(NASA-TM-X-72185) ARBITRARILY CURVED AND TWISTED SPACE BEAMS Ph.D. Thesis - Va. Polytech. Inst. and State Univ. (NASA) 115 p HC $5.25 CSCL 20K
N75-16018
Unclas
G3/39 09467

ARBITRARILY CURVED AND TWISTED SPACE BEAMS

By

William F. Hunter

Dissertation submitted to the Graduate Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

December, 1974

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(ABSTRACT)

A derivation of the equations which govern the deformation of an arbitrarily curved and twisted space beam is presented. These equations differ from those of the classical theory in that extensional effects are included. Other departures from the previous theory are that the strain - displacement relations are derived and that the expressions for the stress resultants are developed from the strain - displacement relations instead of assuming that the resultants are proportional to changes in the curvatures. It is shown that the torsional stress resultant obtained by the classical approach is basically incorrect except when the cross-section is circular.

Using a vector approach the exact expressions for the curvature components of a deformed space beam are developed. Because inextension of the beam is not assumed an additional term appears in each of the linearized curvature expressions. These expressions are utilized in the derivation of the strain - displacement relations. The normal and shearing physical components of the strain tensor are given. These relations are not restricted to beams whose cross-sectional dimensions are very small compared to the radius of curvature. Next, a development of the stress resultants is presented. Effects arising from the

initial twist of the beam are obtained which are not reflected in the classical theory. Finally, the six equilibrium equations are derived using a vector approach.

The governing equations are given in the form of twelve first-order differential equations. A numerical algorithm is given for obtaining the natural vibration characteristics and example problems are presented.

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APPROVED:

J. Counts, Chairman

F. H. Lutze

R. P. McNitt

F. J. Maher

K. L. Reifsnider

December, 1974

Blacksburg, Virginia

II. ACKNOWLEDGMENTS

The author is very grateful to the National Aeronautics and Space Administration for its support of this work. In particular he wishes to thank Mrs. Maria V. Stephens for programing the numerical solution. Also, the author would like to acknowledge that the presented numerical solution method (previously unpublished) was originated by Dr. Robert T. Wingate for determining the lateral vibration characteristics of branched beams.

The author is very appreciative to Dr. Jerry Counts and the other members of the Advisory Committee for the time that they have devoted to this effort and for their valuable suggestions and consultations.

Lastly, the author wishes to express special thanks to his wife, Deanna, for her encouragement and patience.

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V. INTRODUCTION

The work presented herein arose from the need to better understand the elastic and dynamic behavior of twisted and curved propeller blades. It has long been recognized that the stresses in propeller blades can be reduced by designing the blades to be slightly curved so that the centrifugal forces are used to counteract the bending moments arising from lift and drag. This technique can be especially beneficial to the design of wind tunnel blades. However, an adequate theory for accomplishing this has not yet been developed. Also, there was the need to study the effects of curvature on the vibration characteristics of rotating blades. In addition, there is much current interest in analyzing the Darrieus windmill (sometimes called the vertical-axis windmill). Although these specific problems are not treated in this paper, much of the needed analytical development work is presented.

A review of the classical theory for curved and twisted space beams as presented by A. E. H. Love in reference 1 led to an interest to extend the theory to include additional effects and to present a fresh and more rigorous derivation of the governing differential equations.

The classical theory was developed by Kirchoff, Clebsch, Michell, and Basset in the latter part of the last century (in reference 1 Love discusses the development and lists references). However, Love is responsible for presenting the theory in an organized fashion and his work is most often quoted.

In the classical theory, the central line of the beam is assumed to be inextensible and the cross-sectional dimensions are assumed to be very small compared to the radius of curvature. Love gives a development for the curvature expressions of the central line of a deformed beam. Love's work contains an irregularity in that his direction cosines do not satisfy the orthogonality relations. The inextensibility assumption and the direction cosine anomaly, which are related, have been questioned by Waltking (see reference 2) and by Philipson (see reference 3).

In this paper the exact expressions for the curvature components of a deformed beam, which is initially curved in space in any arbitrary manner, are derived using a vector approach. No assumptions are made in the development and, thus, the resulting equations are applicable to large, as well as small, deformations. When the three curvature relations are linearized an additional term, that Love does not obtain, appears in each one. It is shown that these terms arise because extensional deformation is included.

A development is presented for the strain-displacement relations of space beams. Since the strain distribution is not linear over the cross-section, the relations can be applied to beams having much larger curvatures than that permitted by the classical theory. The strain-displacement relations are used to determine the stress resultants.

In the classical theory the strain-displacement relations are not derived. The stress resultants are assumed to be equal to changes in the curvatures times an elastic cross-sectional constant. It is shown

that this approach yields an incorrect stress resultant which can lead to considerable error.

A vector derivation is presented for the six beam equilibrium equations. These equations are essentially the same as Love's equations when linearized for natural vibration solutions.

The analysis developed for curved beams is applied to natural vibration examples. The governing equations consist of twelve first-order differential equations and the solutions are obtained by using a transfer matrix method. Included in the equations are rotary inertia and elastic foundation effects.

Reference 4 is a survey paper by Royster on the subject of curved beam vibrations. Practically all studies have been on rings or beams whose central lines lie in a plane. In fact, in the summary of his report Royster states "Only two papers on the vibrations of a curved beam of double curvature have been published." Also, regarding solutions he lists the conclusion "In general, solutions for the coupled in plane and out of plane vibration problem, as is the case if the central-line of the unstressed beam is not a plane curve, are extremely rare." In references 5 and 6 Volterra applies the Method of Internal Constraints to beams of double curvature. He obtains nine coupled second-order differential equations having variable coefficients with the derivatives being of the nine unknown displacement functions. A solution method is not presented.

In papers pertaining to the vibrations and buckling of circular rings the governing equations are arrived at by other methods, as well

as from Love's work, when the deformation is in the plane of the ring. However, for coupled torsion and out-of-plane bending deformation the equations of Love's classical theory are invariably used. For such deformation the classical theory should be modified because of the previously mentioned error associated with one of the stress resultants.

It is the intent of this dissertation to present a new development of the equations which govern the deformation of space beams rather than to present a study of the vibration characteristics of curved beams. However, a numerical solution method is given for such beams. Comprehensive discussions are presented on the developments of the curvature relations, the strain-displacement relations, and the stress resultants in view of the classical theory according to Love and other literature.

VI. LIST OF SYMBOLS

A	Cross-sectional area of beam
a, b, c	Quantities defined by equations (28)
$[A_i]$	Transfer matrix defined by equation (145)
$[a]$	Matrix appearing in equation (132)
B_i	Cross-sectional constant defined by equations (115), $(i = 1, 2, \ldots 9)$
$[B]$	Product of the transfer matrices as given by equation (147)
$\underset{\sim}{b}$	Binormal vector of the space curve formed by the axis of undeformed beam
$[b]$	Matrix appearing in equation (132)
c_x, c_y, c_z	Elastic foundation constants for linear displacements
$[C]$	Matrix defined by equation (148)
$\underset{\sim}{d}, \underset{\sim}{d}'$	Modified Darboux vector for undeformed and deformed beam, respectively
$[D]$	Matrix defined by equation (149)
d_x, d_y, d_z	Elastic foundation constants for rotational displacements
dM	Mass of differential lengths ds and ds' of beam
dP	Force acting on differential length ds'
dS, dS'	Lengths of an arbitrary incremental line element before and after deformation (see equation (11))
ds, ds'	Differential lengths along elastic axis before and after deformation

$d\mathcal{S}$, $d\mathcal{S}'$	Differential lengths defined by equations (80) and (81)
E	Elastic modulus
e_{ii}	Normal components of the strain tensor
e_{ij}	Shearing components of the strain tensor
$\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, $\underset{\sim}{e}_z$	Orthonormal vectors at point on elastic axis of undeformed beam
$\underset{\sim}{e}'_x$, $\underset{\sim}{e}'_y$, $\underset{\sim}{e}'_z$	Orthonormal vectors at point on elastic axis of deformed beam (defined by the rotation of the imbedded $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, $\underset{\sim}{e}_z$ vectors)
$\underset{\sim}{F}$	Vector of internal forces
F_x, F_y, F_z	Components of $\underset{\sim}{F}$
G	Shear modulus
G_{ij}, g_{ij}	Metric coefficients for deformed and undeformed beam, respectively
$[H]$	Matrix defined by equation (140)
I_{xx}, I_{yy}	Moments of inertia about the cross-sectional axes
I_p	Polar moment of inertia of cross-section about elastic axis
i	Denotes station along beam
$[I]$	Identity matrix
J	Saint-Venant torsional stiffness constant defined by one of equations (115)
k_x, k_y, k_z	Components of curvature of the undeformed beam
k'_x, k'_y, k'_z	Components of curvature of the deformed beam
L_3, M_3, N_3	Direction cosines defined by reference 1

l_i, m_i, n_i	Direction cosines relating $\underset{\sim}{e}'_x$, $\underset{\sim}{e}'_y$, $\underset{\sim}{e}'_z$ to $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, $\underset{\sim}{e}_z$ $(i = 1, 2, 3)$
l_z	Direction cosine of the angle between the z coordinate line and $\underset{\sim}{e}'_z$
$\underset{\sim}{M}$	Vector of the internal moments
M_x, M_y, M_z	Components of $\underset{\sim}{M}$
m, m'	Mass per unit length of the undeformed and deformed beam, respectively
$\underset{\sim}{n}$	Normal vector of the space curve formed by the elastic axis of undeformed beam
P, P'	Denotes position of point before and after deformation, respectively
$\underset{\sim}{p}$	Applied distributed force vector per unit length of deformed beam
p_x, p_y, p_z	Components of $\underset{\sim}{p}$
$\bar{p}_x$, $\bar{p}_y$, $\bar{p}_z$	Applied distributed forces per unit length of undeformed beam
$\underset{\sim}{q}$	Applied distributed moment per unit length of deformed beam
q_x, q_y, q_z	Components of $\underset{\sim}{q}$
$\bar{q}_x$, $\bar{q}_y$, $\bar{q}_z$	Applied distributed moments per unit length of undeformed beam
R	Radius of curvature
$\underset{\sim}{R}$, $\underset{\sim}{R}'$	Position vectors to point on elastic axis before and after deformation, respectively

$\underset{\sim}{r}$, $\underset{\sim}{r}'$	Position vectors to point in cross-section before and after deformation, respectively
s, s'	Coordinate along elastic axis of undeformed and deformed beam, respectively
$\underset{\sim}{t}$	Unit tangent vector of the space curve formed by the elastic axis of the undeformed beam
$[U]$	Matrix defined by equation (151)
u, v, w	Displacements of the elastic axis in the $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, and $\underset{\sim}{e}_z$ directions
$\ddot{u}$	Second partial derivative of u with respect to time
W	Saint-Venant warping function
x, y	Cross-sectional coordinates
$\{Y\}$	Column vector of the variables of the vibration problem, defined by equation (138)
$\{Y'\}$	Column vectors of the variable derivatives, defined by equation (139)
z	Coordinate line defined by the intersection of the coordinate surfaces x = constant and y = constant
α_x, α_y, α_z	Quantities defined by equations (20)
β	Direction cosine of reference 1
γ	Orientation of $\underset{\sim}{e}_x$ relative to $\underset{\sim}{n}$ (see figure 2)
γ_{ij}	Strain tensor
$\Delta\underset{\sim}{R}$	Displacement vector of a point on the elastic axis
$\Delta s'$	Incremental length of deformed beam
ϵ	Tensorial extensional strain of the elastic axis

Θ	Eulerian rotation
Θ^i	Curvilinear coordinate
Θ_{ij}, Θ'_{ij}	Angle between the x^i and x^j coordinate lines before and after deformation, respectively
κ	Curvature of the elastic axis of the undeformed beam
ρ	Mass density of beam
σ_{zz}	Normal stress in direction of the z coordinate line
τ	Torsion of the undeformed beam
ϕ	Rotational displacement of the beam about $\underset{\sim}{e}'_z$
ψ	Eulerian rotation
ω	Natural vibration frequency

VII. ANALYSIS

In this chapter developments will be presented for curvature relations, the strain-displacement relations, the stress resultant expressions, and the equilibrium equations for arbitrarily curved and twisted space beams. Each of the first three developments is followed by a discussion of the development. These discussions are included in this chapter since they are an in-depth analysis of the developments in relation to previous work instead of being of a general discussion nature. Besides being convenient to the reader, it is thought that the placement of the discussions in this chapter will give the reader a better understanding of the developments which follow each.

Also, a summary of the governing differential equations for natural vibration is given. This is followed by a description of a numerical method for solving them.

A. Development of Curvature Relations

Consider a curved beam whose elastic axis forms any general space curve. The elastic axis of the beam before and after deformation is shown in figure 1 where primes denote the deformed state. In this figure, $\underset{\sim}{R}$ is the position vector to point P on the elastic axis, s is the coordinate along the elastic axis, $\Delta\underset{\sim}{R}$ is the displacement vector of point P, and $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, and $\underset{\sim}{e}_z$ are orthonormal vectors with $\underset{\sim}{e}_z$ tangent to the curve s.

The unit vectors $\underset{\sim}{e}_x$ and $\underset{\sim}{e}_y$ are taken to be aligned with the principal axes of the beam's cross-section. The orientation of $\underset{\sim}{e}_x$

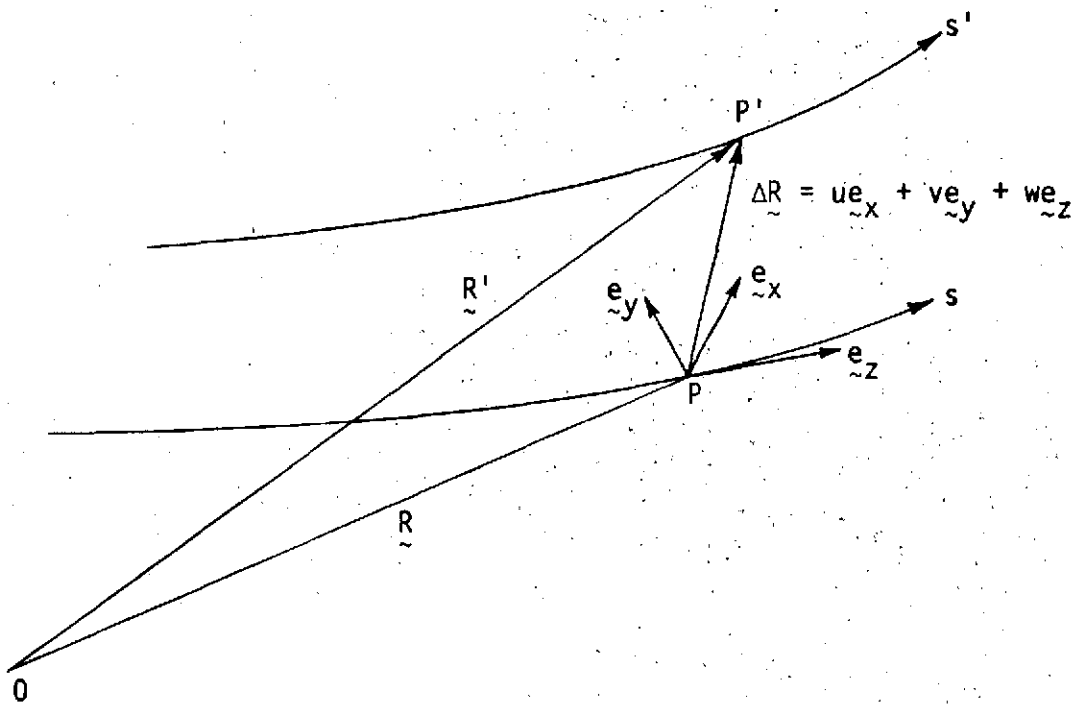

Figure 1.- Elastic axis of beam before and after deformation.

and $\underset{\sim}{e}_y$ is specified relative to the space curve's normal, $\underset{\sim}{n}$, as shown in figure 2. The vectors $\underset{\sim}{t}$, $\underset{\sim}{n}$, and $\underset{\sim}{b}$ are the tangential, normal, and binormal unit vectors, respectively, of the space curve. They are related to $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, and $\underset{\sim}{e}_z$ by the transformation

$$\left.\begin{aligned} \underset{\sim}{n} &= \underset{\sim}{e}_x \cos\gamma - \underset{\sim}{e}_y \sin\gamma \\ \underset{\sim}{b} &= \underset{\sim}{e}_x \sin\gamma + \underset{\sim}{e}_y \cos\gamma \\ \underset{\sim}{t} &= \underset{\sim}{e}_z \end{aligned}\right\} \quad (1)$$

The shape of a space curve can be completely described by two parameters: the curvature $\kappa(s)$ and the torsion $\tau(s)$. The derivatives of $\underset{\sim}{t}$, $\underset{\sim}{n}$, and $\underset{\sim}{b}$ with respect to s are given by the well known Frenet-Serrett formulas as

$$\left.\begin{aligned} \frac{d\underset{\sim}{t}}{ds} &= \kappa\underset{\sim}{n} \\ \frac{d\underset{\sim}{n}}{ds} &= -\kappa\underset{\sim}{t} + \tau\underset{\sim}{b} \\ \frac{d\underset{\sim}{b}}{ds} &= -\tau\underset{\sim}{n} \end{aligned}\right\} \quad (2)$$

Similar expressions can be obtained for the $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, $\underset{\sim}{e}_z$ trihedron. They are most simply obtained by modifying the Darboux vector (see

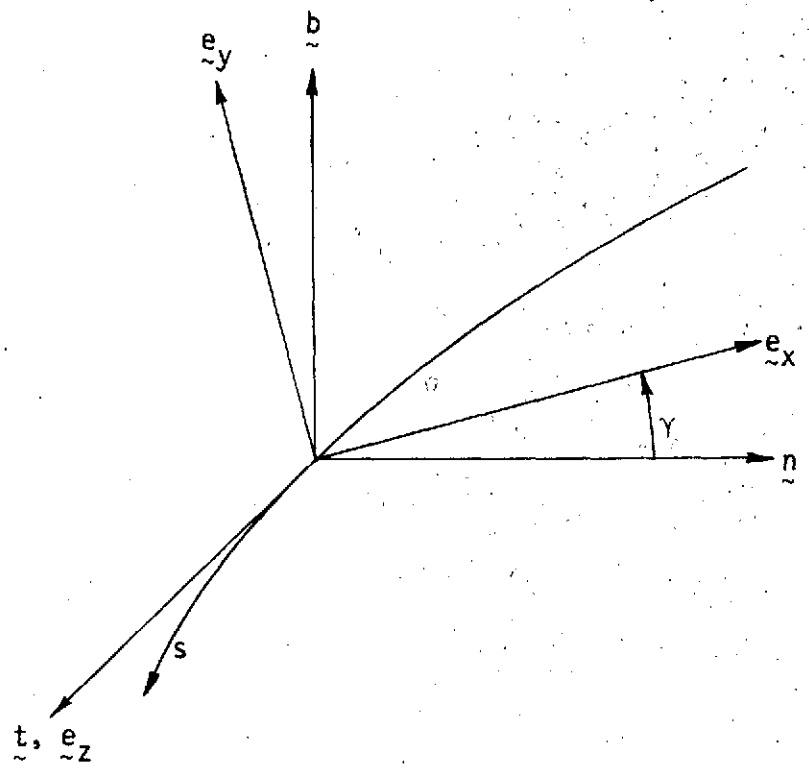

Figure 2.- Orientation of $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, $\underset{\sim}{e}_z$ trihedron relative to $\underset{\sim}{n}$, $\underset{\sim}{b}$, $\underset{\sim}{t}$ trihedron.

