A TIME-DEPENDENT DIFFUSION-CONVECTION MODEL FOR THE LONG TERM MODULATION OF COSMIC RAYS

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ABSTRACT

Incorporation of the effects of time-dependent diffusive propagation of galactic cosmic rays inside a modulating region whose basic parameters are slowly changing in time leads to a new prediction for the modulated density U(t) expected to be observed at a given time t. A first order perturbation analysis shows that if $U_s(t_o)$ is the expected density under completely stationary conditions at time t_o, then the actual density under slowly varying conditions will be given by

$$U(t_o) = U_x[t_o - \tau\ (K)]$$

where $\tau(K)$ is the average time spent by a particle of diffusion coefficient K inside the modulation region. An analysis of the behavior of τ as a function of various modulating parameters in both an idealized one dimensional convective wind and a three dimensional radial wind shows that τ can be greater than 100 days under reasonable values of these parameters. The general behavior is such that in a modulating region characterized by the distance to the modulating boundary R, the convective velocity V, and K, the average time τ is proportional to R^2/K in the limit of large K and R/V in the limit of small K for both geometries. This general behavior is not

appreciably affected by energy loss processes. Since τ is a function of K which is in turn a function of magnetic rigidity R and velocity β this model provides a natural physical explanation for observed rigidity dependent phase lags in modulated spectra sometimes referred to as cosmic ray "hysteresis". If all of the phase lag observed between 500 MeV protons and the Deep River Neutron intensity is attributed to the effects described here, the average distance to the modulating boundary during the last solar cycle is estimated to be 45 - 55 a.u.

I. INTRODUCTION

The diffusion-convection model, first introduced by Parker (1958, 1963) to explain the 11 year modulation of galactic cosmic rays is based on the assumption of time stationary interplanetary conditions. In the basic model, and essentially all variations of this model discussed to date, the solution for the modulated spectra at a particular epoch of the solar cycle is a function of the specific values assigned to the parameters characterizing the interplanetary medium (i.e. solar wind velocity, diffusion coefficient, size of modulation cavity, etc) during that epoch (Jokipii, 1971, Fisk, 1971). The observed time variations in intensity and spectral shape over the whole solar cycle are then explained in terms of a gradual variation in these parameters. Such "quasi-steady" solutions have met with considerable success in explaining the observed time variations although some models have had to introduce parameters beyond those whose physical significance is implicit in the simple diffusion convection picture in order to fit all the observations as is discussed in some detail by Rygg, O'Gallagher, and Earl (1974). Only Parker (1965) has considered the time dependent propagation problem in modulation in any detail and he did not consider the effects on the expected modulated spectrum. Simpson (1964) considered the effect of changes in the modulating region which originate at the sun and are convected outwards but did not consider the diffusive propagation of the particles themselves.

In this paper, a model is developed which incorporates to first order, the direct effects of the time dependent diffusive propagation of interstellar cosmic rays in a slowly changing interplanetary medium. Some concepts basic to this model but limited to a one dimensional convective region were described in a preliminary report (O'Gallagher 1973). Here, these concepts are developed more fully and extended to consideration of a three dimensional radial convective region. The model shows clearly that the effects of time dependent diffusive propagation can be quite significant. Furthermore the model predicts a rigidity dependent time delay or "lag" in the modulated spectra and as such may provide a natural, physically reasonable explanation for the so called "hysteresis effect". (Simpson, 1964; Balasubrahmanyan et al. 1968; Kane and Winckler 1969; O'Gallagher, 1969; Simpson and Wang, 1971, Rygg, et al., 1974). A bonus of this model is that observed hysteresis effects, when interpreted in terms of the model, provide a direct measure of the dynamical features of the modulating cavity which cannot be inferred from time-stationary models. The model is conceptually simple and it is not necessary to introduce any parameters beyond those implicit in the time-stationary model.

The model is best introduced in terms of the usual Fokker-Planck equation for diffusive particle transport in the interplanetary medium.

$$\frac{\partial U}{\partial t} = - \vec{\nabla} \cdot \left[\vec{V}U - K\vec{\nabla}U - \frac{\vec{V}}{3} \frac{\partial}{\partial T} (\alpha TU) \right] - \frac{V}{3} \frac{\partial^2}{\partial r \partial T} (\alpha TU) \qquad (1)$$

Here $U = U(r,t,T)$ is the particle density (of a particular species) with kinetic energy between T and $T + dT$, at heliocentric radius r and time t.

$K = K(\beta,R,r,t)$ is the effective interplanetary diffusion coefficient (which is here assumed to be a scalar, i.e. isotropic diffusion) at r and t as a function of the particle velocity β and magnetic rigidity R.

$\vec{V}$ is the solar wind velocity.

$\alpha = \frac{T + 2T_o}{T + T_o}$ is a factor which compensates for the transition between relativistic and non-relativistic energy regions.

The terms involving derivatives with respect to T on the right hand side of equation 1 incorporate the so-called Compton-Getting effect (Gleeson and Axford, 1968 b). These terms result from the transformation of the propagation equations from a frame moving with the radially diverging solar wind to a frame stationary in the solar system and in effect account for the effects of adiabatic energy losses in the expanding diffusing medium on the density spectrum observed in this stationary frame.

The conventional treatment of equation 1 is to argue that in the long term modulation, changes in U with time are so small that $\partial U/\partial t \approx 0$ and the right hand side can be set identically equal to zero to obtain a solution. The original solution of Parker (1963, 1965) has been modified to obtain approximate solutions including the effects of energy loss explicitly (Fisk and Axford 1969, Gleeson and Axford, 1968) and sophisticated computer methods have been developed to obtain numerical solutions under a wide variety of assumptions in the interplanetary medium (Fisk, 1971, Lezniak and Webber, 1971, Urch and Gleeson 1972). In all of these cases the solutions have been obtained under the assumption of stationary conditions. As we shall see, this assumption may not be strictly valid for some reasonable values of interplanetary medium parameters.

II. ONE DIMENSIONAL MODEL

To incorporate time dependent diffusive effects into the solution for modulated spectra even to first order it is necessary in principle to solve equation (1) as it stands. Since the general solution of equation (1) cannot be obtained in closed form and even in special cases the solution is quite complicated it is most instructive to introduce the basic concepts in the context of a one dimensional limit. In this case the analog of equation (1) is

$$\frac{\partial U}{\partial t}(x,t,T) = K\,\frac{\partial^2 U}{\partial x^2} - V\,\frac{\partial U}{\partial x} \tag{2}$$

where x is the (one) spatial dimension along which particles propagate and K and V have been assumed independent of position for purposes of the model.

It is important to realize in considering the time-dependent modulation problem, both here in the one dimensional case and later in three dimensions, that there are two distinct time-dependent processes involved; a) the time dependent diffusive propagation of the cosmic ray particles themselves and b) the time variations in the medium parameters (K and V) and boundary conditions. In the actual modulation dynamics, the effects of these processes are coupled inseparably. The conventional stationary solutions approximate a solution by ignoring the first process (a) entirely. The model introduced here provides a better approximation by 1) holding the medium parameters and boundary conditions constant and using equation (2) to determine the time scale of the diffusive propagation process under a particular set of stationary conditions and 2) incorporating as a first order correction the effect of cosmic ray propagation with a non-zero diffusion time in a medium whose parameters are slowly changing. Implicitly, this approach assumes separability and neglects the coupling between processes a) and b) above. For instance, taking K and V independent of both time and position is clearly an approximation since, strictly speaking, any time variation in K and/or V will propagate through the medium with velocity V producing a position variation at a given time. However since such changes are assumed to be slow in the formal analysis, the effects which would be produced by this coupling are small and of higher order. For instance, changes in the diffusive propagation time during propagation, or energy loss processes due to differential variation of the solar wind velocity are second order effects and neglected in this model. In effect then, K and V in equation (2) are parameters in a simple model in which the diffusion and convection processes are each represented by a single quantity at a given time which is to be regarded as a characteristic or average value throughout the modulation region. Although not usually discussed in detail, this interpretation of K and V is exactly the interpretation given to modulation parameters in virtually all of the usual stationary treatments.

In the one dimensional case, since the volume element of the diffusive medium is not expanding, individual particle energies remain unaffected and the energy charge terms do not appear. Thus equation (2) is similar to the "classical" diffusion-convection modulation equation neglecting adiabatic energy loss effects (Parker, 1963). This one dimensional analog of solar modulation is represented schematically in Figure 1. A diffusive medium is convected past an observer at $x = 0$ with velocity V to a distance X where there is a boundary beyond which particles move freely without scattering. For purposes of the model the solution to equation (2) is well represented in terms of the distribution function P(t), which is the probability per unit time for finding a particle (A) which crosses the boundary ($x = X$) at $t = 0$, inside the boundary (at $x<X$) between $t(>0)$ and $t+dt$. Parker (1965) has analyzed this problem in some detail and his solution for P at the observer ($x = 0$) can be expressed with a slight change in notation as

$$P(X,V,K,t) = \frac{X}{(4\pi Kt)^{\frac{1}{2}}t} \; e^{-\frac{(Vt+X)^2}{4Kt}} \tag{3}$$

Equation (3) is essentially the Green's function solution for equation (2). In analogy to the real physical modulation problem, the particle density at and beyond the boundary is taken to be a constant, U_o, for all time. Then the density observed at $x = 0$ at a given time t_o is

$$U(0,t_o) = \int_{-\infty}^{t_o} U_o \, P[X,(t_o-t),V,K] \, dt \tag{4}$$

Strictly speaking, equation (4) is valid only in the limit that X, V and K are all constant with respect to time in which case integration of the right hand side of equation (4) (see Appendix B) yields the familiar stationary solution

$$U_s(X,V,K) = U_o e^{-\frac{VX}{K}} \tag{5}$$

If in fact K, X and V are slowly varying in time, then Equation (4) together with equation (3) provides the basis for incorporating the effects of these variations. The effect of variations in K, X and V are coupled to the rate at which changes are occuring, so a perturbation approach has been carried out. The detailed procedure followed is described in Appendix A and the result is discussed in some detail below. However at this point it is appropriate to emphasize the physical basis for the effect produced. This can be seen very simply from the distribution of diffusive propagation times from equation 3 plotted in Figure 2 for 4 different sets of interplanetary (one-dimensional) conditions. The distributions are plotted as $P(t_o-t)$, a function of the time in the "past" at which a particle seen now (at $t=t_o$) crossed the boundary. As a means of characterizing the time scales of the distributions, the time delay, t_m, for which the distribution is a maximum is indicated for each case. The fundamental point is that the characteristic time scales depends on K (as well as X and V) which is a function of R and β. This implies that an observed spectrum of particles having a range of R and β will have sampled the characteristics of the interplanetary medium over a range of time scales in the past. As these characteristics change slowly this produces a rigidity dependent delay in the modulation at some rigidities with respect to other rigidities.

