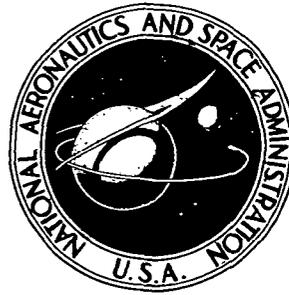


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**INPUT DESIGN FOR IDENTIFICATION OF
AIRCRAFT STABILITY AND CONTROL DERIVATIVES**

Narendra K. Gupta and W. Earl Hall, Jr.

Prepared by
SYSTEMS CONTROL, INC.
Palo Alto, Calif. 94304
for Flight Research Center



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16. Abstract <p>An approach for designing inputs to identify stability and control derivatives from flight test data is presented. This approach is based on finding inputs which provide the maximum possible accuracy of derivative estimates. Two techniques of input specification are implemented for this objective - a time domain technique and a frequency domain technique. The time domain technique gives the control input time history and can be used for any allowable duration of test maneuver, including those where data lengths can only be of short duration. The frequency domain technique specifies the input frequency spectrum, and is best applied for tests where extended data lengths, much longer than the time constants of the modes of interest, are possible.</p> <p>These techniques are used to design inputs to identify parameters in longitudinal and lateral linear models of conventional aircraft. The constraints of aircraft response limits, such as on structural loads, are realized indirectly through a total energy constraint on the input. Tests with simulated data and theoretical predictions show that the new approaches give input signals which can provide more accurate parameter estimates than can conventional inputs of the same total energy. Results obtained indicate that the approach has been brought to the point where it should be used on flight tests for further evaluation.</p>			
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I. INTRODUCTION

An increasingly important measure of the usefulness of aircraft flight test data is its ability to estimate stability and control derivatives. Improvements in the algorithms to accomplish this parameter identification task have advanced to the point where the choice of control inputs may be a limiting factor in the attainable accuracy of these estimated stability and control derivatives. An input design requirement, therefore, has arisen because of the need to improve the efficiency of flight testing by obtaining more accurate estimates from response data in less time.

The importance of choosing appropriate control inputs and exciting specific aircraft modes for extracting stability and control derivatives from aircraft flight testing has long been recognized. As early as 1951, Milliken^[1] summarized the studies conducted in defining good input signals in this statement: "It would appear that an optimal input in a given case is that which best excites the frequency of interest, and, hence, its (the input signal) harmonic content should be examined before the test to ensure that it is suitable". Good inputs could resolve parameter identifiability problems and improve confidences on estimates of stability and control derivatives obtained from the resulting flight test data. In other words, with specially chosen inputs, the same accuracy on parameter estimates can be obtained in much shorter flight test time than with conventional inputs. Shorter flight tests can lead to a saving in time required for stability and control testing and the computation requirements for extraction of aerodynamic derivatives. In addition, these inputs can be chosen specifically to satisfy the ultimate flight test objective such as control systems design, simulator parameter specifications, response prediction, aerodynamic model validation, or handling qualities evaluation.

There are many factors that must be considered when choosing inputs for flight tests. These include:

- (a) Pilot Acceptability. If the flight test is to be carried out with a pilot onboard the aircraft, it is necessary that the control inputs be acceptable to the pilot. The inputs should not maneuver the aircraft into a flight region from which a pilot cannot recover. In addition, the inputs

should be reproducible by the pilot.

- (b) Instrumentation. The inputs must consider specific instruments available, and their dynamic range and accuracy. The primary impact of the instruments on input design is on the signal/noise ratios which the response must have for sufficiently accurate data.
- (c) Parameter Identification Technique. Many parameter extraction methods require a certain class of inputs (e.g., sinusoidal inputs for transfer function identification, random inputs for correlation techniques). More advanced techniques of parameter identification tend not to rely on such specific classes of inputs, but do require inputs which maximize some function of the sensitivity of the output responses to parameters.
- (d) Modeling Assumptions. The inputs that are designed must also consider the model that is assumed. For example, inputs chosen for a linear mode should not cause such large aircraft motions that the assumption of constant stability and control derivatives is invalid.
- (e) Aircraft Structural Constraints. The aircraft maneuvers produced by the inputs should not cause the structural loads to increase beyond the design stresses of various aircraft components.
- (f) Objective of Parameter Identification. This is one of the most important and least understood of input design requirements. It is now known that there may be a significant difference in inputs which allow more accurate estimates of parameters for control system design as opposed to those inputs required for estimates of handling quality coefficients. Unfortunately, there are not extensive systematic techniques for relating the input design to the identification objective.
- (g) Output Sensitivity. Measured aircraft response resulting from these inputs should be most sensitive to the parameters of interest and less sensitive to other, possibly unknown, parameters of interest.

Figure 1.1 illustrates the use of specially designed inputs for flight test design. This procedure is based on specification of overall aircraft characteristics, instrumentation, and parameter identification objectives. The use of an algorithm to design the input may preclude consideration of other possible constraints (due to computational complexity) so some iteration to meet other constraints not considered may be desirable. Once flight tests are completed, and parameter extraction performed, the identification results may be used to design other inputs to further improve accuracy or as a priori estimates to be used in identifying other parameters not originally considered.

The present work is an extension of a previous study [2,3] in which a basic approach to input design was formulated based on optimization of a function of the sensitivity of the aircraft response to the aerodynamic derivations (e.g., the information matrix, M). Though this initial effort did establish the feasibility of the approach, its application to flight test requirements was difficult. To facilitate this application, the present effort was initiated.

The basic objective was to extend the formulation of input design procedure to make it more useful for flight test application and to meet the requirements listed above. This task was to be achieved by extension of the original time domain method and also by development of a frequency domain technique. In the course of the work, it was realized that the original approach could be reformulated by optimizing a function of the inverse of the information matrix (i.e., the dispersion matrix, D) for a significant expansion of capability for the resulting inputs to meet flight test requirements. Implementation of this reformulation in both the time and frequency domain demonstrated both the computational feasibility and the desired improvement. Subsequent computation revealed that the time domain method is more applicable to most flight test objectives and that the frequency domain approach was more useful for test conditions where a steady state condition would be established.

The capabilities afforded by use of these two techniques includes the following:

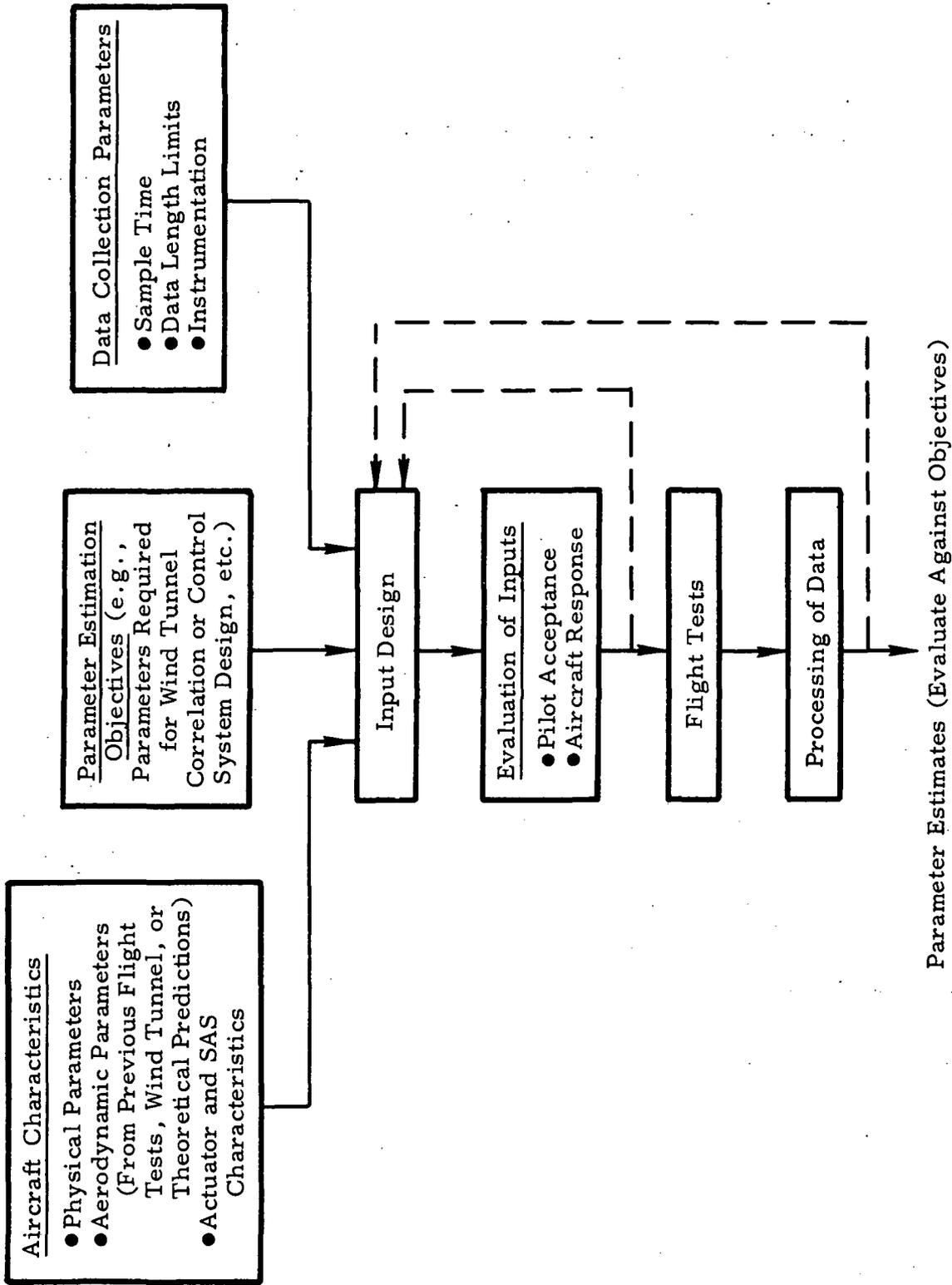


Figure 1.1 Use of Optimal Inputs for Flight Test Design

- (a) Choice of Criteria In Choosing Inputs: Given that the constraints listed above are satisfied, there is still flexibility in selecting certain input objectives. Specifically, it may be necessary that some parameters must be very accurately identified, while others may not be so.* Such an objective is best met by using a weighted trace of the information or dispersion matrix. Alternately, the eventual use of the estimates may not be clearly known, but it is desired to maximize the overall identifiability of the responses for whatever purpose (e.g., handling quality evaluation, control system design, etc.). Such an objective may be achieved by using the determinant of the information or dispersion matrix. The two techniques of this work allow use of any of these options (e.g., $\text{Tr } M$, $\text{Tr } D$, $|M|$, $|D|$). It will be shown that choosing inputs based on the trace of the dispersion matrix option should give the most accurate estimates, in general.
- (b) Ability To Be Able To Identify a Large Number of Parameters: Previously, input design techniques for linear systems were limited by computational requirements to low order aircraft models involving only a few parameters. The techniques developed for this work design inputs for a significantly larger number of parameters to be identified. This capability allows compression of flight test time to acquire the data to estimate the most derivatives.

These capabilities have been evaluated on simulated aircraft data of the Buffalo C-8 longitudinal response and the Lockheed Jet Star lateral response. The methods of evaluation are:

- (1) Comparison of the two input design approaches between each other and also against pulse and doublet-type inputs.

* Alternately, some derivatives, such as $C_{m\alpha}$, may be well known from previous wind tunnel or flight tests, and others, such as C_{l_p} , may not be confidently established. Inputs could then be designed only for the latter group. This situation is not emphasized in this work.

- (2) Comparison of the input designs against such factors as ease of implementation and levels of aircraft response.

The method of comparison is based on the standard deviations of the estimates from one input versus that of another. These criterion values are obtained as the square roots of the diagonal of the dispersion matrix corresponding to the input under evaluation.

The organization of this report is as follows. Chapter II presents a review of the developments of input design for parameter extraction. Chapter III, together with Appendices A, B, C, and D, discuss in detail the theoretical background of the time domain and the frequency domain methods. Chapter IV presents numerical results on optimal flight control inputs obtained using the time domain technique. Similar results, for longitudinal motions of a C-8 aircraft and lateral motions of a Jet Star, are given in Chapter V. Chapter VI evaluates the inputs under off-design conditions and by approximating the inputs by a series of steps. The results and conclusions are summarized in Chapter VII.

II. REVIEW OF INPUT DESIGN TECHNOLOGY

2.1 CONVENTIONAL INPUTS IN AIRCRAFT APPLICATIONS

Since the first efforts of applying parameter extraction technology to aircraft flight test data, many different control inputs have been used. [1,4-8] Many flight tests are presently aimed at determining natural frequency, damping ratio, etc., of a specific mode and steady state gains. [5] Most of the inputs are selected on the needs of simple parameter extraction procedures. One commonly used input is the frequency sweep. [6] In this approach, the aircraft is excited by sinusoidal inputs over a range of frequencies, usually around the natural frequency of the mode, until a steady state is reached at each frequency. The parameters of a suitable linear model are selected to obtain the best fit to the variation with input frequency of the output/input amplitude ratio and phase difference. These inputs work satisfactorily but require much flight test time. With the development of more sophisticated parameter extraction methods, other inputs have been tried. [7,8] Pulse inputs are used sometimes, and the frequency response is obtained by taking the Fourier transform of the output and the input at discrete points. These inputs are limited to simple low order linear systems.

Doublets, steps, and finite duration pulse inputs are generally used to identify aircraft parameters in both linear and nonlinear flight regimes. However, the estimates of certain parameters may be quite poor and, in some cases, a set of parameters may not be identifiable at all. A possible result is that excessive flight test time may be required to get good estimates of all the parameters. Optimal inputs consider the identifiability of each stability and control derivative directly. They can be tuned to obtain better identifiability of the overall parameter set or tuned to identifying only particular parameters of primary interest.

2.2 OPTIMAL INPUT DESIGN METHODS

The use of analytical techniques in input design may be considered to have been initiated by Fisher [9] who gave a quantitative meaning to the knowledge about a certain set of parameters through definition of the information matrix, M .