reference 7 or any other book on differential geometry) and operating with it. The Darboux vector is the rotation rate of the trihedron at point P as P moves along the curve at unit velocity. Or, it can be viewed as rotation per unit length of the curve. Since γ is not taken to be constant, the modified Darboux vector is

$$\underset{\sim}{d} = (\tau + \frac{d\gamma}{ds})\underset{\sim}{t} + \kappa \underset{\sim}{b} \tag{3}$$

Substituting equations (1) into the above gives

$$\underset{\sim}{d} = (\tau + \frac{d\gamma}{ds})\underset{\sim}{e}_z + \kappa \sin \gamma \, \underset{\sim}{e}_x + \kappa \cos \gamma \, \underset{\sim}{e}_y \tag{4}$$

The quantities $(\tau + \frac{d\gamma}{ds})$, $(\kappa \sin \gamma)$, and $(\kappa \cos \gamma)$ are known as the components of curvature. Letting

$$\left.\begin{aligned} k_x &= \kappa \sin \gamma \\ k_y &= \kappa \cos \gamma \\ k_z &= \tau + \frac{d\gamma}{ds} \end{aligned}\right\} \tag{5}$$

the modified Darboux vector may be rewritten as

$$\underset{\sim}{d} = k_x \underset{\sim}{e}_x + k_y \underset{\sim}{e}_y + k_z \underset{\sim}{e}_z \tag{6}$$

In the manner used in dynamics to obtain the derivatives of unit vectors with respect to time, it can be shown that the derivatives of $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, and $\underset{\sim}{e}_z$ with respect to s are

$$\left.\begin{aligned} \frac{d\underset{\sim}{e}_x}{ds} &= \underset{\sim}{d} \times \underset{\sim}{e}_x \\ \frac{d\underset{\sim}{e}_y}{ds} &= \underset{\sim}{d} \times \underset{\sim}{e}_y \\ \frac{d\underset{\sim}{e}_z}{ds} &= \underset{\sim}{d} \times \underset{\sim}{e}_z \end{aligned}\right\} \quad (7)$$

Hence,

$$\left.\begin{aligned} \frac{d\underset{\sim}{e}_x}{ds} &= k_z\underset{\sim}{e}_y - k_y\underset{\sim}{e}_z \\ \frac{d\underset{\sim}{e}_y}{ds} &= -k_z\underset{\sim}{e}_x + k_x\underset{\sim}{e}_z \\ \frac{d\underset{\sim}{e}_z}{ds} &= k_y\underset{\sim}{e}_x - k_x\underset{\sim}{e}_y \end{aligned}\right\} \quad (8)$$

It is noted that in the preceding equations, k_x, k_y, and k_z are the curvature components before deformation. The goal is to arrive at expressions for the curvature components of the deformed beam in terms of the elastic displacements.

As shown in figure 1, position vector to the displaced point P' is

$$\underset{\sim}{R}' = \underset{\sim}{R} + \underset{\sim}{\Delta R} \tag{9}$$

where

$$\underset{\sim}{\Delta R} = u\,\underset{\sim}{e}_x + v\,\underset{\sim}{e}_y + w\,\underset{\sim}{e}_z \tag{10}$$

and u, v, and w are the elastic displacements of point P in the directions of the reference unit vectors of the undeformed beam.

If dS is the length of an arbitrary incremental line element in an elastic solid and dS' is its length after deformation, then (see reference 8 or other book on elasticity or continuum mechanics)

$$(dS')^2 - (dS)^2 = 2\,\gamma_{ij}d\theta^i d\theta^j \quad (i,\ j = 1,\ 2,\ 3) \tag{11}$$

where the γ_{ij} are the components of the strain tensor and the Θ^i are the curvilinear coordinates.

Applying this equation to a curved beam with the incremental line element being chosen to lie along the elastic axis such that $d\theta^1 = d\theta^2 = 0$, $d\theta^3 = ds$, and $\gamma_{33} = \varepsilon$ gives

$$(ds')^2 - (ds)^2 = 2\varepsilon\, ds^2 \tag{12}$$

Or,

$$\varepsilon = \frac{1}{2}\left[\left(\frac{ds'}{ds}\right)^2 - 1\right] \tag{13}$$

where ε is tensorial extensional strain of the centroidal axis. Since

$$(ds')^2 = d\underset{\sim}{R}' \cdot d\underset{\sim}{R}' \tag{14}$$

the above may be rewritten as

$$\varepsilon = \frac{1}{2}\left[\frac{d\underset{\sim}{R}'}{ds} \cdot \frac{d\underset{\sim}{R}'}{ds} - 1\right] \tag{15}$$

Differentiating equation (9) with respect to s,

$$\frac{d\underset{\sim}{R}'}{ds} = \frac{d\underset{\sim}{R}}{ds} + \frac{d(\Delta\underset{\sim}{R})}{ds} \tag{16}$$

Also, it is noted that

$$\frac{d\underset{\sim}{R}}{ds} = \underset{\sim}{e}_z \tag{17}$$

Substituting the above into equation (16) and the result into equation (15) leads to

$$\varepsilon = \frac{d(\Delta\underset{\sim}{R})}{ds} \cdot \underset{\sim}{e}_z + \frac{1}{2}\frac{d(\Delta\underset{\sim}{R})}{ds} \cdot \frac{d(\Delta\underset{\sim}{R})}{ds} \tag{18}$$

Differentiating equation (10) with respect to s and applying equations (8) yields

$$\frac{d(\Delta \underset{\sim}{R})}{ds} = \alpha_x \underset{\sim}{e}_x + \alpha_y \underset{\sim}{e}_y + \alpha_z \underset{\sim}{e}_z \tag{19}$$

where

$$\left.\begin{aligned} \alpha_x &= \frac{du}{ds} - k_z\, v + k_y\, w \\ \alpha_y &= \frac{dv}{ds} + k_z\, u - k_x\, w \\ \alpha_z &= \frac{dw}{ds} - k_y\, u + k_x\, v \end{aligned}\right\} \tag{20}$$

Substituting equation (19) into equation (18) gives

$$\varepsilon = \alpha_z + \frac{1}{2}\left(\alpha_x^2 + \alpha_y^2 + \alpha_y^2\right) \tag{21}$$

which is the nonlinear expression for the extensional strain of the elastic axis. The significance of the above result in view of previous work will be discussed later.

Equation (16) may be rewritten by substituting equations (17) and (19).

$$\frac{d\underset{\sim}{R}'}{ds} = \alpha_x \underset{\sim}{e}_x + \alpha_y \underset{\sim}{e}_y + (1 + \alpha_z)\underset{\sim}{e}_z \tag{22}$$

From equation (12) it is noted that

$$ds' = (1 + 2\varepsilon)^{1/2} ds \tag{23}$$

Hence,

$$\frac{d\underset{\sim}{R}'}{ds'} = (1 + 2\varepsilon)^{-1/2} \frac{d\underset{\sim}{R}'}{ds} \tag{24}$$

Also,

$$\frac{d\underset{\sim}{R}'}{ds'} = \underset{\sim}{e}'_z \tag{25}$$

Combining equations (22), (24), and (25) results in

$$\underset{\sim}{e}'_z = (1 + 2\varepsilon)^{-1/2}[\alpha_x \underset{\sim}{e}_x + \alpha_y \underset{\sim}{e}_y + (1 + \alpha_z)\underset{\sim}{e}_z] \tag{26}$$

The above equation gives the direction of the unit vector tangent to the deformed elastic axis in terms of the displacements, initial curvature, and unit vectors of the undeformed rod.

Equation (26) may be differentiated with respect to s' by applying equation (23) and substituting equations (8). Doing this yields

$$\frac{d\underset{\sim}{e}'_z}{ds'} = (1 + 2\varepsilon)^{-1} (a\, \underset{\sim}{e}_x + b\, \underset{\sim}{e}_y + c\, \underset{\sim}{e}_z) \tag{27}$$

where

$$\left.\begin{aligned} a &= -\alpha_y k_z + (1 + \alpha_z)k_y + \frac{d\alpha_x}{ds} - (1 + 2\varepsilon)^{-1}\,\alpha_x \frac{d\varepsilon}{ds} \\ b &= \alpha_x k_z - (1 + \alpha_z)k_x + \frac{d\alpha_y}{ds} - (1 + 2\varepsilon)^{-1}\,\alpha_y \frac{d\varepsilon}{ds} \\ c &= -\alpha_x k_y + \alpha_y k_x + \frac{d\alpha_z}{ds} - (1 + 2\varepsilon)^{-1}\,(1 + \alpha_z)\frac{d\varepsilon}{ds} \end{aligned}\right\} \quad (28)$$

and from equation (21), $\frac{d\varepsilon}{ds}$ is given by

$$\frac{d\varepsilon}{ds} = \alpha_x \frac{d\alpha_x}{ds} + \alpha_y \frac{d\alpha_y}{ds} + (1 + \alpha_z)\frac{d\alpha_z}{ds} \quad (29)$$

The direction cosines which define the orientation of the trihedrons of the undeformed and deformed beam relative to each other are identified in the table below.

	$\underset{\sim}{e}_x$	$\underset{\sim}{e}_y$	$\underset{\sim}{e}_z$
$\underset{\sim}{e}'_x$	l_1	m_1	n_1
$\underset{\sim}{e}'_y$	l_2	m_2	n_2
$\underset{\sim}{e}'_z$	l_3	m_3	n_3

The direction cosines l_3, m_3, and n_3 have been defined in terms of the displacements by equation (26) and are

$$\left.\begin{aligned} l_3 &= (1 + 2\varepsilon)^{-1/2}\ \alpha_x \\ m_3 &= (1 + 2\varepsilon)^{-1/2}\ \alpha_y \\ n_3 &= (1 + 2\varepsilon)^{-1/2}(1 + \alpha_z) \end{aligned}\right\} \quad (30)$$

In the same manner as before, a modified Darboux vector may also be written for the deformed beam as

$$\underset{\sim}{d}' = k'_x \underset{\sim}{e}'_x + k'_y \underset{\sim}{e}'_y + k'_z \underset{\sim}{e}'_z \quad (31)$$

Using this vector, the derivatives of the unit vectors of the deformed beam are found to be

$$\left.\begin{aligned} \frac{d\underset{\sim}{e}'_x}{ds'} &= k'_z \underset{\sim}{e}'_y - k'_y \underset{\sim}{e}'_z \\ \frac{d\underset{\sim}{e}'_y}{ds'} &= -k'_z \underset{\sim}{e}'_x + k'_x \underset{\sim}{e}'_z \\ \frac{d\underset{\sim}{e}'_z}{ds'} &= k'_y \underset{\sim}{e}'_x - k'_x \underset{\sim}{e}'_y \end{aligned}\right\} \quad (32)$$

From the third of the above equations and equation (27),

$$k'_y \underset{\sim}{e}'_x - k'_x \underset{\sim}{e}'_y = (1 + 2\varepsilon)^{-1}\ (a\ \underset{\sim}{e}_x + b\ \underset{\sim}{e}_y + c\ \underset{\sim}{e}_z) \quad (33)$$

The transformation between the unit vectors in terms of the direction cosines is

$$\left.\begin{aligned} \underset{\sim}{e}_x &= l_1\underset{\sim}{e}'_x + l_2\underset{\sim}{e}'_y + l_3\underset{\sim}{e}'_z \\ \underset{\sim}{e}_y &= m_1\underset{\sim}{e}'_x + m_2\underset{\sim}{e}'_y + m_3\underset{\sim}{e}'_z \\ \underset{\sim}{e}_z &= n_1\underset{\sim}{e}'_x + n_2\underset{\sim}{e}'_y + n_3\underset{\sim}{e}'_z \end{aligned}\right\} \quad (34)$$

Substituting the above into equation (33) and rearranging gives

$$\begin{aligned} k'_y\underset{\sim}{e}'_x - k'_x\underset{\sim}{e}'_y = (1 + 2\varepsilon)^{-1}[&(al_1 + bm_1 + cn_1)\underset{\sim}{e}'_x \\ &+ (al_2 + bm_2 + cn_2)\underset{\sim}{e}'_y \\ &+ (al_3 + bm_3 + cn_3)\underset{\sim}{e}'_z] \end{aligned} \quad (35)$$

Thus,

$$\left.\begin{aligned} k'_x &= -(1 + 2\varepsilon)^{-1}(al_2 + bm_2 + cn_2) \\ k'_y &= (1 + 2\varepsilon)^{-1}(al_1 + bm_1 + cn_1) \\ 0 &= (1 + 2\varepsilon)^{-1}(al_3 + bm_3 + cn_3) \end{aligned}\right\} \quad (36)$$

Two of the curvature components have now been expressed in terms of a, b, and c (defined by equations (28)) and the direction cosines. It can be shown that the third of equations (36) is exactly satisfied by the previously defined a, b, c, l_3, m_3, and n_3. Also, by squaring equations (36), adding, and applying the transformation orthogonality relations it is seen that

$$k_x'^2 + k_y'^2 = (1 + 2\varepsilon)^{-2}(a^2 + b^2 + c^2) \tag{37}$$

From the first of eqs. (32),

$$\frac{d\underset{\sim}{e}_x'}{ds'} \cdot \underset{\sim}{e}_y' = k_z' \tag{38}$$

Using equation (23) and substituting the transformations

$$\left.\begin{aligned} \underset{\sim}{e}_x' &= l_1\underset{\sim}{e}_x + m_1\underset{\sim}{e}_y + n_1\underset{\sim}{e}_z \\ \underset{\sim}{e}_y' &= l_2\underset{\sim}{e}_x + m_2\underset{\sim}{e}_y + n_2\underset{\sim}{e}_z \end{aligned}\right\} \tag{39}$$

into equation (38) gives

$$k_z' = (1 + 2\varepsilon)^{-1/2}\left[\frac{d}{ds}(l_1\underset{\sim}{e}_x + m_1\underset{\sim}{e}_y + n_1\underset{\sim}{e}_z)\right] \cdot [l_2\underset{\sim}{e}_x + m_2\underset{\sim}{e}_y + n_2\underset{\sim}{e}_z]$$

Differentiating the above and substituting equations (8) for the derivatives of the unit vectors results in

$$k_z' = (1 + 2\varepsilon)^{-1/2}[(n_2 m_1 - m_2 n_1)k_x + (l_2 n_1 - n_2 l_1)k_y$$

$$+ (m_2 l_1 - l_2 m_1)k_z + l_2 \frac{dl_1}{ds} + m_2 \frac{dm_1}{ds} + n_2 \frac{dn_1}{ds}] \qquad (40)$$

The table of direction cosines given on page 20 is an orthogonal matrix. The inverse of an orthogonal matrix is equal to its transpose and its determinant is equal to unity. Thus, each element of an orthogonal matrix is equal to its cofactor. Applying this theorem to the last row of the direction cosine matrix yields the following orthogonality relations

$$\left.\begin{aligned} l_3 &= m_1 n_2 - m_2 n_1 \\ m_3 &= -l_1 n_2 + l_2 n_1 \\ n_3 &= l_1 m_2 - l_2 m_1 \end{aligned}\right\} \qquad (41)$$

The above relations may have been obtained by substituting the transformation expressions for $\underset{\sim}{e}_x'$, $\underset{\sim}{e}_y'$, and $\underset{\sim}{e}_z'$ into $\underset{\sim}{e}_x' \times \underset{\sim}{e}_y' = \underset{\sim}{e}_z'$ and carrying out the operation as Novozhilov did in reference 9. Equation (40) can now be rewritten as

$$k_z' = (1 + 2\varepsilon)^{-1/2}(l_3 k_x + m_3 k_y + n_3 k_z + l_2 \frac{dl_1}{ds} + m_2 \frac{dm_1}{ds} + n_2 \frac{dn_1}{ds}) \qquad (42)$$

Equations (36) and (42) give the curvature components in terms of the direction cosines, the initial curvatures, and the quantities a, b, and c. The direction cosines l_3, m_3, and n_3 have been expressed in terms of the displacements u, v, and w. These cosines (l_3, m_3, and n_3) define the orientation of $\underset{\sim}{e}'_z$ relative to the trihedron of the undeformed beam (i.e., $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, and $\underset{\sim}{e}_z$). Besides knowing the orientation of $\underset{\sim}{e}'_z$, one additional angle (or direction cosine) is needed to fully specify the orientation of the trihedron of the deformed beam relative to the trihedron of the undeformed beam (in other words, the orientation of $\underset{\sim}{e}'_x$ and $\underset{\sim}{e}'_y$ needs to be specified). This additional rotation joins u, v, and w as one of the elastic displacements (or variables) of the problem. The direction cosines l_1, m_1, n_1, l_2, m_2, and n_2 can be expressed in terms of the known cosines (l_3, m_3, and n_3) and the new elastic displacement which has not yet been defined.

When the beam is deformed, the principal flexure-torsion trihedron at any general point on the elastic axis undergoes translation and rotation. The rotation may be expressed in terms of Eulerian angles. It has been found that the most suitable Eulerian angle system for this application is that used in aeronautical and aerospace engineering (see reference 10). For this system all three Euler angles are small if small deformations are assumed. This is not the case for the Euler angle system of reference 11.

Let xyz be the initial reference frame aligned with the $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, $\underset{\sim}{e}_z$ trihedron and let $x'y'z'$ be the final reference frame which is aligned with the $\underset{\sim}{e}'_x$, $\underset{\sim}{e}'_y$, $\underset{\sim}{e}'_z$ trihedron. The three rotations will be

taken in the following order:

1. A positive rotation ψ about y axis resulting in XYZ reference
2. A positive rotation Θ about X axis resulting in $X'Y'Z'$ reference
3. A positive rotation ϕ about Z' axis resulting in $x'y'z'$ reference

It is noted that the above is different from reference 10 in that the axes are not named the same. The three transformations are

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix}$$

Substituting and performing the matrix operations yields

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta & -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi \\ -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & -\sin\theta & \cos\theta\cos\psi \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad (43)$$

The above gives the previously discussed direction cosine matrix in terms of ψ, Θ, and ϕ. The angle ϕ will be taken as the needed additional displacement variable. The angle ϕ is the rotation about the z' axis (or elastic axis) that brings the X' and Y' axes to the final x' and y' position.