A particular example of the dependence of the time scale on K is shown by curves (a) and (b) in Figure 2, which have the same values of $V(V_a = V_b = 300$ km/sec) and $X(X_a = X_b = 40$ a.u.) but different $K(K_b = 2K_a = 4 \times 10^{21}$ cm^2/sec). Note first that the most probable time t_m is greater than 100 days in both cases for values of X and V which are not untypical of those in conventional models and a value of K appropriate to $\sim$400 MeV protons based on observed magnetic field power spectra (Jokipii and Coleman 1968). Furthermore note that a difference of a factor of 2 in K corresponds to a difference of more than 40 days in the most probable delay time during which the characteristics of the medium may change slightly and therefore require a correction to the predicted spectrum. In the formal analysis, the first order effect of such corrections will depend on the average time τ which particle spends in the diffusive medium, rather than the most probably time t_m, but the basic behavior of the time scale is well illustrated by t_m.

The parameters X and V of course do not depend on particle parameters and so do not directly affect the observed spectrum in the same way that K does. However they do affect the relevant time scales in a very direct way. Differences in X are illustrated by curves b) and d) which have the same V ($V_b = V_d = 300$ km/sec) and $K(K_b = K_d = 4 \times 10^{21}$ cm^2/sec) but have different $X(X_c = X_b/2 = 20$ a.u.). Particles from the boundary reach the observer much faster in a smaller region so that the characteristic time scale is strongly dependent on X. Differences in V are illustrated by curves (c) and (d) which have the same $X(X_c = X_d = 20$ a.u.) and $K(K_c = K_d = 4 \times 10^{21}$ cm^2/sec) but have different $V(V_c = 2V_d = 600$ km/sec). The larger value of V in case c) causes the exponential decrease of P(t) $\left(\propto e^{-V^2 t/K}\right)$ to set in earlier and shifts the most probable time to <u>shorter</u> values of $(t_o - t)$. The quantitative dependence of the time scales on all three parameters will be evaluated in §IV.

All of the curves in Figure 2 are normalized so that the integral of $P(t_o-t)$ for all $t < t_o$ is equal to the probability that the particle actually arrives at $x = 0$ without being convected back out beyond $x = X$, or simply $e^{-VX/K}$. Note in particular that curves b and c have the same value of VX/K although the differences in the individual parameters give rise to rather different time scales.

Although the function P(X,V,K,t) given by equation (3) and plotted in Figure 2 serves to illustrate the dependence of the time scales on X,V, and K, it is not a completely valid one dimensional physical analog of modulation in that the diffusive-convective medium is assumed to be infinite in extent on the -x direction and in certain limits this leads to non-physical results so that the diffusing medium must be bounded. For simplicity in this model, the effects of diffusion-convection in a finite region, assumed to be roughly symmetric, can be well approximated formally by assuming the distribution is characterized by a scale length equal to X. If the origin is redefined as the center of such a symmetric region then the resultant behavior at the origin is similar to that given by equation 3 but modified by the exponential decay factor e^{-Kt/X^2} resulting from the gradual escape from the finite region yeilding the approximate expression for P(t)

$$P(X,x<<X,V,K,t) \cong \frac{X}{(4\pi K)^{1/2}}\, t^{-3/2} \;\; \exp\left[\frac{(Vt + X)^2}{4Kt} - \frac{Kt}{X^2}\right] \qquad (6)$$

The effect of such a gradual escape is illustrated in Figure 2 by the dashed lines where we now have the additional requirement that the position of the observer at X must be such that $x<<X$. The exact solution of equation (2) for a finite region with free escape boundaries at $x = \pm X$ is derived in an Appendix (c) and yields a more complex expression for P(X,x,V,K,t) given by

$$P(X,x,V,K,t) = \frac{2\pi K}{X^2} e^{-\frac{V(X-x)}{2K}} \sum_{n=0}^{\infty} (n+\tfrac{1}{2})(-1)^n \cos \frac{(n+\frac{1}{2})\pi x}{X} e^{-\left[\frac{(n+\frac{1}{2})^2\pi^2 Kt}{X^2} + \frac{V^2 t}{4K}\right]} \tag{7}$$

but the basic physical concepts and characteristic time scales corresponding to equation (7) are essentially identical to those for equation (6) as will be shown.

For the perturbation analysis it is most convenient to characterize the propagation delay by the average propagation time defined as follows:

$$\tau_x = \frac{\int_0^\infty tP(X,V,K,t)dt}{\int_0^\infty P(X,V,V,Kt)dt} \tag{8}$$

In terms of this parameter it is shown in Appendix A that the solution for the modulated density $U(t_o)$ corrected to first order for variations in modulating parameters is

$$U(t_o) = U_s(t_o) - \tau \frac{d}{dt} [U_s(t_o)] \Big|_{t = to} + \cdots \tag{9}$$

were $U_s(t_o)$ is the solution obtained under stationary conditions using modulating parameters evaluated at $t = t_o$. Furthermore if dU_s/dt is roughly constant over a time τ the results of the analysis can be written in the simple and physically sensible form

$$U(t_o) \cong U_s [t_o - \tau(K)] \tag{10}$$

where we have explicitly noted that τ is a function of K and therefore R and β. At large rigidities, K becomes large and $\tau(K)$ is small so that the stationary solution is quite accurate. At lower rigidities τ can range from a few days to several months with reasonable parameters in this one dimensional model so that equation (10) provides a completely natural qualitative explanation for what is commonly referred to as the "hysteresis effect".

The correction for these time varying effects is thus reduced to a determination of $\tau(K)$ which can be accomplished relatively simply for one dimensional propagation using equations (6) or (7) and (8). Discussion of the detailed results of this determination are deferred to §IV where the three dimensional problem can be discussed at the same time.

III. A THREE DIMENSIONAL RADIAL WIND

Although the mathematical formulation involved for modulation in a three dimensional solar cavity is considerably more complicated, the basic physical concepts and qualitative solutions are identical to those introduced in the one dimensional treatment. If one assumes isotropic diffusion in a sperically symmetric, homogeneous interplanetary medium, equation (1) takes the form

$$\frac{\partial U}{\partial t} = + \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) - \frac{V}{r^2}\frac{\partial}{\partial r}(r^2 U) + \frac{2V}{3r}\frac{\partial}{\partial T}(\alpha T U) \qquad (11)$$

A similar problem has been solved analytically for a special case of anisotropic diffusion subject to boundary and initial conditions appropriate for solar flare particles (Lupton and Stone,1973) but equation (11) has not been solved analytically for the general case with boundary conditions appropriate for solar modulation.

There are two aspects in which equation (11) differs from equation (2) and which make its solution more difficult. These are

a) The three dimensional diffusive propagation described by the first two terms on the right hand side

b) The adiabatic energy losses described by the last term on the right. Since the diffusive process depends on K which in turn depends on energy, these two processes are coupled in a way that precludes analytic solution. Therefore it is necessary to analyze the spatial propagation process separately from the energy loss process. This is accomplished by assuming that the last term in equation (11) does not appreciably affect K in the analyses of the spatial transport problem. This of course is not true at all energies and the effects of the breakdown of this assumption on the results obtained will be considered in Section IV.

