PARAMETER ESTIMATION FOR N714-33292 TERRAIN MODELING FROM GRADIENT DATA

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PARAMETER ESTIMATION FOR TERRAIN MODELING FROM GRADIENT DATA

by

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Analysis and Design of a Capsule Landing System and Surface Vehicle Control System for Mars Exploration

Rensselaer Polytechnic Institute
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The author would like to thank Dr. C. N. Shen for his assistance and supervision throughout the year, and especially in the formulation of this report.
This paper develops a method for modeling terrain surfaces for use on Rensselaer Polytechnic Institute's unmanned Martian roving vehicle. The modeling procedure employs a two-step process which uses gradient as well as height data in order to improve the accuracy of the model's gradient. Least square approximation is used in order to stochastically determine the parameters which describe the modeled surface. A complete error analysis of the modeling procedure is included which determines the effect of instrumental measurement errors on the model's accuracy. Computer simulation is used as a means of testing the entire modeling process which includes the acquisition of data points, the two-step modeling process and the error analysis. Finally, to illustrate the procedure, a numerical example is included.
PART 1

INTRODUCTION

An autonomous navigation system is necessary to allow the Martian rover to safely traverse the unknown Martian surface. A forty-minute time lag in communication between Earth and Mars makes remote control impractical.

One of the tasks involved is the development of a mathematical model to represent the surface terrain in front of the vehicle. This model is for use in the vehicle's path selection system which will decide whether the terrain is passable or impassable. It will then choose the appropriate course of action.

The vehicle has a laser rangefinder which gives all of the data used in modeling. In the proposed system, the laser would scan a specified area in front of the vehicle. This area is then divided into a number of sections and the terrain modeled independently in each one of those sections.
PART 2
DATA ACQUISITION

Surface information used in modeling is obtained by a laser rangefinder attached to a mast extending from the vehicle. The mast was assumed to be 3.0 meters high. The vehicle and its coordinate system are shown in Figure 1. Here the \( h'', a'', b'' \)-coordinate system is attached to the vehicle with the \( h'' \) axis along the mast and the \( b'' \) axis in the forward direction of motion. The \( h,a,b \)-coordinate system is formed by the local vertical and an axis in a plane containing the heading and the local vertical. The two systems are coincident at the origin, and are related by the pitch angle \( \xi \) and roll angle \( \phi \).

The laser beam is transmitted at a specified elevation angle, \( \beta \), and azimuth angle \( \theta \). It returns a range value, \( R \), of the surface data point. The vehicle also measures the corresponding value of \( \xi \) and \( \phi \).

The specific scanning pattern used is illustrated in Figure 2. This shows two "W" shaped scan rows separated by an elevation increment, \( \beta_{inc} \). \( \Delta \beta \) and \( \Delta \theta \) are constant for the scan. Consecutive points in each row are taken within a millisecond of each other, a technique referred to as rapid scan. With this method, the roll and pitch angles of the vehicle essentially do not change for neighboring data points. However, because of the large number of data points in each row, the vehicle does change its angular position between corresponding points on different scan rows.
Figure 1  Figure showing the three measured quantities $\phi$, $R$ and $\Theta$, the two coordinate systems and the transformation angles $\phi$ and $\xi$. 
A. Formation of planes

The first step in the modeling procedure stochastically models planes from sets of four data points, using a previously developed procedure.¹

The four points used for each plane are chosen from the same scan row such as points (1,1), (1,2), (1,3) and (1,4) in Figure 2. By choosing the points in this manner and utilizing the rapid scan technique, the plane can be modeled in the $h''',a''',b'''$-coordinate system. This is valid since the four neighboring points are taken with essentially the same $\xi$ and $\phi$ angles. This would not be true if points were taken from different scan rows, such as points (1,1), (2,1), (1,2), (2,2) in Figure 2 since the points are no longer taken with the same $\xi$ and $\phi$ angles.

In Figure 3, the notation for a modeled section is introduced which will be used in the remainder of this report. Here the superscript 'n' refers to the plane number. In the procedure described below, four planes are modeled for each section; therefore, $n=1,2,3,4$. The double primed superscript ' refers to whether the quantity named is in the $h''',a''',b'''$-coordinate system, or if it is missing, the $h,a,b$-coordinate system.

If the subscript is a number, then the quantity refers to a measured data point. If the subscript is a 'p', then this quantity refers to a modeled center point which is defined later.
Figure 2 Scanning scheme used to obtain data points
Figure 3. Notation used in the modeling process
The procedure is outlined below for finding the equation of the plane 'n'. An equation of the form

\[ h'' = a'' x_1'' + b'' x_2'' + x_3'' \]

is used to describe the plane in the \( h'', a'', b'' \)-coordinate system.

where

\[ x_1'' = \frac{\partial h''}{\partial a''} \]

\[ x_2'' = \frac{\partial h''}{\partial b''} \]

and \( x_3'' \) is the height of the plane at \( a'' = 0, b'' = 0 \). These \( x_i'' \)'s are constant parameters which must be determined.

In order to determine the parameters, first the spherical coordinates of the data points, \( \Theta_i^n, \beta_i^n, R_i^n \) are converted into the \( h'', a'', b'' \)-coordinates by

\[ h_i'' = 3 - R_i^n \sin \beta_i^n \]  \hspace{1cm} (2)

\[ a_i'' = R_i^n \cos \beta_i^n \sin \Theta_i^n \]  \hspace{1cm} (3)

\[ b_i'' = R_i^n \cos \beta_i^n \cos \Theta_i^n \]  \hspace{1cm} (4)

\[ i = 1, 2, 3, 4 \]

A matrix equation is then written utilizing the location of the four data points, and a least square estimation of the parameters is formed by

\[ X'' = (A''^T A'')^{-1} A''^T h'' \]

(5)
where
\[ \mathbf{x}^\text{hn} = \begin{pmatrix} x_1^\text{hn} \\ x_2^\text{hn} \\ x_3^\text{hn} \end{pmatrix} \]
\[ \mathbf{h}^\text{hn} = \begin{pmatrix} h_1^\text{hn} \\ h_2^\text{hn} \\ h_3^\text{hn} \\ h_4^\text{hn} \end{pmatrix} \]

and
\[ \mathbf{A}^\text{hn} = \begin{bmatrix} a_1^\text{hn} & b_1^\text{hn} & 1 \\ a_2^\text{hn} & b_2^\text{hn} & 1 \\ a_3^\text{hn} & b_3^\text{hn} & 1 \\ a_4^\text{hn} & b_4^\text{hn} & 1 \end{bmatrix} \]

The equation of the plane is now determined.

The center point, \( \mathbf{P}^\text{hn} \), (Figure 3) is a point on the modeled plane centrally located between the four data points used in determining the plane. Its \( a^\text{hn}, b^\text{hn}, h^\text{hn} \)-coordinates can be found from

\[ a_p^\text{hn} = \left( a_1^\text{hn} + a_2^\text{hn} + a_3^\text{hn} + a_4^\text{hn} \right) \frac{1}{4} \]  \( \text{(6)} \)

\[ b_p^\text{hn} = \left( b_1^\text{hn} + b_2^\text{hn} + b_3^\text{hn} + b_4^\text{hn} \right) \frac{1}{4} \]  \( \text{(7)} \)

\[ h_p^\text{hn} = x_1^\text{hn} a_p^\text{hn} + x_2^\text{hn} b_p^\text{hn} + x_3^\text{hn} \]  \( \text{(8)} \)

B. Transformation of center point information

The location, height, cross-path and in-path slopes of the center point, \( \mathbf{P}^\text{hn} \), have been found above for the \( a^\text{hn}, b^\text{hn}, h^\text{hn} \)-coordinate system. However, this information must be transformed into the \( h, a, b \)-coordinate system to be used for the terrain model.
The height and location can be transformed by

\[
\begin{bmatrix}
    h_p^n \\
    a_p^n \\
    b_p^n
\end{bmatrix} = C(\phi^n) B(\xi^n) \begin{bmatrix}
    h_p'' \\
    a_p'' \\
    b_p''
\end{bmatrix}
\] (9)

where

\[
C^n = \begin{bmatrix}
    \cos \phi^n & -\sin \phi^n & 0 \\
    \sin \phi^n & \cos \phi^n & 0 \\
    0 & 0 & 1
\end{bmatrix} \quad B^n = \begin{bmatrix}
    \cos \xi^n & 0 & \sin \xi^n \\
    0 & 1 & 0 \\
    -\sin \xi^n & 0 & \cos \xi^n
\end{bmatrix}
\]

In order to find the cross-path and in-path slope of the modeled plane in the unprimed system, the normal to the plane, \( N'^n \), is used. The equation for the plane is rewritten as:

\[
0 = x_4'^n h'' + x_1'^n a'' + x_2'^n b'' + x_3''
\] (10)

where

\[
x_4'' = -1
\]

If the unit vectors \( \hat{i}'' , \hat{j}'' , \hat{k}'' \) are defined as in Figure 3, then \( N'^n \) can be written²

\[
N'^n = x_4'^n \hat{i}'' + x_1'^n \hat{j}'' + x_2'^n \hat{k}''
\] (11)

Since the normal is a vector, the normal in the primed system (\( N''^n \)), must be the same vector as the normal in the unprimed system (\( N^n \)). With the unit vectors shown in Figure 4 the vector \( N^n \) is written as

\[
N^n = N_h^n \hat{i} + N_a^n \hat{j} + N_b^n \hat{k}
\] (12)
Using the transformation from the primed to the unprimed system

\[ N^n = C(\phi^n)B(\xi^n) \]  

(13)

Or rewriting with \[ \chi_4'' = -1 \]

\[
\begin{bmatrix}
N^n_h \\
N^n_a \\
N^n_b
\end{bmatrix} = C(\phi^n)B(\xi^n) \begin{bmatrix}
-1 \\
\chi_1'' \\
\chi_2''
\end{bmatrix}
\]  

(14)

Using the normal \( N^n \) the modeled plane is written in the \( h,a,b \)-coordinate system as

\[ O = N^n_h h + N^n_a a + N^n_b b + K_n \]  

(15)

where \( K_n \) is a constant.

Now using Eqn. 15 as the modeled plane equation in the unprimed system, the cross-path and in-path slopes are found and labeled as:

cross-path slope \[ \chi_1^n = \frac{\partial h}{\partial a} = -\frac{N^n_a}{N^n_h} \]  

(16) 

in-path slope \[ \chi_2^n = \frac{\partial h}{\partial b} = -\frac{N^n_b}{N^n_h} \]  

(17)

Therefore, the location, height, cross-path and in-path slopes are known for the center point \( p^n \).

Repeating this process for a total of four planes results in four center points. This situation is shown in Figure 4 where the location, height and derivatives are found at each center point. Thus, there are twelve known quantities which are used in the terrain modeling process for each section modeled.
Figure 4 Illustration of a modeled surface with the four center points indicated.
A. Surface equation

To represent the terrain, a two-dimensional third order polynomial

\[ h = C_{oo} + C_{10} a + C_{01} b + C_{20} \frac{a^2}{2} + C_{11} a b + C_{02} \frac{b^2}{2} + C_{30} \frac{a^3}{6} + C_{21} \frac{a^2 b}{2} + C_{12} \frac{b^2}{2} + C_{03} \frac{b^3}{6} \quad (18) \]

was chosen where the \( C_{ij} \)'s are unknown parameters which must be determined in order to represent the surface.