From the direction cosine matrix of equation (43),

$$l_3 = \cos\theta\sin\psi$$

$$m_3 = -\sin\theta$$

$$n_3 = \cos\theta\cos\psi$$

Or,

$$\left.\begin{aligned} \theta &= -\sin^{-1} m_3 \\ \psi &= \tan^{-1}\left(\frac{l_3}{n_3}\right) \end{aligned}\right\} \quad (44)$$

By substituting equations(30) the above expressions for θ and ψ may be rewritten in terms of the displacements as

$$\left.\begin{aligned} \theta &= -\sin^{-1}\left[\frac{\alpha_y}{(1 + 2\varepsilon)^{1/2}}\right] \\ \psi &= \tan^{-1}\left(\frac{\alpha_x}{1 + \alpha_z}\right) \end{aligned}\right\} \quad (45)$$

Using the direction cosines defined by equation (43), the expressions for k'_x and k'_y given by equation (36) become

$$\left.\begin{aligned} k'_x &= \frac{-1}{1 + 2\varepsilon}\,[(-\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi)a + (\cos\phi\cos\theta)b \\ &\qquad + (\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)c] \\ k'_y &= \frac{1}{1 + 2\varepsilon}\,[(\cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi)a + (\sin\phi\cos\theta)b \\ &\qquad + (-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi)c] \end{aligned}\right\} \quad (46)$$

From equations (44), (30), and (21), the following expressions for the trigonometric functions in terms of the displacements are obtained:

$$\left.\begin{aligned} \sin\theta &= \frac{-\alpha_y}{(1 + 2\varepsilon)^{1/2}} \\ \cos\theta &= \left(\frac{1 + 2\varepsilon - \alpha_y^2}{1 + 2\varepsilon}\right)^{1/2} \\ \sin\psi &= \frac{\alpha_x}{(1 + 2\varepsilon - \alpha_y^2)^{1/2}} \\ \cos\psi &= \frac{1 + \alpha_z}{(1 + 2\varepsilon - \alpha_y^2)^{1/2}} \end{aligned}\right\} \quad (47)$$

Substituting the above into equations (46) gives

$$k_x' = (1 + 2\varepsilon)^{-3/2}(1 + 2\varepsilon - \alpha_y^2)^{-1/2}\Big\{[(1 + 2\varepsilon)^{1/2}(1 + \alpha_z)\sin\phi + \alpha_x\alpha_y\cos\phi]a$$
$$- [(1 + 2\varepsilon - \alpha_y^2)\cos\phi]b + [-(1 + 2\varepsilon)^{1/2}\alpha_x\sin\phi + \alpha_y(1 + \alpha_z)\cos\phi]c\Big\} \tag{48}$$

$$k_y' = (1 + 2\varepsilon)^{-3/2}(1 + 2\varepsilon - \alpha_y^2)^{-1/2}\Big\{[(1 + 2\varepsilon)^{1/2}(1 + \alpha_z)\cos\phi - \alpha_x\alpha_y\sin\phi]a$$
$$+ [(1 + 2\varepsilon - \alpha_y^2)\sin\phi]b + [-(1 + 2\varepsilon)^{1/2}\alpha_x\cos\phi - \alpha_y(1 + \alpha_z)\sin\phi]c\Big\} \tag{49}$$

The expression for k_z' in terms of the direction cosines is given by equation (42). The last three terms of this equation may be expressed in terms of the Euler angles by substituting the direction cosines as defined by the matrix of equation (43). Performing the operations and combining terms yields the surprisingly simple result

$$l_2\frac{dl_1}{ds} + m_2\frac{dm_1}{ds} + n_2\frac{dn_1}{ds} = \frac{d\phi}{ds} - \sin\theta\frac{d\psi}{ds} \tag{50}$$

Substituting equations (30) and (50) into equation (42) gives

$$k_z' = (1 + 2\varepsilon)^{-1}[k_x\alpha_x + k_y\alpha_y + k_z(1 + \alpha_z)] + (1 + 2\varepsilon)^{-1/2}\left(\frac{d\phi}{ds} - \sin\theta\frac{d\psi}{ds}\right) \tag{51}$$

The last term of equation (51) may be rewritten in terms of the displacements as

$$-\sin\theta \frac{d\psi}{ds} = m_3 \frac{d}{ds}[\tan^{-1}(\frac{l_3}{n_3})]$$

$$= \frac{m_3}{n_3^2 + l_3^2}(n_3 \frac{dl_3}{ds} - l_3 \frac{dn_3}{ds}) \tag{52}$$

Using equations (30) it can be shown that

$$n_3 \frac{dl_3}{ds} - l_3 \frac{dn_3}{ds} = (1 + 2\varepsilon)^{-1}[(1 + \alpha_z) \frac{d\alpha_x}{ds} - \alpha_x \frac{d\alpha_z}{ds}] \tag{53}$$

Substituting the above and equations (30) into equation (52) gives

$$-\sin\theta \frac{d\psi}{ds} = \frac{(1 + 2\varepsilon)^{-1/2}\alpha_y}{(1 + \alpha_z)^2 + \alpha_x^2}[(1 + \alpha_z) \frac{d\alpha_x}{ds} - \alpha_x \frac{d\alpha_z}{ds}] \tag{54}$$

Noting from equation (21) that

$$(1 + \alpha_z)^2 + \alpha_x^2 = 1 + 2\varepsilon - \alpha_y^2$$

and substituting equation (54) into equation (51) gives the third curvature component as

$$k_z' = (1 + 2\varepsilon)^{-1}\Big\{k_x\alpha_x + k_y\alpha_y + k_z(1 + \alpha_z) + (1 + 2\varepsilon)^{1/2} \frac{d\phi}{ds}$$

$$+ \frac{\alpha_y}{1 + 2\varepsilon - \alpha_y^2}[(1 + \alpha_z) \frac{d\alpha_x}{ds} - \alpha_x \frac{d\alpha_z}{ds}]\Big\} \tag{55}$$

Equations (48), (49), and (55) are the equations for the curvature components of the elastic axis of the deformed beam in terms of the

elastic displacements and the initial curvature. The quantities α_x, α_y, α_z, ε, a, b, and c appearing in the equations are defined as functions of the displacements u, v, and w by equations (20), (21), and (28).

It is noted that the expressions derived for k'_x, k'_y, and k'_z are exact. No assumptions whatever have been made.

Also, it is observed that different forms of the curvature components can be obtained by altering the order of the Eulerian rotations. For example, if the rotations are taken in the order

1. A positive rotation Θ about x axis resulting in XYZ reference
2. A positive rotation ψ about Y axis resulting in X'Y'Z' reference
3. A position rotation ϕ about Z' axis resulting in x'y'z' reference

then the curvature components are given by

$$k'_x = (1 + 2\varepsilon)^{-3/2}(1 + 2\varepsilon - \alpha_x^2)^{-1/2}\Big\{[(1 + 2\varepsilon - \alpha_x^2)\sin\phi]a + [-(1 + 2\varepsilon)^{1/2}(1 + \alpha_z)\cos\phi - \alpha_x\alpha_y \sin\phi]b + [(1 + 2\varepsilon)^{1/2}\alpha_y \cos\phi - \alpha_x(1 + \alpha_z)\sin\phi]c\Big\} \tag{56}$$

$$k'_y = (1 + 2\varepsilon)^{-3/2}(1 + 2\varepsilon - \alpha_x^2)^{-1/2}\Big\{[(1 + 2\varepsilon - \alpha_x^2)\cos\phi]a + [(1 + 2\varepsilon)^{1/2}(1 + \alpha_z)\sin\phi - \alpha_x\alpha_y \cos\phi]b + [-(1 + 2\varepsilon)^{1/2}\alpha_y \sin\phi - \alpha_x(1 + \alpha_z)\cos\phi]c\Big\} \tag{57}$$

$$k_z' = (1 + 2\varepsilon)^{-1}\Big\{k_x\alpha_x + k_y\alpha_y + k_z(1 + \alpha_z) + (1 + 2\varepsilon)^{1/2}\frac{d\phi}{ds}$$

$$- \frac{\alpha_x}{(1 + \alpha_z)^2 + \alpha_y^2}\left[(1 + \alpha_z)\frac{d\alpha_y}{ds} - \alpha_y\frac{d\alpha_x}{ds}\right]\Big\} \tag{58}$$

As a check on the expressions given for k_x' and k_y' it can be shown that they do satisfy equation (37).

The exact curvature expressions, which are highly nonlinear, may be approximated to any desired degree by applying the binomial theorem and substituting the trigonometric expansions for $\sin\phi$ and $\cos\phi$. Using equations (48), (49), and (55), the nonlinear approximations which contain terms of order no higher than products of two displacements (or squares of displacements) are

$$k_x' = k_x - k_z\alpha_x - k_x\alpha_z + k_y\phi - \frac{d\alpha_y}{ds} - k_x\alpha_x^2 - \frac{1}{2}k_x\alpha_y^2 + k_x\alpha_z^2 - \frac{1}{2}k_x\phi^2$$

$$+ 2k_z\alpha_x\alpha_z - k_z\alpha_y\phi - k_y\alpha_z\phi + \alpha_y\frac{d\alpha_z}{ds} + 2\alpha_z\frac{d\alpha_y}{ds} + \phi\frac{d\alpha_x}{ds} \tag{59}$$

$$k_y' = k_y - k_z\alpha_y - k_y\alpha_z - k_x\phi + \frac{d\alpha_x}{ds} - \frac{1}{2}k_y\alpha_x^2 - k_y\alpha_y^2 + k_y\alpha_z^2 - \frac{1}{2}k_y\phi^2$$

$$- k_x\alpha_x\alpha_y + 2k_z\alpha_y\alpha_z + k_z\alpha_x\phi + k_x\alpha_z\phi - \alpha_x\frac{d\alpha_z}{ds} - 2\alpha_z\frac{d\alpha_x}{ds} + \phi\frac{d\alpha_y}{ds} \tag{60}$$

$$k_z' = k_z + k_x\alpha_x + k_y\alpha_y - k_z\alpha_z + \frac{d\phi}{ds} - k_z\alpha_x^2 - k_z\alpha_y^2 + k_z\alpha_z^2$$

$$- 2k_x\alpha_x\alpha_z - 2k_y\alpha_y\alpha_z + \alpha_y\frac{d\alpha_x}{ds} - \alpha_z\frac{d\phi}{ds} \tag{61}$$

It is seen from the above that the linear approximations for the curvature components are

$$k_x' = k_x - k_z\alpha_x - k_x\alpha_z + k_y\phi - \frac{d\alpha_y}{ds} \tag{62}$$

$$k_y' = k_y - k_z\alpha_y - k_y\alpha_z - k_x\phi + \frac{d\alpha_x}{ds} \tag{63}$$

$$k_z' = k_z + k_x\alpha_x + k_y\alpha_y - k_z\alpha_z + \frac{d\phi}{ds} \tag{64}$$

It has been found that the order of the Eulerian rotations has no effect on the linearized equations so long as there is a rotation about each of the body axes. Also, it is noted that the linear relations can be obtained most easily from equations (46) and (51) by making small angle assumptions for ψ, Θ, and ϕ.

Using equations (20) the linearized curvature components may be expressed explicitly in terms of u, v, w, and ϕ as

$$k_x' = k_x - \frac{d^2v}{ds^2} - 2k_z\frac{du}{ds} + (k_xk_y - \frac{dk_z}{ds})u$$

$$+ (-k_x^2 + k_z^2)v + (-k_yk_z + \frac{dk_x}{ds})w + k_y\phi \tag{65}$$

$$k_y' = k_y + \frac{d^2u}{ds^2} - 2k_z\frac{dv}{ds} + (k_y^2 - k_z^2)u$$

$$- (k_xk_y + \frac{dk_z}{ds})v + (k_xk_z + \frac{dk_y}{ds})w - k_x\phi \tag{66}$$

$$k_z' = k_z + k_x \frac{du}{ds} + k_y \frac{dv}{ds} - k_z \frac{dw}{ds}$$

$$+ \frac{d\phi}{ds} + 2k_y k_z u - 2k_x k_z v \tag{67}$$

B. Discussion of Curvature Relations

A development is presented for the exact expressions of the curvature components of a deformed beam which is initially curved in any arbitrary manner. The exact equations have not been derived heretofore. The curvature relations are developed using a vector approach. In contrast to nearly all previous work the beam is not assumed to be inextensible. The principal cross-sectional axes are assumed to have a variable orientation relative to the curve's normal and binormal. Also, nonlinear and linear approximations of the curvature equations are given.

In reference 1, Love gives expressions similar to the linearized equations presented herein. However, there is one term in each of equations (62), (63), and (64) that Love does not obtain. The presented vector derivation yields these additional terms which are apparently of the same order of magnitude as the other terms. The new terms in the k_x', k_y', and k_z' expressions are $(-k_x\ \alpha_z)$, $(-k_y\ \alpha_z)$, and $(-k_z\ \alpha_z)$, respectively.

There are two aspects of Love's work on curved beams which have been questioned previously. First, there is the anomaly in Love's work that his direction cosines do not satisfy the orthogonality relationships. In particular, the sum of the squares of the direction

cosines relating $\underset{\sim}{e}_z'$ to $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, and $\underset{\sim}{e}_z$ is not equal to unity. Secondly, the validity and applicability of Love's inextensibility relation for curved beams is questioned.

In the presented development all of the direction cosine orthogonality conditions are, of course, satisfied exactly. For example, it is seen that $l_3^2 + m_3^2 + n_3^2 = 1$ is satisfied exactly by equations (30) after equation (21) is substituted.

The vector approach can be used to explain Love's direction cosine anomaly which has not been fully understood before. It shows that products of deformations must be included in the development if the direction cosines are to be consistent. Also, the reason for the exact formulation yielding additional terms in the linearized equations can be shown.

Love denotes the direction cosines of the angles between $\underset{\sim}{e}_z'$ and each of $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, and $\underset{\sim}{e}_z$ by L_3, M_3, and N_3, respectively. Expressed in the notation used in this paper, Love finds the direction cosines to be

$$L_3 = \alpha_x, \quad M_3 = \alpha_y, \quad N_3 = 1 + \alpha_z$$

It is obvious that these direction cosines do not satisfy $L_3^2 + M_3^2 + N_3^2 = 1$, but give

$$L_3^2 + M_3^2 + N_3^2 = 1 + 2\alpha_z + \alpha_x^2 + \alpha_y^2 + \alpha_z^2$$

$$= 1 + 2\varepsilon$$

In arriving at his curvature expressions, Love applies an inextensibility condition which he takes to mean that the elastic axis is unstrained. It is not thought that such a condition is applicable to curved beams. In general, the elastic axis will extend since loading at one point along the beam will create at some other point an internal force component which is parallel to the elastic axis at that point if the beam is curved. In the presented formulation there was no need for imposing an inextensibility relation.

If a curved beam were assumed to be inextensible, then the inextensibility condition is given by setting ε of equation (21) equal to zero. That is,

$$\varepsilon = \alpha_z + \frac{1}{2}(\alpha_x^2 + \alpha_y^2 + \alpha_z^2) = 0 \tag{68}$$

Love deduced the inextensibility condition to be

$$\alpha_z = 0$$

From equation (68) it is seen that the exact inextensibility assumption actually requires that

$$\alpha_z < 0$$

If the inextensibility condition, $\varepsilon = 0$, were applied to equations (48), (49), and (55), then the linearized equations become

$$k_x' = k_x - k_z\alpha_x + k_x\alpha_z + k_y\phi - \frac{d\alpha_y}{ds}$$

$$k_y' = k_y - k_z\alpha_y + k_y\alpha_z - k_x\phi + \frac{d\alpha_x}{ds}$$

$$k_z' = k_z + k_x\alpha_x + k_y\alpha_y + k_z\alpha_z + \frac{d\phi}{ds}$$

By comparing the above with equations (62) through (64) it is seen that imposing the inextensibility condition $\varepsilon = 0$ changes the sign of the terms containing α_z.

From the definitions of α_x, α_y, and α_z (equations (20)) it appears that the terms $k_x\,\alpha_z$, $k_y\,\alpha_z$, and $k_z\,\alpha_z$ are of the same order of magnitude as the other deformation terms in the curvature expressions. However, if the beam were assumed to be inextensible, then it is seen from equation (68) that α_z would be a higher order term since solving equation (68) for α_z gives

$$\alpha_z \approx -\frac{1}{2}\left(\alpha_x^2 + \alpha_y^2\right) \tag{69}$$

Hence, if the beam were assumed inextensible then the α_z terms could be neglected and the linearized curvature equations reduce to Love's. But there is no reason to impose the inextensibility condition; the curvature equations can be derived without using it. Also, the elastic axes of curved beams do, in fact, extend. In addition, α_z is apparently of the same order as α_x and α_y (see equations (20)). Moreover, the inextensibility assumption imposes a constraint relation

between u, v, and w as is given by equation (69).

Using the notation of this paper, Love gives the direction cosines as

$$L_3 = \frac{du}{ds} - k_z v + k_y w \qquad (= \alpha_x)$$

$$M_3 = \frac{dv}{ds} + k_z u - k_x w \qquad (= \alpha_y)$$

$$N_3 = 1 + \frac{dw}{ds} - k_y u + k_x v \qquad (= 1 + \alpha_z)$$

and states

"The equation $L_3^2 + M_3^2 + N_3^2 = 1$ leads, when we neglect squares and products of u, v, w, to the equation

$$\frac{dw}{ds} - k_y u + k_x v = 0 \ ,$$

which expresses the condition that the central-line is unextended. In consequence of this equation we have $N_3 = 1$".

It appears that Love creates his inextensibility condition to explain away the anomaly that his sum of the cosine squares is not equal to unity. Regardless of the correctness or incorrectness of Love's inextensibility relation, the manner in which it is deduced does not seem to be a logical method for arriving at it. It should come from a strain expression. Love should not have concluded from $L_3^2 + M_3^2 + N_3^2 = 1$ that the elastic axis of a curve beam does not

extend. The problem was that his direction cosines were in slight error.

It is interesting to note that the direction cosines developed herein satisfy $l_3^2 + m_3^2 + n_3^2 = 1$ for the inextensible case as well as for the extensional case.

It was mentioned earlier that Love's direction cosine problem arises because products of deformation are not considered. This can be illustrated by using the presented vector approach with Love's assumptions applied. Neglecting products of deformation in the extensional strain expression of equation (18) gives

$$\varepsilon = \frac{d(\Delta \underset{\sim}{R})}{ds} \cdot \underset{\sim}{e}_z$$

where $\frac{d(\Delta \underset{\sim}{R})}{ds}$ is given by equation (19). Substituting yields

$$\varepsilon = \alpha_z$$

The above result is, of course, Love's inextensibility condition when set equal to zero. Equation (22) is repeated below for convenience.

$$\frac{d\underset{\sim}{R}'}{ds} = \alpha_x \underset{\sim}{e}_x + \alpha_y \underset{\sim}{e}_y + (1 + \alpha_z)\underset{\sim}{e}_z$$

For inextension of the elastic axis, ds' = ds. Thus,

$$\frac{d\underset{\sim}{R}'}{ds} = \frac{d\underset{\sim}{R}'}{ds'} = \underset{\sim}{e}_z'$$

and the above becomes

$$\underset{\sim}{e}_z' = \alpha_x \underset{\sim}{e}_x + \alpha_y \underset{\sim}{e}_y + (1 + \alpha_z)\underset{\sim}{e}_z$$

The above is Love's result in vector notation since the direction cosines defined by the above are the same as Love's. Next, Love reasons that $\alpha_z = 0$ (inextensibility condition) and the above becomes

$$\underset{\sim}{e}_z' = \alpha_x \underset{\sim}{e}_x + \alpha_y \underset{\sim}{e}_y + \underset{\sim}{e}_z$$

Although the orthogonality condition on the direction cosines is still not satisfied exactly, it is now satisfied to a higher order when $\alpha_z = 0$ is assumed. It is seen that Love's inextensibility condition ($\alpha_z = 0$) and his direction cosines are a consequence of neglecting products of deformations and assuming $ds' = ds$. It is noted that the latter expression above for $\underset{\sim}{e}_z'$ is the same that Ojalvo and Newman obtained in reference 12. Ojalvo and Newman derived Love's expressions for the curvature components in the appendix of their paper.