Using notation similar to that in Section II the Green's function solution to equation (11) for the spatial probability density P_r in a radial wind for finding a particle which crosses a modulating boundary at $r = R$ at $t = 0$ and diffuses with constant K inside the boundary at $r < R$ and $t > 0$ has been shown in a slightly modified notation by Parker (1965) to be of the form.

$$P_r(R,r,K,V,t) = \sum_{n=1}^{\infty} a_n Q(w_n, \frac{Vr}{K}) \; e^{\frac{-w_n V^2 t}{K}} \qquad (12)$$

where $Q(w_n, Vr/K)$ are particular solutions chosen such that $P(R,R,V,K,t>0) = 0$ and the a_n are constants chosen to satisfy the initial conditions. This is analogous to the one dimensional solution for $P(X,x,V,K,t)$ given by equation (7). Assuming that the series for P_r converges the conclusions are the same as for the one dimensional case. That is, one defines an average propagation time as

$$\tau_r = \frac{\int_0^\infty t P(R,V,K,r,t)\, dt}{\int_0^\infty P(R,V,K,r,t)\, dt} \qquad (13)$$

and the perturbation analysis in Appendix A leads to

$$U(r,t_o) = U_s(r,t_o) - \tau \frac{\partial}{\partial t} [U_s(r,t_o)] \Big|_{t = to} + \ldots \tag{14}$$

or if $\frac{dU_s}{d\tau} \cong$ constant.

$$U(r,t_o) \cong U_s[r,t_o - \tau_r] \tag{15}$$

The requirement that dU_s/dt be roughly constant means that equation (15) will not be valid for a time interval τ near the reversal in phase of the modulation when higher order effects will need to be included. But for the long periods between solar maximum and solar minimum this is a general result which applies to all stationary models including the most sophisticated computer derived solutions which include the effects of energy loss. As with the one dimensional case the problem reduces to finding τ but here the analysis is complicated both by the radial geometry and energy loss effects since K is not strictly constant through the propagation. Clearly however,

under normal assumptions of spherical symmetry and modulation in a radial wind one must expect non-negligible diffusive propagation times which depend on K and therefore R and β. If these times are sufficiently long they will produce "phase lag" effect on the observed spectrum given by equation (15), which is qualitatively similar to the observed so called (hysteresis effect". In the following sections, the evaluation of $\tau(K)$ in both the one and three dimensional models is discussed in some detail.

IV. EVALUATION OF AVERAGE DIFFUSIVE PROPAGATION TIMES

The behavior of the average propagation time τ_x in one dimension can be determined in a straight forward manner for a given P(t) using equation (8). Consider first the behavior including the effects of escape from a finite region as approximated by equation (6). After substitution in equation (8) both the numerator and denominator can be integrated as described in Appendix B yielding

$$\tau_x(X,V,K) = \left\{\frac{X^2/K}{V^2/K + 4K/X^2}\right\}^{1/2} \tag{16}$$

This function is plotted in Figure 3 to illustrate the predicted dependence of τ_x on K for a fixed convective velocity V = 300 km/sec and a range of values of X. Next consider the average time delay based on the exact solution given by equation (7) as described in Appendix C for one dimensional diffusion convection in a symmetric bounded region for an observer near the origin. Substitution of P(t) from equation (7) into equation (8) and integrating yields

$$\tau_x(X,x<<X,V,K) = \frac{X^2}{K}\;\frac{\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\;(-1)^n\;[(n+\tfrac{1}{2})^2\;\pi^2 + (\frac{VX}{2K})^2]^{-2}}{\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\;(-1)^n\;[(n+\tfrac{1}{2})^2\;\pi^2 + (\frac{VX}{2K})^2]^{-1}} \tag{17}$$

Both the numerator and denominator of equation (17) are alternating series which converge. Values of τ_x calculated for the first 50 terms in each series are indicated in Figure 3 by x's. The agreement between the exact solution and that based on approximate representation is extremely good. Therefore for all intents and purposes the analytical form given by equation (16) can be used to calculate τ_x for any desired values of X,V, and K rather than the much more cumberson representation given by equation (17).

There are two limiting cases of special interest; Case a) the diffusion limit ($V \rightarrow 0$) and Case b) the convection limit ($K \rightarrow 0$). Both equations (16) and (17) give identical results in these limits. In particular if we define τ_D as the propagation time in the diffusion limit and τ_C as the time in the convection limit we find

$$\tau_D(X,K,V\rightarrow 0) = \frac{X^2}{2K} \tag{18}$$

and

$$\tau_C(X,K\rightarrow 0,V) = \frac{X}{V} \quad . \tag{19}$$

This latter result, which is surprisingly simple is a consequence of the fact already mentioned that as V becomes large compared to 2K/X the exponential decrease with time for P(X,V,K,t) in equation (3) prevents there being any substantial contribution to the integral in equation (8) from times much longer than X/V. Looked at more physically the result expressed in equation (19) is a manifestation of the fact that although smaller diffusion coefficients will decrease the diffusive propagation speed (and this might be thought to increase the time) they also decrease the probability that a given particle will reach the observer at $x = 0$ before being convected back across the boundary. Since we are interested only in those fraction ($e^{-VX/K}$) of the original particles which are observed at $x = 0$, these processes set an upper limit on the average time that such an observed particle can spend inside the boundary. This upper limit as indicated by equation (19) is simply equivalent to the time required to convect a given element of the diffusive medium through the distance from the observer to the boundary of the medium.

Note finally that equation (16) which provides an exceedingly good approximation to the exact solution can be written simply as

$$\tau = \left[\frac{1}{\tau_D{}^2} + \frac{1}{\tau_C{}^2}\right]^{-1/2} \tag{20}$$

The average propagation time τ_r in a 3 dimensional, radial convective medium cannot be evaluated for all K and V as was possible in one dimension. Therefore for purposes of this paper, to illustrate the physical validity of the model and to provide a quantitative approximation to the expected behavior we have considered the two limiting cases discussed in the one dimensional propagation mode above. In particular in the diffusion limit of Case a) it is shown in Appendix C that τ_r is given by

$$\tau_{D_r}(R,r,V\to 0,K) = \frac{R^2}{6K} \tag{21}$$

In deriving equation (21) the additional condition that $r \ll R$ has been imposed so that this result will be a good approximation at the orbit of earth only for $R > 5$-10 a.u. In the convection limit (Case b), it is shown in Appendix C that the propagation equation (11) approaches the form of the one dimensional propagation equation in the limit $K \ll Vr$. Therefore from equation (16) or (17) we find

$$\begin{aligned} \tau_{D_r}(R,r,V,K\to 0) &= \frac{R-r}{V} \\ &\cong \frac{R}{V} \end{aligned} \tag{22}$$

which is again an upper limit equal to the characteristic convection time. Not only in the stated limit is equation (22) a very accurate approximation to the three dimensional behavior but since τ is based on an average over all $r < R$, it will be a good approximation as long as $K \ll VR$. Thus equations (21) and (22) represent valid approximate solutions for τ_r in the limits of cases (a) and (b) analogous to the cases for the one dimensional mode above.

To approximate the behavior of τ_r in the transition region between the two limiting cases we have used a form identical to that given by equation (20) which has been shown in the one dimensional case to be virtually identical to that given by the exact solution. The approximate behavior of $\tau_r(V,R,K)$ is plotted in Figure 4 as

$$\tau_r(R,V,K) = \left[\frac{1}{\tau_{Dr}^2} + \frac{1}{\tau_{Cr}^2}\right]^{-1/2} = \left\{\frac{R^2/K}{V^2/K + 36K/R^2}\right\}^{1/2} \qquad (23)$$

which gives precisely the correct behavior in the limiting regions (solid lines) and smoothly connects these limits through a transistion region (dashed lines) where $Vr < K < VR$ where it should be a good approximation. From equation (23) it is apparent that the characteristic break between the two limiting regimes occurs at $K \sim \frac{VR}{6}$. The above discussion shows that the physical concepts illustrated in the one dimensional model are directly applicable to propagation in a radial wind with only small changes in the quantitative behavior to be expected. Again we have an R^2/K dependence of τ at large K. As K becomes smaller this increases until it reaches the limit set by the convection time. Delays greater than 100 days are seen to be predicted for values of R, V and K which are not unreasonable and in fact are commonly used in other stationary solutions of the modulation problem. Clearly the effect of these delays cannot be neglected at low rigidities (small K) and Figure 4 and equation (15) provides a direct way to incorporate these corrections into any stationary model.

Finally we consider again the effect of energy loss processes from the last term in equation (11). Effectively what we have done in obtaining the results in Figure 4 is to determine the approximate propagation time τ_r as if a given particle diffused in from r = R while maintaining the same value of K throughout the process. Clearly if the particle is losing energy during the process and K is in some way dependent on energy this assumption cannot be completely accurate. But it is easily shown that the inclusion of such effects cannot have any appreciable effect on the behavior of $\tau_r(K)$ as approximated by the curves in Figure 4. Consider first that the energy loss process is also characterized by a time scale τ_{loss} given by Parker (1965) as

$$\tau_{loss} = \frac{3R}{2\alpha(T)V} \approx \frac{R}{V} \tag{24}$$

to a good approximation. This time is defined such that the energy E(t) of a particle which crosses the boundary at t = 0 with E(0) = E_o will be

$$E(t) = E_o \exp(-t/\tau_{loss}) \tag{25}$$

Note that $\tau_{loss} = \tau_r(R,V,K << VR/2)$ which is the upper limit on the propagation time. Thus, effectively by definition, the delay time in the large K limit will be a fraction of the characteristic energy loss time so that energy loss effects will be small and the basic assumption of constant K remains valid. At low energies since K is a decreasing function of kinetic energy (Jokipii and Coleman, 1965) down to at least a few tens of MeV the propagation delay will increase and approach the characteristic loss time τ_{loss}. In this limit, energy losses are increasingly important but the propagation delay τ_r is constant independent of K. Thus it will remain unaffected by the energy loss processes and the approximate behavior in Figure 4 will not be modified appreciably.

