A STUDY OF GUST RESPONSE FOR A ROTOR-PROPELLER IN CRUISING FLIGHT

Masahiro Yasue

August 1974

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Prepared under Contract No. NAS2-7262 by

Aeroelastic and Structures Research Laboratory
Department of Aeronautics and Astronautics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

for

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Equations of motion for a rotor-propeller aircraft in cruising flight have been developed and implemented in a computer program. The formulation is based on Galerkin's method using coupled mode shapes for the blade and wing. This procedure is applied to the analysis of two types of rotors, gimballed rotor and hingeless. The results are evaluated by means of eigenvalue analysis of the stability of the system and frequency response analysis of the gust and control response. In general, the results show that:

(a) The choice of mode shape (rigid-body mode with spring restraint at the root, or elastic coupled model) to construct the equations of motion affects the damping of the system significantly. The dependency of the damping on the mode shape is estimated for the first beam bending mode. The frequencies of the system have little dependence on the mode-shape type.

(b) The results of the frequency response are quite similar to those of Johnson, in spite of the use of different models. However, the amplitude of the response is slightly different.

(c) Addition of higher mode degrees-of-freedom has little influence on the stability of the rotor and wing.
FOREWORD

This report has been prepared by the Aeroelastic and Structures Research Laboratory (ASRL), Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts, under NASA Contract No. NAS 2-7262 from the Ames Research Center, National Aeronautics and Space Administration, Moffett Field, California 94035. Mr. John Rabbott and Dr. Wayne Johnson of the Ames Research Center served as technical monitors. The valuable assistance and advice received from these individuals is gratefully acknowledged.

The research described in this report was supervised by Professor Norman D. Ham and Associate Professor Pin Tong. The author would like to express his deep appreciation and gratitude for their invaluable advice and guidance throughout this study. The author is also deeply indebted to Professor John Dugundji and to Dr. Wayne Johnson, who contributed various useful suggestions. Professor E. A. Witmer's advice and assistance in various phases of the work is acknowledged gratefully.
SUMMARY

This study has been devoted to the development and evaluation of a theoretical model of the proprotor on a cantilevered wing, operating in normal cruising flight. This theory expresses the wing and blade motion in coupled form, and can include any number of mode shapes required to describe the motion accurately. It has been applied to the investigation of the dynamic characteristics of the Bell and of the Boeing design. The Bell rotors are gimballed and the Boeing rotors are hingeless. The analysis includes the frequency response to gusts and cyclic pitch, and an eigenvalue analysis of the dynamic system.

Based on the theoretical results included in this study, the following conclusions may be stated:

(a) The choice of mode shape (rigid-body mode or elastic-coupled mode) affects the damping significantly. The dependency of the damping on the mode shape can be estimated for the first beam bending mode. The blade inplane deflection opposing the rotor direction of rotation, accompanied by the forward out-of-plane deflection, increases the damping, comparing it with the rigid-body calculation. The inplane deflection proceeding in the rotor direction of rotation decreases the damping. The mode shape has little influence on the frequencies of the system.

(b) The results of the frequency response are quite similar to those of Johnson, in spite of the difference in the mode shapes. The amplitude of the response is slightly different, since structural damping was not included in the present calculation, and the mode shapes used were different.

(c) The analysis of the eighteen degree-of-freedom system showed that the higher-frequency degrees of freedom have small influence on the basic degrees of freedom.
Stresses or bending moments of the wing or blade can be predicted from the motions of the wing and blade obtained from this analysis. In addition, this analysis may be applied to the development of an automatic control device to alleviate the gust response of the vehicle.
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<td>Aerodynamic loading of the wing due to the rotor aerodynamic forces, corresponding to the ith wing mode shape</td>
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(GJ_w)  Wing torsional rigidity
q_i  Aerodynamic loading of the wing due to the wing aerodynamics, corresponding to the ith wing mode shape
h  Mast height
\bar{h}  Nondimensional mast height: h/R
I  Unit matrix
I_B  Blade mass moment of inertia about the virtual flapping hinge
I_P  Pylon pitching mass moment of inertia
I_R  Pylon rolling mass moment of inertia
I_Y  Pylon yawing mass moment of inertia
I_w  Wing mass moment of inertia about the elastic axis per unit length
K_{pi}  Pitch-flap coupling coefficients corresponding to the ith blade mode shape defined in Eq. 4.26
L  Wing length (semispan)
M_B  Mass of one blade
M_p  Pylon mass
M_Y  Wing resultant pitching moment about the elastic axis per unit length
m  Spanwise mass of the blade per unit length
m_w  Spanwise mass of the wing per unit length
N  Number of blades
n  Blade index
\( P_z \) Resultant force per unit length in the \( z \) direction on the blade

\( P_6 \) Resultant force per unit length in the circumferential direction of the rotating blade

\( P_{6j}, P_{zj} \) Defined in Eq. 4.17

\( P_w \) Wing torsion (positive nose up)

\( p \) Time function corresponding to the wing torsion mode

\( Q_{jo} \) \( j \)th blade collective motion degree of freedom

\( Q_{jc} \) \( j \)th blade cosine cyclic motion degree of freedom

\( Q_{js} \) \( j \)th blade sine cyclic motion degree of freedom

\( q_{nj} \) Time function corresponding to the \( j \)th mode of the \( n \)th blade

\( q_1 \) Time function corresponding to the wing vertical bending

\( q_2 \) Time function corresponding to the wing chordwise bending

\( R \) Blade radius

\( r \) Blade running-spanwise coordinate

\( r_x \) Pylon vertical displacement (positive upward)

\( r_z \) Pylon longitudinal displacement (positive forward)

\( S_o \) Static mass moment of the wing segment about the elastic axis

\( s \) Distance between the elastic axis and local center of gravity of the wing
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<td>Matrix defined in Eq. 4.11</td>
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<td>Centrifugal force of the blade</td>
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<td>t</td>
<td>Time</td>
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<td>U</td>
<td>Blade sectional resultant velocity</td>
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<tr>
<td>U_p</td>
<td>Blade sectional inplane velocity</td>
</tr>
<tr>
<td>U_T</td>
<td>Blade sectional out-of-plane velocity</td>
</tr>
<tr>
<td>U_w</td>
<td>Wing sectional resultant velocity</td>
</tr>
<tr>
<td>U_{xB}</td>
<td>Blade inplane velocity component in the x_B direction</td>
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<tr>
<td>U_{yB}</td>
<td>Blade inplane velocity component in the y_B direction</td>
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<td>u_G</td>
<td>Vertical gust velocity</td>
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<td>u_w</td>
<td>Vertical bending deflection of the wing (positive upward)</td>
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<td>V</td>
<td>Aircraft forward velocity</td>
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<td>V_{ij}</td>
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<td>V^o_j</td>
<td>Jth collective mode shape of the inplane bending of the blade</td>
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<tr>
<td>V^c_j</td>
<td>Jth cyclic mode shape of the inplane bending of the blade</td>
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<tr>
<td>v</td>
<td>Rotor induced flow</td>
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<tr>
<td>v_G</td>
<td>Lateral gust velocity</td>
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<td>v_n</td>
<td>Inplane deflection of the nth blade (positive clockwise)</td>
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\( W_B \)  Pylon vertical translation
\( W_B' \)  Pylon rolling angle
\( W_C \)  Pylon longitudinal translation
\( W_C' \)  Pylon yawing angle
\( W_{ij} \)  Defined in Eq. 4.16
\( w_j \)  Blade jth out-of-plane bending mode shape
\( w_{ij}^o \)  jth collective mode shape of the out-of-plane bending of the blade
\( w_{ij}^c \)  jth cyclic mode shape of the out-of-plane bending of the blade
\( W_{WB} \)  Work done on the wing by the blade aerodynamic forces
\( W_{WW} \)  Work done on the wing by the wing aerodynamic forces
\( w_G \)  Longitudinal gust velocity
\( w_n \)  Out-of-plane deflection of the nth blade (positive forward)
\( w_w \)  Chordwise bending of the wing (positive forward)
\( x \)  Blade nondimensional running spanwise coordinate \( r/R \)
\( x, y, z \)  Coordinates fixed to the wing root (see Fig. 2)
\( x_B, y_B \)  Coordinates fixed to the rotor (see Fig. 6)
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<td>$\alpha$</td>
<td>Blade sectional angle of attack</td>
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<td>$\alpha_o$</td>
<td>Blade sectional angle of attack in the equilibrium state</td>
</tr>
<tr>
<td>$\alpha_w$</td>
<td>Wing sectional angle of attack</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Blade flapping degree of freedom</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Lock number $\frac{\rho a C_B R^4}{I_B}$</td>
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<td>$\gamma_j$</td>
<td>Wing jth vertical bending mode shape</td>
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<td>$\Delta\psi_n$</td>
<td>Phase angle between blades</td>
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<td>$\delta$</td>
<td>Variation</td>
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<td>$\delta_3$</td>
<td>Rotor blade pitch-flap coupling</td>
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<td>$\zeta$</td>
<td>Blade lagging degree of freedom</td>
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<td>$\zeta_j$</td>
<td>Wing jth chordwise bending mode shape</td>
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<td>$\theta_{AT}$</td>
<td>Built-in angle of twist of the blade</td>
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<td>$\theta_B$</td>
<td>Angle of twist of the blade</td>
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<td>$\theta_C$</td>
<td>Blade pitch angle due to pitch control</td>
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<td>Blade pitch angle defined according to the performance analysis</td>
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<td>$\theta_w$</td>
<td>Angle of twist of the wing</td>
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<tr>
<td>$\theta_o$</td>
<td>Blade collective pitch control</td>
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<tr>
<td>$\theta_{1C}$</td>
<td>Blade cyclic pitch control</td>
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LIST OF SYMBOLS CONTINUED

\( \theta_{ls} \) Blade cyclic pitch control

\( \theta_{.75} \) Blade pitch angle at 75% span

\( \kappa \) Integral operator defined in Eq. 4.37

\( A_j \) Wing jth natural frequency

\( \lambda \) Inflow ratio \( V/(\Omega R) \)

\( \lambda_j \) jth natural frequency of the rotating blade

\( \nu_p \) Pylon pitching motion (positive nose up)

\( \nu_R \) Blade rigid-body mode rotations in autorotation case (positive for clockwise rotation, when looking forward)

\( \nu_r \) Pylon rolling motion (positive for clockwise)

\( \nu_y \) Pylon yawing motion (positive for counterclockwise)

\( \pi \) Variational functional

\( \rho \) Air density

\( \tau_i \) Defined in Eq. 3.15

\( \phi \) Pylon pitching angle

\( \phi_i \) Blade sectional inflow angle

\( \phi_j \) Wing jth torsional mode shape

\( \phi_w \) Wing sectional angle formed by the horizontal and vertical velocity components (see Fig. 7)
LIST OF SYMBOLS CONCLUDED

\( \Psi_n \) Azimuth position of the nth blade

\( \Psi_n \) Azimuth position of the nth blade excluding the pylon roll motion

\( \Omega \) Rotational speed of the rotor

\( \bar{\Omega} \) Direction of rotation \( \Omega/|\Omega| \)

\( (\_\_\_)_o \) Value is evaluated in the equilibrium state

\( (',") \) Time derivative \( \partial/\partial t, \partial^2/\partial t^2 \)

\( (\_\_\_), (\_\_\_\_\_\_) \) Nondimensional time derivatives, \( \partial/\partial (|\Omega|t), \partial^2/\partial (|\Omega|t)^2 \)

\( (\_\_\_)_n \) nth blade

Superscript
- \( o \) Collective mode of the rotor
- \( c \) Cyclic mode of the rotor

Subscript
- \( o \) Collective mode of the rotor
- \( c \) Cyclic mode of the rotor
- \( s \) Cyclic mode of the rotor
- \( i, j, k \) Indices
SECTION 1

INTRODUCTION

1.1 General

It has been recognized that the tilting proprotor aircraft, one of the composite aircraft family, is a very promising concept that combines into one aircraft the hover efficiency of the helicopter and the high-speed efficiency of the fixed-wing aircraft (Refs. 1-3).

The typical tilting proprotor aircraft is a twin-engine aircraft with tilting rotors mounted on each wing tip. Its configuration consists of a fuselage, a high swept-forward wing, and an empennage. The empennage has a vertical stabilizer and rudder, and a horizontal stabilizer and elevator. The large diameter rotors are three bladed, hingeless or gimbal-type rotors which are mounted on the rotor shaft. The rotor shaft is connected through the gearbox to each engine in the pylon attached at the wing tip. The conversion system provides the rotation of the rotor pylon from the vertical position to the horizontal position and return, in order to obtain the helicopter mode or airplane mode corresponding to the desired flight regime.

When the aircraft takes off or lands, the rotor pylon is rotated to the vertical position to achieve vertical takeoff or landing similar to the helicopter. The flight controls apply pitch changes to the rotor to provide the longitudinal and directional control corresponding to helicopter rotor cyclic pitch, while the collective pitch controls vertical flight and roll motion.

In high-speed flight, the rotor pylon is rotated to a horizontal position similar to that of the conventional propeller type aircraft. The thrust is produced by the rotor, and the lift by the wing. The flight controls are provided by the conventional aircraft
control surfaces such as the elevator, rudder and aileron.

In addition to the above modes, various conversion modes can be obtained. At that time the rotor pylon tilts to some position between the vertical and horizontal, where it can be safely locked. This makes STOL-type operations possible.

The tilting proprotor is exposed to a severe aerodynamic environment including gusts, the wake of preceding blades, and harmonic airloading like a helicopter. But its dynamic and aeroelastic characteristics are in many ways unique; for example, the large flexible blades with a large amount of twist experience significant coupled out-of-plane (flapping) and inplane (lagging) motion.

As described later in Subsection 1.2, several years of experimental and theoretical analyses have been conducted to establish a fundamental understanding of the dynamic and aeroelastic behavior. However, it is necessary to understand the aeroelastic response of this aircraft to atmospheric turbulence more adequately and to predict it more accurately, since during the preliminary design phase, vibration level prediction is required in order: (a) to evaluate the fatigue life of the blade and wing, (b) to estimate the ride qualities of the vehicle, and, if necessary, (c) to develop suitable gust alleviation devices.

Several design compromise concepts, which make the present analysis distinct from helicopter aeroelastic analysis, are now stated briefly.

In order to obtain high hover efficiency from the rotor, it is desirable to achieve low disc loading, in other words to use large-diameter rotors whose swept discs reach nearly to the fuselage. When the aircraft is operated in high forward speed axial flight in the airplane mode, the rotor is operating at a high inflow ratio (the ratio of axial velocity to blade tip speed). This phenomenon is very different from the helicopter rotor operation which
involves low inflow. High inflow operation requires a large built-in angle of twist for efficient cruising. Therefore, significant coupled out-of-plane (flapping) and in-plane (lagging) motion occurs in the large, flexible and twisted blade. This phenomenon makes analysis more complicated.

The engines and gearboxes are usually located at the wing tip to avoid transmitting high power through a long drive shaft. This leads to low wing natural frequencies and possible resonances in the low frequency range. Also, the center of gravity of the pylon and rotor does not usually coincide with the elastic axis of the wing. Hence, this results in coupled bending and torsion.

1.2 Brief Survey of Past Work

A good review and an elementary description of the dynamic and aeroelastic problems associated with the tilting proprotor aircraft were given by Reed in Ref. 4 and Loewy in Ref. 5. In this report no attempt will be made to repeat the reviews given in Refs. 4 and 5. The only references cited here will be those pertinent to the problem being treated.

The possibility of propeller whirl flutter -- a dynamic instability that can occur in a flexibly mounted aircraft engine-propeller combination -- was first recognized in the late 1930's by Taylor and Browne (Ref. 6). It was not until 1960 that it became a problem of practical concern -- with the appearance of the turboprop aircraft.

Following the two fatal turboprop aircraft accidents, it was established that propeller whirl flutter could have occurred if the nacelle stiffness was severely reduced, for example by a structural failure. Several generalized studies were conducted at NASA-Langley. One of them was carried out by Houbolt and Reed in Ref. 7; it gives an elementary and basic treatment of the equations of motion and propeller aerodynamics for propeller-nacelle whirl flutter.
Because VTOL configurations have unconventional propeller-rotor systems, whirl flutter was a major design consideration on present proprotor aircraft.

The analysis presented in Ref. 8 is for a two-bladed rotor free to tilt on a shaft with two nacelle degrees of freedom (pitch and yaw). No lag or coning degrees of freedom are considered. The analytical method was compared with test results for an existing tilting proprotor aircraft (the Bell XV-3) and of subsequently-tested scale models. They showed good agreement.

Young and Lytwyn in Ref. 9 present a very precise analysis for the whirl stability of a multi-bladed rotor mounted on a nacelle which has pitch and yaw degrees of freedom. Each blade has one flapwise degree of freedom. The blade mode shape is assumed to be a rigid body mode shape. It was concluded that whirl stability is poorest when the nacelle pitch frequency equals the nacelle yaw frequency, but in this situation nacelle damping is quite effective. There is an optimum value of flap bending frequency somewhere between 1.1 and 1.35 for highly stabilized whirl motion.

This analysis neglects (as do Refs. 4 and 8) the effect of coning on proprotor aerodynamics, and flap bending mode shapes other than the rigid blade mode used. Also, autorotation flight must be considered as well as powered flight.

In Ref. 10, Gaffey points out that a highly coupled blade mode has substantial flap bending even if the primary mode involves in-plane motion. This occurs in the case of a highly twisted blade or a blade operating at high geometric pitch angles such as a proprotor blade. The analysis shows that a moderate amount of negative $\delta_3$ (flapping angle at the blade root gives the pitch angle reduction of the amount $\beta \cdot \tan \delta_3$ if $\delta_3$ is positive) has a stabilizing influence on proprotors subject to flap-lag instability at high inflows.

Preliminary design studies of prototype vehicles (Refs. 11 and 12) as a part of the current NASA/ARMY sponsored tilting proprotor
Johnson (in Refs. 13 and 14) derived the equations of motion for a cantilever wing with the rotor at the wing tip. He develops a nine degree-of-freedom model which involves blade flapping motion and lagging motion (each has one collective and two cyclic motions, respectively), wing vertical bending, chordwise bending, and torsion. This model is applied to two proprotor designs and compared with the results of some full-scale wind tunnel tests. It shows reasonable correlation between theory and experiment.

In conclusion, it appears that most of the investigative work has been concerned with whirl flutter. The above review shows that more knowledge is needed for the solution of tilting proprotor aircraft dynamic and aeroelastic problems.

1.3 Objectives of the Present Study

The objective of this study is to establish a verified method of predicting the dynamic and aeroelastic behavior of the tilting proprotor aircraft in order to evaluate the fatigue life of the blades and wings and also to estimate the ride quality.

The equations of motion for a cantilever wing with a rotating rotor at the wing tip will be derived as consistently as possible. The great complexity of rotor blade motion will be included by accounting for blade rotation (i.e., centrifugal and Coriolis forces), significant inplane motion, and the large twist and high pitch angles at high inflows.

The resulting system of equations, obtained using modal analysis, will be applied to the analysis of the two proprotor designs (one is a rigid, soft-inplane type rotor and the other is a gimballed, stiff-inplane rotor).

Finally, the eigenvalues and frequency response of each proprotor design will be determined to establish their dynamic charac-
teristics; the results of the analyses will be compared in terms of eigenvalues for normal modes (coupled elastic modes), assumed modes (uncoupled elastic modes), and rigid body modes, using Galerkin's method.
SECTION 2

THE EQUATIONS OF MOTION

2.1 Model and Coordinate System for the Analysis

The primary interest of this study is in the dynamic and aeroelastic phenomena of the wing, pylon, and rotor of the tilting proprotor aircraft in cruising flight. Hence, the dynamical system considered here consists of a cantilever right wing with a pylon at the wing tip, and an N-bladed rotor mounted on the pylon, as shown in Fig. 1. The model will be restricted to the cantilever wing, since it is sufficient to obtain a basic understanding of the proprotor motion, and many such proprotor models have been tested in wind tunnels.

Therefore, the aircraft rigid body motions are neglected and the wing antisymmetrical modes are also dropped. The left wing motions including the pylon and rotor are given by the mirror image of the right wing.

The wing is assumed to have a high aspect ratio, so that strip theory is used for the wing aerodynamics and beam theory for elastic bending and torsion. Wing sweep and dihedral will not be considered, but angle of attack and angle of twist (built-in twist) will be considered. The elastic axis is assumed to be a straight line. The elastic axis coincides with the y-axis as shown in Fig. 2. The free stream vector coincides with the z-axis. Therefore, the angle between the z-axis and the chordline of the wing results in a wing angle of attack (positive nose up). The wing motion (Fig. 2) consists of elastic bending and elastic torsion. The deflection $w_w$ of the wing elastic axis perpendicular to the y-z plane is called vertical or beamwise bending (positive upward). The deflection $w_w$ parallel to y-z plane is termed chordwise bending (positive forward). Torsion $p_w$ is
defined as pitch angle change (positive nose-up).

A pylon of large mass and moment of inertia is assumed to be rigidly attached to the wing tip. Therefore, the pylon motions (Fig. 2) corresponding to the wing motions are defined as vertical displacement $r_x$ along the x-axis (positive upward), longitudinal displacement $r_z$ along the z-axis (positive forward), pylon yaw $v_y$ about the x-axis (positive counterclockwise), pylon pitch $v_p$ about y-axis (positive nose-up), and pylon roll $v_r$ about the z-axis (positive for clockwise rotation, when looking forward). The above pylon motions are accounted for at the point where the wing elastic axis crosses the plane, parallel to x-Z plane, which includes the rotor shaft. The pylon lateral displacement along the y-axis is neglected as a higher order effect.

The pylon and the rotor shaft are assumed to be parallel to the free stream in equilibrium flight ($r_x$, $r_z$, $v_y$, $v_p$, and $v_r$ are all zero), regardless of the wing angle of attack.

The rotor is located at the distance $h$ (Fig. 2) from the wing tip elastic axis to the rotor-hub (positive forward from the elastic axis). The distance $h$ is termed the mast height. The rotor consists of N blades, whose rotational speed $\Omega$ is defined as positive if clockwise looking forward. The blade (Fig. 2) has out-of-plane (flapping) deflection $w_n$, defined positive for forward displacement from the disc plane (upward in helicopter mode, while the rotor shaft is vertical), and inplane (lagging) deflection $v_n$, defined positive for clockwise deflection regardless of rotor direction of rotation. The lower case letter $n$ means the nth blade, $n=1,2,...N$. Blade torsion is neglected here.

