a parameter of the density profile</td>
<td>ALPHA = α = a parameter of the density profile</td>
<td>UBODY = U = velocity of the vertical flat plate with respect to the experimenter</td>
<td>A = overrelaxation constant</td>
</tr>
</tbody>
</table>

Card 6 contains nine fixed point data inputs, right justified, in fields of six columns each:

| Col. | 1-6 | LL = total number of time incrementations to be made during the computation (should be a multiple of 40) |
7-12 MX = total number of x-grid lines; must be less than or equal to the number of storage locations assigned to X(I) in the DIMENSION statement

13-18 M2NEG = downstream thickness of the obstacle in terms of the number of downstream (x < 0) grid lines it contains

19-24 M2POS = upstream thickness of the obstacle in terms of the number of upstream (x > 0) grid lines it contains

25-30 NZ = total number of z grid lines; must be less than or equal to the number of storage locations assigned to Z(J) in the DIMENSION statement

31-36 MLT1 = 0, if Boussinesq approximation is to be made
             = 1, otherwise

37-42 MLT2 = 0, if Boussinesq approximation is to be made
             = 1, otherwise

43-48 MLT3 = 0, if flow is to be assumed inviscid
             = 1, otherwise

49-54 MLT4 = 0, if flow is to be assumed nondiffusive
             = 1, otherwise

Card 7 contains 12 fixed point data inputs, right justified, in fields of six columns each. These input parameters indicate those results which the user has selected to be printed out. If a parameter is zero, the results corresponding to this parameter are not printed out. If the printout is desired, a nonzero integer must be assigned to the parameter. The following list gives each parameter in the order in which it is read in and the result prints out to which it corresponds:

Col.  1 - 6  IPOK - number of iterations K through SUBROUTINE RELAX
       7 - 12 IPOP - stream function \( \psi \) both initial and at later time
       13 - 18 IPOPC - stream function contour elevations
       19 - 24 IPODI - initial density profile at a typical cross-section
       25 - 30 IPODC - density contour elevations
       31 - 36 IPODN - nondimensional density at later time
       37 - 42 IPOD - dimensional density at later time
       43 - 48 IPOV - vorticity field
       49 - 54 IPOUO - nondimensional u-velocity with respect to the obstacle
55 - 60  IPOWO – nondimensional w-velocity with respect to the obstacle
61 - 66  IPOUE – dimensional u-velocity with respect to the experimenter
67 - 72  IPOWE – dimensional w-velocity with respect to the experimenter

COMPUTER PROGRAM LISTING

The following pages present the program listing.
FLOW ABOUT AN OBSTACLE OF A FLUID WITH A TANH DENSITY PROFILE

DAVID A HURDIS

NOVEMBER, 1972

IMPLICIT REAL*8(A-H,O-Z)

DIMENSION IN(80), X(81), XR(81), XRDIM(81), Z(41), ZR(41), ZRDIM(4)
11, DT1(81,41), DT2(81,41), DDIM(81,41), DSQD(81,41), VT1(81,41), MAN
6
2V1, DT2(81,41), DSQV(81,41), U(81,41), UAV(81,41), UULU(81,41), UR(81, MAN
7
341), URDIM(81,41), W(81,41), WAV(81,41), WOLD(81,41), WR(81,41), WMAN
8
4RDIM(81,41), DUUDX(81,41), DWUDDZ(81,41), DUUDDX(81,41), DWUDDZ(81,41 MAN
9
5), DUUDDX(81,41), DUUDDZ(81,41), DUDDX(81,41), DUDDZ(81,41), BOUS1(8 MAN
10
61,41), BOUS2(81,41), P(81,41), XSEC2(81), ZSEC2(41), H(12,81), PCOMAN
11
7NT(12), DCONT(12)

EQUIVALENCE (UR(1,1),UAV(1,1)), (WR(1,1),WAV(1,1)), (BOUS1(1,1),URMAN
16
1DIM(1,1)), (BOUS2(1,1),WAV(1,1)), (DDIM(1,1),DSQD(1,1),DSQV(1,1) MAN
17
2), (DUDDX(1,1),DUUDDX(1,1),DUUDDZ(1,1),DUUDDX(1,1)), (DUDDZ(1,1),DWUDD MAN
18
19
:WRITE (6,1020)

READ AND WRITE 4 TITLE CARDS

READ (5,1030) IN

WRITE (6,1030) IN

READ AND WRITE INPUT

READ (5,1040) DX,DZ,UT,T,F,RE,EP,DAG,OMEQ,ALPHA,UBODY

READ (5,1050) LL,MX,M2NEG,M2POS,NZ,MLT1,MLT2,MLT3,MLT4

CA=ACRTH((1.0D0)/(1.5D2)*ALPHA)

ALF=ACRTH((1.0D0)/(2.28601)*ALPHA)

WRITE (6,1060) DX,DZ,UT,T

WRITE (6,1070) F,RE,EP

WRITE (6,1080) CA,ALF,DAG,OMEQ,ALPHA,UBODY

WRITE (6,1090) A,LL,MX,M2NEG,M2POS,NZ

WRITE (6,1100) MLT1,MLT2,MLT3,MLT4

IF (MLT1) 10,20,30

10 WRITE (6,1110)

GO TO 1010
20 WRITE (6,*1120)  
GO TO 40  
30 WRITE (6,*1130)  
40 IF (MLT2) 50,60,70  
50 WRITE (6,*1140)  
GO TO 1010  
60 WRITE (6,*1150)  
GO TO 80  
70 WRITE (6,*1160)  
80 IF (MLT3) 90,100,110  
90 WRITE (6,*1170)  
GO TO 1010  
100 WRITE (6,*1180)  
GO TO 120  
110 WRITE (6,*1190)  
120 IF (MLT4) 130,140,150  
130 WRITE (6,*1200)  
GO TO 1010  
140 WRITE (6,*1210)  
GO TO 160  
150 WRITE (6,*1220)  
C   READ AND WRITE THE INPUT PARAMETERS WHICH INDICATE THE USER'S  
C   SELECTION OF RESULTS TO BE PRINTED OUT.  
160 READ (5,*1230) IPOK,IPOR,IP02C,IP02I,IP03C,IP03N,IP03D,IP03V,IP04O,IP05N  
160 IP06O,IP07E,IP07W  
WRITE (6,*1240)  
WRITE (6,*1250)  
WRITE (6,*1260)  
WRITE (6,*1270)  
WRITE (6,*1280)  
WRITE (6,*1290)  
WRITE (6,*1300)  
WRITE (6,*1310)  
WRITE (6,*1320)  
C   MAN 37  
C   MAN 38  
C   MAN 39  
C   MAN 40  
C   MAN 41  
C   MAN 42  
C   MAN 43  
C   MAN 44  
C   MAN 45  
C   MAN 46  
C   MAN 47  
C   MAN 48  
C   MAN 49  
C   MAN 50  
C   MAN 51  
C   MAN 52  
C   MAN 53  
C   MAN 54  
C   MAN 55  
C   MAN 56  
C   MAN 57  
C   MAN 58  
C   MAN 59  
C   MAN 60  
C   MAN 61  
C   MAN 62  
C   MAN 63  
C   MAN 64  
C   MAN 65  
C   MAN 66  
C   MAN 67  
C   MAN 68  
C   MAN 69  
C   MAN 70  
C   MAN 71  
C   MAN 72
WRITE (6, 1330) IPOP, C
WRITE (6, 1340) IPODI C
WRITE (6, 1350) IPODC C
WRITE (6, 1360) IPODN C
WRITE (6, 1370) IPOD C
WRITE (6, 1380) IPOV C
WRITE (6, 1390) IPOOU C
WRITE (6, 1400) IPOWO C
WRITE (6, 1410) IPOUE C
WRITE (6, 1420) IPOWE C