A close examination of the presented derivation shows that the additional linear terms are due to the combination of not setting $ds' = ds$ and not setting $\alpha_z = 0$. Consider, for example, how the term $(- k_y \alpha_z)$ arises in the k_y' relation. In the second of equations (46) there appears the product

$$(1 + 2\varepsilon)^{-1} \cos \phi \cos \psi \, [(1 + \alpha_z)k_y]$$

where the factor inside the brackets comes from the definition of a. The term $(1 + 2\varepsilon)^{-1}$ arises from setting $ds' = (1 + 2\varepsilon)^{1/2}\, ds$ when derivatives were taken. The fourth of equations (47) gives the expression for $\cos\psi$ in terms of the displacements. If the binomial theorem is applied and equation (21) is substituted, it is seen that $\cos\psi$ does not contain a linear term. Assuming ϕ to be small the linear approximation of the above product is

$$
\begin{aligned}
(1 + 2\varepsilon)^{-1} \cos\phi \cos\psi\, [(1 + \alpha_z)k_y] &\approx (1 - 2\varepsilon)(1 + \alpha_z)k_y \\
&\approx (1 - 2\alpha_z)(1 + \alpha_z)k_y \\
&\approx k_y - k_y\alpha_z
\end{aligned}
$$

Thus, it is seen that this additional linear term, $- k_y\, \alpha_z$, comes from setting $ds' = (1 + 2\varepsilon)^{1/2}\, ds$ and not letting $\alpha_z = 0$. If equation (63) is considered to be the correct linearized equation for k_y', then the assumptions of $ds' = ds$ and $\alpha_z = 0$ cause an error in k_y' which is equal to $(+ k_y\, \alpha_z)$ since the term does not appear when such assumptions are made. The additional linear terms do not arise from the Eulerian rotations.

Another aspect of the development which should be mentioned concerns the rotation angle ϕ. In the present derivation the angle ϕ is precisely defined as the Eulerian rotation about the body axis that is tangent to the elastic axis. Love's equations contain the rotational displacement β which, evidently, is equivalent to the

angle ϕ when the exact equations are linearized. Love defines β to be the direction cosine of the angle between $\underset{\sim}{e}'_x$ and $\underset{\sim}{e}_y$. From Love's direction cosines ($L_1 = 1$, $M_1 = \beta$, $L_2 = -\beta$, $M_2 = 1$) it is seen that β is essentially a small rotation about the elastic axis with the direction of the elastic axis unchanged. Other authors (see, for example, references 3 and 12) have followed Love's work in defining the rotation angle β. However, this definition is not satisfactory when considering the exact expressions or the nonlinear approximations for the curvatures.

Now consider an untwisted circular beam of radius R whose normal is in the $\underset{\sim}{e}_x$ direction such that $k_y = 1/R$ and $k_x = k_z = 0$. The linearized curvature of the deformed beam as given by equation (66) reduces to

$$k'_y = \frac{1}{R} + \frac{d^2u}{ds^2} + \frac{u}{R^2} \tag{70}$$

The above is the well known formula for the curvature of a circular beam deformed in plane (see, for example, reference 13). This result is often associated with (or identified as) inextensible deformation. However, since the above is not based on inextension the above formula should be properly regarded as the linearized extensional curvature relation for a deformed circular beam.

Aside, it is noted that if the beam formed a planar curve instead of a circle then the linearized curvature would be

$$k_y' = k_y + \frac{d^2u}{ds^2} + k_y^2 u + w\frac{dk_y}{ds}$$

Love arrives at the same expression (equation (70)) for the curvature of a circular beam and this seems surprising at first in view of the fact that equation (70) was obtained utilizing the additional term which Love's equation for k_y' does not contain. The reason why Love obtained the same result can be explained. Using the notation of this paper, Love's curvature expression for the circular beam reduces to

$$k_y' = \frac{1}{R} + \frac{d\alpha_x}{ds}$$

$$= \frac{1}{R} + \frac{d^2u}{ds^2} + \frac{1}{R}\frac{dw}{ds} \tag{71}$$

Love's inextensibility condition reduces to

$$\frac{dw}{ds} - \frac{u}{R} = 0 \quad \text{or} \quad \frac{dw}{ds} = \frac{u}{R}$$

which he substitutes into k_y' when considering applications. This is the same as subtracting $\frac{1}{R}\left(\frac{dw}{ds} - \frac{u}{R}\right)$, or $k_y\,\alpha_z$, from the right hand side of equation (71). This, of course, Love can do since $\alpha_z = 0$ from his point of view (but from the extensional point of view this could not be done since α_z is not equal to zero and is not of higher order). Subtracting $\frac{1}{R}\left(\frac{dw}{ds} - \frac{u}{R}\right)$, or $k_y\,\alpha_z$, from equation (71) gives

equation (70). It was pointed out earlier that if the extensional linearized expression for k_y' is considered to be correct then Love's expression for k_y' would be in error by the amount $(+ k_y \alpha_z)$ (this is not saying that Love's work is in error - he assumed inextension). Continuing to examine Love's work from an extensional point of view, the subtraction of $\frac{1}{R}(\frac{dw}{ds} - \frac{u}{R})$ causes an error of $(- k_y \alpha_z)$. Thus, the two errors cancel each other and this is the reason the curvature results are the same.

In equations (65) through (67), the linearized curvature components are given in terms of u, v, w and ϕ. Love's inextensional results are presented below in the same notation for comparison.

$$k_x' = k_x - \frac{d^2v}{ds^2} - 2k_z \frac{du}{ds} + k_x \frac{dw}{ds} - \frac{dk_z}{ds} u$$

$$+ k_z^2 v + (-k_y k_z + \frac{dk_x}{ds})w + k_y \phi \tag{72}$$

$$k_y' = k_y + \frac{d^2u}{ds^2} - 2k_z \frac{dv}{ds} + k_y \frac{dw}{ds} - k_z^2 u$$

$$- \frac{dk_z}{ds} v + (k_x k_z + \frac{dk_y}{ds})w - k_x \phi \tag{73}$$

$$k_z' = k_z + k_x \frac{du}{ds} + k_y \frac{dv}{ds} + \frac{d\phi}{ds} + k_y k_z u - k_x k_z v \tag{74}$$

Depending upon the problem, one applying the first two of Love's curvature equations may likely apply Love's inextensional relation which is

$$\alpha_z = \frac{dw}{ds} - k_y u + k_x v = 0$$

But it is seen that subtracting $k_x \alpha_z$ from equation (72) and subtracting $k_y \alpha_z$ from equation (73) yields equations (65) and (66), respectively, which are the linearized extensional results. Also, subtracting $k_z \alpha_z$ from equation (74) gives equation (67); however, one using equation (74) would not be expected to do this since the equation does not contain a $\frac{dw}{ds}$ term.

The reason for Love's expressions for k_x' and k_y' (after Love's inextensibility condition is applied) being identical to the linear extensional expressions is the same as that given earlier for the circular beams. For extensional behavior, Love's equation for k_y', for example, is in error by the amount $k_y \alpha_z$. Then, when Love's inextensibility condition is applied to his equation, k_y' is reduced by the amount $k_y \alpha_z$, which is not zero from the extensional point of view.

It has been shown that Love's expressions for k_x' and k_y' with his inextensibility relation applied are the same as the linearized extensional relations for k_x' and k_y'. Thus, authors who have used Love's equations (with his inextensibility relation applied) in the past have mistakenly assumed that an inextensibility restriction was imposed and that their results described inextensional behavior.

It is noted that although the longitudinal strain ε of the elastic axis is defined using the definition of the nonlinear strain

tensor for curvilinear coordinates (see equation (11)), the knowledge that ε is a strain is not needed in the development since ε is eventually expressed in terms of the displacements (see equations (21) and (20)). The derivation of the curvature relations is purely a geometry problem. The development could have begun with the quantity given by equation (13) being denoted as ε. Although ε does appear in the final expressions for the exact curvature components, it may be eliminated by substituting equation (21). For the curvature expressions to be mathematically correct the extensional strain ε must not be set equal to zero.

C. Development of Strain-Displacement Relations

The strain-displacement relations will be developed with the following assumptions being made regarding the displacement of points within the cross-section.

1. Torsional deformation causes warping of the cross-section and the out-of-plane displacements are proportional to the Saint-Venant warping function.

2. The cross-sectional body-axis coordinates of points off the elastic axis do not change during deformation. This assumption is in agreement with Saint-Venant torsion theory that cross-sections do not change in size or shape during deformation.

3. Shear deformation is a negligible effect. Thus, except for warping due to torsion, cross-sections remain normal to the elastic axis during deformation.

Letting x and y be the cross-sectional body-axis coordinates of a point in the beam and letting $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ be position vectors to the point before and after deformation, respectively, then

$$\underset{\sim}{r}(x,y,s) = \underset{\sim}{R}(s) + x\underset{\sim}{e}_x(s) + y\underset{\sim}{e}_y(s) \tag{75}$$

$$\underset{\sim}{r}'(x,y,s') = \underset{\sim}{R}'(s') + x\underset{\sim}{e}_x'(s') + y\underset{\sim}{e}_y'(s') + W(x,y)\frac{d\phi(s')}{ds'}\underset{\sim}{e}_z'(s') \tag{76}$$

where $W(x,y)$ is the Saint-Venant warping displacement function. Taking the differentials of the above relations and substituting equations (8), (17), (25), and (32) for the derivatives of the vectors gives

$$\begin{aligned} d\underset{\sim}{r} &= (dx - yk_z ds)\underset{\sim}{e}_x + (dy + xk_z ds)\underset{\sim}{e}_y \\ &\quad + (1 - xk_y + yk_x)ds\ \underset{\sim}{e}_z \end{aligned} \tag{77}$$

$$\begin{aligned} d\underset{\sim}{r}' &= (dx - yk_z' ds' + Wk_y'\frac{d\phi}{ds'}ds')\underset{\sim}{e}_x' \\ &\quad + (dy + xk_z' ds' - Wk_x'\frac{d\phi}{ds'}ds')\underset{\sim}{e}_y' \\ &\quad + (ds' - xk_y' ds' + yk_x' ds' \\ &\quad + \frac{\partial W}{\partial x}\frac{d\phi}{ds'}dx + \frac{\partial W}{\partial y}\frac{d\phi}{ds'}dy + W\frac{d^2\phi}{ds'^2}ds')\underset{\sim}{e}_z' \end{aligned} \tag{78}$$

The magnitudes of $d\underset{\sim}{r}$ and $d\underset{\sim}{r}'$ are the lengths of an arbitrary

differential line element in the beam before and after deformation, respectively. If the differential length is denoted by $d\hat{s}$, then

$$\left.\begin{aligned} (d\hat{s})^2 &= d\underset{\sim}{r} \cdot d\underset{\sim}{r} \\ (d\hat{s}')^2 &= d\underset{\sim}{r}' \cdot d\underset{\sim}{r}' \end{aligned}\right\} \quad (79)$$

Substituting equations (77) and (78) into the above and replacing ds', by

$$ds' = (1 + 2\varepsilon)^{1/2}\, ds$$

in the second of the resulting equations and then collecting terms yields

$$(d\hat{s})^2 = dx^2 + dy^2 + (1 - 2xk_y + 2yk_x + x^2k_y^2 - 2xyk_xk_y$$

$$+ y^2k_x^2 + y^2k_z^2 + x^2k_z^2)ds^2 - 2yk_z dxds + 2xk_z dyds \qquad (80)$$

$$
\begin{aligned}
(d\hat{s}')^2 = {} & [1 + (\tfrac{\partial W}{\partial x})^2 (\tfrac{d\phi}{ds})^2 (1 + 2\varepsilon)^{-1}]dx^2 \\
& + [1 + (\tfrac{\partial W}{\partial y})^2 (\tfrac{d\phi}{ds})^2 (1 + 2\varepsilon)^{-1}]dy^2 \\
& + [1 + 2\varepsilon + y^2 k_z'^2 (1 + 2\varepsilon) - 2yWk_y' k_z' \tfrac{d\phi}{ds} (1 + 2\varepsilon)^{1/2} \\
& + W^2 k_y'^2 (\tfrac{d\phi}{ds})^2 + x^2 k_z'^2 (1 + 2\varepsilon) - 2xWk_x' k_z' \tfrac{d\phi}{ds} (1 + 2\varepsilon)^{1/2} \\
& + W^2 k_x'^2 (\tfrac{d\phi}{ds})^2 - 2xyk_y'(1 + 2\varepsilon) + 2yk_x'(1 + 2\varepsilon) + 2W \frac{d^2\phi}{ds^2} \\
& + x^2 k_y'^2 (1 + 2\varepsilon) - 2xyk_x' k_y'(1 + 2\varepsilon) - 2xWk_y' \frac{d^2\phi}{ds^2} \\
& + y^2 k_x'^2 (1 + 2\varepsilon) + 2yWk_x' \frac{d^2\phi}{ds^2} + W^2 (\frac{d^2\phi}{ds^2})^2 (1 + 2\varepsilon)^{-1}]ds^2 \\
& + [2 \frac{\partial W}{\partial x} \frac{\partial W}{\partial y} (\tfrac{d\phi}{ds})^2 (1 + 2\varepsilon)^{-1}]dxdy \\
& + [-2yk_z'(1 + 2\varepsilon)^{1/2} + 2Wk_y' \frac{d\phi}{ds} + 2 \frac{\partial W}{\partial x} \frac{d\phi}{ds} - 2x \frac{\partial W}{\partial x} k_y' \frac{d\phi}{ds} \\
& + 2y \frac{\partial W}{\partial x} k_x' \frac{d\phi}{ds} + 2W \frac{\partial W}{\partial x} \frac{d\phi}{ds} \frac{d^2\phi}{ds^2} (1 + 2\varepsilon)]dxds \\
& + [2xk_z'(1 + 2\varepsilon)^{1/2} - 2Wk_x' \frac{d\phi}{ds} + 2 \frac{\partial W}{\partial y} \frac{d\phi}{ds} - 2x \frac{\partial W}{\partial y} k_y' \frac{d\phi}{ds} \\
& + 2y \frac{\partial W}{\partial y} k_x' \frac{d\phi}{ds} + 2W \frac{\partial W}{\partial y} \frac{d\phi}{ds} \frac{d^2\phi}{ds^2} (1 + 2\varepsilon)]dyds \qquad (81)
\end{aligned}
$$

Equations (80) and (81) define the metric coefficients g_{ij} and G_{ij} for the undeformed and deformed beam, respectively, since the metric tensors g_{ij} and G_{ij} are, by definition, given by

$$\left.\begin{aligned}(d\hat{s})^2 &= g_{ij}dx^i dx^j \\ (d\hat{s}')^2 &= G_{ij}dx^i dx^j\end{aligned}\right\} \quad (82)$$

Being that

$$(d\hat{s}')^2 - (d\hat{s})^2 = (G_{ij} - g_{ij})dx^i dx^j$$

is a measure of the deformation, the covariant strain tensor is defined to be

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}) \quad (83)$$

However, the physical components of the infinitesimal strain tensor are needed for expressing the stress resultants in terms of the displacements.

The normal strain components are defined by

$$e_{ii} = \frac{d\hat{s}'(i) - d\hat{s}(i)}{d\hat{s}(i)} = \frac{\sqrt{G_{ii}}dx^i - \sqrt{g_{ii}}dx^i}{\sqrt{g_{ii}}\, dx^i} \quad (84)$$

and the shearing strain components are given by

$$e_{ij} = \theta_{ij} - \theta'_{ij} \quad (i \neq j) \quad (85)$$

where Θ_{ij} and Θ'_{ij} are the angles between the x^i and x^j

coordinate curves before and after deformation, respectively. These angles are related to the metric tensor components by

$$\left.\begin{aligned} \cos\theta_{ij} &= g_{ij}/\sqrt{g_{ii}g_{jj}} \\ \cos\theta'_{ij} &= G_{ij}/\sqrt{G_{ii}G_{jj}} \end{aligned}\right\} \quad (86)$$

It is noted that in equations (84) and (86) and in the following expressions the repeated subscripts do not denote summation. Beginning with the definitions of equations (84) and (85) it is shown in reference 8 and other texts that the normal and shearing strains are given by

$$e_{ii} = \gamma_{ii}/g_{ii} \tag{87}$$

$$e_{ij} = [2\gamma_{ij} - g_{ij}(\frac{\gamma_{ii}}{g_{ii}} + \frac{\gamma_{jj}}{g_{jj}})]\Big/\sqrt{g_{ii}g_{jj} - (g_{ij})^2} \tag{88}$$

Using equations (83) and (87) the above may be rewritten as

$$e_{ii} = \frac{1}{2}(G_{ii} - g_{ii})/g_{ii} \tag{89}$$

$$e_{ij} = [G_{ij} - g_{ij} - g_{ij}(e_{ii} + e_{jj})]\Big/\sqrt{g_{ii}g_{jj} - (g_{ij})^2} \tag{90}$$

As noted earlier, G_{ij} and g_{ij} are defined by equations (80) and (81). In the infinitesimal strain theory, products of deformation are neglected in arriving at equations (87) and (88). Thus, deformation

products will not be included in the expressions for the metric coefficients. The linearized expressions for the curvature components k_x', k_y', and k_z' are given by equations (62), (63), and (64). Also, ε is defined by equation (21). These equations may be substituted into equation (81) to give $(d\hat{s}')^2$ in terms of the displacements α_x, α_y, α_z, and ϕ. Letting

$$dx = dx^1$$

$$dy = dx^2$$

$$dz = dx^3$$

and comparing equations (80) and (81) with equations (82), the metric coefficients for the undeformed and deformed beam are found to be

$$\left.\begin{aligned}
g_{11} &= 1 \\
g_{22} &= 1 \\
g_{33} &= 1 + 2(y\,k_x - x\,k_y) + (y\,k_x - x\,k_y)^2 + (x^2 + y^2)\,k_z^2 \\
g_{12} &= 0 \\
g_{13} &= -y\,k_z \\
g_{23} &= x\,k_z
\end{aligned}\right\} \quad (91)$$

$$\left.\begin{aligned}
G_{11} &= 1 \\
G_{22} &= 1
\end{aligned}\right\}$$

$$\left.\begin{aligned}
G_{33} &= g_{33} + 2(1 + yk_x - xk_y)(\alpha_z + xk_x\phi + yk_y\phi \\
&\quad - x\frac{d\alpha_x}{ds} - y\frac{d\alpha_y}{ds} + W\frac{d^2\phi}{ds^2}) \\
&\quad + 2(-y + x^2k_x + xyk_y)k_z\alpha_x + 2(x + y^2k_y + xyk_x)k_z\alpha_y \\
&\quad + 2(x^2 + y^2 - xk_xW - yk_yW)k_z \\
G_{12} &= 0 \\
G_{13} &= g_{13} - y(k_x\alpha_x + k_y\alpha_y + \frac{d\phi}{ds}) \\
&\quad + (1 + yk_x - xk_y)\frac{\partial W}{\partial x}\frac{d\phi}{ds} + Wk_y\frac{d\phi}{ds} \\
G_{23} &= g_{23} + x(k_x\alpha_x + k_y\alpha_y + \frac{d\phi}{ds}) \\
&\quad + (1 + yk_x - xk_y)\frac{\partial W}{\partial y}\frac{d\phi}{ds} - Wk_x\frac{d\phi}{ds}
\end{aligned}\right\} \quad (92)$$