B. Coordinate shifting

The polynomial used for the surface equation is centered around the origin and it models well near the origin and worsens as the distance from the origin increases. Therefore, the polynomial should be centered as close as possible to the section in which it is used. This can be accomplished by shifting the coordinate axis. A new coordinate system \( a^+, b^+ \) is formed by shifting the axis so that the point \((a_p^+, b_p^+)\) is located at \((0,0)\) in the \( a^+, b^+ \) system (Figure 4). This is accomplished by the transformation

\[ a^+ = a - a_p^+ \quad (19a) \]

\[ b^+ = b - b_p^+ \quad (19b) \]

A new set of parameters is used to write the surface equation

\[ h = C_{oo}^+ + C_{10}^+ a^+ + C_{01}^+ b^+ + C_{20}^+ \frac{(a^+)^2}{2} + C_{11}^+ a^+ b^+ \]

\[ + C_{02}^+ \frac{(b^+)^2}{2} + C_{30}^+ \frac{(a^+)^3}{6} + C_{21}^+ \frac{(a^+)^2 b^+}{2} + C_{12}^+ \frac{a^+ (b^+)^2}{2} + C_{03}^+ \frac{(b^+)^3}{6} \quad (20) \]
Or this can be written in matrix notation as:

\[ h = H M C \]  

where

\[ H M = H M (a^t, b^t) = \begin{bmatrix} 1, a^t, b^t, \frac{(a^t)^2}{2}, a^t b^t, \frac{(a^t)^3}{6}, \frac{(a^t)^2}{2}, b^t, \frac{(b^t)^2}{2}, \frac{(b^t)^3}{6} \end{bmatrix} \]  

\[ C^* = \begin{bmatrix} C_{00}, C_{01}, C_{02}, C_{10}, C_{11}, C_{12}, C_{20}, C_{21}, C_{22}, C_{23} \end{bmatrix}^T \]  

Expressions can easily be found for \( \frac{\partial h}{\partial a^t} \) and \( \frac{\partial h}{\partial b^t} \) from Eqn. 20. These are written in matrix notation as:

\[ \frac{\partial h}{\partial a^t} = V C^* \]  

\[ \frac{\partial h}{\partial b^t} = Y C^* \]  

where

\[ V = \begin{bmatrix} 0, 1, 0, a^t, b^t, 0, (a^t)^2, a^t b^t, (b^t)^2, 0 \end{bmatrix} \]

\[ Y = \begin{bmatrix} 0, 0, 1, 0, a^t, b^t, 0, (a^t)^2, a^t b^t, (b^t)^2 \end{bmatrix} \]

Since at each center point the location, height, \( \frac{\partial h}{\partial a^t} \), and \( \frac{\partial h}{\partial b^t} \), are known, 12 eqns. can be written relating the known quantities to the unknown parameters.

\[ h^n_p = H M (a^t_n, b^t_n) C^* \]  

\[ (\frac{\partial h}{\partial a^t})^n_p = X^1_n = V (a^t_n, b^t_n) C^* \]  

\[ (\frac{\partial h}{\partial b^t})^n_p = X^2_n = Y (a^t_n, b^t_n) C^* \]  

\( n = 1, 2, 3, 4 \)

These 12 eqns. are written in a single matrix equation by
\[ \hat{W} = \hat{T} \hat{C}^T \]

where \( \hat{W} = \begin{bmatrix} h_p^1 x_1^1, x_2^1, h_p^2 x_1^2, x_2^2, h_p^3 x_1^3, x_2^3, h_p^4 x_1^4, x_2^4 \end{bmatrix}^T \)

\[
\hat{T} = \begin{bmatrix}
    \text{HM}(a_{p}^{t1}, b_{p}^{t1}) \\
    \text{V}(a_{p}^{t1}, b_{p}^{t1}) \\
    \text{Y}(a_{p}^{t1}, b_{p}^{t1}) \\
    \text{HM}(a_{p}^{t2}, b_{p}^{t2}) \\
    \text{V}(a_{p}^{t2}, b_{p}^{t2}) \\
    \text{Y}(a_{p}^{t2}, b_{p}^{t2}) \\
    \vdots \\
    \text{HM}(a_{p}^{t4}, b_{p}^{t4}) \\
    \text{V}(a_{p}^{t4}, b_{p}^{t4}) \\
    \text{Y}(a_{p}^{t4}, b_{p}^{t4})
\end{bmatrix}
\]

C. Matrix order reduction

Because of the coordinate shift, by definition the \( a, b \) -coordinates of center point number one are \((0, 0)\). This allows three parameters to be written immediately by utilizing Eqns. 23. Thus

\[
\begin{align*}
h_p^1 &= C_{00}^t \\
x_1^1 &= C_{10}^t \\
x_2^1 &= C_{01}^t
\end{align*}
\]
By utilizing the above values for the first three parameters and eliminating the first three rows of Eqn. 24, a new matrix equation is written as

$$W = \mathbf{T} \mathbf{C} \mathbf{1}^t$$

where

$$\mathbf{W} = 
\begin{bmatrix}
    h_p^2 - h_p^1 - a_p x_1^1 - b_p x_2^1 \\
    x_1^2 - x_1^1 \\
    x_2^2 - x_2^1 \\
    h_p^3 - h_p^1 - a_p x_1^1 - b_p x_2^1 \\
    x_1^3 - x_1^1 \\
    x_2^3 - x_2^1 \\
    h_p^4 - h_p^1 - a_p x_1^1 - b_p x_2^1 \\
    x_1^4 - x_1^1 \\
    x_2^4 - x_2^1
\end{bmatrix}$$

and

$$\mathbf{T} = 
\begin{bmatrix}
    \frac{(a_p^1)^2}{2} & a_p b_p a_p^2 & (b_p^2)^2 & \frac{(a_p^2)^3}{6} & \frac{(a_p^1)^2}{2} b_p a_p^2 & (b_p^2)^2 & (b_p^2)^3 \\
    a_p^2 & b_p^2 & 0 & \frac{(a_p^1)^2}{2} & a_p^2 b_p^2 & (b_p^2)^2 & 0 \\
    0 & a_p^2 & b_p^2 & 0 & \frac{(a_p^1)^2}{2} & a_p^2 b_p^2 & \frac{(b_p^2)^3}{2} \\
    0 & 0 & a_p^4 & b_p^4 & 0 & \frac{(a_p^4)^2}{2} & a_p^4 b_p^4 & \frac{(b_p^4)^2}{2}
\end{bmatrix}$$
This manipulation reduces $\hat{W} (12 \times 1)$ to $\hat{W} (9 \times 1)$, $\hat{T} (12 \times 10)$ to $T (9 \times 7)$, and $\hat{C} (10 \times 1)$ to $\hat{C}_1 (7 \times 1)$.

D. Stochastic fitting

Both $W$ and $T$ in Eqn. 26 are known quantities and $C_1^\dagger$ is the unknown vector to be determined. Since the order of $W$ is higher than that of $C_1^\dagger (9$ to $7$) this system of equations is overdetermined. Therefore, a stochastic fit must be used. The method of least squares estimation was chosen to perform the stochastic fit. This is accomplished by the matrix equation

$$C_1^\dagger = (T^T T)^{-1} T^T W$$

Eqn. 27 requires the inversion of $(T^T T)$ a $7 \times 7$ matrix.

Once the $C_1^\dagger$ vector is determined, the $C^\dagger$ vector is also determined. Thus, the surface polynomial, Eqn. 20, can be written. This allows the calculation of the modeled height, cross-path and in-path slopes for any location $(a,b)$. With this information, the gradient at location $(a,b)$ can be calculated by

$$\text{Gradient} = SG = \left[ \left( \frac{\partial h}{\partial a} \right)^2 + \left( \frac{\partial h}{\partial b} \right)^2 \right]^{1/2}$$
A. **Covariance matrix for the data points**

Because of instrumental inaccuracies in measuring $\Theta$, $\beta$ and $R$, there is error involved in the determination of $a^n$, $b^n$ and $h^n$ for each measured data point. This error can be expressed by an error covariance matrix for each data point.¹

\[
M^n_l = E \left\{ \begin{bmatrix}
\delta h^n_l \\
\delta a^n_l \\
\delta b^n_l
\end{bmatrix} \begin{bmatrix}
\delta h^n_l & \delta a^n_l & \delta b^n_l
\end{bmatrix} \right\}
\]

\[
= G^n_l \begin{bmatrix}
E(\delta R)^2 & 0 & 0 \\
0 & E(\delta \beta)^2 & 0 \\
0 & 0 & E(\delta \Theta)^2
\end{bmatrix} G^n_l^T
\]

where

\[
G^n_l = \begin{bmatrix}
-\sin \beta^n_l & -R^n_l \cos \beta^n_l & 0 \\
\cos \beta^n_l \sin \Theta^n_l & -R^n_l \sin \beta^n_l \sin \Theta^n_l & R^n_l \cos \beta^n_l \cos \Theta^n_l \\
\cos \beta^n_l \cos \Theta^n_l & -R^n_l \sin \beta^n_l \cos \Theta^n_l & -R^n_l \cos \beta^n_l \sin \Theta^n_l
\end{bmatrix}
\]

where $E$ denotes expected value and $\delta R, \delta \beta, \delta \Theta$ are assumed uncorrelated. This equation relates the standard deviations of $h^n_l, a^n_l, b^n_l$ for each data point to the standard deviation of $R, \beta$ and $\Theta$ which are known quantities. There are $16$ of these matrices.

B. **Covariance matrix of the slopes in the primed system**
Because of inaccuracies in measuring the data points, the modeled plane is also subject to inaccuracies. The error covariance matrix for the slopes can be expressed as a function of the covariances of the four data points which are used in modeling the plane.\(^1\)

\[
E \left\{ \begin{bmatrix} \delta x_1^n \\ \delta x_2^n \\ \delta x_3^n 
\end{bmatrix} \right\} = F \begin{bmatrix} \delta h^n \\ \delta h^n^T \end{bmatrix}
\]

where

\[
F = (A''_n A'')^{-1} A''_n^T
\]

These values can easily be evaluated and they relate the covariance matrix of the slopes to the standard deviation of the measured quantities. There are four of these matrices for each section modeled.