The azimuth position $\psi_n$ (Fig. 3) of the nth blade is defined as:
\( \Phi_n = \Omega t + \psi + \Delta \psi_n \) \hspace{1cm} (2.1)

where \( \psi_n \) is measured from the vertical and \( t \) is time. The phase angle between blades, \( \Delta \psi_n \) is defined as

\[ \Delta \psi_n = (n-1) \frac{2\pi}{N} \]

(2.2)

The azimuth position of the \( n \)th blade excluding the pylon roll motion is denoted as

\[ \Phi_n = \Omega t + \Delta \psi_n \]

(2.3)

2.2 Basic Formulation for Powered Flight

The governing linear equations of motion are derived in this subsection. A more complete and detailed derivation is given in Appendix A.

The equations involve ten unknowns: \( v_n, w_n, r_x', r_z', v_y', v_p', \)
\( r_r', r_w', r_n' \) (actually \( v_n \) and \( w_n \) represent \( N \) unknowns, respectively, but for convenience they may be treated as one unknown without inconsistency). The equations consist of three categories: the blade equations, the wing equations, and matching conditions between rotor and pylon.

The blade equations are complicated by the pylon motions which produce the centrifugal forces and Coriolis forces. The wing equations are derived from beam theory. The rotor is rigidly attached to the pylon for the powered flight case and, therefore, rotor motion is related to pylon motion; this gives the matching conditions between the rotor and the pylon.

(a) Blade Equations

\[
\frac{\partial^2}{\partial r^2} \left[ \left\{ (EI_c \sin^2 \theta_b + (EI_b \cos^2 \theta_b) \right\} \frac{\partial^2 w_n}{\partial r^2} \right] \\
+ \frac{\partial^2}{\partial r^2} \left[ \left\{ (EI_c - (EI_b) \sin \theta_b \cos \theta_b \right\} \frac{\partial^2 v_n}{\partial r^2} \right]
\]
\[-\frac{\partial}{\partial r}(r \frac{\partial \omega_n}{\partial r}) + m \left\{ \ddot{\omega}_n + \ddot{r}_y + 2 \Omega r \dot{y}_f \sin \Phi_n \\
- r \dot{y}_f \cos \Phi_n + 2 \Omega r \dot{y}_f \cos \Phi_n \\
+ r \dot{y}_f \sin \Phi_n \right\} = (P_z)_n \] (2.4)

\[\frac{\partial^2}{\partial r^2} \left[ \left\{ (EI)_C \cos^2 \theta_b + (EI)_B \sin^2 \theta_b \right\} \frac{\partial^2 \omega_r}{\partial r^2} \right] \]

\[+ \frac{\partial^2}{\partial r^3} \left[ \left\{ (EI)_c - (EI)_b \right\} \sin \theta_b \cos \theta_b \frac{\partial^2 \omega_r}{\partial r^2} \right] \]

\[- \frac{\partial}{\partial r}(r \frac{\partial \omega_r}{\partial r}) + m \left\{ \ddot{\omega}_r - \Omega^2 \omega_n - \dot{r}_y \sin \Phi_n \\
+ r \dot{y}_f - \dot{\lambda} \dot{y}_f \sin \Phi_n - \dot{\lambda} \dot{y}_f \cos \Phi_n \right\} \]

\[= (P_\theta)_n \] (2.5)

Where:

- \(r\) running spanwise coordinate for the blade from the axis of rotation.
- \(\theta_b\) angle formed by the rotor disc plane and the blade sectional chordline, usually including built-in angle of twist and collective pitch.
- \((EI)_C\) bending stiffness in the blade sectional chordline direction.
- \((EI)_B\) bending stiffness in the direction perpendicular to the chordline.
- \(m\) spanwise mass of the blade per unit length.
centrifugal force at $r$ expressed as:

$$T = \Omega^2 \int_r^8 m \, r \, dr$$

resultant force per unit length in the $z$ direction on the blade (positive forward).

resultant force per unit length in the circumferential direction on the blades (positive clockwise when looking forward) about $n$th blade.

(b) Wing Equations

$$\frac{\partial^2}{\partial y^2} \left[ \left\{ (EI_w)_c \sin^2 \theta_w + (EI_w)_b \cos^2 \theta_w \right\} \frac{\partial^2 \bar{w}_w}{\partial y^2} \right] + \frac{\partial}{\partial y} \left[ \left\{ (EI_w)_c - (EI_w)_b \right\} \sin \theta_w \cos \theta_w \frac{\partial^2 \bar{w}_w}{\partial y^2} \right]$$

$$+ m_w \ddot{u}_w + S_n \ddot{p}_w = F_x$$

(2.6)

$$\frac{\partial^2}{\partial y^2} \left[ \left\{ (EI_w)_c \cos^2 \theta_w + (EI_w)_b \sin^2 \theta_w \right\} \frac{\partial^2 \bar{w}_w}{\partial y^2} \right] + \frac{\partial}{\partial y} \left[ \left\{ (EI_w)_c - (EI_w)_b \right\} \sin \theta_w \cos \theta_w \frac{\partial^2 \bar{w}_w}{\partial y^2} \right]$$

$$+ m_w \ddot{w}_w = F_z$$

(2.7)
\[
\frac{\partial}{\partial y} \left\{ (GJ_w) \frac{\partial R_y}{\partial y} \right\} + I_w \ddot{p}_w + S_\alpha \ddot{u}_w = M_y \quad (2.8)
\]

where

- \( \theta_w \) angle between the wing sectional chordline and free-stream direction.
- \((EI_w)_c\), \((EI_w)_B\) wing bending stiffness defined similarly as for the blade.
- \((GJ_w)\) wing torsional rigidity.
- \(m_w\) spanwise wing mass per unit length.
- \(I_w\) wing mass moment of inertia about the elastic axis per unit length.
- \(s_\alpha\) static mass moment of wing segment about the elastic axis defined as

\[
s_\alpha = ms
\]

where \( s \) is a distance between the center of gravity and the elastic axis of the wing and positive if center of gravity is ahead of the elastic axis.

- \(F_x, F_z\) resultant wing force per unit length in the \(x\) and \(z\) directions, respectively.
- \(M_y\) resultant wing pitching moment per unit length.

The wing cross-sectional geometry is shown in Fig. 4, and pylon motions are expressed with wing deflections as

\[
\begin{bmatrix}
\gamma_x \\
\gamma_z \\
\gamma_y \\
\gamma_r
\end{bmatrix} =
\begin{bmatrix}
u_w \\
w_w \\
\frac{\partial u_w}{\partial y} \\
-\frac{3u_w}{y^2}
\end{bmatrix} \quad y=L
\quad (2.9)
\]
(c) Matching Conditions between the Blade and Pylon

\[ M_p \ddot{r}_z - \frac{\partial}{\partial y} \left[ \left( (E I_w)_c \sin^2 \Theta_w + (E I_w)_b \cos^2 \Theta_w \right) \frac{\partial u_y}{\partial y} \right]_{y=L} \]

\[ - \frac{\partial}{\partial y} \left[ \left( (E I_w)_c - (E I_w)_b \right) \sin \Theta_w \cos \Theta_w \frac{\partial u_y}{\partial y} \right]_{y=L} \]

\[ + \sum_{n=1}^{N} \left[ \int_{0}^{R} m \left\{ \ddot{r}_z + \ddot{w}_n - \left( \nu_n \sin \Theta_n \right) \right\} dr \right]_{n} \]

\[ = \sum_{n=1}^{N} \left[ \int_{0}^{R} \left( -P_0 \sin \Theta_n + \nu_P P_z \right) dr \right]_{n} \]  \hspace{1cm} (2.10)

\[ M_p \ddot{z}_z - \frac{\partial}{\partial y} \left[ \left( (E I_w)_c \cos^2 \Theta_w + (E I_w)_b \sin^2 \Theta_w \right) \frac{\partial u_y}{\partial y} \right]_{y=L} \]

\[ - \frac{\partial}{\partial y} \left[ \left( (E I_w)_c - (E I_w)_b \right) \sin \Theta_w \cos \Theta_w \frac{\partial u_y}{\partial y} \right]_{y=L} \]

\[ + \sum_{n=1}^{N} \left[ \int_{0}^{R} m \left( \ddot{z}_z + \ddot{w}_n \right) dr \right]_{n} \]

\[ = \sum_{n=1}^{N} \left[ \int_{0}^{R} \left( \nu_P P_0 \sin \Theta_n + \nu_P P_z \cos \Theta_n + P_z \right) dr \right]_{n} \]  \hspace{1cm} (2.11)

\[ I_p \ddot{z}_y + \left[ \left( (E I_w)_c \cos^2 \Theta_w + (E I_w)_b \sin^2 \Theta_w \right) \frac{\partial u_y}{\partial y} \right]_{y=L} \]

\[ + \left[ \left( (E I_w)_c - (E I_w)_b \right) \sin \Theta_w \cos \Theta_w \frac{\partial u_y}{\partial y} \right]_{y=L} \]

\[ + \sum_{n=1}^{N} \left[ \int_{0}^{R} m \left\{ \left( r^2 + \frac{r^4}{2} \right) \ddot{z}_y + \Omega r^2 \dot{v}_r + r \left( \nu_n \sin \Theta_n \right) \right\} dr \right]_{n} \]

\[ = 2 \Omega r \left( \nu_n \cos \Theta_n - \nu_P P_z \right) + r \nu_P P_0 + r P_z \sin \Theta_n \]

13
\[ I_p \ddot{v}_p + \left[ (G_{I_{\theta \phi}}) \frac{\partial P_\theta}{\partial y} \right]_{y=L} + \sum_{n=1}^{\infty} \left[ \int_0^R m \left\{ -\dot{v}_n \sin \Phi_n + \dot{v}_n \sin \Phi_n + \frac{1}{2} \omega \cdot \omega \right\} \, dr \right]_n \]

\[ = \sum_{n=1}^{\infty} \left[ \int_0^R \{ -P_\theta \sin \Phi_n + \dot{v}_n \sin \Phi_n \} \, dr \right]_n \quad (2.12) \]

\[ I_{\theta r} \dot{\varphi}_r - \left[ \{ (E I_{w})_c \sin^2 \Theta \varphi + (E I_{w})_b \cos^2 \Theta \varphi \} \frac{\partial^2 u_x}{\partial y^2} \right]_{y=L} \]

\[ - \left[ \{ (E I_{w})_c - (E I_{w})_b \} \sin \Theta \varphi \cos \Theta \varphi \frac{\partial^4 u_x}{\partial y^4} \right]_{y=L} \]

\[ + \sum_{n=1}^{\infty} \left[ \int_0^R m(r^2 \ddot{v}_r + r \dddot{v}_n) \, dr \right]_n \]

\[ = \sum_{n=1}^{\infty} \left[ \int_0^R (\gamma P_\theta) \, dr \right]_n \quad (2.14) \]

where

- \( M_p \) Pylon mass
- \( I_{\theta y} \) Pylon yawing mass moment of inertia
- \( I_{\theta p} \) Pylon pitching mass moment of inertia
- \( I_{\theta r} \) Pylon rolling mass moment of inertia

\("\),\(" \) Time derivative applied to entire formula in the parentheses
The above equations are derived independently of Refs. 7, 9, and 13. However, if the wing motions are eliminated, the equations are similar to the basic whirl flutter equations derived in Ref. 7 or Ref. 9.

2.3 Supplement for Autorotational Flight

Autorotation may be defined as the condition of flight where there is no restraint of the rotor rotation about the rotor shaft. Therefore, no rotor torque is transmitted to the shaft, and no pylon roll motion is transmitted to the rotor. This means that rigid body rotation of the entire rotor about the shaft will be produced. This rigid body rotation is designated as $v_R$ (positive clockwise).

The equations of motion for autorotational flight are almost the same as those for powered flight. The equations: 2.4, 2.6, 2.7, 2.8, 2.10, 2.11, 2.12 and 2.13 are the same. In Eq. 2.5 $v_x$ must be replaced by $v_R$. In lieu of Eq. 2.14 the fact that the rotor inplane motion is independent of the pylon roll motion in autorotation flight results in Eq. 2.15 as follows:

\[ \sum_{n=1}^{N} \left[ \int_{0}^{R} m (r^2 \ddot{\theta}_n + r \dddot{\theta}_n) \, dr \right]_n = \sum_{n=1}^{N} \left[ \int_{0}^{R} (r P_\theta) \, dr \right]_n \]

This yields a new degree of freedom.

The angle in Eq. 2.1 is also expressed as

\[ \Phi_n = \Omega t + v_R + \Delta \Phi_n \]

in this case.
3.1 Rotor Aerodynamic Forces

The rotor aerodynamic forces will be evaluated next. The analysis is almost the same as in the helicopter blade element theory. The significant difference is that it is impossible to assume small angles for the angle of twist, the collective pitch and the angle of attack of the blade operating in the high inflow. The basic idea is presented by Young and Lytwyn (Ref. 9).

The section aerodynamic lift and drag forces \( dL \), \( dD \) yield the resultant forces \( P_z \) and \( P_\theta \) as follows (in Fig. 5),

\[
P_z = dL \cos \phi_i - dD \sin \phi_i
\]

\[
P_\theta = -dD \cos \phi_i - dL \sin \phi_i
\]

where

\[
dL = \frac{1}{2} \rho c_B U^2 a
\]

\[
dD = \frac{1}{2} \rho c_B U^2 C_{DO}
\]

and the angle \( \phi_i \) is defined as,

\[
\sin \phi_i = \frac{U_\pi}{U}
\]

\[
\cos \phi_i = \frac{|U_T|}{U}
\]

The blade section inplane velocity (positive counterclockwise direction) is \( U_T \) and \( U_P \) is the blade section out-of-plane velocity (positive for negative z direction). Therefore, the resultant air velocity is expressed as,

\[
U = \sqrt{U_P^2 + U_T^2}
\]

In cruise flight,

\[
U_T = \Omega r
\]

\[
U_P = V + v
\]
where $V$ is the forward aircraft speed (axial velocity) and $v$ is the induced inflow velocity. The induced velocity is very small compared with the forward velocity $V$ in the high inflow operation. Therefore the induced velocity is neglected in this entire study.

The inflow ratio $\lambda$ is defined as the ratio of the axial velocity to tip speed of the blade:

$$\lambda = \frac{V}{\Omega R}$$

The section effective angle of attack is

$$\alpha = \theta_B + \theta_{PF} - \tan^{-1} \frac{U_p}{|U_r|} + \theta_c$$

The angle $\theta_B$ formed by the disc plane and the blade chordline includes the angle of twist and collective pitch. Therefore

$$\theta_B = \theta_{AT} + \theta_{.75}$$

where $\theta_{AT}$ is the built-in section angle of twist of the blade, which is zero at 75% span of the blade, and $\theta_{.75}$ is the collective pitch to be obtained from the performance calculation. Usually $\theta_{.75}$ is:

$$\theta_{.75} = \tan^{-1} \left( \frac{V}{\frac{3}{2} R \Omega} \right) + \theta_D$$

The first term express the inflow at 75% span and $\theta_D$ depends on the proprotor design to obtain the optimum cruising performance. Pitch-flap coupling $\theta_{PF}$ and blade pitch control $\theta_c$ will be discussed later. Finally:

$$\alpha = \theta_{AT} + \theta_{.75} + \theta_{PF} - \tan^{-1} \frac{U_p}{|U_r|} + \theta_c$$
Substituting Eqs. 3.2 and 3.3 into Eq. 3.1 yields

\[ P_x = \frac{1}{2} \rho C_b a U|U_r| \alpha - \frac{1}{2} \rho C_b C_{b_0} U U_r \]

\[ P_\theta = \Omega \left[ -\frac{1}{2} \rho C_b C_{b_0} U|U_r| - \frac{1}{2} \rho C_b a U U_r \alpha \right] \]  

(3.11)

where

\[ \text{sign } (\Omega) = \frac{\Omega}{|\Omega|} \]

(3.12)

Next the perturbation method is applied to Eq. 3.11 to derive the aerodynamic forces of the rotor in disturbed motion:

\[ P_x = (P_x)_0 + \left( \frac{\partial P_x}{\partial U_r} \right)_0 S U_r + \left( \frac{\partial P_x}{\partial U_t} \right)_0 S U_t + \left( \frac{\partial P_x}{\partial \alpha} \right)_0 S \alpha \]

\[ P_\theta = (P_\theta)_0 + \left( \frac{\partial P_\theta}{\partial U_r} \right)_0 S U_r + \left( \frac{\partial P_\theta}{\partial U_t} \right)_0 S U_t + \left( \frac{\partial P_\theta}{\partial \alpha} \right)_0 S \alpha \]  

(3.13)

In these equations, \( (\quad)_0 \) means those values are evaluated in the equilibrium state, given by the following expressions:

\[ (P_x)_0 = \frac{1}{2} \rho C_b a (\Omega R)^2 \left[ \alpha_0 \tau_3 - \frac{C_{b_0}}{\alpha} \bar{x} \tau_2 ight. \]

\[ \left. + \bar{x}^2 \alpha_0 \tau_1 - \frac{C_{b_0}}{\alpha} \bar{x}^3 \tau_0 \right] \]

\[ \left( \frac{\partial P_x}{\partial U_r} \right)_0 = \frac{1}{2} \rho C_b a \Omega R \left[ 2 \alpha_0 \tau_2 + \bar{x} \left( 1 - \frac{C_{b_0}}{\alpha} \right) \tau_1 + \bar{x}^2 \alpha_0 \tau_0 \right] \Omega \]

\[ \left( \frac{\partial P_x}{\partial U_t} \right)_0 = \frac{1}{2} \rho C_b a \Omega R \left[ \left( 1 + \frac{C_{b_0}}{\alpha} \right) \tau_2 + \bar{x} \alpha_0 \tau_1 - 2 \bar{x}^2 \frac{C_{b_0}}{\alpha} \tau_0 \right] \]

\[ \left( \frac{\partial P_x}{\partial \alpha} \right)_0 = \frac{1}{2} \rho C_b a (\Omega R)^2 \left[ \tau_3 + \bar{x}^2 \tau_1 \right] \]  

(3.14a)
\[
(P_e)_0 = \frac{1}{2} \rho C_b a (\Omega R)^2 \left[ -\frac{C_{pe}}{\alpha} \tau_3 - \overline{\lambda} \alpha_0 \tau_2 - \chi^2 \frac{C_{pe}}{\alpha} \tau_1 - \chi^2 \lambda \alpha_0 \tau_0 \right] \Omega \\
\left( \frac{\partial P_e}{\partial \alpha} \right) = \frac{1}{2} \rho C_b a |\Omega| R \left[ -2 \frac{C_{pe}}{\alpha} \tau_2 - \overline{\lambda} \alpha_0 \tau_1 - \chi^2 \left( 1 + \frac{C_{pe}}{\alpha} \right) \tau_0 \right] \\
\left( \frac{\partial P_e}{\partial \tau_0} \right) = \frac{1}{2} \rho C_b a |\Omega| R \left[ -\alpha_0 \tau_2 + \overline{\lambda} \left( 1 - \frac{C_{pe}}{\alpha} \right) \tau_1 - \chi^2 \alpha_0 \tau_0 \right] \Omega \\
\left( \frac{\partial P_e}{\partial \tau_2} \right) = \frac{1}{2} \rho C_b a (\Omega R)^2 \left[ -\overline{\lambda} \tau_2 - \chi^2 \alpha_0 \tau_0 \right] \Omega 
\] (3.14b)

where

\[
\tau_K = \frac{x_K}{\sqrt{x^2 + \chi^2}} \quad (K = 0, 1, 2, 3) \] (3.15)

\[
\alpha = r/R \] (3.16)

\[
\overline{\lambda} = |\lambda| \] (3.17)

\[
\alpha_0 = \theta_{AT} + \theta_{TS} - \tan^{-1} \frac{U_p}{|U_r|} \] (3.18)

The perturbation quantities \( \delta U_T, \delta U_P \) and \( \delta \alpha \) will be considered next. The perturbation velocities \( \delta U_T \) and \( \delta U_P \) consist of rotor and pylon motion and gust velocities.

The velocity at the position of the blade axis in \( x, y, z \) coordinate (Fig. 2, see details in Appendix A) can be written as
Wind velocities at the blade element in the disc plane $U_T$ are then obtained by applying the transformation matrix due to pylon motion to change the $x,y,z$ coordinate system to blade disc coordinate system. Wind velocities $U_{xB}$ and $U_{yB}$ are $x_B$, $y_B$ components of the inplane velocity of the blade in the $x_B$-$y_B$ coordinate system fixed to the rotor hub (Fig. 6). Gust velocity as well as the aircraft forward velocity is included.

\[
\begin{align*}
\dot{x} &= -\Omega r \sin \phi_n - \Omega v_n \sin \phi_n - \dot{v}_n \sin \phi_n + \dot{r}_x \\
\dot{y} &= \Omega r \cos \phi_n - \dot{v}_n + \nu_r \cos \phi_n - \Omega \nu_r \cos \phi_n
\end{align*}
\]

\[
\dot{z} = \dot{w}_n + \dot{r}_z - \nu_p \cos \phi_n + \Omega \nu_p \sin \phi_n + \Omega r \nu_y \cos \phi_n
\]

Wind velocities are expressed as

\[
\begin{bmatrix}
U_{xB} \\
U_{yB} \\
\nu_p
\end{bmatrix}
= \begin{bmatrix}
1 & \nu_r & -\nu_p \\
-\nu_r & 1 & \nu_y \\
\nu_p & -\nu_y & 1
\end{bmatrix}
\begin{bmatrix}
\dot{x} + u_a \\
\dot{y} + u_q \\
\nu + \dot{z} + w_q
\end{bmatrix}
\]

Tangential velocity $U_T$ is expressed as

\[
U_T = -U_{xB} \sin \phi_n + U_{yB} \cos \phi_n
\]

After higher order terms are neglected,

\[
U_T = r\Omega - u_a \sin \phi_n + v_a \cos \phi_n + \nu_n - \dot{r}_x \sin \phi_n - \dot{r}_z \sin \phi_n - \nu_p \sin \phi_n - \nu_y \cos \phi_n
\]
Finally the perturbation velocities \( \delta U_T \) and \( \delta U_P \) are given as

\[
\delta U_T = \Omega R \left[ -\lambda \frac{u}{V} \sin \phi_n + \Lambda \frac{\dot{u}}{V} \cos \phi_n + \frac{\dot{v}_n}{R} - \frac{\dot{v}_3}{R} \sin \phi_n \right.
\]

\[
\left. + \frac{\dot{v}_3}{R} \cos \phi_n + \Lambda \left( \dot{v}_p \sin \phi_n + \dot{v}_5 \cos \phi_n \right) \right]
\]

\[
\delta U_P = \Omega R \left[ \lambda \frac{u}{V} + \frac{\dot{v}_3}{R} + \frac{\dot{v}_5}{R} - \lambda \frac{\dot{v}_p}{R} \cos \phi_n + \lambda \frac{\dot{v}_5}{R} \sin \phi_n \right]
\]

where

\[
( \dot{v}_3 ) = \frac{\partial}{\partial (|\Omega| t)}
\]

\[
( \dot{v}_5 ) = \frac{\dot{v}_5}{R}
\]

Pitch-flap coupling gives a change in blade pitch angle proportional to the flapping angle at the root. It is defined as

\[
(\theta_{PF})_n = -\frac{\partial \omega_n}{\partial r} \tan \delta_3
\]

The angle \( \delta_3 \) is a design factor to yield the optimum pitch angle gain to prevent blade motion instability. Blade pitch control consists of collective pitch control and cyclic pitch control.

\[
(\theta_C)_n = \theta_{o}(t) + \theta_{ic}(t) \cos \phi_n + \theta_{is}(t) \sin \phi_n
\]

Therefore, the perturbation quantity associated with the angle of attack of the blade is simply derived from Eq. 3.7 as

\[
\delta \alpha = \theta_{PF} + \theta_C
\]
3.2 Wing Aerodynamic Forces

Wing aerodynamic forces are derived by the blade aerodynamic perturbation method. Aerodynamic coefficients are obtained from strip theory (Ref. 15).