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Later, Cramer and Rao^[10] showed that the inverse of the information matrix, referred to as the dispersion matrix, is a bound on parameter error covariances, i.e.,

$$\text{cov} \{ (\theta - \hat{\theta}) (\theta - \hat{\theta})^T \} \geq M^{-1} \Delta D \quad (2.1)$$

where θ is the actual value of a parameter and $\hat{\theta}$ is the estimate. This is called the Cramer-Rao lower bound. The dispersion matrix is extremely useful because it relates to physically meaningful quantities (parameter estimation errors) and gives a method of comparing different experiments. Most of the analytical methods in input design use a function of information or dispersion matrix as the extremizing criterion. It is assumed that an efficient parameter identification algorithm, which can extract all available information about the parameters from data, is used and the Cramer-Rao lower bound is met with equality. This is important since this makes it possible to design inputs independent of the parameter extraction procedure.

2.2.1 Inputs for Regression Systems

Statistical and optimization concepts for selection of inputs were first used for regression experiments (e.g., static systems). Suppose it is possible to take noisy measurement of linear combinations of a set of unknown parameters, θ , i.e.,

$$y = F(u)\theta + n \quad (2.2)$$

where y is the observations and n is random noise. The set of possible linear combinations is defined by the control u . The maximum number of observations is N . The input design problem consists of finding a set of u 's and the number of observations at each u to get the "best" estimate of parameters, θ , under the above constraints. Some of the earliest work in this field was done by Kiefer and Wolfowitz,^[11] where they proved the equivalence of two extremum problems. They showed that optimizing a certain criterion in parameter space (e.g., sum of covariance of parameter estimates) is equivalent to optimizing a certain other criterion in sample or output space. This is an important conclusion and resulted in a number of significant contributions, e.g., Kiefer,^[12] and Karlin and Studden.^[13] An excellent summary of these methods is given in Fedorov.^[14]

b. 2.2.2 Inputs for Dynamic Systems

Earliest work in the field of input design for dynamic systems was done by Levin^[15] and Litman and Huggins.^[16] These authors designed inputs for unknown parameters in system impulse response based on least square estimation. In addition, Litman and Huggins considered an infinite observation time. Levadi^[17] was able to put the problem in a more general framework. In his approach, the observation time is finite and there are constraints on total input energy. By using a variational procedure, the trace of the error covariance matrix is minimized leading to a nonlinear Fredholm equation.

Aoki and Staley^[18] designed inputs for single-input, single-output systems based on maximizing the trace of the information matrix. They considered discrete time representation of a dynamic system and showed that the energy bounded optimal input is the eigenvector of a certain matrix corresponding to its maximum eigenvalue. Nahi and Wallis^[19] have also considered this problem but did not come up with a general algorithm. Mehra^[2] proposed an algorithm which maximizes the trace of the information matrix for multi-input, multi-output systems. Several methods for solving the resulting two point boundary value problem were also given. One method required solving a Riccati equation and was tried with partial success. The designed inputs, using this method, are unsuitable because, in general, maximizing the trace of the information matrix does not ensure that the covariances on parameter estimates are small. In extreme circumstances, it may give a singular information matrix, which gives infinite covariance in certain directions in the parameter space. Reid^[20] and Goodwin et al.^[21] design inputs based on the trace of the dispersion matrix (sum of parameter error covariances). They use direct gradient procedures, which require excessive computation time even for simple systems. The method, they propose, is unusable for multi-input, multi-output systems and for systems with more than a few unknown parameters.

Mehra^[22], in a novel approach, uses the steady state assumption to convert a linear constant coefficient system into its frequency domain representation. He demonstrates the procedure for determining optimal input spectra to minimize the

determinant, trace or any of a variety of functions of the dispersion matrix. Viort^[23] also considers a similar problem.

Until now, inputs have been computed for only a few simple practical systems. There have been two major barriers limiting the determination of optimal inputs. Most techniques, developed to date, can handle only the trace of the information matrix as the optimizing criterion and in many cases these inputs produce either marginal or no improvement in parameter estimation accuracy. Secondly, the computation time required is so large for high order systems (e.g., more than three states) with many unknown parameters (e.g, ten) that it makes the actual determination of inputs infeasible. This is because largely brute force methods have been used in the past.^[2,20,21] In aircraft applications, Stepner and Mehra^[3] computed inputs which maximize the trace of the information matrix for identifying five parameters in the longitudinal short period mode using the two state approximation. The computed input gave better estimates of three parameters but poorer estimates of two parameters as compared to a conventional doublet input. Swanson and Bellville^[24] have used some of these techniques to design inputs to identify parameters in certain biological systems.

In summary, previous input design techniques for dynamic systems have demonstrated the potential of improving the capability to identify aerodynamic derivatives from flight data. These techniques have, however, been limited in the flexibility they allow to meet important flight test requirements such as obtaining high accuracy for specific derivatives. In addition, such previous approaches have not demonstrated the capability to provide inputs for estimation of a large number of derivatives within reasonable computation limits. In the following chapter, an input design method is discussed which is directed toward alleviating these problems.

III. METHOD OF TIME AND FREQUENCY INPUT DESIGN FOR DYNAMIC SYSTEMS

3.1 INTRODUCTION

This chapter describes a method consisting of two different techniques for design of input signals which provide estimates of unknown parameters in linear time-invariant systems. The first technique uses the time domain representation of system dynamics and develops methods to compute the time history of control input sequence for any duration of the experiment. In the second technique, the system is assumed to be in oscillatory steady state and a frequency domain representation is utilized. This gives the optimal control input spectrum. The corresponding time history is "optimal" only for long experiments. The time domain approach is computationally much more complicated than the frequency domain approach. The two approaches are, therefore, complementary. The frequency domain approach is suitable for long experiments and the time domain approach should be used for short and medium experiments.

A computation algorithm based on eigenvalue-eigenvector decomposition is developed to solve the time domain problem. A new sensitivity functions reduction method affords considerable savings in computation time by decreasing the order of the problem. These two algorithms have made implementation of the time domain algorithm more feasible for practical systems because they allow design of inputs for much higher order systems than previously reported.

3.2 PROBLEM STATEMENT

Consider a linear, time-invariant, dynamic system following the differential equation:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}u \\ \mathbf{x}(0) &= 0 \qquad 0 \leq t \leq T \end{aligned} \tag{3.1}$$

where

x is an $n \times 1$ state vector,

u is a $q \times 1$ input vector, and

F and G are appropriate matrices which depend on m unknown parameters θ .

Let there be continuous noisy measurements of p linear combinations of state variables.

$$y = Hx + v \quad (3.2)$$

where

y is a $p \times 1$ measurement vector,

v is a $p \times 1$ white noise vector with zero mean and power spectral density R , and

H is a $p \times n$ matrix which is a function of parameters θ .

The problem is to choose u from a class of inputs to obtain "best" estimates of the unknown parameters. A total energy constraint is imposed on the input to limit state and control input excursions. This method of state deviation constraint is indirect. The more desirable and direct method of including quadratic penalty on state in the cost function is a difficult analytical and computational extension to the present approach of limiting total input energy alone,

$$\int_0^T u^T u dt = E \quad (3.3)$$

3.3 CRITERIA OF OPTIMALITY

It is usually not possible to find an input which gives better estimates of all parameters in a given system than any other input. The optimal input is determined by giving suitable importance to different parameters. Let M be the infor-

mation matrix for parameters θ resulting from an input u . Then the dispersion matrix is defined as

$$D \triangleq M^{-1} \quad (3.4)$$

and the Cramer-Rao lower bound (Equation (2.1)) gives

$$\text{cov}(\hat{\theta}) \geq D \quad (3.5)$$

where $\hat{\theta}$ is the estimated value of θ . In general, it is necessary to make a trade-off among the accuracies on estimates of unknown parameters in the system.

Based on Equation (3.5), several optimality criteria have been proposed and used. Some of these criteria are not altogether appropriate in that they do not ensure small estimation errors, but have been used because it is easier to compute the corresponding "optimal" input. There are three classes of criteria which have received special attention in the past.

- (a) Linear functional of the information matrix.

$$J_1 = \max_u \mathcal{L}(M) \quad (3.6)$$

- (b) The determinant of the dispersion matrix.

$$J_2 = \min_u |D| \quad (3.7)$$

- (c) Linear functional of the dispersion matrix.

$$J_3 = \min_u \mathcal{L}(M) \quad (3.8)$$

\mathcal{L} is such that for two positive semi-definite matrices A and B and a constant c

- (a) $\mathcal{L}(A) \geq 0$
- (b) $\mathcal{L}(A + B) = \mathcal{L}(A) + \mathcal{L}(B)$ (3.9)
- (c) $\mathcal{L}(cA) = c \mathcal{L}(A)$

Examples of linear operator \mathcal{L} are the trace and the weighted trace.

J_1 maximizes the total or partial sum of information of all the parameters or of a linear transformation of parameters. If the linear operator is the trace, the total information of all parameters is maximized. This may, however, lead to an almost singular information matrix with large terms on the diagonal. Then the dispersion matrix, which is a lower bound on parameter error covariances, has large diagonal terms. Therefore, this optimality criterion is not very suitable. It is used mainly because of its simplicity.

The dispersion matrix which is positive-definite, in the light of the Cramer-Rao lower bound, can be looked upon as a hyperellipsoid of uncertainty in the parameter space. J_2 works with the determinant of the dispersion matrix and minimizes the volume of the uncertainty ellipsoid.

J_3 minimizes a weighted sum of covariances of parameter estimates (or some linear combinations of parameters). The weighting matrix serves two purposes. Since the covariances of different parameters have different units, it converts each term in the sum to the same units. Secondly, the weighting matrix offers a tremendous flexibility because it is possible to assign varying importance to parameters, through weights on their nondimensional covariance. This is considered to be one of the most suitable performance criteria since it works with parameter estimate covariances directly.

In the next sections, we indicate how these different criteria can be handled in the time domain approach and the frequency domain approach.

3.4 TIME DOMAIN INPUT DESIGN TECHNIQUE FOR DYNAMIC SYSTEMS

In the past it has been possible to work only with a linear function of the information matrix, in particular, the trace. Under the present effort, methods have been developed which make it possible to optimize the determinant or weighted trace of the dispersion matrix. The details of the theory behind these methods is given in Appendix A. Here, we describe the algorithms and then indicate numerical procedures which are used to solve the resulting equations.

3.4.1 Weighted Trace of the Information Matrix

It is shown in Appendix A that maximizing the trace of the information leads to an eigenvalue problem of a positive self adjoint function. It is possible to reformulate it as a two-point boundary value problem. [2]

$$\frac{d}{dt} \begin{bmatrix} x_{\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} F_{\theta} & -\mu G_{\theta} G_{\theta}^T \\ H_{\theta}^T R_{\theta}^{-1} H_{\theta} & -F_{\theta}^T \end{bmatrix} \begin{bmatrix} x_{\theta} \\ \lambda \end{bmatrix} = \mathcal{H} \begin{bmatrix} x_{\theta} \\ \lambda \end{bmatrix} \quad (3.10)$$

$$x_{\theta}(0) = 0 \quad \lambda(T) = 0$$

$$u_{\text{opt}} = -\mu G_{\theta}^T \lambda \quad (3.11)$$

The smallest value of constant μ for a nontrivial solution to the two-point boundary value problem gives the optimal control input. The matrices x_{θ} , F_{θ} , G_{θ} , H_{θ} and R_{θ} are

$$x_{\theta} = \begin{bmatrix} x \\ \hline \frac{\partial x}{\partial \theta_1} \\ \hline \frac{\partial x}{\partial \theta_2} \\ \vdots \\ \hline \frac{\partial x}{\partial \theta_m} \end{bmatrix} \quad (3.12)$$

$$H_{\theta} = \begin{bmatrix} \frac{\partial H}{\partial \theta_1} & H & O & \dots & O \\ \frac{\partial H}{\partial \theta_2} & O & H & \dots & O \\ \vdots & & & & \\ \frac{\partial H}{\partial \theta_m} & O & \dots & \dots & H \end{bmatrix} \quad (3.13)$$

$$R_{\theta} = \begin{bmatrix} R & O & \dots & O \\ O & \cdot & & \\ \vdots & & \cdot & \\ \vdots & & & \cdot \\ O & \dots & \dots & R \end{bmatrix} \quad (3.14)$$

$$F_{\theta} = \begin{bmatrix} F & O & \dots & O \\ \frac{\partial F}{\partial \theta_1} & F & \dots & O \\ \vdots & \cdot & & \\ \vdots & & \cdot & \\ \frac{\partial F}{\partial \theta_m} & O & \dots & F \end{bmatrix}, \quad G_{\theta} = \begin{bmatrix} G \\ \hline \frac{\partial G}{\partial \theta_1} \\ \vdots \\ \frac{\partial G}{\partial \theta_m} \end{bmatrix} \quad (3.15)$$

This two point boundary value problem is solved as described in Section 3.4.3 and in Appendix C.

A weighted trace of the information matrix is maximized by defining a new set of m parameters φ related to θ as

$$\theta = C\bar{\varphi} \quad (3.16)$$

The information matrices for φ and θ are related to each other as

$$M_{\varphi} = C^T M_{\theta} C \quad (3.17)$$

Therefore,

$$\begin{aligned} \text{Tr}(M_{\varphi}) &= \text{Tr}(C^T M_{\theta} C) \\ &= \text{Tr}(CC^T M_{\theta}) \end{aligned} \quad (3.18)$$

Thus, maximizing the trace of M_{φ} is equivalent to maximizing a weighted trace of the information matrix for parameters θ . The sensitivities of the state vector to parameters φ and θ are related by the following transformation:

$$\left(\frac{\partial \mathbf{x}}{\partial \varphi} \right) = \left(\frac{\partial \mathbf{x}}{\partial \theta} \right) C^T \quad (3.19)$$

It is clear that the parameter transformation of Equation (3.16) enables us to maximize a weighted trace of the information matrix, when the weighting matrix is symmetric and positive semi-definite. The symmetry is not a restriction and positive semi-definiteness is required in the light of condition (a), Equation (3.9).

3.4.2 Determinant or Weighted Trace of the Dispersion Matrix

The idea behind minimizing the determinant or weighted trace of the dispersion matrix is presented in Appendix A, Sections A.4 and A.5. We present here the algorithm used in the computation of optimal input. It is an iterative pro-

cedure with convergence to a stationary point.