From the above relations and equations (89) and (90) it is immediately seen that

$$e_{11} = e_{22} = e_{12} = 0 \tag{93}$$

It is assumed that the curvatures are small such that the product of a cross-sectional coordinate and a curvature component is much less than unity (i.e., $x_i\ k_j << 1$). In other words, x and y are small compared to the radius of curvature and the angle obtained by multiplying the twist curvature k_z (radians per unit length of beam) by x (or y) is much less than one radian. With this assumption the

quantity $\frac{1}{g_{33}}$ appearing in equation (89) when $i = 3$ can be approximated using the binomial theorem to give

$$\frac{1}{g_{33}} = 1 - 2(yk_x - xk_y) + 3(yk_x - xk_y)^2 - (x^2 + y^2)k_z^2$$

$$- 4(yk_x - xk_y)^3 + 4(yk_x - xk_y)(x^2 + y^2)k_z^2 + O(x_i^4 k_j^4) \qquad (94)$$

Similarly, when e_{13} and e_{23} are determined, the inverse of the denominator of equation (90) is approximated by

$$\left.\begin{aligned} [g_{11}g_{33} - (g_{13})^2]^{-1/2} &= 1 - (yk_x - xk_y) + (yk_x - xk_y)^2 - \tfrac{1}{2}x^2k_z^2 + O(x_i^3k_j^3) \\ [g_{22}g_{33} - (g_{23})^2]^{-1/2} &= 1 - (yk_x - xk_y) + (yk_x - xk_y)^2 - \tfrac{1}{2}y^2k_z^2 + O(x_j^3k_j^3) \end{aligned}\right\} \qquad (95)$$

Besides giving more accurate strain expressions, the retaining of higher order terms allows the analysis to be applied to problems having larger curvatures. In the strain expressions terms of order $x_i^3\ k_j^3\ \alpha_k$ (or ϕ) will be kept. Equations (20) and equations (62) through (64) indicate that $k_i\ \alpha_j$ is of the same order of magnitude as $\frac{d\alpha_i}{ds}$. Thus, for terms involving derivatives of α_x, α_y, or ϕ, terms of order $x_i^3\ k_j^2$ will be kept. Also, W will be taken to be $O(x_i^2)$ since for all known solutions for the warping function, W is of the same order of magnitude as the product of the cross-sectional coordinates or smaller. Substituting equations (91), (92), (94), and (95) into equations (89) and (90) gives the strains $e_{zz}(=e_{33})$, and $e_{xz}(=e_{13})$, and $e_{yz}(=e_{23})$ as

$$
\begin{aligned}
e_{zz} = {} & [-y - xyk_y + 2y^2k_x + x^2k_x - x^2yk_y^2 - 3y^2k_x^2 \\
& + 2x^3k_xk_y - 2x^2yk_x^2 + 4xy^2k_xk_y + y(x^2 + y^2)k_z^2]k_z\alpha_x \\
& + [x - xyk_x + 2x^2k_y + y^2k_y + xy^2k_x^2 + 3x^3k_y^2 \\
& - 2y^3k_xk_y + 2xy^2k_y^2 - 4x^2yk_xk_y - x(x^2 + y^2)k_z^2]k_z\alpha_y \\
& + [1 + xk_y - yk_x + x^2k_y^2 - 2xyk_xk_y + y^2k_x^2 - (x^2 + y^2)k_z^2 \\
& + x^3k_y^3 - 3x^2yk_xk_y^2 + 3xy^2k_x^2k_y - y^3k_x^3 + 3(x^2 + y^2)(yk_x - xk_y)k_z^2]\alpha_z \\
& + [xk_x + yk_y + x^2k_xk_y + xyk_y^2 - xyk_x^2 - y^2k_xk_y + x^3k_xk_y^2 + x^2yk_y^3 \\
& - 2x^2yk_x^2k_y - 2xy^2k_xk_y^2 + xy^2k_x^3 + y^3k_x^2k_y - (xk_x + yk_y)(x^2 + y^2)k_z^2]\phi \\
& - x[1 - (yk_x - xk_y) + (yk_x - xk_y)^2 - (x^2 + y^2)k_z^2]\frac{d\alpha_x}{ds} \\
& - y[1 - (yk_x - xk_y) + (yk_x - xk_y)^2 - (x^2 + y^2)k_z^2]\frac{d\alpha_y}{ds} \\
& + [(1 - 2yk_x + 2xk_y)(x^2 + y^2) - (xk_x + yk_x)W]k_z\frac{d\phi}{ds} + W\frac{d^2\phi}{ds^2} \qquad (96)
\end{aligned}
$$

$$
\begin{aligned}
e_{xz} = {} & -y[k_x - (yk_x - xk_y)k_x + yk_z^2 + (yk_x - xk_y)^2 k_x \\
& \qquad + (-\tfrac{3}{2}x^2k_x + 2xyk_y - 3y^2k_x)k_z^2]\alpha_x \\
& - y[k_y - (yk_x - xk_y)k_y - xk_z^2 + (yk_x - xk_y)^2 k_y \\
& \qquad + (-\tfrac{7}{2}x^2k_y + 2xyk_x - y^2k_y)k_z^2]\alpha_y \\
& + yk_z[1 - 2(yk_x - xk_y) + 3(yk_x - xk_y)^2 - (\tfrac{3}{2}x^2 + y^2)k_z^2]\alpha_z \\
& + yk_z[xk_x + yk_y + 2(x^2 - y^2)k_xk_y + 2xy(k_y^2 - k_x^2)]\phi \\
& - xyk_z[1 - 2(yk_x - xk_y)]\frac{d\alpha_x}{ds} - y^2k_z[1 - 2(yk_x - xk_y)]\frac{d\alpha_y}{ds} \\
& - y[1 - yk_x + xk_y + (yk_x - xk_y)^2 - (\tfrac{3}{2}x^2 + y^2)k_z^2]\frac{d\phi}{ds} \\
& + (1 - \tfrac{1}{2}x^2k_z^2)\frac{\partial W}{\partial x}\frac{d\phi}{ds} + (1 - yk_x + xk_y)Wk_y\frac{d\phi}{ds} + yWk_z\frac{d^2\phi}{ds^2}
\end{aligned}
\tag{97}
$$

$$e_{yz} = x[k_x - (yk_x - xk_y)k_x + yk_z^2 + (yk_x - xk_y)^2 k_y$$

$$+ (-\tfrac{7}{2} y^2 k_x + 2xyk_y - x^2 k_x)k_z^2]\alpha_x$$

$$+ x[k_y - (yk_x - xk_y)k_y - xk_z^2 + (yk_x - xk_y)^2 k_y$$

$$+ (-\tfrac{3}{2} y^2 k_y + 2xyk_x - 3x^2 k_y)k_z^2]\alpha_y$$

$$- xk_z[1 - 2(yk_x - xk_y) + 3(yk_x - xk_y)^2 - (x^2 + \tfrac{3}{2} y^2)k_z^2]\alpha_z$$

$$- xk_z[xk_x + yk_y + 2(x^2 - y^2)k_x k_y + 2xy(k_y^2 - k_x^2)]\phi$$

$$+ x^2 k_z[1 - 2(yk_x - xk_y)] \frac{d\alpha_x}{ds} + xyk_z[1 - 2(yk_x - xk_y)] \frac{d\alpha_y}{ds}$$

$$+ x[1 - yk_x + xk_y + (yk_x - xk_y)^2 - (x^2 + \tfrac{3}{2} y^2)k_z^2] \frac{d\phi}{ds}$$

$$+ (1 - \tfrac{1}{2} y^2 k_z^2)\frac{\partial W}{\partial y} \frac{d\phi}{ds} - (1 - yk_x + xk_y)Wk_x \frac{d\phi}{ds} - xWk_z \frac{d^2\phi}{ds^2} \qquad (98)$$

In the above strain-displacement relations terms have been collected on each variable. If the curvatures are very small it is obvious that the majority of the terms are negligible. When retaining only the largest terms associated with each variable the strain expressions are approximated by

$$e_{zz} = \alpha_z - x(-k_z\alpha_y - k_y\alpha_z - k_x\phi + \frac{d\alpha_x}{ds})$$

$$+ y(-k_z\alpha_x - k_x\alpha_z + k_y\phi - \frac{d\alpha_y}{ds}) + (x^2 + y^2)k_z \frac{d\phi}{ds} + W \frac{d^2\phi}{ds^2} \tag{99}$$

$$e_{xz} = -y(k_x\alpha_x + k_y\alpha_y - k_z\alpha_z + \frac{d\phi}{ds}) + \frac{\partial W}{\partial x}\frac{d\phi}{ds}$$

$$- xyk_z \frac{d\alpha_x}{ds} - y^2k_z \frac{d\alpha_y}{ds} + yk_z(xk_x + yk_y)\phi + yk_z W \frac{d^2\phi}{ds^2} \tag{100}$$

$$e_{yz} = x(k_x\alpha_x + k_y\alpha_y - k_z\alpha_z + \frac{d\phi}{ds}) - \frac{\partial W}{\partial y}\frac{d\phi}{ds}$$

$$+ x^2k_z \frac{d\alpha_x}{ds} + xyk_z \frac{d\alpha_y}{ds} - xk_z(xk_x + yk_y)\phi - xk_zW \frac{d^2\phi}{ds^2} \tag{101}$$

D. Discussion of Strain-Displacement Relations

In the expression for e_{zz} as given by equation (99), the first term is the linearized form of the strain of the elastic axis that was given in the development of the curvatures. The quantities inside the first two sets of parentheses are equal to $(k_y' - k_y)$ and $(k_x' - k_x)$, respectively, and these are the changes in the curvatures of the elastic axis due to deformation. The term $(x^2 + y^2)k_z\, d\phi/ds$ is due to the initial twist of the beam. This term has been obtained previously by Houbolt and Brooks (see reference 14) in their analysis of a straight pretwisted propeller. The last term in equation (99) arises because uniform torsion has not been assumed.

The quantity enclosed in the first set of parentheses in each of equation (100) and (101) is equal to the change in the torsional curvature component (i.e., $k_z' - k_z$). The terms $\frac{\partial W}{\partial x}\frac{d\phi}{ds}$ and $\frac{\partial W}{\partial y}\frac{d\phi}{ds}$ contribute to the Saint-Venant torsional stiffness constant that appears in the torsional stress resultant. Based on the previous observation that $k_i\ \alpha_j$ may be of the same order of magnitude as $\frac{d\alpha_j}{ds}$ and the assumption that $k_i\ x_j << 1$, the remaining four terms in each of equations (100) and (101) may be smaller than those mentioned above. However, these terms cannot be assumed negligible since there may be instances where, for example, α_x may be zero while its derivative is nonzero. The last four terms in each of equations (100) and (101) appear because of the beam's initial twist curvature.

It is noted that since shear deformation was neglected the strains e_{xz} and e_{yz} do not reflect the strain arising from transverse shear. The expressions for e_{xz} and e_{yz} are used to determine the torsional moment stress resultant. The omission of the strain due to transverse shear does not affect this stress resultant since moments are taken about the shear center of the cross-section.

In the development of the strain relations it was assumed that the warping displacement was proportional to the Saint-Venant warping function W. The solution for the Saint-Venant warping function of a cross-section assumes that the torsion is uniform (constant torque, constant $d\phi/ds$). Since this analysis is not restricted to uniform torsion there is an inconsistency. However, for many problems in which the torsion is nonuniform, it is common practice to determine the

torsional stiffness using the Saint-Venant theory. This is especially true for natural vibration problems. The effect of nonuniform torsion on the torsional stress resultant becomes important for flanged thin-wall open sections such as I-beams. For such beams the theory has been developed to correct the stress-resultant expression for nonuniform torsion (see reference 15). For beams which are not thin-walled, a general theory for nonuniform torsion has not yet been developed. For this analysis, it will be assumed that the torques acting on the beam are distributed loadings (such as inertial loads) and that the derivatives of $d\phi/ds$ are small such that the effects of nonuniform torsion are negligible. With this assumption, the terms proportional to $d^2\phi/ds^2$ in the strain expressions may be dropped. However, for the examples presented herein these terms have no effect on the stress resultants since the cross-section is assumed to be doubly symmetric.

The strain-displacement relations have been developed for the purpose of obtaining expressions for the stress resultants in terms of the displacements. In the classical theory for curved space beams the strain-displacement relations are not derived. The stress-resultants are assumed to be equal to the changes in the curvatures (without the extensional effects included) times a cross-sectional stiffness property. The importance of using the strain-displacement relations instead of the classical approach will be seen when the stress resultants are compared since the classical approach can lead to considerable error.

It was noted earlier that the accuracy of the strain expressions depends upon the number of terms kept in the binomial expansion of the denominators of equations (89) and (90). The strain e_{zz} as given by equation (96) can be compared with the Winkler theory for curved beams. The Winkler theory, which is valid for large curvatures, applies to untwisted beams lying in a plane, having a constant radius of curvature, and with the deformations being in-plane. Thus, considering a beam of constant curvature which lies in and is displaced in the x, z plane such that $k_y = 1/R$ and $k_x = k_z = \gamma = v = \phi = \alpha_y = 0$, equation (96) gives

$$e_{zz} = (1 + \frac{x}{R} + \frac{x^2}{R^2} + \frac{x^3}{R^3})\alpha_z - x(1 + \frac{x}{R} + \frac{x^2}{R^2}) \frac{d\alpha_x}{ds} \tag{102}$$

Substituting equations (20) for α_x and α_z yields

$$e_{zz} = \frac{dw}{ds} - (1 + \frac{x}{R} + \frac{x^2}{R^2} + \frac{x^3}{R^3}) \frac{u}{R} - x(1 + \frac{x}{R} + \frac{x^2}{R^2}) \frac{d^2u}{ds^2} \tag{103}$$

It is noted that according to the sign convention established by figure 2 and equations (1), x and u are positive in the direction toward the center of curvature when $\gamma = 0$.

In reference 16 Langhaar gives the following expression (notation and signs have altered to conform to that presented herein) for the strain according to the Winkler theory

$$e_{zz} = \frac{dw}{ds} - \frac{u/R}{1 - x/R} - \frac{x\, d^2u/ds^2}{1 - x/R} \tag{104}$$

The above gives a parabolic distribution of strain over the cross-section instead of the familiar linear distribution associated with the bending of straight beams. This distribution is brought about because equation (104) was obtained by taking into account the differences in the initial lengths of the longitudinal fibers in a differential segment of the beam. The strain distribution given in equation (103) appears because approximations were made for the complicated denominators of the strain expressions for space beams.

From the binomial theorem it is seen that equations (103) and (104) are essentially the same when x is small compared to R. In fact, since the higher order terms were kept, the strain expressions can be applied to beams having moderately large curvatures. For example, suppose that $x = 0.2\ R$. Then, equation (103) gives

$$e_{zz} = \frac{dw}{ds} - 1.248\ \frac{u}{R} - 0.248\ R\ \frac{d^2u}{ds^2}$$

From equation (104), which is the expression for large curvatures, it is found for $x/R = 0.2$ that

$$e_{zz} = \frac{dw}{ds} - 1.250\ \frac{u}{R} - 0.250\ R\ \frac{d^2u}{ds^2}$$

If $x = 0.2\ R$ is largest cross-sectional coordinate of a point in the beam, then it is seen by comparing the two above results that the largest error in the strain as given by equation (103) is less than one percent (the same error is obtained at $x = -0.2\ R$). Equation (103)

is extremely accurate for $x/R \leq 0.1$ and does not give grossly inaccurate results until x is about one-half of R.

Equation (99) gave the expression for e_{zz} with higher order terms dropped. For the example being considered it reduces to

$$e_{zz} = \frac{dw}{ds} - \left(1 + \frac{x}{R}\right) \frac{u}{R} - x \frac{d^2u}{ds^2}$$

At $x = 0.2\ R$ the above becomes

$$e_{zz} = \frac{dw}{ds} - 1.2 \frac{u}{R} - 0.2 \frac{d^2u}{ds^2}$$

When compared to the result of the Winkler theory, the last term is seen to be 20% in error.

In the development of the stress resultants the more accurate strain expressions containing the higher order terms will be used (i.e., equations (96), (97), and (98)).

It is recalled that in the classical theory the elastic axis is assumed to be inextensible such that $\alpha_z = 0$. From equation (102) it is observed that the extensional strain α_z of the elastic axis must be included. Otherwise, the strain-displacement relation bears no resemblance to what it should be.

In reference 17 Oden gives an expression for the strain of a curved planar beam of radius R. His result is different from that given by Langhaar and from that presented herein. Oden arrives at the total strain by summing strains that arise from three types of

displacement (i.e., extensional, radial, and rotational). The strain due to the radial displacement is given as

$$(\epsilon_s)_1 = \frac{-y}{1 - y/R} \frac{v}{R^2}$$

where v is the inward radial displacement. The strain due to a radial displacement is actually the second term of equation (104). By considering the strain due to the radial displacement of a circular ring it is obvious that the above result is in error. The error of reference 17 was caused by taking the final length of an incremental fiber at a distance y from the centroid to be $\Delta s[1 - y(R - v)]$ instead of $\Delta s[1 - (y + v)/R]$ as it should be.

E. Development of Stress Resultants

The stress resultants are determined in the usual manner by summing over the cross-section the stresses that are acting on it. But first it is noted that in general the x, y, and z curvilinear coordinates (s and z are used interchangeably) and not orthogonal. When the beam is twisted ($k_z \neq 0$) the coordinate lines are orthogonal only at points along the elastic axis. The x and y coordinate lines are orthogonal at every point in the cross-section before deformation ($g_{12} = 0$) and after deformation ($G_{12} = 0$ when neglecting product of deformation). But the z coordinate line passing through a point in the cross-section off the elastic axis is not normal to the xy plane ($g_{13} \neq 0$, $g_{23} \neq 0$, $G_{13} \neq 0$, $G_{23} \neq 0$). This can be visualized for a

twisted beam since the z coordinate line is defined by the intersection of the coordinate surfaces x = constant and y = constant.

Figure 3 shows the stress σ_{zz} acting at point P in the cross-section and in the direction of the z coordinate line passing through that point. Also shown are the components of σ_{zz} in the directions of $\underset{\sim}{e}'_z$, $\underset{\sim}{e}'_y$, and $\underset{\sim}{e}'_z$. The components of σ_{zz} in the $\underset{\sim}{e}'_x$, $\underset{\sim}{e}'_y$, and $\underset{\sim}{e}'_z$ directions contribute to the torsional moment stress resultant. This effect, which occurs in twisted beams, is discussed by Goodier in reference 15. The bending moment stress resultants about the x and y axes and the axial force stress resultant are determined from the component of σ_{zz} in the $\underset{\sim}{e}'_z$ direction.