The wing element lift, drag and pitching moment (Fig. 7) per unit span length can be written as

\[
\begin{align*}
\frac{dL}{dS} &= \frac{1}{2} \rho U_w^2 C_{l\alpha} \alpha_w \\
\frac{dD}{dS} &= \frac{1}{2} \rho U_w^2 C_{d\alpha} \\
\frac{dM_{\alpha}}{dS} &= \frac{1}{2} \rho U_w^2 (C_{m\alpha} + C_{m\alpha} \alpha_w) C_w^2
\end{align*}
\]

These values are transformed to the \(x,y,z\) coordinate system:

\[
\begin{align*}
F_x &= \frac{dL}{dS} \cos \phi_w - \frac{dD}{dS} \sin \phi_w \\
&= \frac{1}{2} \rho C_{l\alpha} U_w^2 \alpha_w \cos \phi_w - \frac{1}{2} \rho C_{d\alpha} U_w^2 \alpha_w \sin \phi_w \\
F_y &= -\frac{dD}{dS} \cos \phi_w - \frac{dL}{dS} \sin \phi_w \\
&= -\frac{1}{2} \rho C_{l\alpha} C_{m\alpha} U_w^2 \cos \phi_w - \frac{1}{2} \rho C_{d\alpha} \alpha_w U_w^2 \alpha_w \sin \phi_w \\
M_y &= \frac{dM_{\alpha}}{dS} + \frac{dL}{dS} \bar{e} C_w \\
&= \frac{1}{2} \rho C_{l\alpha}^2 (C_{m\alpha} + C_{m\alpha} \alpha_w) U_w^2 \\
&\quad + \frac{1}{2} \rho C_{d\alpha} \bar{e} \alpha_w U_w^2 \alpha_w
\end{align*}
\]

where

\[
\begin{align*}
U_w^2 &= (V + \omega_w + \omega_q)^2 + (\dot{u}_w + u_q)^2 \\
\alpha_w &= \theta_w + \rho_w - \phi_w \\
tan \phi_w &= \frac{\dot{u}_w + u_q}{V + \dot{\omega}_w + \omega_q}
\end{align*}
\]
and $\bar{e}$ is the distance (nondimensionalized by wing chord) between the elastic axis and the quarter-chord line (positive if the quarter-chord line is ahead of the elastic axis).

Hereafter the small angle assumption may be used for the wing aerodynamic forces because of the small angle of attack and angle of twist. Then it is approximated that $\sin \phi_w = \phi_w$, $\cos \phi_w = 1$, etc.

Perturbation velocities $\dot{u}_w$, $\dot{w}_w$, $u_G$ and $w_G$ are chosen to express the aerodynamic forces. The perturbation equations are then given by

$$F_x = (F_x)_0 + \left( \frac{\partial F_x}{\partial u_w} \right)_0 \dot{u}_w + \left( \frac{\partial F_x}{\partial w_w} \right)_0 \dot{w}_w + \left( \frac{\partial F_x}{\partial P_z} \right)_0 P_z$$

$$+ \left( \frac{\partial F_x}{\partial u_g} \right)_0 u_g + \left( \frac{\partial F_x}{\partial w_g} \right)_0 w_g$$

$$F_z = (F_z)_0 + \left( \frac{\partial F_z}{\partial u_w} \right)_0 \dot{u}_w + \left( \frac{\partial F_z}{\partial w_w} \right)_0 \dot{w}_w + \left( \frac{\partial F_z}{\partial P_z} \right)_0 P_z$$

$$+ \left( \frac{\partial F_z}{\partial u_g} \right)_0 u_g + \left( \frac{\partial F_z}{\partial w_g} \right)_0 w_g$$

$$M_y = (M_y)_0 + \left( \frac{\partial M_y}{\partial u_w} \right)_0 \dot{u}_w + \left( \frac{\partial M_y}{\partial w_w} \right)_0 \dot{w}_w + \left( \frac{\partial M_y}{\partial P_z} \right)_0 P_z$$

$$+ \left( \frac{\partial M_y}{\partial u_g} \right)_0 u_g + \left( \frac{\partial M_y}{\partial w_g} \right)_0 w_g$$

(3.31)

This study is concerned only with deviation from the equilibrium state. Then the steady state aerodynamic forces $(F_x)_0$, $(F_z)_0$ and $(M_y)_0$ may be dropped from the equations without inconsistency although the blade steady state aerodynamic forces $(P_z)_0$ and $(P_0)_0$ cannot be neglected since they influence the pylon motion.

Applying Eq. 3.31 to Eq. 3.29, the following expressions for the wing aerodynamic forces due to wing motion are obtained

23
\[
F_x = \begin{bmatrix}
-\frac{1}{2} \rho c_w V a_w (1 + \frac{C_{Dw}}{a_w}) \\
\rho c_w V a_w a_w \\
0
\end{bmatrix}^T \begin{bmatrix}
\dot{u}_w \\
\dot{w}_w \\
\dot{p}_w
\end{bmatrix} \\
+ \begin{bmatrix}
0 \\
0 \\
\frac{1}{2} \rho c_w V^2 a_w \\
-\frac{1}{2} \rho c_w V a_w (1 + \frac{C_{Dw}}{a_w}) \\
0 \\
\rho c_w V a_w a_w
\end{bmatrix}^T \begin{bmatrix}
\ddot{u}_w \\
\ddot{v}_w \\
\ddot{w}_w
\end{bmatrix}
\] 

(3.32a)

\[
F_z = \begin{bmatrix}
-\frac{1}{2} \rho c_w V a_w a_w \\
\rho c_w V C_{Dw} \\
0
\end{bmatrix}^T \begin{bmatrix}
\ddot{u}_w \\
\ddot{w}_w \\
\ddot{p}_w
\end{bmatrix} \\
+ \begin{bmatrix}
0 \\
0 \\
-\rho c_w V C_{Dw}
\end{bmatrix}^T \begin{bmatrix}
\ddot{u}_w \\
\ddot{v}_w \\
\ddot{w}_w
\end{bmatrix}
\] 

(3.32b)
where the angle of attack in equilibrium state is given as

\[ \alpha_{\omega_0} = \Theta_\omega \]  

(3.33)
SECTION 4

MODAL ANALYSES

4.1 Variational Functional

In order to apply Galerkin's method the variational functional is derived from Eqs. 2.4, 2.5, 2.6, 2.7, 2.8, 2.10, 2.11, 2.12, 2.13, and 2.14.

\[
\Pi = - \sum_{n=1}^{N} \left[ \frac{1}{2} \int_{0}^{R} \left( (EI_c \sin^2 \Theta + (EI_b \cos^2 \Theta \right) \left( \frac{3w}{\alpha^2} \right)^2 \right) dr \right]_n \\
- \sum_{n=1}^{N} \left[ \frac{1}{2} \int_{0}^{R} \left( (EI_c \cos^2 \Theta + (EI_b \sin^2 \Theta \right) \left( \frac{3w}{\alpha^2} \right)^2 \right) dr \right]_n \\
- \sum_{n=1}^{N} \left[ \frac{1}{2} \int_{0}^{R} \left( (EI_c - (EI_b) \sin \Theta \cos \Theta \right) \frac{3w}{\alpha^2} \frac{3w}{\alpha^2} \right) dr \right]_n \\
- \frac{1}{2} \sum_{n=1}^{N} \left[ \int_{0}^{R} \left( T \left( \frac{\partial \Phi}{\partial r} \right) + T \left( \frac{\partial \Phi}{\partial \alpha} \right) \right) dr \right]_n \\
+ \frac{1}{2} \sum_{n=1}^{N} \left[ \int_{0}^{R} m \left( \dot{x}_n^2 + \dot{y}_n + \Omega^2 \dot{r}_n^2 \right) dr \right]_n \\
+ \frac{1}{2} \sum_{n=1}^{N} \left[ \int_{0}^{R} m \left( r \dot{u}_n \dot{v}_n - (\dot{u}_n \sin \phi_n) (\dot{r}_n + \dot{\phi}_n) \right. \right. \\
+ \dot{w}_n \dot{r}_n \right. \right. \\
\left. \left. - \Omega (\dot{u}_n \cos \phi_n) \dot{y}_n + r (\dot{u}_n \sin \phi_n) \dot{y}_n \right) \right. \right. \\
\left. \left. - (\dot{w}_n \cos \phi_n) \dot{r}_n \right) \right. \right. \\
\left. \left. - 2 \Omega \dot{w}_n \dot{y}_n \cos \phi_n \right) \right. \right. \\
\left. \left. - 2 \Omega \dot{w}_n \dot{r}_n \dot{y}_n \sin \phi_n \right) \right]_n \\
+ \frac{M_{bN}}{2} \left\{ (\dot{r}_n + \dot{\phi}_n)^2 + \dot{r}_n^2 + \dot{r}_n^2 \right\} \\
+ \frac{I_{bN}}{2} \left\{ \frac{\dot{r}_n^2}{2} + \frac{\dot{r}_n^2}{2} + \frac{\dot{r}_n^2}{2} + 2 \Omega \dot{\phi}_n \dot{y}_n \right\}
\]
\[-\frac{1}{2} \int_0^L \left\{ (EI_c) \sin^2 \theta_{xy} + (EI_y) \cos^2 \theta_{xy} \right\} \left( \frac{\partial^2 u_y}{\partial y^2} \right)^2 \, dy \]

\[-\frac{1}{2} \int_0^L \left\{ (EI_c) \cos^2 \theta_{xy} + (EI_y) \sin^2 \theta_{xy} \right\} \left( \frac{\partial^2 u_x}{\partial y^2} \right)^2 \, dy \]

\[-\frac{1}{2} \int_0^L \left\{ (EI_c) - (EI_y) \sin \theta_{xy} \cos \theta_{xy} \right\} \frac{\partial^2 u_y}{\partial y^2} \frac{\partial^2 u_x}{\partial y^2} \, dy \]

\[-\frac{1}{2} \int_0^L (G J_t \left( \frac{\partial R_x}{\partial y} \right)^2 \right) \, dy \]

\[+ \frac{1}{2} \int_0^L m_w \ddot{u}_x \, dy + \frac{1}{2} \int_0^L m_w \ddot{u}_y \, dy \]

\[+ \frac{1}{2} \int_0^L J w \ddot{\theta}_x \, dy + \int_0^L S_x \ddot{u}_x \, dy \]

\[+ \frac{1}{2} M_e \ddot{\gamma}_x^2 + \frac{1}{2} M_e \ddot{\gamma}_y^2 + \frac{1}{2} I_{xR} \ddot{\theta}_y^2 \]

\[+ \frac{1}{2} I_{yR} \ddot{\theta}_x^2 + \frac{1}{2} I_{zR} \ddot{\theta}_z^2 \]

\[+ \frac{1}{2} \sum_{n=1}^N \left[ \int_0^R (P_x \ddot{u}_n \) \, dr \right]_n \]

\[+ \frac{1}{2} \sum_{n=1}^N \left[ \int_0^R (P_y \ddot{v}_n \) \, dr \right]_n \]

\[+ \sum_{n=1}^N \left[ \int_0^R (-P_x \sin \theta_n + \nu \ddot{P}_x) \, dr \right]_n \]

\[+ \sum_{n=1}^N \left[ \int_0^R (-\nu \nu \ddot{P}_x \sin \theta_n + \nu \ddot{P}_x \cos \theta_n + \ddot{P}_x) \, dr \right]_n \]

\[+ \sum_{n=1}^N \left[ \int_0^R \left\{ -\ddot{\theta}_x \left( \ddot{P}_x \cos \theta_n - \nu \ddot{P}_x \right) + \nu \nu \ddot{P}_x \right\} \, dr \right]_n \]

\[+ \nu P_x \sin \theta_n + \ddot{P}_x \ddot{u}_n \cos \theta_n \]

\[-\ddot{\theta}_x \ddot{P}_x \cos \theta_n + \ddot{P}_x \ddot{u}_n \sin \theta_n \]

\[-P_x \ddot{u}_n \cos \theta_n \]
4.2 Free Vibration of the Blade and Wing

In preparation for modal analysis of the basic equations, the free vibration of the rotating blade and the cantilever wing will be considered. It is very advantageous to use the natural frequency of the free vibrating beam to represent the quite complicated stiffness terms of the beam.

The variational functional for free vibration of the rotating blade is derived from Eqs. 4.1, neglecting aerodynamic forces, pylon and wing motion.

\[
\mathcal{F}_\theta = - \sum_{n=1}^N \left[ \frac{1}{2} \int_0^L \left( (EI)_c \sin^2 \Theta_b + (EI)_b \cos^3 \Theta_b \right) \left( \frac{\partial^2 w_b}{\partial r^2} \right)^2 \, dr \right]_n
\]
Assume a series solution in terms of normal modes which express the coupled motion of inplane and out-of-plane motion

\[
\begin{align*}
\{w^r_n\} &= \sum_{j} \left\{ \frac{w_j(r)}{V_j(r)} \right\} q_{nj}(\theta) \\
\{v^r_n\} &= \sum_{j} \left\{ \frac{v_j(r)}{V_j(r)} \right\} q_{nj}(\theta)
\end{align*}
\]  

Substituting Eq. 4.3 into Eq. 4.2 and applying Lagrange's equations as follows:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_{ni}(\theta)} \right) - \frac{\partial U}{\partial q_{ni}} = 0
\]  

yields

\[
s\int \left[ \left\{ (EI)_c \sin^2 \Theta + (EI)_o \cos^2 \Theta \right\} \left( \frac{d^2 w_i}{dr^2} \right) \left( \frac{d^2 w_i}{dr^2} \right) \right]
\]

\[
+ \left\{ (EI)_c \cos^2 \Theta + (EI)_o \sin^2 \Theta \right\} \left( \frac{d^2 v_i}{dr^2} \right) \left( \frac{d^2 v_i}{dr^2} \right)
\]
where \( \lambda_j \) denotes the jth rotating undamped natural frequency of the blade. Simplification of Eq. 4.5 will result from use of the orthogonality condition (Ref. 15) of the normal modes which is expressed as

\[
\int_0^R m (W_i W_j + V_i V_j) \, dr = \begin{cases} \int_0^R (W_i^2 + V_i^2) \, dr & j = i \\ 0 & j \neq i \end{cases} \tag{4.6}
\]

The amplitude of the mode shape is normalized to unity as

\[
\int_0^R m (W_i^2 + V_i^2) \, dr = 1 \tag{4.7}
\]

Hence, Eq. 4.4 results in

\[
\int_0^R \left\{ (EI)_c \sin^2 \theta_\theta + (EI)_b \cos^2 \theta_\theta \right\} \left( \frac{d^4 W_i}{dr^4} \right) \left( \frac{d^4 W_j}{dr^4} \right) + \left\{ (EI)_b \cos^2 \theta_\theta + (EI)_b \sin^2 \theta_\theta \right\} \left( \frac{d^4 V_i}{dr^4} \right) \left( \frac{d^4 V_j}{dr^4} \right) + \frac{1}{2} \left\{ (EI)_c - (EI)_b \right\} \sin \theta_\theta \cos \theta_\theta \left( \frac{d^2 W_i}{dr^2} \frac{d^2 V_j}{dr^2} + \frac{d^2 W_j}{dr^2} \frac{d^2 V_i}{dr^2} \right)
\]
In the similar way the free vibration of the wing will be analyzed, treating the rotor and pylon as lumped masses at the wing tip. The wing deflection can be expressed as

\[
+ T \frac{dW_i}{dr} \frac{dW_j}{dr} + T \frac{dV_i}{dr} \frac{dV_j}{dr} + \Omega^2 m v_i v_j \right] \, \text{d}r
\]

\[
\begin{cases} 
\lambda_i^2 & \text{if } j = i \\
0 & \text{if } j \neq i 
\end{cases}
\quad (4.8)
\]

Then, pylon motion is described by

\[
\begin{pmatrix} u_w \\ w_w \\ \phi_w \end{pmatrix} = \sum_j \begin{pmatrix} \xi_j(y) \\ \zeta_j(y) \\ \phi_j(y) \end{pmatrix} q_j(t) 
\quad (4.9)
\]

Then, pylon motion is described by

\[
\begin{pmatrix} \gamma_x/R \\ \gamma_y/R \\ \nu_y \\ \nu_p \end{pmatrix} = \sum_j \begin{pmatrix} \xi_j/R \\ \zeta_j/R \\ \phi_j \end{pmatrix} a_j(t) 
\quad (4.10)
\]

\[
= \begin{bmatrix} T \end{bmatrix} \begin{pmatrix} a(t) \end{pmatrix} 
\quad (4.10)
\]
where

\[
[T] = \begin{bmatrix}
\delta_1 & \delta_2 & \cdots & \delta_n \\
\dot{\delta}_1 & \dot{\delta}_2 & \cdots & \dot{\delta}_n \\
d_1/dy & d_2/dy & \cdots & d_n/dy \\
\phi_1 & \phi_2 & \cdots & \phi_n \\
-d_1/dy & -d_2/dy & \cdots & -d_n/dy
\end{bmatrix}
\]  \tag{4.11}

\[
\{a(\omega)\} = \begin{bmatrix}
a_1(\omega) \\
a_2(\omega) \\
\vdots \\
a_n(\omega)
\end{bmatrix}
\]

The wing coupled modes are normalized as

\[
\int_0^L \left\{ m \pi (\delta_j^2 + \dot{\delta}_j^2) + I_w \dot{\phi}_j^2 + 2 S \delta_j \phi_j \right\} dy \\
+ \left[ M_p (\delta_j^2 + \dot{\delta}_j^2) + I_{by} \left( \frac{d\delta_j}{dy} \right)^2 + I_{pz} \phi_j^2 + I_{pr} \left( \frac{d\phi_j}{dy} \right)^2 \right]_{y=L} \\
+ NM \left[ (\dot{\delta}_j + \kappa \phi_j)^2 + \kappa^2 \left( \frac{d\delta_j}{dy} \right)^2 + \delta_j^2 \right]_{y=L} \\
+ \frac{N I_0}{2} \left[ 2 \left( \frac{d\delta_j}{dy} \right)^2 + \left( \frac{d\dot{\delta}_j}{dy} \right)^2 + \phi_j^2 \right]_{y=L}
= 1
\]  \tag{4.12}
Hence,

\[
\int_0^L \left[ \left( EI_w \right)_c \sin^2 \Theta \omega + (EI_w)_b \cos^2 \Theta \omega \left( \frac{d^2 \phi_i}{dy^2} \right) \left( \frac{d^2 \phi_i}{dy^2} \right) 
+ \left( (EI_w)_c \cos^2 \Theta \omega + (EI_w)_b \sin^2 \Theta \omega \right) \left( \frac{d^2 \phi_i}{dy^2} \right) \left( \frac{d^2 \phi_i}{dy^2} \right) 
+ \frac{1}{2} \left( (EI_w)_c -(EI_w)_b \right) \sin \Theta \omega \cos \Theta \omega \left( \frac{d^2 \phi_i}{dy^2} \right) \left( \frac{d^2 \phi_i}{dy^2} \right) + \frac{1}{2} \left( GJ_0 \right) \frac{d\phi_i}{dy} \frac{d\phi_i}{dy} \right] dy 
= \begin{cases} 
\Lambda_j^2 & i = j \\
0 & i \neq j 
\end{cases} \tag{4.13}
\]

where \( \Lambda_j \) is the \( j \)th wing undamped natural frequency.

4.3 The Variational Functional Described in Modal Form

The simplified variational functional expressed in terms of natural frequencies is

\[
\Pi = -\frac{1}{2} \sum_{j=1}^N \left[ \sum_{i=1}^N \left( \frac{\dot{\phi}_j \dot{\phi}_i}{x_{ij}^2} \right) \right] + \frac{1}{2} \sum_{j=1}^N \left[ \sum_{i=1}^N \left( \frac{\dot{\phi}_j \dot{\phi}_i}{x_{ij}^2} \right) \right] 
+ \sum_{j=1}^N \left[ \frac{1}{2} \int_0^L \left( \sum_{i=1}^N \dot{\phi}_j \phi_i \dot{\phi}_i \right) \right] - \sum_i \left( \frac{d\phi_i}{dy} \right) \frac{d\phi_i}{dy} 
- \left\{ \sum_i \dot{\phi}_j \left( g_{ij} \sin \phi \right) \right\} \left\{ \sum_i \left( \dot{\phi}_i + K_\phi \dot{\phi}_i \right) \right\} 
+ \left\{ \sum_i \dot{\phi}_j \left( \dot{\phi}_i \right) \right\} \left\{ \sum_i \left( \dot{\phi}_i \right) \right\} 
+ \left\{ \sum_i \dot{\phi}_j \left( \dot{\phi}_i \right) \right\} \left\{ \sum_i \left( \dot{\phi}_i \right) \right\}
\]
\[ + r \left\{ \sum_{j} w_{j} (q_{mj} \sin q_{m}) \left( \sum \frac{d_{i}}{dy} \dot{a}_{i} \right) \right\}_{mm} \\
- r \left\{ \sum_{j} w_{j} (q_{mj} \cos q_{m}) \left( \sum \phi_{i} \dot{a}_{i} \right) \right\}_{mm} \\
- 2r \int \left( \sum_{j} w_{j} \delta_{nj} \right) \left( \sum \frac{d_{i}}{dy} \dot{a}_{i} \right) \cos q_{m} \, dr \right\}_{m} \\
- 2r \int \left( \sum_{j} w_{j} \delta_{nj} \right) \left( \sum \phi_{i} \dot{a}_{i} \right) \sin q_{m} \, dr \right\}_{m} \\
- \frac{1}{2} \sum_{l} \left( \frac{d_{l}}{dy} \dot{a}_{l}^{2} \right) + \frac{1}{2} \sum_{l} \ddot{a}_{l}^{2} + I_{B} \Omega \left( \sum_{l} \phi_{l} \dot{a}_{l} \right) \left( \sum \frac{d_{i}}{dy} \dot{a}_{i} \right) \\
+ \sum_{n=1}^{N} \left[ \int_{c}^{R} \left( \sum_{j} w_{j} \delta_{nj} \right) \, dr \right]_{n} \\
+ \sum_{n=1}^{N} \left[ \int_{c}^{R} \left( \sum_{j} \psi_{j} \delta_{nj} \right) \, dr \right]_{n} \\
+ W_{WB} + W_{WW} \]  

(4.14)

where \( W_{WB} \) denotes the work on the wing done by rotor aerodynamic forces and \( W_{WW} \) the work on the wing done by wing aerodynamic forces. Details are discussed in Subsection 4.5.