Algorithm (see flowchart in Figure 3.1):

- (1) Choose any input $u(t)$ with energy E which gives a nonsingular information matrix, M_o .
- (2) Find an input $u_m(t)$ with energy E to maximize $\varphi(u)$, such that

$$\int_0^T u_o^T(t) u_m(t) dt \geq 0$$

where,

$$\varphi = \text{Tr}(M_o^{-1}M) \text{ to minimize } |D| \quad (3.20)$$

$$= \text{Tr}(WM_o^{-1}M M_o^{-1}) \text{ to minimize } \text{Tr}(WD) \quad (3.21)$$

Both these criteria can be recast as maximizing a weighted trace of the information matrix. The matrix C of Equations (3.16) to (3.19) is

$$C = M_o^{-1/2} \text{ to minimize } |D| \quad (3.22)$$

$$= W^{1/2}M_o^{-1} \text{ to minimize } \text{Tr}(WD) \quad (3.23)$$

- (3) The information matrix for input $\alpha u_o(t) + \beta u_m(t)$ is

$$M_1 = \alpha^2 M_o + \beta^2 M_m + 2\alpha\beta M_{om} \quad (3.24)$$

where M_m is the information matrix for input $u_m(t)$ and M_{om} is a "cross information matrix" for inputs $u_o(t)$ and $u_m(t)$ and is defined in Appendix A. The energy constraint on the input requires

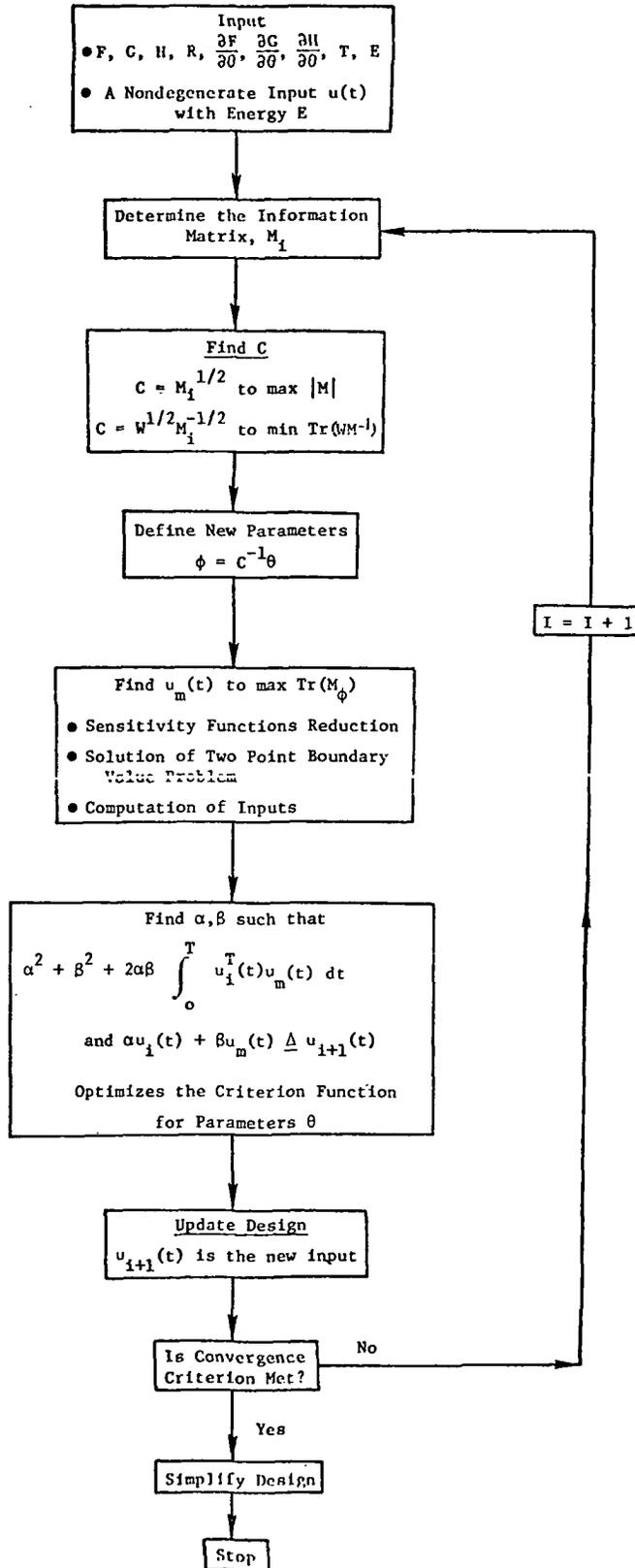


Figure 3.1 Flowchart for Input Design for Dynamic Systems

$$\alpha^2 + \beta^2 + 2\alpha\beta \int_0^T u_0^T(t) u^m(t) dt = 1 \quad (3.25)$$

Use Equations (3.24) and (3.25) to find a β between 0 and 1 which optimizes the criterion function for M_1 . It is shown in Appendix A that if the input $u_0(t)$ is not optimal, it is always possible to bring about an improvement in the performance index for this choice of β .

- (4) Check to see if the termination criterion is met. One of the following can be used.
 - (a) The information matrix does not change substantially from one step to the next, or if the optimizing function is not improving significantly.
 - (b) The value of β which optimizes the value of the desired function is approaching zero. In other words, very little power is being placed at newly chosen frequencies.
 - (c) The maximum value of the function Φ is not much higher than the maximum value for the optimum design (i.e., m to minimize $|D|$ and $\text{Tr}(WD^*)$ to minimize $\text{Tr}(WD)$). If the termination criterion is not met, repeat from step (2).
- (5) Check if the design is globally optimum.

It is clear that, in this technique it is necessary to maximize a weighted trace of the information matrix in each iteration.

Unless otherwise mentioned, in all computations reported in Chapter IV, trace or weighted trace of the dispersion matrix is used as the optimality criterion. This has proved to be a more useful criterion in general than the trace or weighted trace of the information matrix.

3.4.3 Sensitivity Functions Reduction

The two point boundary value problem requires a solution of $n(m+1)$ sensitivity equations and $n(m+1)$ adjoint equations. It is shown in Appendix D that the state sensitivities to system parameters are not independent of each other, in general. The number of sensitivity equations can, therefore, be reduced by a proper linear mapping and propagating only the independent equations. The maximum number of equations to obtain all sensitivity functions is $n(q+1)$. In many practical cases, it is smaller.

The theory of how the sensitivity functions reductions can be carried out is given by Gupta^[25] and in Appendix D. It is necessary to work only with the controllable part of (F_θ, G_θ) to obtain all sensitivities (F_θ and G_θ are defined in (3.15)). The uncontrollable subspace of (F_θ, G_θ) is dropped. Also, the states unobservable through (F_θ, H_θ) do not affect the performance index (i.e., the trace of the information matrix). For example, when all parameters are in G , the system states, x , are unobservable through H_θ and can be discarded. This happens rarely and is not incorporated in the algorithm.

The implemented algorithm is given here and the flowchart is illustrated in Figure 3.2.

1. The linearly dependent columns in G are merged. Then the structurally uncontrollable states in (F, G) are dropped.
2. Matrices F_θ and G_θ are formed. k_1, k_2, \dots, k_q of Equation (D.22) are determined and are used to choose $(q+1)n$ appropriate columns from the controllability matrix of (F_θ, G_θ) .
3. The dimension k_i^1 of state space controllable from each input u_i alone is determined. If for any i

$$2k_i^1 < n + k_i$$

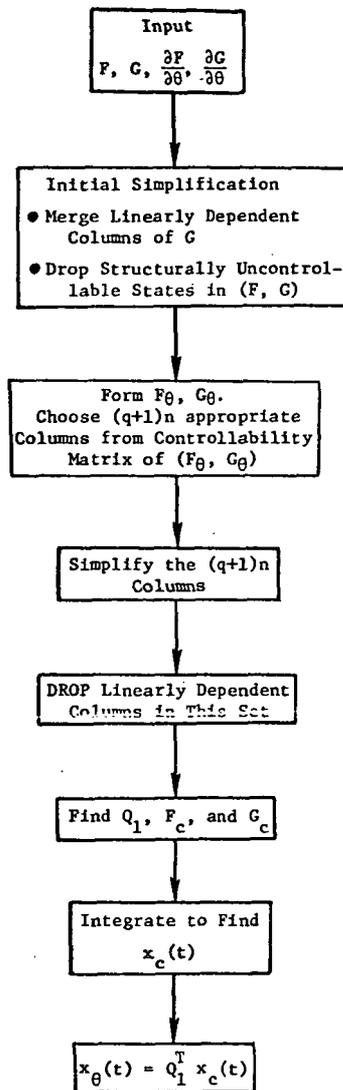


Figure 3.2 Sensitivity Function Reduction

only the first $2k_1!$ columns involving G_1 in the controllability matrix are considered.

4. The remaining columns are checked for linear independence. The Gram-Schmidt procedure is used to drop columns, which are linearly dependent on other columns. The set of remaining columns is Q_1 .
5. Any pseudo-inverse of Q_1 is determined. Equation (D.7) is used to compute F_c and G_c .
6. Equation (D.6) is solved for $x_c(t)$ and Equation (D.8) is used to find $x_\theta(t)$ at the desired points.

3.4.4 Solution to the Two Point Boundary Value Problem [26]

Several solution techniques have been suggested to solve the two point boundary value problem of Equation (3.10). The Riccati equation method suggested by Mehra [2] has been tried with limited success. In this method, a μ is chosen and a certain Riccati equation solved to determine the experiment duration where the elements of the Riccati matrix become large (theoretically infinite). The parameter μ corresponding to the desired experiment time is determined iteratively. The problem with this method is that it is difficult to determine numerically the time at which the elements become large. Also, it usually does not give good insight into the nature of optimal inputs.

A new method, which uses the symplectic properties of the Hamiltonian matrix \mathcal{H} of the two point boundary value problem, has been developed and is described in Appendix C. The eigenvalues of the Hamiltonian occur in pairs \mathcal{P}_+ and $-\mathcal{P}_+$. Let the corresponding eigenvector matrix be

$$\begin{bmatrix} X_+ & X_- \\ \Lambda_+ & \Lambda_- \end{bmatrix} \quad (3.26)$$

with normalization

$$\Lambda_-^T X_+ - X_-^T \Lambda_+ = I \quad (3.27)$$

Then the two point boundary value problem has a nontrivial solution if

$$U = \Lambda_+^{-1} \Lambda_- e^{-\mathcal{L}_+^T} X_-^{-1} X_+ e^{-\mathcal{L}_+^T} \quad (3.28)$$

has at least one eigenvalue equal to one.

A computer program has been written to solve Equation (3.10) using eigenvector decomposition. It consists of the following steps:

- (1) A reasonable μ_0 is chosen. It can be shown that

$$\frac{\partial J}{\partial E} = \frac{1}{\mu} \quad (3.29)$$

And since μ does not depend upon energy E in u ,

$$\text{Tr}(M) = E/\mu \quad (3.30)$$

So μ_0 can be selected from a knowledge of the energy and the expected value of $\text{Tr}(M)$. Alternatively, one could find the value of the performance index for a reasonable input which satisfies the energy constraint and then apply a suitable correction factor to μ determined using Equation (3.29). Choosing a good initial μ_0 is important in obtaining quick convergence.

- (2) Eigenvalues and eigenvectors of the Hamiltonian are determined and $\Lambda_+^{-1} \Lambda_-$ and $X_-^{-1} X_+$ are computed. Starting from $T = 0$, eigenvalues of U are determined for increasing T in steps of ΔT until a point T_0 is reached where one eigenvalue of U is "close" to 1.

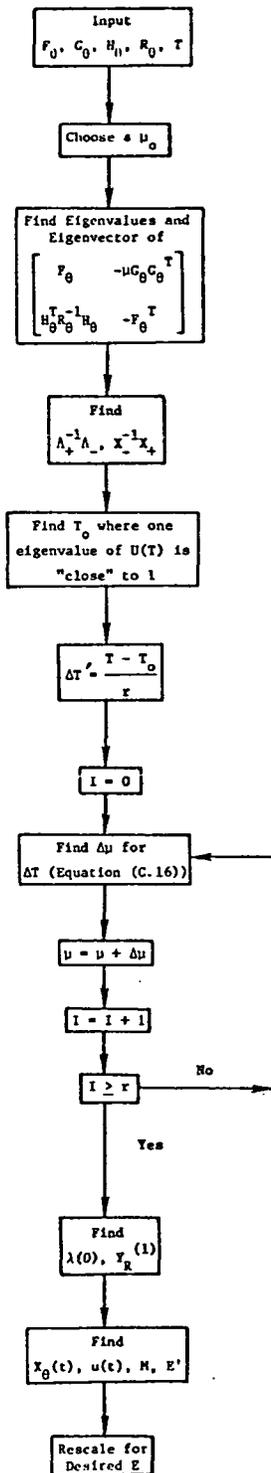


Figure 3.3 Solution of Two Point Boundary Value Problem Using Eigenvalue-Eigenvector Decomposition

(3) μ is updated for small changes, ΔT , until the desired T is reached. Thereafter, a correction in μ is applied to bring the eigenvalue very close to one.

(4) $y_R^{(1)}$, the right eigenvector of U corresponding to the eigenvalue close to one, and $\lambda(0)$ are determined. The states $x_\theta(t)$ and input $u(t)$ are obtained from the expressions

$$x_\theta(t) = -X_+ e^{-\mathcal{L}_+(T-t)} y_R^{(1)} + X_- e^{-\mathcal{L}_+ t} X_+^T \lambda(0) \quad (3.31)$$

$$u(t) = \mu \{ G_\theta^T \Lambda_+ e^{-\mathcal{L}_+(T-t)} y_R^{(1)} - G_\theta^T \Lambda_- e^{-\mathcal{L}_+ t} X_+^T \lambda(0) \} \quad (3.32)$$

(5) The input energy and information matrix are computed. Since the system is linear, the states, inputs, and information matrix can be scaled for desired energy in control inputs.

Remarks

1. In most cases which have been tried, only a few eigenvalues of the Hamiltonian are oscillatory or have low damping. Other eigenvalues can usually be discarded since they give additional eigenvalues of U close to zero. This reduces the size of U . If all but one eigenvalue of \mathcal{H} are highly damped, it is usually possible to compute T for a μ through knowledge of \mathcal{L}_+ and S , by hand calculation.

