MATHEMATICAL MODEL FOR THE SIMULATION OF DYNAMIC DOCKING TEST SYSTEM (DDTS) ACTIVE TABLE MOTION
MATHEMATICAL MODEL FOR THE SIMULATION OF DYNAMIC DOCKING TEST SYSTEM (DDTS)
ACTIVE TABLE MOTION

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ABSTRACT

This document describes the mathematical model developed to describe the three-dimensional motion of the Dynamic Docking Test System (DDTS) active table. The active table is modeled as a rigid body supported by six flexible hydraulic actuators which produce the commanded table motions.

Key Words

Docking Simulator
Dynamic Docking Test System (DDTS)
Equations of Motion
Hydraulic Actuator
Mathematical Model
Motion Simulator
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1. D2-118544-2, "Dynamic Docking Test System (DDTS) Active Table Computer Program NASA Advanced Docking System (NADS)," August 30, 1974.

1.0 INTRODUCTION

The development of the three-dimensional mathematical model and computer program which simulates the dynamic motion of the DDTS active table in response to table motion commands is documented in two volumes. Volume 1 presents the derivation of the mathematical model and Volume 2 (Reference 1) describes the resulting computer program, "NASA Advanced Docking System (NADS)."

The active table shown in Figure 1-1 is modeled as a rigid body, and each of the six actuators is modeled as a flexible rod with pinned ends. The model includes nonlinear hydraulic equations for the hydraulic actuators and a mathematical representation of the electronic control system for each actuator. Actuator position, velocity, and differential pressure across the hydraulic piston are used as feedback signals in the control system. The nomenclature used in the equations is shown in the Appendix.
Figure 1-1. DDT Simulator Facility
D2-118544-1

2.0 COORDINATE SYSTEMS AND TRANSFORMATIONS

2.1 INERTIAL COORDINATES \((x_I, y_I, z_I)\)

The inertial simulator coordinate system is an orthogonal, right-handed coordinate system whose origin is on the simulator centerline in the plane of the floor swivel joints. The \(y_I\) and \(z_I\) axes form a horizontal plane, and the \(x_I\) axis is positive down (see Figure 2-1).

2.2 TABLE COORDINATES \((x_T, y_T, z_T)\)

The table coordinate system is an orthogonal, right-handed coordinate system whose origin is at the center of gravity of the simulator table. The \(y_T\) and \(z_T\) axes lie in the plane of the table, and the \(x_T\) axis is positive "down" (see Figure 2-1).

2.3 ACTUATOR COORDINATES \((x_{s_i}, y_{s_i}, z_{s_i})\)

Each actuator has its own coordinate system. The \(x_{s_i}\) axis is colinear with the actuator centerline. The \(y_{s_i}\) axis is perpendicular to the \(x_{s_i}\) axis and the inertial gravity vector. The \(z_{s_i}\) axis is perpendicular to both \(x_{s_i}\) and \(y_{s_i}\) and is positive "up" as shown in Figure 2-1.

2.4 TRANSFORMATION FROM INERTIAL TO TABLE COORDINATES

Euler angles shown in Figure 2-2 are used to transform from inertial coordinates to table coordinates.
Figure 2-1. Active Table Coordinate Systems
2.4 (Continued)

Figure 2-2. Euler Angles

The order of rotation is $\theta$, $\psi$, $\phi$ which corresponds to rotation about the $y$, $z$, $x$ axes, respectively. The transformation from table to inertial coordinates is:

$$
\begin{bmatrix}
    x_I \\
    y_I \\
    z_I
\end{bmatrix} =
[A]
\begin{bmatrix}
    x_T \\
    y_T \\
    z_T
\end{bmatrix}
$$

Therefore:

$$
\begin{bmatrix}
    x_T \\
    y_T \\
    z_T
\end{bmatrix} =
[A]^T
\begin{bmatrix}
    x_I \\
    y_I \\
    z_I
\end{bmatrix}
$$
2.4 (Continued)

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\psi} \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
0 & \frac{\cos \phi}{\cos \psi} & -\frac{\sin \phi}{\cos \psi} \\
0 & \sin \phi & \cos \phi \\
1 & -\cos \phi \tan \psi & \sin \phi \tan \psi
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
\] (3)

Where:

\[
[A] =
\begin{bmatrix}
C\phi \cdot C\psi & -C\phi \cdot C\theta \cdot S\psi + S\theta \cdot S\phi & S\phi \cdot C\theta \cdot S\psi + C\phi \cdot S\theta \\
S\psi & C\phi \cdot C\psi & -S\phi \cdot C\psi \\
-S\theta \cdot C\psi & C\phi \cdot S\theta \cdot S\psi + S\phi \cdot C\theta & -S\phi \cdot S\theta \cdot S\psi + C\phi \cdot C\theta
\end{bmatrix}
\] (4)

C = cosine
S = sine

2.5 TRANSFORMATION FROM ACTUATOR TO INERTIAL COORDINATES

The transformation from actuator coordinates to inertial coordinates uses the following angles:

\( \theta_A \) - the angle between the horizontal plane through the floor joint and the actuator \( x_s \) axis (Figure 2-3)

\( \beta_A \) - the angle between the inertial \( z_I \) axis and the projection of the actuator \( x_s \) axis in the \( y_I \) - \( z_I \) plane (Figure 2-3)

\[
\sin \theta_A = -\frac{r_{s_x}}{l_{p_I}}
\] (5)

\[
\cos \theta_A = \frac{\sqrt{r_{s_y}^2 + r_{s_z}^2}}{l_{p_I}}
\] (6)
Figure 2-3. Actuator Transformation Angles
2.5 (Continued)

\[
\sin \beta_i = \frac{r_s y_i}{\sqrt{r_s^2 + r_{s2}^2}} \\
\cos \beta_i = \frac{r_s z_i}{\sqrt{r_s^2 + r_{s2}^2}}
\]

where \( r_s \), \( r_{s2} \), and \( r_{s3} \) are vector components of actuator length, \( z_i \), \( y_i \), and \( x_i \), in the inertial coordinate system.

The transformation from actuator to inertial coordinates is then:

\[
\begin{bmatrix}
x_i \\
y_i \\
z_i
\end{bmatrix} = \begin{bmatrix}
x_{s1} \\
y_{s1} \\
z_{s1}
\end{bmatrix} [T_{i_1}]
\]

where:

\[
[T_{i_1}] = \begin{bmatrix}
-S^A_i & 0 & -C^A_i \\
C^A_i \cdot S^A_i & C^A_i & -S^A_i \cdot S^A_i \\
C^A_i \cdot C^A_i & -S^A_i & -C^A_i \cdot C^A_i
\end{bmatrix}
\]
2.5 (Continued)

The equivalent "Euler angle" rotations for this transformation (from inertial to actuator coordinates) would:

a. Rotate $-90^\circ$ about $Y_I$

b. Rotate $\beta_{A_i}$ about the intermediate $z$ axis

c. Rotate $-\theta_{A_i}$ about the intermediate $y$ axis

2.6 TRANSFORMATION FROM INDIVIDUAL ACTUATOR COORDINATES TO TABLE COORDINATES

Using the previous transformations, the transformation from individual actuator coordinate systems to the table coordinate system becomes:

$$
\begin{pmatrix}
x_T \\
y_T \\
z_T
\end{pmatrix} =
[T_i]^T
\begin{pmatrix}
x_{S_i} \\
y_{S_i} \\
z_{S_i}
\end{pmatrix}
$$

where

$$
[T_i]^T = [A]^T [T_{I_i}]
$$

Likewise:

$$
\begin{pmatrix}
x_{S_i} \\
y_{S_i} \\
z_{S_i}
\end{pmatrix} =
[T_i]
\begin{pmatrix}
x_T \\
y_T \\
z_T
\end{pmatrix}
$$

where:

$$
[T_i] = [T_{I_i}]^T [A]
$$
3.0 TABLE MOTION COMMANDS

Table commands are specified in the inertial coordinate system. Actuator commands for two types of commands will be discussed: sinusoidal position commands and constant velocity commands.