The principle of virtual work is applied to Eq. 4.14 to obtain the modal equation for the blade and wing.

For the blade the modal equation can be expressed as

\[ \ddot{\delta}_{nj} + \lambda_{j} \delta_{nj} + V_{ij} \ddot{v}_{r} - V_{ij} \left( \frac{R_{r}}{R} + \bar{R} \bar{v}_{p} \right) \sin q_{m} \]
\[ \begin{align*}
+ W_{ij} \frac{\ddot{e}}{R} - R \dot{V}_{ij} \dot{Y}_j \cos \psi_n + W_{ij} \dot{Y}_j \sin \psi_n \\
- W_{ij} \dot{V}_p \cos \psi_n + 2 \Omega W_{ij} \dot{Y}_j \cos \psi_n \\
+ 2 \Omega W_{ij} \dot{V}_p \sin \psi_n \\
= P_{ej} + P_{xj}
\end{align*} \] (4.15)

where

\[ \begin{align*}
W_{ij} &= R^{1-i} \left[ \int_0^R m r^i W_j \, dr \right] \\
&= R^{1-i} \left[ \int_0^R m r^i V_j \, dr \right] \\
&= R^{1-i} \left[ \int_0^R m r^i V_j \, dr \right] \\
P_{ej} &= \int_0^R P_e V_j \, dr \\
P_{xj} &= \int_0^R P_x W_j \, dr
\end{align*} \] (4.16) (4.17)

For the wing modal equation can be expressed as
\[ \dddot{a}_i + \Lambda_i^2 a_i + I_j N \Omega \left[ \sum_j \left( \frac{d^3}{dy} \Phi_j - \Phi_i \frac{d^3}{dy} \right) \dot{a}_i \right]_{\text{fr}} + \sum_1^n \sum_j \left[ -V_j \dddot{q}_{nj} \left( \frac{d^3}{dy} \right)_{\text{fr}} - V_j \left( \delta_{nj} \sin \Phi_n \right) \left( \frac{d^3}{dy} \right)_{\text{fr}} \right]_{\text{fr}} + W_{nj} \dddot{q}_{nj} \left( \frac{d^3}{dy} \right)_{\text{fr}} - W_{nj} \left( \delta_{nj} \cos \Phi_n \right) \left( \frac{d^3}{dy} \right)_{\text{fr}} \right]_{\text{fr}} = f_i + g_i \] (4.18)

where \( f_i \) denotes the loading due to the rotor aerodynamic forces derived from the work \( W_{WB} \), and \( g_i \) the loading due to wing aerodynamic forces derived from the work \( W_{WB} \).

### 4.4 Rotor Non-Rotating Coordinate System

#### 4.4.1 Rotor Non-Rotating Coordinate System

The equations of motion 4.15 and 4.18 of the blade and wing have periodic coefficients because of the appearance of \( \sin \psi_n \) and \( \cos \psi_n \). To avoid difficulties due to these periodic coefficients, the Fourier-type coordinate system will be chosen to describe the motion in the non-rotating system instead of the rotating system (Ref. 16).

(a) For the hingeless rotor, the Fourier coordinate system is described in general as

\[ q_{nj} = Q_j o + \sum_m \left[ Q_{j(m)} \cos (m \Phi_n) + Q_{j(m)} \sin (m \Phi_n) \right] + Q_{j(n)} (1)^n \] (4.19)
where $Q_{j0}$ is the collective motion, $Q_{jc(m)}$ and $Q_{js(m)}$ are cyclic motion of $m$th order, defining tilting or warping of the rotor plane and $Q_{js}$ is differential collective motion (only occurring for even $N$-bladed rotors).

Since many recent proprotor designs have three-bladed rotors, this study is concentrated on the odd $N$-bladed rotor model. Therefore, the last term $Q_{js}$ of Eq. 4.19, the differential collective motion, will be dropped hereafter. In addition to this, the higher order terms $Q_{jc(2)}$, $Q_{js(2)}$, ..., $Q_{jc(m)}$, $Q_{js(m)}$ are not coupled with pylon motion. They represent internal rotor motion and usually make only a small contribution to rotor motion. Therefore they are also truncated in this study. Hence,

$$g_{ij} = Q_{j0} + Q_{jc} \cos q_n + Q_{js} \sin q_n$$  \hspace{1cm} (4.20)

By inverting the system of Eq. 4.20, the new degrees of freedom of the non-rotating system can be obtained:

$$Q_{j0} = \frac{1}{N} \sum_{k=1}^{N} g_{kj}$$

$$Q_{jc} = \frac{2}{N} \sum_{k=1}^{N} g_{kj} \cos q_n$$

$$Q_{js} = \frac{2}{N} \sum_{k=1}^{N} g_{kj} \sin q_n$$  \hspace{1cm} (4.21)

The same approach may be applied for the rotor generalized force due to periodic aerodynamic forces.

$$(P_{e0} + P_{eij})_0 = \frac{1}{N} \sum_{k=1}^{N} (P_{e0} + P_{eij})$$

$$(P_{e0} + P_{eij})_c = \frac{2}{N} \sum_{k=1}^{N} (P_{e0} + P_{eij}) \cos q_n$$

$$(P_{e0} + P_{eij})_s = \frac{2}{N} \sum_{k=1}^{N} (P_{e0} + P_{eij}) \sin q_n$$  \hspace{1cm} (4.22)
For the gimballed rotor, special care must be taken since it has both collective mode shapes (symmetrical modes with clamped boundary condition at the root) and cyclic mode shapes (antisymmetrical mode shapes with hinged boundary condition), while in the hingeless rotor case both collective mode shapes and cyclic mode shapes are the same. Therefore, collective motion is described with collective mode shapes $W_j^0, V_j^0$ and cyclic motion with cyclic mode shapes $W_j^c, V_j^c$.

$$\begin{bmatrix} W_n \\ V_n \end{bmatrix} = \sum_j \left[ \begin{bmatrix} W_j^0 \\ V_j^0 \end{bmatrix} Q_{j0} + \begin{bmatrix} W_j^c \\ V_j^c \end{bmatrix} (Q_{jc} \cos \phi_n + Q_{jc} \sin \phi_n) \right]$$

This expression is consistent with the derivation of equations of motion up to now, if mode shapes $W_j, V_j$ of the coefficients of the blade collective motion are replaced by the collective mode shapes $W_j^0, V_j^0$ in Eq. 4.15 and 4.18, and those of cyclic motion are replaced by the cyclic mode shapes $W_j^c, V_j^c$. Therefore, the derivation of equations of motion hereafter is based on the hingeless rotor case, and those equations can be applied to the gimballed rotor case as stated before without confusion. The coefficients associated with collective mode shapes are denoted by superscript 0 and those with cyclic mode shapes by superscript c.

The truncation of higher order terms and of the differential collective motion in Eq. 4.19 is also applied to the gimballed rotor case by the same reasoning as for the hingeless rotor case.

The orthogonality condition for the gimballed rotor case is different from the hingeless orthogonality condition. The entire rotor system including periodic functions $\sin \psi_n$ and $\cos \psi_n$ must be considered. For example, the orthogonality condition between the collective mode and the cyclic sine mode is described as
Since $\psi_n$ is the phase angle defined in Eq. 2.3, the summation over $n=1, \ldots, N$ of $\sin \psi_n$ becomes zero.

4.4.2 Pitch-Flap Coupling in the Non-Rotating Coordinate System

Pitch-Flap Coupling is expressed in Eq. 3.25 as

$$\Theta_{PF} = -\frac{\partial \omega}{\partial v} \bigg|_{r=x} \tan \delta_3$$

(3.25)

The substitution of mode shapes into this expression gives

$$\Theta_{PF} = \sum_i \left[ -\frac{dW_i}{dv} \bigg|_{r=x} \tan \delta_3 \{Q_{i_0} + Q_{i_c} \cos \psi_n + Q_{i_s} \sin \psi_n \} \right]$$

$$= \sum_i K_{pi} \{Q_{i_0} + Q_{i_c} \cos \psi_n + Q_{i_s} \sin \psi_n \}$$

(4.25)

where

$$K_{pi} = -\frac{dW_i}{dv} \bigg|_{r=x} \tan \delta_3$$

(4.26)
For the hingeless rotor case considered here, $K_{pi}$ is always zero because the slopes of blade mode shapes at the root are zero, due to the clamped boundary condition.

For the gimbaled rotor case, Eq. 4.25 is rewritten as

$$\theta_{PF} = \sum_i \left[ K_{pi}^o Q_{i0} + K_{pi}^c (Q_{c} \cos \phi_m + Q_{i3} \sin \phi_m) \right]$$

(4.27)

where

$$K_{pi}^o = -\frac{dW_i^o}{dr} \bigg|_{r=0} \tan \delta_3$$

$$K_{pi}^c = -\frac{dW_i^c}{dr} \bigg|_{r=0} \tan \delta_3$$

(4.28)

From the definition of the collective mode shapes, $K_{pi}^o$ is zero. Therefore, for the gimbaled rotor case the pitch-flap coupling contributes only to the cyclic motion of the rotor, not to the collective motion.

4.5 Aerodynamic Forces Described in Modal Form

4.5.1 Blade Aerodynamic Forces

From Eq. 4.17 blade aerodynamic force is described as

$$P_{ej} + P_{zj} = \int_0^s (P_{e} V_j + P_{z} W_j) \, dr$$

(4.29)

The application of Eq. 4.22, accompanied by the substitution Eq. 4.21 into Eq. 3.13 yields
\[
(P_{ej} + P_{ej})_o = \frac{1}{2} r I_o \Omega^2 \left[ \sum_i G_{ji} \dot{Q}_{io} + G_{ji} \ddot{Q} \right.
+ G_{ji} \frac{\dot{\theta}_i}{R} + \lambda G_{ji} \frac{\dot{\theta}_i}{V} + G_{ij} \sum_i K_{pi} Q_{io} \\
+ G_{ij} \theta_o \right]
\]
(4.30)

\[
(P_{ej} + P_{ej})_c = \frac{1}{2} r I_o \Omega^2 \left[ \sum_i G_{ji} \dot{Q}_{ic} + \Omega \sum_i G_{ji} (Q_{is} - \ddot{Q}) \\
- \bar{K} G_{ij} \ddot{\gamma} + \lambda G_{ij} \gamma - G_{ij} \dot{\theta}_i \\
+ \lambda G_{ij} \frac{\dot{\theta}_i}{V} + G_{ij} \sum_i K_{pi} Q_{ic} \\
+ G_{ij} \theta_c \right]
\]
(4.31)

\[
(P_{ej} + P_{ej})_s = \frac{1}{2} r I_s \Omega^2 \left[ \sum_i G_{ji} \dot{Q}_{is} \\
- \Omega \sum_i G_{ji} Q_{ic} - G_{ij} \frac{\dot{\theta}_i}{R} - \bar{K} G_{ij} \dot{\theta}_i \\
+ \lambda G_{ij} \beta + G_{ij} \dot{\gamma} \\
+ \lambda G_{ij} \frac{\dot{\theta}_i}{V} + G_{ij} \sum_i K_{pi} Q_{is} \\
+ G_{ij} \theta_{is} \right]
\]
(4.32)
where

\[ G_{ji} = \int_{0}^{1} \left( F_{01} \frac{V_{i} V_{i}}{R^2} + F_{02} \frac{W_{i} V_{i}}{R^2} + F_{02} \frac{V_{i} W_{i}}{R^2} + F_{x2} \frac{W_{i} W_{i}}{R^2} \right) \, dz \]

\[ G_{Ij} = \int_{0}^{1} \left( F_{01} \frac{V_{i}}{R} + F_{x1} \frac{W_{i}}{R} \right) \, dx \]

\[ G_{IIj} = \int_{0}^{1} \left( F_{01} \frac{V_{i}}{R} + F_{x1} \frac{W_{i}}{R} \right) \, dx \]

\[ G_{IIIj} = \int_{0}^{1} \left( F_{02} \frac{V_{i}}{R} + F_{x2} \frac{W_{i}}{R} \right) \, dx \]

\[ G_{IVj} = \int_{0}^{1} \left( F_{02} \frac{V_{i}}{R} + F_{x2} \frac{W_{i}}{R} \right) \, dx \]

\[ G_{Vj} = \int_{0}^{1} \left( F_{03} \frac{V_{i}}{R} + F_{x3} \frac{W_{i}}{R} \right) \, dx \]

(4.33)

\[ F_{00} = \frac{P_{00}}{\left[ \frac{1}{2} \chi I_{0} \left( \frac{\nu}{R^2} \right)^3 \right]} \]

\[ F_{01} = \frac{\delta P_{0}}{\partial U_{10}} \left[ \frac{1}{2} \chi I_{0} \left( \frac{\nu}{R^2} \right)^3 \right] \]

\[ F_{02} = \frac{\delta P_{0}}{\partial U_{p0}} \left[ \frac{1}{2} \chi I_{0} \left( \frac{\nu}{R^2} \right)^3 \right] \]

\[ F_{03} = \frac{\delta P_{0}}{\partial \theta_{0}} \left[ \frac{1}{2} \chi I_{0} \left( \frac{\nu}{R^2} \right)^3 \right] \]

\[ F_{x0} = \frac{P_{x0}}{\left[ \frac{1}{2} \chi I_{0} \left( \frac{\nu}{R^2} \right)^3 \right]} \]

\[ F_{x1} = \frac{\delta P_{x}}{\partial U_{10}} \left[ \frac{1}{2} \chi I_{0} \left( \frac{\nu}{R^2} \right)^3 \right] \]

\[ F_{x2} = \frac{\delta P_{x}}{\partial U_{p0}} \left[ \frac{1}{2} \chi I_{0} \left( \frac{\nu}{R^2} \right)^3 \right] \]

\[ F_{x3} = \frac{\delta P_{x}}{\partial \theta_{0}} \left[ \frac{1}{2} \chi I_{0} \left( \frac{\nu}{R^2} \right)^3 \right] \]

(4.34)
4.5.2 Wing-Loading Due to Blade Aerodynamic Forces

The virtual work done by wing loading due to blade aerodynamic forces is described as

\[
\delta W_{\text{wing}} = \begin{bmatrix} \delta r_z & \delta r_x & \delta v_y & \delta v_r & \delta v_p & \delta v_r \end{bmatrix}^T \mathbf{x}
\]

\[
= \begin{bmatrix} -P_0 \sin \theta_m + \nu_r \rho_b \\
\nu_p \rho_o \sin \theta_m + \nu_y \rho_o \cos \theta_m + \rho_b \\
-\rho_r (\rho_o \cos \theta_m - \nu_y \rho_b) + \nu_r \rho_o + \rho_b \nu_o \sin \theta_m + \nu_r \rho_o \cos \theta_m - \rho_b \nu_o \cos \theta_m \\
-\rho_r (\rho_o \sin \theta_m + \nu_y \rho_b) - \rho_o \nu_o \sin \theta_m + \nu_r \rho_o \sin \theta_m - \rho_b \nu_o \cos \theta_m \\
\nu_r \rho_o \end{bmatrix}
\]

(4.35)

The application of the blade mode shapes and non-rotating coordinate system gives

\[
\delta W_{\text{wing}} = \Omega^T \begin{bmatrix} \delta r_x/R \\ \delta r_y/R \\ \delta v_x \\ \delta v_y \\ \delta v_p \\ \delta v_r \end{bmatrix} \begin{bmatrix} \kappa \left[ A_{\omega}^0 \right] \left\{ r_{\omega}/R \right\} \\
\kappa \left[ A_{\omega}^1 \right] \left\{ r_{\omega}/R \right\} \\
\kappa \left[ A_{\phi} \right] \left\{ \phi_o \right\} \\
\kappa \left[ A_{\psi} \right] \left\{ \psi_o \right\} \\
\kappa \left[ A_{\omega} \right] \left\{ \theta_{i}\right\} \end{bmatrix} + \sum_i \left[ A_{\omega i} \right] \left\{ Q_i \right\} + \sum_i \left[ A_{\phi i} \right] \left\{ Q_i \right\}
\]

\[
= \Omega^T \begin{bmatrix} \delta r_x/R \\ \delta r_y/R \\ \delta v_x \\ \delta v_y \\ \delta v_p \\ \delta v_r \end{bmatrix} \begin{bmatrix} w_x/\nu \\
w_y/\nu \\
\omega/\nu \\
\phi_o \\
\psi_o \\
\theta_{i}\end{bmatrix} + \kappa \left[ A_{\omega} \right] \left\{ \theta_{i}\right\}
\]

(4.36)
where $\kappa$ is an integral operator defined as

$$\kappa = \frac{N}{2} \epsilon I_0 \int_0^1 d\chi \tag{4.37}$$

and integrand matrices are

$$[A_{\alpha}] = \begin{bmatrix}
\frac{1}{2}F_{\alpha_1} & 0 & -\frac{1}{2}F_{\alpha_2} \\
0 & F_{\alpha_3} & 0 \\
-\frac{1}{2}xF_{\alpha_1} & 0 & \frac{1}{2}(x^2F_{\alpha_1} + x^2F_{\alpha_3}) \\
\frac{1}{2}xF_{\alpha_1} & 0 & \frac{1}{2}x(-F_{\alpha_2} + F_{\alpha_1}) \\
0 & xF_{\alpha_3} & 0
\end{bmatrix}$$

$$[A_{\beta}] = \begin{bmatrix}
\frac{1}{2}x^2F_{\alpha_1} & 0 \\
0 & x^2F_{\alpha_1} \\
\frac{1}{2}x^2F_{\alpha_2} & 0 \\
0 & x^2F_{\alpha_1}
\end{bmatrix} \tag{4.38a}$$

$$[A_{\gamma}] = \begin{bmatrix}
0 & 0 & 0 & -\frac{\lambda}{2}F_{\alpha_1} + F_{\alpha_0} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\lambda}{2}(F_{\alpha_0} - \frac{\lambda}{2}F_{\alpha_1}) & \frac{\lambda}{2}xF_{\alpha_1} + xF_{\alpha_0} & 0 \\
0 & 0 & -\frac{\lambda}{2}xF_{\alpha_1} & \frac{\lambda}{2}(F_{\alpha_0} - \frac{\lambda}{2}F_{\alpha_1}) & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{4.38b}$$
\[
[A^\alpha_{\omega}] = \begin{bmatrix}
0 & 0 & -\frac{1}{2}(\frac{V_i}{R} F_{e1} + \frac{W_i}{R} F_{e2}) \\
-\frac{1}{2} \left( V_i F_{e1} + \frac{W_i}{R} F_{e2} \right) & 0 & -\frac{3}{2}(\frac{V_i}{R} F_{e1} + \frac{W_i}{R} F_{e2}) \\
-\frac{3}{2}(\frac{V_i}{R} F_{e1} + \frac{W_i}{R} F_{e2}) & -\frac{1}{2} \left( V_i F_{e1} + \frac{W_i}{R} F_{e2} \right) & 0 \\
\alpha(\frac{V_i}{R} F_{e1} + \frac{W_i}{R} F_{e2}) & 0 & 0 \\
\end{bmatrix}
\]

(4.38c)

\[
[A^\omega_{\phi}] = \begin{bmatrix}
0 & \frac{K_p}{2} (\frac{V_i}{R} F_{e1} + \frac{W_i}{R} F_{e2}) & \frac{1}{2} K_p \cdot F_{e3} \\
\frac{1}{2} \left[ (\frac{V_i}{R} F_{e1} - \frac{W_i}{R} F_{e2}) \right] & 0 & 0 \\
0 & -\frac{1}{2} K_p \cdot x F_{e3} & \frac{K_p}{2} \cdot x F_{e3} \\
K_p \cdot x F_{e3} & 0 & 0 \\
\end{bmatrix}
\]

(4.38d)

\[
[A^\phi_{\omega}] = \begin{bmatrix}
\frac{K_p}{2} F_{e1} & 0 & 0 & 0 & 0 & \frac{1}{2} F_{e3} \\
0 & \frac{K_p}{2} F_{e3} & F_{e3} & 0 & 0 & 0 \\
-\frac{1}{2} x F_{e1} & -\frac{1}{2} x F_{e1} & 0 & 0 & \frac{1}{2} F_{e3} & \frac{1}{2} x F_{e3} \\
\frac{1}{2} x F_{e1} & -\frac{1}{2} x F_{e1} & 0 & 0 & \frac{1}{2} F_{e3} & -\frac{1}{2} x F_{e3} \\
p_{\omega} & 0 & 0 & \alpha x F_{e3} & 2 F_{e3} & 0 \\
\end{bmatrix}
\]

(4.38e)
4.5.3 Wing Aerodynamic Forces

The virtual work done by wing aerodynamic forces is expressed by

\[
\mathbf{SW}_{ww} = \{ \mathbf{S} \mathbf{u}_w, \mathbf{S} \mathbf{w}_w, \mathbf{S} \mathbf{p}_w \} \begin{bmatrix} F_x \\ F_y \\ M_y \end{bmatrix}
\]  (4.39)

The displacements \( u_w, w_w, p_w \) are described in terms of the mode shapes as

\[
\begin{bmatrix} u_w \\ w_w \\ p_w \end{bmatrix} = [S] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}
\]  (4.40)

where

\[
[S] = \begin{bmatrix}
\ddot{z}_1(y) & \ddot{z}_2(y) & \ldots & \ddot{z}_n(y) \\
\ddot{\xi}_1(y) & \ddot{\xi}_2(y) & \ddots & \ddot{\xi}_n(y) \\
\ddot{\phi}_1(y) & \ddot{\phi}_2(y) & \ldots & \ddot{\phi}_n(y)
\end{bmatrix}
\]  (4.41)

Then, the substitution of Eq. 4.40 into 4.39 gives

\[
\mathbf{SW}_{ww} = \Omega^2 \{ \mathbf{S} \mathbf{a}_1, \mathbf{S} \mathbf{a}_2, \ldots, \mathbf{S} \mathbf{a}_n \} \begin{bmatrix}
\mathbf{A}_{\delta}^{(s)} & 0 \\
0 & \mathbf{A}_{\delta}^{(s)} \\
0 & 0
\end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}
\]  (4.42)
where

\[
\begin{align*}
\left[ A_{\alpha}^{(a)} \right] &= \frac{1}{|\Omega|} \left[ S \right]^T \left[ A_{\alpha} \right] \left[ S \right] dy \\
\left[ A_{\alpha}^{(b)} \right] &= \frac{1}{|\Omega|} \left[ S \right]^T \left[ A_{\alpha} \right] \left[ S \right] dy \\
\left[ A_{\alpha}^{(c)} \right] &= \frac{1}{|\Omega|} \left[ S \right]^T \left[ A_{\alpha} \right] dy 
\end{align*}
\] (4.43)

The matrices \( A_w, A_w \) and \( A_G \) are derived from Eq. 3.32 as follows:

\[
\begin{align*}
\left[ A_w \right] &= \begin{bmatrix}
-\frac{1}{2} \rho C_w V \alpha_w (1 + \frac{C_w}{\rho}) & \rho C_w V \alpha_w \alpha_w & 0 \\
-\frac{1}{2} \rho C_w V \alpha_w \alpha_w & -\rho C_w V C_w & 0 \\
-\frac{1}{2} \rho C_w^2 V (C_w + \alpha_w \bar{c}) & \rho C_w^2 V (C_w + \alpha_w \bar{c}) & 0
\end{bmatrix} \\
\left[ A_w \right] &= \begin{bmatrix}
0 & 0 & \frac{1}{2} \rho C_w V^2 \alpha_w \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{2} \rho C_w^2 V^2 (C_w + \alpha_w \bar{c})
\end{bmatrix} \\
\end{align*}
\] (4.44a) (4.44b)
4.6 Final Equations of Motion

In this part the final equations of motion will be derived. They will be expressed in general format, so that any number of mode shapes both for the blade and wing can be chosen to describe the rotor or wing motion as accurately as required.