C DOUBLE PRECISION CONSTANTS
D0 = 0.0D0 MAN 73
DP25 = 2.5D-1 MAN 74
DP5 = 5.0D-1 MAN 75
DP75 = 7.5D-1 MAN 76
D1 = 1.0D0 MAN 77
D2 = 2.0D0 MAN 78
D3 = 3.0D0 MAN 79
D4 = 4.0D0 MAN 80
D5 = 5.0D0 MAN 81
D6 = 8.0D0 MAN 82

C EVALUATION OF 'STRETCHED' REAL COORDINATES, (XR, ZR) AND
C OF TRANSFORMED COORDINATES, (X, Z)
M1 = MX - 1 MAN 83
M2 = (MX + 1) / 2 MAN 84
M3 = M2 - M2NEG MAN 85
M4 = M2 + M2POS MAN 86
WRITE (6, 1430) MAN 87
DO 170 I = 1, MX MAN 88
X(I) = DX * DFLOAT(I - M2) MAN 89
XSEC2(I) = D1 - X(I) ** 2 MAN 90
Y = X(I) MAN 91
XR(I) = (D1 / CA) * ARCTNH(Y) MAN 92
XR(I) = (-1.0D5) * ALPHA
XR(MX) = (1.0D5) * ALPHA

C XRDM=DIMENSIONAL FORM OF XR
XRDM(I) = XR(I) / ALPHA
WRITE (6,1440) I, XR(I), XRDM(I), XSEC2(I)

170 CONTINUE
N1 = NZ - 1
N11 = N1 - 1
N2 = (NZ + 1) / 2
WRITE (6,1450)
DO 180 J = 1, NZ
Z(J) = DZ*DFLOAT(J - N2)
ZSEC2(J) = DI - Z(J)**2
Y = Z(J)
ZR(J) = (DI/ALF)*ARCTNH(Y)
ZR(I) = (-1.0D5) * ALPHA
ZR(NZ) = (1.0D5) * ALPHA

C ZRDM=DIMENSIONAL FORM OF ZR
ZRDM(J) = ZR(J) / ALPHA
WRITE (6,1460) J, Z(J), ZR(J), ZRDIM(J), ZSEC2(J)

180 CONTINUE
C DETERMINATION OF GRID AREA OCCUPIED BY OBSTACLE
ER = (1.2700) * ALPHA
E = DTANH(ALF*ER)
EGRD = E/DZ
EGRD1 = EGRD*DPS
EGRD = DFLOAT(IDINT(EGRD))
EGRD1 = DFLOAT(IDINT(EGRD1))
IF (EGRD*LT. EGRD1) EGRD = EGRD1
E = EGRD*DZ
N3 = N2 - IDINT(EGRD)
N4 = N2 + IDINT(EGRD)
N31 = N3 - 1
N41 = N4 + 1
C FIRST GUESS FOR INITIAL STREAM FUNCTION, P(I,J)
DO 190 I=1,MX
DO 190 J=1,NZ
190 P(I,J)=-ZR(J)
DO 220 I=M3,M4
DO 200 J=N3,N4
200 P(I,J)=0
DO 210 J=1,N31
210 P(I,J)=((E+Z(J))/(E-D1))*(-ZR(J))
DO 220 J=N41,NZ
220 P(I,J)=((E-Z(J))/(E-D1))*(-ZR(J))

C OVER-RELAXATION FOR INITIAL STREAM FUNCTION, P(I,J)
KK=31
DO 230 I=1,MX
DO 230 J=1,NZ
C FLOW IS INITIALLY IRROTATIONAL, HENCE...
230 VT1(I,J)=0
IF (IPOK.EQ.0.AND.IPOP.EQ.0) GO TO 240
WRITE (6,1470)
240 CONTINUE
CALL RELAX (AX,Z,ZR,DX,DZ,CA,ALF,P,VT1,KK,MX,M3,M4,NZ,N3,N4,IPOK)

C FLOW OUT OF INITIAL TIME
DIMT=T/(ALPHA*UBODY)
IF (IPOP.EQ.0) GO TO 260
WRITE (6,1600) T,DIMT
C WRITE INITIAL STREAM FUNCTION, P(I,J)
WRITE (6,1480)
DO 250 I=1,MX
250 WRITE (6,1490) I,P(I,J),J=1,NZ
260 CONTINUE
C COMPUTE CONTOUR ELEVATIONS FOR INITIAL STREAMLINES
IF (IPOPC.EQ.0) GO TO 320
MESH=5
KZ=2*MESH+2
DO 270 K=1,KZ
PCONT(K)=0
C
DO 270 I=1, MX
270 H(K,I)=D0
NPM=NZ-IDINT((DFLOAT(NZ)-D1)/D8)
PMAK=DFLOAT(IDINT(-D1*P(1,NPM)))
DO 280 J=1,4
IF (DABS(P(1,NPM)+PMAK).LT.DP25) GO TO 290
PMAK=PMAK+DP25
280 CONTINUE
290 DP=PMAK/DFLOAT(MESH)
CALL CONTR (MX,M3,M4,NZ,N3,N4,MESH,Z,DZ,P,DP,D0,PCONT,H)
C
PUT CONTOUR ELEVATION, H, IN DIMENSIONAL FORM
DO 300 K=1,KZ
DO 300 I=1, MX
Y=H(K,I)
300 H(K,I)=(D1/ALF)*ARCTNH(Y)/ALPHA
WRITE (6,1600) T,DIMT
C
PRINT-OUT OF CONTOUR ELEVATIONS FOR INITIAL STREAMLINES
DO 310 K=1,KZ
WRITE (6,1500) PCONT(K),MX
310 WRITE (6,1510) (H(K,I),I=1, MX)
320 CONTINUE
C
INITIAL SOLUTION FOR DENSITY FIELD. DT1=DENSITY AT OLD TIME, AND MAN 200
C
DT2=DENSITY AT NEW TIME
DO 330 I=1, MX
DO 330 J=2, N1
330 DT1(I,J)=D1-OMEG*DTANH(ZR(J))
DO 340 I=1, MX
DT1(I,1)=DT1(I,2)
340 DT1(I,NZ)=DT1(I,N1)
DO 350 I=1, MX
DO 350 J=1, NZ
DT2(I,J)=DT1(I,J)
350 CONTINUE
C CALCULATION AND PRINT-OUT OF INITIAL DIMENSIONAL DENSITY FIELD AT MAN 212
C A TYPICAL CROSS-SECTION.
IF (IPODI.EQ.0) GO TO 370
WRITE (6,1470)
WRITE (6,1600) T,DIMT
WRITE (6,1520)
WRITE (6,1530)
DO 360 J=1,NZ
       J1=NZ+1-J
       DDIM(2,J1)=DT1(2,J1)*DAVG
360 WRITE (6,1540) J1,DDIM(2,J1)
370 CONTINUE
C COMPUTE CONTOUR ELEVATIONS FOR INITIAL LINES OF CONSTANT DENSITY MAN 224
IF (IPODC.EQ.0) GO TO 410
MESH=5
KZ=2*MESH+2
DO 380 K=1,KZ
       DCONT(K)=0
       DO 380 I=1,MX
       H(K,I)=0
       DD=(DT1(1,7)-D1)/DFLOAT(MESH)
       CALL CONTUR (MX,M3,M4,NZ,N3,N4,MESH,Z,DZ,DT1,DD,U1,DCONT,H)
380 H(K,I)=(D1/ALF)*ARCTNH(Y)/ALPHA
       WRITE (6,1600) T,DIMT
C PRINT-OUT OF CONTOUR ELEVATIONS FOR INITIAL LINES OF
C CONSTANT DENSITY
DO 400 K=1,KZ
WRITE (6,1550) DCONT(K),MX
400 WRITE (6,1560) (H(K,I),I=1,MX)
C INITIAL SOLUTION FOR VORTICITY FIELD. VT1=VORTICITY AT OLD TIME,
C AND VT2=VORTICITY AT NEW TIME
DO 420 I=1,MX
DO 420 J=1,NZ
VT2(I,J)=VT1(I,J)
420 CONTINUE
C INITIAL SOLUTION FOR VELOCITY FIELD. UOLD=DP/DZ WOLD=-DP/DX
DO 430 I=2,M1
DO 430 J=3,N11
UOLD(I,J)=(P(I,J+1)-P(I,J-1))/(D2*DZ)
WOLD(I,J)=(P(I-1,J)-P(I+1,J))/(D2*DX)
430 CONTINUE
DO 440 I=1,MX
WOLD(I,1)=0
WOLD(I,2)=0
WOLD(I,NZ)=0
440 WOLD(I,N11)=0
DO 450 J=3,N11
UOLD(MX,J)=(P(MX,J+1)-P(MX,J-1))/(D2*DZ)
WOLD(MX,J)=WOLD(M1,J)
UOLD(1,J)=(P(1,J+1)-P(1,J-1))/(D2*DZ)
450 WOLD(1,J)=WOLD(2,J)
DO 460 I=1,MX
UOLD(I,2)=(ZSEC2(3)/ZSEC2(2))*UOLD(I,3)
460 UOLD(I,N11)=(ZSEC2(N11)/ZSEC2(N11))*UOLD(I,N11)
C SET VELOCITY WITHIN REGION OCCUPIED BY SOLID BODY EQUAL TO ZERO.
DO 470 I=M3,M4
DO 470 J=N3,N4
UOLD(I,J)=0
470 WOLD(I,J)=0
C CALCULATION AND PRINT-OUT OF INITIAL DIMENSIONAL U VELOCITY WITH RESPECT TO THE EXPERIMENTER