C. Covariance matrix for the center points in the primed system

Using perturbation technique, the error covariance matrix for the center points in the primed system can be evaluated. Since there are six variables of interest, the covariance matrix for the center points is a 6 x 6 matrix. Because some of the quantities in this matrix are a function of other quantities in the same matrix, the covariance matrix may best be evaluated by partitions, as shown in Eqn. 31.
Block IV can be evaluated as shown in Appendix A and yields

$$
E\begin{bmatrix}
\delta a_p^{un} \\
\delta b_p^{un}
\end{bmatrix}
= Q E\begin{bmatrix}
\delta a_1^{un} \\
\vdots \\
\delta a_n^{un} \\
\delta b_4^{un}
\end{bmatrix} Q^T
$$

(32)

where

$$
Q = 
\begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{bmatrix}
$$

The separate terms in Block V can be evaluated as

$$
E(\delta x_j^{un} \delta a_p^{un}) = 
\frac{1}{4} \left[ I, -x_1^{un}, -x_2^{un} \right] \left\{ f_{j1} \begin{bmatrix}
E(\delta h_1^{un} \delta a_1^{un}) \\
E(\delta a_1^{un} \delta a_1^{un}) \\
E(\delta b_1^{un} \delta a_1^{un})
\end{bmatrix} + f_{j2} \begin{bmatrix}
E(\delta h_2^{un} \delta a_2^{un}) \\
E(\delta a_2^{un} \delta a_2^{un}) \\
E(\delta b_2^{un} \delta a_2^{un})
\end{bmatrix} + \cdots \right\}
$$

(33)

$$
E(\delta x_j^{un} \delta b_p^{un}) = 
\begin{bmatrix}
E(\delta h_1^{un} \delta a_1^{un}) \\
E(\delta a_1^{un} \delta a_1^{un}) \\
E(\delta b_1^{un} \delta a_1^{un})
\end{bmatrix}
$$

and

$$
E(\delta x_j^{un} \delta b_p^{un}) = 
\begin{bmatrix}
E(\delta h_2^{un} \delta a_2^{un}) \\
E(\delta a_2^{un} \delta a_2^{un}) \\
E(\delta b_2^{un} \delta a_2^{un})
\end{bmatrix}
$$

(33)
\[ \frac{1}{4} \left[ 1, -x_1, -x_2 \right] \begin{bmatrix} 0 \\ E(\delta h_1^{m} \delta b_1^{m}) \\ E(\delta a_1^{m} \delta b_1^{m}) \\ E(\delta b_1^{m} \delta b_1^{m}) \end{bmatrix} + f_{j1} \begin{bmatrix} 0 \\ E(\delta h_2^{m} \delta b_2^{m}) \\ E(\delta a_2^{m} \delta b_2^{m}) \\ E(\delta b_2^{m} \delta b_2^{m}) \end{bmatrix} + \ldots \} (34) \]

where \( f_{j1} \) are elements of \( F = (A^T A)^{-1} A^T \).

These terms are derived in Appendix B and can be calculated directly from Eqn. 29.

Because the covariance matrix (Eqn. 31) is symmetric, Block VI is just the transpose of Block V.

Block VII is equal to the covariance matrix found in Eqn. 30.

Block II can be expressed as

\[
E \left\{ \begin{bmatrix} \delta a_p^{m} \\ \delta b_p^{m} \\ \delta x_1^{m} \\ \delta x_2^{m} \\ \delta x_3^{m} \end{bmatrix} \right\} = E \left\{ \begin{bmatrix} \delta a_p^{m} \\ \delta b_p^{m} \\ \delta x_1^{m} \\ \delta x_2^{m} \\ \delta x_3^{m} \end{bmatrix} \right\} S^n \]

where

\[
S^n = \begin{bmatrix} x_1^{m} & x_2^{m} & a_p^{m} & b_p^{m} & 1 \end{bmatrix} \]

This equation is derived in Appendix C. Note that the expected value matrix used in Eqn. 35 is a subset of the covariance matrix in Eqn. 31. This subset \((M1^n)\) is composed of blocks IV, VI, V, and VII. These blocks have been calculated previously; therefore, Eqn. 35 can be calculated. Similarly, Block III is the transpose of Block II.

Finally, the last block of Eqn. 31 can be calculated by the equation

\[
\]
\[ E \left\{ [\delta h_p^n][\delta h_p^n] \right\} = S^n M1''^n S^{nT} \]  

(36)

where the quantities \( S^n \) and \( M1''^n \) are defined in Eqn. 35.

Thus, the covariance matrices in the primed system can be calculated for the four center points. This matrix relates the standard deviations of the quantities at the center points to the standard deviations of \( R, \theta, \) and \( \beta \).

D. Covariance matrix of the center points in the unprimed system

In the modeling process, the center points were transformed into the unprimed system before the polynomial parameters were evaluated. Therefore, the error covariance matrix of the center points must also be transformed into the unprimed system. In this stage, the error in roll and pitch measurement is introduced into the model.

The covariance matrix of the center points in the unprimed system is defined as:

\[
M_p^n = E \left\{ \begin{bmatrix} \delta h_p^n \\ \delta a_p^n \\ \delta b_p^n \\ \delta x_1^n \\ \delta x_2^n \end{bmatrix} \begin{bmatrix} \delta h_p^n & \delta a_p^n & \delta b_p^n & \delta x_1^n & \delta x_2^n \end{bmatrix} \right\} 
\]

(37)

Note the absence of any \( \delta x_3^n \) terms. These are not included since they are not used in the modeling process. Again breaking this matrix into blocks...
Block A may be evaluated by using the expression

\[
E \begin{pmatrix}
\delta h_p^n \\
\delta a_p^n \\
\delta b_p^n
\end{pmatrix} \begin{pmatrix}
\delta h_p^n \\
\delta a_p^n \\
\delta b_p^n
\end{pmatrix} = \ldots
\]
where

\[
D^n = 
\begin{bmatrix}
-h_p \sin \phi^n \cos \xi^n - a_p^n \cos \phi^n - b_p^n \sin \phi^n \sin \xi^n \\
-h_p^n \cos \phi^n \cos \xi^n - a_p^n \sin \phi^n + b_p^n \cos \phi^n \sin \xi^n \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-h_p \cos \phi^n \sin \xi^n + b_p^n \cos \phi^n \cos \xi^n \\
-h_p \sin \phi^n \sin \xi^n + b_p^n \sin \phi^n \cos \xi^n \\
-h_p^n \cos \xi^n - b_p^n \sin \xi^n
\end{bmatrix}
\]

and \(B^n\) and \(C^n\) are defined in Eqn. 9.

The derivation for the equation is shown in Appendix D. Notice that this expression is a function of the standard deviation of pitch and roll and also of the covariance matrix of the center points in the primed system, Eqn. 31.

This will be the case for all of the blocks in Eqn. 38.

For evaluating Block A, the expression

\[
E \left\{ \begin{bmatrix} \delta x_1^n \\ \delta x_2^n \end{bmatrix} \begin{bmatrix} \delta x_1^n \\ \delta x_2^n \end{bmatrix} \right\} =
\]

\[
\begin{bmatrix} E(\delta \phi)^2 & 0 \\ 0 & E(\delta \xi)^2 \end{bmatrix} \begin{bmatrix} \delta x_1^n \\ \delta x_2^n \end{bmatrix} \begin{bmatrix} \delta x_1^n \\ \delta x_2^n \end{bmatrix}
\]

\[
U_D^n U_C^n E \begin{bmatrix} \delta x_1^n \\ \delta x_2^n \end{bmatrix} B^n C^n
\]
where

\[ U^n = \begin{bmatrix}
\frac{N_o^n}{(N_h^n)^2} & -\frac{1}{N_h^n} & 0 \\
\frac{N_b^n}{(N_h^n)^2} & 0 & -\frac{1}{N_h^n}
\end{bmatrix} \]

and

\[ D^n_x = \begin{bmatrix}
\sin\Phi^n \cos\xi^n - x_1^n \cos\Phi^n - x_2^n \sin\Phi^n \sin\xi^n \\
-\cos\Phi^n \cos\xi^n - x_1^n \sin\Phi^n + x_2^n \cos\Phi^n \sin\xi^n \\
0 \\
\cos\Phi^n \sin\xi^n + x_2^n \cos\Phi^n \cos\xi^n \\
\sin\Phi^n \sin\xi^n + x_2^n \sin\Phi^n \cos\xi^n \\
\cos\xi^n - x_2^n \sin\xi^n
\end{bmatrix} \]

is found. The derivation is shown in Appendix E. Since the covariance matrix Eqn. 38 is symmetric, Block C is just the transpose of Block B.

Finally, Block D is found by squaring Eqn. E-10 and taking the expected value.
Now eliminating non-correlated terms:

\[
E \left\{ \left( U^n D^n_x [\delta \phi] + U^n C^n B^n \begin{bmatrix} 0 \\ [\delta x_1^n] \\ [\delta x_2^n] \end{bmatrix} \right) \right\} = \left\{ \left( U^n D^n_x [\delta \phi] + U^n C^n B^n \begin{bmatrix} 0 \\ [\delta x_1^n] \\ [\delta x_2^n] \end{bmatrix} \right) \right\}^T
\]

Therefore, the error covariance matrices for the four center points in the unprimed coordinate system can be determined.
E. Covariance matrix of the modeling parameters

The covariance matrix of the $\delta C_{ij}$ parameters used in the surface polynomial must be found. This $10 \times 10$ matrix is defined as

$$
M_c = E \begin{bmatrix}
\delta C_{00}^t \\
\delta C_{10}^t \\
\delta C_{01}^t \\
\vdots \\
\delta C_{03}^t
\end{bmatrix}
$$

Matrix $M_c$ may be broken down into blocks as

$$
M_c = E \begin{bmatrix}
\begin{bmatrix}
\delta C_{00}^t \\
\delta C_{10}^t \\
\delta C_{01}^t \\
\delta C_{11}^t
\end{bmatrix} & \begin{bmatrix}
\delta C_{00}^t \\
\delta C_{10}^t \\
\delta C_{01}^t \\
\delta C_{11}^t
\end{bmatrix} \\
\begin{bmatrix}
\delta C_{00}^t \\
\delta C_{10}^t \\
\delta C_{01}^t \\
\delta C_{11}^t
\end{bmatrix} & \delta C_{11}^t
\end{bmatrix}^{1/2}
$$

$$
M_c = E \begin{bmatrix}
M_cI & M_cIII \\
M_cII & M_cIV
\end{bmatrix}
$$

(43)
Covariance Block $M_c I$ may be easily determined as
(See Appendix F)

$$M_c I = R M_p^T R^T$$

(45)

where

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Block $M_c II$ can be evaluated by the expression derived in Appendix G

$$M_c II = -Z \left\{ \begin{bmatrix} \Omega_{22} \\ \Omega_{33} \\ \Omega_{44} \end{bmatrix} \right\} M_p^T R^T$$

(46)

where

$$Z = (T^T T)^{-1} T^T$$

and $\Omega_{ij}$ is defined by Eqns. G-I6a, G-19, and G-11a.

$M_p^n$ is the covariance matrix for the center point found in Part III d.

Block $M_c III$ is the transpose of Block $M_c II$ since $M_c$ is a symmetric matrix.

The expression for Block III is derived in Appendix H and is given by
Thus the covariance matrix of the $C_{ij}$ parameters can be determined.

