LIQUID JET PUMPED BY RISING GAS BUBBLES

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ABSTRACT

From observations of a stream of gas bubbles rising through a liquid, a two-phase mathematical model is proposed for calculating the induced turbulent vertical liquid flow. The bubbles provide a large buoyancy force and the associated drag on the liquid moves the liquid upward. The liquid pumped upward consists of the bubble wakes and the liquid brought into the jet region by turbulent entrainment. The expansion of the gas bubbles as they rise through the liquid is taken into account. The continuity and momentum equations are solved numerically for an axisymmetric air jet submerged in water. Water pumping rates are obtained as a function of air flow rate and depth of submersion. Comparisons are made with limited experimental information in the literature.

NOMENCLATURE

- a: local outer radius of liquid jet region
- ac: local radius of bubbly core region
- K: turbulent jet entrainment coefficient
- g: ratio of bubble wake volume to bubble volume
- L: depth of jet origin below liquid surface
- N: mass flow rate
- p: pressure
- Pa: atmospheric pressure
- Q: volume flow rate
- r: radial coordinate, origin is at jet axis
- R: perfect gas constant
- T: absolute temperature
- u: velocity in x direction
- um: terminal velocity of single bubble rising in infinite liquid region
- v: velocity in radial direction
- x: vertical coordinate along jet axis, origin is at gas release orifice

Subscripts:
- p: fluid density
- g: surface tension

Subscripts:
- c: jet core region
- g: gas phase
- l: inside and outside of a particular jet region
- l: liquid phase

INTRODUCTION

Liquid pumping can be obtained by utilizing the buoyant force of gas bubbles rising through the liquid as shown in Fig. 1. This is a free convection type of process using two immiscible fluids that have a large difference in density \( \rho \). Because of the large density difference made possible by using a gas and a liquid, large amounts of liquid pumping can be obtained. The large local density difference is in contrast to ordinary thermally driven single phase free convection where the density variations are usually small.

Figure 1 - Configuration of liquid jet induced by rising column of gas bubbles.
An interesting application of bubble pumping is for ice prevention in lakes (2) (4) (5). Since the maximum density of water is at 4°C the water in the bottom regions of a quiescent lake can be as much as 4°C warmer than the surface regions when the air temperature is below freezing. Ice formation on the lake can be diminished or prevented by pumping the warmer bottom water toward the surface by means of rising air bubbles. Bubble pumping can also be used in fluidized beds, and for aeration in water purification and waste treatment plants. The flow induced by a curtain of rising air bubbles has also been considered as a breakwater for oncoming water waves (1) (2).

The purpose of this paper is to formulate a two-phase model and analyze the turbulent liquid pumping by a rising discharge of gas bubbles. By means of the drag on the liquid, the buoyancy force of the bubble imparts an upward momentum to the liquid. There is turbulent entrainment of liquid into the rising jet as indicated in fig. 1, and the result is that substantial vertical liquid pumping can result. The extent of the pumping is illustrated by experimental results in refs. (4) and (5) which show that for bubbles rising through 1.68 m (5.5 ft) of water, an air flow of 0.000472 m³/sec (1.66 ft/sec) produces a pumped liquid volume 120 times that of the gas. In ref. (6) the volume of water pumped per unit volume of air for an orifice at a depth of 4.5 m (14.7 ft) varied from about 60 to 175 depending on the air flow rate. The analysis given here will provide the liquid pumping rate as a function of bubble rise distance and gas flow rate.

Experimental measurements of a rising water jet pumped by air were made by Koubas (6) at several air flow rates for an orifice depth of 4.5 m. A theory was given utilizing the experimentally measured flow to specify the entrainment term in the continuity equation. In ref. (7) an analysis was made of the bubble-pumped jet in a manner analogous to a turbulent convective plane such as in ref. (6). This procedure leads to a singularity where the fluid velocity becomes infinite at the jet origin, making it necessary to define an apparent origin in order to compare the analytical results with experiment. The data of Koubas (6) was used in ref. (7) to determine the coefficients governing the flow such as the jet spreading rate and the turbulent entrainment coefficient. In order to obtain better agreement between the analysis and the experiment of Koubas, it was found necessary to decrease the entrainment coefficient substantially as the air flow decreased.