IF (IPOUE.EQ.0) GO TO 510
DO 480 I=1,MX
DO 480 J=2,N1

480 URDIM(I,J)=((ALF*ZSEC2(J)*VOLD(I,J)*UBODY)*UBODY
DO 490 I=1,MX
URDIM(I,1)=0

490 URDIM(I,NZ)=D0
WRITE (6,1470)
WRITE (6,1600) I,DIMG
WRITE (6,1570)
DO 500 I=1,MX

500 WRITE (6,1690) I,(URDIM(I,J),J=1,NZ)
510 CONTINUE

C START OF TIME INCREMENTATION OUTER DO LOOP
LI=40
IF (IPOK.EQ.0) GO TO 520
WRITE (6,1580)

520 CONTINUE
DO 1000 L=1,LL
T=T+DT

C EVALUATION OF VELOCITY FIELD. U=DP/DZ  W=-DP/DX
DO 530 I=2,M1
DO 530 J=3,N11
U(I,J)=(P(I,J+1)-P(I,J-1))/(D2*DZ)
W(I,J)=(P(I-1,J)-P(I+1,J))/(D2*DX)

530 CONTINUE
DO 540 I=1,MX
W(I,1)=D0
W(I,2)=D0
W(I,NZ)=D0

540 W(I,N11)=D0
DO 550 J=3,N11
U(MX,J)=(P(MX,J+1)-P(MX,J-1))/(D2*DZ)
W(MX,J)=W(M1,J)
U(I,J)=(P(I,J+1)-P(I,J-1))/(D2*DZ)

550 W(I,J)=W(2,J)
DO 560 I=1,MX
U(I,2)=(ZSEC2(3)/ZSEC2(2))*U(I,3)
560 U(I,N1)=(ZSEC2(N11)/ZSEC2(N1))*U(I,N11)
C SET VELOCITY WITHIN REGION OCCUPIED BY SOLID BODY EQUAL TO ZERO.
DO 570 I=M3*M4
DO 570 J=N3*N4
U(I,J)=0
570 W(I,J)=0
C EVALUATION OF NON-LINEAR CONVECTIVE TERMS IS BY THE SPECIAL
C THREE-POINT NON-CENTRAL DIFFERENCES METHOD.
C NOMENCLATURE...
C DUUDX = D(U**2)/DX          VELOCITY
C DWUDZ = D(W**2)/DZ          VELOCITY
C DUWDX = D(U*W)/DX           VELOCITY
C DWWDZ = D(W**2)/DZ          VELOCITY
C DUVDX =D(U*VT1)/DX          VORTICITY
C DWVDZ = D(W*VT1)/DZ         VORTICITY
C DUDDX = D(U*DT1)/DX         DENSITY
C DWDDZ = D(W*DT1)/DZ         DENSITY
DO 580 I=1,M1
DO 580 J=2*N11
UAV(I,J)=(U(I+1,J)+U(I,J))/2
580 WAV(I,J)=(W(I,J+1)+W(I,J))/2
DO 590 I=2,M1
DO 590 J=3,N11
CALL NONLIN (DUDDX,UAV,DT1,DX,DWDDZ,WAV,DT1,DZ,I+J)
590 CONTINUE
C TIME ADVANCEMENT OF DENSITY
DO 600 I=2,M1
DO 600 J=3,N11
CALL DELSQR (USQD,CA,X,DT1,DX,ALF,Z,DZ,I,J)
DT2(I,J)=DT1(I,J)+DT*(-CA*XSEC2(I)*ALF*ZSEC2(J)*(DUDDX(I,J)+DWDDZ(MAN
I,J)))*(EPS/RE)*DSQD(I,J)*DFLOAT(MLT4))
600 CONTINUE
DO 610 J=3,N11
DT2(I,J)=DT2(2,J)
DT2(MX,J)=DT2(M1,J)
610 CONTINUE
C DENSITY BOUNDARY CONDITION AT OBSTACLE...DT2=1.00; D(DT2)/DN=0.
DO 620 I=M3,M4
DO 620 J=N3,N4
620 DT2(I,J)=01
DO 630 I=M3,M4
DT2(I,N3-1) = DP75*DP25*DT2(I,N3-2)
630 DT2(I,N4+1) = DP75*DP25*DT2(I,N4+2)
DO 640 J=N3,N4
DT2(M3-1,J) = DP75*DP25*DT2(M3-2,J)
640 DT2(M4+1,J) = DP75*DP25*DT2(M4+2,J)
C TIME ADVANCEMENT OF VORTICITY
DO 650 I=2,M1
DO 650 J=3,N11
CALL NONLIN (DUDWX,UAU,U,DX,DWUDZ,WAV,U,DZ,I,J)
C1=ALF*XSEC2(J)
C2=C1
C3=CA*XSEC2(I)*C1*C1
C4=C3/ZSEC2(J)
BOUS1(I,J)=C1*((D1(I,J*1)-DT1(I,J-1))/(D2*DZ))*(C2*(W(I,J)-UOLD(1MAN 370
1,J))/DT+C3*DUDWX(I,J)+C4*(ZSEC2(J)*DWUDZ(I,J)-D2*U(I,J)*W(I,J)*Z(JMAN 371
))
) CALL NONLIN (DUDWX,UAU,W,DX,DWUDZ,WAV,W,DZ,I,J)
C1=-CA*XSEC2(I)
C2=-C1
C3=CA*C2*ALF*ZSEC2(J)
C4=C3*XSEC2(I)
BOUS2(I,J)=C1*((D1(I,J)+1)-DT1(I,J-1))/(D2*DZ))*(C2*(W(I,J)-WOLD(1MAN 378
1,J))/DT+C3*(XSEC2(I)*DUDWX(I,J)-D2*U(I,J)*W(I,J)*X(I))+C4*DUDWX(I,J)
2J))
) CALL NONLIN (DUDWX,UAU,V1,DX,DWVDZ,WAV,V1,DZ,I,J)
CALL DELSQR (DSQV,CA,X,V1,DX,ALF,Z,DZ,I,J)
C14