F. Standard deviation of height

Once the covariance matrix of the parameters have been determined, the standard deviation of height can be found. This value is a function of location. From Eqn. 21 the height of any point $(a, b)$ can be determined. Perturbing Eqn. 21 yields

$$\delta h = HM \delta C$$  \hspace{1cm} (48)$$

Finding the expected value of Eqn. 48

$$E(\delta h \delta h^T) = HM \delta C^T \delta C HMT$$  \hspace{1cm} (49)$$
However, from Eqn. 43, which defines $\delta C^T \delta C^T$ as $M_c$:

$$E(\delta h)^2 = HM(M_c)H^T$$

(50)

From the definition of standard deviation,

$$\sigma_H = \left\{HM(M_c)H^T\right\}^{\frac{1}{2}}$$

(51)

where $\sigma_H = \left\{E(\delta h)^2\right\}^{\frac{1}{2}}$ standard deviation of height.

G. Standard deviation of gradient

The values of $\delta \left(\frac{\partial h}{\partial a} \right)$ and $\delta \left(\frac{\partial h}{\partial b} \right)$ can be found by perturbing Eqns. 22a and 22b.

$$\begin{bmatrix} \delta \left(\frac{\partial h}{\partial a} \right) \\ \delta \left(\frac{\partial h}{\partial b} \right) \end{bmatrix} = \begin{bmatrix} \nabla \delta C^T \\ \gamma \delta C^T \end{bmatrix}$$

(52)

The covariance matrix of the slopes is defined as

$$M_{SLOPE} = E \left\{ \begin{bmatrix} \delta \left(\frac{\partial h}{\partial a} \right) \\ \delta \left(\frac{\partial h}{\partial b} \right) \end{bmatrix} \begin{bmatrix} \delta \left(\frac{\partial h}{\partial a} \right) & \delta \left(\frac{\partial h}{\partial b} \right) \end{bmatrix} \right\}$$

(53)
By using Eqn. 52, this becomes

\[
\mathbf{M}_{\text{slope}} = E \left\{ \begin{bmatrix} V \delta \mathbf{C}^T \\ Y \delta \mathbf{C}^T \end{bmatrix} \begin{bmatrix} \delta \mathbf{C}^T \mathbf{V}^T & \delta \mathbf{C}^T \mathbf{Y}^T \\ \end{bmatrix} \right\}
\]

\[
= E \left\{ \begin{bmatrix} V \delta \mathbf{C}^T \delta \mathbf{C}^T \mathbf{V}^T & V \delta \mathbf{C}^T \delta \mathbf{C}^T \mathbf{Y}^T \\ Y \delta \mathbf{C}^T \delta \mathbf{C}^T \mathbf{V}^T & Y \delta \mathbf{C}^T \delta \mathbf{C}^T \mathbf{Y}^T \end{bmatrix} \right\}
\]

(54)

However, from Eqn. 43

\[
\mathbf{M}_{\text{slope}} = \begin{bmatrix} V \mathbf{M}_c \mathbf{V}^T & V \mathbf{M}_c \mathbf{Y}^T \\ Y \mathbf{M}_c \mathbf{V}^T & Y \mathbf{M}_c \mathbf{Y}^T \end{bmatrix}
\]

(55)

From Eqn. 28, the value for gradient, \( S_g \), is given as

\[
\text{Gradient} = S_g = \left[ \left( \frac{\partial h}{\partial a} \right)^2 + \left( \frac{\partial h}{\partial b} \right)^2 \right]^{1/2}
\]

(56)

Perturbing Eqn. 56 yields

\[
\delta S_g = \left[ \left( \frac{\partial h}{\partial a} \right)^2 + \left( \frac{\partial h}{\partial b} \right)^2 \right]^{-\frac{1}{2}} \left( \frac{\partial h}{\partial a} \right) \delta \left( \frac{\partial h}{\partial a} \right)
\]

\[+ \left[ \left( \frac{\partial h}{\partial a} \right)^2 + \left( \frac{\partial h}{\partial b} \right)^2 \right]^{-\frac{1}{2}} \left( \frac{\partial h}{\partial b} \right) \delta \left( \frac{\partial h}{\partial b} \right)
\]

(57)
Writing in matrix notation

\[
\delta S_G = \frac{1}{S_G} L \begin{bmatrix}
\delta \left( \frac{\partial h}{\partial a} \right) \\
\delta \left( \frac{\partial h}{\partial b} \right)
\end{bmatrix}
\]  \hspace{1cm} (58)

Where

\[
\frac{1}{S_G} = \left[ \left( \frac{\partial h}{\partial a} \right)^2 + \left( \frac{\partial h}{\partial b} \right)^2 \right]^{-\frac{1}{2}}
\]

\[
L = \begin{bmatrix}
\left( \frac{\partial h}{\partial a} \right) & \left( \frac{\partial h}{\partial b} \right)
\end{bmatrix}
\]

Finding the expected value of gradient from Eqn. 58

\[
\sigma_{S_G}^2 = E \left\{ (\delta S_G)(\delta S_G)^T \right\} = \frac{1}{S_G} L \left[ \begin{bmatrix} M_{SLOPE} \end{bmatrix} L^T \right] \frac{1}{S_G}
\]

or finally

\[
\sigma_{S_G} = \frac{1}{S_G} \left\{ L \left[ \begin{bmatrix} M_{SLOPE} \end{bmatrix} L^T \right] \right\}^{\frac{1}{2}}
\]  \hspace{1cm} (60)

Thus, the standard deviation of any point on the modeled surface can be calculated.
NUMERICAL RESULTS

The modeling procedure explained in Parts 2-5 was simulated, using a computer program. This program simulates the scanning process, models the polynomial and performs the error analysis. These results vary with every terrain configuration and scanning parameters. Therefore, only one detailed example will be presented here as an illustration of the developed modeling procedure.

The example terrain surface is shown in Figure 5. Here the vehicle is located on level ground, traveling towards the center of a mound located 23 meters away from the front of the vehicle. The mound is a gaussian hill larger in width than in depth, with a maximum height of two meters. The equation of this hill is

$$h = 2e^{-0.08 (b-23)^2} - 0.05a^2$$

where $h, a, b$ refer to the inertial coordinate system.

Enough data points were taken to allow ten polynomial sections to be modeled. Only one section will be looked at in detail. The data points for this section are shown graphically in Figure 5. Shown here, also, are the two "W" shaped scan rows. Although there is roll and pitch of the vehicle between each row, the points in a row are assumed to be taken so fast that the vehicle essentially does not roll or pitch between data points.

The actual scan was carried out by using constant
Figure 5  Illustration of gaussian hill used for the example
Δβ and Δθ increments. The Δβ angle between successive points in a scan row was \(0.03261\) rad., while the \(β_{inc}\) angle between corresponding points in the two scan rows was \(0.06523\) rad. The Δθ spacing between successive points in both rows was \(0.016305\) rad. The actual data for the 16 data points is shown in Table 1. Here the data point number refers to the row number (first column) and the number of the data point left to right (second column). For each measured point, the vehicle transmits the laser beam at a certain elevation and azimuth angle and receives the range measurement. It also measures the roll and pitch angle corresponding to that point.

The first step in the modeling process is to model planes in the vehicle's coordinate system from sets of four data points. Here no data point overlap was used. The data points which were used in each plane are listed below:

- Plane 1 - (1,1), (1,2), (1,3), (1,4)
- Plane 2 - (1,5), (1,6), (1,7), (1,8)
- Plane 3 - (2,1), (2,2), (2,3), (2,4)
- Plane 4 - (2,5), (2,6), (2,7), (2,8)

The height, location, cross-path and in-path slopes of the center points were found in the \(h",a",b"\)-coordinate system. These quantities were then transformed into the \(h,a,b\)-coordinate system.

The numerical results are shown in Table 2, where HP is the height of the center point, AP and BP are the a and b coordinates of the center point and XP1 and XP2 are the cross-path and in-path slopes respectively.

The next step is forming the surface equation...
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<th>DATA POINT</th>
<th>ELEVATION</th>
<th>AZIMUTH</th>
<th>RANGE</th>
<th>ROLL</th>
<th>PITCH</th>
<th>HEIGHT</th>
<th>A</th>
<th>B</th>
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Table 1  Data points used for terrain model
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<td>HP = 1.53408</td>
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<tr>
<td>AP = -0.55481</td>
<td>AP = 0.84657</td>
</tr>
<tr>
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<td>BP = 21.42360</td>
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<td>XP2 = 0.39474</td>
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<td>HP = 0.48101</td>
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<td>XP2 = 0.30909</td>
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Table 2 Center point information

| C00 = 1.534870 |
| C10 = 0.128256 |
| C01 = 0.394753 |
| C20 = 0.212926 |
| C11 = 0.029409 |
| C02 = -0.066491 |
| C30 = 0.023077 |
| C21 = -0.056413 |
| C12 = -0.005039 |
| C03 = -0.075679 |

Table 3 Modeled polynomial parameters
polynomial. Here the $C_{ij}$ parameters were computed using the information at the four center points. The values which were found are shown in Table 3.

Since the model parameters are determined, the height and gradient for any point can be found. In order to find the modeled surface shape, 100 test points were taken in an area bounded by the four center points. The location of these points is shown in Table 4.

The height of the modeled surface corresponding to the 100 test points is shown in Table 5. Also shown is the actual height and the error of the modeled height compared to the actual height. The maximum error for this section was 6.6 cm. Graphs of four different cross sections of the hill are shown in Figure 6. Here the modeled surface can be seen to approximate the actual surface shape very well.

Tables of the modeled gradient, actual gradient, and gradient error are given in Table 6, again for the 100 test points in Table 4. Figure 7 shows the gradient graphically for four different cross sections of the hill. The modeled gradient can be seen to approximate the actual gradient very well, deviating from the true gradient by only 2.4°, which is fairly small. However, since the maximum slope which the vehicle can climb is 25°, the actual surface indicates an impassable object while the model indicates it passable. Therefore, the maximum threshold must be lowered to compensate for modeling error.

For the error analysis, the standard deviation for
### Table 4

#### A Location of Test Points

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<th>-0.06466</th>
<th>0.16803</th>
<th>0.22374</th>
<th>0.37445</th>
<th>0.5516</th>
<th>0.69627</th>
<th>0.84628</th>
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#### B Location of Test Points

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Table 6: Values of gradient modeled, gradients actual, and gradient error for the test points.
Figure 6: Plot of modeled cross section at A = -0.55481 m vs. distance for actual cross sections.
Figure 6b: Cross section at A = -0.08766 m.
Figure 6c Cross section at $A = 0.37945\, \text{m}$. 

Plot of predicted height vs. actual height for $A = 0.37945$

- Curve for actual height
- 1-Curve for predicted height

Distance from vehicle (meters)
Figure 6d Cross section at A = 0.84658 m.
Figure 7
Plot of modeled gradient vs. actual gradient for a = const = -0.55481

- Curve for actual gradient
- Curve for modeled gradient

Cross section at A = -0.55481 m.