In ref. (8) analytical expressions are given from an approximate analysis by I. M. Konовал of the water pumped by air released from a submerged perforated pipe. The present situation is axisymmetric, whereas the perforated pipe provides a jet that is two dimensional in rectangular coordinates. The empirical constants in the theory were evaluated from laboratory observations for depths of the perforated pipe up to about 1 m and for air flow rates up to about 0.00139 m³/sec per meter of pipe length. Some limited full scale tests were also made at depths up to 10 m. Some data for a two-dimensional plane are also given in ref. (6). The analysis here is concerned with localized discharges spaced apart along the length of a submerged pipe; hence an axisymmetric jet is being considered rather than the perforated pipe.

The present work is an attempt to model the gas driven jet as a two-phase flow. A simplified two-phase model is constructed consisting of a bubbly core and an outer liquid flow. An entrainment model is constructed to account for the contributions to the entrainment by the outer liquid flow, the bubble wakes, and the rising gas bubbles.

ANALYSIS

Model Description

A glass tube having an opening about 1-1/2 mm in diameter was placed at the bottom of a tank 0.5 m deep and 0.31 m square in horizontal cross section. Motion pictures were made at several rates of air flow from the tube, and typical configurations of rising bubbles are shown in fig. 2. Based on these observations, a model was constructed which was a compromise between mathematical difficulty and physical realism. In the model, as shown in fig. 3, the jet is considered axisymmetrical with two regions. There is a central core surrounding the jet axis consisting of large bubbles rising in a chain bubble fashion and separated by liquid wakes that are carried along by the bubbles. Surrounding the core is an outer region of entrained

![Figure 2](image1.png)

![Figure 3](image2.png)

Experimental results obtained with help of Dr. Y. Y. Hau.
liquid being carried upward by the drag on the liquid arising from the buoyancy force of the bubbles. This model permits the different relations for the entrainment by the gas and by the liquid regions. The flow field is assumed to be steady, isothermal and fully turbulent. The liquid density is assumed constant, but the gas density varies with the local pressure in accordance with the perfect gas law, thus the bubbles expand as they rise through the liquid. The bubbles are assumed to be sufficiently large that their drag is fully turbulent and hence they rise at a constant terminal velocity relative to the liquid. The local bubble velocity is assumed equal to the local liquid velocity plus the bubble terminal velocity. It is assumed that the gas leaves the orifice with negligible upward momentum. After the gas is released, there is an adjustment region within which the bubbles achieve their terminal velocity. For an installation in a lake or river this region would be small compared with the total rise height and hence this region is not taken into account.

As shown in fig. 1 the rising jet flow will turn at the liquid surface and move radially outward. At the surface the vertical velocity component has to go to zero. However, to incorporate this condition requires coupling the solutions for the jet and the turning zone which is a difficult analysis. The turning zone is not accounted for here; it is assumed that the jet continues to the surface as shown by the dotted extrapolation lines in fig. 1. Because of their strong vertical buoyancy force, the bubbles do not turn very much with the flow, hence they exert their pumping effect all the way to the surface. Also there is zero shear at the liquid surface which facilitates the turning and tends to minimize the influence of the turning zone in the rising plume. These same types of assumptions are discussed in ref. (10) for a free convection plume above a heated cylinder. Consequently an analysis without the effect of the turning zone should yield reasonable vertical pumping rates.

**Gas Continuity**

For a single rising bubble, let the ratio of the wake volume to the bubble volume be a quantity $K$. From ref. (11), for bubble Reynolds numbers greater than about 200, which would be reasonable for the large bubbles in the present application, the $K$ is essentially constant and equals about 1.5. Then in the core region of local radius $a_0(x)$ in fig. 3, on the average $1/(K + 1)$ of the vertical height is occupied by gas bubbles and $K/(K + 1)$ is occupied by wake regions. This assumes for mathematical simplicity that the bubbles are rising in a chain-bubble fashion in which the bubbles are vertically spaced by the liquid wakes. Strictly speaking such a configuration exists only over a certain range of flow rates. However, the results from this model may apply to other bubble regimes. This is because, as discussed later, the turbulent entrainment of liquid by gas is minor compared with the entrainment by liquid. This tends to diminish the importance of the exact configuration of the rising gas. Since the gas weight flow rate is constant,

$$M_g = \int_{0}^{a_c} \frac{2\pi r}{K + 1} \rho_g u dr = \text{constant} \quad (1)$$