2. $\lambda(0)$ can also be computed using the following expression:

$$\lambda(0) = e^{-\mathcal{L}_+^T T} X_-^{-T} y_{IR} \quad (3.33)$$

It is easier to use this expression since the matrices on the right hand side are available in the program.

3. Equation (3.32) can be simplified to give analytic expressions for optimal inputs.

3.5 FREQUENCY DOMAIN INPUT DESIGN

If the duration of the experiment is much longer than the system characteristic time, it is possible to design inputs based on a variety of criteria quickly by making the assumption of steady state. In an ingenious approach, Mehra^[22] converts a linear time-invariant system into its frequency domain representation. This eliminates the dynamics of the system. The parameter estimation problem becomes a regression problem in which input frequencies and the power in each frequency are the control parameters. These parameters, which define the input design, are chosen by an iterative procedure.

Consider the state space representation of a discrete-time linear system

$$\mathbf{x}(k+1) = \varphi \mathbf{x}(k) + \mathbf{G}u(k) \quad k=1,2,3,\dots,N \quad (3.34)$$

and the noisy measurements

$$y(k) = \mathbf{H}\mathbf{x}(k) + v(k) \quad (3.35)$$

φ , \mathbf{G} , and \mathbf{H} are appropriate matrices and contain m unknown parameters θ . Fourier transform (3.34) and (3.35) to get

$$\tilde{y}(n) = \mathbf{H} \left(e^{-j \frac{2\pi n}{N}} - \varphi \right)^{-1} \mathbf{G} \tilde{u}(n) + \tilde{v}(n) \quad (3.36)$$

$$\underline{\Delta} \mathbf{T}(n, \theta) \tilde{u}(n) + \tilde{v}(n)$$

As the number of sample points increases, the information matrix per sample approaches^[22]

$$\mathbf{M} = \frac{1}{2\pi} \operatorname{Re} \int_{-\pi}^{\pi} \frac{\partial \mathbf{T}^*}{\partial \theta} \mathbf{S}_{vv}^{-1}(\omega) \frac{\partial \mathbf{T}}{\partial \theta} dF_{uu}(\omega) \quad (3.37)$$

where F_{uu} is the spectral distribution function of u and S_{vv} is the spectral density of v and superscript '*' denotes the conjugate transpose of a matrix.

Based on Appendix B, the following algorithm can be shown to converge to the optimum input. See Figure 3.4 for flowchart.

- (1) Choose a nondegenerate input $f_o(\omega)$ (i.e., consisting of more than $\frac{m}{2p}$ frequencies, with a finite power in each frequency).
- (2) Compute the function $\psi(\omega, f_o)$ and find the value of ω where it is maximum. Call it ω_o , where

$$\psi(\omega, f_o) = \text{Re} \left[\text{Tr} S_{vv}^{-1}(\omega) \frac{\partial T(\omega)}{\partial \theta} D(f_o) \frac{\partial T^*(\omega)}{\partial \theta} \right] \quad (3.38)$$

to minimize $|D|$

and

$$\psi(\omega, f) = \mathcal{L} \left[D(f_o) \frac{\partial T^*(\omega)}{\partial \theta} S_{vv}^{-1}(\omega) \frac{\partial T(\omega)}{\partial \theta} D(f_o) \right] \quad (3.39)$$

to minimize a linear function \mathcal{L} of D .

- (3) Evaluate the normalized information matrix at ω_o . Call it $M(\omega_o)$.
- (4) Update the design

$$f_1 = (1 - \alpha_o) f_o + \alpha_o f(\omega_o) \quad 0 < \alpha_o < 1 \quad (3.40)$$

α_o is chosen to minimize $|D(f)|$ or $[\mathcal{L}(D(f))]$ where

$$M(f_1) = (1 - \alpha_o) M(f_o) + \alpha_o M(\omega_o), \quad 0 < \alpha_o < 1 \quad (3.41)$$

It can be shown that such an α_o exists.

- (5) Repeat steps (2) - (4) until the desired accuracy is obtained.

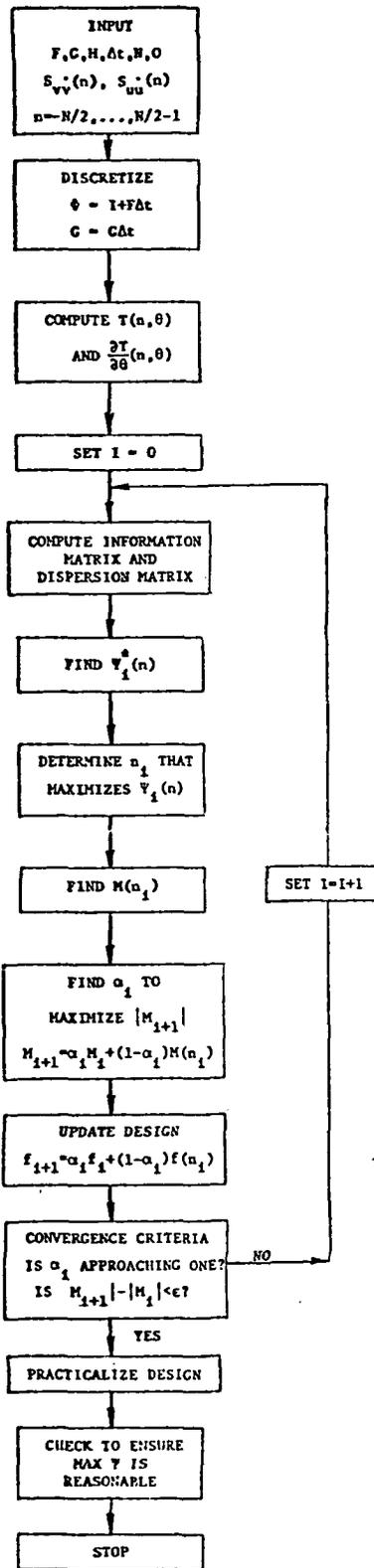


Figure 3.4 Input Design in Steady State

Remarks

1. If k is the number of frequencies in the optimal design, then

$$\left\lceil \frac{m}{2p} \right\rceil \leq k \leq \frac{m(m+1)}{2} \quad (3.42)$$

where $\left\lceil \frac{m}{2p} \right\rceil$ is the smallest integer higher than $\frac{m}{2p}$.

2. The function ψ has many local maxima. It is computationally time consuming to find ω_0 where $\psi(\omega, f)$ is maximum. In the computer implementation of the algorithm, we consider finite but large N and search through all values of $\psi(n)$ to find the maximum. Most stable systems of interest are low pass filters. Thus, in most cases, it is possible to find a subset of $[-\pi, \pi]$, where the search need be carried out.
3. The termination criteria are the same as in Section 3.3.2 for the time domain case.

Practical Considerations in the Computation of Optimal Input^[27]

The above algorithm will produce an optimal input design with a sufficient number of iterations. However, at each iteration, the procedure adds one point to the spectrum of the input. There are many inputs with unit power leading to the same information matrix. From a practical point of view, it is desirable to have as few frequencies in the optimal input as possible. The fewer the frequencies, the easier it is for the aircraft pilot or input generating system to reproduce the desired input. During the computation, a few steps can be taken to reduce the number of points in the spectrum. Suppose the normalized input at any stage has k frequencies ω_i with power α_i ($i = 1, 2, \dots, k$).

- (1) All frequencies which are "close" to each other can be lumped into one frequency. We consider two frequencies close to each other if

they are less than $\Delta\omega$ apart. Suppose q frequencies ω_i^* are within a band $\Delta\omega$ wide. Then they can be replaced by one frequency ω^* with power α^* where

$$\alpha^* = \sum_{i=1}^q \alpha_i^* \quad (3.43)$$

and

$$\omega^* = \frac{1}{\alpha^*} \sum_{i=1}^q \alpha_i^* \omega_i^* \quad (3.44)$$

- (2) From this new normalized input, all frequencies $\omega_i^!$ with power less than a threshold $\alpha^!$ are chosen. These frequencies are not within $\Delta\omega$ of any other frequency in the normalized input. They are dropped. The remaining frequencies do not form a normalized design, so the design is renormalized.

Steps (1) and (2) are carried out ensuring that the design does not become degenerate. This "practicalization" requires judgment of $\Delta\omega$ and α .

3.6 OTHER CONSIDERATIONS IN THE DESIGN OF OPTIMAL INPUTS

The stability and control derivatives determined from aircraft parameter identification are used for several purposes, for example, handling qualities specification, control system design, and simulator aerodynamic coefficient values. The ultimate objective of parameter identification enters into the considerations for the choice of optimal inputs to identify unknown system parameters.

3.6.1 Primary and Secondary Parameters

In many situations of practical interest, it is desirable to obtain accurate estimates of only a subset of all unknown parameters. Let the first k parameters be of primary interest and the remaining $m-k$ parameters be of secondary interest. There is no direct incentive to obtain good estimates of the last $m-k$ parameters. Inputs which give outputs sensitive to primary parameters should

be used. The secondary parameters are estimated only to the extent that they help reduce uncertainty in primary parameters resulting from errors in secondary parameters.

This consideration is simple to handle when the optimality criterion is a weighted trace of the dispersion matrix. The weighting matrix should be chosen as

$$W_{ps} = \text{diag} \left[\underbrace{1, 1, \dots, 1}_k, \underbrace{0, 0, \dots, 0}_{m-k} \right] \quad (3.45)$$

In the determinant criterion by selecting the function φ ,

$$\varphi_{ps} = \text{Tr}(\hat{W} M) \quad (3.46)$$

$$\hat{W} = \begin{bmatrix} D_{kk} & D_{k\ell} \\ D_{\ell k} & D_{\ell\ell} - M_{\ell\ell}^{-1} \end{bmatrix}, \quad k + \ell = m \quad (3.47)$$

the $|D_{kk}|$ is minimized. In other words, the input is such that it minimizes the area of the cross section of uncertainty ellipsoid made by a k dimensional hyperplane in the space of primary parameters.

3.6.2 Technique for Evaluating Parameter Identifying Inputs

Various techniques can be used to determine how effective an input is in identifying system parameters. The accuracy with which the parameters can be estimated depends not only on the inputs but also on the data reduction technique used.

The most definitive way of determining the usefulness of an input is to use it with the system for which it is designed, and then to compare the resulting parameter estimates with the ones obtained using other inputs. If this is not

possible, simulation data must be used. There are two more basic methods to evaluate the simulation data:

- (1) Monte Carlo Simulation. This method consists of generating typical measurement time histories using simulations of system equations. The parameters are determined from each of these time histories by a minimum variance estimator. The mean and variance of the estimated parameters can be determined.
- (2) Information Matrix Method. It is known that the diagonal terms on the inverse of the information matrix are the covariance of the parameter estimates if an efficient estimator is used. Information matrix can predict the parameter estimation errors without actually generating a number of time histories.

Monte Carlo simulation is usually expensive because the parameters must be estimated for each simulation using a minimum variance estimator (this is necessary so that the effects of the data reduction method do not mask the identifiability of a certain input). The information matrix method can accomplish the same task much more quickly. Any difference in the covariance of parameter estimates predicted by the two methods is only due to the inefficiency of the data reduction method used in the Monte Carlo simulation.

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IV. TIME DOMAIN SYNTHESIS OF OPTIMAL INPUTS

4.1 INTRODUCTION

The time domain computer program determines the optimal inputs' time history for a specified length of flight test. In general, the optimal input depends on planned flight test duration, available instruments and their accuracies, the best a priori estimates of unknown parameter values, and many other factors. The techniques, developed in Chapter III, can be used to design the optimal input based on a variety of criteria.

The elevator deflection sequence, which gives good estimates of parameters in the short period mode of a C-8 aircraft, ^[3] is computed under a variety of circumstances. These inputs are evaluated against conventional inputs and against each other, with simulated data. The accelerations and velocities are determined over a simulated flight test to ensure that the inputs are safe and implementable.

Rudder inputs are designed to identify five parameters in the lateral dynamics model of a Jet Star aircraft. ^[3] Combined rudder and aileron inputs are also determined and compared with rudder inputs alone.

The weighted trace of the dispersion matrix is considered as one of the best criteria and is used throughout this chapter. Its comparison with other criteria for practical examples is given.

4.2 INPUTS FOR LONGITUDINAL SYSTEM

The longitudinal perturbation motions of a Buffalo C-8 aircraft about a steady trim condition obey the following differential equations (units: meters, deg, sec).

$$\frac{d}{dt} \begin{bmatrix} u \\ \theta \\ q \\ \alpha \end{bmatrix} = \begin{bmatrix} -.02 & -.171 & .001 & .179 \\ 0 & 0 & 1 & 0 \\ .0984 & 0 & \underline{-1.588} & \underline{-.562} \\ -.131 & 0 & 1 & \underline{-.737} \end{bmatrix} \begin{bmatrix} u \\ \theta \\ q \\ \alpha \end{bmatrix} + \begin{bmatrix} 0.0 \\ 0.0 \\ \underline{-1.66} \\ \underline{.005} \end{bmatrix} \delta_s \quad (4.1)$$

where

- u is forward speed,
- θ is pitch angle,
- q is pitch rate,
- α is angle-of-attack, and
- δ is stabilator deflection.

The poles of this system are at

$$\omega_{sp} = -1.16 \pm .62j \quad \text{short period mode}$$

$$\omega_{ph} = -.0153 \pm .088j \quad \text{phugoid mode}$$

The first pair of complex roots corresponds to the fast, highly damped, short period mode and the second to the slow, weakly damped, phugoid mode.

It is assumed that there are five unknown parameters, all in the short period mode (underlined). It is well known that the two state (pitch rate and angle-of-attack) model of an aircraft is a good representation of the short period motion. This approximation is used to find the elevator deflection time history. The equations of motion become

$$\begin{bmatrix} \dot{q} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} \underline{-1.588} & \underline{-.562} \\ 1 & \underline{-.737} \end{bmatrix} \begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} \underline{-1.66} \\ \underline{.005} \end{bmatrix} \delta_s \quad (4.2)$$

with poles at

$$\omega_{sp} = -1.16 \pm .62j$$

These are noisy measurements of q and α

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} v_q \\ v_\alpha \end{bmatrix} \quad (4.3)$$

The measurement noise is assumed white. The root mean square errors in the measurements of q and α are $.70 \text{ deg sec}^{-1}$ and 1.0 deg , respectively, and the sampling rate is 25 per second.