4.6.1 Equations of Motion in General Format

The substitution of Eqs. 4.20 and Eq. 4.10 into Eq. 4.15 gives the \( j \)th blade modal equations of motion. Note that time derivatives are normalized by rotational speed as in Eq. 3.24.

For the \( j \)th blade mode

\[
\{\ddot{Q}_j\} + \bar{\alpha} [\mathbf{M}_b]\{Q_j\} + \left[\mathbf{M}_{o,j}\right]\{Q_j\} \\
+ \left[\mathbf{M}_j[\mathbf{T}]\hat{a}\right] + \bar{\alpha}\left[\mathbf{C}_j[\mathbf{T}]\hat{a}\right] = \{\mathbf{P}_j\}
\]  

(4.45)
where

\[
\{ Q_j \} = \begin{pmatrix} Q_{j_o} \\ Q_{j_c} \\ Q_{j_s} \end{pmatrix}, \quad \{ a \} = \begin{pmatrix} a_o \\ a_c \\ \vdots \end{pmatrix}
\]

(4.46a)

\[
[M_0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}
\]

(4.46b)

\[
[M_{\omega}] = \begin{bmatrix} \left( \frac{\lambda_j}{\Omega} \right)^2 & 0 & 0 \\ 0 & \left( \frac{\lambda_j}{\Omega} \right)^2 - 1 & 0 \\ 0 & 0 & \left( \frac{\lambda_j}{\Omega} \right)^2 - 1 \end{bmatrix}
\]

(4.46c)

\[
[M_j] = \begin{bmatrix} 0 & W_{oj} & 0 & 0 & V_j \\ 0 & 0 & -k V_{oj} & -W_j & 0 \\ -V_{oj} & 0 & W_j & -k V_{oj} & 0 \end{bmatrix}
\]

(4.46d)

\[
[C_j] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2W_j & 0 & 0 \\ 0 & 0 & 0 & 2W_j & 0 \end{bmatrix}
\]

(4.46e)

\[
\{ P_j \} = \frac{1}{\Omega^4} \begin{pmatrix} (P_{o,j} + P_{c,j})_o \\ (P_{o,j} + P_{c,j})_c \\ (P_{o,j} + P_{c,j})_s \end{pmatrix}
\]

(4.47)

and the matrix \([T]\) is defined in Eq. 4.11.
Substituting Eqs. 4.21 and 4.10 into Eq. 4.18 yields the wing equation.

For the wing mode

\[
\begin{align*}
\{\ddot{a}\} + \bar{\Omega} [W_a] \{a\} + [W_a] \{a\} \\
+ \sum_j \left([T]^T[M_j]^T[K]\right) \{Q_j\} \\\n- \tilde{\Omega} [T]^T[C_j]^T[K]\{Q_j\} = \{F\}
\end{align*}
\]  

(4.48)

where

\[
[W_a] = N I_{I_a} \begin{bmatrix}
\frac{\partial x}{\partial Y} \\
\frac{\partial y}{\partial Y} \\
\vdots
\end{bmatrix} \begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots
\end{bmatrix}_{y=L} - \begin{bmatrix}
\frac{\partial x}{\partial Y} \\
\frac{\partial y}{\partial Y} \\
\vdots
\end{bmatrix}_{y=L}
\]

\[
[W_a] = \begin{bmatrix}
(N_\alpha \rho)^2 & 0 \\
0 & (N_\alpha \rho)^2 \\
0 & 0
\end{bmatrix} : \text{diagonal matrix}
\]

(4.49)

\[
[K] = \begin{bmatrix}
N & 0 & 0 \\
0 & N_2 & 0 \\
0 & 0 & N_2
\end{bmatrix}
\]
Loading matrix \( \{F\} \) for the wing consists of aerodynamic forces \( f_i \) due to the rotor, and aerodynamic forces \( g_i \) due to wing motion itself.

\[
\{ F \} = \begin{bmatrix}
  f_1 + g_1 \\
  f_2 + g_2 \\
  \vdots \\
  \vdots 
\end{bmatrix}
\]

(4.50)

Similarly, aerodynamic forces are derived.

For rotor aerodynamic forces

\[
\{ P_j \} = \frac{1}{\Omega^2} \begin{bmatrix}
  (P_{0j} + P_{2j})_o \\
  (P_{0j} + P_{2j})_c \\
  (P_{0j} + P_{2j})_s 
\end{bmatrix}
\]

\[
= \sum_i [A_{\alpha i}]_j \{ Q_i \} + \sum_i [A_{\Omega i}]_j \{ Q_i \} \\
+ [A_{\dot{\alpha}}]_j \{ \dot{a} \} + [A_{\Omega}]_j \{ \Omega \} \\
+ [A_{\varepsilon}]_j \{ \varepsilon \}
\]

(4.51)

where
\[
\begin{align*}
\left[ A_{\omega}^0 \right]_{ij} &= \frac{1}{2} \gamma I_\omega \\
&= \begin{bmatrix}
G_{ji} & 0 & 0 \\
0 & G_{ji} & 0 \\
0 & 0 & G_{ji}
\end{bmatrix} \\
\left[ A_{\phi}^0 \right]_{ij} &= \frac{1}{2} \gamma I_\omega \\
&= \begin{bmatrix}
K_{pi} & G_{ij} & 0 & 0 \\
0 & K_{pi} & G_{ij} & \tilde{\Omega} G_{ji} \\
0 & -\tilde{\Omega} G_{ji} & K_{pi} & G_{ij}
\end{bmatrix} \\
\left[ A_{\alpha}^0 \right]_{ij} &= \frac{1}{2} \gamma I_\omega \\
&= \begin{bmatrix}
0 & G_{ij} & 0 & 0 & G_{ij} \\
0 & 0 & -\tilde{\kappa} G_{ij} & -G_{ij} & 0 \\
-\tilde{G}_{ij} & 0 & G_{ij} & -\tilde{\kappa} G_{ij} & 0
\end{bmatrix} \\
\left[ A_{\xi}^0 \right]_{ij} &= \frac{1}{2} \gamma I_\omega \\
&= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda G_{ij} & 0 & 0 \\
0 & 0 & 0 & \lambda G_{ij} & 0
\end{bmatrix}
\end{align*}
\]
The excitation vector \( \{ e \} \) is expressed as

\[
[A_0]_j = \frac{1}{2} \gamma R_j \begin{bmatrix}
0 & 0 & \lambda G_{ij} & G_{ij} & 0 & 0 \\
0 & -\lambda G_{ij} & 0 & 0 & G_{ij} & 0 \\
-\lambda G_{ij} & 0 & 0 & 0 & 0 & G_{ij}
\end{bmatrix}
\] (4.52e)

For wing aerodynamic forces

From Eq. 4.50

\[
\{ f \} = \{ f + g \}
\] (4.54)

Loading to the wing due to rotor aerodynamic forces \( f_i \) corresponding to \( a_i \) is expressed from Eq. 4.36 as
\[
\begin{align*}
\{f\} &= \kappa [T]^T [A_a] [\alpha] \{a\} + \kappa [T]^T [A_\phi] [\alpha] \{a\} + \sum_i \kappa [T]^T [A_{\phi_i}] \{Q_i\} \\
&+ \sum_i \kappa [T]^T [A_{\phi_s}] \{Q_i\} + \kappa [T]^T [A_e] \{e\}
\end{align*}
\]

(4.55)

Loading due to wing aerodynamic forces \(q_1\) is described from Eq. 4.42 as
\[
\{g\} = [A_a] \{a\} + [A_\alpha] \{a\} + [A_e] \{e\}
\]

(4.56)

4.6.2 Equations of Motion for Nine Degrees of Freedom

For a typical case, nine degrees of freedom are considered here. These degrees are rotor flapping motion (collective flapping motion \(Q_{10}\), cyclic flapping motion \(Q_{1c}\) and \(Q_{1s}\)), rotor lagging motion (collective lagging motion \(Q_{20}\), cyclic lagging motion \(Q_{2c}\) and \(Q_{2s}\)), wing vertical bending motion \(a_1\), wing chordwise bending motion \(a_2\) and wing torsion \(a_3\).

These mode shapes are natural modes, therefore they are coupled.

The final equations in matrix form are
\[
[A] \{x\} + [B] \{x\} + [C] \{x\} = [D] \{e\}
\]

(4.57)
where

$$\{x\} = \begin{pmatrix} Q_{10} \\ Q_{1c} \\ Q_{1s} \\ Q_{20} \\ Q_{2c} \\ Q_{2s} \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \{e\} = \begin{pmatrix} \frac{\psi}{\sqrt{v}} \\ \frac{\psi}{\sqrt{v}} \\ \frac{\psi}{\sqrt{v}} \\ \theta_0 \\ \theta_c \\ \theta_s \end{pmatrix} \quad (4.58)$$

The coefficient matrices $[A]$, $[B]$ and $[C]$ are $9 \times 9$ matrices, and $[D]$ is a $9 \times 6$ matrix, as given below.

$$[A] = \begin{bmatrix} [I] & 0 & [M_1][T] \\ 0 & [I] & [M_2][T] \\ [T^T][M_1][K] & [T^T][M_2][K] & [I] \end{bmatrix} \quad (4.59a)$$
\[ [B] = \begin{bmatrix}
\tilde{\Omega}[^{M\alpha}] & \tilde{\Omega}[^{A\alpha}] & \tilde{\Omega}[^{C\beta}][T] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{M\alpha}] & \tilde{\Omega}[^{C\beta}][T] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{M\alpha}] & \tilde{\Omega}[^{C\beta}][T] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{M\alpha}] & \tilde{\Omega}[^{C\beta}][T] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{M\alpha}] & \tilde{\Omega}[^{C\beta}][T] & \tilde{\Omega}[^{A\alpha}][T]
\end{bmatrix}
\]

(4.59b)

\[ [C] = \begin{bmatrix}
[^{M\alpha}] & \tilde{\Omega}[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}][T] \\
-[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}] & \tilde{\Omega}[^{A\alpha}][T]
\end{bmatrix}
\]

(4.59c)
In order to have a better understanding of tilting prop-rotor dynamics, a simple special case was considered. This case is based on the assumption that the mode shapes are uncoupled. Therefore, rotor motion is expressed as

\[
\begin{bmatrix}
  w_n^r \\
  v_n^r 
\end{bmatrix} = \begin{bmatrix} w_i \\ 0 \end{bmatrix} (Q_{10} + Q_{1c} \cos \phi_n + Q_{1s} \sin \phi_n) + \begin{bmatrix} 0 \\ v_2 \end{bmatrix} (Q_{20} + Q_{2c} \cos \phi_n + Q_{2s} \sin \phi_n) \tag{4.60}
\]

The wing motion is also,\n
\[
\begin{bmatrix}
  u_w \\
  w_w \\
  f_w 
\end{bmatrix} = \begin{bmatrix} \gamma_i \\ 0 \\ 0 \end{bmatrix} a_1 + \begin{bmatrix} 0 \\ \delta_2 \\ 0 \end{bmatrix} a_2 + \begin{bmatrix} 0 \\ 0 \\ \phi_3 \end{bmatrix} a_3 \tag{4.61}
\]

The substitution of Eq. 4.60 and 4.61 into Eq. 4.59 gives the following matrices for the same nine degrees of freedom
equation as Eq. 4.57.

\[ [A] = \]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & W_0 \frac{W_0}{R} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -W_{i1} \Phi \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & W_{ii} W_c' & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -V_{12} W_0' & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -V_{21} W_0' & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{W_{i1}}{R} W_0' & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{W_{i1}}{R} W_0' & 0 \\
0 & 0 & 0 & \frac{N W_{01}}{R} & \frac{N W_{01}}{R} & 0 & 0 & \frac{N W_{01} W_c}{R} & 0 \\
0 & 0 & -\frac{N W_{01} W_c}{R} & 0 & 0 & 0 & 0 & -\frac{N W_{01} W_c}{R} & 0 \\
0 & -\frac{N W_{01} W_c}{R} & 0 & 0 & 0 & 0 & 0 & -\frac{N W_{01} W_c}{R} & 0 \\
\end{bmatrix}
\]

(4.62a)
\[ [B] = \]

\[
\begin{bmatrix}
F_{\phi \phi} & 0 & 0 & F_{3 \phi} & 0 & 0 & -F_{\phi \phi}^* W_0 + F_{\phi \phi}^* W_0 & 0 \\
0 & F_{\phi \phi} & 2 \Omega & 0 & F_{3 \phi} & 0 & 0 & 2 \Omega W_0 W_0' - F_{\phi \phi}^* W_0 \\
0 & -2 \Omega & F_{\phi \phi} & 0 & 0 & F_{3 \phi} & -F_{\phi \phi}^* W_0 & -F_{\phi \phi}^* W_0 \\
L_{\phi \phi} & 0 & 0 & L_{33} & 0 & 0 & -L_{\phi \phi} W_0^* & L_{\phi \phi}^* W_0^* \\
0 & L_{\phi \phi} & 0 & 0 & L_{33} & 2 \Omega & 0 & -\tilde{L}_{\phi \phi}^* W_0^* - L_{\phi \phi}^* W_0^* \\
0 & 0 & L_{\phi \phi} & 0 & -2 \Omega & L_{33} & -L_{\phi \phi} W_0^* & L_{\phi \phi}^* W_0^* \\
-N L_{\phi \phi} W_0^* & 0 & -N L_{\phi \phi}^* W_0 & -N L_{33}^* W_0 & 0 & -N L_{\phi \phi} W_0^* & C_{\alpha \alpha} + C_{\alpha \alpha} & C_{\alpha \alpha} + C_{\alpha \alpha} \\
-N F_{\phi \phi} W_0^* & -N F_{33} W_0^* & N F_{33}^* W_0^* & N F_{\phi \phi}^* W_0^* & 0 & -N F_{\phi \phi} W_0^* & C_{\alpha \alpha} + C_{\alpha \alpha} & C_{\alpha \alpha} + C_{\alpha \alpha} + \tilde{W}_{\alpha \alpha} W_0^* \\
\end{bmatrix}
\]

\[(4.62b)\]
\[ [c] = \]

\[
\begin{bmatrix}
\frac{\lambda^2}{\ell^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\lambda^2}{\ell^2} - 1 + k_F L \delta & \bar{\Omega} F_{\delta \delta} & 0 & 0 & \bar{\Omega} F_{\beta \beta} & 0 & \lambda F_{\beta \bar{\delta}} W_{\bar{\delta}} & 0 \\
0 & -\bar{\Omega} F_{\beta \beta} & \frac{\lambda^2}{\ell^2} - 1 + k_F L \delta & 0 & -\bar{\Omega} F_{\delta \delta} & 0 & 0 & 0 & \lambda F_{\beta \bar{\delta}} W_{\bar{\delta}} \\
0 & 0 & 0 & \frac{\lambda^2}{\ell^2} & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\Omega} L \delta & \bar{\Omega} L \bar{\delta} & 0 & \frac{\lambda^2}{\ell^2} - 1 & \bar{\Omega} L \delta & 0 & \lambda L \delta W_{\delta} & 0 \\
0 & -\bar{\Omega} L \bar{\delta} & \bar{\Omega} L \delta & 0 & -\bar{\Omega} L \delta & \frac{\lambda^2}{\ell^2} - 1 & 0 & 0 & \lambda L \delta W_{\delta} \\
0 & \frac{1}{2} N L \delta + \frac{1}{2} D L_\beta \delta & \frac{1}{2} N L \bar{\delta} & 0 & \frac{1}{2} N L \alpha \delta & \frac{1}{2} N L \alpha \bar{\delta} & 0 & \frac{(\Lambda)^2}{\ell^2} & 0 & \frac{(\Lambda)^2}{\ell^2} \\
0 & \frac{1}{2} L \delta + \frac{1}{2} L_\beta \delta & \frac{1}{2} L \bar{\delta} & 0 & \frac{1}{2} L \alpha \delta & \frac{1}{2} L \alpha \bar{\delta} & 0 & (\Lambda)^2 + C_{\alpha \alpha} & C_{\alpha \beta} & C_{\alpha \bar{\beta}} \\
0 & \frac{1}{2} L_\delta + \frac{1}{2} L_\beta \delta & \frac{1}{2} L_\delta + \frac{1}{2} L_\bar{\delta} & 0 & \frac{1}{2} L_\alpha \delta & \frac{1}{2} L_\alpha \bar{\delta} & 0 & (\Lambda)^2 + C_{\bar{\alpha} \alpha} & C_{\bar{\alpha} \beta} & C_{\bar{\alpha} \bar{\beta}} \\
\end{bmatrix}
\]

(4.62c)
\[
[D] =
\begin{bmatrix}
0 & 0 & \bar{\lambda} F_{T_0} & F_{e_0} & 0 & 0 \\
0 & \bar{\lambda} F_{T_0} & 0 & 0 & 0 & F_{e_0} \\
-\bar{\lambda} F_{T_0} & 0 & 0 & 0 & 0 & F_{e_0} \\
0 & 0 & \bar{\lambda} L_{T_3} & L_{e_3} & 0 & 0 \\
0 & 0 & \bar{\lambda} L_{T_3} & 0 & 0 & L_{e_3} \\
-\bar{\lambda} L_{T_3} & 0 & 0 & 0 & 0 & L_{e_3} \\
\frac{N L_{T_3} W_0}{2} + C_{e_3} & 0 & -N L_{T_3} W_0 & -N L_{T_3} W_0 & 0 & \frac{N L_{T_3} W_0}{2} \\
-\frac{N L_{T_3} W_0}{2} + C_{e_3} & 0 & -N L_{T_3} W_0 & -N L_{T_3} W_0 & N L_{T_3} W_0 & \frac{N L_{T_3} W_0}{2} \\
\frac{N L_{T_3} W_0}{2} + C_{e_3} & 0 & -N L_{T_3} W_0 & -N L_{T_3} W_0 & N L_{T_3} W_0 & \frac{N L_{T_3} W_0}{2} \\
\end{bmatrix}
\]

(4.62d)
New symbols will be defined below, but all these symbols can be derived by applying the uncoupled mode shapes to the general equations.