This can be seen more formally by representing the average effect of the energy loss process in terms of the propagation of a particle whose diffusion co-efficient is a monotonically decreasing function of time during diffusion. Specifically consider the behavior of a particle which crosses the boundary at $r = R$ at $t = 0$ with $K = K_o$ and is observed at $r<R$ at $t>0$ with $K = K_r$ after undergoing propagation with a diffusion co-efficient $K_p(t)$ such that $K_o \geq K_p(t) \leq K_r$ at all times. First compare the average time τ_r for two idealized cases:

a) Propagation with constant diffusion coefficient $K_p = K_o$ so that τ_r is given by equations (21-23) and

b) Propagation with $K_p(t) = K_o$ for $t<t_1$ and $K_p(t) = K_1 (K_1<K_o)$ for $t \geq t_1$. The behavior in the two cases is identical for $t<t_1$. In particular at $t = t_1$ the expected intermediate distribution in position r' $P(R,r',K_o,V,t_1)$ is identical. Furthermore a lower limit for the distribution in propagation times from any r' to the observer at r is given by $P(r',r,K_p,v,t-t_1)$. Thus a lower limit to the average time from any r' to r is given by equations (21-23) (with substitution of r' for R and $t-t_1$ for t). Substitution of $K_1<K_o$ into equations (21-23) shows that the average propagation time from r' to r is greater in Case (b) than in Case (a) for all r'. From this it follows that $\tau(K_b) > \tau(K_a)$ where $K_b < K_a$ for any time during the diffusion process. Thus for monotorically decreasing $K_p(\tau)$ with time we

conclude that in general $\tau[K_p(t) \leq K_o] \geq \tau(K_o)$. Conversely if $K_p(t) \geq K_r$ it follows that $\tau[K_p(t) \geq K_r] \leq \tau(K_r)$. If we apply these inequalities to particles which enter a modulating region with a spectrum in K_o and are observed with a spectrum in K_r we have the following cases of interest:

1) $K_r >> \frac{VR}{6}$. In this case we apply equation (21) for K_r and find that $\tau_r << \tau_{loss}$. Thus we know that $K_r \cong K_o \simeq K_p$ = constant so that $\tau \approx R^2/6K_r$ is accurate.

2) $K_o << VR/6$. In this case $K_r<K_o$ we apply equation (22) and find that $\tau(K_o) \approx \frac{R}{V} \approx \tau(K_r)$ and by the inequalities above we must have $\tau(Kp) \approx \frac{R}{V}$.

Thus $\tau = \tau(K_r)$ as before is accurate in this limit also.

3) $K_o \sim VR/6$ and $K_r \sim VR/6$ (with of course $K_r < K_o$). Application of equation (20) and the inequalities above in this transition region yields $\frac{R}{V} \gtrsim \tau(Kp) \lesssim \frac{R}{\sqrt{2V}}$ so that $\tau = \tau(K_r)$ is a reasonable approximation.

In all three cases the propagation delays obtained for a given K_r without considering the effects of energy loss are not appreciably altered by inclusion of these effects. It must be emphasized however that an accurate application of the entire model should include energy loss effects in calculating the stationary solutions in equation (13).

V. SUMMARY AND DISCUSSION

In the preceding sections of this paper, the effect of time dependent diffusive propagation in a medium whose characteristics are slowly changing, has been examined for the first time. These effects are found to be appreciable for some reasonable values of modulation parameters. A model which incorporates these effects as a first order perturbation on the usual time independent solution has been developed. The model predicts a "phase lag" between the response of cosmic ray intensities at different rigidities to changes in modulation parameters which is strikingly similar to observed hysteresis effects. In brief the most important conclusions of this analysis as they apply to modulation in a radial convective region are as follows:

(1) The effects of time dependent diffusion propagation can be formally analyzed in terms of the average time τ_r spent by a particle inside the modulation region.

(2) This average time is a function of the particle diffusion coefficient K.

(3) The solution for modulated density of cosmic rays observed at time t corrected to first order for variations in modulation parameters is

$$U(t) = U_s(t - \tau_r(K))$$

where U_s is the density predicted by the stationary solution for K.

(4) In the limit of large diffusion coefficients $\tau_r(K)$ is inversely proportional to K and directly proportional to R^2.

(5) In the limit of small diffusion coefficients the propagation time becomes equal to R/V, which is an upper limit on τ_R independent of K as long as K << VR/6.

(6) Propagation times the order of 100 days or more are consistent with reasonable values of V,R, and K.

(7) This behavior is not appreciably affected by the action of adiabatic energy loss processes.

In recent years much effort has been spent in developing modulation models which can explain the so called "hysteresis effects". Considerable success has been achieved by such authors as Van Hollebeke et al. (1972, 1973), Burger and Swannenburg (1973) Bedijn et al (1973) by introducing changes in the rigidity dependence of K or incorporating rigidity dependence into other time variables parameters such as R. Processes which could be described by such phenomenological models may in fact be operative but there is as yet no independent evidence (other than hysteresis) that they are important as pointed out by Rygg, et al (1974).

Finally, we consider a simple example of the application of this model to the interpretation of an observed "hysteresis effect" between the intensities at two different energies. To illustrate the essential simplicity of the model and its basic features we have restricted both energies under consideration to the region T $\gtrsim$ 300 MeV/nucleon where energy loss effects are not severe and the simple solution of Parker (1963, 1965) remains an excellent approximation. Therefore, we can write the stationary solution for the density of a given species characterised by magnetic rigidity R and velocity $c\beta$ at a given time t simply as

$$U(R,\beta,t) = U_o \; e^{-\eta(t)/f(R,\beta)} \tag{26}$$

under the assumption that $K(R,\beta,r,t) = K_o(r,t)f(R,\beta)$, where K_o is the diffusion coefficient at some reference value of R and β. Here $\eta(t)$ incorporates all the time variations of the modulating parameters and in the particular case that K_o is independent of r, we have $\eta(t) = V(t)R(t)/K_o(t)$. So long as this assumption of separability holds it follows directly for all R and β for which equation (26) is valid that

$$\ln \left[\frac{U_1(R_1,\beta_1,t)}{U_1(R_1,\beta_1,t_o)}\right] = A\ln \left[\frac{U_2(R_2,\beta_2,t)}{U_2(R_2,\beta_2,t)}\right] \tag{27}$$

where $A = f(R_2,\beta_2)/f(R_1,\beta_1)$ is a constant, independent of time. From Equation 27 we see that under these conditions one expects U_1 to be a single valued function of U_2.

In Figure 5, the intensity of protons from 260 to 720 MeV (average energy $\sim$ 500 MeV) observed on a series of balloon flights of the same instrument between 1965 and 1972 (Rygg and Earl, 1971, Rygg, O'Gallagher and Earl, 1974) is plotted versus the Deep River neutron intensity which monitors cosmic rays of an average energy $\sim$ 10 GeV. The observations are not consistent with the expected single valued relation illustrated by the solid curve in the figure which is determined from equation (27) with the parameters adjusted so that the curve lies approximately half way between the points before solar maximum (solid) and those after (open). The effect of a true time lag between the intensities at these two energies has been calculated assuming (1) the time variation between maximum and minimum in the Deep River intensity is sinusoidal with an eleven-year period and

(2) the stationary solution at any time for the 500 MeV proton intensity corresponds to the solid curve in Figure 5. Thus the expected proton intensity for a given neutron intensity including the hysteresis lag given by equation (15) is found from the point in the solid curve corresponding to the Deep River intensity at a time τ earlier under assumption (1). Such "loops" are shown superposed on the data in Figure 5 for values of τ = 90, 180, 270 days. The separation between the rising and falling portions of the loops can be interpreted as being produced by a value of τ between 180 and 270 days. Of course, this will be so only if there has been no time variation in $f(R,\beta)$ some of which may of course occur (and in fact it is just such variations which are proposed to explain hysteresis in all stationary models). If we interpret τ strictly in terms of this model neglecting all other sources of "hysteresis" and higher order effects (variations in τ over the solar cycle, etc.) discussed elsewhere, it yields an estimate of R. With V = 300 Kilometers/second and K(500 MeV protons) $\approx 4 \times 10^{21}$ cm^2/sec (Jokipii and Coleman, 1968) we find from equation (23) R = 45-55 a.u. which, although somewhat larger than some earlier estimates (O'Gallagher, 1968, Simpson and Wang, 1970) is consistent with current ideas and with the small radial cosmic ray gradients observed to date on Pioneer 10 (Lentz, et al, 1973, Teegarden, et al, 1973, Van Allen 1972). More detailed calculations based on this model, incorporating the effects of energy loss with computer generated solutions for the complete stationary equation have been undertaken and will be reported. The simple example above, however, illustrates the utility of the model.

In contrast to stationary models we see that the model developed here provides a physical basis from which a "hysteresis effect" emerges naturally without the introduction of any new independent modulation parameters. The fact that the phase parameter τ_r is not independent of R, V and K, thus provides a powerful tool for the study of the modulation process itself. Observed values of hysteresis phase lags provide direct measures of R and K which are not attainable in models based on stationary solutions.