Distance from vehicle (meters)
Figure 7b Cross section at A = -0.08768 m.
Figure 7d Cross section at $A = 0.84658$ m.
elevation and azimuth angles were set at one arc minute, while the standard deviation of range was set at 5 cm. The standard deviation of roll (\(\sigma_{\mathrm{roll}}\)) and pitch (\(\sigma_{\mathrm{pitch}}\)) angles was set at \(0^\circ, 0.25^\circ, 0.5^\circ\) and \(1.0^\circ\). The covariance matrices of the parameters, Eqn. 43, are shown in Table 7 for the four values of \(\sigma_{\mathrm{roll}}\) and \(\sigma_{\mathrm{pitch}}\). The values of this matrix increase with an increase in \(\sigma_{\mathrm{roll}}\) and \(\sigma_{\mathrm{pitch}}\) as expected.

The standard deviation of height (\(\sigma_H\)) can now be found for the 100 test points in Table 4 for the four different values of \(\sigma_R\) (here \(\sigma_R = \sigma_{\mathrm{roll}} = \sigma_{\mathrm{pitch}}\)). These values are shown in Table 8. These values are also plotted for four different cross sections of the hill (Figure 8). In order to keep \(\sigma_H\) below a value such as 20 cm would require a \(\sigma_R\) of 0.5° or less.

The standard deviation of gradient, \(\sigma_{SG}\), is likewise calculated for the 100 test points and for the four values of \(\sigma_R\). These are shown in Table 9. Figure 9 shows plots of modeled \(\sigma_{SG}\) and actual \(\sigma_{SG}\) vs. distance for four different hill cross sections. For \(\sigma_{SG}\) to be below a value such as 6° would require a \(\sigma_R\) of 0.5° or less. For \(\sigma_{SG}\) to be below 3° requires \(\sigma_R\) to be 0.25° or less.
| Table 7 | Covariance matrices of the parameter for |}

- **Covariance Matrix of the Parameters for Sigma Roll = 0.0 Deg Sigma Pitch = 0.0 Deg**

| Parameter | 0.00071 | -0.00007 | -0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 |

- **Covariance Matrix of the Parameters for Sigma Roll = 0.2 Deg Sigma Pitch = 0.2 Deg**

| Parameter | 0.00071 | -0.00007 | -0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 |

- **Covariance Matrix of the Parameters for Sigma Roll = 0.5 Deg Sigma Pitch = 0.5 Deg**

| Parameter | 0.00071 | -0.00007 | -0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 |

- **Covariance Matrix of the Parameters for Sigma Roll = 1.0 Deg Sigma Pitch = 1.0 Deg**

| Parameter | 0.00071 | -0.00007 | -0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 | -0.00000 | 0.00000 |
Table 8: Standard Deviation of Height (DH) for

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Note: The table continues with similar entries for various values of $\phi$.
Figure 8

Plot of standard deviation of height (σh) for four hill cross sections

8a Cross section at A = 0.55481 m.
Figure 8b. Cross section at A = 0.08768 m.

Distance from Vehicle (Meters)
Figure 8c Cross section at $A = 0.37945$ m.
Figure 8d: Cross section at A = 0.84658 m.
Figure 9a: Cross section at $A = 0.55481$ m.

Plot of standard deviation of gradient $(\sigma_g)$ for four values of $g$ for four hill cross sections.
Figure 9b. Cross section at A = -0.0876 m.

Distance from vehicle (meters)

Plot of standard deviation of gradient for #100ST.

- Curve 1: Sigma roll = 0.0 deg, sigma pitch = 0.0 deg.
- Curve 2: Sigma roll = 0.250 deg, sigma pitch = 0.500 deg.
- Curve 3: Sigma roll = 0.250 deg, sigma pitch = 0.250 deg.
Figure 9c Cross section at A = 0.37945 m.
Figure 9d
Cross section at A = 0.84658 m.
This report has developed in detail a two-step terrain modeling procedure, using gradient and height information obtained via a laser rangefinder. The formulated terrain model is composed of third order polynomials which approximate the actual terrain surface. Two steps are used instead of one so that the developed models are less sensitive to instrumental error. The use of gradient information in the modeling process results in a model that follows the actual gradient closer than one using height data only. This is important since the gradient is an important factor in determining whether the terrain is passable or impassable by the vehicle. Also, this method uses few data points, thus saving time in the scanning process. This could allow the vehicle to save energy as well as increase its rate of forward travel.

From the simulation results, this method of terrain modeling seems to have the potential for usefully portraying the actual terrain contour and gradient, although much work is still needed to refine the modeling method.

The use of two scan rows for the terrain model may not prove practical unless the standard deviation of the roll and pitch measurement is reduced to about $0.25^\circ$. If this is not feasible, this modeling method may still be used if the scanning scheme can be changed so that enough points can be
found rapid from one row of scan instead of the present two rows of scan.

This scheme could also be used in conjunction with an edge detection scheme. The plan here is for the edge detection scheme to locate the boundary of an obstacle. Once the boundary has been located, the terrain modeling procedure would be used to determine the frontal shape of the obstacle. By examining the terrain model, a decision could be made as to whether or not the obstacle was passable.

Another plan would be a data point saving technique, where the surface is scanned, using few data points. The terrain modeling procedure would then be used to determine a rough picture of the terrain. If any questionable areas were present, more data points could be taken of these areas to obtain a detailed model, or the area avoided completely.

APPENDIX A

DERIVATION OF COVARIANCE BLOCK IV

The quantities $a_p^n$ and $b_p^n$ are related to the measured quantities of the four data points used to determine the plane by:

$$a_p^n = \left( a_1^n + a_2^n + a_3^n + a_4^n \right) \frac{1}{4} \quad (A-1)$$

$$b_p^n = \left( b_1^n + b_2^n + b_3^n + b_4^n \right) \frac{1}{4} \quad (A-2)$$

Perturbing (A-1) and (A-2) yields

$$\delta a_p^n = \left( \delta a_1^n + \delta a_2^n + \delta a_3^n + \delta a_4^n \right) \frac{1}{4} \quad (A-3)$$

$$\delta b_p^n = \left( \delta b_1^n + \delta b_2^n + \delta b_3^n + \delta b_4^n \right) \frac{1}{4} \quad (A-4)$$

Rewriting this in matrix form

$$\begin{bmatrix} \delta a_p^n \\ \delta b_p^n \end{bmatrix} = Q \begin{bmatrix} \delta a_1^n & \delta a_2^n & \delta a_3^n & \delta a_4^n & \delta b_1^n & \delta b_2^n & \delta b_3^n & \delta b_4^n \end{bmatrix}^T \quad (A-5)$$

where

$$Q = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$
Then multiplying \((A-5)\) by its transpose yields

\[
\begin{bmatrix}
\delta a_p^{\nu n} \\
\delta b_p^{\nu n}
\end{bmatrix}
\begin{bmatrix}
\delta a_p^{\nu n} \\
\delta b_p^{\nu n}
\end{bmatrix}^T = Q
\begin{bmatrix}
\delta a_1^{\nu n} \\
\delta a_2^{\nu n} \\
\vdots \\
\delta b_4^{\nu n}
\end{bmatrix}
\begin{bmatrix}
\delta a_1^{\nu n} \\
\delta a_2^{\nu n} \\
\vdots \\
\delta b_4^{\nu n}
\end{bmatrix}^T
\]  

\[ (A-6) \]

The expected value of \((A-6)\) can now be taken.

Since all of the data points are measured independently, all of the terms on the right of Eqn. \((A-6)\) with non-matching subscripts are non-correlated. Therefore, their expected value is zero. Finally, this yields

\[
E\left\{ \begin{bmatrix}
\delta a_p^{\nu n} \\
\delta b_p^{\nu n}
\end{bmatrix} \right\} =
\begin{bmatrix}
E(\delta a_1^{\nu n})^2 & 0 & 0 & 0 & E(\delta a_1^{\nu n}\delta b_1^{\nu n}) & 0 & 0 & 0 \\
0 & E(\delta a_2^{\nu n})^2 & 0 & 0 & 0 & E(\delta a_2^{\nu n}\delta b_2^{\nu n}) & 0 & 0 \\
0 & 0 & E(\delta a_3^{\nu n})^2 & 0 & 0 & 0 & E(\delta a_3^{\nu n}\delta b_3^{\nu n}) & 0 \\
0 & 0 & 0 & E(\delta a_4^{\nu n})^2 & 0 & 0 & 0 & E(\delta a_4^{\nu n}\delta b_4^{\nu n}) \\
E(\delta a_1^{\nu n}\delta b_1^{\nu n}) & 0 & 0 & 0 & E(\delta b_1^{\nu n})^2 & 0 & 0 & 0 \\
0 & E(\delta a_2^{\nu n}\delta b_2^{\nu n}) & 0 & 0 & 0 & E(\delta b_2^{\nu n})^2 & 0 & 0 \\
0 & 0 & E(\delta a_3^{\nu n}\delta b_3^{\nu n}) & 0 & 0 & 0 & E(\delta b_3^{\nu n})^2 & 0 \\
0 & 0 & 0 & E(\delta a_4^{\nu n}\delta b_4^{\nu n}) & 0 & 0 & 0 & E(\delta b_4^{\nu n})^2
\end{bmatrix}
\]  

\[ (A-7) \]
The expected value quantities in (A-7) are calculated in Eqn. 29. Therefore, Eqn. (A-7) can be evaluated directly.
The perturbed values of the slopes can be written as

\[
\begin{bmatrix}
\delta x_1''^n \\
\delta x_2''^n \\
\delta x_3''^n
\end{bmatrix} = F^n (\delta h''^n - \delta A''^n \mathbf{x}''^n)
\]  

(B-1)

where

\[ F^n = (A''^T A'')^{-1} A''^T \]

\[ A'' = \begin{bmatrix}
a_1''^n & b_1''^n & 1 \\
a_2''^n & b_2''^n & 1 \\
a_3''^n & b_3''^n & 1 \\
a_4''^n & b_4''^n & 1
\end{bmatrix} \]

\[ \delta A''^n = \begin{bmatrix}
\delta a_1''^n x_1''^n + \delta b_1''^n x_2''^n \\
\delta a_2''^n x_1''^n + \delta b_2''^n x_2''^n \\
\delta a_3''^n x_1''^n + \delta b_3''^n x_2''^n \\
\delta a_4''^n x_1''^n + \delta b_4''^n x_2''^n
\end{bmatrix} \]

\[ \delta h''^n = \begin{bmatrix}
\delta h_1''^n \\
\delta h_2''^n \\
\delta h_3''^n \\
\delta h_4''^n
\end{bmatrix}^T \]

Then the vector \((\delta h''^n - \delta A''^n \mathbf{x}''^n)\) can be written expressly by

\[
(\delta h''^n - \delta A''^n \mathbf{x}''^n) = \begin{bmatrix}
\delta h_1''^n - \delta a_1''^n x_1''^n - \delta b_1''^n x_2''^n \\
\delta h_2''^n - \delta a_2''^n x_1''^n - \delta b_2''^n x_2''^n \\
\delta h_3''^n - \delta a_3''^n x_1''^n - \delta b_3''^n x_2''^n \\
\delta h_4''^n - \delta a_4''^n x_1''^n - \delta b_4''^n x_2''^n
\end{bmatrix}
\]  