**Perfect Gas Law**

The pressure at height $x$ above the nozzle is $\rho_g(L - x)$. Then from the perfect gas law of the gas density at $x$ is

$$\rho_g = \frac{1}{R T} \left[ P_a + \rho_g(L - x) \right] \quad (2)$$

where $P_a$ is atmospheric pressure at the liquid surface.

**Liquid Continuity**

The liquid continuity equation accounts for the liquid carried into the jet by turbulent entrainment. The entrained liquid goes into the liquid region surrounding the core, or into the bubble wakes which are growing as the rising bubbles expand,

$$\frac{d}{dx} \left[ \frac{K}{K + 1} \int_{0}^{a_c} 2\pi r \rho_g u^2 dr + \int_{a_c}^{b_c} 2\pi r \rho L u^2 dr \right] = -(2\nu \rho_f L^2) \frac{dx}{x} \quad (3)$$

The term on the right will be expressed later in terms of a turbulent entrainment relation.

**Gas and Liquid Momentum**

In most applications involving air driven jets, the surrounding body of water is large; hence for simplicity in the present analysis, the surrounding liquid region will be assumed infinite. The analysis of a jet in a small container would be much more complicated because of liquid recirculation and the interaction with the container walls.

The upward buoyancy force of the bubbles produces a change in momentum (usually quite small) of the gas bubbles, a momentum change of the liquid in the bubble wakes, and a momentum change of the liquid in the region surrounding the bubble core. This yields the momentum equation as,

$$\int_{0}^{a_c} \frac{g}{K + 1} \left[ (c_1 - \rho_g)2\pi r^2 dr \right] = \frac{d}{dx} \left[ \int_{0}^{a_c} \frac{2\pi r \rho_g u^2 dr}{K + 1} \right] + \left[ \int_{0}^{a_c} \frac{K}{K + 1} \left( \frac{2\pi r \rho L u^2 dr}{K + 1} \right) \right] \quad (4)$$
**Top Hat** Distributions

To integrate eqs. (1), (3), and (4), "top hat" velocity profiles are assumed which have yielded good results for free convection plumes (g),

\[ u_g(x,t) = u(x) \quad 0 < x < a_o(x) \]  

(5a)

\[ u_l(x,t) = \begin{cases} u_l(x) & \text{if } a(x) < x < a_l(x) \\ 0 & \text{if } a_l(x) < x \end{cases} \]  

(5b)

The \( u_g \) and \( u_l \) are related by

\[ u_g(x) = u_l(x) + u_w \]  

(6)

where \( u_w \) is the terminal velocity of a single bubble rising in a large region of quiescent liquid.

Insert eq. (5) into eqs. (1), (3), and (4) to obtain

\[ \kappa_{g} = \frac{\tau_0}{k + 1} \frac{\partial u_g}{\partial x} = \text{constant} \]  

(7)

\[ \kappa_{l} = \frac{\tau_0}{k + 1} \frac{d}{dx} \left[ \left( \frac{a^2}{u} \right) u \right] \]  

(8)

\[ \kappa_{w} = \frac{\tau_0}{k + 1} \frac{d}{dx} \left( \frac{a^2}{u} \right) = \frac{2\kappa_0}{k + 1} \]  

(9)

Entrainment Function

An expression is now needed for the entrainment on the right side of eq. (8). As discussed by Morton (12) for free convection plumes above fires, the buoyant single phase jet the entrainment depends on an additional variable which is the density in the plume relative to the density of the surrounding fluid. The entrainment is reduced when the plume density is smaller than the density of the surrounding fluid. The present situation involves two distinct phases that each maintain their separate identities; hence there will be two types of entrainment, liquid-liquid and gas-liquid. For the latter there is a lack of entrainment information when the density ratio is as small as that for air to water. As soon as some liquid is in motion, much of the entrainment is by liquid entraining additional liquid which is reasonably well understood.