The component of σ_{zz} in the $\underset{\sim}{e}'_x$ direction, for example, is $\sigma_{zz} \cdot \cos \Theta'_{13}$ where $\cos \Theta'_{13}$ is defined in terms of the metric coefficients G_{ij} by the second of equations (86). However, since products of deformations will not be included in the stress resultant expressions, the reference of the undeformed beam will be used. If the subscripts x_o, y_o, and z_o are used to denote the components of σ_{zz} at point P in the $\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$, and $\underset{\sim}{e}_z$ directions, respectively, then

$$\left.\begin{aligned} (\sigma_{zz})_{x_o} &= \sigma_{zz} \cos \theta_{13} = \sigma_{zz} g_{13}/\sqrt{g_{33}} \\ (\sigma_{zz})_{y_o} &= \sigma_{zz} \cos \theta_{23} = \sigma_{zz} g_{23}/\sqrt{g_{33}} \end{aligned}\right\} \quad (105)$$

since $g_{11} = g_{22} = 1$. Letting l_z be the direction cosine of the angle between σ_{zz} and $\underset{\sim}{e}_z$, the orthogonality relation

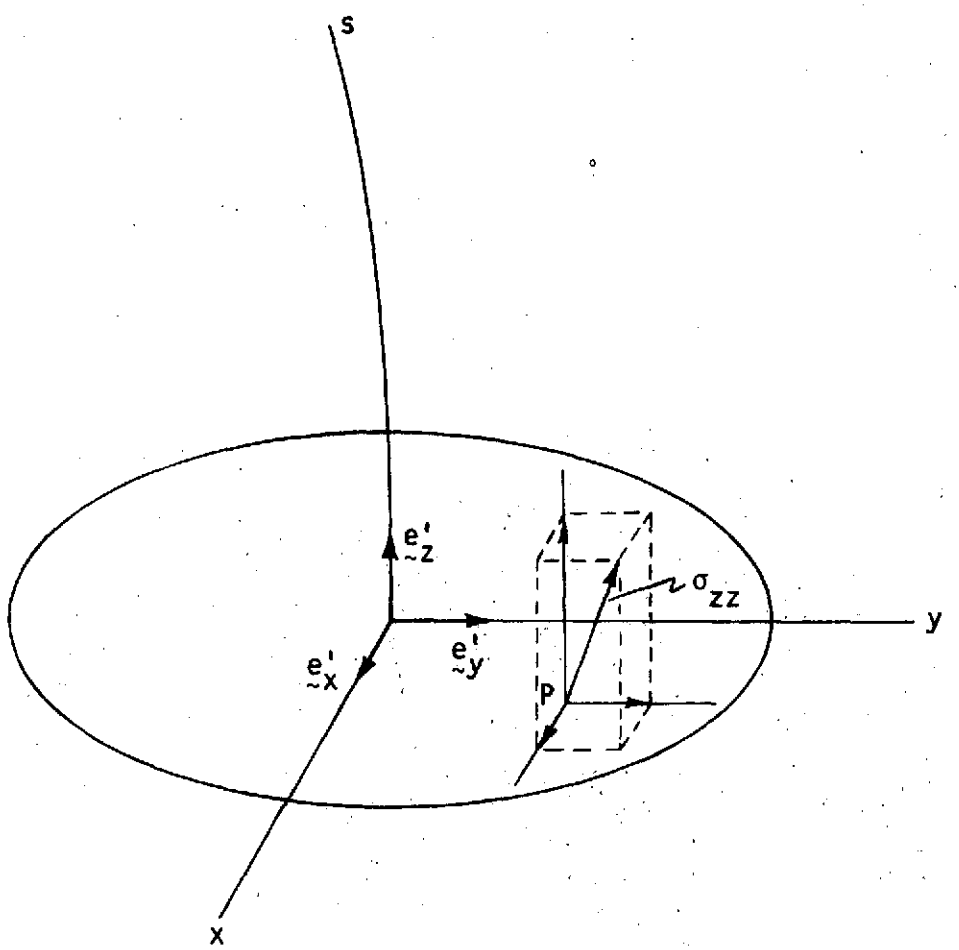

Figure 3.- The stress σ_{zz} at a point in the cross-section.

$$\cos^2 \theta_{13} + \cos^2 \theta_{23} + l_z^2 = 1$$

gives

$$l_z = \left(1 - \frac{g_{13}^2 + g_{23}^2}{g_{33}}\right)^{1/2}$$

Thus,

$$(\sigma_{zz})_{z_o} = \sigma_{zz} \left(1 - \frac{g_{13}^2 + g_{23}^2}{g_{33}}\right)^{1/2} \tag{106}$$

From these components of σ_{zz} and the shear stresses acting in the x, y plane, the stress resultants are given by the following integrals.

$$M_x = \int_A E\, y\left(1 - \frac{g_{13}^2 + g_{23}^2}{g_{33}}\right)^{1/2} e_{zz} dA \tag{107}$$

$$M_y = -\int_A E\, x\left(1 - \frac{g_{13}^2 + g_{23}^2}{g_{33}}\right)^{1/2} e_{zz} dA \tag{108}$$

$$M_z = \int_A G(-y e_{xz} + x e_{yz}) dA + \int_A E \left(-\frac{y g_{13}}{\sqrt{g_{33}}} + \frac{x g_{23}}{\sqrt{g_{33}}}\right) e_{zz} dA \tag{109}$$

$$F_z = \int_A E \left(1 - \frac{g_{13}^2 + g_{23}^2}{g_{33}}\right)^{1/2} e_{zz}\, dA \tag{110}$$

Using the binomial theorem, it is found from equations (91) and (96) that

$$\left(1 - \frac{g_{13}^2 + g_{23}^2}{g_{33}}\right)^{1/2} e_{zz}$$

$$= [-y - xyk_y + 2y^2k_x + x^2k_x - x^2yk_y^2 - 3y^3k_x^2$$

$$+ 2x^3k_xk_y - 2x^2yk_x^2 + 4xy^2k_xk_y + \frac{3}{2}\, y(x^2 + y^2)k_z^2]k_z\alpha_x$$

$$+ [x - xyk_x + 2x^2k_y + y^2k_y + xy^2k_x^2 + 3x^3k_y^2$$

$$- 2y^3k_xk_y + 2xy^2k_y^2 - 4x^2yk_xk_y - \frac{3}{2}\, x(x^2 + y^2)k_z^2]k_z\alpha_y$$

$$+ [1 + xk_y - yk_x + x^2k_y^2 - 2xyk_xk_y + y^2k_x^2$$

$$- \frac{3}{2}\,(x^2 + y^2)k_z^2 + x^3k_y^3 - 3x^2yk_xk_y^2 + 3xy^2k_x^2k_y$$

$$- y^3k_x^3 + \frac{9}{2}\,(x^2 + y^2)(yk_x - xk_y)k_z^2]\alpha_z$$

$$+ [xk_x + yk_y + x^2k_xk_y + xyk_y^2 - xyk_x^2 - y^2k_xk_y$$

$$+ x^3k_xk_y^2 + x^2yk_y^3 - 2x^2yk_x^2k_y - 2xy^2k_xk_y^2$$

$$+ xy^2k_x^3 + y^3k_x^2k_y - \frac{3}{2}\,(x^2 + y^2)(xk_x + yk_y)k_z^2]\phi$$

$$- x[1 - yk_x + xk_y + (yk_x - xk_y)^2 - \frac{3}{2}\,(x^2 + y^2)k_z^2]\,\frac{d\alpha_x}{ds}$$

$$- y[1 - yk_x + xk_y + (yk_x - xk_y)^2 - \frac{3}{2}\,(x^2 + y^2)k_z^2]\,\frac{d\alpha_y}{ds}$$

$$+ [(x^2 + y^2)(1 - 2yk_x + 2xk_y) - (xk_x + yk_y)W]k_z\,\frac{d\phi}{ds} + W\,\frac{d^2\phi}{ds^2} \qquad (111)$$

$$\left(-\frac{yg_{13}}{\sqrt{g_{33}}} + \frac{xg_{23}}{\sqrt{g_{33}}}\right) e_{zz} = \frac{(x^2 + y^2)k_z}{\sqrt{g_{33}}} e_{zz}$$

$$= (-y - 2xyk_y + 3y^2k_x + x^2k_x)(x^2 + y^2)k_z^2\alpha_x$$

$$+ (x - 2xyk_x + 3x^2k_y + y^2k_y)(x^2 + y^2)k_z^2\alpha_y$$

$$+ [1 + 2xk_y - 2yk_x + 3x^2k_y^2 - 6xyk_xk_y$$

$$+ 3y^2k_x^2 - \frac{3}{2}(x^2 + y^2)k_z^2](x^2 + y^2)k_z\alpha_z$$

$$+ (xk_x + yk_y)(1 - 2yk_x + 2xk_y)(x^2 + y^2)k_z\phi$$

$$- x(1 - 2yk_x + 2xk_y)(x^2 + y^2)k_z \frac{d\alpha_x}{ds}$$

$$- y(1 - 2yk_x + 2xk_y)(x^2 + y^2)k_z \frac{d\alpha_y}{ds}$$

$$+ (x^2 + y^2)^2 k_z^2 \frac{d\phi}{ds} + (x^2 + y^2)Wk_z \frac{d^2\phi}{ds^2} \qquad (112)$$

From the above equations it is seen that the stress resultant expressions will be rather lengthy when the above and the shear strains are substituted. Solely for the purpose of reducing the size of the stress resultant expressions it is assumed that the cross-section has two axes of symmetry. Thus, the following integrals are zero.

$$\left.\begin{aligned} &\int_A x\,dA = \int_A y\,dA = \int_A xy\,dA = \int_A x^3\,dA = \int_A y^3\,dA = 0 \\ &\int_A x^2y\,dA = \int_A xy^2\,dA = \int_A x^3y\,dA = \int_A xy^3\,dA = 0 \end{aligned}\right\} \quad (113)$$

Also, the warping function $W(x, y)$ is an odd function of both x and y since the warping displacement must be antisymmetric with respect to both the x and y axes for doubly symmetric cross-sections. Thus,

$$\int_A W\,dA = \int_A xW\,dA = \int_A yW\,dA = \int_A x^2W\,dA = \int y^2W\,dA = 0 \qquad (114)$$

Even if the cross-section is not symmetric the second and third integrals above are zero since x and y are measured from the shear center (or center of twist). This was proven by Goodier in reference 18.

It is assumed that the elastic modulus E and the shear modulus G are constant over the cross-section. The integrals which appear in the stress resultants are denoted as follows

$$
\left.
\begin{aligned}
A &= \int_A dA \\
I_{xx} &= \int_A y^2 dA \\
I_{yy} &= \int_A x^2 dA \\
I_p &= \int_A (x^2 + y^2) dA \\
J &= \int_A \left(x^2 + y^2 + x \frac{\partial W}{\partial y} - y \frac{\partial W}{\partial x}\right) dA \\
B_1 &= \int_A x^4 dA \\
B_2 &= \int_A y^4 dA \\
B_3 &= \int_A x^2 y^2 dA \\
B_4 &= \int_A x^2 (x^2 + y^2) dA \\
B_5 &= \int_A y^2 (x^2 + y^2) dA \\
B_6 &= \int_A (x^2 + y^2)^2 dA \\
B_7 &= \int_A xyW dA \\
B_8 &= \int_A x^2 y \frac{\partial W}{\partial x} dA \\
B_9 &= \int_A xy^2 \frac{\partial W}{\partial y} dA
\end{aligned}
\right\} \quad (115)
$$

In the above, J is the well-known Saint-Venant torsional stiffness constant.

Substituting equations (97), (98), (111), and (112) into equations (107) through (110) and applying equations (113) through (115) and replacing α_z by its definition

$$\alpha_z = \frac{dw}{ds} - k_y u + k_x v$$

yields the following expressions for the stress resultants.

$$\begin{aligned}
M_x = E[&-(I_{xx} + B_3k_y^2 + 3B_2k_x^2 + 2B_3k_x^2 - \tfrac{3}{2} B_5k_z^2)k_z\alpha_x \\
&- 2(B_2 + 2B_3)k_xk_yk_z\alpha_y \\
&- (I_{xx} + B_3k_y^2 + B_2k_x^2 - \tfrac{9}{2} B_5k_z^2)k_x \frac{dw}{ds} \\
&+ (I_{xx} + B_3k_y^2 + B_2k_x^2 - \tfrac{9}{2} B_5k_z^2)k_xk_yu \\
&- (I_{xx} + B_3k_y^2 + B_2k_x^2 - \tfrac{9}{2} B_5k_z^2)k_x^2v \\
&+ (I_{xx} + B_3k_y^2 - 2B_3k_x^2 + B_2k_x^2 - \tfrac{3}{2} B_5k_z^2)k_y\phi \\
&+ 2B_3k_xk_y \frac{d\alpha_x}{ds} - (I_{xx} + B_3k_y^2 + B_2k_x^2 - \tfrac{3}{2} B_5k_z^2) \frac{d\alpha_y}{ds} \\
&- (2B_5 + B_7)k_xk_z \frac{d\phi}{ds}]
\end{aligned} \tag{116}$$

$$
\begin{aligned}
M_y = E[&-2(B_1 + 2B_3)k_x k_y k_z \alpha_x \\
&- (I_{yy} + B_3 k_x^2 + 3B_1 k_y^2 + 2B_3 k_y^2 - \tfrac{3}{2} B_4 k_z^2) k_z \alpha_y \\
&- (I_{yy} + B_1 k_y^2 + 3B_3 k_x^2 - \tfrac{9}{2} B_4 k_z^2) k_y \frac{dw}{ds} \\
&+ (I_{yy} + B_1 k_y^2 + 3B_3 k_x^2 - \tfrac{9}{2} B_4 k_z^2) k_y^2 u \\
&- (I_{yy} + B_1 k_y^2 + 3B_3 k_x^2 - \tfrac{9}{2} B_4 k_z^2) k_x k_y v \\
&- (I_{yy} + B_1 k_y^2 - 2B_3 k_y^2 + B_3 k_x^2 - \tfrac{3}{2} B_4 k_z^2) k_x \phi \\
&+ (I_{yy} + B_1 k_y^2 + B_3 k_x^2 - \tfrac{3}{2} B_4 k_z^2) \frac{d\alpha_x}{ds} \\
&- 2B_3 k_x k_y \frac{d\alpha_y}{ds} - (2B_4 - B_7) k_y k_z \frac{d\phi}{ds}]
\end{aligned} \tag{117}
$$

$$
\begin{aligned}
M_z = G\Big\{&[I_p + B_4 k_y^2 + B_5 k_x^2 - (B_1 + 3B_2 + 5B_3) k_z^2 + \tfrac{E}{G}(3B_5 + B_4) k_z^2] k_x \alpha_x \\
&+ [I_p + B_4 k_y^2 + B_5 k_x^2 - (3B_1 + B_2 + 5B_3) k_z^2 + \tfrac{E}{G}(3B_4 + B_5) k_z^2] k_y \alpha_y \\
&+ [-I_p - 3B_4 k_y^2 - 3B_5 k_x^2 + (B_3 + B_6) k_z^2 \\
&\qquad + \tfrac{E}{G}(I_p + 3B_4 k_y^2 + 3B_5 k_x^2 - \tfrac{3}{2} B_6 k_z^2)] k_z \frac{dw}{ds} \\
&- [-I_p - 3B_4 k_y^2 - 3B_5 k_x^2 + (B_3 + B_6) k_z^2 \\
&\qquad + \tfrac{E}{G}(I_p + 3B_4 k_y^2 + 3B_5 k_x^2 - \tfrac{3}{2} B_6 k_z^2)] k_y k_z u \\
&+ [-I_p - 3B_4 k_y^2 - 3B_5 k_x^2 + (B_3 + B_6) k_z^2 \\
&\qquad + \tfrac{E}{G}(I_p + 3B_4 k_y^2 + 3B_5 k_x^2 - \tfrac{3}{2} B_6 k_z^2)] k_x k_z v \\
&+ 2[B_2 - B_1 + \tfrac{E}{G}(B_4 - B_5)] k_x k_y k_z \phi
\end{aligned}
$$

$$+ 2(1 - \frac{E}{G})B_4 k_y k_z \frac{d\alpha_x}{ds} - 2(1 - \frac{E}{G})B_5 k_x k_z \frac{d\alpha_y}{ds}$$

$$+ [J + (B_4 - B_7)k_y^2 + (B_5 + B_7)k_x^2 - (B_3 + B_6 - \frac{1}{2} B_8 + \frac{1}{2} B_9)k_z^2$$

$$+ \frac{E}{G} B_6 k_z^2] \frac{d\phi}{ds}\Big\} \qquad (118)$$

$$F_z = E[(I_p + I_{xx})k_x k_z \alpha_x + (I_p + I_{yy})k_y k_z \alpha_y$$

$$+ (A + I_{yy}k_y^2 + I_{xx}k_x^2 - \frac{3}{2} I_p k_z^2) \frac{dw}{ds}$$

$$- (A + I_{yy}k_y^2 + I_{xx}k_x^2 - \frac{3}{2} I_p k_z^2)k_y u$$

$$+ (A + I_{yy}k_y^2 + I_{xx}k_x^2 - \frac{3}{2} I_p k_z^2)k_x v$$

$$+ (I_{yy} - I_{xx})k_x k_y \phi - I_{yy}k_y \frac{d\alpha_x}{ds} + I_{xx}k_x \frac{d\alpha_y}{ds} + I_p k_z \frac{d\phi}{ds}] \qquad (119)$$

F. Discussion of Stress Resultants

Equations (116) through (119) give the stress resultants in terms of the displacements. These expressions include extensional effects which have not been taken into account previously in stress resultants for space beams. Also, the expressions contain terms which allow the equations to be applied to beams having curvatures larger than would otherwise be permitted.

It was noted earlier that in the classical approach the stress resultants are not derived from strain-displacement relations but are assumed to be proportional to the changes in the curvatures. Thus, according to the classical theory, which also assumes the elastic axis to be inextensible ($\alpha_z = 0$), the three moment stress resultants are given by

$$\left.\begin{aligned} M_x &= EI_{xx}(-k_z\alpha_x + k_y\phi - \frac{d\alpha_y}{ds}) \\ M_y &= EI_{yy}(-k_z\alpha_y - k_x\phi + \frac{d\alpha_x}{ds}) \\ M_z &= GJ(k_x\alpha_x + k_y\alpha_y + \frac{d\phi}{ds}) \end{aligned}\right\} \quad (120)$$

It is obvious that the more rigorous approach for determining the stress resultant yields many additional terms. In equations (116) through (119) the dw/ds, u, and v terms appear because of extension of the elastic axis. The terms involving the B_i cross-sectional constants arise because of the more accurate representation of the strain distribution over the cross-section. The first two of equations (120) essentially assume a linear strain distribution. When the cross-sectional dimensions are very, very small compared to the radius of curvature (i.e., $x_i\ k_j$ is much, much less than unity) most of the terms in the stress resultants which contain a B_i constant can be neglected.

A comparison of the third of equations (120) with equation (118) points out that a very common mistake, which has not been noted before, has been made in previous analyses that dealt with torsional and out-of-plane deformations. The comparison shows that in the third of equations (120) only the displacement $d\phi/ds$ should be multiplied by the torsional stiffness J. The variables α_x and α_y appearing in this equation should be multiplied by the polar moment of inertia I_p. Unless the cross-section is circular, the use of the stress resultant M_z as given by the classical approach can lead to considerable error

since I_p may be many times larger than J when the cross-section is not compact. Consider, for example, a rectangular cross-section. The ratio of I_p to J is given below for various ratios of the cross-sectional dimensions a and b.

a/b	I_p/J
1	1.19
2	1.82
3	3.17
4	5.04
5	7.44
10	26.98

The easily made mistake described above has often been made in papers pertaining to torsional and out-of-plane vibrations and out-of-plane buckling (for example, see references 12, 19, 20, 21, 22, and 23). In the classical formulation the fact that the α_x and α_y terms should be multiplied by I_p instead of J is not at all obvious. This error shows the need for determining the stress resultants from strain-displacement relations.