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APPENDIX A

We wish to determine the effects of small changes in the modulating parameters X, V, K with time by means of a perturbation analysis of the modulated density expression in the form of Equation 4. To do this we express the probability distribution function, which is implicitly a slowly varying function of time, in terms of its form evaluated at $t = t_o$ which we write as $P_o(t)$ plus a small perturbation

$$P(X,V,K,t) = P_o(t) + \delta P_o(t) + \ldots \qquad \text{A.1}$$

where

$$\delta P_o(t) = \frac{\delta P_o}{\delta t_o} \cdot [t-t_o]$$

$$= \left[\frac{\partial P_o}{\partial x}\frac{\Delta X}{\Delta t_o} + \frac{\partial P_o}{\partial V}\frac{\Delta V}{\Delta t_o} + \frac{\partial P_o}{\partial K}\frac{\Delta K}{\Delta t_o}\right] \cdot [t-t_o] \qquad \text{A.2}$$

in which

$$\frac{\Delta X}{\Delta t}, \frac{\Delta V}{\Delta t} \text{ and } \frac{\Delta K}{\Delta t}$$

are the slow rates of change in each of the modulation parameters. Substituting from equations A.1 and A.2 into Equation (4) we obtain

$$U(t_o) = \int_{-\infty}^{t_o} U_o P_o(t_o - t)\,dt - \int_{-\infty}^{t_o} U_o \cdot [t_o - t]\,\frac{\delta P_o}{\delta t_o}\,(t_o - t)\,dt + \ldots \qquad \text{A.3}$$

The first term on the right hand side is simply by definition the stationary solution $U_s(t=t_o)$ evaluated at $t = t_o$. The second term can be evaluated in this first order perturbation treatment by requiring simply that

$$\frac{1}{P_o}\frac{\delta P_o}{\delta t} << \frac{1}{\tau_x}$$

Physically this is equivalent to the assumption that most of the contribution to the integral takes place on time scales which, although non-zero, remain short compared to the scale of the 11-year modulation. Clearly this is a better assumption than to assume that

$$\frac{1}{P}\frac{\delta P_o}{\delta t_o} \equiv 0$$

which is implicit in all stationary solutions. With this assumption the derivative with respect to t_o in equation A.3 can be taken outside the integral and the second term becomes:

$$\frac{\delta}{\delta t_o}\left| U_o \int_{-\infty}^{t_o} [t_o - t]\ P_o(t_o - t)\,dt \right| = \frac{\delta}{\delta t_o}\left| \tau_x(t_o) \int_{-\infty}^{t_o} U_o P_o(t_o - t)\,dt \right| \qquad \text{A.4}$$

from the definition of τ_x in equation 6. Finally this term can be written as

$$\frac{\delta}{\delta t_o}[\tau_x(t_o) U_s(t_o)] = \tau_x(t_o)\frac{\delta U_s}{\delta t_o}(t_o) + U_s(t_o)\frac{\delta \tau_x(t_o)}{\delta t_o} \qquad \text{A.5}$$

$$\cong \tau_x(t_o)\frac{\delta U_s}{\delta t_o}(t_o)$$

neglecting <u>changes</u> in $\tau_x(t_o)$ with time since these are second order effects. Substituting back into equation A.3 one obtains:

$$U(t_o) \cong U_s(t_o) - \tau_x(t_o)\frac{\delta}{\delta t_o}[U_s(t_o)] \qquad \text{A.6}$$

$$= U_s(t_o - \tau_x)$$

if $\delta U_s / \delta t_o$ is approximately constant over the time τ_x.

Since the exact dependence of $P_o(t)$ on the modulated parameters X, V, K was not used in any of the above treatment, the result given in equation A.6 is quite general and will be valid in the context of any modulation model. In particular with a slight change in notation we can perform a perturbation analysis on the three dimensional model where the general form for the time dependent solution is

$$U(R,V,K,r,t_o) = \int_{-\infty}^{o} U_o P(R,V,K,r,t_o - t)\,dt \qquad \text{A.7}$$

analogous to equation 4 in the one dimensional model and P(R,V,K,r,t) is given by equation 10. Thus equation (12) and (13) in the text follow directly from the above arguments.

APPENDIX B

The behavior of τ_x is given in terms of integrals of the form

$$\int_0^\infty t^{-n/2}\, e^{-\beta/t\ -\gamma t}\, dt$$

where n is an integer and β and γ are coefficients depending on X, V, and K. Definite integrals of this form diverge in general if either β or γ is 0 but otherwise they converge to the solution.

$$\int_0^\infty t^{\nu-1}\, e^{\,-\beta/t\ -\ \gamma t}\, dt = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}) \qquad \text{B.1}$$

(Gradshetyn and Ryzhik, 1965, p. 340) where K_ν is a generalized Bessel function of imaginary argument.

Consider first the normalization integral

$$\int_0^\infty P(t)dt = \frac{X}{(4\pi K)^{\frac{1}{2}}} \cdot \int_0^\infty t^{-3/2}\, e^{-\frac{(Vt+X)^2}{4Kt}}\, dt \quad . \qquad \text{B.2}$$

Equation B.1 with $\nu = -\frac{1}{2}$, $\beta = X^2/4K$, and $\gamma = V^2/4K$ yields

$$\int_0^\infty P(t)dt = \frac{Xe^{-VX/2K}}{(4\pi K)^{\frac{1}{2}}} \cdot 2\left(\frac{X^2}{V^2}\right)^{-\frac{1}{4}} \cdot \sqrt{\frac{\pi K}{XV}}\, e^{-VX/2K} \qquad \text{B.3}$$

$$= e^{-VX/K}$$

Similarly the expression for the average time τ_x spent inside a finite one dimensional region of characteristic scale X from equations (6) and (8) is

$$\tau_x = \frac{\int_0^\infty t^{-\frac{1}{2}} \exp\left[- \left(\frac{V^2}{4K} + \frac{K}{X^2}\right) t - \frac{X^2}{4Kt} \right] dt}{\int_0^\infty t^{-3/2} \exp\left[- \left(\frac{V^2}{4K} + \frac{K}{X^2}\right) t - \frac{X^2}{4Kt} \right] dt} \qquad \text{B. 4}$$

which can be evaluated in terms of equation B.1 with $\beta = X^2/4K$ and $\gamma = (V^2/4K + K/X^2)$ and yields equation (16) in the text directly.

APPENDIX C

The solution for diffusive propagation and escape from a finite convecting region in both one and three dimensions can be analyzed in terms of the classical probability density $w(x,t)$ (or $w(r,t)$) following the notation of Parker (1965). Consider just a finite one dimensional diffusing region with free escape boundaries at $x = \pm X$ and a symmetric convective wind of velocity $|V|$ originating at $x = o$ and directed in the $-x$ direction for $x < o$ and in the $+x$ direction for $x > o$. To be directly analogous to the physical modulation problem, particles can be introduced at both boundaries so that the entire problem is completely symmetric about the origin.

Thus the diffusion convection equation (2) for $w(x,t)$ is,

$$\frac{\partial w(x,t)}{\partial t} = K \frac{\partial^2 w(x,t)}{\partial x^2} - V \frac{\partial w}{\partial x} \qquad \text{C.1}$$

is to be solved in the domain $o < x < X$ subject to the boundary conditions

$$\text{and} \quad \left.\begin{aligned} w(X,t) &= 0 \\ \frac{\partial w}{\partial x}(0,t) &= 0 \end{aligned}\right\} \qquad \text{C.2}$$

Define the variables $s = V^2t/K$ and $\zeta = \frac{Vx}{K}$ and set $w(x,t) = S(s) \cdot L(\zeta)$ so that equation C.1 is separable into

$$S'(s) + \omega S(s) = 0$$

$$P''(\zeta) - P'(\zeta) + \omega P(\zeta) = 0 \qquad \text{C.3}$$

where ω is the separation parameter. The general solutions of C.3 compatible with the boundary conditions are,

$$w(x,t) = \sum_{n=o}^{\infty} C_n \cos(b_n\zeta) \cdot e^{\zeta/2} e^{-\omega_n S} \qquad \text{C.4}$$

with $\omega_n = b_n^2 + \frac{1}{4}$ and $b_n = (n + \frac{1}{2})\pi(\frac{K}{VX})$ to satisfy C.2. The coefficients

Cn are determined from the initial condition

$$w(x,0) = \delta\left[x - (X-h)\right] \qquad \text{C.5}$$

where h is the distance inside the boundary where the first scattering takes place ($h \ll x$) by multiplying C.4 λ(at $t = 0$) and C.5 by $e^{-\zeta/2}\cos(b_n\zeta)$ and integrating from $\zeta = 0$ to $\zeta = VX/K$ yielding

$$C_n = \frac{2}{X} e^{VX/2K} \cos\left[(n + \tfrac{1}{2})\pi \frac{h}{X}\right] \qquad \text{C.6}$$

$$\cong \frac{2}{X} e^{VX/2K} (n + \tfrac{1}{2})\pi \; (-1)^n\left(\frac{h}{X}\right)$$

Parker (1965) has shown that h can be expressed as $h = 4K/v$, where v is the particle velocity so that the complete solution for the probability density $P(x,t) = \frac{v}{4} \cdot w(x,t)$ is given by

$$P(X,x,V,K,t) = \frac{2\pi K}{X^2} e^{-\frac{V(X-x)}{2K}} \sum_{n=0}^{\infty}(n + \tfrac{1}{2})(-1)^n \cos\left[\frac{(n + \frac{1}{2})\pi x}{X}\right] \cdot \exp\left[-\frac{(n + \frac{1}{2})^2\pi^2 Kt}{X^2} - \frac{V^2 t}{4K}\right] \qquad \text{C.7}$$

The solution for the similar problem of radial diffusion-convection in a three dimensional region inside a free escape boundary of radius R has been treated in some detail by Parker (1965) and has the form given by equation (12). However the particular radial solutions Q_n cannot be analytically described as was possible in C.4 above for the one dimensional case. However the behavior in the limits of large and small K are easily treated.