A doublet input, shown in Figure 4.1, is used conventionally to identify these five parameters. Starting from this doublet, the input design program is used to determine the optimal input for a 6 sec long experiment with $100 \text{ deg}^2 \text{ sec}$ total input energy. The performance index is the trace of the dispersion matrix. The input at the end of each iteration step is shown in Figure 4.2. Fairly good convergence is obtained in three steps. Table 4.1 shows the standard deviation in parameter estimates for each of these inputs.* Also shown is the value of β which determines the component of new input added to the old input (see Chapter III). β was taken to be one if the decrease in $\text{Tr}(D)$ was more than 50%. It is clear from Table 4.1 that the optimal input should give much better parameter estimates (e.g., smaller standard deviations) than the conventional doublet input.

*The standard deviation format of Table 4.1 is used throughout this report. These quantities are the square roots of the diagonal elements of the Cramer-Rao lower bound (Equation (2.1), p. 8), and represent the lowest possible value of the parameter estimate standard deviations which can be attained using an unbiased and efficient parameter identification procedure. As explained in Section 3.7, these lowest bounds rather than parameter estimates based on individual runs are a meaningful comparison of different inputs, a better input giving a smaller lower bound. The nominal parameter values are given for reference, in terms of dimensional coefficients, C_j ($j = m_q, m_{\delta_e}$, etc.), to simplify the notation.

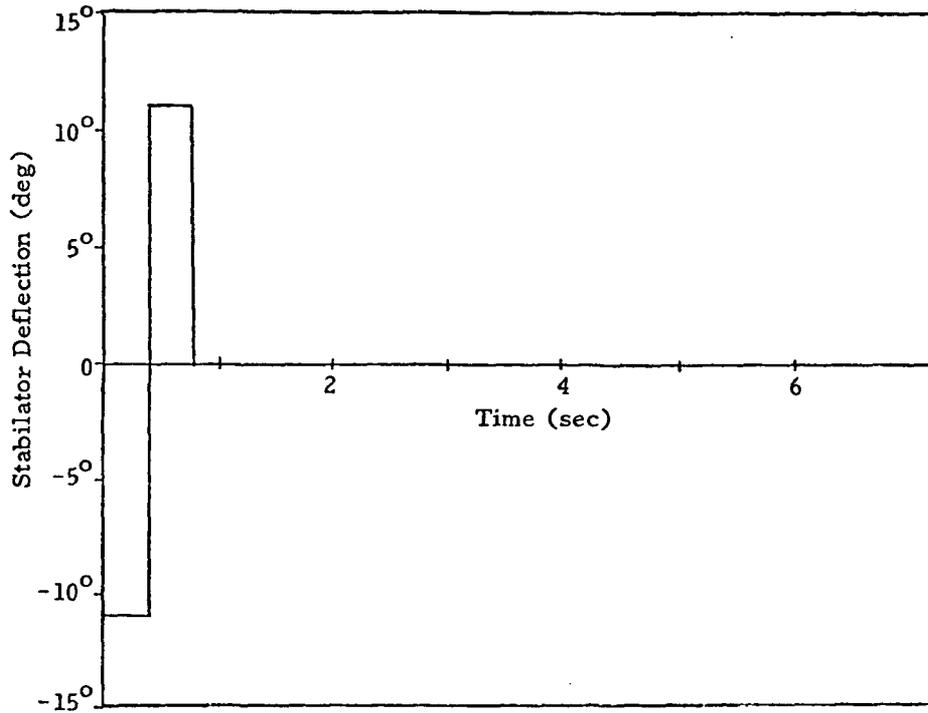


Figure 4.1 Conventional Doublet Input

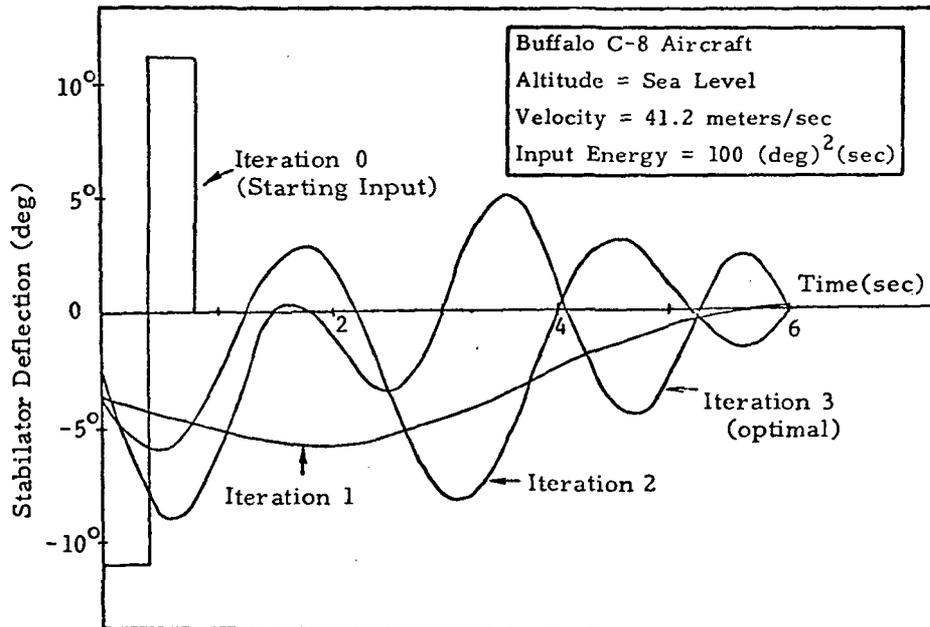


Figure 4.2 Input at the End of Each Iteration

Table 4.1. Standard Deviations of Parameter Estimates for Inputs at the End of Each Iteration

Length of Simulated Flight Test = 6 sec

Total Input Energy = 100 deg² sec

	Standard Deviations					Tr(D)	β
	C_{m_q}	C_{m_α}	C_{Z_α}	$C_{m_{\delta_s}}$	C_{Z_δ}		
Iteration 0 (Doublet)	.219	.362	.326	.0978	.0957	.304	
Iteration 1	.294	.0703	.0529	.193	.0452	.134	1.0
Iteration 2	.113	.0729	.0595	.0620	.0421	.0272	.61
Iteration 3	.113	.0676	.0561	.0672	.0400	.0264	.43
Parameter Value	-1.588	-.562	-.737	-1.66	0.0		

[All in units of deg, sec]

Figure 4.3 shows pitch rate, angle-of-attack, pitch acceleration, and vertical acceleration time histories for the optimal input and compares them to the corresponding time histories resulting from a doublet input. Peak values of accelerations are higher for the doublet input. However, high accelerations last for a longer time when the optimal input is used. Also, the excursions in angle-of-attack are much higher for the optimal input.

4.2.1 Primary/Secondary Considerations

An input is designed considering C_{m_q} as the primary parameter of interest. The variance on the estimate of C_{m_q} is weighted 100 times more heavily than the variance on C_{m_α} , C_{Z_α} , $C_{m_{\delta_s}}$ and C_{Z_δ} . The starting input is the optimal input when all parameters are equally important. The input obtained after one iteration

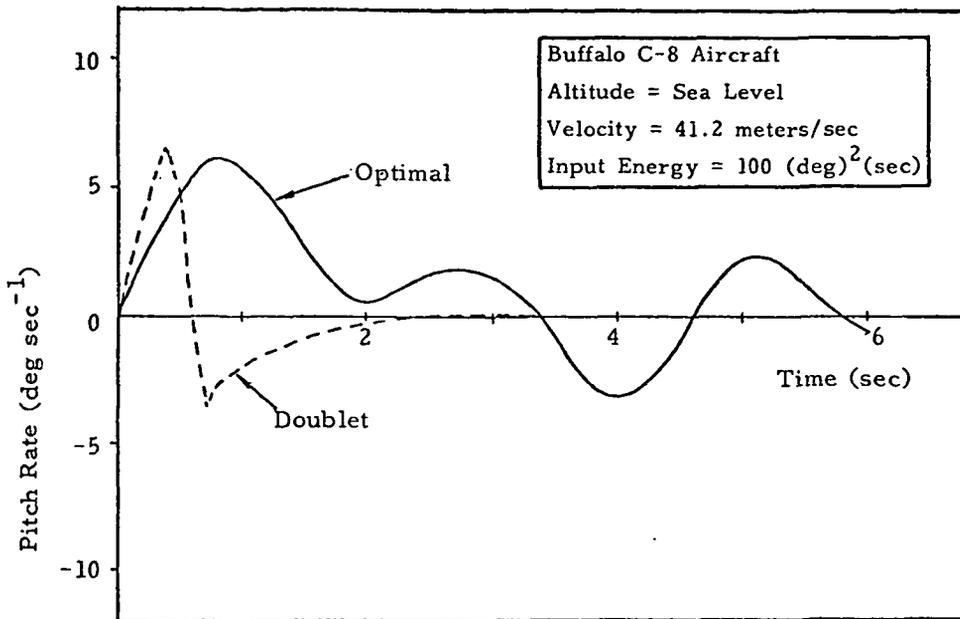


Figure 4.3a Comparison of Pitch Rates for Doublet and Optimal Control (Simulation)

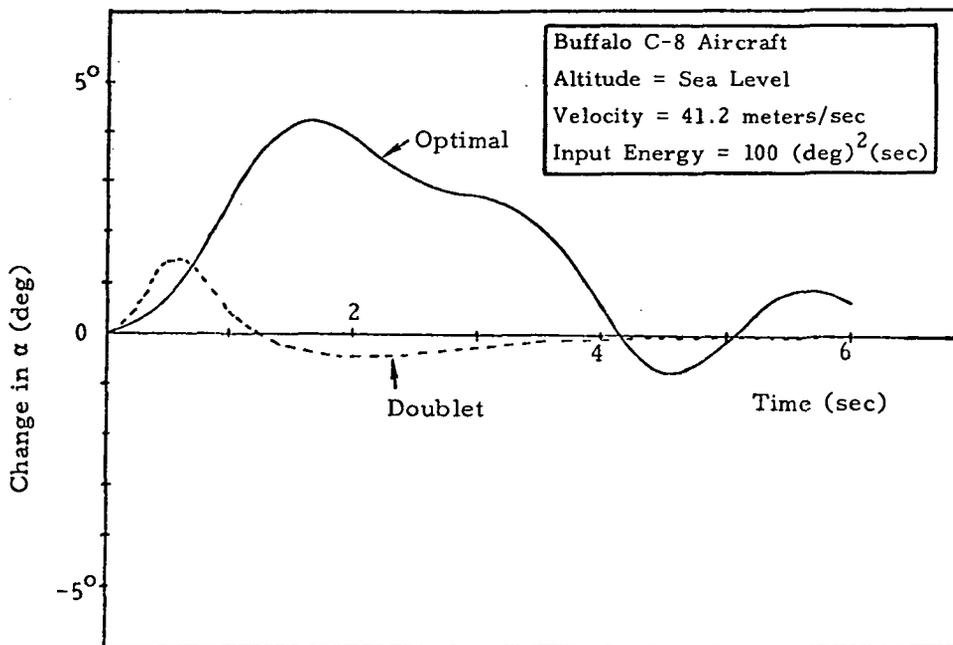


Figure 4.3b Comparison of Angle-of-Attack Variation for Doublet and Optimal Input (Simulation)

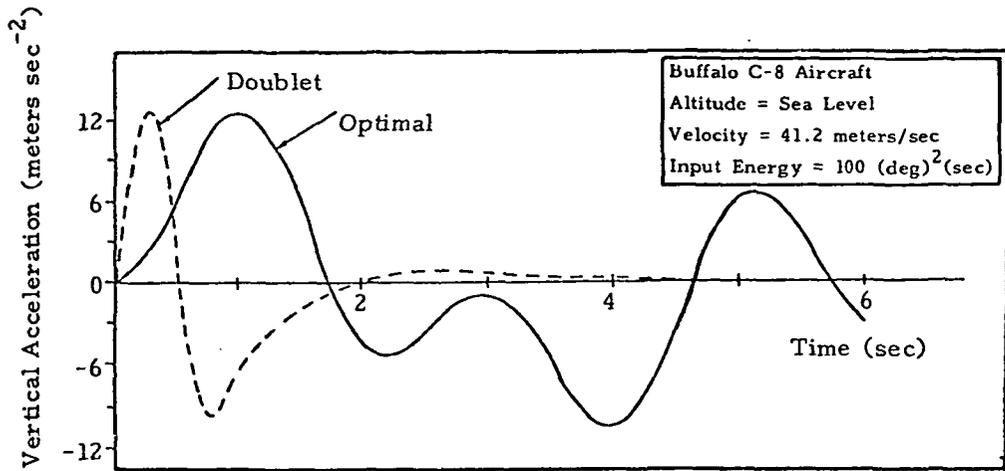


Figure 4.3c Comparison of Vertical Acceleration with Doublet and Optimal Input (Simulation)

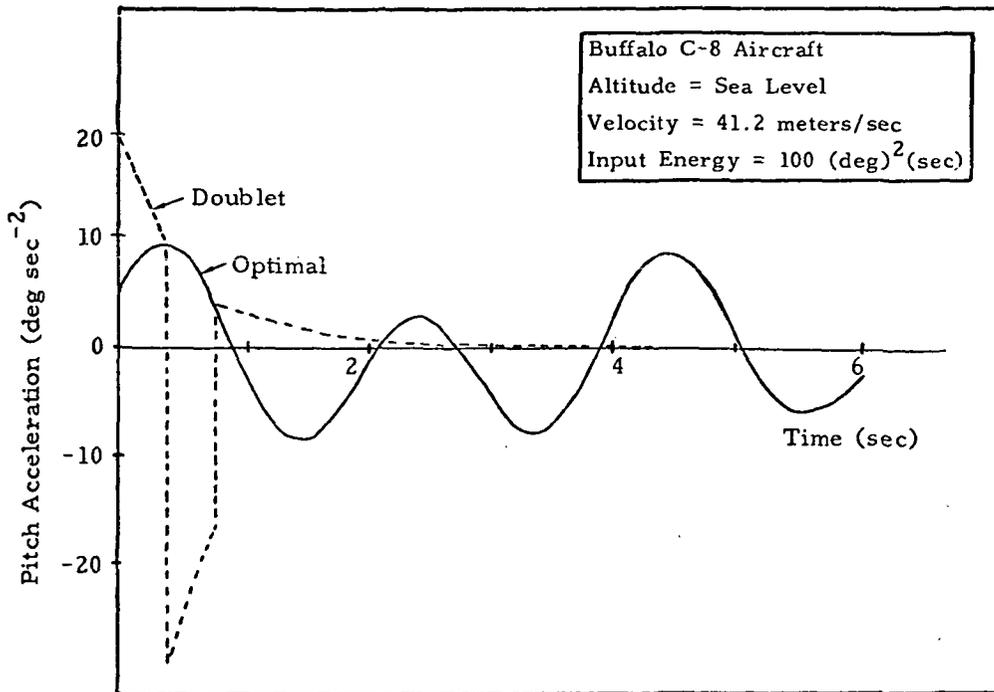


Figure 4.3d Comparison of Pitch Acceleration for Doublet and Optimal Input (Simulation)

is shown in Figure 4.4. It looks similar to the input when all parameters are weighted equally. The standard deviations of parameter estimates for these two inputs are compared in Table 4.2. The standard deviations of C_{m_q} and $C_{m_{\delta_s}}$ decrease by about 3% while those on other parameters increase. This shows that to get a good estimate of C_{m_q} it is necessary to have a good estimate of $C_{m_{\delta_s}}$ also. Also, C_{m_q} is an important parameter in the system. Even when all the parameters are to be identified, almost the best possible estimate of C_{m_q} is obtained. The considerations of primary and secondary parameters may be more useful when the two sets of parameters affect different modes.