3.1 SINUSOIDAL POSITION COMMANDS

Let \( \Delta x_I \), \( \Delta y_I \), and \( \Delta z_I \) be the amplitude of commanded sinusoidal table motion in the inertial coordinate system. The total inertial commands are then obtained by adding the commanded sinusoidal motion to the initial inertial position of the table.

\[
\begin{align*}
\dot{x}_I &= x_{I_0} + \Delta x_I \sin \omega_c t \\
\dot{y}_I &= y_{I_0} + \Delta y_I \sin \omega_c t \\
\dot{z}_I &= z_{I_0} + \Delta z_I \sin \omega_c t \\
\dot{\theta}_I &= \theta_{I_0} + \Delta \theta \sin \omega_c t \\
\dot{\psi}_I &= \psi_{I_0} + \Delta \psi \sin \omega_c t \\
\dot{\phi}_I &= \phi_{I_0} + \Delta \phi \sin \omega_c t \\
\dot{x}_I &= \Delta x_I \omega_c \cos \omega_c t \\
\dot{y}_I &= \Delta y_I \omega_c \cos \omega_c t \\
\dot{z}_I &= \Delta z_I \omega_c \cos \omega_c t \\
\dot{\theta}_I &= \Delta \theta \omega_c \cos \omega_c t \\
\dot{\psi}_I &= \Delta \psi \omega_c \cos \omega_c t \\
\dot{\phi}_I &= \Delta \phi \omega_c \cos \omega_c t
\end{align*}
\]
3.2 CONSTANT VELOCITY COMMANDS

For constant velocity commands ($\Delta x_I$, $\Delta y_I$, $\Delta z_I$, $\Delta \theta$, $\Delta \psi$, $\Delta \phi$), the total inertial commands are:

\[
\begin{align*}
    x_{Ic} &= x_{Io} + \Delta x_I t \\
    y_{Ic} &= y_{Io} + \Delta y_I t \\
    z_{Ic} &= z_{Io} + \Delta z_I t \\
    \theta_c &= \theta_o + \Delta \dot{\theta} t \\
    \psi_c &= \psi_o + \Delta \dot{\psi} t \\
    \phi_c &= \phi_o + \Delta \dot{\phi} t \\
    \dot{x}_{Ic} &= \Delta \dot{x}_I \\
    \dot{y}_{Ic} &= \Delta \dot{y}_I \\
    \dot{z}_{Ic} &= \Delta \dot{z}_I \\
    \ddot{\theta}_c &= \Delta \ddot{\theta} \\
    \ddot{\psi}_c &= \Delta \ddot{\psi} \\
    \ddot{\phi}_c &= \Delta \ddot{\phi}
\end{align*}
\]

(16)

3.3 ACTUATOR COMMANDS

The total inertial position and velocity commands are transformed to individual actuator commands.

Let $[A_c]$ be the transformation from the table coordinate system to the inertial coordinate system (equation 4) with the Euler angles replaced by the commanded Euler angles. Then the commanded inertial velocities of the actuator/table attachment points are:
3.3 (Continued)

\[
\begin{align*}
\begin{bmatrix}
\dot{r}_{sx_1} \\
\dot{r}_{sy_1}
\end{bmatrix}
&= \begin{bmatrix}
\dot{x}_{I_c} \\
\dot{y}_{I_c}
\end{bmatrix}
+ \left[ A_c \right]
\begin{bmatrix}
0 & -\omega_z c & \omega y_c \\
\omega_z c & 0 & -\omega_x c \\
-\omega y c & \omega x c & 0
\end{bmatrix}
\begin{bmatrix}
\dot{r}_{xa_1} \\
\dot{r}_{ya_1}
\end{bmatrix}
\end{align*}
\]

(17)

where

\[
\begin{align*}
\begin{bmatrix}
\omega_x c \\
\omega y c \\
\omega z c
\end{bmatrix}
&= \begin{bmatrix}
1 & 0 & S \psi_c \\
0 & S \psi_c & C \psi_c \cdot C \phi_c \\
0 & C \phi_c & -C \psi_c \cdot S \phi_c
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_c \\
\dot{\psi}_c \\
\dot{\theta}_c
\end{bmatrix}
\end{align*}
\]

(18)

The commanded inertial components of actuator length are:

\[
\begin{align*}
\begin{bmatrix}
r_{sx_1} \\
r_{sy_1}
\end{bmatrix}
&= \begin{bmatrix}
x_{I_c} \\
y_{I_c}
\end{bmatrix}
+ \left[ A_c \right]
\begin{bmatrix}
r_{xa_1} \\
r_{ya_1}
\end{bmatrix}
- \begin{bmatrix}
y_{f_1} \\
z_{f_1}
\end{bmatrix}
\end{align*}
\]

(19)

Commanded actuator lengths are then:

\[
l_{c_1} = \sqrt{r_{sx_1}^2 + r_{sy_1}^2 + r_{sz_1}^2}
\]

(20)

3-3
3.3 (Continued)

and commanded actuator velocities are:

\[
\mathbf{c}_i = \frac{1}{\mathbf{c}_i} \left[ r_{sx_i} \cdot \dot{r}_{sx_i} + r_{sy_i} \cdot \dot{r}_{sy_i} + r_{sz_i} \cdot \dot{r}_{sz_i} \right] (21)
\]
4.0 SERVO ELECTRONICS

The servo electronics consist of actuator position and rate command signals; forward loop compensation network; servo valve; and position, rate, and differential pressure feedback loops as shown in Figure 4-1. The forward loop compensation network, the valve dynamics representation, and the position feedback filter are designed to be optional in the computer program. For example, if \( \beta < 1 \), then the forward loop compensation network is not included in the simulation. Similarly, the valve dynamics and position feedback filter are neglected if \( \omega_v < 1 \) and \( \omega_s < 1 \), respectively.

4.1 ACTUATOR COMMANDS

Define

\[
\delta_i = K_f \dot{l}_{c_i} + K_{rc} \ddot{l}_{c_i}
\]

\[
\ddot{\delta}_i = K_f \ddot{l}_{c_i} + K_{rc} \dddot{l}_{c_i}
\]

where \( l_{c_i} \), \( \dot{l}_{c_i} \), and \( \ddot{l}_{c_i} \) are the commanded actuator length, velocity, and acceleration, respectively.

4.2 POSITION FEEDBACK FILTER

The differential equation for the actuator position feedback filter is:

If \( \omega_s \geq 1 \)

\[
x_s^{\prime\prime} = \omega_s^2 \left( l_p^{\prime} - \frac{2\zeta_s}{\omega_s} x_s^{\prime} - x_s \right)
\]

(23)

If \( \omega_s < 1 \)

\[
x_s^{\prime\prime} = l_p^{\prime}
\]

\[
x_s^{\prime} = l_p^{\prime}
\]

\[
x_s = l_p
\]

(24)
Figure 4-1. Servo Electronics Block Diagram

\[ g = \frac{s}{s/\omega_s + 1} \]

\[ a = \frac{s}{s/\omega_{pf2} + 1} \]

\[ a_0 = \frac{s}{s/\omega_{pf1} + 1} \]

\[ \frac{1}{\left(\frac{s}{\omega_s}\right)^2 + 2\frac{\zeta_s}{\omega_s}s + 1} \]
4.3 DIFFERENTIAL PRESSURE FEEDBACK COMPENSATION

Pressure feedback compensation consists of two parts—a first-order lag filter and a high-pass filter. The first-order lag attenuates the higher frequency pressure fluctuations, while the high-pass filter eliminates the static differential pressure caused by unequal piston areas. The differential equations for these filters are:

\[
\begin{align*}
\dot{a}_{o_i} &= \omega_{pf1} \left[ K_{pf} \left( p_{1_i} - p_{2_i} \right) - a_{o_i} \right] \\
\dot{a}_{l_i} &= \omega_{pf2} \left( \dot{a}_{o_i} - a_{l_i} \right)
\end{align*}
\]

where \( K_{pf} \) is the pressure feedback gain.