**Pylon Motions**

\[
\begin{align*}
W_a &= \delta_1 \quad \text{at} \quad y = L \\
W_c &= \delta_2 \quad \text{at} \quad y = L \\
W_c' &= \frac{d\delta_2}{dy} \quad \text{at} \quad y = L \\
\Phi &= \phi_3 \quad \text{at} \quad y = L \\
W_b' &= \frac{d\delta_1}{dy} \quad \text{at} \quad y = L
\end{align*}
\] (4.63)

Therefore the matrix \([T]\) in Eq. 4.10 is described as

\[
[T] = \begin{bmatrix}
\frac{W_a}{R} & 0 & 0 \\
0 & \frac{W_c}{R} & 0 \\
0 & \frac{W_c'}{R} & \Phi \\
-W_b' & 0 & 0
\end{bmatrix}
\] (4.64)

**Aerodynamic Forces**

(a) Rotor Aerodynamic Forces

\[
\begin{align*}
F_{\phi_1} &= -\frac{1}{2} \gamma I_b \int_0^1 (F_{x_1}) \frac{W_x^2}{R^2} \, dx \\
F_{\phi_2} &= -\frac{1}{2} \gamma I_b \int_0^1 (F_{x_2}) \frac{W_x V_x}{R^2} \, dx \\
L_{\phi_1} &= -\frac{1}{2} \gamma I_b \int_0^1 (F_{\phi_1}) \frac{W_x V_x}{R^2} \, dx \\
L_{\phi_2} &= -\frac{1}{2} \gamma I_b \int_0^1 (F_{\phi_1}) \frac{V_x^2}{R^2} \, dx
\end{align*}
\] (4.65a)
\begin{align*}
F_{\tau \phi} &= -\frac{1}{2} Y I_o \int_0^1 (F_{\tau \phi}) \frac{W_1}{R} \, dz \\
F_{\rho \phi} &= -\frac{1}{2} Y I_o \int_0^1 (F_{\rho \phi}) \frac{W_1}{R} \, dz \\
F_{\tau \phi}^+ &= -\frac{1}{2} Y I_o \int_0^1 (F_{\tau \phi}^+) \frac{W_1}{R} \, dz \\
F_{\rho \phi}^+ &= -\frac{1}{2} Y I_o \int_0^1 (F_{\rho \phi}^+) \frac{W_1}{R} \, dz \\
F_{\rho \phi} &= -\frac{1}{2} Y I_o \int_0^1 (F_{\rho \phi}) \frac{W_1}{R} \, dz
\end{align*}
\quad \text{(4.65b)}

\begin{align*}
F_{\tau \phi} &= -\frac{1}{2} Y I_o \int_0^1 (F_{\tau \phi}) \frac{V_2}{R} \, dz \\
F_{\tau \phi}^+ &= -\frac{1}{2} Y I_o \int_0^1 (F_{\tau \phi}^+) \frac{V_2}{R} \, dz
\end{align*}
\quad \text{(4.65c)}

\begin{align*}
L_{\tau \phi} &= -\frac{1}{2} Y I_o \int_0^1 (F_{\tau \phi}) \frac{V_2}{R} \, dz \\
L_{\rho \phi} &= -\frac{1}{2} Y I_o \int_0^1 (F_{\rho \phi}) \frac{V_2}{R} \, dz \\
L_{\tau \phi}^+ &= -\frac{1}{2} Y I_o \int_0^1 (F_{\tau \phi}^+) \frac{V_2}{R} \, dz \\
L_{\rho \phi}^+ &= -\frac{1}{2} Y I_o \int_0^1 (F_{\rho \phi}^+) \frac{V_2}{R} \, dz
\end{align*}
\quad \text{(4.65d)}

\begin{align*}
L_{\rho \phi} &= -\frac{1}{2} Y I_o \int_0^1 (F_{\rho \phi}) \frac{W_1}{R} \, dz \\
L_{\rho \phi}^+ &= -\frac{1}{2} Y I_o \int_0^1 (F_{\rho \phi}^+) \frac{W_1}{R} \, dz
\end{align*}
\quad \text{(4.65e)}
where in general $F, L$ expresses the forces corresponding to the direction of flapping motion and lagging motion respectively. Subscripts $\beta$ and $\xi$ describe the mode shape contributions to the aero-dynamic forces; $T$ and $P$ show the derivatives for the perturbation expression with respect to inplane velocity $U_T$ and out-of-plane velocity $U_P$; superscript $+$ expresses the moment at the blade root.

$$
F_{o} = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zo}) \, dr
$$

$$
F_T = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zT}) \, dr
$$

$$
F_T^+ = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zT}^+) \, dr
$$

$$
F_P = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zP}) \, dr
$$

$$
F_P^+ = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zP}^+) \, dr
$$

$$
F_\theta = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{z\theta}) \, dr
$$

$$
F_\theta^+ = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{z\theta}^+) \, dr
$$

$$
L_0 = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{z0}) \, dr
$$

$$
L_0^+ = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{z0}^+) \, dr
$$

$$
L_T = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zT}) \, dr
$$

$$
L_T^+ = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zT}^+) \, dr
$$

$$
L_P = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zP}) \, dr
$$

$$
L_P^+ = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{zP}^+) \, dr
$$

$$
L_\theta = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{z\theta}) \, dr
$$

$$
L_\theta^+ = \frac{1}{2} \gamma I_0 \int_{r_o}^{r_i} (F_{z\theta}^+) \, dr
$$

\begin{align}
C_{\phi_{o}} &= -N \left[ \frac{1}{2} L_T \left( \frac{N_C}{R} \right)^2 - L_T^+ \left( W_B \right)^2 \right] \\
C_{\phi_{12}} &= -N \left[ -L_P \frac{N_C}{R} \frac{W_B}{R} - \frac{1}{2} L_P^+ \frac{N_C}{R} \frac{W_B}{R} \right] \\
C_{\phi_{13}} &= -N \left[ \frac{1}{2} \bar{R} L_T \frac{W_B}{R} W_C \right] \\
C_{\phi_{21}} &= -N \left[ \frac{1}{2} F_T^+ \frac{W_C}{R} W_B - F_T \frac{W_C}{R} W_B \right] \\
C_{\phi_{22}} &= -N \left[ F_P \left( \frac{N_C}{R} \right)^2 + \frac{1}{2} \left( \bar{R}^2 L_T + F_P^+ \right) \right] (W_B^2) \\
C_{\phi_{23}} &= -N \left[ \frac{1}{2} \bar{R} \left( L_T^+ - F_T^+ \right) W'_C \right] \\
C_{\phi_{31}} &= -N \left[ \frac{1}{2} \bar{R} L_T \frac{W_B}{R} W_C \right] (4.67a)
\end{align}
\[
\begin{align*}
C_{a_{31}} &= -N \left[ \frac{1}{2} R \left( -F_p + F^+_p \right) \omega c \right] \\
C_{a_{33}} &= -N \left[ \frac{1}{2} \left\{ R^2 L_T + F_{p+} \right\} \omega^2 \right] \\
C_{a_{13}} &= -N \left[ \{ F_0 - \frac{3}{2} L_T \} \omega \right] \\
C_{a_{22}} &= -N \left[ R \{ F_0 - \frac{3}{2} L_T \} (\omega')^{2} \right] \\
C_{a_{23}} &= -N \left[ \left( \frac{3}{2} F_p + L_T \right) \omega c \right] \\
C_{a_{32}} &= -N \left[ -\frac{3}{2} F^+_p \omega c \right] \\
C_{a_{33}} &= -N \left[ R \{ F_0 - \frac{3}{2} L_T \} \omega^2 \right] \\
\end{align*}
\] (4.67b)

(b) Wing Aerodynamic Forces

\[
\begin{align*}
C_{s_{11}} &= -\frac{1}{15\Omega} \int_{0}^{L} A\omega_{s_{1}}(s_{1}) \, dy \\
C_{s_{21}} &= -\frac{1}{15\Omega} \int_{0}^{L} A\omega_{s_{2}} \delta_{1} \, dy \\
C_{a_{21}} &= -\frac{1}{15\Omega} \int_{0}^{L} A\omega_{s_{2}} \delta_{2} \, dy \\
C_{a_{31}} &= -\frac{1}{15\Omega} \int_{0}^{L} A\omega_{s_{1}}(s_{1})^{2} \, dy \\
\end{align*}
\] (4.68a)
\[
\begin{align*}
C_{\alpha_1} &= -\frac{1}{i\Omega_1} \int_0^L A_\omega \phi_1 \phi_1 \, dy \\
C_{\alpha_2} &= -\frac{1}{i\Omega_1} \int_0^L A_\omega \phi_2 \phi_2 \, dy \\
C_{\alpha_4} &= -\frac{i}{\Omega_2} \int_0^L A_{\omega_4} \phi_4 \phi_4 \, dy \\
C_{a_{23}} &= -\frac{i}{\Omega_2} \int_0^L A_{\omega_2} (\phi_3)^2 \, dy
\end{align*}
\] (4.68b)

\[
\begin{align*}
C_{a_{11}} &= \frac{i}{\Omega_2} \int_0^L A_{\alpha_1} \phi_1 \, dy \\
C_{a_{13}} &= \frac{i}{\Omega_2} \int_0^L A_{\alpha_3} \phi_1 \, dy \\
C_{a_{21}} &= \frac{i}{\Omega_2} \int_0^L A_{\alpha_2} \phi_2 \, dy \\
C_{a_{23}} &= \frac{i}{\Omega_2} \int_0^L A_{\alpha_2} \phi_3 \, dy \\
C_{a_{31}} &= \frac{i}{\Omega_2} \int_0^L A_{\alpha_3} \phi_1 \, dy \\
C_{a_{33}} &= \frac{i}{\Omega_2} \int_0^L A_{\alpha_3} \phi_3 \, dy
\end{align*}
\] (4.68c)
where $A_{w_{ij}}$, $A_{w_{ij}}$, and $A_{G_{ij}}$ are components of matrix $[A_w]$, $[A_w]$, and $[A_G]$ respectively in Eq. 4.44.

**Pitch-Flap Coupling**

The pitch-flap coupling coefficient is expressed as

$$k_p = -\frac{\partial W_1}{\partial r}\bigg|_{r_0} \tan \delta_3 \tag{4.69}$$

The rotor collective mode shape has zero slope at the root. Therefore, pitch-flap coupling appears only in cyclic motion.

**Natural frequencies**

- $\lambda_8$: blade flapping natural frequency
- $\lambda_5$: blade lagging natural frequency
- $\lambda_1$: wing vertical bending natural frequency
- $\lambda_2$: wing chordwise bending natural frequency
- $\lambda_3$: wing torsion natural frequency

**4.7 Equations for Gimbaled Rotor**

As discussed in Subsection 4.4, equations of motion (Eqs. 4.45 and 4.48) are applicable to the gimbaled rotor. The significant difference between the gimbaled rotor and the hingeless rotor lies in the application of the blade mode shapes to the rotor motion. The gimbaled rotor has independent collective mode shapes for the collective motion and cyclic mode shapes for the cyclic motion. Therefore, the above distinction is found in the following matrices:

For blade equations
For Rotor Aerodynamic Forces

\[
[M_{0j}] = \begin{bmatrix}
\left(\frac{x_j^2}{h^2}\right)^2 & 0 & 0 \\
0 & \left(\frac{x_j^2}{h^2}\right)^2 - 1 & 0 \\
0 & 0 & \left(\frac{x_j^2}{h^2}\right)^2 - 1
\end{bmatrix}
\]

\[
[M_j] = \begin{bmatrix}
0 & w_{0j} & 0 & 0 & v_{0j}^w \\
0 & 0 & -\bar{r} v_{0j}^w & -w_{ij}^w & 0 \\
-\nu_{0j}^w & 0 & w_{ij}^w & -\bar{r} v_{ij}^w & 0
\end{bmatrix}
\]

(4.70)

\[
[C_j] = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 2w_{0j} \\
0 & 0 & 2w_{ij}
\end{bmatrix}
\]

\[
[A_{a_1}^{(i)}]_j = \frac{1}{2} \delta I_B \begin{bmatrix}
\xi^{(i)}_j & 0 & 0 \\
0 & \xi^{(i)}_j & 0 \\
0 & 0 & \xi^{(i)}_j
\end{bmatrix}
\]

(4.71a)

\[
[A_{a_2}^{(i)}]_j = \frac{1}{2} \delta I_B \begin{bmatrix}
K_{p_i}^{(i)} G_{ij}^{(i)} & 0 & 0 \\
0 & K_{p_i}^{(i)} G_{ij}^{(i)} & \bar{a} G_{ij}^{(i)} \\
0 & -\bar{a} G_{ij}^{(i)} & K_{p_i}^{(i)} G_{ij}^{(i)}
\end{bmatrix}
\]

(4.71b)
\[ [A_{\alpha}^{(0)}] = \frac{1}{2} \gamma I_8 \begin{bmatrix}
0 & G_{ij}^{(0)} & 0 & 0 & 0 \\
0 & 0 & -\bar{\gamma} G_{ij}^{(0)} & -G_{ij}^{(3)} & 0 \\
-G_{ij}^{(3)} & 0 & G_{ij}^{(0)} & -\bar{\gamma} G_{ij}^{(3)} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{bmatrix} \] (4.71c)

\[ [A_{\alpha}^{(0)}] = \frac{1}{2} \gamma I_8 \begin{bmatrix}
0 & 0 & \lambda G_{ij}^{(c)} & 0 & 0 \\
0 & 0 & 0 & \lambda G_{ij}^{(c)} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{bmatrix} \] (4.71d)

\[ [A_{\alpha}^{(0)}] = \frac{1}{2} \gamma I_8 \begin{bmatrix}
0 & 0 & \lambda G_{ij}^{(c)} & G_{ij}^{(0)} & 0 \\
0 & 0 & -\lambda G_{ij}^{(c)} & 0 & G_{ij}^{(0)} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{bmatrix} \] (4.71e)

\[ [A_{\alpha\delta}] = \begin{bmatrix}
0 & 0 & -\frac{1}{2} \left( \frac{V_{i}^{(0)}}{R} F_{e1} + \frac{W_{i}^{(0)}}{R} F_{e2} \right) \\
\frac{V_{i}^{(0)}}{R} F_{z1} + \frac{W_{i}^{(0)}}{R} F_{z2} & 0 & 0 & \frac{1}{2} \left( \frac{V_{i}^{(0)}}{R} F_{e1} + \frac{W_{i}^{(0)}}{R} F_{e2} \right) \\
0 & -\frac{1}{2} \bar{\gamma} \left( \frac{V_{i}^{(0)}}{R} F_{e1} + \frac{W_{i}^{(0)}}{R} F_{e2} \right) & -\frac{1}{2} \bar{\gamma} \left( \frac{V_{i}^{(0)}}{R} F_{e1} + \frac{W_{i}^{(0)}}{R} F_{e2} \right) & \frac{1}{2} \bar{\gamma} \left( \frac{V_{i}^{(0)}}{R} F_{e1} + \frac{W_{i}^{(0)}}{R} F_{e2} \right) \\
\frac{1}{2} \left( \frac{V_{i}^{(0)}}{R} F_{e1} + \frac{W_{i}^{(0)}}{R} F_{e2} \right) & \frac{1}{2} \left( \frac{V_{i}^{(0)}}{R} F_{e1} + \frac{W_{i}^{(0)}}{R} F_{e2} \right) & -\frac{1}{2} \bar{\gamma} \left( \frac{V_{i}^{(0)}}{R} F_{e1} + \frac{W_{i}^{(0)}}{R} F_{e2} \right) & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix} \] (4.71f)
\[
[A_{Q_{2}}] =
\begin{bmatrix}
0 & \frac{\bar{a}}{2} (\frac{V^{(4)}}{R} F_{e0} + \frac{w^{(4)}}{R} F_{e3}) \\
K_{p_{1}}^{(c)} F_{e3} & 0 \\
0 & \frac{1}{2} \left[ (\frac{V^{(4)}}{R} F_{e0} - \frac{w^{(4)}}{R} F_{e3}) - \bar{a} \chi (\frac{V^{(4)}}{R} F_{e0} + \frac{w^{(4)}}{R} F_{e3}) \right] - \frac{1}{2} \bar{R} K_{p_{1}}^{(c)} F_{e3} \\
K_{p_{1}}^{(c)} \chi F_{e3} & 0 \\
\frac{1}{2} K_{p_{1}}^{(c)} F_{e3} & 0 \\
0 & \frac{\bar{a}}{2} \bar{R} \left( \frac{V^{(4)}}{R} F_{e0} + \frac{w^{(4)}}{R} F_{e3} \right) + \frac{1}{2} K_{p_{1}}^{(c)} \chi F_{e3} \\
0 & \frac{1}{2} \left[ (\frac{V^{(4)}}{R} F_{e0} - \frac{w^{(4)}}{R} F_{e3}) - \bar{a} \chi (\frac{V^{(4)}}{R} F_{e0} + \frac{w^{(4)}}{R} F_{e3}) \right] - \frac{1}{2} \bar{R} K_{p_{1}}^{(c)} F_{e3} \\
0 & 0
\end{bmatrix}
\]
(4.71g)
SECTION 5

NATURAL FREQUENCIES AND MODE SHAPES
OF THE ROTOR AND WING

5.1 Rigid Rotor

In order to obtain the natural frequencies and corresponding mode shapes of the free vibration of the rotating blade, the equations of motion are derived neglecting aerodynamic forces and pylon motions in Eqs. 2.3 and 2.4.

\[
\frac{\partial^2}{\partial t^2} \left[ \left( EI_c \sin^2 \theta + (EI)_0 \cos^2 \theta \right) \frac{\partial \theta}{\partial t} \right] \\
+ \frac{\partial}{\partial r} \left[ \left( (EI)_c - (EI)_0 \right) \sin \theta \cos \theta \frac{\partial \theta}{\partial r} \right] \\
- \frac{\partial}{\partial r} \left( T - \frac{\partial \omega}{\partial r} \right) + m \ddot{r} = 0
\]

\[
\frac{\partial^2}{\partial t^2} \left[ \left( EI_c \cos^2 \theta + (EI)_0 \sin^2 \theta \right) \frac{\partial \theta}{\partial t} \right] \\
+ \frac{\partial}{\partial r} \left[ \left( (EI)_c - (EI)_0 \right) \sin \theta \cos \theta \frac{\partial \theta}{\partial r} \right] \\
- \frac{\partial}{\partial r} \left( T \frac{\partial \theta}{\partial r} \right) + m \ddot{r} - m \Omega^2 \dot{r} = 0 \tag{5.1}
\]

These equations of motion are rewritten as a usual eigenvalue problem to find eigenvalues and/or eigenvectors.

Boundary conditions for the rigid rotor are

at the root:

\[
\dot{r} \quad \frac{\partial \theta}{\partial r} = 0 \\
\dot{\theta} = 0 \\
\frac{\partial \theta}{\partial r} = 0 \tag{5.2}
\]
As a typical hingeless rotor, the Boeing rotor is chosen in this study. Mass, stiffness distributions and built-in angle of twist are shown in Fig. 8.

Nonrotating natural frequencies of the blade are shown in Fig. 9. Almost no influence of the collective pitch change on the natural frequency is found in the case of the rigid rotor.

In Figs. 10a and 10b, the first and second natural frequencies of the rotating blade are shown versus rotor rotational speed and inflow ratio. Inflow ratio is related to the collective pitch as described in Eq. 3.9. At normal rotational speed (in this case \( \Omega = 386 \text{ R.P.M.} \)) the natural frequency variation due to collective pitch change is shown in Fig. 11. Typical mode shapes up to the 4th mode are shown in Figs. 12a through 12d. Blade mode shapes are normalized with respect to the blade radius at the point of maximum deflection in either out-of-plane or inplane bending.

5.2 Gimballed Rotor

In the case of the gimballed rotor, rotor motion is expressed by collective modes and cyclic modes (in Ref. 17 and 18). Hence, based on Eq. 5.1, the eigenvalue problem should be solved with boundary conditions for collective modes and cyclic modes for the powered flight, respectively.

Boundary conditions for collective modes are defined to yield symmetrical modes for flapping and lagging. Therefore,

\[
\begin{align*}
\frac{\partial w}{\partial r} &= 0 \\
\frac{\partial^2 w}{\partial r^2} &= 0 \\
\frac{\partial v}{\partial r} &= 0 \\
\frac{\partial^2 v}{\partial r^2} &= 0
\end{align*}
\]  

(5.3)

At the tip:

\[
\begin{align*}
\frac{\partial w}{\partial r} &= 0 \\
\frac{\partial^2 w}{\partial r^2} &= 0 \\
\frac{\partial v}{\partial r} &= 0 \\
\frac{\partial^2 v}{\partial r^2} &= 0
\end{align*}
\]  

(5.4)
Boundary conditions for cyclic modes which consist of anti-symmetrical modes for flapping motion and symmetrical modes for lagging motion are expressed as:

at the root:

\[
\begin{align*}
\frac{\partial^2 \hat{w}}{\partial r^2} &= 0 \\
\frac{\partial \hat{v}}{\partial r} &= 0 \\
\hat{w} &= 0 \\
\frac{\partial \hat{v}}{\partial r} &= 0
\end{align*}
\]  
(5.6)

at the tip:

\[
\begin{align*}
\frac{\partial^2 \hat{w}}{\partial r^2} &= 0 \\
\frac{\partial \hat{v}}{\partial r} &= 0 \\
\hat{w} &= 0 \\
\frac{\partial \hat{v}}{\partial r} &= 0
\end{align*}
\]  
(5.7)

As a typical case, the Bell design is considered here. Mass, stiffness distributions and built-in angle of twist are shown in Fig. 8.

Blade nonrotating natural frequencies for the gimballed rotor are shown in Fig. 13. In Fig. 13b, the first natural frequency for rigid body flapping motion has non-zero natural frequency due to the rubber hub spring, which is intended to increase control power and damping.

In Figs. 14 and 15 it is shown that collective pitch change has a large influence on the natural frequency variation except for the first cyclic mode natural frequency (rigid body mode). Mode shapes for the gimballed rotor are shown in Figs. 16 and 17. The normalization system is the same as that of the rigid rotor.
5.3 Wing

The equations of motion for free vibration of the wing are the same as Eq. 2.5, 2.6, and 2.7 when the aerodynamic forces \( F_x, F_y, \) and \( M_y \) are eliminated from these equations. Boundary conditions are as follows:

(a) at the root:
\[
\begin{align*}
\dot{u}_w &= 0 \\
\ddot{u}_w &= 0 \\
\ddot{u}_w &= 0
\end{align*}
\]
\[
\frac{\partial u}{\partial y} = 0
\]
\[
\frac{\partial u}{\partial y} = 0
\]
\[
(5.8)
\]

(b) at the wing tip:
\[
\begin{align*}
M_p \ddot{\bar{y}}_x - \frac{\partial}{\partial y} \left[ \left( (E_1)_{y} \sin^2 \theta_w + (E_1)_{y} \cos^2 \theta \right) \frac{\partial u_w}{\partial y} \right]_{y=L} & = 0 \\
- \frac{\partial}{\partial y} \left[ \left( (E_1)_{y} - (E_1)_{y} \right) \sin \theta_w \cos \theta_w \frac{\partial u_w}{\partial y} \right]_{y=L} & = 0 \\
+ N M_{y} \ddot{y} & = 0
\end{align*}
\]
\[
(5.9a)
\]
\[
\begin{align*}
M_p \ddot{\bar{y}}_y - \frac{\partial}{\partial y} \left[ \left( (E_1)_{y} \cos^2 \theta_w + (E_1)_{y} \sin^2 \theta_w \right) \frac{\partial u_w}{\partial y} \right]_{y=L} & = 0 \\
- \frac{\partial}{\partial y} \left[ \left( (E_1)_{y} - (E_1)_{y} \right) \sin \theta_w \cos \theta_w \frac{\partial u_w}{\partial y} \right]_{y=L} & = 0 \\
+ N M_{y} \ddot{y} & = 0
\end{align*}
\]
\[
(5.9b)
\]
\[
\begin{align*}
I_{y} \ddot{y} + \left[ \left( (E_1)_{y} \cos^2 \theta_w + (E_1)_{y} \sin^2 \theta_w \right) \frac{\partial u_w}{\partial y} \right]_{y=L} & = 0 \\
+ \left[ \left( (E_1)_{y} - (E_1)_{y} \right) \sin \theta_w \cos \theta_w \frac{\partial u_w}{\partial y} \right]_{y=L} & = 0 \\
+ N M_{y} \ddot{y} + \frac{N}{2} I_{y} \ddot{y} & = 0
\end{align*}
\]
\[
(5.9c)
\]
\[ I_{PP} \ddot{y}_p + \left[ \frac{\partial^2 P}{\partial y} \right]_{y=L} + N M_b \ddot{x} + N M_b \ddot{y}_p + \frac{N}{2} I_b \ddot{y}_p = 0 \]  

\[ I_{p}\ddot{y}_r - \left[ \frac{(EI_c) \sin^2 \Theta_r + (EI_d) \cos^2 \Theta_r}{2} \right]_{y=L} - \left[ \frac{(EI_c) - (EI_d) \sin \Theta_r \cos \Theta_r}{2} \right]_{y=L} + N I_b \ddot{y}_r = 0 \]  

(5.9d)  

(5.9e)

The relationship between pylon motions and wing deflections is defined in Eq. 2.8. The rotor and the pylon are treated as lumped masses at the wing tip in these equations; as in the actual structural dynamic test, the proprotor blades will be removed and replaced by equivalent weights. Note that in Eq. 5.9 the blade is treated as a lumped mass which has equivalent weight and equivalent blade flapping inertia \( I_B \). This flapping inertia leads to lower wing frequencies, especially in torsion, than when the blade is treated as a lumped mass with equivalent mass and without equivalent blade flapping inertia.