Using the new time domain method, the entire idea of primary/secondary derivatives could be recast in a more general framework where parameters are assigned different levels of importance through weights on their covariance.

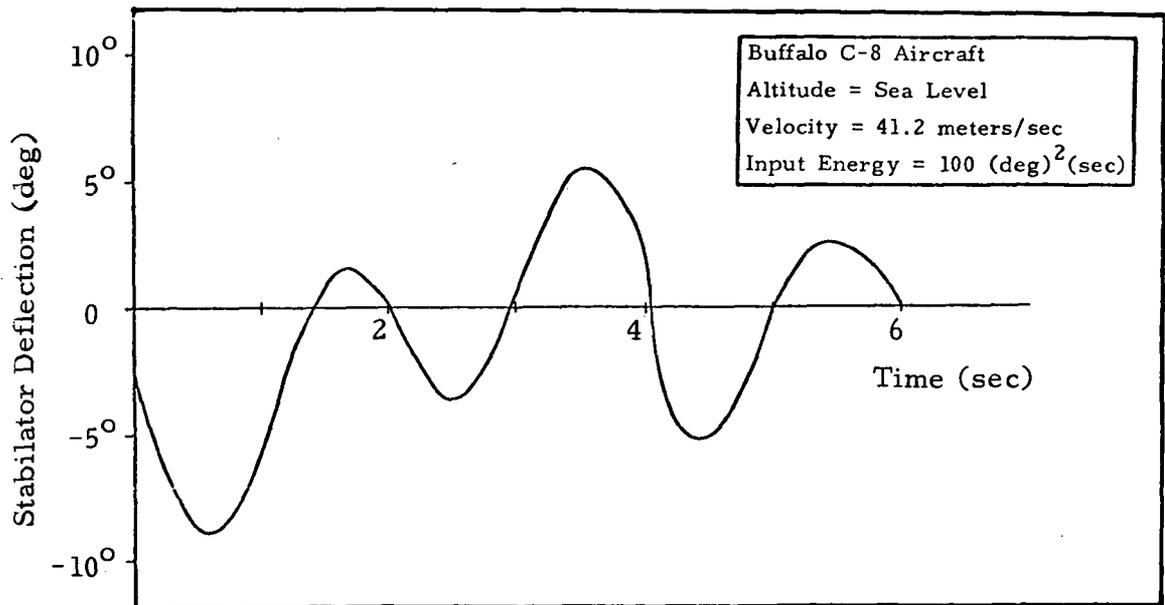


Figure 4.4 Optimal Elevator Deflection (C_{m_q} Only Primary Derivative)

Table 4.2 Comparison of Standard Deviations on Parameter Estimates With All Parameters Equally Important vs. C_{m_q} Primary

Length of Simulated Flight Test = 6 sec

Total Input Energy = 100 deg² sec

	Standard Deviations					Tr(D)
	C_{m_q}	C_{m_α}	C_{Z_α}	$C_{m_{\delta_s}}$	$C_{Z_{\delta_s}}$	
Parameters Equally Important	.113	.0676	.0561	.0672	.0400	.0264
C_{m_q} Primary	.110	.0747	.0615	.0651	.0415	.0275
Parameter Value	-1.588	-.562	-.737	-1.66	0.0	

[All in units of deg, sec]

4.2.2 Comparison to Inputs Based on Maximizing the Trace of the Information Matrix

Before the computer program was written to minimize the trace of the dispersion matrix, many inputs were designed using the trace of the information matrix as the optimality criterion. Three such inputs 4, 6, and 12 sec long, each with total input energy of 100 deg² sec, are shown in Figure 4.5. Table 4.3 compares these three inputs with each other and with the 6 sec. long time domain input obtained based on the Tr(D) criterion. It is apparent that the trace of the dispersion matrix is a much superior criterion.

4.3 INPUTS FOR THE LATERAL SYSTEM

The equations of motion for lateral motions of one version of a Jet Star flying at 573.7 meters/sec. at 6,096 meters are [25] (all in units of deg, sec)

Table 4.3 Comparison of Tr(D) and Tr(M) Criteria

Total Input Energy = 100 deg² sec

Criterion for Designing Input	Duration of Simulated Flight Test	Standard Deviations					Tr(D)
		C _m q	C _m α	C _Z α	C _m δ _s	C _Z δ _s	
Max Tr(M)	4 sec.	.289	.111	.0606	.165	.0433	.128
	6 sec.	.316	.0732	.0526	.207	.0450	.153
	12 sec.	.264	.0762	.0654	.191	.0609	.121
Min Tr(D)	6 sec.	.113	.0676	.0561	.0672	.0400	.0264
Reference Parameter Value		-1.588	-.562	-.737	-1.66	0.0	

[All in units of deg, sec.]

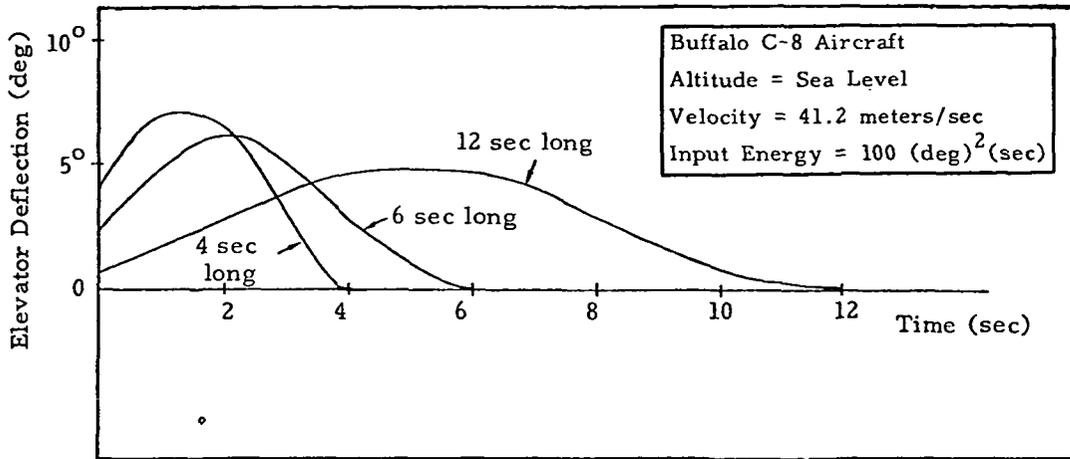


Figure 4.5 Longitudinal Inputs for Different Durations of Flight Test to Maximize $\text{Tr}(M)$

$$\frac{d}{dt} \begin{bmatrix} \beta \\ \varphi \\ p \\ r \end{bmatrix} = \begin{bmatrix} \underline{-0.119} & 0.0565 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \underline{-4.43} & 0 & -0.935 & 0.124 \\ \underline{2.99} & 0 & -0.119 & \underline{-0.178} \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ p \\ r \end{bmatrix} + \begin{bmatrix} 0 & 0.0289 \\ 0 & 0 \\ 2.88 & 1.40 \\ 0.0 & \underline{-1.55} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \quad (4.4)$$

where β is sideslip angle, φ is roll angle, and p and r are roll and yaw rates, respectively. Aileron and rudder are two control inputs. These are noisy measurements of the four states

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ p \\ r \end{bmatrix} + \begin{bmatrix} \eta_\beta \\ \eta_\varphi \\ \eta_p \\ \eta_r \end{bmatrix} \quad (4.5)$$

The noise in measurements is white and Gaussian with root-mean-square values .

$$\begin{array}{ll} \eta_\beta & 1 \text{ deg} \\ \eta_\varphi & .5 \text{ deg} \\ \eta_p & .71 \text{ deg sec}^{-1} \\ \eta_r & .71 \text{ deg sec}^{-1} \end{array} \quad (4.6)$$

and the sampling rate is 25 per second. The poles of this system are at

$$\begin{array}{ll} \underline{-.0511 + 1.78 j} & \text{Dutch-roll} \\ -1.12 & \text{Roll} \\ -.00667 & \text{Spiral} \end{array} \quad (4.7)$$

The inputs are designed to identify the parameters which predominantly affect the Dutch-roll mode (underlined).

4.3.1 Rudder and Aileron Inputs When All Parameters Are Equally Important

The optimal rudder input to minimize the sum of dispersions of these five parameters is determined and is shown in Figure 4.6. The duration of the simulated test is 8 sec. and the input energy is $100 \text{ deg}^2 \text{ sec}$. Figure 4.7 shows a conventional doublet input with equal energy. The comparison of standard deviation on parameters resulting from these inputs is given in Table 4.4. The optimal input results in better parameter estimates than the doublet, based on comparing the standard deviations.

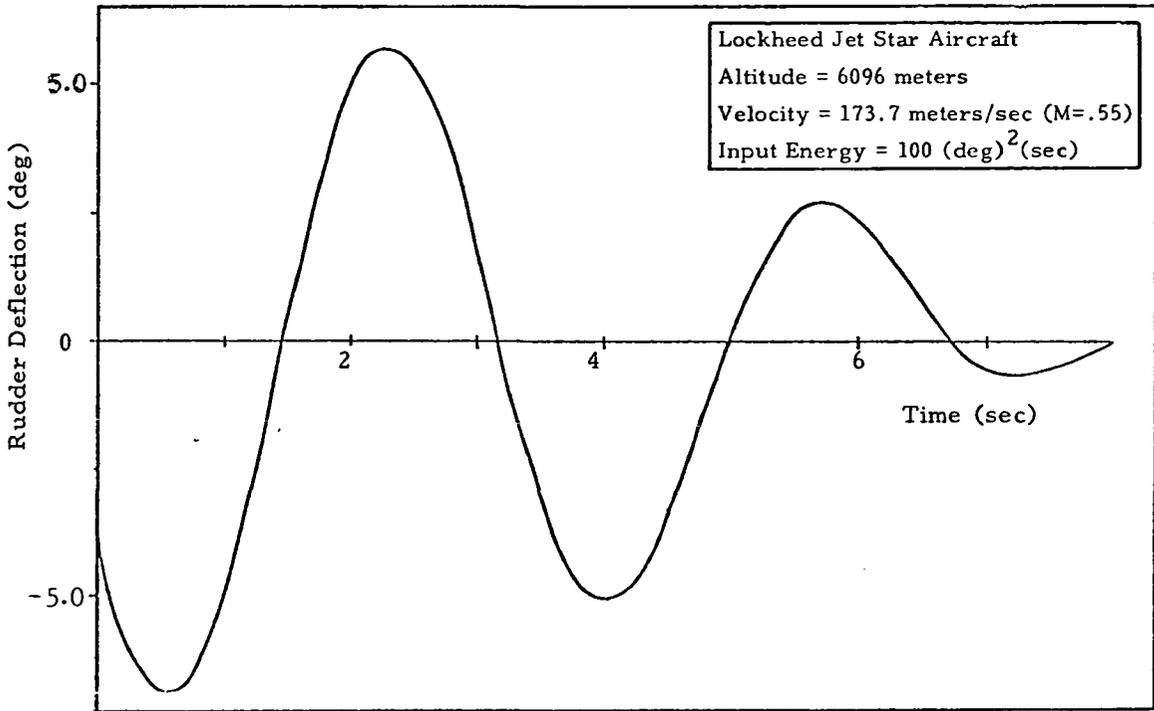


Figure 4.6 Optimal Rudder Input to Identify Five Parameters in the Lateral Motion of an Aircraft

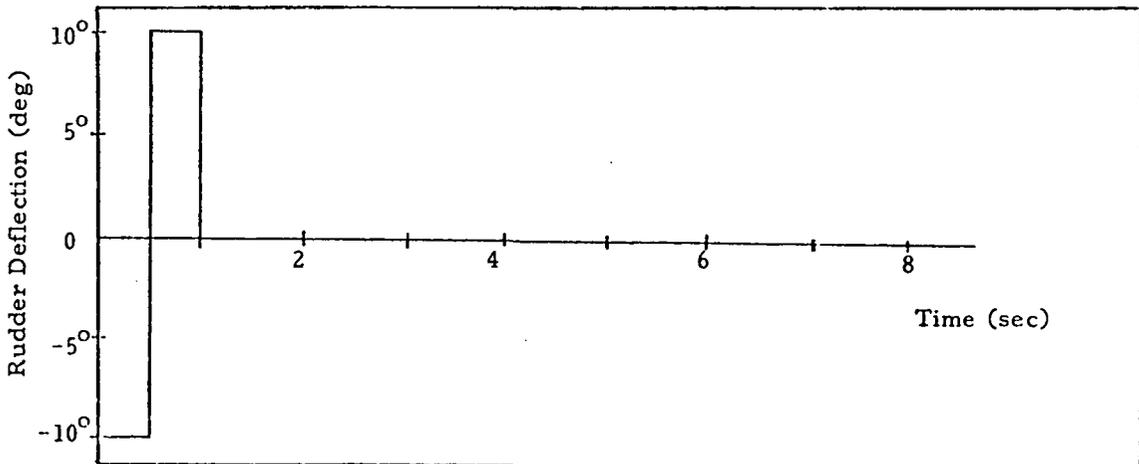


Figure 4.7 Conventional Rudder Input

Table 4.4 Parameter Estimate Standard Deviations for Different Lateral Inputs

Duration of Simulated Flight Test = 8 sec.