It was deduced in ref. (12) by using the information in ref. (11), that for a single phase jet the usual jet entrainment coefficient \( E \) used when the jet and outer fluid have the same density, should be modified when the jet and outer fluid densities are different. The \( E \) is multiplied by the density ratio \( \rho_j/\rho_o \) where \( \rho_j \) is the density in the jet and \( \rho_o \) is outside the jet. For the present situation the liquid-liquid entrainment terms will therefore have a unity ratio factored out, while the gas-liquid terms will contain \( \rho_j/\rho_o \) raised to 1/2. Since the entrainment is so different for the two phases, the two-phase nature of the jet will be retained rather than trying to assign an average density to the entire jet.

The entrainment also depends on the velocity of the jet relative to its surroundings, and on the interfacial area between the jet and the surrounding region. These velocities and areas are different for the gas and liquid portions. The liquid region moving at velocity \( u_l \) as shown in fig. 3, entrains liquid from the quiescent fluid around it. The higher velocity bubbly region is moving at velocity \( u_g \) relative to its surrounding liquid, and hence should enhance the entrainment process. A well defined interfacial area bounding the bubbly region can only be obtained by utilizing a simplified model such as that in fig. 3, and this area is used to obtain the magnitude of the entrainment. However, if this entrainment were assumed to be retained in the core, then the core would become a liquid jet containing a bubbly core and the difficulty of defining the entrainment for this two-phase inner jet would become the same as that for the original two-phase problem. Although the concept of a bubbly core is used to provide a well defined interfacial area, it is realized that the bubbly motion is actually more random. With these considerations in mind and the absence of any better information on such a two-stage entrainment process, it is assumed that the vigorous action of the bubbly region increases the turbulence in the liquid surrounding the bubbles and thereby enhances the total entrainment into the moving region. This is in accord with the results in ref. (13) that the entrainment is a function of the excess momentum flux in the jet. Thus in the present model, as the jet grows with increasing height above the source of gas the core will retain its identity as being composed of only bubbles and air wakes. Hence the total entrainment will be taken as the entrainment by the moving liquid outside the bubbly core, augmented by the entrainment effects of the liquid wakes and gas within core, and is given by

\[ \left( \frac{2\kappa_0}{k + 1} \right) = 2\kappa_0 \frac{\rho_j}{\rho_o} \frac{u_l}{u_w} + 2\kappa_0 \frac{\rho_j}{\rho_o} \frac{K}{1 + K} \]  

(10)

Solution for \( u_l \) and \( a_l \)

To obtain the amount of liquid being pumped by the rising gas, the liquid velocity and jet radius must be obtained. The liquid mass flow rate in the wake regions is known from the specified gas flow rate as \( \kappa_0 \rho_g \). The mass flow rate of liquid is then

\[ \kappa_0 = \pi \left( \frac{a^2}{u} \right) \]  

(11)

Since \( \kappa_0 \), in eq. (7) is a constant, eqs. (8) and (9) can be simplified by using eq. (7) to eliminate some of the \( u_l \). Eq. (10) is then substitut- ed for the right side of eq. (8). The result is

\[ \frac{K}{1 + K} \]  

(12)
The $u_2$ can be eliminated in terms of $u_1$ by using eq. (6), and the $a_2$ is given in terms of $x$ by eq. (2). Thus eqs. (12) and (13) are reduced to equations for the unknown $u_1(x)$ and $a(x)$. To obtain numerical results, the differentiations were carried out analytically and then the equations combined to eliminate $du_1/dx$ and $da/dx$. The result was the following set of simultaneous differential equations that were solved by the Runge-Kutta method (14),

$$\frac{du_1}{dx} = \left[ 2K \frac{d}{dx} \left( \frac{\rho_g u_2}{\rho_a u_1} \right) + \frac{\rho_g}{\rho_a} \left( \frac{K+1}{u_1 + u_2} \right) \right]^{1/2} \left( K + \frac{\rho_a}{\rho_g} \right) - \frac{\rho_g a_2}{\rho_a} \left( 2 + \frac{K}{u_1 + u_2} \right) \right] F(u_1, a, x) \tag{14}$$