(B-3)
In matrix form as

\[ (\delta h^n - \delta A^n x^n) = \begin{bmatrix} \delta h_1^n & \delta a_1^n & \delta b_1^n \\ \delta h_2^n & \delta a_2^n & \delta b_2^n \\ \delta h_3^n & \delta a_3^n & \delta b_3^n \\ \delta h_4^n & \delta a_4^n & \delta b_4^n \end{bmatrix} \begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix} \] (B-3)

If \( f_{ij} \) is an element of \( F^n \) then the \( j \)th row of \( \text{Eqn.} (B-1) \) is

\[ \delta x_j^n = \begin{bmatrix} f_{j1}^n \\ f_{j2}^n \\ f_{j3}^n \\ f_{j4}^n \end{bmatrix} (\delta h^n - \delta A^n x^n) \] (B-4)

or

\[ \delta x_j^n = f_{j1}^n \begin{bmatrix} 1 \\ -x_1^n \\ -x_2^n \end{bmatrix} \begin{bmatrix} \delta h_1^n \\ \delta a_1^n \\ \delta b_1^n \end{bmatrix} + f_{j2}^n \begin{bmatrix} 1 \\ -x_1^n \\ -x_2^n \end{bmatrix} \begin{bmatrix} \delta h_2^n \\ \delta a_2^n \\ \delta b_2^n \end{bmatrix} + f_{j3}^n \begin{bmatrix} 1 \\ -x_1^n \\ -x_2^n \end{bmatrix} \begin{bmatrix} \delta h_3^n \\ \delta a_3^n \\ \delta b_3^n \end{bmatrix} + f_{j4}^n \begin{bmatrix} 1 \\ -x_1^n \\ -x_2^n \end{bmatrix} \begin{bmatrix} \delta h_4^n \\ \delta a_4^n \\ \delta b_4^n \end{bmatrix} \] (B-5)

The terms in Block V can be written out as

\[ E \begin{bmatrix} \delta x_1^n \\ \delta x_2^n \\ \delta x_3^n \end{bmatrix} \begin{bmatrix} \delta a_p^n \\ \delta b_p^n \end{bmatrix} = \begin{bmatrix} E(\delta x_1^n \delta a_p^n) & E(\delta x_1^n \delta b_p^n) \\ E(\delta x_2^n \delta a_p^n) & E(\delta x_2^n \delta b_p^n) \\ E(\delta x_3^n \delta a_p^n) & E(\delta x_3^n \delta b_p^n) \end{bmatrix} \] (B-6)
For the first column the terms are now expressed using Eqn. (B-4) and Eqn. (A-1)

\[ E(\delta x_j^n \delta a_p^n) = E \left\{ \left[ f_{j_1} f_{j_2} f_{j_3} f_{j_4} \right] \left[ \delta h_1^n - \delta A x^n \right] \left[ \delta a_1^n \delta a_2^n \delta a_3^n \delta a_4^n \right] \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right] \right\} \left( \frac{1}{4} \right) \] (B-7)

and rewriting this using Eqn. (B-5)

\[ E(\delta x_j^n \delta a_p^n) = E \left\{ f_{j_1} \left[ \begin{array}{c} 1 -X_1^n -X_2^n \\ \delta a_1^n \delta a_2^n \delta a_3^n \delta a_4^n \end{array} \right] \left[ \delta b_1^n \right] \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right] \right\} \left( \frac{1}{4} \right) + \left\{ f_{j_2} \left[ \begin{array}{c} 1 -X_1^n -X_2^n \\ \delta a_1^n \delta a_2^n \delta a_3^n \delta a_4^n \end{array} \right] \left[ \delta b_2^n \right] \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right] \right\} \left( \frac{1}{4} \right) + \ldots \} \] (B-8)

Because terms in the right-hand side of Eqn. (B-8) are uncorrelated if their subscripts do not match, the expected value of these terms is zero. Combining terms and simplifying Eqn. (B-8) results in
\[ E(\delta x_j^n \delta a_p^n) = \]
\[ \frac{1}{4} \left[ 1, -x_1^n, -x_2^n \right] \left\{ f_{j1} \begin{bmatrix} E(\delta h_1^n \delta a_1^n) \\ E(\delta a_1^n \delta a_1^n) \\ E(\delta b_1^n \delta a_1^n) \end{bmatrix} + f_{j2} \begin{bmatrix} E(\delta h_2^n \delta a_2^n) \\ E(\delta a_2^n \delta a_2^n) \\ E(\delta b_2^n \delta a_2^n) \end{bmatrix} \right\} \]
\[ j = 1, 2, 3 \] (B-9)

A similar analysis can be done for the \( E(\delta x_j^n \delta b_p^n) \) terms yielding

\[ E(\delta x_j^n \delta b_p^n) = \]
\[ \frac{1}{4} \left[ 1, -x_1^n, -x_2^n \right] \left\{ f_{j1} \begin{bmatrix} E(\delta h_1^n \delta b_1^n) \\ E(\delta a_1^n \delta b_1^n) \\ E(\delta b_1^n \delta b_1^n) \end{bmatrix} + f_{j2} \begin{bmatrix} E(\delta h_2^n \delta b_2^n) \\ E(\delta a_2^n \delta b_2^n) \\ E(\delta b_2^n \delta b_2^n) \end{bmatrix} \right\} \]
\[ j = 1, 2, 3 \] (B-10)
Eqn. 8, the height of $h_p^{\prime\prime n}$, can be perturbed which yields

$$\delta h_p^{\prime\prime n} = x_1^{\prime\prime n} \delta a_p^{\prime\prime n} + x_2^{\prime\prime n} \delta b_p^{\prime\prime n} + \delta x_1^{\prime\prime n} + \delta x_2^{\prime\prime n} + \delta x_3^{\prime\prime n} \quad (C-1)$$

This can be rewritten as a matrix equation

$$\delta h_p^{\prime\prime n} = S^n \begin{bmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \\ \delta x_1^{\prime\prime n} \\ \delta x_2^{\prime\prime n} \\ \delta x_3^{\prime\prime n} \end{bmatrix} \quad \text{(C-2)}$$

where

$$S^n = \begin{bmatrix} x_1^{\prime\prime n} & x_2^{\prime\prime n} & a_p^{\prime\prime n} & b_p^{\prime\prime n} & 1 \end{bmatrix}$$

Therefore, Eqn. 35 can be written directly by

$$E \left\{ \begin{bmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \\ \delta x_1^{\prime\prime n} \\ \delta x_2^{\prime\prime n} \\ \delta x_3^{\prime\prime n} \end{bmatrix} \right\} E^T = \begin{bmatrix} \delta a_p^{\prime\prime n} \\ \delta b_p^{\prime\prime n} \\ \delta x_1^{\prime\prime n} \\ \delta x_2^{\prime\prime n} \\ \delta x_3^{\prime\prime n} \end{bmatrix} S^n^T \quad \text{(C-3)}$$

$$= M^{\prime\prime n} S^n^T$$
The transformation of the center point in the primed system to the unprimed system

\[
\begin{bmatrix}
  h_p^n \\
  a_p^n \\
  b_p^n
\end{bmatrix} = C^n B^n 
\begin{bmatrix}
  h_p''^n \\
  a_p''^n \\
  b_p''^n
\end{bmatrix}
\]  

(D-1)

can be perturbed

\[
\begin{bmatrix}
  \delta h_p^n \\
  \delta a_p^n \\
  \delta b_p^n
\end{bmatrix} = \delta C^n B^n 
\begin{bmatrix}
  h_p''^n \\
  a_p''^n \\
  b_p''^n
\end{bmatrix} + C^n \delta B^n 
\begin{bmatrix}
  h_p^n \\
  a_p^n \\
  b_p^n
\end{bmatrix} + C^n B^n 
\begin{bmatrix}
  \delta h_p''^n \\
  \delta a_p''^n \\
  \delta b_p''^n
\end{bmatrix}
\]  

(D-2)

This equation can be reduced to yield

\[
\begin{bmatrix}
  \delta h_p^n \\
  \delta a_p^n \\
  \delta b_p^n
\end{bmatrix} = D^n \begin{bmatrix}
  \delta \phi \\
  \delta \xi
\end{bmatrix} + C^n B^n 
\begin{bmatrix}
  \delta h_p''^n \\
  \delta a_p''^n \\
  \delta b_p''^n
\end{bmatrix}
\]  

(D-3)

where \( D^n \) is written in Eqn. 39.

Multiplying Eqn. (D-3) by its transpose and taking the expected value

\[
E \left\{ \begin{bmatrix}
  \delta h_p^n \\
  \delta a_p^n \\
  \delta b_p^n
\end{bmatrix} [\delta h_p^n \delta a_p^n \delta b_p^n] \right\} = E \left\{ D^n \begin{bmatrix}
  \delta \phi \\
  \delta \xi
\end{bmatrix} \begin{bmatrix}
  \delta \phi \\
  \delta \xi
\end{bmatrix} D^n^T \right\}
\]

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\[ D^n \left[ \frac{\delta \phi}{\delta \xi} \right] \begin{bmatrix} \delta h_p^n & \delta a_p^n & \delta b_p^n \end{bmatrix} B^n C^{n^T} + C^n B^n \begin{bmatrix} \delta h_p^n \\ \delta a_p^n \\ \delta b_p^n \end{bmatrix} \begin{bmatrix} \delta \phi \\ \delta \xi \end{bmatrix} D^{n^T} \]

\[ + C^n B^n \begin{bmatrix} \delta h_p^n \\ \delta a_p^n \\ \delta b_p^n \end{bmatrix} \begin{bmatrix} \delta h_p^n & \delta a_p^n & \delta b_p^n \end{bmatrix} B^n C^{n^T} \quad \text{(D-4)} \]

Since \( \delta \phi \) and \( \delta \xi \) are not correlated with themselves or \( \delta h_p^n, \delta a_p^n, \delta b_p^n \) then Eqn. (D-4) reduces to Eqn. 39.
From Eqns. 16 and 17, the expressions for $X_1^n$ and $X_2^n$ are written

$$X_1^n = - \frac{N_a^n}{N_h^n} \quad (E-1)$$

$$X_2^n = - \frac{N_b^n}{N_h^n} \quad (E-2)$$

Perturbing Eqns. (E-1) and (E-2)

$$\delta X_1^n = - \frac{\delta N_a^n N_h^n - N_a^n \delta N_h^n}{(N_h^n)^2}$$

$$= \frac{N_a^n}{(N_h^n)^2} \delta N_h^n - \frac{1}{N_h^n} \delta N_a^n \quad (E-3)$$

$$\delta X_2^n = - \frac{\delta N_b^n N_h^n - N_b^n \delta N_h^n}{(N_h^n)^2}$$

$$= \frac{N_b^n}{(N_h^n)^2} \delta N_h^n - \frac{1}{N_h^n} \delta N_b^n \quad (E-4)$$