VT2(I,J) = VT1(I,J) + DT*(-CA*XSEC2(I)*ALF*ZSEC2(J)*(DUVDX(I,J) + DWVDZ(MAN 383
11,J)) + (D1/DT1(I,J))*(-DFLOAT(MLT1)*BOUS1(I,J) - DFLOAT(MLT2)*BOUS2(IMAN 384
2,J) + (CA/(F+F))*XSEC2(I)*(DT1(I+1,J) - DT1(I-1,J))/(D2*DX) + (D1/RE) *DFMAN 385
3LOT(MLT3)*DSQV(I,J)))  
650 CONTINUE  
DO 660 J=3,N11  
VT2(1,J) = VT2(2,J)  
VT2(MX,J) = VT2(M1,J)  
660 CONTINUE  
DO 660 I=1,MX  
670 VT2(I,N2) = DU  
C VORTICITY BOUNDARY CONDITION AT OBSTACLE  
DO 680 I=M3,M4  
VT2(I,N3) = (ALF*ZSEC2(N3)/DZ)**2*(-P(I,N3-3)*D4*P(I,N3-2) - D5*P(I,MAN 396
1N3-1) + D2*P(I,N3))  
680 VT2(I,N4) = (ALF*ZSEC2(N4)/DZ)**2*(-P(I,N4+3)*D4*P(I,N4+2) - D5*P(I,MAN 398
1N4+1) + D2*P(I,N4))  
DO 690 J=N3,N4  
VT2(M3,J) = (CA*XSEC2(M3)/DX)**2*(-P(M3-3,J) + D4*P(M3-2,J) - D5*P(M3-MAN 401
11,J) + D2*P(M3,J))  
690 VT2(M4,J) = (CA*XSEC2(M4)/DX)**2*(-P(M4+3,J) + D4*P(M4+2,J) - D5*P(M4+MAN 403
11,J) + D2*P(M4,J))  
C OVER-RELAXATION FOR STREAM FUNCTION, P(I,J)  
KK=11  
CALL RELAX (A,X,Z,SR,DX,DZ,CA,ALF,P,VT1,KK,MX,M3,M4,NZ,N3,N4,IPOK)  
C CONVERT NEW TIME VARIABLE TO OLD  
DO 700 I=1,MX  
DO 700 J=1,NZ  
DT1(I,J) = DT2(I,J)  
VT1(I,J) = VT2(I,J)  
UOOLD(I,J) = U(I,J)  
WOLD(I,J) = W(I,J)  
700 CONTINUE
C CHECK NUMBER OF ITERATIONS, L, THROUGH TIME INCREMENTATION DO LOOPMAN 416
C IF L=L1, PRINT OUT RESULTS, INCREMENT LI, AND REPEAT THE
C TIME INCREMENTATION AND PERIODIC PRINT-OUT UNTIL L=L1=L1.
C IF (L-L1) 1000, 710, 710
C 710 LI=LI+40
C CALCULATION OF DIMENSIONAL TIME
C DIMT=T/(ALPHA*U*BODY)
C C PRINT-OUT OF RESULTS
C C IF (IPD=EQ.0) GO TO 730
C WRITE (6,1590)
C WRITE (6,1600) T,DIMT
C C PRINT-OUT NON-DIMENSIONAL DENSITY FIELD
C WRITE (6,1610)
C DO 720 I=1,MX
C 720 WRITE (6,1620) I, (DT2(I,J),J=1,NZ)
C 730 CONTINUE
C CALCULATE AND PRINT-OUT DIMENSIONAL DENSITY FIELD
C IF (IPD=EQ.0) GO TO 760
C DO 740 I=1,MX
C DO 740 J=1,NZ
C 740 DDIM(I,J)=DT2(I,J)*DAVG
C WRITE (6,1590)
C WRITE (6,1600) T,DIMT
C WRITE (6,1630)
C DO 750 I=1,MX
C 750 WRITE (6,1620) I, (DDIM(I,J),J=1,NZ)
C 760 CONTINUE
C PRINT-OUT NON-DIMENSIONAL VORTICITY FIELD
C IF (IPV=EQ.0) GO TO 780
C WRITE (6,1590)
C WRITE (6,1600) T,DIMT
C WRITE (6,1640)
DO 770 I=1,Mx
770 WRITE (6,1650) I,(VT2(I,J),J=1,NZ)
780 CONTINUE
C PRINT-OUT NON-DIMENSIONAL STREAM FUNCTION
IF (IPOP.EQ.0) GO TO 800
WRITE (6,1590)
WRITE (6,1600) T,DIMT
WRITE (6,1660)
DO 790 I=1,Mx
790 WRITE (6,1670) I,(P(I,J),J=1,NZ)
800 CONTINUE
C REFER THE CALCULATED VELOCITIES TO THE 'REAL' COORDINATE SYSTEM
DO 810 I=1,Mx
DO 810 J=2,N1
UR(I,J)=ALF*ZSEC2(J)*U(I,J)
WR(I,J)=CA*XSEC2(I)*W(I,J)
810 CONTINUE
DO 820 I=1,Mx
UR(I,1)=-D1
UR(I,NZ)=-D1
WR(I,1)=D0
WR(I,NZ)=D0
820 CONTINUE
C PRINT-OUT NON-DIMENSIONAL VELOCITY FIELD
IF (IPOV0.EQ.0) GO TO 840
WRITE (6,1590)
WRITE (6,1600) T,DIMT
WRITE (6,1680)
DO 830 I=1,Mx
830 WRITE (6,1690) I,(UR(I,J),J=1,NZ)
840 CONTINUE
IF (IPOW0.EQ.0) GO TO 860
WRITE (6,1590)
WRITE (6,1600) T,DIMT
WRITE (6,1700)
DO 850 I=1, MX
850  WRITE (6,1690) I, (WR(I,J), J=1,NZ)
860  CONTINUE