In reference 15 Goodier showed that for a straight twisted beam the pretwist contributes to the torsional stiffness since the displacement $d\phi/ds$ causes a longitudinal stress which opposes the motion. In the notation of this paper, Goodier gives the stress resultant as

$$M_z = (GJ + EB_6k_z^2)\frac{d\phi}{ds} + EI_pk_z\frac{dw}{ds}$$

These terms, which appear in equation (118), are shown to be significant for slender cross-section, such as propellers, even when k_z times the largest cross-sectional coordinate is as small as 0.1. In the stress resultant development, the inclination of σ_{zz} causes terms other than those above to appear in M_z since the treated beam also has the curvature components k_x and k_y.

In reference 24 Den Hartog gives an interesting observation that he made in his experiments with bending of pretwisted straight beams and challenges the reader for an explanation. It was found that the bending stiffness of doubly symmetric beams having equal cross-sectional moments of inertia is reduced when the beam is pretwisted. The cruciform cross-section is cited as an example. According to the conventional theory for straight pretwisted beams (for example, see reference 25), the pretwist has no effect on the bending stiffness (nor on the bending displacements) when the moments of inertia are equal. In reference 26 the reduction in stiffness is attributed to distortion of the cross-section and anticlastic effects. Equations (116) and (117) show that there is also a reduction in the bending stiffness due to the pretwist itself. For a straight pretwisted beam having only the displacements u and v, the equations give

$$M_x = -E(I_{xx} - \frac{3}{2} B_5 k_z^2) \frac{d\alpha_y}{ds}$$

$$M_y = E(I_{yy} - \frac{3}{2} B_4 k_z^2) \frac{d\alpha_x}{ds}$$

It can be shown that for the cruciform cross-section, B_4 and B_5 are equal, as well as I_{xx} and I_{yy}, and do not depend upon the orientation chosen for the x and y axes. Thus, it is seen that the pretwist does indeed affect the bending stiffness even when the moments of inertia are equal.

G. Development of Equilibrium Equations

The equilibrium equations are derived by considering the forces and moments acting on an incremental segment of the beam. The equations are obtained using a very expedient vector approach.

The internal moments (or moment stress resultants) acting in the $\underset{\sim}{e}'_x$, $\underset{\sim}{e}'_y$, and $\underset{\sim}{e}'_z$ directions were previously denoted by M_x, M_y, and M_z, respectively. Similarly, let F_x, F_y, and F_z denote the internal forces acting in these directions. Also, let p_x, p_y, p_z, q_x, q_y, and q_z be the applied distributed forces and moments acting in the $\underset{\sim}{e}'_x$, $\underset{\sim}{e}'_y$, and $\underset{\sim}{e}'_z$ directions per unit length of the extended rod. From these components mentioned above the following vectors may be defined.

$$\left.\begin{aligned}
\underset{\sim}{F} &= F_x\underset{\sim}{e}'_x + F_y\underset{\sim}{e}'_y + F_z\underset{\sim}{e}'_z \\
\underset{\sim}{M} &= M_x\underset{\sim}{e}'_x + M_y\underset{\sim}{e}'_y + M_z\underset{\sim}{e}'_z \\
\underset{\sim}{p} &= p_x\underset{\sim}{e}'_x + p_y\underset{\sim}{e}'_y + p_z\underset{\sim}{e}'_z \\
\underset{\sim}{q} &= q_x\underset{\sim}{e}'_x + q_y\underset{\sim}{e}'_y + q_z\underset{\sim}{e}'_z
\end{aligned}\right\} \quad (121)$$

Figure 4 shows an incremental segment of the beam of length $\Delta s'$. Thus, the applied force and applied moment vectors acting on the increment are $\underset{\sim}{p}\,\Delta s'$ and $\underset{\sim}{q}\,\Delta s'$. The changes in the internal force and internal moment vectors from one side of increment to other are denoted by $\Delta\underset{\sim}{F}$ and $\Delta\underset{\sim}{M}$ as shown. To establish the sign convention, consider an imaginary cross-sectional cut of the beam taken at $s' = s_1'$. The sign convention is chosen such that the internal force and moment components acting on the cross-sectional face of the material on the $s' < s_1'$ side of the cut are positive in the $\underset{\sim}{e}_x'$, $\underset{\sim}{e}_y'$, and $\underset{\sim}{e}_z'$ directions.

The summation of forces and the summation of moments about the left hand end of the increment give

$$\underset{\sim}{F} + \Delta\underset{\sim}{F} + \underset{\sim}{p}\,\Delta s' - \underset{\sim}{F} = 0$$

$$\underset{\sim}{M} + \Delta\underset{\sim}{M} + \underset{\sim}{q}\,\Delta s' + (\Delta s'\underset{\sim}{e}_z') \times (\underset{\sim}{F} + \Delta\underset{\sim}{F})$$

$$+ (\tfrac{1}{2}\,\Delta s'\underset{\sim}{e}_z') \times (\underset{\sim}{p}\,\Delta s') - \underset{\sim}{M} = 0$$

Dividing each of the above equations by $\Delta s'$ and taking the limit as $\Delta s'$ approaches zero gives the equilibrium equations in vector form as

$$\frac{d\underset{\sim}{F}}{ds'} + \underset{\sim}{p} = 0 \tag{122}$$

$$\frac{d\underset{\sim}{M}}{ds'} + \underset{\sim}{e}_z' \times \underset{\sim}{F} + \underset{\sim}{q} = 0 \tag{123}$$

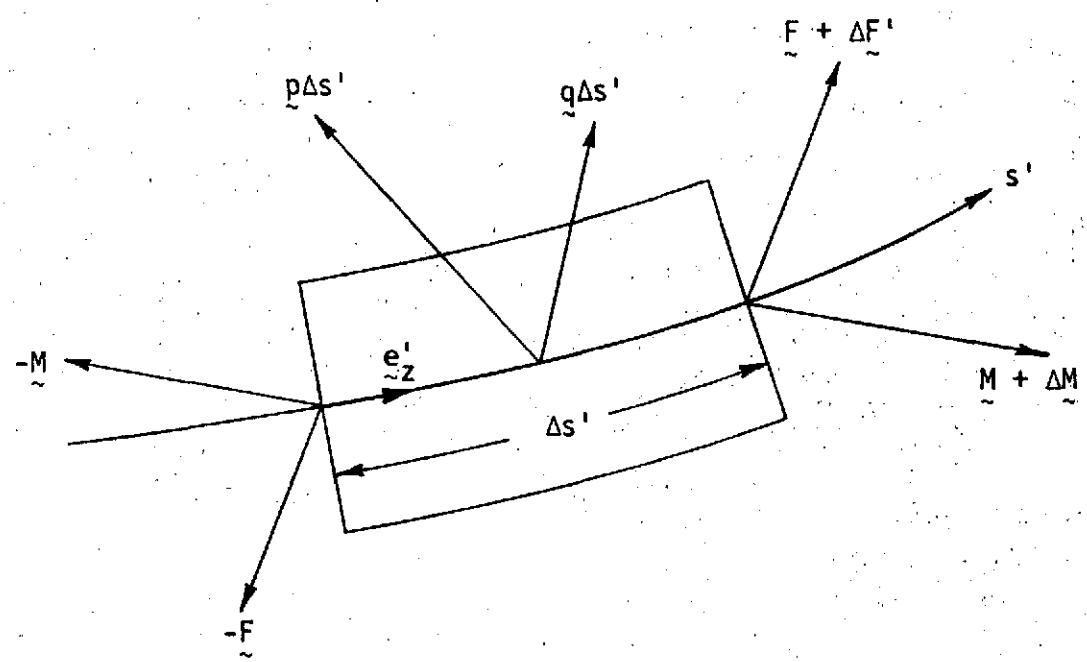

Figure 4.- Incremental segment of beam.

Substituting equations (121) into the above, performing the indicated operations, and substituting equations (32) for the derivatives of the unit vectors yields

$$\left.\begin{aligned} &\frac{dF_x}{ds'} - k_z' F_y + k_y' F_z + p_x = 0 \\ &\frac{dF_y}{ds'} + k_z' F_x - k_x' F_z + p_y = 0 \\ &\frac{dF_z}{ds'} - k_y' F_x + k_x' F_y + p_z = 0 \end{aligned}\right\} \quad (124)$$

$$\left.\begin{aligned} &\frac{dM_x}{ds'} - k_z' M_y + k_y' M_z - F_y + q_x = 0 \\ &\frac{dM_y}{ds'} + k_z' M_x - k_x' M_z + F_x + q_y = 0 \\ &\frac{dM_z}{ds'} - k_y' M_x + k_x' M_y + q_z = 0 \end{aligned}\right\} \quad (125)$$

The above equilibrium equations are the same as those given by Love in reference 1 except that ds' appears in the above derivatives instead of ds since extensional behavior is considered. Replacing ds' by

$$ds' = (1 + 2\,\varepsilon)^{1/2}\, ds$$

in the above gives equations of the type

$$\frac{dF_x}{ds} + (1 + 2\varepsilon)^{1/2}(-k_z'F_y + k_y'F_z + p_x) = 0 \qquad (126)$$

As noted before, p_x is the force acting in the $\underset{\sim}{e}_x'$ direction per unit length of the extended rod. Let dP_x be the force acting on ds'. That is,

$$p_x = \frac{dP_x}{ds'}$$

Therefore,

$$(1 + 2\varepsilon)^{1/2}\, p_x = (1 + 2\varepsilon)^{1/2}\, \frac{dP_x}{ds'}$$
$$= \frac{dP_x}{ds}$$

Let dP_x/ds, which is the force acting in the $\underset{\sim}{e}_x'$ direction per unit length of the undeformed beam, be denoted by $\bar{p}_x$. Thus,

$$(1 + 2\varepsilon)^{1/2}\, p_x = \bar{p}_x \qquad (127)$$

There are similar relations for p_y, p_z, q_x, q_y, and q_z. Hence, the equilibrium equations as typified by equation (126) take the form

$$\frac{dF_x}{ds} + (1 + 2\varepsilon)^{1/2}(-k_z'F_y + k_y'F_z) + \bar{p}_x = 0 \qquad (128)$$

Suppose that p_x is the inertial load given by $(-m'\ddot{u})$ where m' is the distributed mass of the extended beam. Then,

$$(1 + 2\varepsilon)^{1/2}\, p_x = -(1 + 2\varepsilon)^{1/2}\, m'\ddot{u}$$

Let dM be the mass of the beam in the differential lengths ds and ds'. Using equation (127), the above becomes

$$\bar{p}_x = -(1 + 2\varepsilon)^{1/2} \frac{dM}{ds'}\, \ddot{u}$$

$$= -\frac{dM}{ds}\, \ddot{u}$$

$$\bar{p}_x = -m\ddot{u}$$

where m is the distributed mass of the undeformed beam. Therefore, when considering inertia loads the distributed mass in the $\bar{p}_x$, $\bar{p}_y$, etc., expressions is that of the undeformed beam.

In this analysis only linear solutions are being considered. For free vibration problems the internal forces and moments are proportional to the displacement variables. In the k_x', k_y', and k_z' expressions, the initial curvatures are the only constant terms. Thus, from equation (128) and those similar to it, the linear equilibrium equations for natural vibrations are found to be

$$\left.\begin{aligned}
&\frac{dF_x}{ds} - k_z F_y + k_y F_z + \bar{p}_x = 0 \\
&\frac{dF_y}{ds} + k_z F_x - k_x F_z + \bar{p}_y = 0 \\
&\frac{dF_z}{ds} - k_y F_x + k_x F_y + \bar{p}_z = 0
\end{aligned}\right\} \quad (129)$$

$$\left.\begin{aligned}
&\frac{dM_x}{ds} - k_z M_y + k_y M_z - F_y + \bar{q}_x = 0 \\
&\frac{dM_y}{ds} + k_z M_x - k_x M_z + F_x + \bar{q}_y = 0 \\
&\frac{dM_z}{ds} - k_y M_x + k_x M_y + \bar{q}_z = 0
\end{aligned}\right\} \quad (130)$$

H. Summary of Governing Differential Equations for Natural Vibrations

All of the equations needed for solution have now been developed. It is planned for the natural vibration solution to be obtained using a transfer matrix technique. Thus, the first-order equations that have been derived will not be combined to form higher-order equations. The problem has twelve variables: u, v, w, ϕ, α_x, α_y, M_x, M_y, M_z, F_x, F_y, and F_z. The twelve equations are: the first two of equations (20) which relate α_x and α_y to u, v, and w; equations (116) through (119) which are the four stress resultant relations; and equations (129) and (130) which are the six equilibrium equations.

The first two of equations (20) may be rewritten as

$$\left.\begin{aligned} \frac{du}{ds} &= k_z v - k_y w + \alpha_x \\ \frac{dv}{ds} &= -k_z u + k_x w + \alpha_y \end{aligned}\right\} \quad (131)$$

The four stress resultant relations can be written in matrix form as

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \\ F_z \end{Bmatrix} = [a] \begin{Bmatrix} u \\ v \\ \phi \\ \alpha_x \\ \alpha_y \end{Bmatrix} + [b] \begin{Bmatrix} \frac{dw}{ds} \\ \frac{d\phi}{ds} \\ \frac{d\alpha_x}{ds} \\ \frac{d\alpha_y}{ds} \end{Bmatrix} \quad (132)$$

where the matrix elements a_{ij} and b_{ij} are defined by equations (116) through (119). It can be shown that the matrix [b] is always non-singular. Thus, the above becomes

$$\begin{Bmatrix} \frac{dw}{ds} \\ \frac{d\phi}{ds} \\ \frac{d\alpha_x}{ds} \\ \frac{d\alpha_y}{ds} \end{Bmatrix} = - [b]^{-1}[a] \begin{Bmatrix} u \\ v \\ \phi \\ \alpha_x \\ \alpha_y \end{Bmatrix} + [b]^{-1} \begin{Bmatrix} M_x \\ M_y \\ M_z \\ F_z \end{Bmatrix} \tag{133}$$

The distributed applied loads $\bar{p}_x$, $\bar{q}_x$, etc., appearing in the equilibrium equations consist of inertial loads and elastic foundation forces and moments. Rotary inertia and elastic foundation effects are included in the equations since they do not complicate the transfer matrix solution in any way.

The distributed mass m is equal to ρA where ρ is the mass density and A is the cross-sectional area of the beam. The reversed effective inertial force in the x-direction, for example, is $(-\rho A\ \partial^2 u/\partial t^2)$. Thus, for harmonic motion

$$\left.\begin{aligned} \bar{p}_x &= \omega^2 \rho A u - c_x u \\ \bar{p}_y &= \omega^2 \rho A v - c_y v \\ \bar{p}_z &= \omega^2 \rho A w - c_z w \end{aligned}\right\} \tag{134}$$

where ω is the natural vibration frequency and the c_i are the elastic foundation constants for translational displacements.

In the development of the curvature relations the Eulerian rotations about the x, y, and z axes were denoted by Θ, ψ, and ϕ, respectively. The rotations Θ and ψ are given in terms of the elastic axis displacements by equations (45). In the linearized analysis Θ and ψ are small and are approximated by

$$\theta = -\alpha_y$$

$$\psi = \alpha_x$$

The rotary inertias about the x, y, and z axes, respectively, are

$$\int_A \rho y^2 dA = \rho I_{xx}$$

$$\int_A \rho x^2 dA = \rho I_{yy}$$

$$\int_A \rho(x^2 + y^2) dA = \rho I_p$$

Also, it is assumed that the elastic foundation can resist rotations about the x, y, and z axes, and these foundation constants are denoted by d_i. Thus, using the above relations for the rotations Θ and ψ, the applied moments are given by

$$\left.\begin{aligned}
\bar{q}_x &= -\omega^2\rho I_{xx}\alpha_y + d_x\alpha_y \\
\bar{q}_y &= \omega^2\rho I_{yy}\alpha_x - d_y\alpha_x \\
\bar{q}_z &= \omega^2\rho I_p\phi - d_z\phi
\end{aligned}\right\} \quad (135)$$

Substituting equations (134) and (135) into the equilibrium relations as given by equations (129) and (130) and rearranging yields

$$\left.\begin{aligned}
\frac{dM_x}{ds} &= k_zM_y - k_yM_z + F_y + (\omega^2\rho I_{xx} - d_x)\alpha_y \\
\frac{dM_y}{ds} &= -k_zM_x + k_xM_z - F_x - (\omega^2\rho I_{yy} - d_y)\alpha_x \\
\frac{dM_z}{ds} &= k_yM_x - k_xM_y - (\omega^2\rho I_p - d_z)\phi
\end{aligned}\right\} \quad (136)$$

$$\left.\begin{aligned}
\frac{dF_x}{ds} &= k_zF_y - k_yF_z - (\omega^2\rho A - c_x)u \\
\frac{dF_y}{ds} &= -k_zF_x + k_xF_z - (\omega^2\rho A - c_y)v \\
\frac{dF_z}{ds} &= k_yF_x - k_xF_y - (\omega^2\rho A - c_z)w
\end{aligned}\right\} \quad (137)$$

Equations (131), (133), (136), and (137) are the twelve first-order differential equations which describe the natural vibration motion of space beams. The motion consists of bending in two

directions, extension, and torsion. The degree to which these motions are coupled depends upon the application. It is noted that the usual beam bending-torsion coupling arising from the noncoincidence of the elastic and centroidal axes does not appear because of the assumption of double symmetry for the cross-section.

I. Numerical Solution Method

As noted earlier the solutions for the natural vibration characteristics will be obtained using a transfer matrix method. This method, which is based on second-order Runge-Kutta integration, does not discretize (i.e., lump masses and assume constant elastic properties for a beam segment) the beam as most other transfer matrix methods do. Instead, distributed mass values and elastic properties at selected stations are used. The method assumes that the mass and all other properties vary linearly from station to station. Also, any discontinuities in the beam's properties are conveniently handled by assigning two stations to the point at which the discontinuity occurs. In addition, the method can be applied to rings as well as to beams.

Matrix differential equation.- Let {Y} be the column vector consisting of the twelve variables. That is,

$$\{Y(s)\} = \begin{Bmatrix} u \\ v \\ w \\ \phi \\ \alpha_x \\ \alpha_y \\ M_x \\ M_y \\ M_z \\ F_x \\ F_y \\ F_z \end{Bmatrix} \tag{138}$$

Also, let $\{Y'\}$ be a similar vector defined by the derivatives of the variables. The set of differential equations given by equations (131), (133), (136), and (137) may be expressed in matrix notation as

$$\{Y'(s)\} = [H(s, \omega)]\{Y(s)\} \tag{139}$$

Subscripts will be used to denote stations. Thus, at station i the above is written as

$$\{Y_i'\} = [H_i]\{Y_i\} \tag{140}$$

<u>Development of transfer matrix</u>.- The value of $\{Y\}$ at station $(i + 1)$ may be approximated as follows.

$$\{Y_{i+1}\} = \{Y_i\} + \frac{\Delta s_i}{2}\left\{\{Y_i'\} + \{Y_{i+1}'\}\right\} \tag{141}$$

Equation (139) may be written at station (i + 1) and substituted, along with equation (140), into equation (141) to give

$$\{Y_{i+1}\} = \{Y_i\} + \frac{\Delta s_i}{2}\left\{[H_i]\{Y_i\} + [H_{i+1}]\{Y_{i+1}\}\right\} \qquad (142)$$

Now, the $\{Y_{i+1}\}$ on the right hand side of the above equation may be approximated as

$$\{Y_{i+1}\} = \{Y_i\} + \Delta s_i\{Y_i'\}$$

$$= \{Y_i\} + \Delta s_i[H_i]\{Y_i\}$$

Substituting into equation (142) gives

$$\{Y_{i+1}\} = \{Y_i\} + \frac{\Delta s_i}{2}\left\{[H_i]\{Y_i\} + [H_{i+1}]\left\{\{Y_i\} + \Delta s_i[H_i]\{Y_i\}\right\}\right\} \qquad (143)$$

The above may be rewritten as

$$\{Y_{i+1}\} = [A_i]\{Y_i\} \qquad (144)$$

where

$$[A_i] = [I] + \frac{1}{2}\Delta s_i\left[[H_i] + [H_{i+1}]\right] + \frac{1}{2}\Delta s_i^2\,[H_{i+1}][H_i] \qquad (145)$$

2

The above matrix, $[A_i]$, is the desired transfer matrix. It is used to relate the variables defined by $\{Y\}$ evaluated at station $(i + 1)$ to those at station i.