Total Input Energy = 100 deg² sec.

Input	Standard Deviations						Tr(D)
	$C_{y\beta}$	$C_{\ell\beta}$	$C_{n\beta}$	C_{n_r}	$C_{n\delta_r}$		
Rudder Doublet	.0238	.0576	.00635	.0240	.0207		.00492
Optimal Rudder Input (Figure 4.8)	.00880	.0204	.00277	.00880	.00860		.000648
Optimal Rudder Input (Computed with Numerical Simplification)	.00867	.0204	.00274	.00897	.00850		.000646
Optimal Rudder + Aileron Input (Figure 4.11)	.00840	.0201	.00266	.00877	.00822		.000632
Parameter Value	-.119	-4.43	2.99	-.178	-1.55		

[All in units of deg, sec.]

The state and lateral acceleration simulated time histories are compared in Figure 4.8. The optimal input results in large excursions in states and accelerations. This is because the Dutch-roll mode has low damping and optimal input continuously excites the responses.

New simultaneous rudder and aileron inputs are designed to identify these parameters. The inputs with combined energy of $100 \text{ deg}^2 \text{ sec}$ are shown in Figure 4.9. The aileron input amplitude is very small. The estimates resulting from this simultaneous input are presented in Table 4.4. There is a very small improvement over the single rudder input case, as would be expected since the rudder input is much more effective in estimating these five parameters than the aileron input. Larger aileron deflections can be obtained by placing separate energy constraints on the aileron and the rudder.

There are eight linearly independent columns in the controllability matrix of (F_θ, G_θ) . Therefore, eight linear differential equations must be solved to obtain all sensitivity functions (see Section 3.3.3 and Appendix D). However, the last column is almost linearly dependent on the remaining seven. The optimal input is designed through propagating just seven equations. The standard deviations of parameters resulting from this input are given in Table 4.4. There is an insignificant difference in this input and the input based on using all eight sensitivity equations. This approximation brings about considerable saving in computation time. It could prove extremely useful in computation of optimal inputs for practical high order systems.

4.3.2 Primary/Secondary Derivatives

A rudder deflection sequence is determined to primarily identify C_{n_β} , C_{n_r} and $C_{n_{\delta_r}}$ and consider C_{y_β} and C_{l_β} as unknown secondary coefficients. As in the previous case, this is accomplished by weighting the error covariance on primary parameters 100 times as on secondary parameters. The input is shown in Figure 4.10 (one iteration starting from the optimal input of Figure 4.7). The inputs considering all parameters primary and considering only C_{n_r} , C_{n_β} , and $C_{n_{\delta_r}}$

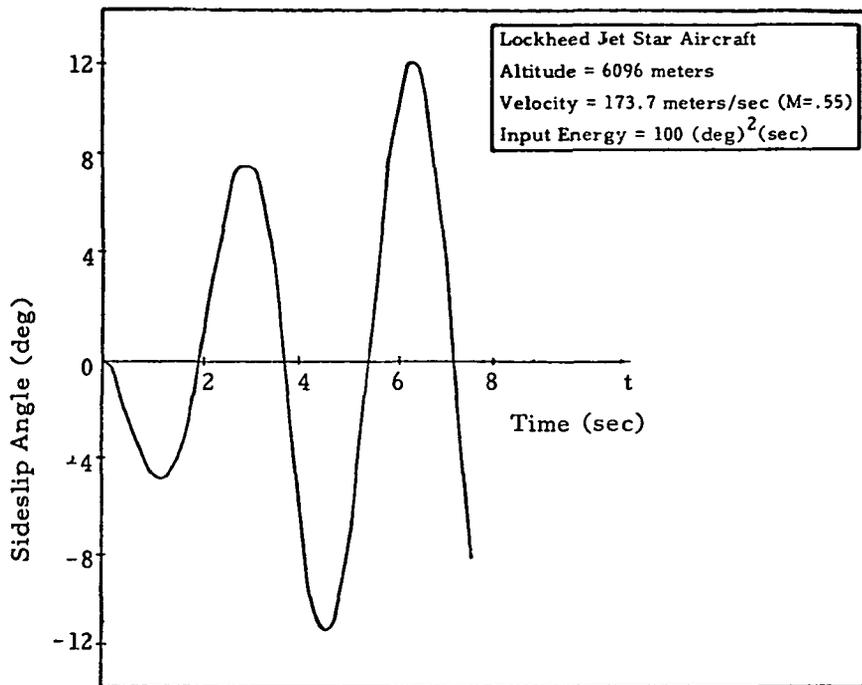


Figure 4.8a Sideslip Angle for the Optimal Input (Simulation)

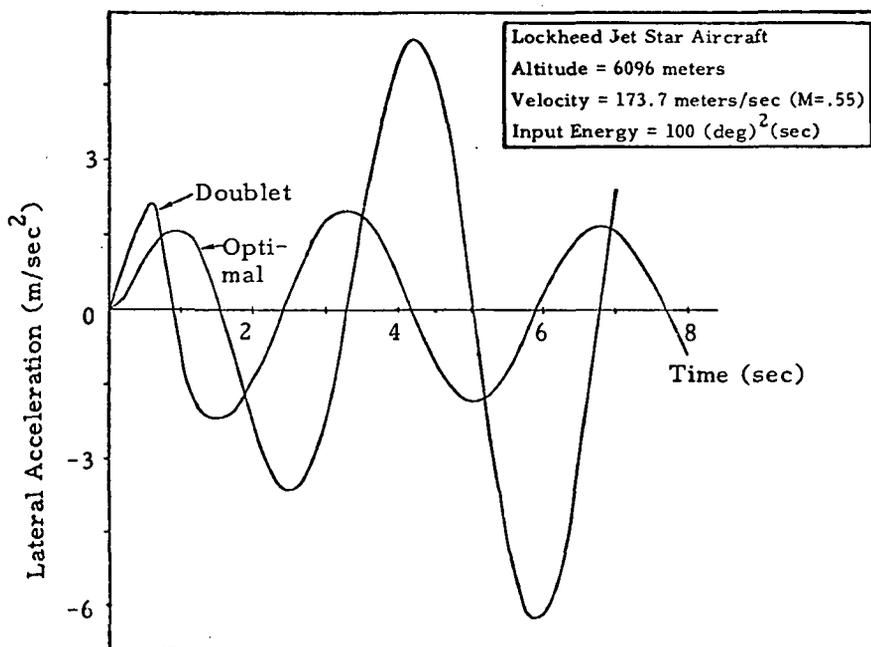


Figure 4.8b Lateral Acceleration for Doublet and Optimal Input (Simulation)

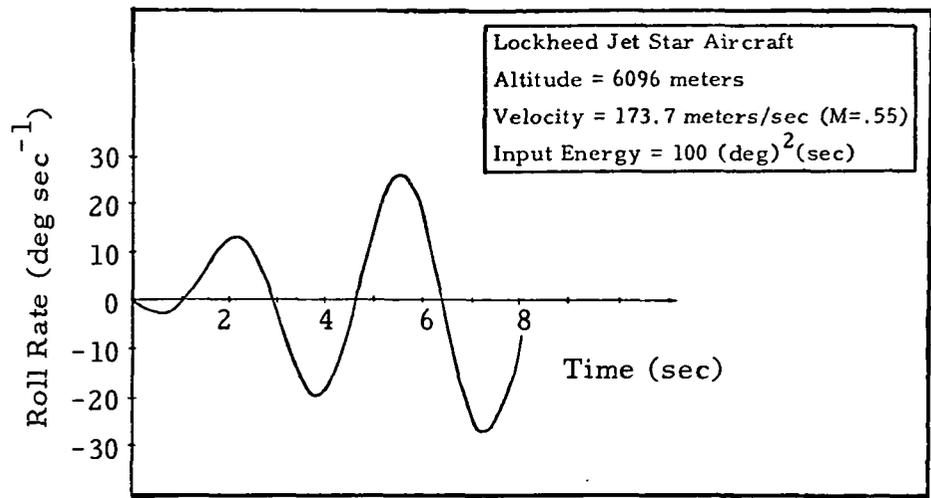


Figure 4.8c Roll Rate for the Optimal Rudder Input (Simulation)

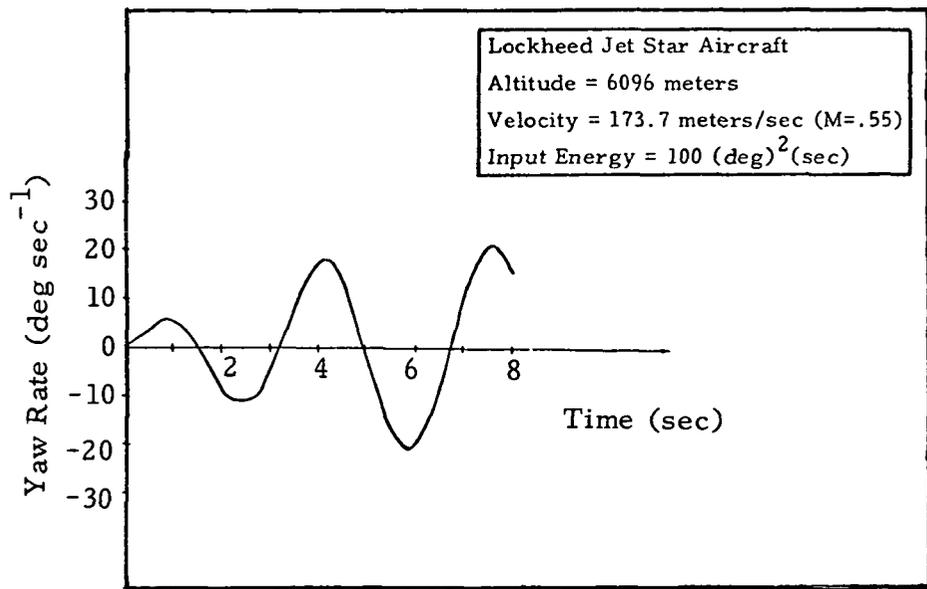


Figure 4.8d Yaw Rate for the Optimal Rudder Input (Simulation)

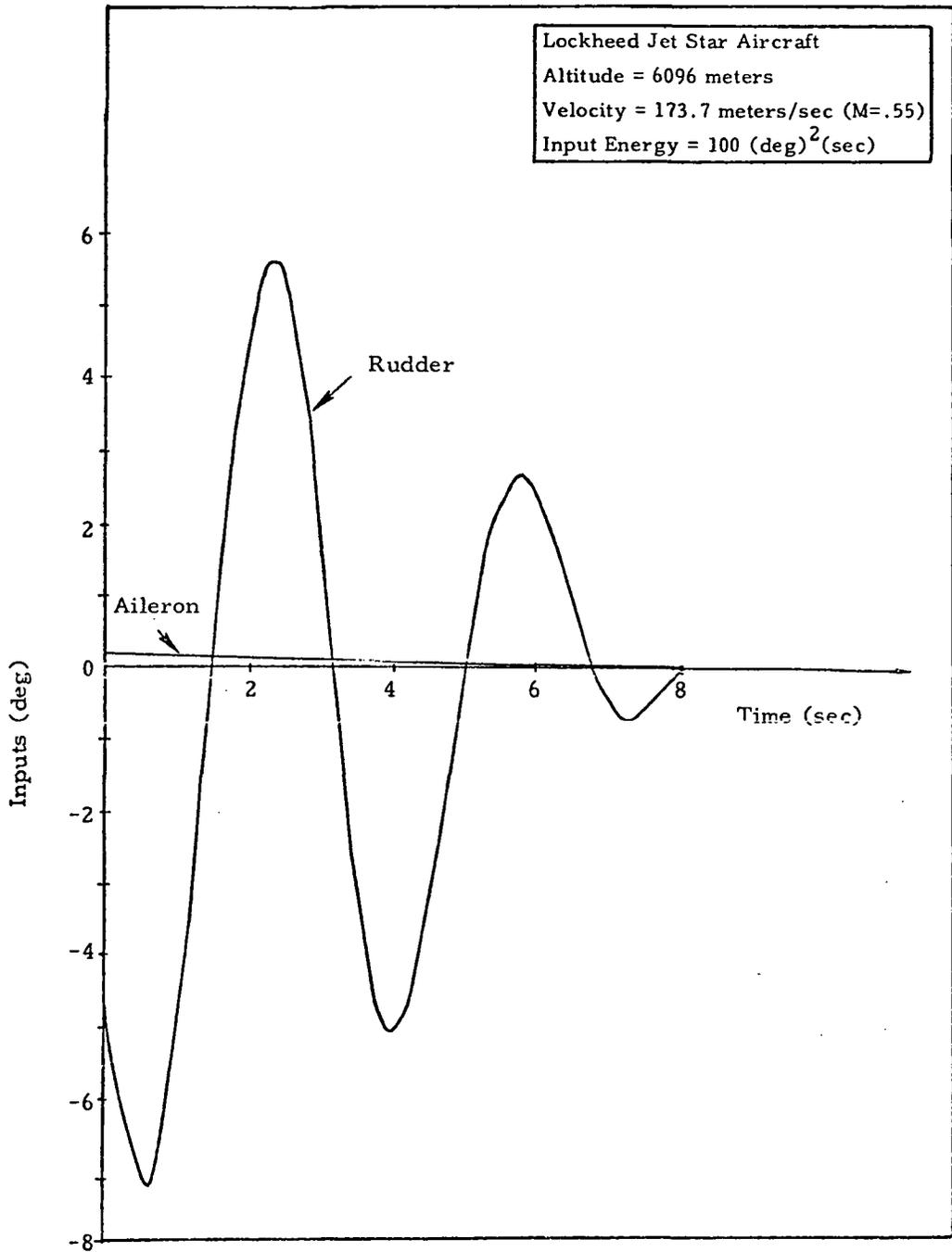


Figure 4.9 Simultaneous Rudder and Aileron Inputs to Identify Five Lateral Parameters