Rewriting Eqns. (E-3) and (E-4) as a single matrix equation
\[
\begin{bmatrix}
\delta X_1^n \\
\delta X_2^n
\end{bmatrix} = U^n \begin{bmatrix}
\delta N_h^n \\
\delta N_a^n \\
\delta N_b^n
\end{bmatrix}
\] (E-5)

where

\[
U^n = \begin{bmatrix}
\frac{N_0^n}{(N_h^n)^2} & - \frac{1}{N_h^n} & 0 \\
\frac{N_b^n}{(N_h^n)^2} & 0 & - \frac{1}{N_h^n}
\end{bmatrix}
\]

Now from Eqn. 14

\[
\begin{bmatrix}
N_h^n \\
N_a^n \\
N_b^n
\end{bmatrix} = C^n B^n \begin{bmatrix}
-1 \\
x_1^n \\
x_2^n
\end{bmatrix}
\] (E-6)

Perturbing Eqn. (E-6)

\[
\begin{bmatrix}
\delta N_h^n \\
\delta N_a^n \\
\delta N_b^n
\end{bmatrix} = \delta C^n B^n \begin{bmatrix}
-1 \\
x_1^n \\
x_2^n
\end{bmatrix} + C^n \delta B^n \begin{bmatrix}
-1 \\
x_1^n \\
x_2^n
\end{bmatrix} + C^n B^n \begin{bmatrix}
0 \\
\delta x_1^n \\
\delta x_2^n
\end{bmatrix}
\] (E-7)

The first two right-hand terms can be written as

\[
\left\{ \begin{array}{c}
C_A^n B^n \begin{bmatrix}
-1 \\
x_1^n \\
x_2^n
\end{bmatrix} \\
C^n B_A^n \begin{bmatrix}
-1 \\
x_1^n \\
x_2^n
\end{bmatrix}
\end{array} \right\} \frac{\delta \phi}{\delta \xi}
\] (E-8)
where

\[
C_A = \begin{bmatrix}
-\sin \phi^n & -\cos \phi^n & 0 \\
\cos \phi^n & -\sin \phi^n & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B_A = \begin{bmatrix}
-\sin \xi^n & 0 & \cos \xi^n \\
0 & 0 & 0 \\
-\cos \xi^n & 0 & -\sin \xi^n
\end{bmatrix}
\]

Substituting the values for the quantities indicated in Eqn. (E-8) and performing the operations reduces the first two terms in Eqn. (E-7) to

\[
D^n_x \begin{bmatrix}
\delta \phi \\
\delta \xi
\end{bmatrix}
\]

(E-9)

where \( D^n_x \) is defined in Eqn. 40.

Substituting Eqns. (E-9) and (E-7) into Eqn. (E-5) yields the expression

\[
\begin{bmatrix}
\delta X_1^n \\
\delta X_2^n
\end{bmatrix} = U^n D^n_x \begin{bmatrix}
\delta \phi \\
\delta \xi
\end{bmatrix} + U^n C^n B^n \begin{bmatrix}
0 \\
\delta X_1'^{un} \\
\delta X_2'^{un}
\end{bmatrix}
\]

(E-10)

Multiplying Eqn. (E-10) by the transpose of Eqn. (E-3) and taking the expected value, one obtains the expression for Block B.

\[
E \left\{ \begin{bmatrix}
\delta X_1^n \\
\delta X_2^n
\end{bmatrix} \begin{bmatrix}
\delta h_p^n & \delta a_p^n & \delta b_p^n
\end{bmatrix} \right\} = E \left\{ U^n D^n_x \begin{bmatrix}
\delta \phi \\
\delta \xi
\end{bmatrix} \begin{bmatrix}
\delta \phi & \delta \xi
\end{bmatrix} D^{nt} \right\}
\]
The two middle terms are eliminated since they are not correlated. Finally, Eqn. (E-11) reduces to Eqn. 40 in the text.
APPENDIX F
DERIVATION OF COVARIANCE BLOCK $M_{i1}$

From Eqn. 25
\[
\begin{bmatrix}
C_{00}^+ \\
C_{10}^+ \\
C_{01}^+
\end{bmatrix} =
\begin{bmatrix}
 h_p^+ \\
 x_1^+ \\
 x_2^+
\end{bmatrix}
\] (F-1)

Perturbing Eqn. (F-1) results in
\[
\begin{bmatrix}
\delta C_{00}^+ \\
\delta C_{10}^+ \\
\delta C_{01}^+
\end{bmatrix} =
\begin{bmatrix}
\delta h_p^+ \\
\delta x_1^+ \\
\delta x_2^+
\end{bmatrix}
\] (F-2)

or in another form
\[
\begin{bmatrix}
\delta C_{00}^+ \\
\delta C_{10}^+ \\
\delta C_{01}^+
\end{bmatrix} = R \Phi_1
\] (F-3)

where
\[
R =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and
\[
\Phi_1 =
\begin{bmatrix}
\delta h_p^+ \delta x_1^+ \delta x_1^+ \delta x_2^+
\end{bmatrix}^T
\]

Multiplying Eqn. (F-3) by its transpose and taking the expected value results in
\[
E \left\{ \begin{bmatrix}
\delta C_{00}^i \\
\delta C_{10}^i \\
\delta C_{01}^i
\end{bmatrix}
\begin{bmatrix}
\delta C_{00}^i & \delta C_{10}^i & \delta C_{01}^i
\end{bmatrix} \right\} = M_c I = R \Phi I \Phi^T R^T
\]  

(F-4)

However, \( \Phi I \Phi^T \) is just the covariance matrix of the center point number 1. Therefore,

\[
M_c I = R M_p^i R^T
\]

(F-5)
APPENDIX G

DERIVATION OF COVARIANCE BLOCK NII

The system equation is given as Eqn. 26 for the $C_1$ parameters. Perturbing this equation yields

$$\delta W = T \delta C_1^T + \delta T C_1$$  \hspace{1cm} (G-1)

Rearranging terms in Eqn. (G-1)

$$(\delta W - \delta T C_1^T) = T \delta C_1^T$$  \hspace{1cm} (G-2)

Taking the least square estimate of the perturbed parameters yields

$$\delta C_1^T = (T^T T)^{-1} T^T (\delta W - \delta T C_1^T)$$  \hspace{1cm} (G-3)

or

$$\delta C_1^T = Z (\delta W - \delta T C_1^T)$$  \hspace{1cm} (G-4)

where

$$Z = (T^T T)^{-1} T^T$$

The next step is to find an expression for the right-hand side of Eqn. (G-4). The first step in this process is to perturb the equation for $W$, Eqn. 26b. This yields

$$\delta W = \begin{bmatrix}
\delta h_p^2 - \delta h_p^1 - \delta a_p^{12} x_1 - a_p^{12} \delta x_1^1 - \delta b_p^{12} x_2 - b_p^{12} \delta x_2^1 \\
\delta x_1^2 - \delta x_1^1 \\
\delta x_2^2 - \delta x_2^1 \\
\delta h_p^3 - \delta h_p^1 - \delta a_p^{13} x_1 - a_p^{13} \delta x_1^1 - \delta b_p^{13} x_2 - b_p^{13} \delta x_2^1 \\
\vdots \\
\delta x_2^4 - \delta x_2^1
\end{bmatrix}$$  \hspace{1cm} (G-5)
However, this is written in terms of the transformed coordinates. The transformation equation was given before as

\[ a_p^+ = a_p^n - a_p^1 \]
\[ b_p^+ = b_p^n - b_p^1 \]  \hspace{1cm} (G-6)

Now perturbing Eqn. (G-6) yields

\[ \delta a_p^+ = \delta a_p^n - \delta a_p^1 \]
\[ \delta b_p^+ = \delta b_p^n - \delta b_p^1 \]  \hspace{1cm} (G-7)

Substituting the results of Eqn. (G-7) into Eqn. (G-5) yields

\[
\delta W = \begin{bmatrix}
(\delta h_p^2 - \delta h_p^1 - \delta a_p^2 x_1' + \delta a_p^1 x_2') - a_p^{\frac{4}{2}} \delta x_1' - \delta b_p^2 x_2' \\
(\delta x_1^2 - \delta x_1^1) \\
(\delta x_2^2 - \delta x_2^1) \\
(\delta h_p^3 - \delta h_p^1 - \delta a_p^3 x_1' + \delta a_p^1 x_2') - a_p^{\frac{3}{2}} \delta x_1' - \delta b_p^3 x_2' \\
(\delta x_1^3 - \delta x_1^2) \\
\vdots \\
(\delta x_2^4 - \delta x_2^3)
\end{bmatrix}
\]  \hspace{1cm} (G-8)

This matrix can be split into two parts. One part is a function of the perturbed variables for center point 1. The other part is a function of the perturbed variables for the center points 2, 3, 4.
The matrixies in Eqn. (G-9) can be written as partitioned matrixies in the compact form.

\[
\delta W = \begin{bmatrix}
\delta h_p^2 - \delta a_p^2 x'_1 - \delta b_p^2 x'_2 \\
\delta X_1 \\
\delta X_2 \\
\delta h_p^3 - \delta a_p^3 x'_1 - \delta b_p^3 x'_2 \\
\vdots \\
\delta X_4
\end{bmatrix}
\]

\[
\delta W = \begin{bmatrix}
\delta h_p' - \delta a_p x'_1 - \delta b_p x'_2 - a_p^2 \delta X_1 + b_p^2 \delta X_2 \\
\delta X_1 \\
\delta X_2 \\
\delta h_p' - \delta a_p x'_1 - \delta b_p x'_2 - a_p^3 \delta X_1 + b_p^3 \delta X_2 \\
\vdots \\
\delta X_2
\end{bmatrix}
\]

\[
\text{(G-9)}
\]
where
\[
\Pi_n = \begin{bmatrix}
1 & -x_1^n & -x_2^n & a_p^n & b_p^n \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] (G-11a)
\[
\Phi_n = \begin{bmatrix}
\delta h_p^n & \delta a_p^n & \delta b_p^n & \delta x_1^n & \delta x_2^n
\end{bmatrix}^T
\] (G-11b)