The same wing is used for the Boeing and the Bell design; mass and stiffness properties are shown in Fig. 18. The differences appear in the pylon and blade mass properties shown in Table 1. No built-in angle of twist, dihedral angle, and sweep angle are considered here.

Natural frequencies of both cases are listed in Table 1, and mode shapes are described in Fig. 19 for the Boeing case and in
Fig. 20 for the Bell case. Wing mode shapes are normalized with respect to the wing semispan at the point of maximum deflection in either vertical bending or in chordwise bending. However, the mode shape corresponding to the third natural frequency of the wing, which is predominantly one of wing torsional deflection, is normalized at the point of maximum torsional deflection.
SECTION 6
PROPROTOR DYNAMIC CHARACTERISTICS

6.1 Introduction

In this section the eigenvalues and frequency response of the system will be discussed. The results are compared with those of Ref. 13. The analyses were conducted for two types of prop-rotor design: the hingeless rotor Boeing design, and the gimbaled rotor Bell design. The case selected to be investigated is cruising flight in the airplane mode for each design. The data for the calculation are listed in Table 1. One should notice that the equations of motion shown in Eq. 4.57 were derived based on mass-normalized coupled modes. However, for the sake of aiding physical understanding of the frequency response and eigenvalue analysis, those results are based on physical mode shapes, as presented in Figs. 12, 16, 17, 19, and 20.

The primary differences between the Ref. 13 analysis and this report are tabulated below.

<table>
<thead>
<tr>
<th></th>
<th>This Report</th>
<th>Ref. 13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural Frequencies</td>
<td>Calculated Data</td>
<td>Experimental Data</td>
</tr>
<tr>
<td>Mode Shapes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wing</td>
<td>Coupled Modes</td>
<td>Uncoupled Modes</td>
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<tr>
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<td>Coupled Modes</td>
<td>Uncoupled Elastic</td>
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<td></td>
<td></td>
<td>Modes for Inertia</td>
</tr>
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<td>Structural Damping</td>
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<td>Yes</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Rigid-Body</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mode for Aerodynamic</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Terms</td>
</tr>
</tbody>
</table>
The case studies done in the present study are tabulated below:

<table>
<thead>
<tr>
<th>Degree-of-Freedom</th>
<th>9 Degree-of-Freedom</th>
<th>18 Degree-of-Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode Shapes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blade</td>
<td>Rigid Body</td>
<td>Uncoupled</td>
</tr>
<tr>
<td></td>
<td>Elastic</td>
<td>Coupled</td>
</tr>
<tr>
<td>Wing</td>
<td>Coupled</td>
<td>Coupled</td>
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<tr>
<td></td>
<td>Coupled</td>
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<td>Eigenvalue</td>
<td>Eigenvalue</td>
</tr>
<tr>
<td></td>
<td>and Eigen-vector</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Response to</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$u_G$, $v_G$, $w_G$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta_{ls}$</td>
<td></td>
</tr>
</tbody>
</table>

The general symbols $Q_{1o}$, $Q_{jc}$, $Q_{js}$, and $a_i$ for blade and wing motions are convenient for the theoretical derivation of the equations, but from the physical understanding aspect, it would be better to choose other symbols. For this section, except for the eigenvalue analysis, the following system will be used to express blade and wing motion for the 9 degree-of-freedom system: flapping and lagging motions which are designated as $\phi$ and $\zeta$, respectively, and wing motions are $q_1$, $q_2$, and $p$, which express the vertical bending, chordwise bending, and torsion, respectively.

The correspondence between these sets of symbols is tabulated below:
### 6.2 Eigenvalues and Eigenvectors

The system stability characteristics are shown in Figs. 21 through 24.

#### 6.2.1 Addition of Higher Mode Degrees of Freedom

The eigenvalue variation with number of degrees-of-freedom considered is shown in Fig. 21. The 9 degrees-of-freedom include $Q_{10}, Q_{1c}, Q_{ls}, Q_{20}, Q_{2c}, Q_{2s}, a_1, a_2, a_3 (\beta_0, \beta_{1c}, \beta_{1s}, \xi_0, \xi_{1c}, \xi_{ls}, q_1, q_2, \text{and } p)$ corresponding to the blade's first two natural frequencies and the wing's first three natural frequencies. This system has 9 eigenvalues and 9 corresponding eigenvectors. The 9 eigenvalues are denoted by their frequencies, and the participation of the degrees-of-freedom in the eigenvectors is indicated below:

<table>
<thead>
<tr>
<th>New Symbols</th>
<th>General Symbols</th>
<th>Bell Design</th>
<th>Boeing Design</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>Predominant Flapping Motion</td>
<td>$Q_{20}$</td>
<td>$Q_{20}$</td>
</tr>
<tr>
<td>$\beta_{1c}$</td>
<td>Flapping Motion</td>
<td>$Q_{1c}$</td>
<td>$Q_{2c}$</td>
</tr>
<tr>
<td>$\beta_{1s}$</td>
<td>Flapping Motion</td>
<td>$Q_{1s}$</td>
<td>$Q_{2s}$</td>
</tr>
<tr>
<td>$\xi_0$</td>
<td>Predominant Lagging Motion</td>
<td>$Q_{10}$</td>
<td>$Q_{10}$</td>
</tr>
<tr>
<td>$\xi_{1c}$</td>
<td>Lagging Motion</td>
<td>$Q_{2c}$</td>
<td>$Q_{1c}$</td>
</tr>
<tr>
<td>$\xi_{1s}$</td>
<td>Lagging Motion</td>
<td>$Q_{2s}$</td>
<td>$Q_{1s}$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>Wing Vertical Bending</td>
<td>$a_1$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>Wing Chordwise Bending</td>
<td>$a_2$</td>
<td>$a_2$</td>
</tr>
<tr>
<td>$p$</td>
<td>Wing Torsion</td>
<td>$a_3$</td>
<td>$a_3$</td>
</tr>
</tbody>
</table>
Present

<table>
<thead>
<tr>
<th>Eigenvalue Symbol</th>
<th>Participation</th>
<th>Bell</th>
<th>Boeing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1^-$</td>
<td>Low frequency $Q_1$</td>
<td>$\beta-1$</td>
<td>$\zeta-1$</td>
</tr>
<tr>
<td>$Q_1^0$</td>
<td>Collective $Q_1$</td>
<td>$\zeta$</td>
<td>$\zeta$</td>
</tr>
<tr>
<td>$Q_1^+$</td>
<td>High frequency $Q_1$</td>
<td>$\beta+1$</td>
<td>$\zeta+1$</td>
</tr>
<tr>
<td>$Q_2^-$</td>
<td>Low frequency $Q_2$</td>
<td>$\zeta-1$</td>
<td>$\beta-1$</td>
</tr>
<tr>
<td>$Q_2^0$</td>
<td>Collective $Q_2$</td>
<td>$\beta$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>$Q_2^+$</td>
<td>High frequency $Q_2$</td>
<td>$\zeta+1$</td>
<td>$\beta+1$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>The first wing</td>
<td>$q_1$</td>
<td>$q_1$</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>The second wing</td>
<td>$q_2$</td>
<td>$q_2$</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_3$</td>
<td>The third wing</td>
<td>$p$</td>
<td>$p$</td>
</tr>
<tr>
<td></td>
<td>frequency</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the 18 degree-of-freedom system two higher elastic modes for the blade and three higher elastic modes for the wing are added to the 9 degree-of-freedom system. The same designation system as used for the 9 degree-of-freedom is employed for the higher eigenvalues. The results of the 18 degree-of-freedom analysis show that the addition of more degrees of freedom does not substantially influence the basic system eigenvalues since the added degrees of freedom have large natural frequencies in comparison with the original values (see Table 1). The 18 degree-of-freedom system eigenvalue locations are shown in Fig. 22.
6.2.2 The Sensitivity to Mode Shape

Eigenvalues and damping ratios (the fraction of the critical damping) for three different types of mode shapes for the blade are shown in Fig. 23. One is the rigid-body mode (with spring restraint at the root), another is the elastic uncoupled mode, and the third is the elastic coupled mode, while the wing mode shapes considered are restricted to the elastic coupled mode shapes only.

The results tell that mode-shape types make negligible difference in the frequency of the eigenvalues; however, they influence the damping a lot.

In the Bell design, there is almost no difference between the rigid mode and the elastic mode in the damping ratio. This is due to the similarity between the rigid-body mode and the elastic uncoupled mode. In the elastic coupled mode, a slightly higher damping is obtained.

In the Boeing design, each mode-shape type gives a different damping. For the first natural frequency mode of the blade, the rigid-body mode calculation is conservative rather than that of the elastic coupled mode-shape type, and for the second natural frequency mode, it is nonconservative. This is because the first mode has both positive out-of-plane and inplane deflection, while the second has a positive out-of-plane and a negative inplane deflection as shown in Figs. 12 (a) and (b). The corresponding positive out-of-plane deflection of the first mode increases the damping, while the associated negative inplane deflection of the second mode reduces the damping. This explanation also holds for the Bell case. The damping of the second collective elastic coupled mode of the Bell rotor is lower than that of the rigid-body mode. The second collective mode has a positive out-of-plane deflection and a small positive inplane deflection. However, the rotation direction of the rotor is different for the Bell and the Boeing.
designs. Therefore, the positive inplane deflection mode of the Bell rotor is physically the same motion as that of the first mode of the Boeing rotor. The coordinate system related to this is shown in Fig. 5, and the contribution to the damping of the coupled mode shape can be seen in the term $G_{ji}$ in Eq. 4.33.

In conclusion, the use of the elastic coupling mode affects the damping significantly. The coupled mode with a combination of forward out-of-plane deflection (upward in helicopter mode) and inplane deflection opposing the rotor direction of rotation, or vice versa, increases the damping over that of the rigid-body-mode calculation. The coupled mode which has forward out-of-plane deflection and inplane deflection proceeding with rotor direction of rotation, or vice versa, decreases the damping. This holds only for the first beam bending mode; in other words, the coupled mode without nodal points between the root and the tip.

6.3 Frequency Response

The frequency response analysis is dealt with in this subsection. These calculations are all based on mass-normalized coupled modes for both the blade and the wing. However, for convenience in physical understanding, the results of the frequency response calculation are presented in terms of length-normalized mode shapes in Figs. 25 through 32.

6.3.1 Frequency Response to the Gust

Frequency responses to the vertical gust $u_G$, the lateral gust $v_G$, and the longitudinal gust $w_G$ for both the Bell and the Boeing designs are shown in Figs. 25 through 30.

As a whole, the behavior of the frequency response is quite similar to that of Ref. 13 in spite of the differences in the theoretical model stated in Subsection 6.1.

The detail characteristics of the frequency response are discussed next.
6.3.1.1 Blade Collective Motion Response ($\beta_0$, $\gamma_0$)

Collective motion responses have a close relationship with the wing motions (vertical bending $q_1$, chordwise bending, $q_2$, and torsion, $p$). They have strong resonances with these modes. In the case of $u_G$ and $v_G$ inputs, the static response ($\omega$ equal to zero) and the lower-frequency-range response are negligible. However, the response to the longitudinal gust $w_G$ has a significant static and lower frequency range response. Resonances of the collective modes occur in the response to the longitudinal gust input. Comparing the Boeing design with the Bell design, the Boeing has a larger response in collective responses to each gust input.

6.3.1.2 Blade Cyclic Flapping Motion Response ($\beta_{1c}$ and $\beta_{1s}$)

For the cyclic inputs ($u_G$ and $v_G$) there are not strong resonances between cyclic flapping motions and the wing vertical bending motion ($q_1$). In the upper frequency range, the high frequency flapping mode ($\beta+1$ mode) has a significant resonance appearing in the flapping motion. In the lower frequency range, there is a resonance of the low frequency flapping mode ($\beta-1$ mode) for the Boeing design and the low frequency lagging mode ($\zeta-1$ mode) for the Bell design.

For the collective input ($w_G$) the response of the cyclic flapping motion has resonances with the wing motion modes ($q_1$, $q_2$, and $p$) in both designs.

6.3.1.3 Blade Cyclic Lagging Motion Response ($\zeta_{1c}$ and $\zeta_{1s}$)

In the Bell design there is an evident resonance in the cyclic lagging motion response ($\zeta_{1c}$ and $\zeta_{1s}$) to the vertical gust input ($u_G$) in proximity to the wing vertical bending mode ($q_1$). To the lateral gust input ($v_G$), a low frequency lagging mode ($\zeta-1$ mode) resonance appears in the lagging motion response.

In the Boeing design there cannot be seen such obvious resonances in the response to the vertical and lateral gust inputs.
The static response is much larger than that of the Bell design due to the soft inplane design (the blade lagging natural frequency is less than unity).

For the collective input ($w_c$), the responses of cyclic lagging motion have resonances with the wing motion modes ($q_1$, $q_2$, and $p$) in both designs.

### 6.3.1.4 Wing Motion Response ($q_1$, $q_2$, and $p$)

The wing motion includes resonances in each wing mode. Although the response magnitude is quite large, it is expected to become rather small if structural damping is included.

### 6.3.2 Frequency Response to Control Pitch Angle

Frequency responses to longitudinal cyclic pitch input are shown in Figs. 31 and 32.
7.1 Conclusions

This study has been devoted to the development and evaluation of a theoretical model of the proprotor on a cantilever wing, operating in normal cruising flight. This theory expresses the wing and blade motions in coupled form, and can include any number of mode shapes required to express the motions accurately. It has been applied to an investigation of the dynamic characteristics of the Bell and the Boeing designs.

Based on the theoretical results included in this study, the following conclusions may be stated:

(a) The choice of mode shape (rigid-body mode or elastic-coupled mode) affects the damping significantly. The dependency of the damping on the mode shape can be estimated for the first beam bending mode. The blade inplane deflection opposing the rotor direction of rotation, accompanied by the forward out-of-plane deflection, increases the damping, comparing it with the rigid-body calculation. The inplane deflection proceeding in the rotor direction of rotation decreases the damping. The mode shape has little influence on the frequencies of the system.

(b) The results of the frequency response are quite similar to those of Ref. 13, in spite of the difference in the mode shapes. The amplitude of the response is slightly different, since structural damping was not included in the present calculation, and the mode shapes used were different.
The analysis of the eighteen degree-of-freedom system showed that the higher-frequency degrees of freedom have small influence on the basic degrees of freedom.

7.2 Suggestions for Future Research

A direct and useful extension of the present study would be the stability analysis of the proprotor aircraft, with respect to air resonance and flutter. This theory will be very powerful because the eigenvector components can be compared directly without any adjustment between components. Mach number effects of the blade should be included for the flutter analysis.

Stresses or bending moments of the wing or blade can be predicted from the motions of the wing and blade obtained from this analysis. In addition, this analysis may be applied to the development of an automatic control device to alleviate the gust response of the vehicle.
REFERENCES


10. Gaffey, T.M., "The Effect of Positive Pitch-Flap Coupling (Negative δ3) on Rotor Blade Motion Stability and Flapping".


<table>
<thead>
<tr>
<th>Description</th>
<th>Bell</th>
<th>Boeing</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type</strong></td>
<td>gimbaled, stiff</td>
<td>cantilever, soft</td>
</tr>
<tr>
<td><strong>Number of blades, ( N )</strong></td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td><strong>Radius, ( R )</strong></td>
<td>156 in.</td>
<td>150 in.</td>
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<tr>
<td><strong>Chord, ( C_B )</strong></td>
<td>18.9 in.</td>
<td>14 in.</td>
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<tr>
<td><strong>Lock number, ( \gamma )</strong></td>
<td>3.83</td>
<td>4.04</td>
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<td><strong>Solidity, ( \sigma )</strong></td>
<td>0.089</td>
<td>0.115</td>
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<td><strong>Pitch/flap coupling, ( \delta_3 )</strong></td>
<td>-15 deg.</td>
<td>0</td>
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<td><strong>Collective pitch, ( \theta_D )</strong></td>
<td>1.25 deg.</td>
<td>1.0 deg.</td>
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<td><strong>Lift-curve slope, ( a )</strong></td>
<td>5.7</td>
<td>5.7</td>
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<tr>
<td><strong>Drag Coefficient, ( C_D )</strong></td>
<td>0.0065</td>
<td>0.0065</td>
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<tr>
<td><strong>Rotor rotation direction, ( \Omega )</strong></td>
<td>+1</td>
<td>-1</td>
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<td><strong>Inflow ratio, ( \lambda )</strong></td>
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<td>0.7</td>
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<td>**Rotational speed, (</td>
<td>\Omega</td>
<td>)**</td>
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<tr>
<td></td>
<td>48.9 rad/sec</td>
<td>40.4 rad/sec</td>
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<td><strong>Blade Natural Frequencies</strong></td>
<td></td>
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</tr>
<tr>
<td>first, ( \lambda_1/</td>
<td>\Omega</td>
<td>)</td>
</tr>
<tr>
<td>second, ( \lambda_2/</td>
<td>\Omega</td>
<td>)</td>
</tr>
<tr>
<td>third, ( \lambda_3/</td>
<td>\Omega</td>
<td>)</td>
</tr>
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<td>ROTOR (cont'd)</td>
<td>BELL</td>
<td>BOEING</td>
</tr>
<tr>
<td>---------------</td>
<td>------</td>
<td>--------</td>
</tr>
<tr>
<td>fourth, $\lambda_4/</td>
<td>\Omega</td>
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<td>Collective Natural Frequency</td>
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<tr>
<td>first, $\lambda_1^{(o)}/</td>
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<tr>
<td>second, $\lambda_2^{(o)}/</td>
<td>\Omega</td>
<td>$</td>
</tr>
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<td>third, $\lambda_3^{(o)}/</td>
<td>\Omega</td>
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</tr>
<tr>
<td>fourth $\lambda_4^{(o)}/</td>
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<td>Blade flapping inertia, $I_B$</td>
<td>105 slug-ft²</td>
<td>150 slug-ft²</td>
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<td>One blade weight, $M_B$</td>
<td>133 lb</td>
<td>124 lb</td>
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<td>WING</td>
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<tr>
<td>Semispan, $L$</td>
<td>200 in.</td>
<td>200 in.</td>
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<tr>
<td>Chord, $c_w$</td>
<td>62.2 in.</td>
<td>62.2 in.</td>
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<td>Mast height, $h$</td>
<td>51.3 in.</td>
<td>51.3 in.</td>
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<tr>
<td>Sweep</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Dihedral</td>
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<td>0</td>
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<td>Lift-curve slope, $a_w^{(o)}$</td>
<td>5.7</td>
<td>5.7</td>
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<td>Drag coefficient, $C_{D_{ow}}$</td>
<td>0.004</td>
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<td>Moment coefficient $C_{mo}$</td>
<td>-0.005</td>
<td>-0.005</td>
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<td>Aerodynamic center, $c = x_A/w$</td>
<td>0.01</td>
<td>0.01</td>
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<td>Angle of attack, $\alpha_{wo}$</td>
<td>2.0 deg</td>
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TABLE 1 CONCLUDED

WING (cont'd)

Natural Frequencies

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<tr>
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<th>BELL</th>
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</tr>
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<tbody>
<tr>
<td>first, ( \Lambda_1 /</td>
<td>\Omega</td>
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</tr>
<tr>
<td>second, ( \Lambda_2 /</td>
<td>\Omega</td>
<td>)</td>
</tr>
<tr>
<td>third, ( \Lambda_3 /</td>
<td>\Omega</td>
<td>)</td>
</tr>
<tr>
<td>fourth, ( \Lambda_4 /</td>
<td>\Omega</td>
<td>)</td>
</tr>
<tr>
<td>fifth, ( \Lambda_5 /</td>
<td>\Omega</td>
<td>)</td>
</tr>
<tr>
<td>sixth, ( \Lambda_6 /</td>
<td>\Omega</td>
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PYLON

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<tr>
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<td>Weight, ( M_p )</td>
<td>1420 lb</td>
<td>2000 lb</td>
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<td>Yaw inertia, ( I_{py} )</td>
<td>164.8 slug-ft(^2)</td>
<td>250.0 slug-ft(^2)</td>
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<td>Pitch inertia, ( I_{pp} )</td>
<td>190.0 slug-ft(^2)</td>
<td>250.0 slug-ft(^2)</td>
</tr>
<tr>
<td>Roll inertia, ( I_{pr} )</td>
<td>42.4 slug-ft(^2)</td>
<td>30.0 slug-ft(^2)</td>
</tr>
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</table>

FLIGHT CONDITION FOR CALCULATIONS, \( \lambda = 0.7 \)

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<thead>
<tr>
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<th>BELL</th>
<th>BOEING</th>
</tr>
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<tbody>
<tr>
<td>Cruising speed, ( V )</td>
<td>250 kt</td>
<td>218 kt</td>
</tr>
<tr>
<td>Cruising altitude</td>
<td>sea level</td>
<td>sea level</td>
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FIG. 1 CONFIGURATION OF ANALYTICAL MODEL, PROPROTOR ON A CANTILEVER WING OPERATING IN THE AIRPLANE MODE
FIG. 2 PROPROTOR GEOMETRY AND MOTION WITH GUST VELOCITY DEFINITION
AFTER DEFORMATION

FIG. 3 BLADE GEOMETRY
FIG. 4 WING GEOMETRY
FIG. 5 VELOCITIES AT THE BLADE ELEMENT AND RESULTING AERODYNAMIC FORCES WHEN LOOKING OUTBOARD
FIG. 6 WIND VELOCITIES IN THE DISC PLANE OF THE ROTOR, LOOKING FORWARD
FIG. 7 VELOCITIES AT THE WING ELEMENT AND RESULTING AERODYNAMIC FORCES WHEN LOOKING OUTBOARD
FIG. 8 STRUCTURAL CHARACTERISTICS OF TWO PROPROTOR BLADES

(a) Section Mass Distribution
(b) Section Flapwise Bending Stiffness Distribution

FIG. 8 CONTINUED
(c) Section Chordwise Bending Stiffness Distribution

FIG. 8 CONTINUED
(d) Angle of Twist

FIG. 8 CONCLUDED
FIG. 9 BLADE NATURAL FREQUENCIES, NONROTATING, FOR BOEING ROTOR
FIG. 10  BLADE ROTATING NATURAL FREQUENCIES FOR BOEING ROTOR
(a) 1st Natural Frequency

\[ \Omega = 193 \text{ RPM} \]

\[ \Omega = 300 \text{ RPM} \]

\[ \Omega = 386 \text{ RPM (NORMAL } \Omega) \]

\[ \Omega = 550 \text{ RPM} \]
(b) 2nd Natural Frequency

Fig. 10 Concluded
FIG. 11 BLADE NATURAL FREQUENCIES FOR BOEING ROTOR AT CONSTANT ROTATIONAL SPEED ($\Omega = 386$ RPM)
(a) First Mode (Frequency $\lambda_1/\Omega = 0.827$)