C CALCULATE AND PRINT-OUT DIMENSIONAL VELOCITIES WITH RESPECT TO
C AN OBSERVER WHO SEES THE OBSTACLE MOVING IN THE +X-DIRECTION
C WITH VELOCITY EQUAL TO UBODY. (I.E. THE VELOCITIES ARE MADE
C DIMENSIONAL AND A GALILEAN TRANSFORMATION OF THE U VELOCITY IS
C MADE.)
DO 870 I=1, MX
DO 870 J=1,NZ
URDIM(I,J)=(UR(I,J)*UBODY)+UBODY
870  WRDIM(I,J)=WR(I,J)*UBODY
IF (IPOUE.EQ.0) GO TO 890
WRITE (6,1590)
WRITE (6,1600) T,DIMT
WRITE (6,1710)
DO 880 I=1, MX
880  WRITE (6,1690) I, (URDIM(I,J), J=1,NZ)
890  CONTINUE
IF (IPOWE.EQ.0) GO TO 910
WRITE (6,1590)
WRITE (6,1600) T,DIMT
WRITE (6,1720)
DO 900 I=1, MX
900  WRITE (6,1690) I, (WRDIM(I,J), J=1,NZ)
910  CONTINUE

C COMPUTE COUTOUR ELEVATIONS FOR STREAMLINES
IF (IPOC.EQ.0) GO TO 950
MESH=5
KZ=2*MESH+2
DO 920 K=1,KZ
PCONT(K)=0
DO 920 I=1, MX
920  H(K,I)=0
CALL CONTUR (MX,M3,M4,NZ,N3,N4,MESH,Z,DZ,P,DP,D0,PCONT,H)
C PUT CONTOUR ELEVATION, H, IN DIMENSIONAL FORM
DO 930 K=1,KZ
DO 930 I=1,MX
Y=H(K,I)
930 H(K,I)=(D1/ALF)*ARCTNH(Y)/ALPHA
WRITE (6*1600) T,DIMT
C PRINT-OUT OF CONTOUR ELEVATIONS FOR STREAMLINES
DO 940 K=1,KZ
WRITE (6*1730) PCONT(K),MX
940 WRITE (6*1740) (H(K,I),I=1,MX)
950 CONTINUE
C COMPUTE CONTOUR ELEVATIONS FOR LINES OF CONSTANT DENSITY
IF (IPODC.EQ.0) GO TO 990
MESH=5
KZ=2*MESH+2
DO 960 K=1,KZ
DCONT(K)=0
DO 960 I=1,MX
960 H(K,I)=0
CALL CONTR (MX,M3,M4,NZ,N3,N4,MESH,Z,DZ,DT2,DL,D1,DCONT,H)
C PUT CONTOUR ELEVATION, H, IN DIMENSIONAL FORM
DO 970 K=1,KZ
DO 970 I=1,MX
Y=H(K,I)
970 H(K,I)=(D1/ALF)*ARCTNH(Y)/ALPHA
WRITE (6*1600) T,DIMT
C PRINT-OUT OF CONTOUR ELEVATIONS FOR LINES OF CONSTANT DENSITY
DO 980 K=1,KZ
WRITE (6*1750) DCONT(K),MX
980 WRITE (6*1760) (H(K,I),I=1,MX)
990 CONTINUE
1000 CONTINUE
1010 STOP
C
1020 FORMAT (1H1)
1030 FORMAT (20A4/,20A4/,20A4/,20A4/)
1050 FORMAT (9I6) MAN 558
1060 FORMAT ('ODX= ',F6.3,' DZ= ',F6.3,' UT= ',F6.3,' T= ',F6.3) MAN 559
1070 FORMAT ('OF= ',F8.5,' RE= ',F6.1,' EPS= ',F8.5) MAN 560
1080 FORMAT ('OCA= ',F6.3,' ALF= ',F6.3,' DAVG= ',F6.4,' OMEG= ',F8.5) MAN 561
L, 'ALPHA= ',F6.2,' UBODY= ',F6.2) MAN 562
1090 FORMAT ('OA= ',F6.2,' LL= ',I6,' MX= ',I6,' M2NEG= ',I6,' MAN 563
L M2POS= ',I6,' NZ= ',I6) MAN 564
1100 FORMAT ('OMLT1= ',I5,' MLT2= ',I5,' MLT3= ',I5,' MLT4= ',I5) MAN 565
1110 FORMAT ('0INPUT DATA IS IN ERROR. MLT1 CANNOT BE NEGATIVE') MAN 566
1120 FORMAT ('OSINCE MLT1=0, FIRST BOUSSINESQ TERM WILL BE NEGLECTED') MAN 567
1130 FORMAT ('OSINCE MLT1=1, FIRST BOUSSINESQ TERM WILL BE INCLUDED') MAN 568
1140 FORMAT ('0INPUT DATA IS IN ERROR. MLT2 CANNOT BE NEGATIVE') MAN 569
1150 FORMAT ('OSINCE MLT2=0, SECOND BOUSSINESQ TERM WILL BE NEGLECTED') MAN 570
1160 FORMAT ('OSINCE MLT2=1, SECOND BOUSSINESQ TERM WILL BE INCLUDED') MAN 571
1170 FORMAT ('0INPUT DATA IS IN ERROR. MLT3 CANNOT BE NEGATIVE') MAN 572
1180 FORMAT ('OSINCE MLT3=0, VORTICITY DIFFUSION TERM WILL BE NEGLECTED') MAN 573
L '1') MAN 574
1190 FORMAT ('OSINCE MLT3=1, VORTICITY DIFFUSION TERM WILL BE INCLUDED') MAN 575
L 1) MAN 576
1200 FORMAT ('0INPUT DATA IS IN ERROR. MLT4 CANNOT BE NEGATIVE') MAN 577
1210 FORMAT ('OSINCE MLT4=0, DENSITY DIFFUSION TERM WILL BE NEGLECTED') MAN 578
1220 FORMAT ('OSINCE MLT4=1, DENSITY DIFFUSION TERM WILL BE INCLUDED') MAN 579
1230 FORMAT (12I6) MAN 580
1240 FORMAT (1H0) MAN 581
1250 FORMAT ('0THE FOLLOWING INPUT PARAMETERS INDICATE THOSE RESULTS WHICH THE USER HAS SELECTED TO BE PRINTED OUT.') MAN 582
11260 FORMAT ('0THE 12 PARAMETERS ARE READ FROM THE INPUT CARD AS INTEGE MAN 584
LRS IN 12 FIELDS OF 6 COLUMNS EACH.') MAN 584
1270 FORMAT ('0IF A PARAMETER = 0, (I.E., BLANK OR 0 ON THE INPUT CARD) THE RESULTS CORRESPONDING TO THIS PARAMETER') MAN 587
1280 FORMAT ('0ARE NOT PRINTED OUT. IF THE PRINT-OUT IS DESIRED, A NON-ZERO INTEGER MUST BE ASSIGNED TO THE PARAMETER') MAN 589
1290 FORMAT ('0THE FOLLOWING LIST GIVES EACH PARAMETER IN THE ORDER IN WHICH IT IS READ IN, THE RESULT PRINT-OUT(S)') MAN 590
1300 FORMAT ('*TO WHICH IT CORRESPONDS, AND THE VALUE CHOSEN FOR IT FORMAN 592
1 T H I S RUN.*) MAN 593
1310 FORMAT ('*OIPOK -- NUMBER OF ITERATIONS, K, THROUGH SUBROUTINE RELAMAN 594
1X ------ IPOK=**,13) MAN 595