$$\frac{da}{dx} = \left[ \frac{1}{a_1} \left( \frac{u_2}{u_1 + u_2} \right) \right]^{1/2} \left( K + \frac{\rho_a}{\rho_g} \right)$$

$$+ \frac{\rho_a}{2 \rho_g} \left( \frac{K+1}{u_1 + u_2} \right) - \left( \frac{a_2}{u_1 + u_2} \right)^2 F(u_1, a, x) \tag{15}$$

At $x = 0$, $a = a_0$, and $u_2 = 0$ so that $u_2 = u_1$. Then the initial conditions at $x = 0$, to begin the integration are from eq. (7),

$$a(x = 0) = \left[ \frac{N (1 + K)}{2u_1 (p_e + p_L)} \right]^{1/2}$$

and $u_1(0) = 0$. The latter condition however causes starting difficulty in the integration since it is in the denominator of a few terms, so a small value was used, $u_1(0) = 0.001$. The calculations were tested using smaller $u_1(0)$ and no significant changes were found.

**Figure 15**

To compute the induced liquid flow, a number of quantities must be specified. The results that follow are with regard to the prevention of ice in lakes by raising warm bottom water to the surface. For these conditions the bulk water is a few degrees $K$ above freezing so in what follows $T = 275 K$ ($495^oR$). The terminal velocity $u_m$ of a single bubble in undisturbed fluid was obtained from the relation (15) (16)

$$u_m = 1.53 \left( \frac{N (1 + p_g)}{\rho_a} \right)^{1/4}$$

which yielded $u_m = 0.252$ m/sec (0.825 ft/sec) for air bubbles in water.

A significant source of uncertainty is in specifying the entrainment coefficient $E_m$. In ref. (17) which is concerned with the penetration of a condensing vapor jet into a liquid, a "top hat" velocity profile is used and from 0.06 to 0.12 is given as being found in the literature. For entrainment of various gas jets in still air with density ratios in the range $\rho_1/\rho_a = 0.66$ to 14.5, the work in ref. (15) yielded $E_m = 0.08$ based on considerations of the excess momentum in the jet. The entrainment is proportional to a characteristic velocity in the jet. If a Gaussian velocity profile is used, the centerline velocity is used as the characteristic velocity which is larger than the average velocity used in the "top hat" profile. As a result in ref. (18) $E_m = 0.082$ is recommended for use with a Gaussian profile and $E_m = 0.112$ for a "top hat" profile. In view of all these considerations it was decided to obtain two sets of calculations using $E_m = 0.08$ and 0.112. The results will be discussed in the next section.

Equations (14) and (15) were integrated to $x = L$, the surface of the water, to obtain $u_1(L)$ and $a(L)$. By using eqs. (6) and (7) $a_0(L)$ is found as the upward flow of liquid obtained from $a_0$. (11). In reality the upward flow will begin to turn as it approaches the surface, but the influence of the water surface was not accounted for here, as previously discussed. Figure 4 gives the mass flow rate of liquid being pumped to the surface as a function of the mass flow rate of air, for the air being introduced at various depths below the surface. In terms of the same quantities fig. 5 shows the radius of the jet region at $x = L$ and fig. 6 gives the liquid velocity at $x = L$. Some of the results in fig. 5 have been cross-plotted in fig. 7 to show the trend of the jet radius with orifice depth for fixed gas mass flow rates.

**Discussion**

As shown by fig. 4 the liquid pumped to the surface increases with the depth of the orifice below the surface and with the gas flow rate. For a fixed weight flow rate, the volume of the gas introduced at the orifice decreases as the orifice depth increases. Hence it seems better to present the results in terms of gas weight flow rate than in terms of volume flow. For each orifice depth the results in fig. 4 lie fairly well along a straight line on a log-log plot. The slope decreases somewhat as $L$ is increased but the results vary essentially as $N = N^{1/4}$. A cross plot of the results shows that for a given $N$ the $M$ variation with $L$ is also as a power function.
When $N_1$ and $N_2$ are in kg/sec and $L$ is in m, then $C = 365$ for $E_0 = 0.08$ and $C = 550$ for $E_0 = 0.116$. When $N_1$ and $N_2$ are in #/sec and $L$ is in ft, then $C = 111$ for $E_0 = 0.08$ and $C = 172$ for $E_0 = 0.116$.