The solution in the diffusion limit ($V \to 0$) is

$$P(R,K,r,t) = \frac{2\pi K}{Rr} \sum_{n=1}^{\infty} (-1)^{n-1}\, n \sin\left(\frac{n\pi r}{R}\right) e^{-\frac{n^2\pi^2 Kt}{R^2}} \qquad \text{C.8}$$

which is given by Parker (1965) in a slightly different notation. Substituting in equation (13) this yields

$$\tau_r(R,K,r) = \frac{2\pi K}{Rr} \sum_{n=1}^{\infty} (-1)^{n-1}\, n \sin\left(\frac{n\pi r}{R}\right) \int_0^{\infty} t\, e^{-\frac{n^2\pi^2 Kt}{R}}\, dt \qquad \text{C.9}$$

After evaluation of the integral on the right hand side, equation C.9 converges to

$$\tau_r(R,K,V \to o) = \frac{R^2}{6K} \qquad \text{C.10}$$

in the limit $r << R$.

Behavior in the limit of small K can be examined in terms of the radial diffusion-convection (equation 11) which when written out explicitly takes the form

$$\frac{\partial U}{\partial t} = K \frac{\partial^2 U}{\partial r^2} + \left(\frac{2K}{r} - V\right) \frac{\partial U}{\partial r} - \frac{V}{r} U \qquad \text{C.12}$$

where the energy loss term has been left out since we are considering only spatial propagation. If we further make use of the condition that

$$\frac{1}{U} \frac{\partial U}{\partial t} << \frac{1}{\tau_r}$$

appropriate to the perturbation analysis in Appendix A the gradient can be approximate by its stationary value $\frac{\partial U}{\partial r} = \frac{VU}{K}$

so that in the limit $K << Vr$ the terms $\frac{2K}{r}(\partial U/\partial r)$ and VU/r are small compared to $V\, \partial U/\partial r$ in equation which then takes the approximate form

$$\frac{\partial U}{\partial t} \approx K \frac{\partial^2 U}{\partial r^2} - \frac{V \partial U}{\partial r} \qquad \text{C.13}$$

which is identical to the one dimensional expression (Equation 2) so that equations 16 through 19 apply with appropriate substitutions.

List of References

Balasubrahmanyan, V. K., Hagge, D. E., and McDonald, F. B., 1968, Canadian J. Phys., 46, S887.

Bedijn, P. J., J. J., and Swannenburg, B. N., 1973, Thirteenth International Cosmic Ray Conference, Conference Papers, 5, 3106.

Burger, J. J., and Swannenburg, B. N., 1973, J. Geophys. Res., 78, 292.

Fisk, L. A., 1971, J. Geophys. Res., 76, 221.

Fisk, L. A., and Axford, W. I., 1969, J. Geophys. Res., 74, 4973.

Gleeson, L, J., and Axford, W. I., 1968 a) Ap. J., 154, 1011.

Gleeson, L. J., and Axford, W. I., 1968 b) Astrophys. and Space Sci., 2, 431.

Gradshteyn, I. S., and Ryzhik, I. M., 1965, Table of Integrals, Series and Products, (New York, Academic Press), page 414.

Jokipii, J. R., 1971, Rev. of Geophys. and Space Phys., 9, 27.

Jokipii, J. R., and Coleman, P. J., Jr., 1968, J. Geophys. Res., 73, 5495.

Kane, S. R., and Winckler, J. R., 1969, J. Geophys. Res., 74, 6247.

Lentz, G. A., McKibben, R. B., O'Gallagher, J. J., Perkins, M., Simpson, J. A., and Tuzzolino, A. J., 1973, Thirteenth International Conference on Cosmic Rays, Conference Papers, 2, 743.

Lezniak, J. A., and Webber, W. R., 1971, J. Geophys. Res., 76, 1605

Lupton, J. E., and Stone, E. C. 1972, J. Geophys. Res., 78, 1007.

O'Gallagher, J. J., 1968, Canadian J. Phys., 46, S946.

O'Gallagher, J. J., 1969, J. Geophys. Res., 74, 43.

O'Gallagher, J. J., 1973, Thirteenth International Cosmic Ray Conference, Conference Papers, 2, 1135.

Parker, E. N., 1958, Phys. Rev., 110, 1445.

Parker, E. N., 1963, Interplanetary Dynamical Processes, (New York, John Wiley).

Parker, E. N., 1965, Planet. and Space Sci., 13, 9.

Rygg, T. A. and Earl, J. A., 1971, J. Geophys. Res., 76, 7445.

Rygg, T. A., O'Gallagher, J. J., and Earl, J. A., 1974, in J. Geophys. Res. (in press).

Simpson, J. A., 1964, Proc. Internat. Conf. on Cosmic Rays, (Jaipur), 2, 155.

Simpson, J. A., and Wang, J. R., 1970, Ap. J., 161, 265.

Teegarden, B. J., McDonald F. B., Trainor, J. H., Roelof, E. C., and Webber, W. R., 1973, Ap. J. (Letters), 185, L155.

Urch, I. H., and Gleeson, L. J., 1972, Ap. Space Sci., 17, 426.

Van Allen, J. A., Ap. J. (Letters), 177, L49.

Van Hollebeke, M. A. I., Wang, J. R., and McDonald, F. B., 1972, J. Geophys. Res., 77, 6881.

Van Hollebeke, M. A. I., Wang, J. R., and McDonald, F. B., 1973, Thirteenth International Conference on Cosmic Rays, Conference Papers, 2, 1298

Figure Captions

Figure 1. Schematic representation of a one dimensional analog of solar modulation. A particle A crosses the boundary ($x = X$) of a diffusive-convective region at $t = 0$ and is detected by an observer at $x = 0$ at a later time t. The problem is analyzed in terms of $P(X,V,K,t)$, the probability per unit time for observing the particle at $x = 0$ between given t and t+dt.

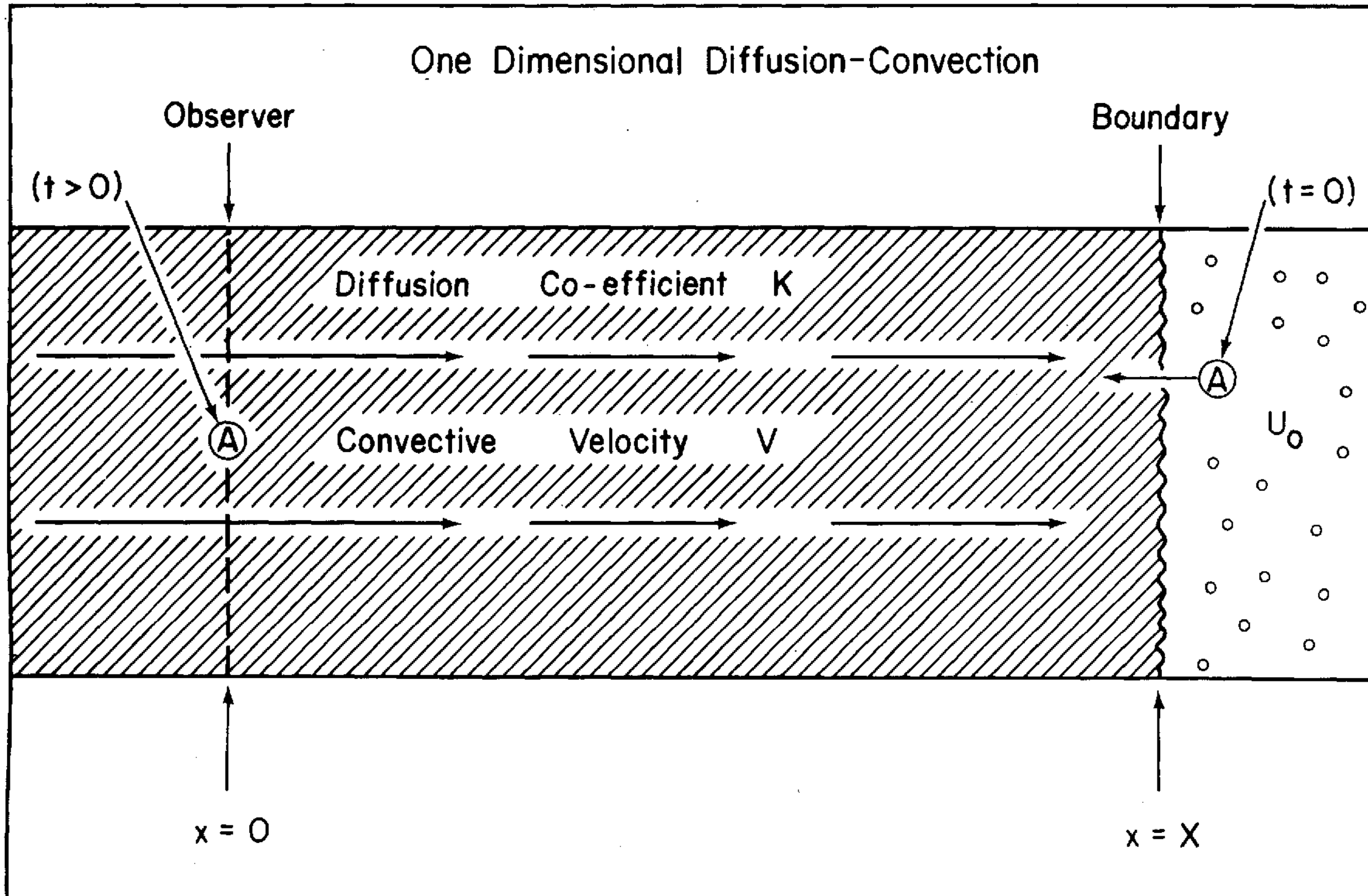
Figure 2. The relative probability distribution as a function of time in the "past" ($t < t_o$) for a particle seen now ($t = t_o$) at the origin is plotted for four sets of "interplanetary" conditions; a) $X = 40$ a.u., $V = 300$ km/sec, $K = 2 \times 10^{21}$ cm^2/sec; b) $X = 40$ a.u., $V = 300$ km/sec, $K = 4 \times 10^{21}$ cm^2/sec; c) $X = 20$ a.u., $V = 600$ km/sec, $K = 4 \times 10^{21}$ cm^2/sec, and d) $X = 20$ a.u., $V = 300$ km/sec, $K = 4 \times 10^{21}$ cm^2/sec. The gradual decay produced by escape from a finite region characterized by a spatial scale length X is indicated by the dashed lines.