4.4 FORWARD LOOP COMPENSATION NETWORK

The forward loop compensation network consists of a lead-lag filter with corner frequencies \( \alpha \) and \( \beta \).

For \( \beta \geq 1 \):

\[
\begin{align*}
\dot{e}_i &= \beta \left[ \frac{1}{\alpha} \left( \dot{a}_{l_i} - a_{l_i} - K_r \dot{x}_{s_i} - K_f x_{s_i} \right) \\
&\quad + (\dot{a}_{l_i} - a_{l_i} - K_r \dot{x}_{s_i} - K_f x_{s_i}) - e_i \right]
\end{align*}
\]

where \( \dot{a}_{l_i}, a_{l_i}, \dot{x}_{s_i}, \dot{x}_{s_i}, \) and \( x_s \) are signals from the feedback loops.

For \( \beta < 1 \):

\[
\begin{align*}
\dot{e}_i &= 0 \\
\dot{e}_i &= \delta_i - a_{l_i} - K_r \dot{x}_{s_i} - K_f x_{s_i}
\end{align*}
\]

where \( K_f \) and \( K_r \) are the displacement and rate feedback gains, respectively.
4.5 SERVO VALVE DYNAMICS

The dynamics of the servo valve are represented by a single-degree-of-freedom system with a natural frequency $\omega_v$ and damping ratio $\zeta_v$.

If $\omega_v \geq 1$:

$$\ddot{Q}_{o_i} = \omega_v^2 \left( K_g e_i - \frac{2\zeta_v}{\omega_v} \dot{Q}_{o_i} - Q_{o_i} \right)$$  \hspace{1cm} (28)$$

where $K_g$ is the forward loop amplifier gain and $Q_o$ is the no-load flow through the valve.

If $\omega_v < 1$:

$$\ddot{Q}_{o_i} = \dot{Q}_{o_i} = 0$$  \hspace{1cm} (29)$$

$$Q_{o_i} = K_g e_i$$

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5.0 ACTUATOR MODEL

Each actuator is modeled as a flexible rod with pinned ends. Hydraulic forces are calculated using nonlinear hydraulic flow equations and unequal push and pull piston areas. Actuator control system electronics are modeled and include differential pressure, velocity, and position feedback. The actuator geometry and nomenclature are shown in Figure 5-1.

5.1 ACTUATOR MASS AND INERTIA CHARACTERISTICS

The mass moment of inertia of the piston rod is:

\[ I_p = \frac{m_p l_r^2}{12} \]  

(30)

where \( m_p \) is the driven mass (piston and piston rod). The mass moment of inertia of the entire actuator assembly about the floor pivot is:

\[ I_{A_i} = I_{A_c} + I_p + m_p \left( \frac{1}{2} l_{p_i}^2 - \frac{1}{2} l_r^2 \right) \]  

(31)

where:

\[ I_{A_c} = \text{Mass moment of inertia of the cylinder structure about the floor pivot (excludes driven mass, } m_p). \]

The effective rigid lateral mass of each actuator assembly for use in the equations of motion is then:

\[ m_{L_i} = \frac{I_{A_i}}{l_{p_i}^2} \]  

(32)

5.2 ACTUATOR FLEXIBILITY

The dynamic bending characteristics of each actuator are calculated assuming that the cylinder is rigid compared to the piston rod and that the effective dynamic mass is lumped at the rod end seal of the cylinder. The bending characteristics are also assumed to be identical for each of the two bending planes of the actuator.
Figure 5-1. Hydraulic Actuator
5.2 (Continued)

The effective dynamic mass lumped at the cylinder end is approximated as:

\[ m_{q_i} = \left[ I_{A_c} / l_c^2 \right] + \left[ m_p / 2 \right] \]  \hspace{1cm} (33)

Assuming pinned joints between the cylinder and piston rod, the piston rod stiffness is:

\[ k_{r_i} = \frac{3 (EI)_r l_r}{l_{r1_i} l_{r2_i}} \]  \hspace{1cm} (34)

where \( l_{r1_i} \) and \( l_{r2_i} \) are defined as follows:

\[ l_{r1_i} = l_{p1_i} - l_c \]
\[ l_{r2_i} = l_r - l_{r1_i} \]

The effective lateral stiffness of the actuator with a rigid cylinder is:

\[ k_{e_i} = \frac{3 (EI)_r \left( l_{r1_i} l_{r1_i} l_{p1} \right) \left( l_r + l_{p1} \right) \left( l_c - l_{p1} \right)} {l_{r1_i} l_{r2_i} l_c l_r} \]  \hspace{1cm} (35)

The actuator bending frequency is then:

\[ \omega_{e_i}^2 = \frac{k_{e_i}}{m_{q_i}} \]  \hspace{1cm} (35)

5.3 HYDRAULIC FLOW EQUATIONS

The nonlinear hydraulic flow equations are based on the derivations presented in Reference 2 for a double-acting hydraulic piston. A schematic of the hydraulic servo valve and actuator is shown in Figure 5-2.
Figure 5-2. Hydraulic Servo Valve Schematic
5.3 (Continued)

The flow continuity equations are:

\[ Q_1 = C_p (p_1 - p_2) + \dot{V}_1 + \frac{V_1}{\beta_e} \dot{p}_1 \]

\[ Q_2 = C_p (p_1 - p_2) - C_{e_p} p_2 - \dot{V}_2 - \frac{V_2}{\beta_e} \dot{p}_2 \]

where:

\[ Q_1 = Q_o - 2 K_c p_1 \]
\[ Q_2 = Q_o + 2 K_c p_2 \]

and

\[ Q_o = \text{The no-load flow of the valve} \]
\[ K_c = \text{Valve pressure flow coefficient} \]
\[ C_p = \text{Leakage coefficient across the piston} \]
\[ C_{e_p} = \text{Leakage coefficient past the piston rod seal} \]

The volume-stroke relationships are:

\[ V_1 = V_{01} + A_1 (l_p - l_o) \]
\[ V_2 = V_{02} - A_2 (l_p - l_o) \]
\[ \dot{V}_1 = A_1 \dot{l}_p \]
\[ \dot{V}_2 = -A_2 \dot{l}_p \]

where \( V_{01} \) and \( V_{02} \) are the hydraulic volumes at zero stroke.
5.3 (Continued)

Therefore, neglecting piston rod seal leakage, the hydraulic flow equations for each actuator are:

\[
\dot{p}_1 = \frac{\beta e}{V_1} \left[ Q_0 - 2k_c p_1 - C_p (p_1 - p_2) - A_1 \dot{p}_1 \right]
\]

\[
\dot{p}_2 = \frac{\beta e}{V_2} \left[ -Q_0 - 2k_c p_2 + C_p (p_1 - p_2) + A_2 \dot{p}_1 \right]
\]

5.4 ACTUATOR FORCES

Actuator forces \( F_p \) are calculated from the differential pressure across the piston. In addition to the viscous damping forces associated with the actuators, coulomb friction is also included.

\[
F_p = A_1 p_1 - A_2 p_2 - B_p \dot{p}_1 - C_F F_f
\]

A velocity "bandwidth" for coulomb friction is used to prevent a discontinuity at zero velocity.