1320 FORMAT ('*OIPOP -- STREAM FUNCTION, P, BOTH INITIAL AND AT LATER TIMAN 596
1ME ----- IPOP=**,13) MAN 597
1330 FORMAT ('*OIPOC -- STREAM FUNCTION CONTOUR ELEVATIONS ---------------MAN 598
1------- IPOPC=**,12) MAN 599
1340 FORMAT ('*OIPON -- INITIAL DENSITY PROFILE AT A TYPICAL CROSS SECTMAN 600
1ION --- IPODN=**,12) MAN 601
1350 FORMAT ('*OIPOC -- DENSITY CONTOUR ELEVATIONS ---------------------MAN 602
1------- IPODN=**,12) MAN 603
1360 FORMAT ('*OIPON -- NON-DIMENSIONAL DENSITY AT LATER TIME ----------MAN 604
1------- IPODN=**,12) MAN 605
1370 FORMAT ('*OIPOR -- DIMENSIONAL DENSITY AT LATER TIME ---------------MAN 606
1------- IPOD=**,13) MAN 607
1380 FORMAT ('*OPOV -- VORTICITY FIELD ---------------------------------MAN 608
1------- IPOV=**,13) MAN 609
1390 FORMAT ('*OPOUO -- NON-DIMENSIONAL U VELOCITY WITH RESPECT TO OBSTMAN 610
1ACU -- IPOUO=**,12) MAN 611
1400 FORMAT ('*OPOWO -- NON-DIMENSIONAL W VELOCITY WITH RESPECT TO OBSTMAN 612
1ACW -- IPOWO=**,12) MAN 613
1410 FORMAT ('*OPOUE -- DIMENSIONAL U VELOCITY WITH RESPECT TO EXPERIMAN 614
1INTER -- IPOUE=**,12) MAN 615
1420 FORMAT ('*OPOWE -- DIMENSIONAL W VELOCITY WITH RESPECT TO EXPERIMAN 616
1INTER -- IPOWE=**,12) MAN 617
1430 FORMAT (1H1) MAN 618
1CM,** (1.-X(I)**2)=**,F7.4) MAN 620
1450 FORMAT (1H1) MAN 621
1CM,** (1.-Z(J)**2)=**,F7.4) MAN 623
1470 FORMAT (1H1) MAN 624
1480 FORMAT ('*OINITIAL NON-DIMENSIONAL STREAM FUNCTION*) MAN 625
1490 FORMAT ('* I=**,I3/** (8F15.3)) MAN 626
1500 FORMAT ('*OCONTOUR ELEVATIONS IN CENTIMETERS FOR THE INITIAL STREAMAN 627
1LINE P=**,F6.2,** FOR I=**,I3,**.*) MAN 628
1510 FORMAT ('IH0', (8F15.2)) MAN 629
1520 FORMAT ('*INITIAL DIMENSIONAL DENSITY FIELD AT A TYPICAL CROSS-SECTION', MAN 630
ITION, (GM./CU. CM.)) MAN 631
1530 FORMAT ('IH0') MAN 632
1540 FORMAT ('* J= ', 'I3', 'F10.4') MAN 633
1550 FORMAT ('*CONTOUR ELEVATIONS IN CENTIMETERS FOR THE DT1=' ', 'F6.4', ' IMAN 634
INITIAL LINE OF CONSTANT NON-DIMENSIONAL DENSITY I=1', 'I2', 'I') MAN 635
1560 FORMAT ('IH0', (8F15.2)) MAN 636
1570 FORMAT ('*INITIAL DIMENSIONAL U VELOCITY WITH RESPECT TO THE EXPERIMAN 637
INTER, (CM./SEC.)) MAN 638
1580 FORMAT ('IH1') MAN 639
1590 FORMAT ('IH1') MAN 640
1600 FORMAT ('*ONON-DIMENSIONAL TIME= ', 'F7.3', ' DIMENSIONAL TIME= ', 'F7.3MAN 641
1', ' SECONDS') MAN 642
1610 FORMAT ('*ONON-DIMENSIONAL DENSITY FIELD') MAN 643
1620 FORMAT ('* I= ', 'I3', ' (8F15.4)') MAN 644
1630 FORMAT ('*ONOMERDIMENTAL DENSITY FIELD, (GM./CU. CM.)) MAN 645
1640 FORMAT ('*ONON-DIMENSIONAL VORTICITY FIELD') MAN 646
1650 FORMAT ('* I= ', 'I3', ' (8F15.3)') MAN 647
1660 FORMAT ('*ONON-DIMENSIONAL STREAM FUNCTION') MAN 648
1670 FORMAT ('*' I= ', 'I3', ' (8F15.3)') MAN 649
1680 FORMAT ('*ONON-DIMENSIONAL U VELOCITY WITH RESPECT TO THE OBSTACLE MAN 650
1) MAN 651
1690 FORMAT ('*' I= ', 'I3', ' (8F15.2)') MAN 652
1700 FORMAT ('*ONON-DIMENSIONAL W VELOCITY WITH RESPECT TO THE OBSTACLE MAN 653
1) MAN 654
1710 FORMAT ('*ONOMERDIMENTAL U VELOCITY WITH RESPECT TO THE EXPERIMENTER, MAN 655
1 (CM./SEC.)) MAN 656
1720 FORMAT ('*ONOMERDIMENTAL U VELOCITY WITH RESPECT TO THE EXPERIMENTER, MAN 657
1 (CM./SEC.)) MAN 658
1730 FORMAT ('*CONTOUR ELEVATIONS IN CENTIMETERS FOR THE P= ', 'F6.2', ' STRMAN 659
IEMLINE; I=1', 'I3', 'I') MAN 660
1740 FORMAT ('IH0', (8F15.2)) MAN 661
1750 FORMAT ('*CONTOUR ELEVATIONS IN CENTIMETERS FOR THE DT2=' ', 'F6.4', ' LMAN 662
LINE OF CONSTANT NON-DIMENSIONAL DENSITY I=1', 'I3', 'I') MAN 663
1760 FORMAT ('IH0', (8F15.2)) END MAN 664
SUBROUTINE RELAX (A*X,Z,ZR,DX,DZ,CA,ALF,P,ETA,KK,MX,M4,NZ,N3,N4RLX)
1,1POK)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION X(MX), Z(NZ), ZR(NZ), P(MX,NZ), PA(81,41), ETA(MX,NZ)
D0=0.0D0
D4=4.0D0
M1=MX-1
M31=M3+1
M32=M3-2
M41=M4+1
M42=M4+2
N1=NZ-1
N11=N1-1
N2=(NZ+1)/2
N21=N2+1
N42=N4+2
D0 150 K=1,150
D0 10 I=1, MX
10 P(I,N2)=D0
DO 20 I=M3,M4
DO 20 J=N3,N4
20 P(I,J)=D0
DO 30 I=M3,M4
P(I,N3-1)=P(I,N3-2)/D4
30 P(I,N4+1)=P(I,N4+2)/D4
DO 40 J=N3,N4
P(M3-1,J)=P(M3-2,J)/D4
40 P(M4+1,J)=P(M4+2,J)/D4
D0 50 I=1, MX
P(I,1)=-ZR(1)
P(I,2)=-ZR(2)
P(I,N1)=-ZR(N1)
P(I,NZ)=-ZR(NZ)
D0 60 I=2,M32
D0 60 J=N21,N11
CALL PSI (A*I,J*P*X,Z,CA,DX,DZ,ALF,ETA)
CALL END