Beam solution.- Letting $i = 0$ and $i = n$ denote the stations at the ends of the beam, $\{Y_n\}$ may be related to $\{Y_o\}$ by repeated use of equation (144). Hence,

$$\{Y_n\} = [B]\{Y_o\} \tag{146}$$

where

$$[B] = [A_{n-1}][A_{n-2}] \cdot \cdot \cdot [A_2][A_1][A_o] \tag{147}$$

The six homogeneous boundary conditions at each end of the beam may be expressed in matrix form as

$$[C]\{Y_n\} = \{0\} \tag{148}$$

$$[D]\{Y_o\} = \{0\} \tag{149}$$

where [C] and [D] are 6×12 matrices. Equation (146) may be substituted into equation (148) and the result combined with equation (149) to give

$$[U]\{Y_o\} = \{0\} \tag{150}$$

where

$$[U] = \begin{bmatrix} [C][B] \\ \text{--------} \\ [D] \end{bmatrix} \tag{151}$$

For a non-trivial solution to equation (150) the determinant of [U] must be zero. In the usual fashion for the transfer matrix method, the values of ω which make

$$|[U]| = 0 \tag{152}$$

are found by iteration. For each natural frequency the solution for $\{Y_i\}$ at every station is obtained by first setting one element of $\{Y_0\}$ equal to unity and solving equation (150) for the remaining elements of $\{Y_0\}$. Then the $\{Y_i\}$ are given by equation (144). Also, the derivatives, $\{Y_i'\}$, at each station may be computed from equation (140).

Ring Solution.- For rings and other curved beams which do not have endpoints the "boundary conditions" are applied in a different manner than that above. One arbitrary point on the ring is chosen to be both station $i = 0$ and $i = n$. The variables of $\{Y\}$ at these stations are related by

$$\{Y_n\} = [E]\{Y_0\} \tag{153}$$

where E is a 12 × 12 matrix. When the total twist of a ring is zero or a multiple of 2π radians the matrix [E] is the identity matrix. If the total twist of a ring is an odd numbered multiple of π radians then the matrix [E] is the negative of the identity matrix. Substituting equation (146) into the above gives

$$[B]\{Y_0\} = [E]\{Y_0\}$$

Or,

$$\Big[[B] - [E]\Big]\{Y_0\} = \{0\} \tag{154}$$

For solutions other than the trivial solution,

$$|[B] - [E]| = 0 \tag{155}$$

As before, the values of ω which make the above determinant equal to zero are found by iteration. The modal characteristics are determined using equations (154), (144), and (140).

VIII. NUMERICAL EXAMPLES

Two numerical examples are given which utilize the deformation theory and numerical method presented in this dissertation. A twisted curved beam and a twisted ring are considered. It is thought that natural vibration solutions have not been obtained previously for beams which are both curved and twisted.

Twisted curved beam.- The natural vibration characteristics were computed for a beam whose elastic axis forms a semi-circle as shown in figure 5. Both ends of the beam have cantilever boundary conditions ($u = v = w = \phi = \alpha_x = \alpha_y = 0$). The radius of curvature is chosen to be 10 inches ($\kappa = 1/R = 0.1$) and the torsion τ is zero since the beam lies in a plane. The beam is uniformly twisted and has a total twist of π radians (in other words, as the beam is traveled from one end to the other the cross-section rotates about the elastic axis a total of 180 degrees). Thus, from equations (5) the curvature components of the undeformed beam are

$$k_x = (0.1 \text{ in}^{-1}) \sin \gamma$$

$$k_y = (0.1 \text{ in}^{-1}) \cos \gamma$$

$$k_z = 0.1 \text{ rad/in}$$

where the angle γ defines the orientation of $\underset{\sim}{e}_x$ relative to the curve's normal $\underset{\sim}{n}$ (see figure 2). As shown in figure 5 the beam is

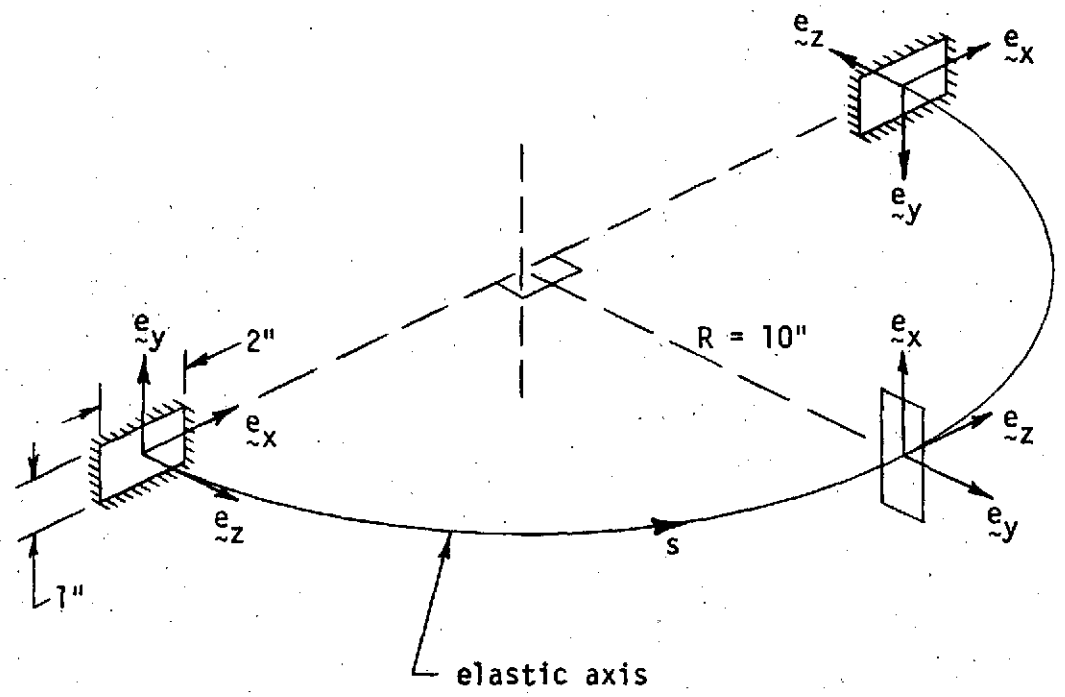

Figure 5.- Twisted curved beam.

assumed to have a rectangular cross-section of dimensions 1 inch by 2 inches. The elastic moduli are taken to be $E = 10^7$ lb/in^2 and $G = 4 \times 10^6$ lb/in^2. The Saint-Venant warping function and torsional stiffness constant for a rectangular cross-section as given by Wang in reference 27 were used in the computations.

The solutions were obtained using a computer program based on the previously described transfer matrix technique. Twenty-one equally spaced stations (including the end-points) along the beam's length were utilized. The first four natural vibration frequencies were found to be the following:

Mode	Frequency
First	939 rad/sec
Second	3169 rad/sec
Third	4490 rad/sec
Fourth	7535 rad/sec

Because of the twist the modal displacements u and v (displacements in $\underset{\sim}{e}_x$ and $\underset{\sim}{e}_y$ directions) are coupled. The mode shapes are easier visualized by considering the displacements in the directions of the normal $\underset{\sim}{n}$ and binormal $\underset{\sim}{b}$ of the elastic axis (see figure 2). Letting η and ζ be the displacements of the elastic axis in the $\underset{\sim}{n}$ and $\underset{\sim}{b}$ directions, respectively, then η is the inward radial displacement and ζ is the out-of-plane displacement. The first mode has predominantly out-of-plane motion. The modal displacement ζ is given in figure 6 as a function of the beam's axial coordinate s. The

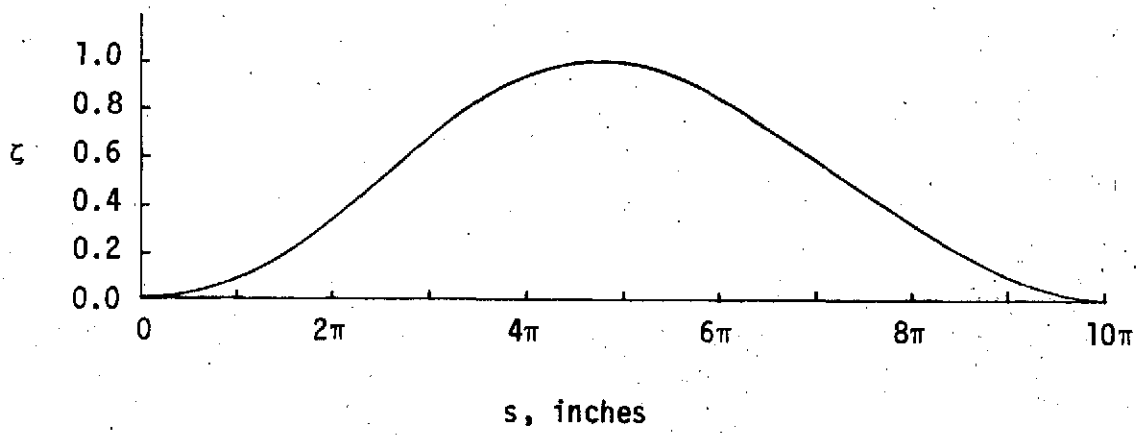

Figure 6.- Modal displacement ζ for first mode of twisted curved beam example.

other displacements are not shown since they are very small compared to ζ.

Since there is no experimental or calculated data for a beam such as the one considered, the results can only be compared to those of an untwisted beam. In reference 28 Den Hartog gives expressions for the first out-of-plane frequency and for the first in-plane frequency of an untwisted cantilevered ring segment. For an untwisted beam having the same dimensions as those of the numerical example and having its cross-section oriented the same as the one at the ends of the beam of figure 5, the first out-of-plane frequency is 1049 rad/sec and the first in-plane frequency is 5218 rad/sec according to reference 28. For the same untwisted beam the analysis and solution method presented herein gives the first out-of-plane frequency as 981 rad/sec and the first in-plane frequency as 4670 rad/sec.

Twisted ring.- The natural vibration characteristics were also computed for a twisted beam whose elastic axis forms a complete circular ring. This beam has the same cross-sectional dimensions and the same curvatures as the previous example. The total twist of the ring is 2π radians. Also, forty-one stations were utilized in the computations. The first four natural frequencies were found to be the following:

Mode	Frequency
First	1981 rad/sec
Second	2307 rad/sec
Third	5782 rad/sec
Fourth	5927 rad/sec

Figure 7 gives the modal displacements for the first mode.

As before, the calculated frequencies may be compared with those of an untwisted ring. For an untwisted ring the out-of-plane and in-plane motions are not coupled. Consider an untwisted ring having the same dimensions as the numerical example and with the longer sides of the cross-section parallel to the plane of the ring. According to reference 15 the first two out-of-plane and in-plane frequencies for the untwisted ring are

Mode	Out-of-plane frequency	In-plane frequency
First	1485 rad/sec	3044 rad/sec
Second	4252 rad/sec	8610 rad/sec

Discussion.- For an untwisted curved beam lying in a plane such that $k_x = k_z = 0$ and $k_y \neq 0$, the twelve governing differential equations uncouple into two sets of equations with each set consisting of six coupled equations. The variables associated with each set of equations are

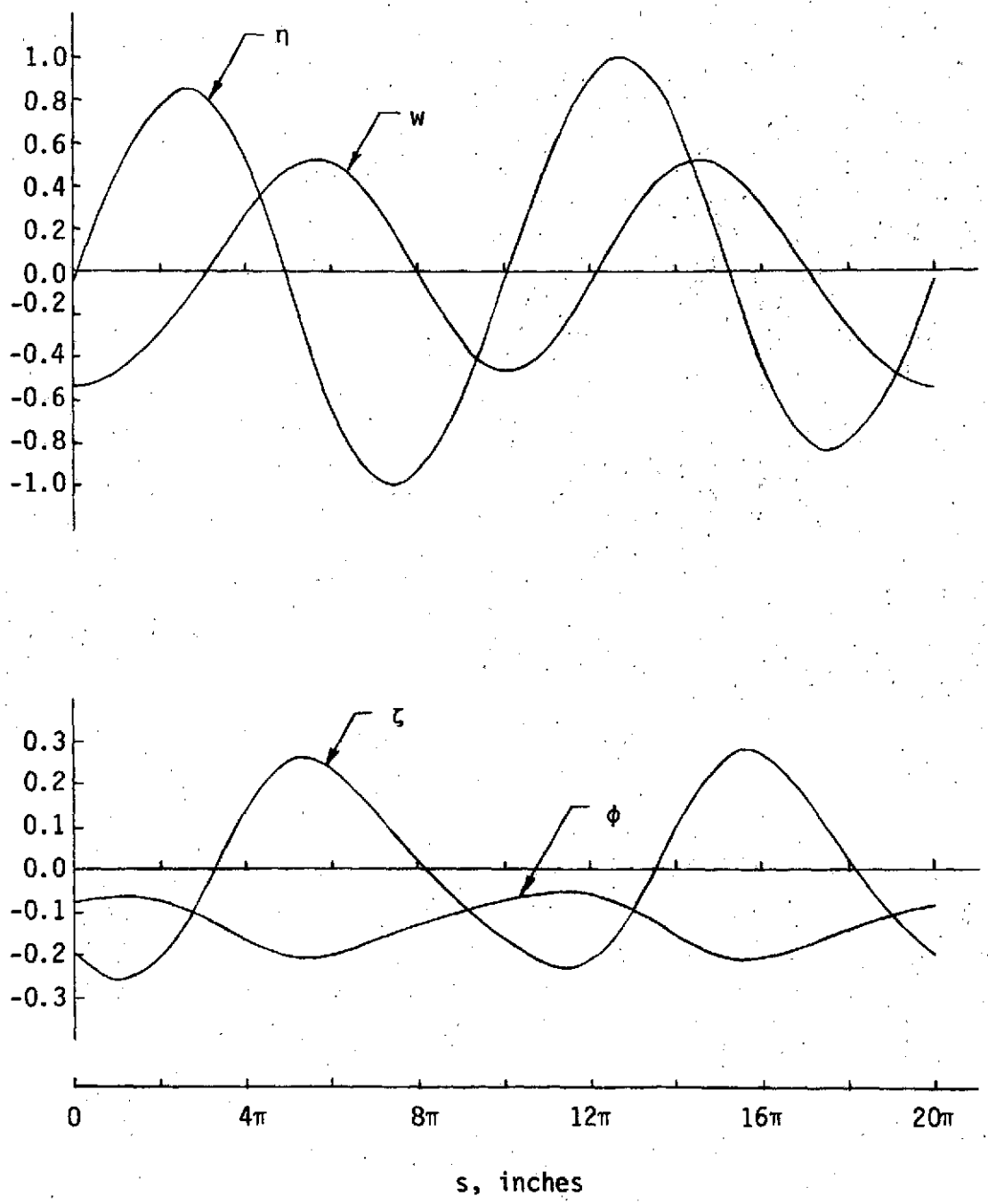

Figure 7.- Modal displacements for first mode of twisted ring example.

(a): $u, w, \alpha_x, M_y, F_x, F_z$

and

(b): $v, \phi, \alpha_y, M_x, M_z, F_y$

The variables of set (a) describe the in-plane bending and extensional vibration modes. The out-of-plane bending and torsional vibration modes are described by the variables of set (b).

For such a beam the uncoupling of the twelve first-order equations can cause trouble in the numerical solution for the modal functions. As noted earlier, after determining a frequency the first step for obtaining the associated modal displacements is to set one element (or variable) of $\{Y_o\}$ equal to unity and then solve equation (150) for the remaining elements. For example, if the last element of $\{Y_o\}$, which is F_z, is set equal to unity then equation (150) may be written as

$$\left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] \left\{\begin{array}{c} \bar{Y}_o \\ \hline 1 \end{array}\right\} = \left\{ 0 \right\}$$

where U_{11} is an 11×11 matrix. The first eleven equations contained in the above may be written as

$$[U_{11}]\{\bar{Y}_o\} = -\{U_{12}\}$$

which can be solved for $\{\bar{Y}_o\}$, the remaining elements of $\{Y_o\}$. For the case of the untwisted beam the variable which is set equal to unity

must be non-zero. For example, if the frequency has been determined for an out-of-plane mode, then the variable which is chosen to be equal to unity must be one of the variables of set (b) (setting $F_z = 1$ for this case yields modal displacements which do not satisfy the boundary conditions). However, the computer program cannot be told ahead of time which frequencies will be in-plane mode frequencies and which will be out-of-plane mode frequencies. Instead of writing separate computer programs for the in-plane and out-of-plane modes, it is thought that for cases in which the equations uncouple the problem is best remedied by obtaining two modal solutions for each frequency. This is done by setting equal to unity, in turn, one variable from each of the variable sets (a) and (b). Of course, only one of the modal solutions obtained for each frequency is valid.

IX. SUMMARY

Developments have been presented for the extensional equations which govern the deformation of curved and twisted space beams. Also, discussions of the equations in view of previous work were given.

First, the exact curvature equations for the deformed beams were derived. It is shown that allowing extension of the elastic axis instead of assuming inextension results in an additional term in each of the linearized curvature expressions. These equations were used in the development of the normal and shearing components of the strain tensor. The stress resultants were derived from the strain-displacement relations instead of assuming the resultants to be proportional to changes in the curvatures. It was shown that the classical approach could lead to considerable error in the torsional stress resultant. Also, for the same cross-section the presented stress resultants are applicable to beams having larger curvatures than those allowed by the classical theory. Moreover, by developing the stress resultants from the strain - displacement relations, additional terms appear due to the initial twist of the beam. A vector derivation of the equilibrium equations was given.

The governing equations for natural vibrations were summarized in the form of twelve first - order differential equations. A transfer matrix method was described for obtaining the solutions. Numerical examples were presented which illustrate the effect that twist has on the natural vibration frequencies of curved beams.

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XI. VITA

The author was born in [redacted] He graduated from R. J. Reynolds High School of Winston-Salem, North Carolina in 1958. The same year he entered North Carolina State University, Raleigh, North Carolina, and four years later received the degree of Bachelor of Science in Mechanical Engineering. Since June 1962 he has been employed by the National Aeronautics and Space Administration at the Langley Research Center in Hampton, Virginia. Under the NASA graduate program, he attended VPI & SU during the 1965-66 academic year and was awarded the degree of Master of Science in Engineering Mechanics in June 1967. During the 1970-71 academic year he was in residency at VPI & SU to continue his graduate work toward a Doctor of Philosophy degree. He is a member of the Phi Kappa Phi Honorary Society. He is presently employed as an Aerospace Engineer in the Structural Analysis Section of the Systems Engineering Division. He currently resides in Newport News, Virginia, with his wife, the former Deanna S. [redacted].

William F. Hunter