The coulomb friction force is:

\[
F_{CF} = -C_F F_f
\]
5.3 (Continued)

where \( C_F \) is a coefficient which is a function of actuator velocity:

If \( |\dot{V}_p| \geq V_{bw} \), then \( C_F = \frac{\dot{V}_p}{|\dot{V}_p|} \) \hspace{1cm} (43)

If \( |\dot{V}_p| < V_{bw} \), then \( C_F = \frac{\dot{V}_p}{V_{bw}} \)
6.0 EQUATIONS OF MOTION

Table and actuator equations of motion are written in the body fixed table coordinates in the following form:

\[ \{\ddot{X}\} = [M]^{-1} \{C\} \]  \hspace{1cm} (44)

where: \( \{\ddot{X}\} \) is a column of accelerations for each degree of freedom (six degrees of freedom for the table and two elastic degrees of freedom for each actuator)

\([M]\) is the 18 x 18 coupled mass matrix

\(\{C\}\) is a column of generalized forces for each degree of freedom

The mass coupling effects of the actuators due to table motions are derived by Lagrange's method. The three-dimensional rigid motions of the actuators are completely constrained (i.e., they are dependent upon the motions of the table). These constraints are expressed by the velocity substitutions in the energy expressions.

The final equations are much simplified when compared with the equations which would result from a rigorous derivation. Due to the nonorthogonality between actuator and table motions, a large number of nonlinear velocity coupling terms results. All of these terms were assumed negligible since, for expected table velocities, they are quite small and their omission prevents the equations from becoming unwieldy.

6.1 MASS MATRIX

The kinetic energy of the rigid table and actuators is:

\[ T = \frac{1}{2} m_t \left( \dot{r}_T \cdot \dot{r}_T \right) + \frac{1}{2} \omega_T \cdot (I_T \omega_T) + \frac{1}{2} \sum_{p=1}^{6} m_p \dot{r}_p^2 \]

\[ + \frac{1}{2} \sum_{i=1}^{6} \int_0^1 p_i (\dot{x}_{a_i}^2 + \dot{z}_{a_i}^2) \, dm_i \]  \hspace{1cm} (45)

6-1
6.1 (Continued)

The first two terms in this expression are the energy of the rigid table; the third term is the energy of the piston rod due to stroking; and the fourth term is the energy of the actuator assembly due to rotation about the floor pivot.

Using the transformation from table coordinates to actuator coordinates, \([T_i]_\), shown in equation (14):

\[
\dot{p}_i = T_{i11} (\dot{x}_T - \omega_z r_{ya_i} + \omega_y r_{za_i}) + T_{i12} (\dot{y}_T + \omega_z r_{xa_i} - \omega_x r_{za_i}) + T_{i13} (\dot{z}_T - \omega_y r_{xa_i} + \omega_x r_{ya_i})
\]

\[
= T_{i11} \dot{x}_a_i + T_{i12} \dot{y}_a_i + T_{i13} \dot{z}_a_i
\]

\[
\dot{y}_a_i = \left[ T_{i21} (\dot{x}_T - \omega_z r_{ya_i} + \omega_y r_{za_i}) + T_{i22} (\dot{y}_T + \omega_z r_{xa_i} - \omega_x r_{za_i}) + T_{i23} (\dot{z}_T - \omega_y r_{xa_i} + \omega_x r_{ya_i}) \right] \frac{x_s}{p_i} \]

\[
+ \dot{y}_e_i \phi_i (x_s)
\]

\[
= \dot{y}_{r_i} \frac{x_s}{p_i} + \dot{y}_e_i \phi_i (x_s)
\]
6.1 (Continued)

\[
\ddot{z}_{ai} = \left[ T_{i31} \left( \dot{x}_T - \omega_z y_{ai} + \omega_y r_{za_i} \right) + T_{i32} \left( \dot{y}_T + \omega_z r_{xa_i} - \omega_x r_{za_i} \right) + T_{i33} \left( \dot{z}_T - \omega_y r_{xa_i} + \omega_x r_{ya_i} \right) \right] \frac{x_s}{1_{pi}} + \dot{z}_{ei} \phi_i (x_s)
\]

\[
= \dot{z}_{ri} \frac{x_s}{1_{pi}} + \dot{z}_{ei} \phi_i (x_s)
\]

where:

\( \phi_i (x_s) \) is the actuator bending mode shape as a function of \( x_s \), and \( y_{ei} \) and \( z_{ei} \) are bending velocities of the actuators.

The elastic bending modes of each actuator are assumed to be a simple mode shape normalized to unity at the upper end of the cylinder \( x_s = 1_c \).

There are two identical modes for each actuator. The generalized mass for each mode is assumed to be lumped at the upper end of the cylinder; thus, the mass distribution terms can be integrated.

E.g.

\[
\int_0^{1_{pi}} x_s^2 \ dm = \frac{1}{2} I_{A_i} = \text{mass moment of total actuator assembly about the floor pivot.}
\]
6.1 (Continued)

and:

\[ \int_0^{l_p} p_i x_s (x_s) \, dm_i = c \, m_i \]

where \( m_i \) is the generalized mass of \( i^{th} \) actuator for each bending mode.

Lagrange's equation requires the determination of \( \frac{d}{dt} \frac{\partial T}{\partial \dot{Q}_j} \) where \( Q_j \) is the \( j^{th} \) generalized coordinate in the equation of motion. In this simulation:

\[
\begin{align*}
\dot{Q}_1 &= x_T \\
\dot{Q}_2 &= y_T \\
\dot{Q}_3 &= z_T \\
\dot{Q}_4 &= \omega_T \times x \\
\dot{Q}_5 &= \omega_T \times y \\
\dot{Q}_6 &= \omega_T \times z \\
\dot{Q}_7 &= y_{e1} \\
\dot{Q}_8 &= z_{e1} \\
\dot{Q}_{17} &= y_{e6} \\
\dot{Q}_{18} &= z_{e6}
\end{align*}
\]

Then:

\[
\frac{\partial T}{\partial Q_j} = \frac{1}{2} m_j \frac{\partial}{\partial Q_j} \left( \dot{r}_T \cdot \dot{r}_T \right) + \frac{1}{2} \frac{\partial}{\partial Q_j} \left[ \frac{\partial}{\partial \dot{Q}_j} \left( \dot{r}_T \cdot (I \cdot \omega_T) \right) \right]
\]

\[
+ m \sum_{i=1}^{6} \frac{\partial}{\partial Q_j} \left( \frac{a y_i}{Q_j} \dot{y}_i + \frac{a z_i}{Q_j} \dot{z}_i \right) \, dm_i
\]

6-4
6.1 (Continued)

For \( j \leq 6 \):

\[
\frac{\dot{a}y_{ai}}{\dot{\xi}_j} = x_s \frac{1}{p_i} c_{y_{ij}}
\]

where \( c_{y_{ij}} \) are coefficients from equation (47). e.g.

\[ c_{y_{ii}} = T_{i21} \]

\[ c_{y_{i4}} = -T_{i22} r_{za_i} + T_{i23} r_{ya_i} \]

\[ \vdots \]

etc.