Figure 3. The average propagation time delay τ_x in a one dimensional finite region of scale size X is illustrated as a function of K and X. The delay is proportional to X^2/K at large K and is proportional to X/V for small K. The crosses indicate values of τ_x calculated from the series for the exact solution for particles observed at the origin in a symmetric region with boundaries at $\pm$ X.

Figure 4. In a three dimensional radial convective modulating region, the average propagation time from a boundary R to an observer at $r \ll R$ can be analytically evaluated only in the limits of large and small K indicated by the solid lines where the behavior is similar to the one dimensional case. The behavior for intermediate K has been represented by a smooth function (dashed lines) which joins the two limiting regimes and has a form identical to that which is known to be an excellent approximation in one dimension.

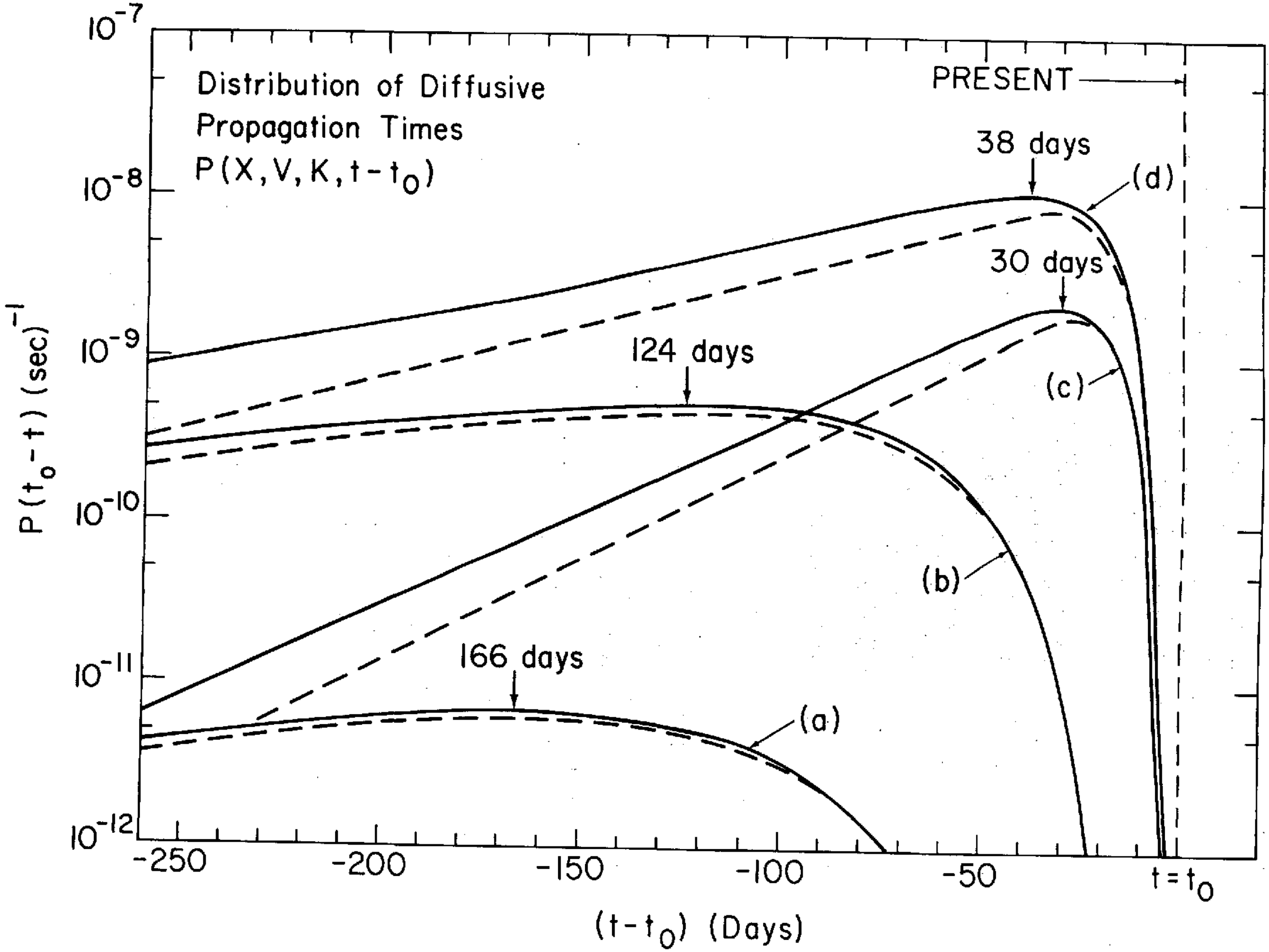
Typical times of the order of a few months are predicted for some values of R,V and K.

Figure 5. The "hysteresis effect" observed for 260-720 MeV protons with respect to the Deep River neutron intensity. The data is from Rygg and Earl, 1971 and Rygg et al, 1974. The behavior before solar maximum (in 1969) (solid points) and after solar maximum (open points) is not consistent with a single valued relation (heavy solid line) but can be interpreted as resulting from time lag in the response of the protons of between 180 and 270 days.



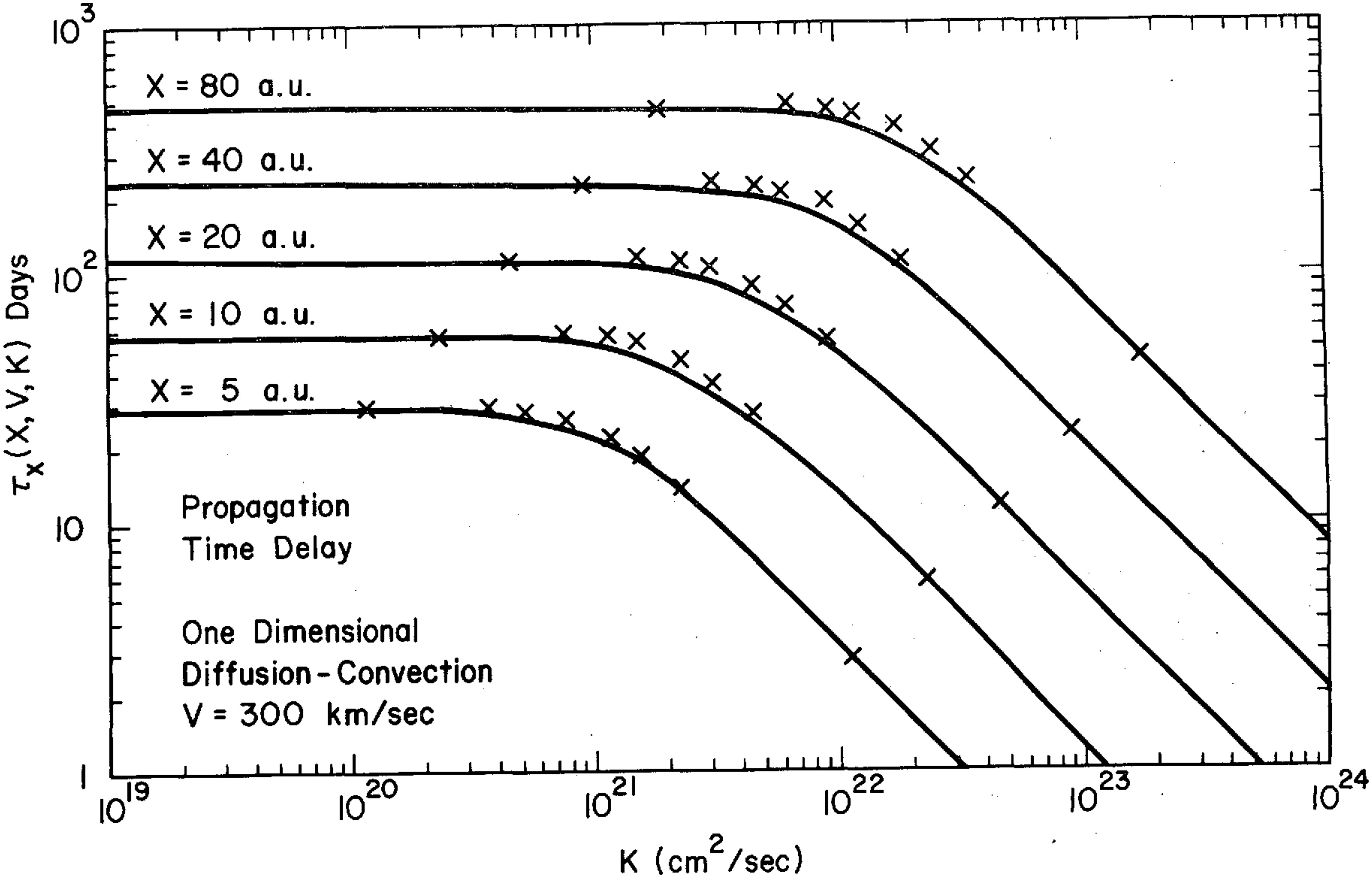
One Dimensional Diffusion-Convection
Observer
Boundary
(t > 0)
(t = 0)
Diffusion Co-efficient K
A
A
Convective Velocity V
U_0
x = 0
x = X