FIG. 12 MODE SHAPES FOR BOEING ROTOR AT $\lambda = 0.7$ AND $\Omega = 386$ RPH
(b) Second Mode (Frequency $\lambda_2/\Omega = 1.32$)

FIG. 12 CONTINUED
(c) Third Mode (Frequency $\lambda_3/\Omega = 3.40$)

FIG. 12 CONTINUED

109
(d) Fourth Mode (Frequency $\lambda_4/\Omega = 6.77$)

**FIG. 12 CONCLUDED**
BLADE COLLECTIVE PITCH AT 75% SPAN (0.75)
(a) Collective Mode

FIG. 13 BLADE NATURAL FREQUENCIES, NONROTATING, FOR BELL ROTOR
FIG. 13 CONCLUDED

(b) Cyclic Mode

BLADE COLLECTIVE PITCH AT 75% SPAN (0.75)

1st (OUT-OF-PLANE)

2nd (INPLANE)

3rd (OUT-OF-PLANE)

4th (INPLANE)

5th (INPLANE)
FIG. 14 BLADE ROTATING NATURAL FREQUENCIES FOR BELL ROTOR
(b) Second Collective Mode

FIG. 14 CONTINUED
(c) First Cyclic Mode (Out-of-Plane Mode)

FIG. 14 CONTINUED
(d) Second Cyclic Mode (Inplane Mode)

FIG. 14 CONCLUDED
FIG. 15 BLADE NATURAL FREQUENCIES FOR BELL ROTOR AT CONSTANT ROTATIONAL SPEED ($\Omega = 458$ RPM)

(a) Collective Modes
(b) Cyclic Modes

FIG. 15 CONCLUDED
FIG. 16 COLLECTIVE MODE SHAPES FOR BELL ROTOR AT
\( \lambda = 0.7 \) AND \( \Omega = 458 \) RPM
\( \frac{w(0)}{R} \) \& \( \frac{v(0)}{R} \)

(b) Second Mode (Frequency \( \lambda_2^{(o)}/|\Omega| = 2.12 \))

FIG. 16 CONTINUED
FIG. 16 CONTINUED

(c) Third Mode (Frequency \( \lambda_3^{(o)} / |\Omega| = 4.93 \))
Fourth Mode (Frequency $\lambda_4(0) = 10.6$)

FIG. 16 CONCLUDED
FIG. 17 CYCLIC MODE SHAPES FOR BELL ROTOR AT
\( \lambda = 0.7 \) AND \( \Omega = 458 \) RPM

(a) First Mode (Frequency \( \lambda_{1}^{(c)}/|\Omega| = 1.016 \)
FIG. 17 CONTINUED

(b) Second Mode (Frequency $\lambda_2/\lambda_1 = 1.34$)
(c) Third Mode (Frequency $\chi_3^{(c)} / |\Omega| = 4.35$)

FIG. 17 CONTINUED
FIG. 17 CONCLUDED
(a) Mass and Cross-Sectional Moment of Inertia Distribution

FIG. 18  STRUCTURAL CHARACTERISTICS OF THE WING (THE SAME WING IS USED BY BOTH BELL AND BOEING)
(b) Stiffness Distribution: Vertical Bending Stiffness \((EI_w)_{B}'\), Chordwise Bending Stiffness \((EI_w)_{C}'\), and Torsional Rigidity \(GJ\)

FIG. 18 CONCLUDED
Fig. 19 Mode shapes for Boeing wing

(a) First mode (2.35 Hz)

Fig. 19 Mode shapes for Boeing wing
(b) Second Mode (4.18 Hz)

FIG. 19 CONTINUED
Fig. 19 continued

(c) Third Mode (7.11 Hz)
FIG. 19 CONTINUED

(d) Fourth Mode (15.9 Hz)
(e) Fifth Mode (25.4 Hz)

FIG. 19 CONTINUED
(f) Sixth Mode (89.5 Hz)

FIG. 19 CONCLUDED
FIG. 20 MODE SHAPES FOR BELL WING
FIG. 20 CONTINUED

WING STATION y/L
(b) Second Mode (4.72 Hz)

FIG. 20 CONTINUED
FIG. 20 CONTINUED

(c) Third Mode (6.3 Hz)
(d) Fourth Mode (8.1 Hz)

FIG. 20 CONTINUED
FIG. 20 CONTINUED

(e) Fifth Mode (28.8 Hz)
(f) Sixth Mode (80.8 Hz)

FIG. 20 CONCLUDED
FIG. 21 INFLUENCE OF ADDITION OF HIGHER DEGREES OF FREEDOM ON EIGENVALUES COMPARED WITH BASIC NINE DEGREES OF FREEDOM

(a) Bell Proprotor
(b) Boeing Proprotor

FIG. 21 CONCLUDED
(a) Bell Propotor

FIG. 22 EIGENVALUES FOR 18 DEGREES OF FREEDOM
(b) Boeing Proprotor

FIG. 22 CONCLUDED
Influence of blade mode shape types on proprotor system eigenvalues and damping ratios for coupled elastic mode shapes, compared with those based on rigid-body mode shapes and uncoupled elastic mode shapes.
Δ BLADE COUPLED ELASTIC MODE
□ BLADE UNCOUPLED ELASTIC MODE
○ BLADE RIGID-BODY MODE

(b) Boeing Proprotor Eigenvalues

FIG. 23 CONTINUED
FIG. 23 CONTINUED

(c) Bell Proprotor Damping Ratios
DAMPING RATIO ($\zeta$)

BLADE COUPLED ELASTIC MODE
BLADE UNCOPLED ELASTIC MODE
BLADE RIGID-BODY MODE

(d) Boeing Proprotor Damping Ratios

FIG. 23 CONCLUDED
FIG. 24 EIGENVALUES AND EIGENVECTORS FOR NINE DEGREES OF FREEDOM
FIG. 24 CONTINUED
\[ \lambda = -0.0136 + 0.6101 \]
\[ \lambda = -0.0244 + 0.347i \]
\[ \lambda = -0.0449 + 1.27i \]

(a) Concluded

FIG. 24 CONCLUDED
Q1 MODE (ζ - 1 MODE)
\[ \lambda = -0.271 + 0.168i \]

Q2 MODE (ζ + 1 MODE)
\[ \lambda = -0.238 + 0.837i \]

(b) Boeing Proprotor

FIG. 24 CONTINUED
\[ \lambda = -0.108 + 0.222i \]

**Q^+ \text{ MODE (} \beta + 1 \text{ MODE) }**
\[ \lambda = -0.0836 + 2.241i \]

**Q^- \text{ MODE (} \beta - 1 \text{ MODE) }**
\[ \lambda = -0.106 + 1.28i \]

(b) Continued

FIG. 24 CONTINUED
\[ \lambda = -0.0228 + 0.361i \]

\[ \lambda = -0.0580 + 1.221i \]

(b) Concluded

FIG. 24 CONCLUDED
(a) Collective Flapping Motion $\beta_0$ Response to $u_G$

FIG. 25 FREQUENCY RESPONSE OF BELL ROTOR TO VERTICAL GUST $u_G$
INPUT AT FREQUENCY $\omega$
(b) Cyclic Flapping Motion $\beta_{lc}$ Response to $u_G$

FIG. 25 CONTINUED
FIG. 25 CONTINUED

(c) Cyclic Flapping Motion $\beta_{ls}$ Response to $u_G$
(d) Collective Lagging Motion $\zeta_o$ Response

FIG. 25 CONTINUED
FIG. 25 CONTINUED

(e) Cyclic Lagging Motion $\zeta_{1c}$ Response to $u_G$
(f) Cyclic Lagging Motion $\zeta_{1s}$ Response to $u_G$

FIG. 25 CONTINUED
FIG. 25 CONTINUED

(g) Wing Vertical Bending $q_1$ Response to $u_G$

$\frac{q_1}{(u_G/V)}$

$\omega/\Omega$

0.01 0.1 1.0 10.0
FIG. 25 CONTINUED
(i) Wing Torsion p Response to $u_G$

FIG. 25 CONCLUDED
FIG. 26 FREQUENCY RESPONSE OF BELL ROTOR TO LATERAL GUST \( v_G \) INPUT AT FREQUENCY \( \omega \)

(a) Collective Flapping Motion \( \beta_o \) Response to \( v_G \)
(b) Cyclic Flapping Motion $\beta_{1c}$ Response to $v_G$

FIG. 26 CONTINUED
(c) Cyclic Flapping Motion $\beta_{1s}$ Response to $v_G$

FIG. 26 CONTINUED
Fig. 26 continued

(d) Collective Lagging Motion $\tau_o$ Response to $v_G$

$\frac{\zeta_0}{(v_G/v)}$

$0.01 \quad 0.1 \quad 1.0 \quad \omega/\Omega \quad 10.0$

$0.01 \quad 0.1 \quad 1.0$

FIG. 26 CONTINUED
(e) Cyclic Lagging Motion $\xi_{1c}$ Response to $v_G$

FIG. 26 CONTINUED
(f) Cyclic Lagging Motion $\tau_{ls}$ Response to $v_G$

FIG. 26 CONTINUED
(g) Wing Vertical Bending $q_1$ Response to $v_G$

FIG. 26 CONTINUED
FIG. 26 CONTINUED

(h) Wing Chordwise Bending $q_2$ Response to $v_G$
(i) Wing Torsion \( p \) Response to \( v_G \)

FIG. 26 CONCLUDED
FIG. 27 FREQUENCY RESPONSE OF BELL ROTOR TO LONGITUDINAL GUST $w_G$
INPUT AT FREQUENCY $\omega$

(a) Collective Flapping Motion $\beta_o$ Response to $w_G$
FIG. 27 CONTINUED

(b) Cyclic Flapping Motion $\frac{\beta_{1c}}{(w_G/V)}$ Response to $w_G$
FIG. 27 CONTINUED

(c) Cyclic Flapping Motion \( \beta_{ls} \) Response to \( w_G \)
FIG. 27 CONTINUED
(d) Collective Lagging Motion \( \frac{\zeta_0}{(w_G/V)} \) Response to \( w_G \)
(e) Cyclic Lagging Motion $\zeta_{lc}$ Response to $w_G$

FIG. 27 CONTINUED
FIG. 27 CONTINUED

(f) Cyclic Lagging Motion $\zeta_{ls}$ Response to $w_G$

$\frac{\zeta_{ls}}{(w_G/V)}$
FIG. 27 CONTINUED

(g) Wing Vertical Bending $q_1$ Response to $w_G$
FIG. 27 CONTINUED

(h) Wing Chordwise Bending $q_2$ Response to $w_G$
(i) Wing Torsion p Response to \( w_G \)

FIG. 27 CONCLUDED
(a) Collective Flapping Motion $\beta_o$ Response

FIG. 28 FREQUENCY RESPONSE OF BOEING ROTOR TO VERTICAL GUST $u_G$
INPUT AT FREQUENCY $\omega$
FIG. 28 CONTINUED

(b) Cyclic Flapping Motion \[ \frac{\phi_{1c}}{u_c/\omega} \] Response to \[ u_c \]
(c) Cyclic Flapping Motion $\beta_{ls}$ Response to $u_G$

FIG. 28 CONTINUED
(d) Collective Lagging Motion $\zeta_o$ Response

FIG. 28 CONTINUED
\( \frac{\zeta_{1c}}{(u_G/V)} \)

(e) Cyclic Lagging Motion \( \zeta_{1c} \) Response to \( u_G \)

FIG. 28 CONTINUED
(f) Cyclic Lagging Motion $\frac{\zeta_{ls}}{(u_G/V)}$ Response to $u_G$

FIG. 28 CONTINUED
FIG. 28 CONTINUED

(g) Wing Vertical Bending $q_1$ Response to $u_G$
FIG. 28 CONTINUED

(h) Wing Chordwise Bending $q_2$ Response to $u_G$
Wing Torsion p Response to $u_G$

FIG. 28 CONCLUDED
FIG. 29 FREQUENCY RESPONSE OF BOEING ROTOR TO LATERAL GUST $v_G$
INPUT AT FREQUENCY $\omega$

(a) Collective Flapping Motion $\beta_o$ Response
(b) Cyclic Flapping Motion $\beta_{1c}$ Response to $v_G$

FIG. 29 CONTINUED
FIG. 29 CONTINUED

(c) Cyclic Flapping Motion $\beta_{ls}$ Response to $v_G$
FIG. 29 CONTINUED

(d) Collective Lagging Motion $\xi_0$ Response to $v_G$
(e) Cyclic Lagging Motion $\zeta_{lc}$ Response to $v_G$

FIG. 29 CONTINUED
FIG. 29 CONTINUED

(f) Cyclic Lagging Motion $\zeta_{ls}$ Response to $v_G$
(g) Wing Vertical Bending $q_1$ Response to $v_G$

FIG. 29 CONTINUED
FIG. 29 CONTINUED
(i) Wing Torsion $p$ Response to $v_C$

FIG. 29 CONCLUDED
FIG. 30 FREQUENCY RESPONSE OF BOEING ROTOR TO LONGITUDINAL GUST $w_G$ INPUT AT FREQUENCY $w$

(a) Collective Flapping Motion $\beta_o$ Response
(b) Cyclic Flapping Motion $\beta_{1C}$ Response to $w_G$

FIG. 30 CONTINUED
FIG. 30 CONTINUED

(c) Cyclic Flapping Motion $\beta_{ls}$ Response to $w_G$
(d) Collective Lagging Motion $\tau_o$ Response to $w_G$

FIG. 30 CONTINUED
(e) Cyclic Lagging Motion $\xi_{lc}$ Response to $w_G$

FIG. 30 CONTINUED
Cyclic Lagging Motion \( \xi_{ls} \) Response to \( w_G \)

FIG. 30 CONTINUED
(g) Wing Vertical Bending $q_1$ Response to $w_G$

FIG. 30 CONTINUED
(h) Wing Chordwise Bending $q_2$ Response to $w_G$

FIG. 30 CONTINUED
(i) Wing Torsion p Response to $w_G$

FIG. 30 CONCLUDED
(a) Cyclic Flapping Motion $\beta_{lc}$ Response to $\theta_{ls}$

FIG. 31 FREQUENCY RESPONSE OF BELL ROTOR TO LONGITUDINAL CYCLIC PITCH $\theta_{ls}$ INPUT AT FREQUENCY $\omega$
(b) Cyclic Flapping Motion $\beta_{ls}$ Response to $\theta_{ls}$

FIG. 31 CONCLUDED
FIG. 32 FREQUENCY RESPONSE OF BOEING ROTOR TO LONGITUDINAL CYCLIC PITCH $\theta_{1s}$ INPUT AT FREQUENCY $\omega$
FIG. 32 CONCLUDED
A.1 Blade Motion

The blade coordinate system is described in Section 2, and in Figs. 2 and 3. The position vector \( \mathbf{r}_n \) defining an arbitrary point on the \( r \)-axis of the \( n \)-th blade with respect to the inertial \( x-y-z \) system becomes

\[
\mathbf{r}_n = \mathbf{r}_o + \mathbf{r} \begin{bmatrix} v_p \\ -v_y \\ 0 \end{bmatrix}
\]

\[
+ \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos v_p & -\sin v_p \\
0 & \sin v_p & \cos v_p
\end{bmatrix}
\begin{bmatrix}
\cos v_p & 0 & \sin v_p \\
0 & 1 & 0 \\
-\sin v_p & 0 & \cos v_p
\end{bmatrix}
\begin{bmatrix}
\rho \cos \Phi_n - v_y \sin \Phi_n \\
\rho \sin \Phi_n + v_y \cos \Phi_n \\
\mathcal{W}_n
\end{bmatrix}
\]

(A.1)

where \( \mathbf{r}_o \) is the position vector which expresses translation at the wing tip as

\[
\mathbf{r}_o = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}
\]

(A.2)

The square of the velocity at an arbitrary point on the \( r \)-axis is expressed as

\[
\dot{\mathbf{r}}_n \cdot \dot{\mathbf{r}}_n = (r \Omega)^2 + \dot{x}_n^2 + \dot{y}_n^2 + (\Omega \dot{x}_n)^2 + \dot{z}_n^2 \\
+ \dot{v}_n^2 + r^2 \dot{v}_n^2 + (\Omega^2 + r^2 \cos^2 \phi_n) \dot{v}_n^2 \\
+ (\Omega^2 + r^2 \sin^2 \phi_n) \dot{y}_n^2 + 2r \Omega \dot{\phi}_n - 2r \Omega \dot{v}_n \sin \phi_n
\]

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The total kinetic energy $T_E$ for $N$ blades is expressed as

$$T_E = \frac{1}{2} \sum_{n=1}^{N} \int_{V} m \mathbf{\dot{r}_n} \cdot \mathbf{\dot{r}_n} \, dV$$  \hspace{1cm} (A.4)$$

where $m$ is the blade mass per unit length.
The potential energy $U_E$ of $N$ blades is

$$U_E = \frac{1}{2} \sum_{n=1}^{N} \int_{0}^{R} \left\{ [(E)_{o} \cos^2 \theta_{o} + (E)_{c} \sin^2 \theta_{o}] \left( \frac{\partial^2 u_n^r}{\partial \gamma^2} \right)^2 
+ [(E)_{o} \sin^2 \theta_{o} + (E)_{c} \cos^2 \theta_{o}] \left( \frac{\partial^2 u_n^r}{\partial \gamma^2} \right)^2 
+ [(E)_{c} - (E)_{o}] \sin \theta_{o} \cos \theta_{o} \frac{\partial u_n^r}{\partial \gamma} \frac{\partial^2 u_n^r}{\partial \gamma^2} \right\} d\gamma \quad (A.5)$$

The virtual work done by the centrifugal forces $\delta W_{CF}$ is described as

$$\delta W_{CF} = \sum_{n=1}^{N} \int_{0}^{R} \left[ \frac{\partial}{\partial \gamma} \left( \frac{\partial^2 u_n^r}{\partial \gamma^2} \right) \delta u_n^r + \frac{\partial}{\partial \gamma} \left( \frac{\partial^2 u_n^r}{\partial \gamma^2} \right) \delta u_n^r \right] d\gamma \quad (A.6)$$

A.2 The Work Done by Blade Aerodynamic Forces

Aerodynamic forces acting on the $n$th blade $(\vec{f})_n$ are derived in the $z-y-z$ coordinate system from the $P_z$ and $P_\theta$ components as

$$\vec{f}_n = \begin{bmatrix} 1 & -\nu_r & \nu_r \\ \nu_r & 1 & -\nu_y \\ -\nu_y & \nu_y & 1 \end{bmatrix} \begin{bmatrix} -P_\theta \sin \psi_n \\ P_\theta \cos \psi_n \\ P_z \end{bmatrix} \quad (A.7)$$

Then the virtual work done by the aerodynamic forces of all blades $\delta W_{AF}$ is described as

$$\delta W_{AF} = \sum_{n=1}^{N} \int_{0}^{R} \delta \vec{r}_n \cdot \vec{f}_n \, d\gamma \quad (A.8)$$
A.3 Lagrange's Equations

In addition to the above, the kinetic energy and potential energy of the wing derived from the simple beam theory are appended.

Then, Lagrange's equations are given by

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$  \hspace{1cm} (A.9)

where $T$, $U$ and $Q_i$ are respectively, the total kinetic energy, potential energy and generalized forces of the system, and

$$Q_i = \omega_x, \, \omega_y, \, \nu_x, \, \nu_y, \, \nu_z, \, \nu_r, \, \omega_r, \, \omega_w, \, \omega_w$$  \hspace{1cm} (A.10)

Finally, the above represents the derivation of Eqs. 2.4, 2.5, 2.6, 2.7, 2.8, 2.10, 2.11, 2.12, 2.13, and 2.14.
In this appendix, the method of deriving the stresses in the rotor blades and in the wing due to the gust are briefly stated. If the deformation of an elastic structure has been determined, it is a straightforward procedure to determine the stresses corresponding to this deformation. The mode acceleration method discussed in Ref. 15 is very suitable because the stresses can be determined accurately and directly when the deformation has been computed in terms of displacements of normal modes.

Equation 4.57 is written here again in matrix form as

\[
[A] \ddot{x} + [B] \dot{x} + [C] x = [D] e + [F]
\]

where \([A]\), \([B]\) and \([C]\) are square matrices to define the coefficients of equations derived from mass, stiffness and aerodynamic forces, and the \([D]\) matrix is the excitation. The generalized coordinates \(\{x\}\) and excitation input \(\{e\}\) are

\[
\{x\} = \begin{bmatrix} \Theta_0 \\ \Theta_{1c} \\ \Theta_{2s} \\ \vdots \\ \Theta_s \\ \Theta_a \\ \Theta_{1s} \\
\end{bmatrix}, \quad \{e\} = \begin{bmatrix} u_r \\ v_r \\ w_r \\ \Theta_0 \\ \Theta_{1c} \\ \Theta_{1s} \end{bmatrix}
\]

The static force \(\{F\}\) from the lift or drag of the wing and rotor in steady flight may be included if the total stresses are required. When only the additional stress due to the gust or control input varying with time is required, the \(\{F\}\) matrix may be dropped.
In general, the stress at a particular point \( p \) in the rotor wing structure is expressed by

\[
\{\sigma_p\} = [A_p] \{x\}
\]

(B.3)

where the \([A_p]\) matrix is a constant matrix that represents the stress at the point \( p \) due to a unit displacement in the normal mode.

The static mode displacements are given by setting \( \{\ddot{x}\} = 0 \) and \( \{\dot{x}\} = 0 \) in Eq. B.1.

\[
\{\chi\}_{\text{static}} = [C]^{-1} [D] \{x\} + [C]^{-1} \{F\}
\]

(B.4)

Therefore, the static stress becomes

\[
\{\sigma_p\}_{\text{static}} = [A_p] \{x\}_{\text{static}}
\]

(B.5)

When the rotor and wing are vibrating, the total displacements of the rotor or wing can be expressed as

\[
\{x\} = [C]^{-1} [D] \{e\} + [C]^{-1} \{F\} - [C]^{-1} [A] \{\ddot{x}\} - [C]^{-1} [B] \{\dot{x}\}
\]

(B.6)

Substituting Eqs. B.5 and B.6 into Eq. B.3, the total stress becomes

\[
\{\sigma_p\} = \{\sigma_p\}_{\text{static}} - [A_p] [C]^{-1} [A] \{\ddot{x}\} - [A_p] [C]^{-1} [B] \{\dot{x}\}
\]

(B.7)

This result gives the stress in the rotor or wing at any instant in terms of the static stress and an additional stress due to the vibration.