For \( j = 7, 9, 11, 13, 15, 17 \):

\[
\frac{\dot{a}y_{ai}}{\dot{\xi}_j} = \phi_i = 1.
\]

and, for \( j = 8, 10, 12, 14, 16, 18 \)

\[
\frac{\dot{a}y_{ai}}{\dot{\xi}_j} = 0
\]

Likewise:

\[
\frac{\ddot{a}z_{ai}}{\dot{\xi}_j} = x_s \frac{1}{p_i} c_{z_{ij}}
\]

for \( j \leq 6 \)

\[ = 0 \quad \text{for } j = 7, 9, 11, 13, 15, 17 \]

\[ = 1 \quad \text{for } j = 8, 10, 12, 14, 16, 18 \]
Therefore, for \( j \leq 6 \), the last term in equation (50) becomes:

\[
\frac{\partial T}{\partial Q_j} = \sum_{i=1}^{6} \left[ \left( \frac{x_s}{p_i} \right)^2 \hat{y}_{r_i} \ c_{yij} + \frac{x_s}{p_i} \ c_{yij} \ \hat{y}_{e_i} \ \phi_i \ (x_s) \right.
\]

\[
+ \left( \frac{x_s}{p_i} \right)^2 \hat{z}_{r_i} \ c_{zij} + \frac{x_s}{p_i} \ c_{zij} \ \hat{z}_{e_i} \ \phi_i \ (x_s) \right] \ dm_i
\]

\[
= \sum_{i=1}^{6} \left[ \frac{IA}{1-p_i^2} \left( \hat{y}_{r_i} \ c_{yij} + \hat{z}_{r_i} \ c_{zij} \right) + m_i \ \frac{1}{p_i} c_{yij} \ \hat{y}_{e_i} + c_{zij} \ \hat{z}_{e_i} \right] \quad \text{(58)}
\]

where, from equations (47) and (48):

\[
\hat{y}_{r_i} = T_{i21} \dot{x}_{a_i} + T_{i22} \ \dot{y}_{a_i} + T_{i23} \ \dot{z}_{a_i} \quad \text{(59)}
\]

\[
\hat{z}_{r_i} = T_{i31} \dot{x}_{a_i} + T_{i32} \ \dot{y}_{a_i} + T_{i33} \ \dot{z}_{a_i} \quad \text{(60)}
\]

and, from equation (46):

\[
\dot{x}_{a_i} = \dot{x}_{T} - \omega z \ r_{ya_i} + \omega y \ r_{za_i} \quad \text{(61)}
\]

\[
\dot{y}_{a_i} = \dot{y}_{T} + \omega z \ r_{xa_i} - \omega x \ r_{za} \quad \text{(62)}
\]

\[
\dot{z}_{a_i} = \dot{z}_{T} - \omega y \ r_{xa_i} + \omega x \ r_{ya_i} \quad \text{(63)}
\]
6.1 (Continued)

Also, for \( j = 7, 9, 11, 13, 15, 17 \), the last term in equation (50) becomes:

\[
\frac{\partial T}{\partial Q_j} = \left( \hat{y}_r_i \frac{x_s}{T_p_i} \phi_i \left( x_s \right) + \dot{y}_e_i \phi_i^2 \left( x_s \right) \right) \text{d}m_i 
\]

(64)

\[
= m_{q_i} \left( \dot{y}_r_i \frac{1}{T_p_i} + \dot{y}_e_i \right)
\]

(65)

Likewise, for \( j = 8, 10, 12, 14, 16, 18 \):

\[
\frac{\partial T}{\partial Q_j} = m_{q_i} \left( \ddot{z}_r_i \frac{1}{T_p_i} + \ddot{z}_e_i \right)
\]

(66)

Differentiating equations (50), (58), (65), and (66) to obtain \( \frac{d}{dt} \left( \frac{\partial T}{\partial Q_j} \right) \):

for \( j \leq 6 \)

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial Q_j} \right) = \frac{d}{dt} \left\{ \frac{1}{2} m_T \frac{\partial}{\partial Q_j} (\ddot{r}_i \cdot \ddot{r}_i) + \frac{1}{2} \frac{\partial}{\partial Q_j} \left[ \ddot{\omega}_T \cdot (\dot{I}_T \cdot \dot{\omega}_T) \right] \right\}
\]

\[
+ m_p \sum_{i=1}^{6} \left[ \frac{\partial}{\partial Q_j} \ddot{i}_p_i \right] + \ddot{i}_p_i \frac{d}{dt} \frac{\partial}{\partial Q_j} \frac{\partial}{\partial Q_j} p_i
\]

\[
+ \sum_{i=1}^{6} \left[ \frac{1}{p_i} \left( I_{A_i}^2 \hat{I}_{A_i} - 2 I_A \hat{I}_p_i \right) \right] \left( \hat{y}_r_i \hat{c}_{y_{ij}} + \hat{z}_r_i \hat{c}_{z_{ij}} \right)
\]

\[
+ \frac{I_{A_i}}{p_i} \left( \hat{y}_r_i \hat{c}_{y_{ij}} + \hat{z}_r_i \hat{c}_{z_{ij}} \right) + \frac{\hat{y}_r_i}{p_i} \hat{c}_{y_{ij}} + \frac{\hat{z}_r_i}{p_i} \hat{c}_{z_{ij}}
\]

6-7
6.1 (Continued)

\[\begin{align*}
&+ \frac{1}{l^2_{p_i}} \left( l_{p_i} l_c m_{q_i} - m_{q_i} l_c \dot{l}_{p_i} \right) \left( c_{y_{ij}} \dot{y}_{e_i} + c_{z_{ij}} \dot{z}_{e_i} \right) \\
&+ m_{q_i} \frac{l_c}{l^2_{p_i}} \left( c_{y_{ij}} \ddot{y}_{e_i} + c_{z_{ij}} \ddot{z}_{e_i} + \ddot{c}_{y_{ij}} \dot{y}_{e_i} + \ddot{c}_{z_{ij}} \dot{z}_{e_i} \right) \\
\end{align*}\]

(67)

for \( j = 7, 9, 11, 13, 15, 17 \):

\[\begin{align*}
\frac{d}{dt} \left( \frac{\delta T}{\delta q_j} \right) &= m_{q_i} \left( r_{y_{i}} \frac{l_c}{l_{p_i}} + \dot{y}_{e_i} \right) + m_{q_i} \left( c_{y_{i}} \frac{l_c}{l_{p_i}} + \ddot{y}_{e_i} \right) \\
&- y_{r_{i}} \frac{l_c}{l_{p_i}} \\
\end{align*}\]

(68)

and, for \( j = 8, 10, 12, 14, 16, 18 \):

\[\begin{align*}
\frac{d}{dt} \left( \frac{\delta T}{\delta q_i} \right) &= m_{q_i} \left( r_{z_{i}} \frac{l_c}{l_{p_i}} + \dot{z}_{e_i} \right) \\
&+ m_{q_i} \left( c_{z_{i}} \frac{l_c}{l_{p_i}} + \ddot{z}_{e_i} - y_{r_{i}} \frac{l_c}{l_{p_i}} \right) \\
\end{align*}\]

(69)

At this point, it is convenient to redefine several quantities to simplify the notation.

Let:

\[\begin{align*}
\begin{pmatrix}
\dot{x}_{a_i} \\
\dot{y}_{a_i} \\
\dot{z}_{a_i}
\end{pmatrix}
&= \begin{pmatrix}
\dot{r}_{a_{i1}} \\
\dot{r}_{a_{i2}} \\
\dot{r}_{a_{i3}}
\end{pmatrix} \\
\begin{pmatrix}
\ddot{x}_{a_i} \\
\ddot{y}_{a_i} \\
\ddot{z}_{a_i}
\end{pmatrix}
&= \begin{pmatrix}
\ddot{r}_{a_{i1}} \\
\ddot{r}_{a_{i2}} \\
\ddot{r}_{a_{i3}}
\end{pmatrix}
\end{align*}\]

(70)
6.1 (Continued)

also:

\[ C_{y_{1j}} = T_{1j} \]  
\[ C_{z_{1j}} = T_{3j} \]  

Then equations (46), (59), and (60) become:

\[ \dot{p}_i = \sum_{k=1}^{3} T_{1k} \dot{r}_{a_{ki}} \]  
\[ \dot{y}_i = \sum_{k=1}^{3} T_{2k} \dot{r}_{a_{ki}} \]  
\[ \dot{z}_i = \sum_{k=1}^{3} T_{3k} \dot{r}_{a_{ki}} \]