Figure 1



Distribution of Diffusive
Propagation Times
P(X, V, K, t - t0)
PRESENT
38 days
(d)
30 days
(c)
124 days
(b)
166 days
(a)
10-7
10-8
10-9
10-10
10-11
10-12
P(to-t) (sec)-1
-250
-200
-150
-100
-50
t = t0
(t - t0) (Days)

Figure 2



X = 80 a.u.
X = 40 a.u.
X = 20 a.u.
X = 10 a.u.
X = 5 a.u.
Propagation
Time Delay
One Dimensional
Diffusion - Convection
V = 300 km/sec
$\tau_x(X,V,K)$ Days
K (cm^2/sec)
10^3
10^2
10
1
10^{19}
10^{20}
10^{21}
10^{22}
10^{23}
10^{24}

Figure 3

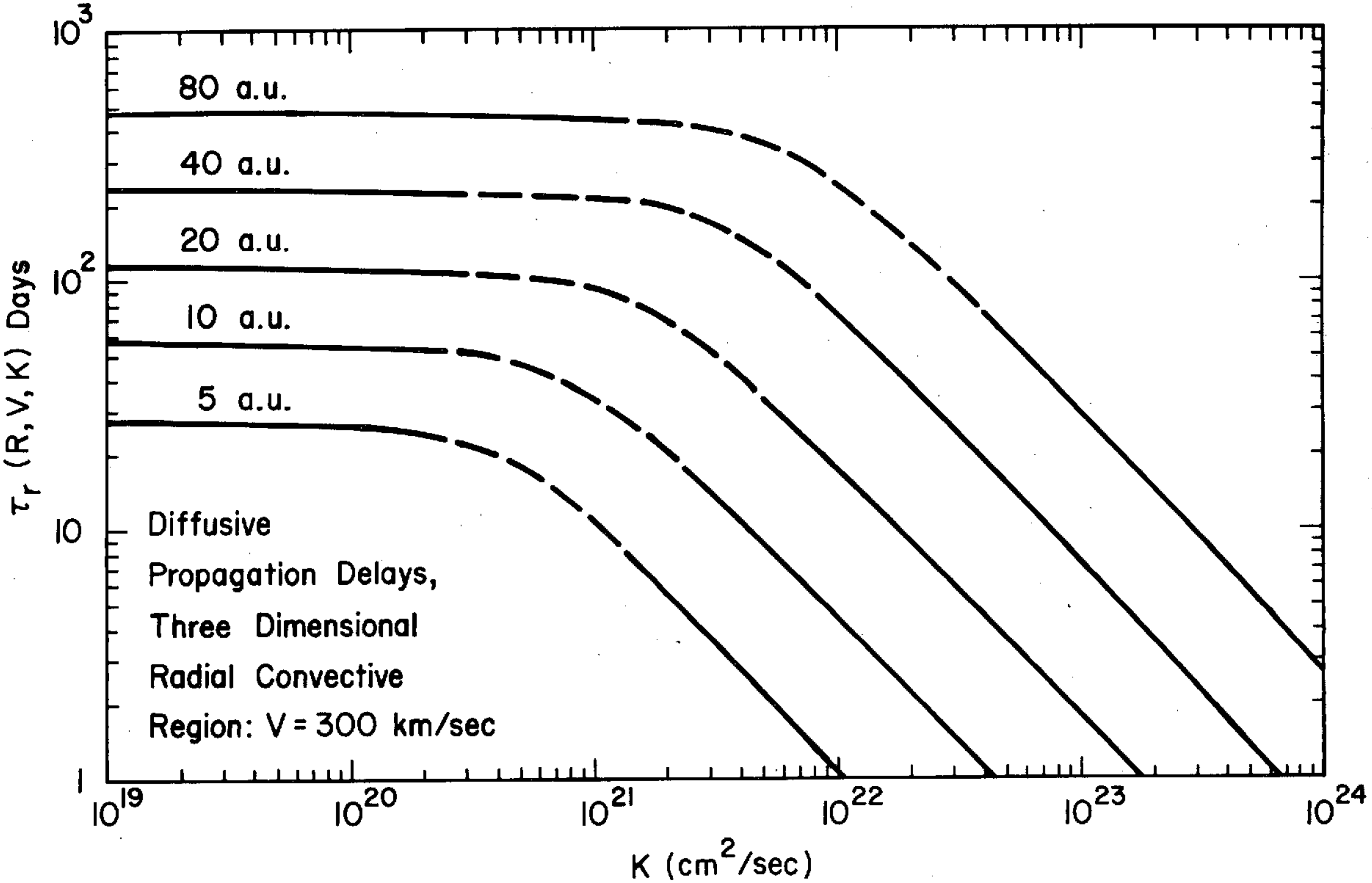
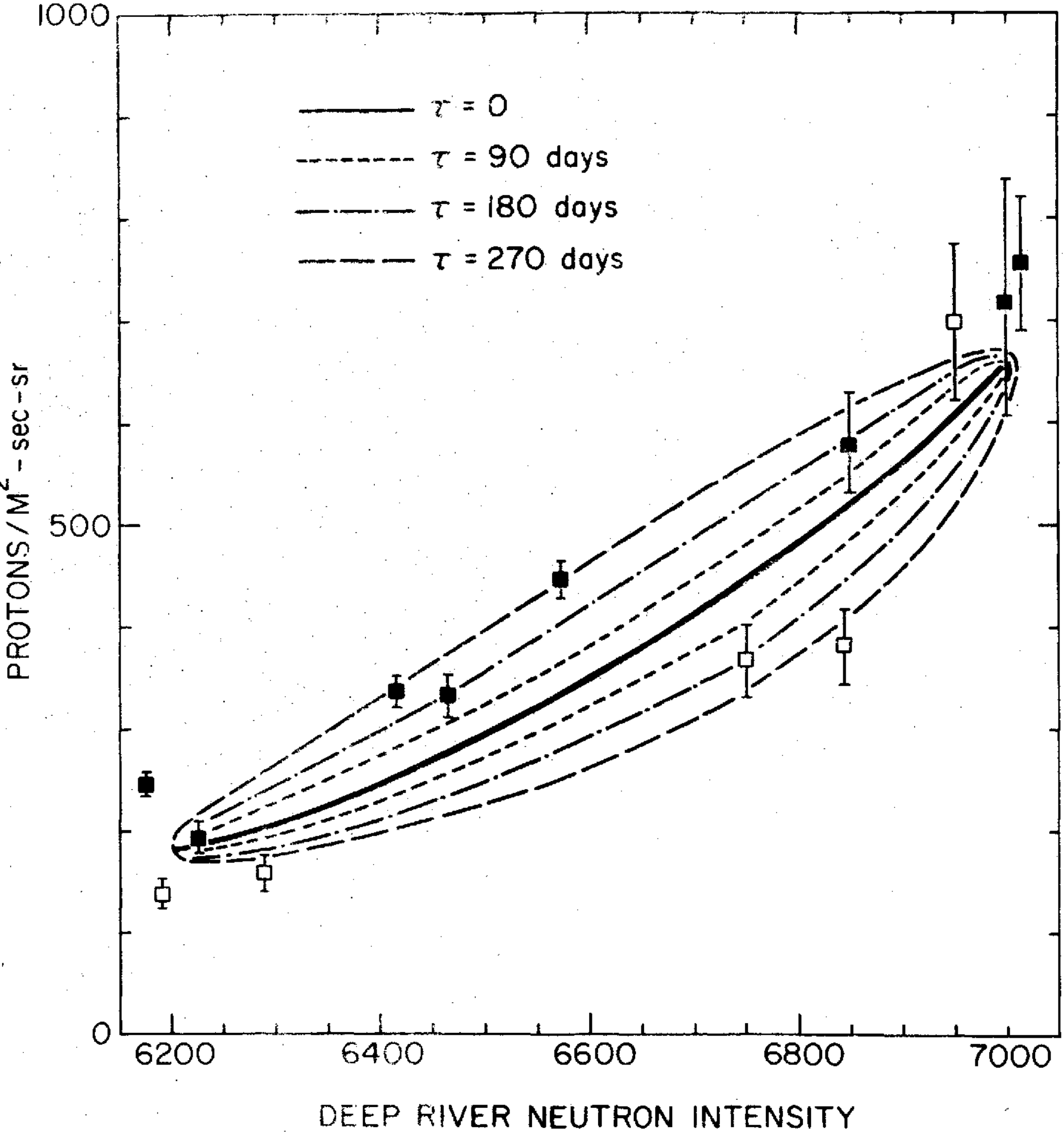

80 a.u.
40 a.u.
20 a.u.
10 a.u.
5 a.u.
Diffusive
Propagation Delays,
Three Dimensional
Radial Convective
Region: V = 300 km/sec
τ_r (R, V, K) Days
10^3
10^2
10
1
10^{19}
10^{20}
10^{21}
10^{22}
10^{23}
10^{24}
K (cm^2/sec)

Figure 4



1000
500
0
6200
6400
6600
6800
7000
PROTONS / M^2 - sec - sr
DEEP RIVER NEUTRON INTENSITY
τ = 0
τ = 90 days
τ = 180 days
τ = 270 days

Figure 5