In order to evaluate Eqn. (G-4), an expression for \( \delta T \, C_1^\pm \) must also be found. Multiplying the equation for \( T \) (Eqn. 26b) by \( C_1^\pm \) and perturbing

\[
\delta T \, C_1^\pm = \begin{bmatrix}
(a_p^{t_2} \delta a_p^{t_2} C_{20}^\pm + \delta a_p^{t_2} b_p^{t_2} C_{11}^\pm + a_p^{t_2} \delta b_p^{t_2} C_{11}^\pm + b_p^{t_2} \delta b_p^{t_2} C_{20}^\pm + \frac{(a_p^{t_2})^2}{2} \delta a_p^{t_2} C_{30}^\pm + a_p^{t_2} b_p^{t_2} \delta a_p^{t_2} C_{21}^\pm + \frac{(a_p^{t_2})^2}{2} \delta b_p^{t_2} C_{21}^\pm + \delta a_p^{t_2} \frac{(b_p^{t_2})^2}{2} C_{12}^\pm + a_p^{t_2} b_p^{t_2} \delta b_p^{t_2} C_{12}^\pm + \frac{(b_p^{t_2})^2}{2} \delta b_p^{t_2} C_{21}^\pm + \frac{(b_p^{t_2})^2}{2} \delta b_p^{t_2} C_{30}^\pm
\end{bmatrix}
\] (G-12)
\[ \delta TC_1 = \begin{bmatrix}
\delta h_p^2 & \delta a_p^2 & \delta b_p^2 & \delta x_1^2 & \delta x_2^2 \\
\delta h_p^3 & \delta a_p^3 & \delta b_p^3 & \delta x_1^3 & \delta x_2^3 \\
\delta h_p^4 & \delta a_p^4 & \delta b_p^4 & \delta x_1^4 & \delta x_2^4 
\end{bmatrix}^T \\
\begin{bmatrix}
TK_2 & TK_3 & TK_4
\end{bmatrix}
\]

where

\[ TK_n = \begin{bmatrix}
0 & TK_n(1,2) & TK_n(1,3) & 0 & 0 \\
0 & TK_n(2,2) & TK_n(2,3) & 0 & 0 \\
0 & TK_n(3,2) & TK_n(3,3) & 0 & 0
\end{bmatrix}
\]

\[ TK_n(1,2) = a_p^{\text{fn}} C_{20} + b_p^{\text{fn}} C_{11} + \frac{(a_p^{\text{fn}})^2}{2} C_{30} + a_p^{\text{fn}} b_p^{\text{fn}} C_{21} + \frac{(b_p^{\text{fn}})^2}{2} C_{12} \]

\[ TK_n(1,3) = a_p^{\text{fn}} C_{11} + b_p^{\text{fn}} C_{02} + \frac{(a_p^{\text{fn}})^2}{2} C_{21} + a_p^{\text{fn}} b_p^{\text{fn}} C_{12} + \frac{(b_p^{\text{fn}})^2}{2} C_{03} \]

\[ TK_n(2,2) = C_{20} + a_p^{\text{fn}} C_{30} + b_p^{\text{fn}} C_{21} \]

\[ TK_n(2,3) = C_{11} + a_p^{\text{fn}} C_{21} + b_p^{\text{fn}} C_{12} \]

\[ TK_n(3,2) = C_{11} + a_p^{\text{fn}} C_{21} + b_p^{\text{fn}} C_{12} \]

\[ TK_n(3,3) = C_{02} + a_p^{\text{fn}} C_{12} + b_p^{\text{fn}} C_{03} \]

However, since \( \delta a_p^{\text{fn}} \) and \( \delta b_p^{\text{fn}} \) are defined in Eqn. (G-7), this matrix must be broken into two matrices, one a function of the perturbed variables for the center points 2, 3, and 4, the other a function of the perturbed variables for center point 1. Using the matricies defined by Eqn. 48.
\[ \delta T \Phi^* = \begin{bmatrix} TK_2 & TK_3 & TK_4 \\ \Phi_2 & \Phi_3 & \Phi_4 \end{bmatrix} - \begin{bmatrix} TK_2 & TK_3 \\ \Phi_2 & \Phi_3 \\ TK_4 & TK_4 \end{bmatrix} \]  \hspace{1cm} (G-14)

Therefore, Eqn. (G-14) may be expressed as

\[ \delta \Phi^* = Z \left\{ \begin{bmatrix} \Pi_1 \Phi_2 \\ \Pi_2 \Phi_1 \\ \Pi_3 \Phi_1 \\ \Pi_4 \Phi_1 \end{bmatrix} - \begin{bmatrix} \Pi_1 \Phi_3 \\ \Pi_3 \Phi_3 \\ \Pi_4 \Phi_3 \\ \Pi_4 \Phi_4 \end{bmatrix} \right\} - \begin{bmatrix} TK_2 \Phi_2 \\ TK_3 \Phi_3 \\ TK_4 \Phi_4 \end{bmatrix} \]  \hspace{1cm} (G-15)

Combining terms yields

\[ \delta \Phi^* = Z \left\{ \begin{bmatrix} \Omega_{12} \Phi_2 \\ \Omega_{13} \Phi_3 \\ \Omega_{14} \Phi_4 \end{bmatrix} - \begin{bmatrix} \Omega_{22} \Phi_1 \\ \Omega_{33} \Phi_1 \\ \Omega_{44} \Phi_1 \end{bmatrix} \right\} \]  \hspace{1cm} (G-16)

where

\[ \Omega_{ij} = (\Pi_i - TK_j) \]  \hspace{1cm} (G-16a)

To find the expression for Block \( M_{c,II} \), Eqn. (G-16) must be multiplied by the transpose of Eqn. (F-3) and the expected value taken. This results in

\[ M_{c,II} = E \left\{ \delta \Phi^* \left[ \begin{bmatrix} \delta C_{00}^* \\ \delta C_{10}^* \\ \delta C_{01}^* \end{bmatrix} \right] \right\} \]  \hspace{1cm} (G-17)
\[ M_{cII} = E \left\{ Z \left\{ \begin{bmatrix} \Omega_{12} \Phi_2 + \Phi_1^T \\ \Omega_{13} \Phi_3 + \Phi_1^T \\ \Omega_{14} \Phi_4 + \Phi_1^T \end{bmatrix} - \begin{bmatrix} \Omega_{22} \Phi_1 + \Phi_1^T \\ \Omega_{33} \Phi_1 + \Phi_1^T \\ \Omega_{44} \Phi_1 + \Phi_1^T \end{bmatrix} \right\} \right\} \]  

(G-17)

Since the \( \Phi_n \) matrices represent perturbed quantities, then the expected value of \( \Phi_i \Phi_j^T \) is equal to the covariance matrix \( M_p^i \) if \( i=j \), or equals 0 if \( i \neq j \).

\[ E \left\{ \Phi_i \Phi_j^T \right\} = \left\{ \begin{array}{ll} 0 & \text{if } i=j \\ M_p^i & \text{if } i \neq j \end{array} \right\} \]  

(G-18)

Multiplying terms in Eqn. (G-17) yields

\[ M_{cII} = E \left\{ Z \left\{ \begin{bmatrix} \Omega_{12} \Phi_2 + \Phi_1^T \\ \Omega_{13} \Phi_3 + \Phi_1^T \\ \Omega_{14} \Phi_4 + \Phi_1^T \end{bmatrix} - \begin{bmatrix} \Omega_{22} \Phi_1 + \Phi_1^T \\ \Omega_{33} \Phi_1 + \Phi_1^T \\ \Omega_{44} \Phi_1 + \Phi_1^T \end{bmatrix} \right\} R^T \} \]  

(G-19)

Using Eqn. (G-18) and the fact that the expected value of a constant is equal to itself

\[ M_{cII} = -Z \left\{ \begin{bmatrix} \Omega_{22} \\ \Omega_{33} \\ \Omega_{44} \end{bmatrix} \right\} M_p^i R^T \]  

(G-20)
APPENDIX H

DERIVATION OF COVARIANCE BLOCK $M_{III}$

This block is defined by the quantity

$$M_{III} = E \left\{ \delta C_1^\dagger \delta C_1^{\dagger T} \right\} \quad (H-1)$$

Since the expression for $\delta C_1^\dagger$ was derived in Appendix G (Eqn. G-16), Eqn. (H-1) may be evaluated by

$$M_{III} = E \left\{ Z \left\{ \begin{bmatrix} \Omega_{12} \Phi_2 \\ \Omega_{13} \Phi_3 \\ \Omega_{14} \Phi_4 \end{bmatrix} - \begin{bmatrix} \Omega_{22} \Phi_1 \\ \Omega_{33} \Phi_1 \\ \Omega_{44} \Phi_1 \end{bmatrix} \right\} \right\} \quad (H-2)$$

Multiplying out the terms yields

$$M_{III} = E \left\{ Z \left\{ \begin{bmatrix} \Omega_{12} \Phi_2 \\ \Omega_{13} \Phi_3 \\ \Omega_{14} \Phi_4 \end{bmatrix} - \begin{bmatrix} \Omega_{22} \Phi_1 \\ \Omega_{33} \Phi_1 \\ \Omega_{44} \Phi_1 \end{bmatrix} \right\} \right\}^T \quad (H-2)$$

$$M_{III} = E \left\{ Z \left\{ \begin{bmatrix} \Omega_{12} \Phi_2 \\ \Omega_{13} \Phi_3 \\ \Omega_{14} \Phi_4 \end{bmatrix} \begin{bmatrix} \Omega_{12} \Phi_2 \\ \Omega_{13} \Phi_3 \\ \Omega_{14} \Phi_4 \end{bmatrix} \right\} - \left\{ \begin{bmatrix} \Omega_{12} \Phi_2 \\ \Omega_{13} \Phi_3 \\ \Omega_{14} \Phi_4 \end{bmatrix} \begin{bmatrix} \Omega_{22} \Phi_1 \\ \Omega_{33} \Phi_1 \\ \Omega_{44} \Phi_1 \end{bmatrix} \right\} \right\}$$

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\[
\begin{bmatrix}
\Omega_{22} \Phi_1 \\
\Omega_{33} \Phi_1 \\
\Omega_{44} \Phi_1
\end{bmatrix}
\begin{bmatrix}
\Omega_{12} \Phi_2 \\
\Omega_{13} \Phi_3 \\
\Omega_{14} \Phi_4
\end{bmatrix}^T
\begin{bmatrix}
\Omega_{22} \Phi_1 \\
\Omega_{33} \Phi_1 \\
\Omega_{44} \Phi_1
\end{bmatrix}
\begin{bmatrix}
\Omega_{12} \Phi_2 \\
\Omega_{13} \Phi_3 \\
\Omega_{14} \Phi_4
\end{bmatrix}
\begin{bmatrix}
\Omega_{22} \Phi_1 \\
\Omega_{33} \Phi_1 \\
\Omega_{44} \Phi_1
\end{bmatrix}
\begin{bmatrix}
\Omega_{12} \Phi_2 \\
\Omega_{13} \Phi_3 \\
\Omega_{14} \Phi_4
\end{bmatrix}
\begin{bmatrix}
\Omega_{22} \Phi_1 \\
\Omega_{33} \Phi_1 \\
\Omega_{44} \Phi_1
\end{bmatrix}
\]

Using Eqn. (G-18) to combine terms and eliminate non-correlated terms yields

\[
M_{cIII} = Z \begin{bmatrix}
\Omega_{12} M_p^2 \Omega_{12} & 0 & 0 \\
0 & \Omega_{13} M_p^3 \Omega_{13} & 0 \\
0 & 0 & \Omega_{14} M_p^4 \Omega_{14}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\Omega_{22} M_p^1 \Omega_{22} & \Omega_{22} M_p^1 \Omega_{33} & \Omega_{22} M_p^1 \Omega_{44} \\
\Omega_{33} M_p^1 \Omega_{22} & \Omega_{33} M_p^1 \Omega_{33} & \Omega_{33} M_p^1 \Omega_{44} \\
\Omega_{44} M_p^1 \Omega_{22} & \Omega_{44} M_p^1 \Omega_{33} & \Omega_{44} M_p^1 \Omega_{44}
\end{bmatrix} Z^T
\]

\[\text{(H-4)}\]