Then, for \( j = 1, 2, 3 \):

\[ \frac{\partial \dot{p}_i}{\partial Q_j} = T_{1j} \]  
\[ \dot{p}_i = \sum_{k=1}^{3} \ddot{T}_{1k} \ddot{r}_{a_{ik}} + T_{1k} \dddot{r}_{a_{ik}} \]  
\[ \dot{y}_i = \sum_{k=1}^{3} \ddot{T}_{2k} \ddot{r}_{a_{ik}} + T_{2k} \dddot{r}_{a_{ik}} \]  
\[ \dot{z}_i = \sum_{k=1}^{3} \ddot{T}_{3k} \ddot{r}_{a_{ik}} + T_{3k} \dddot{r}_{a_{ik}} \]
6.1 (Continued)

Therefore, for \( j = 1, 2, 3 \):

\[
\frac{d}{dt} \left( \frac{\partial \mathbf{T}}{\partial \mathbf{q}_j} \right) = \left[ j^{\text{th}} \text{ row of } m \right] \left( \ddot{\mathbf{r}}_T + \mathbf{a}_T \times \dot{\mathbf{r}}_T \right)
\]

\[
+ m_p \sum_{i=1}^{6} \left[ T_{i1j} \sum_{k=1}^{3} \left( \ddot{r}_{a_{1k}} + T_{i1k} \ddot{r}_{a_{1k}} \right) \right]
\]

\[
+ \sum_{i=1}^{3} \left[ \frac{1}{p_i} \left( I_{p_i} \ddot{T}_{a_i} \right) \right] \sum_{j=1}^{3} \left( T_{i2j} \sum_{k=1}^{3} \left( \ddot{r}_{a_{2k}} + T_{i2k} \ddot{r}_{a_{2k}} \right) \right)
\]

\[
+ T_{i3j} \sum_{k=1}^{3} \left( \ddot{r}_{a_{3k}} + T_{i3k} \ddot{r}_{a_{3k}} \right)
\]

\[
+ \sum_{i=1}^{3} \sum_{j=1}^{3} \left( \ddot{r}_{a_{ik}} + T_{i3k} \ddot{r}_{a_{ik}} \right)
\]

\[
+ \frac{1}{2} \left( I_{p_i} \ddot{\mathbf{m}_{q_i}} - m_{q_i} \ddot{\mathbf{p}_i} \right) \left( T_{i2j} \ddot{y}_{e_i} + T_{i3j} \ddot{z}_{e_i} \right)
\]

\[
+ \frac{1}{2} \left( I_{p_i} \ddot{\mathbf{m}_{q_i}} - m_{q_i} \ddot{\mathbf{p}_i} \right) \left( T_{i2j} \ddot{y}_{e_i} + T_{i3j} \ddot{z}_{e_i} \right)
\]

The general equations are extremely complex, particularly because of all the centrifugal and coriolis acceleration terms. These terms can be shown.
6.1 (Continued)

to be small (less than 0.01 g) for the expected table velocities. Neglecting these terms, equation (79) becomes (for \( j = 1, 2, 3 \)):

\[
\frac{d}{dt} \left( \frac{\ddot{a}_{T}}{a_{Q_{j}}} \right) = \left[ \text{\( j^{th} \) row of} \ m_{T} \left( \ddot{\tau}_{T} + \dddot{\omega}_{T} \times \dddot{\tau}_{T} \right) \right] \\
\begin{align*}
&+ m_{p} \sum_{i=1}^{6} T_{i_{1j}} \sum_{k=1}^{3} T_{i_{1k}} \dddot{a}_{ik} \\
&+ \sum_{i=1}^{6} \frac{I_{A_{i}}}{2} \left( T_{i_{2j}} \dddot{a}_{ik} + T_{i_{3j}} \dddot{a}_{ik} \right) \\
&+ \sum_{i=1}^{6} m_{q_{i}} \frac{l_{c}}{T_{p_{i}}} \left( T_{i_{2j}} \dddot{y}_{e_{i}} + T_{i_{3j}} \dddot{z}_{e_{i}} \right)
\end{align*}
\]

Likewise for \( j = 4, 5, 6 \):

\[
\frac{d}{dt} \left( \frac{\ddot{a}_{T}}{a_{Q_{j}}} \right) = \left[ \text{\( j^{th} \) row of} \ I_{T} \left( \ddot{\omega}_{T} \times (\dddot{\omega}_{T} \times \dddot{\omega}_{T}) \right) \right] \\
\begin{align*}
&+ m_{p} \sum_{i=1}^{6} \frac{\dddot{a}_{p_{i}}}{a_{Q_{i}}} \dddot{p}_{i} + \sum_{i=1}^{6} \frac{I_{A_{i}}}{2} \left( y_{r_{i}} c y_{j} + z_{r_{i}} c z_{j} \right) \\
&+ \sum_{i=1}^{6} m_{q_{i}} \frac{l_{c}}{T_{p_{i}}} \left( c y_{j} \dddot{y}_{e_{i}} + c z_{j} \dddot{z}_{e_{i}} \right)
\end{align*}
\]

where:

\[
\begin{align*}
\frac{\dddot{a}_{p_{i}}}{a_{Q_{4}}} &= -T_{i_{12}} r_{za_{1}} + T_{i_{13}} r_{ya_{1}} = C_{xx_{i}} \\
\frac{\dddot{a}_{p_{i}}}{a_{Q_{5}}} &= T_{i_{11}} r_{za_{1}} - T_{i_{13}} r_{xa_{1}} = C_{xy_{i}}
\end{align*}
\]
Define:

\[
C_{yi4} = -T_{i12} \cdot r_{za_i} + T_{i12} \cdot r_{ya_i} = C_{yx_i}
\]

(85)

\[
C_{zi4} = -T_{i13} \cdot r_{za_i} + T_{i13} \cdot r_{ya_i} = C_{zx_i}
\]

(86)

\[
C_{yi5} = T_{i21} \cdot r_{za_i} - T_{i23} \cdot r_{xa_i} = C_{yy_i}
\]

(87)

\[
C_{yi6} = -T_{i21} \cdot r_{ya_i} + T_{i22} \cdot r_{xa_i} = C_{yz_i}
\]

(88)

Then, using equations (72), (73), and (74), equation (81) becomes:

\[
\frac{d}{dt} \left( \frac{\partial \dot{r}_i}{\partial q_j} \right) = \left[ \text{row (j-3) of } [\dot{\dot{\gamma}}] \right] \left\{ \dot{\omega}_T \right\} + \dot{\omega}_T \times (\dot{\gamma} \cdot \dot{\omega}_T)
\]

\[
+ m_p \sum_{i=1}^{6} \left( \frac{\partial \dot{p}_i}{\partial q_j} \right) \sum_{k=1}^{3} T_{i1k} \ddot{r}_{a_{ik}}
\]

\[
+ \sum_{i=1}^{6} \frac{l_{A_i}}{l_{p_i}} \left( C_{yi,j} \sum_{k=1}^{3} T_{i2k} \ddot{r}_{a_{ik}} + C_{zi,j} \sum_{k=1}^{3} T_{i3k} \ddot{r}_{a_{ik}} \right)
\]

\[
+ \sum_{i=1}^{6} \frac{m_{q_i}}{l_{p_i}} \left( C_{yi,j} \ddot{y}_{e_i} + C_{zi,j} \ddot{z}_{e_i} \right)
\]

(89)

Simplifying equation (68), for j = 7, 9, 11, 13, 15, 17:

\[
\frac{d}{dt} \left( \frac{\partial \dot{r}_i}{\partial q_j} \right) \approx m_{q_i} \left( \frac{l_c}{l_{p_i}} \ddot{y}_{r_i} + \ddot{y}_{e_i} \right)
\]
6.1 (Continued)

\[ m_{q_i} \ddot{e}_i + m_{q_i} \frac{1}{p_i} \sum_{k=1}^{3} T_{i2k} \ddot{r}_{a_{ik}} \]  