60 CONTINUE
   DO 70 I=M31,M41
   DO 70 J=N42,N11
   CALL PSI (A,I,J,P,X,Z,CA,DX,DZ,ALF,ETA)
70 CONTINUE
   DO 80 I=M42,M1
   DO 80 J=N21,N11
   CALL PSI (A,I,J,P,X,Z,CA,DX,DZ,ALF,ETA)
80 CONTINUE
C TEST FOR CONVERGENCE
   IF (K-KK) 150,90,110
90 DO 100 I=1,MX
   DO 100 J=N2,NZ
100 PA(I,J)=P(I,J)
   GO TO 150
110 DO 120 I=1,MX
   DO 120 J=N2,NZ
   IF (OABS(P(I,J)-PA(I,J))-0.0005) 120,120,140
120 CONTINUE
C TAKING ADVANTAGE OF SYMMETRY ABOUT THE HORIZONTAL CENTER LINE
   DO 130 I=1,MX
   DO 130 J=2,N2
130 P(I,N2-J+1)=-P(I,N2+J-1)
   IF (IPOK.EQ.0) GO TO 160
   WRITE (6,170) K
   GO TO 160
140 KK=KK+10
150 CONTINUE
   WRITE (6,180) K
160 CONTINUE
   RETURN
C
170 FORMAT (' K= ',I3)
180 FORMAT ('OTHER OVER-RELAXATION DOES NOT CONVERGE. K=',I3)
END
SUBROUTINE PSI (A, I, J, P, X, Z, CA, DX, DZ, ALF, ETA)  
IMPLICIT REAL*8 (A-H, O-Z)  
DIMENSION X (a1), XSEC2 (81), Z (41), ZSEC2 (41), P (81, 41), ETA (81, 41)  
D = DX/DZ  
DP5 = 5.0D-1  
D1 = 1.0D0  
XSEC2 (I) = D1 - X (I) ** 2  
ZSEC2 (J) = D1 - Z (J) ** 2  
C1 = DP5 * A / ((CA * XSEC2 (I)) ** 2) + ((ALF * ZSEC2 (J) * D) ** 2)  
C2 = -DX * DX  
C3 = (CA * XSEC2 (I)) ** 2  
C4 = -CA * CA * X (I) * XSEC2 (I) * DX  
C5 = (ALF * ZSEC2 (J) * D) ** 2  
C6 = -ALF * ALF * Z (J) * ZSEC2 (J) * DX * DX / DZ  
RETURN  
END
SUBROUTINE NONLIN (DDX, AV1, FN1, DX, DDZ, AV2, FN2, DZ, I, J)

IMPLICIT REAL*8 (A-H, O-Z)

DIMENSION DDX(81,41), AV1(81,41), FN1(81,41), DDZ(81,41), AV2(81,41)

10 D0=0.000

IF (AV1(I, J) .GE. D0 .AND. AV1(I-1, J) .GE. D0) GO TO 10

IF (AV1(I, J) .LT. D0 .AND. AV1(I-1, J) .LT. D0) GO TO 20

IF (AV1(I, J) .GE. D0 .AND. AV1(I-1, J) .LT. D0) GO TO 30

DDX(I, J) = (AV1(I, J) * FN1(I+1, J) - AV1(I-1, J) * FN1(I-1, J)) / DX

GO TO 40

10 DDX(I, J) = (AV1(I, J) * FN1(I+1, J) - AV1(I-1, J) * FN1(I-1, J)) / DX

GO TO 40

20 DDX(I, J) = (AV1(I, J) * FN1(I+1, J) - AV1(I-1, J) * FN1(I-1, J)) / DX

GO TO 40

30 DDX(I, J) = (AV1(I, J) * FN1(I+1, J) - AV1(I-1, J) * FN1(I-1, J)) / DX

GO TO 40

40 IF (AV2(I, J) .GE. D0 .AND. AV2(I, J-1) .GE. D0) GO TO 50

IF (AV2(I, J) .LT. D0 .AND. AV2(I, J-1) .LT. D0) GO TO 60

IF (AV2(I, J) .GE. D0 .AND. AV2(I, J-1) .LT. D0) GO TO 70

DDZ(I, J) = (AV2(I, J) * FN2(I, J+1) - AV2(I, J-1) * FN2(I, J-1)) / DZ

GO TO 80

50 DDZ(I, J) = (AV2(I, J) * FN2(I, J) - AV2(I, J-1) * FN2(I, J-1)) / DZ

GO TO 80

60 DDZ(I, J) = (AV2(I, J) * FN2(I, J+1) - AV2(I, J-1) * FN2(I, J-1)) / DZ

GO TO 80

70 DDZ(I, J) = (AV2(I, J) * FN2(I, J) - AV2(I, J-1) * FN2(I, J-1)) / DZ

GO TO 80

80 RETURN

END
SUBROUTINE DEFSQ (DSQFN, CA, X, FN, DX, ALF, Z, DZ, I, J)

IMPLICIT REAL*8 (A-H, O-Z)

DIMENSION X(81), XSEC2(81), Z(41), ZSEC2(41), DSQFN(81,41), FN(81,81)

D1=1.0D0
D2=2.0D0
XSEC2(I)=D1-X(I)**2
ZSEC2(J)=D1-Z(J)**2
C1=(CA*XSEC2(I))**2
C2=-D2*CA*CA*X(I)*XSEC2(I)
C3=(ALF*ZSEC2(J))**2
C4=-D2*ALF*ALF*Z(J)*ZSEC2(J)

DSQFN(I, J)=C1*(FN(I+1, J)-D2*FN(I, J)+FN(I-1, J))/(DX**2)*C2*(FN(I+1, DSQ 13
1, J)-FN(I-1, J))/(D2*DX)+C3*(FN(I, J+1)-D2*FN(I, J)+FN(I, J-1))/(DZ**2)+DSQ 14
2C4*(FN(I, J+1)-FN(I, J-1))/(DZ**2)