To examine in more detail the effect of the various entrainment terms in eq. (20) calculations were made with either the liquid wake term and/or the gas

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The photographic results of fig. 2 indicate an increase of cone angle with gas flow rate. In fig. 2(a) the cone angle is about 140°, while in fig. 2(d) it is about 120°. Equation (20) is plotted in fig. 7 and compares reasonably well with calculated results for an entrainment coefficient of \( E_0 = 0.06 \). For a larger total included angle of say 180°, which is characteristic of figs. 2(c) and (d), eq. (20) becomes \( a(L) = 0.158 L \). As shown by fig. 7, this provides reasonable agreement with the values computed with \( E_0 = 0.116 \).

CONCLUSIONS

A tractable mathematical model was formulated for computing the liquid carried upward in an axisymmetric jet driven by a rising stream of gas bubbles. The model geometry was based on observations of air rising through water 0.6 m deep. Liquid is carried into the jet by turbulent entrainment and a difficulty in the analysis is in specifying the proper value of the entrainment coefficient. Calculations were made for two values of the entrainment coefficient within the range given in the literature. The large buoyancy resulting from the large density difference between the gas and liquid produced considerable liquid movement compared to the theoretical.

To provide a little more information on bubble pumping consider briefly some results reported in ref. (2) from an approximate analysis made in 1946 by Evonov. The pumping in a geometry that is two dimensional in rectangular coordinates as produced by air released from a submerged perforated pipe. The empirical constants in the theory were evaluated from laboratory observations for depths of the perforated pipe up to about 1 m and air flow rates up to 0.00138 \( m^3/sec \) per meter of pipe length. There were also some limited full scale tests run at depths up to 10 m.

The results for orifice depths greater than 1 m are given by the correlation

\[
Q_x = 0.75 \left( 10 + 1 \right)^{2/3} \ln \left( 1 + \frac{L}{10} \right)^{1/3} Q_0 \tag{19}
\]

where \( Q_x \) and \( Q_0 \) are in \( m^3/sec \) and \( L \) is in meters.

Because of the difference in geometry precise comparisons cannot be made, but its interest to see how the predictions of liquid volume pumped per unit gas volume \( Q_x/Q_x \) compare for the release from an orifice as obtained from eq. (17), and from a meter length of perforated pipe, eq. (19). Results are given in table II and is it seen that the pumping rates are of the same magnitude. The \( Q_x \) was assumed to be at the discharge location, although this was not clearly specified in the reference.

In ref. (16) it is mentioned that the bubbly region of the axisymmetric jet is approximately contained within a cone having a total included angle of 150°. Although the present mathematical model considers all the gas to be in the form of large bubbles contained in the core of the jet, in the physical case there are small bubbles that break off from the large ones as shown in fig. 2. The turbulent motion diffuses the small bubbles within the flow so that they probably extend throughout most of the entrained region. However the total bubbly region should give an indication of the extent of the jet region. Using a total included cone angle of 120°, the jet radius at the surface is

\[
\phi = \frac{1}{L} \arctan \frac{x}{L} = 0.158 L \tag{20}
\]

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TABLE I - VOLUMETRIC FLOW RATES FOR DIAMETER 4.5 M M.E.G. ORIFICE SUBMERGED AT A LOCATION 3.5 M ABOVE ORIFICE SUBMERGED.

<table>
<thead>
<tr>
<th>Flow Rate (m³/s)</th>
<th>Orifice Flow Rate (m³/s)</th>
<th>Total Flow Rate (m³/s)</th>
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<tbody>
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Note: The total flow rate is the sum of the orifice flow rate and the flow rate through the submerged portion of the flow passages.