(90)

and, simplifying equation (69), for \( j = 8, 10, 12, 14, 16, 18 \):

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial q_j} \right) \approx m_{q_i} \ddot{e}_i + m_{q_i} \frac{1}{p_i} \sum_{k=1}^{3} T_{i3k} \ddot{r}_{a_{ik}} \]  

(91)

Expanding some of the summations in equations (89), (90), and (91):

\[ \sum_{k=1}^{3} T_{i1k} \ddot{r}_{a_{ik}} = T_{i11} \left( \dddot{x}_T - \dot{\omega}_z r_y + \dot{y}_z r_{za} \right) \]

\[ + T_{i12} \left( \dddot{y}_T + \dot{\omega}_z r_x - \dot{\omega}_x r_{za} \right) \]  

\[ + T_{i13} \left( \dddot{z}_T - \dot{\omega}_y r_x + \dot{\omega}_x r_{ya} \right) \]  

(92)

\[ \sum_{k=1}^{3} T_{i2k} \ddot{r}_{a_{ik}} = T_{i21} \left( \dddot{x}_T - \dot{\omega}_z r_y + \dot{y}_z r_{za} \right) \]

\[ + T_{i22} \left( \dddot{y}_T + \dot{\omega}_z r_x - \dot{\omega}_x r_{za} \right) \]  

\[ + T_{i23} \left( \dddot{z}_T - \dot{\omega}_y r_x + \dot{\omega}_x r_{ya} \right) \]  

(93)

\[ \sum_{k=1}^{3} T_{i3k} \ddot{r}_{a_{ik}} = T_{i31} \left( \dddot{x}_T - \dot{\omega}_z r_y + \dot{y}_z r_{za} \right) \]

(94)
6.1 (Continued)

\[ + T_{i32} \left( y_T + \omega z r_x a_i - \omega x r_z a_i \right) \]

\[ + T_{i33} \left( z_T - \omega y r_x a_i + \omega x r_y a_i \right) \]  \hspace{1cm} (94)

Using the definitions in equations (82) through (88), these equations reduce to:

\[ \sum_{k=1}^{3} T_{i1k} \ddot{r}_{aik} = T_{i11} x_T + T_{i12} y_T + T_{i13} z_T + C_{xxi} \dot{\omega}_x \]

\[ + C_{xyi} \dot{\omega}_y + C_{xz_i} \dot{\omega}_z \]  \hspace{1cm} (95)

\[ \sum_{k=1}^{3} T_{i2k} \ddot{r}_{aik} = T_{i21} x_T + T_{i22} y_T + T_{i23} z_T + C_{yyi} \dot{\omega}_x \]

\[ + C_{yyi} \dot{\omega}_y + C_{yz_i} \dot{\omega}_z \]  \hspace{1cm} (96)

\[ \sum_{k=1}^{3} T_{i3k} \ddot{r}_{aik} = T_{i31} x_T + T_{i32} y_T + T_{i33} z_T + C_{zi} \dot{\omega}_x \]

\[ + C_{zyi} \dot{\omega}_y + C_{zz_i} \dot{\omega}_z \]  \hspace{1cm} (97)

The mass matrix shown in upper triangular form shown in Figure 6-1 is obtained by combining equations (95) through (97) and (82) through (88) with equations (80), (89), (90), and (91).

6.2 GENERALIZED FORCES

The generalized forces are obtained by considering the work required to produce a unit displacement for each degree of freedom.
Figure 6-1. Mass Matrix
6.2 (Continued)

\[ \delta W_j = Q_j (\delta q_j) \]  \hspace{3cm} (98)

where:

- \( Q_j \) = Generalized force
- \( \delta q_j \) = Unit displacement

Let \( F_p \) be the net piston force along the local \( x_s \) actuator axis. The work done is then:

\[ \delta W_i = F_{p_i} x_{s_i} \]  \hspace{3cm} (99)

But, since:

\[
\begin{bmatrix}
  x_{s_i} \\
  y_{s_i} \\
  z_{s_i}
\end{bmatrix} =
\begin{bmatrix}
  T_i
\end{bmatrix}
\begin{bmatrix}
  x_T \\
  y_T \\
  z_T
\end{bmatrix}
\]  \hspace{3cm} (100)

then:

\[ \delta W_i = F_{p_i} \begin{bmatrix}
  T_{i11} & T_{i12} & T_{i13}
\end{bmatrix}
\begin{bmatrix}
  x_T \\
  y_T \\
  z_T
\end{bmatrix} \]  \hspace{3cm} (101)

The generalized force for the table translational degrees of freedom are obtained by letting \( x_T = 1 \), \( y_T = z_T = 0 \) and then \( y_T = 1 \), \( x_T = z_T = 0 \), etc.
The displacements at the table swivel joints due to rotations of the table are:

\[
\begin{pmatrix}
\Delta x_i \\
\Delta y_i \\
\Delta z_i
\end{pmatrix}
= \begin{bmatrix}
0 & -\Delta \theta_z & \Delta \theta_y \\
\Delta \theta_z & 0 & -\Delta \theta_x \\
-\Delta \theta_y & \Delta \theta_x & 0
\end{bmatrix}
\begin{pmatrix}
r_{x,i} \\
r_{y,i} \\
r_{z,i}
\end{pmatrix}
\]  

\text{(103)}

For \( \Delta \theta_x = 1, \Delta \theta_y = \Delta \theta_z = 0: \)

\[
\begin{pmatrix}
\Delta x_i \\
\Delta y_i \\
\Delta z_i
\end{pmatrix}
= \begin{pmatrix}
0 \\
-r_{z,i} \\
r_{y,i}
\end{pmatrix}
\]  

\text{(104)}

For \( \Delta \theta_y = 1, -\Delta \theta_x = \Delta \theta_z = 0: \)

\[
\begin{pmatrix}
\Delta x_i \\
\Delta y_i \\
\Delta z_i
\end{pmatrix}
= \begin{pmatrix}
r_{z,i} \\
0 \\
-r_{x,i}
\end{pmatrix}
\]  

\text{(105)}
6.2 (Continued)

For \( \Delta \theta = 1, \Delta \theta_x = \Delta \theta_y = 0 \):

\[
\begin{bmatrix}
\Delta x_i \\
\Delta y_i \\
\Delta z_i
\end{bmatrix}_T =
\begin{bmatrix}
-r_{y_i}
\\
r_{x_i}
\\
0
\end{bmatrix}
\]  

(106)

Transforming these displacements to the servo actuator coordinate system:

\[
\Delta x_{s_i} = -T_{112} r_{z_i} a_i + T_{113} r_{y_i} a_i \quad \text{for } \Delta \theta_x = 1
\]

(107)

\[
\Delta x_{s_i} = T_{111} r_{z_i} a_i - T_{113} r_{x_i} a_i \quad \text{for } \Delta \theta_y = 1
\]

\[
\Delta x_{s_i} = -T_{111} r_{y_i} a_i + T_{112} r_{x_i} a_i \quad \text{for } \Delta \theta_z = 1
\]

Then:

\[
\delta W = \sum_{i=1}^{6} F_p_i \Delta x_{s_i}
\]

(108)

or:

\[
\begin{bmatrix}
M_{H_x} \\
M_{H_y} \\
M_{H_z}
\end{bmatrix} =
\begin{bmatrix}
\sum_{i=1}^{6} F_p_i (-T_{112} r_{z_i} a_i + T_{113} r_{y_i} a_i) \\
\sum_{i=1}^{6} F_p_i (T_{111} r_{z_i} a_i - T_{113} r_{x_i} a_i) \\
\sum_{i=1}^{6} F_p_i (-T_{111} r_{y_i} a_i + T_{112} r_{x_i} a_i)
\end{bmatrix}
\]