RETURN

END
SUBROUTINE CONTUR (MX, M3, M4, NZ, N3, N4, MESH, Z, DZ, F, DF, FAV, FCONT, H) CON 1
IMPLICIT REAL*8 (A-H,O-Z) CON 2
DIMENSION Z(NZ), F(MX,NZ), FCONT(12), DFDZ(81,41), DDFDZ(81,41), HCON 3
1(12,81) CON 4
C CARRIAGE CONTROL CON 5
WRITE (6,240) CON 6
C DOUBLE PRECISION CONSTANTS CON 7
D0=0.000 CON 8
D1=1.000 CON 9
D2=2.000 CON 10
D3=3.000 CON 11
D4=4.000 CON 12
D5=5.000 CON 13
C COMPUTE DELIMITERS CON 14
M31=M3-1 CON 15
M41=M4+1 CON 16
N1=NZ-1 CON 17
N11=N1-1 CON 18
C COMPUTE 1ST AND 2ND DERIVATIVES OF F WITH RESPECT TO Z CON 19
DO 10 I=1,MX CON 20
DO 10 J=3,N11 CON 21
DFDZ(I,J)=(F(I,J+1)-F(I,J-1))/(D2*Z) CON 22
10 DDFDZ(I,J)=(F(I,J+1)-D2*F(I,J)+F(I,J-1))/(D2*Z) CON 23
C DERIVATIVES AT UPPER AND LOWER BOUNDARIES OF OBSTACLE CON 24
DO 20 I=M3,M4 CON 25
C BACKWARD DIFFERENCE AT LOWER BOUNDARY, J=N3 CON 26
DFDZ(I,N3)=(D3*F(I,N3)-D4*F(I,N3-1)+F(I,N3-2))/(D2*Z) CON 27
DDFDZ(I,N3)=(-F(I,N3-3)+D4*F(I,N3-2)-D5*F(I,N3-1)+D2*F(I,N3))/(D2*Z) CON 28
1DZ) CON 29
C FORWARD DIFFERENCE AT UPPER BOUNDARY, J=N4 CON 30
DFDZ(I,N4)=(-D3*F(I,N4)+D4*F(I,N4+1)-F(I,N4+2))/(D2*Z) CON 31
20 DDFDZ(I,N4)=(-F(I,N4+3)+D4*F(I,N4+2)-D5*F(I,N4+1)+D2*F(I,N4))/(D2*Z) CON 32
1DZ) CON 33
C COMPUTATION OF THE VALUES OF F FOR WHICH THE CONTOUR ELEVATIONS CON 34
C ARE TO BE DETERMINED CON 35
KZ=2*MESH+2 CON 36
C    K3=CENTER LINE AND LOWER BOUNDARY OF OBSTACLE
    K3=KZ/2
C    K4=CENTER LINE AND UPPER BOUNDARY OF OBSTACLE
    K4=K3+1
    DO 30 K=1,K3
      FCONT(K)=FAV+DF*(K3-K)
    DO 40 K=K4,KZ
      FCONT(K)=FAV+DF*(K4-K)
C    EVALUATION OF THE CONTOUR ELEVATION OF F(I,J)=FCONT(K) FOR
C    K=1,KZ AND I=1,MX.
C    DO 50 I=1,M31
      H(K3,I)=0
    50 H(K4,I)=0
    DO 60 I=M3,M4
      H(K3,I)=Z(N3)
    60 H(K4,I)=Z(N4)
    DO 70 I=M41,MX
      H(K3,I)=0
    70 H(K4,I)=0
C    DO 220 K=1,KZ
      IF (K.EQ.K3.OR.K.EQ.K4) GO TO 220
      DO 220 I=1,MX
    DO 80 J=2,N1
    IF (F(I,J).GE.FCONT(K)) GO TO 80
      GO TO 90
    80 CONTINUE
    90 J=J-1
      IF (F(I,J).NE.FCONT(K)) GO TO 100
      H(K,I)=Z(J)
      GO TO 220
C    TO AVOID USING THE DERIVATIVES AT J=2, SPECIAL TREATMENT IS GIVEN
C    TO THIS CASE.
100 IF (J.EQ.2) GO TO 190
   DISCR=DFDZ(I,J)**2-D2*DDFDZ(I,J)*(F(I,J)-FCONT(K))
   IF (DISCR.GE.0.0) GO TO 110
   WRITE (6,250) I,J,K,FAV
   WRITE (6,260) FCONT(K),F(I,J),DDFDZ(I,J),DDFDZ(I,J),DISCR
   GO TO 230
   CON 71
   CON 72
   CON 73
   CON 74
   CON 75
   CON 76
110 IF (DABS(DDFDZ(I,J)).GE.1.0D-7) GO TO 120
   H(K,I)=Z(J)*(FCONT(K)-F(I,J))/DFDZ(I,J)
   GO TO 220
   CON 77
   CON 78
   CON 79
120 ROOT1=(01/DFDZ(I,J))*(-DFDZ(I,J)+DSQRT(DISR))
   ROOT2=(01/DFDZ(I,J))*(-DFDZ(I,J)-DSQRT(DISR))
   CON 80
   CON 81
   C DETERMINE WHICH ROOT MAKES PHYSICAL SENSE.
   CHEK1=1.0D3
   CHEK2=1.0D3
   IF (ROOT1.GE.0.0.AND.ROOT1.LT.DZ) CHEK1=ROOT1
   IF (ROOT2.GE.0.0.AND.ROOT2.LT.DZ) CHEK2=ROOT2
   CON 82
   CON 83
   CON 84
   CON 85
   CON 86
130 IF (CHEK1.EQ.ROOT1.AND.CHEK2.EQ.ROOT2) GO TO 140
   GO TO 150
   CON 87
   CON 88
140 IF (ROOT1.NE.ROOT2) GO TO 160
   GO TO 170
   CON 89
   CON 90
150 IF (CHEK1.EQ.ROOT1.AND.CHEK2.EQ.1.0D3) GO TO 170
   IF (CHEK1.EQ.1.0D3.AND.CHEK2.EQ.ROOT2) GO TO 180
   CON 91
   CON 92
160 WRITE (6,270)
   WRITE (6,280) I,J,K,FAV
   WRITE (6,290) ROOT1,ROOT2
   CON 93
   CON 94
   CON 95
170 H(K,I)=Z(J)+ROOT1
   GO TO 220
   CON 96
   CON 97
180 H(K,I)=Z(J)+ROOT2
   GO TO 220
   CON 98
   CON 99
190 J=3
   DISCR=DFDZ(I,J)**2-D2*DDFDZ(I,J)*(F(I,J)-FCONT(K))
   IF (DISCR.GE.0.0) GO TO 200
   WRITE (6,250) I,J,K,FAV
   WRITE (6,260) FCONT(K),F(I,J),DDFDZ(I,J),DDFDZ(I,J),DISCR
   GO TO 230
   CON 100
   CON 101
   CON 102
   CON 103
   CON 104
   CON 105
200 IF (DABS(DDFDZ(I,J)).GE.1.0D-7) GO TO 210
   H(K,I)=Z(J)*(FCONT(K)-F(I,J))/DFDZ(I,J)
   GO TO 220
   CON 106
   CON 107
   CON 108
210 ROOT1=(D1/DDFDZ(I,J))*(-DFDZ(I,J)+DSQRT(DISCR))
ROOT2=(D1/DDFDZ(I,J))*(-DFDZ(I,J)-DSQRT(DISCR))
C
DETERMINE WHICH ROOT MAKES PHYSICAL SENSE
CHEK1=1.0D3
CHEK2=1.0D3
IF (ROOT1.LE.D0.AND.DABS(ROOT1).LT.DZ) CHEK1=ROOT1
IF (ROOT2.LE.D0.AND.DABS(ROOT2).LT.DZ) CHEK2=ROOT2
GO TO 130
220 CONTINUE
230 RETURN
C
240 FORMAT (1H1)
250 FORMAT ('0DISCRIMINANT IN SUBROUTINE CONTUR LESS THAN ZERO FOR I=',CON 120
1,'J=',J,'K=',K,' WITH FAV=',F7.4)
260 FORMAT ('0DISCRIMINANT IN SUBROUTINE CONTUR: BOTH ROOTS ARE BOTH PHYSICALLY
1LY POSSIBLE SOLUTIONS WHEN...'),
270 FORMAT ('0I=',I,'J=',J,'K=',K,' WITH FAV=',F7.4)
280 FORMAT ('0ROOT1=',F7.5,' ROOT2=',F7.5,' ...SO BEWARE!'),
END
FUNCTION ARCTNH (Y)                      ATH  1
IMPLICIT REAL*8(A-H,O-Z)                  ATH  2
ARCTNH=0.0D0                               ATH  3
TERM=0.0D0                                 ATH  4
DO 10 N=1,1000                             ATH  5
TERM=TERM+(1.0D0)                          ATH  6
ATNH=(Y**(2*N-1))/(2.*TERM-1.0D0)         ATH  7
IF (DABS(ATNH).LE.(1.0D-13)) GO TO 20      ATH  8
ARCTNH=ARCTNH+ATNH                         ATH  9
10 CONTINUE                                 ATH 10
20 RETURN                                  ATH 11
END                                        ATH 12-
"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."
—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons. Also includes conference proceedings with either limited or unlimited distribution.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include final reports of major projects, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION OFFICE
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Washington, D.C. 20546