(109)

These generalized forces are combined with the \( m_T (\omega_T \times \overset{\cdot}{\omega}_T) \) terms from equation (80), the \( \omega_T \times (\overset{\cdot}{r}_T \cdot \overset{\cdot}{\omega}_T) \) terms from equation (81), the actuator
6.2 (Continued)

damping and stiffness terms and the external forces and moments to obtain the \( \{C\} \) matrix of equation (44):

\[
\begin{bmatrix}
\ddot{x}_T \\
\ddot{y}_T \\
\ddot{z}_T \\
\omega_x \\
\omega_y \\
\omega_z \\
ye_i \\
z_i
\end{bmatrix}
= [M]^{-1}
\begin{bmatrix}
0 & -\omega_z & \omega_y \\
-\omega_z & 0 & -\omega_x \\
\omega_y & \omega_x & 0 \\
-\omega_y & -\omega_x & -\omega_z \\
-2\zeta \omega_i e_1 m_q \ddot{e}_i - \omega_i^2 m_q e_i \\
-2\zeta \omega_i e_1 m_q \ddot{e}_i - \omega_i^2 m_q e_i \\
-2\zeta \omega_i e_1 m_q \ddot{e}_i - \omega_i^2 m_q e_i \\
\end{bmatrix}
\begin{bmatrix}
\dot{x}_T \\
\dot{y}_T \\
\dot{z}_T \\
\omega_x \\
\omega_y \\
\omega_z \\
ye_i \\
z_i
\end{bmatrix}
+ \begin{bmatrix}
F_{Hx} \\
F_{Hy} \\
F_{Hz} \\
F_{Ex} \\
F_{Ey} \\
F_{Ez}
\end{bmatrix}
\]
7.0 CALCULATION OF ACTUATOR VELOCITIES AND POSITIONS

Actuator lengths and velocities are calculated from the equations of motion variables for use in the servo loop feedbacks and for the determination of actuator friction forces.

Actuator velocities are calculated by first determining the velocities of the actuator/table attachment points in table coordinates:

\[
\begin{pmatrix}
\dot{r}_{a_{i1}} \\
\dot{r}_{a_{i2}} \\
\dot{r}_{a_{i3}}
\end{pmatrix} =
\begin{pmatrix}
\dot{x}_T \\
\dot{y}_T \\
\dot{z}_T
\end{pmatrix} +
\begin{bmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & -\omega_x \\
-\omega_y & \omega_x & 0
\end{bmatrix}
\begin{pmatrix}
r_{xa_i} \\
r_{ya_i} \\
r_{za_i}
\end{pmatrix}
\]

(111)

These velocities are then transformed to actuator coordinates to obtain:

\[
\dot{p}_i = \sum_{k=1}^{3} T_{ik} \dot{r}_{a_{ik}}
\]

(112)

Actuator lengths are calculated by first obtaining the components of actuator length in the inertial coordinate system.

\[
\begin{pmatrix}
r_{sx_{i1}} \\
r_{sy_{i}} \\
r_{sz_{i}}
\end{pmatrix} =
\begin{pmatrix}
x_I \\
y_I \\
z_I
\end{pmatrix} +
[A]
\begin{pmatrix}
r_{xa_i} \\
r_{ya_i} \\
r_{za_i}
\end{pmatrix} -
\begin{pmatrix}
y_{f_{i}} \\
z_{f_{i}}
\end{pmatrix}
\]

(113)

where \(y_{f_{i}}\) and \(z_{f_{i}}\) are the inertial coordinates of the floor swivel joints of each actuator.
7.0 (Continued)

Then the actuator lengths are calculated as follows:

\[ l_{p_i} = \sqrt{r_{s_{x_i}}^2 + r_{s_{y_i}}^2 + r_{s_{z_i}}^2} \]  \hspace{1cm} (114)

These actuator lengths and velocities are used in the feedback loops in the servo electronics shown in Figure 4-1.
**APPENDIX**  

**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[A]$</td>
<td>Transformation matrix from table to inertial coordinates</td>
</tr>
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<td>$A_1, A_2$</td>
<td>&quot;Push&quot; and &quot;pull&quot; stroke working areas of actuators</td>
</tr>
<tr>
<td>$\dot{a}_0$</td>
<td>Output of the pressure feedback first-order lag filter</td>
</tr>
<tr>
<td>$\ddot{a}_1$</td>
<td>Output of the pressure feedback high-pass filter</td>
</tr>
<tr>
<td>$B_p$</td>
<td>Viscous damping coefficient of actuator</td>
</tr>
<tr>
<td>${C}$</td>
<td>Column of generalized forces for equations of motion solution</td>
</tr>
<tr>
<td>$C_p$</td>
<td>Leakage coefficient across piston seals</td>
</tr>
<tr>
<td>$e$</td>
<td>Output of the forward loop compensation network</td>
</tr>
<tr>
<td>$EI_r$</td>
<td>Bending modulus of piston rod</td>
</tr>
<tr>
<td>$F_{EXT}$</td>
<td>External force</td>
</tr>
<tr>
<td>$F_f$</td>
<td>Coulomb friction force of actuator</td>
</tr>
<tr>
<td>$F_H$</td>
<td>Total hydraulic and friction forces acting on pistons</td>
</tr>
<tr>
<td>$F_P$</td>
<td>Net forces on actuator piston</td>
</tr>
<tr>
<td>$\hat{I}$</td>
<td>Inertia tensor of the active table</td>
</tr>
<tr>
<td>$I_A$</td>
<td>Mass moment of inertia of entire actuator assembly about floor pivot</td>
</tr>
<tr>
<td>$I_{AC}$</td>
<td>Mass moment of inertia of cylinder (excluding the mass of the piston) about floor swivel joint</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Mass moment of inertia of the piston rod</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$I_{xx}, I_{yy}$</td>
<td>Moment and products of inertia</td>
</tr>
<tr>
<td>$I_{zz}, I_{xy}, I_{xz}, I_{yz}$</td>
<td></td>
</tr>
<tr>
<td>$k_e$</td>
<td>Effective lateral stiffness of the actuator</td>
</tr>
<tr>
<td>$k_r$</td>
<td>Piston rod stiffness</td>
</tr>
<tr>
<td>$K_c$</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$K_g$</td>
<td>Electronics and valve forward loop gain</td>
</tr>
<tr>
<td>$K_{pf}$</td>
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</tr>
<tr>
<td>$K_r$</td>
<td>Velocity feedback loop gain</td>
</tr>
<tr>
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<td>Velocity command gain</td>
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<tr>
<td>$l_c$</td>
<td>Distance from floor swivel to center line of piston rod seal at end of cylinder</td>
</tr>
<tr>
<td>$l_{ci}$</td>
<td>Commanded actuator length</td>
</tr>
<tr>
<td>$i_{ci}$</td>
<td>Commanded actuator velocity</td>
</tr>
<tr>
<td>$\ddot{i}_{ci}$</td>
<td>Commanded actuator acceleration</td>
</tr>
<tr>
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</tr>
<tr>
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<tr>
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<tr>
<td>$l_p$</td>
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<tr>
<td>$\dot{l}_p$</td>
<td>Actuator velocity</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$l_p$</td>
<td>Actuator acceleration</td>
</tr>
<tr>
<td>$m_l$</td>
<td>Effective rigid lateral mass of actuator assembly</td>
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<tr>
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<td>Kinetic energy of the system</td>
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<tr>
<td>Symbol</td>
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<tr>
<td>$\beta$</td>
<td>Break frequency of first order filter</td>
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</tr>
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<tr>
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<td>The angle between the $y_I$-$z_I$ plane and the actuator $x_s$ axis</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
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<tr>
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</table>