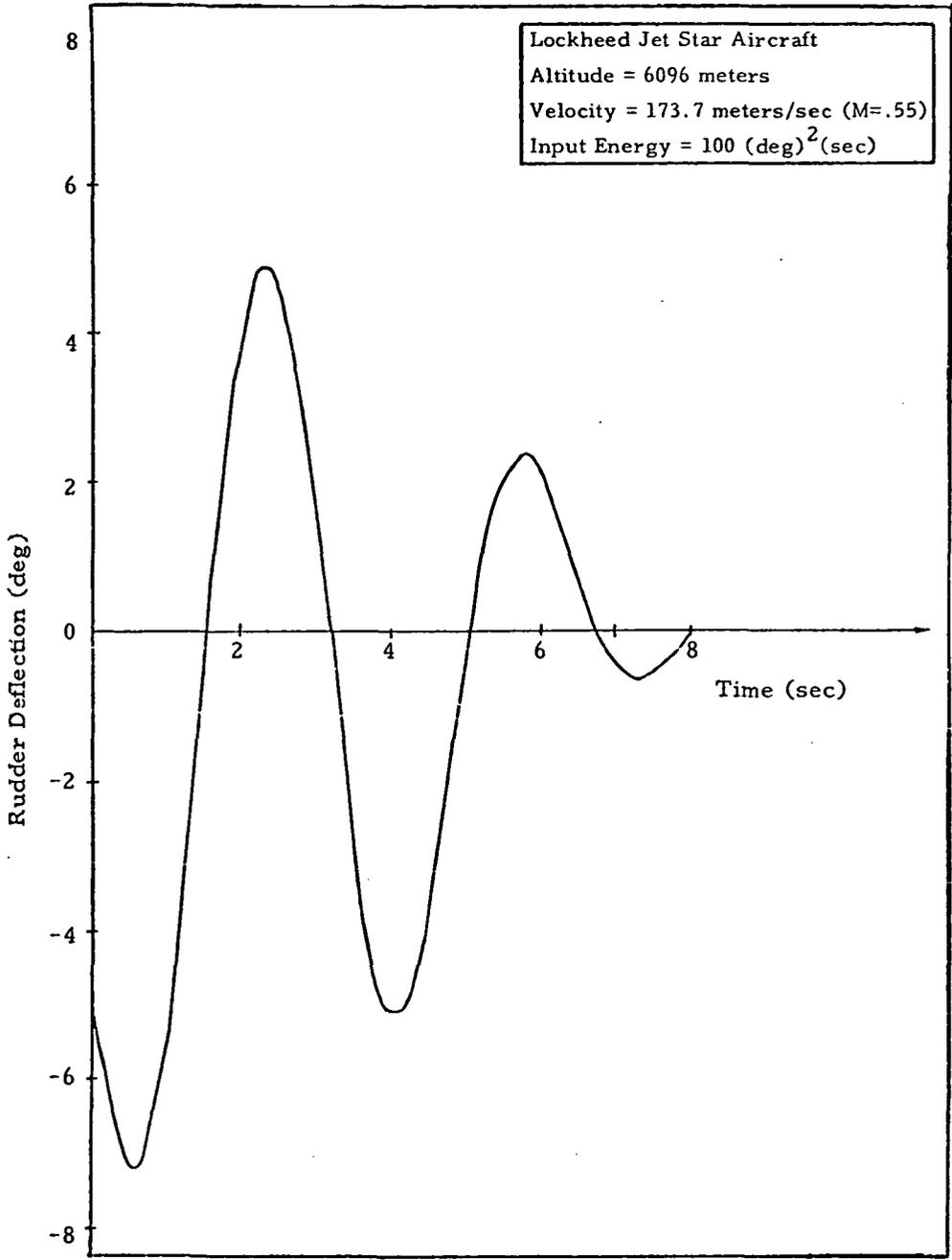


Figure 4.10 Optimal Input Considering $C_{n\beta}$, C_{nr} , $C_{n\delta_r}$ as Primary Parameters

primary are quite similar. The parameter error covariances are compared in Table 4.5. There is only a moderate improvement in the accuracy of the primary parameters.

4.4 CONCLUSIONS

Inputs are designed for both longitudinal and lateral systems based on a variety of criterion functions. For these inputs, the following conclusions are reached:

1. Inputs based on the trace of the dispersion matrix criterion give lower derivative estimate error standard deviations than those based on the trace of the information matrix criterion.
2. Low order approximations to system representations can usually be made in the computation of the optimal input (two state representation of short period mode gives almost as low estimate error variances as the more complete, four state longitudinal model).
3. For weakly damped systems, it may be necessary to place direct state constraints to avoid excessive state excursions. It is noted, however, that weakly damped modes causing large aircraft motions are acceptable as long as the pilot can tolerate and control them.
4. The inputs designed to identify primary parameters alone did not show much improvement in the estimation accuracy of these parameters over the inputs designed to identify all parameters. A more careful separation of primary and secondary parameters may be required in the design of optimal input to gain maximum benefits from these considerations. In the procedure used here, it is assumed that there is no knowledge about the secondary parameters before the flight test. Further work is required to include cases in which partial information about the secondary parameters can be obtained from an independent source.

Table 4.5 Comparison of Standard Deviations on Parameter Estimates for Inputs
 (All Parameters Equally Important vs. $C_{n\beta}$, C_{n_r} , and $C_{n\delta_r}$ Primary)

Duration of Simulated Flight Test = 8 sec
 Total Energy in Input = $100 \text{ deg}^2 \text{ sec}$

Input	Standard Deviations					Tr(D)
	$C_{y\beta}$	$C_{l\beta}$	$C_{n\beta}$	C_{n_r}	$C_{n\delta_r}$	
All Parameters Equally Important (Figure 4.9)	.00880	.0204	.00277	.00880	.00860	.000648
$C_{n\beta}$, C_{n_r} and $C_{n\delta_r}$ Primary (Figure 4.12)	.00907	.0233	.00264	.00812	.00812	.000761
Parameter Value	- .119	-4.43	2.99	-.178	1.55	

[All in units of deg, sec]

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V. FREQUENCY DOMAIN SYNTHESIS OF OPTIMAL INPUTS

5.1 INTRODUCTION

The frequency domain inputs are designed on the assumption of steady state and are "optimal" only when the flight test duration is long as compared to the time constants of modes of interest. In aircraft applications, this may not be true. Nevertheless, frequency domain methods play an important role in the design of optimal inputs even for "short" experiments. The input spectrum, which specifies steady state optimal input, can be computed quickly. Subsequently, the time trace obtained by adding different frequency components with a random phase relationship is usually a good first pass at the optimal input time history for short experiments. It can be used as such if the flight test is not too short and is an excellent starting input for the time domain program.

A detailed study is made of the effect of short data length on the optimality of frequency domain inputs. Input sequences based on different experiment durations are generated from the spectrum and evaluated on the system, which starts from zero initial condition and does not reach steady state.

5.2 FREQUENCY DOMAIN OPTIMAL INPUTS [27]

Under the present effort, a computer program is written to design optimal inputs in frequency domain for single input systems. It is used to find optimal elevator and rudder input spectra to estimate parameters in longitudinal and lateral modes, respectively.

5.2.1 Longitudinal System

As a first example, an optimum elevator deflection spectrum is derived to identify parameters in the longitudinal short period mode of a C-8 aircraft. The state and measurement equations are given in Section 4.2. Two criteria of optimality are tried. They are: (a) $\min |D|$, and (b) $\min \text{Tr}(D)$. To use the computer

program, the discretization step is chosen as .02 sec and the number of points is 4000. Since there are five parameters and two outputs, $\frac{m}{2p}$ is two. Thus, the minimum number of frequencies for a nondegenerate design is two and the maximum number of frequencies required in an optimal design is $\frac{5 \times 6}{2}$, i.e., fifteen (Equation (3.42)).

To minimize the determinant of the dispersion matrix, the initial input is selected to have two frequencies at 0 cps and at .125 cps with equal power. Figure 5.1 shows the spectrum of the elevator deflection input after each iteration. Notice that during some iteration steps, the program puts more power at already chosen frequencies. After eight iterations, the change in the determinant of the dispersion matrix is less than .1% from the previous step. There is a total of seven frequencies in the final input spectrum.

This input has interesting characteristics. The spectrum is divided into two parts: A low frequency input to identify gains and a high frequency input to identify natural frequency and damping, etc. The higher frequency input occurs around the natural frequency, which is reasonable. Table 5.1 shows the standard deviations (lower bound) on parameter estimates for an average 12 sec long experiment when the system is in steady state and the total energy in u during this time is $200 \text{ deg}^2 \text{ sec}$. Also shown are the trace and determinant of the information matrix and the trace of the dispersion matrix. Next, the frequencies close to each other are lumped and ones with too little power are dropped to get the spectrum of Figure 5.2. The standard deviations on parameter estimates for this simplified design in steady state for the same experiment duration and input energy are also shown in Table 5.1. There has been an improvement in the determinant of M and trace of D , showing the value of the simplification.

The frequency domain inputs are designed with the assumption of steady state. If the flight testing time is short, the aircraft does not reach a steady state. To find the true information matrix for a 12 sec long test starting from zero initial conditions, a time domain input based on the frequency spectrum and same average power is generated and is shown in Figure 5.3. This time domain input is not unique since the initial phase relationship between the sinusoidal

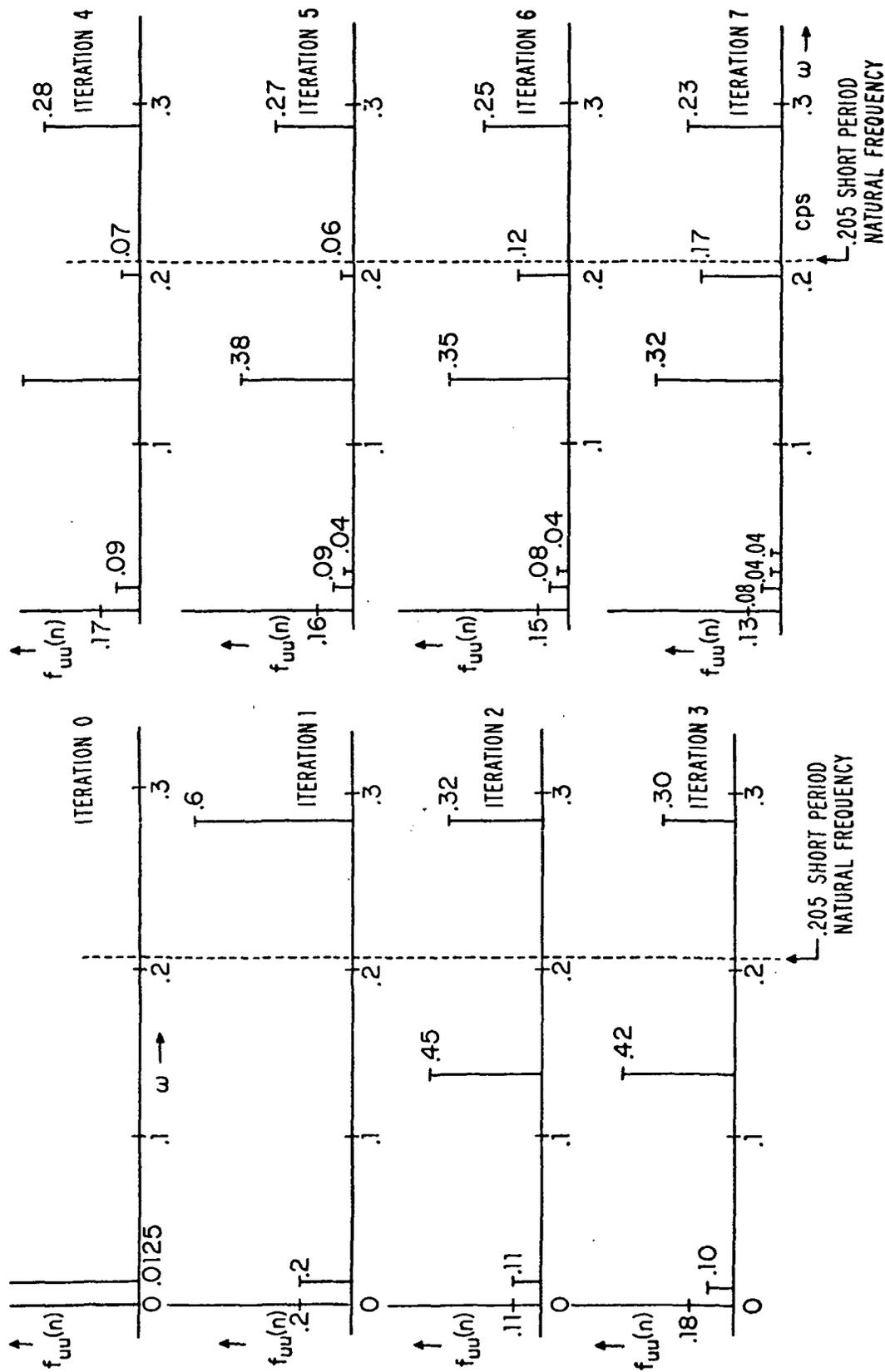


Figure 5.1.1 Steps in the Computation of Optimal Input ($\min |D|$) to Identify Parameters in the Short Period Mode of a C-8 Aircraft

Table 5.1 Errors in Parameter Estimates Using $|D|$ as the Optimality Criterion

Duration of Simulated Flight Test = 12 sec

Total Energy in $u = 200 \text{ deg}^2 \text{ sec}$

Input	Standard Deviations					Tr(M)	M	Tr(D)
	C_{m_q}	C_{m_α}	C_{Z_α}	$C_{m_{\delta_s}}$	$C_{Z_{\delta_s}}$			
Frequency Domain Input (Figure 5.1), Steady State Value	.0855	.0367	.0310	.0571	.0246	1.29×10^4	2.07×10^{15}	.0135
Simplified Freq. Domain Input (Figure 5.2), Steady State Value	.0850	.0371	.0310	.0566	.0246	1.28×10^4	2.08×10^{15}	.0134
Time Domain Input From Freq. Spectrum (Figure 5.3)	.0929	.0429	.0354	.0594	.0261	1.15×10^4	$.921 \times 10^{15}$.0159
Parameter Value	-1.588	-.562	-.737	-1.66	.005			

[All in units of deg, sec]

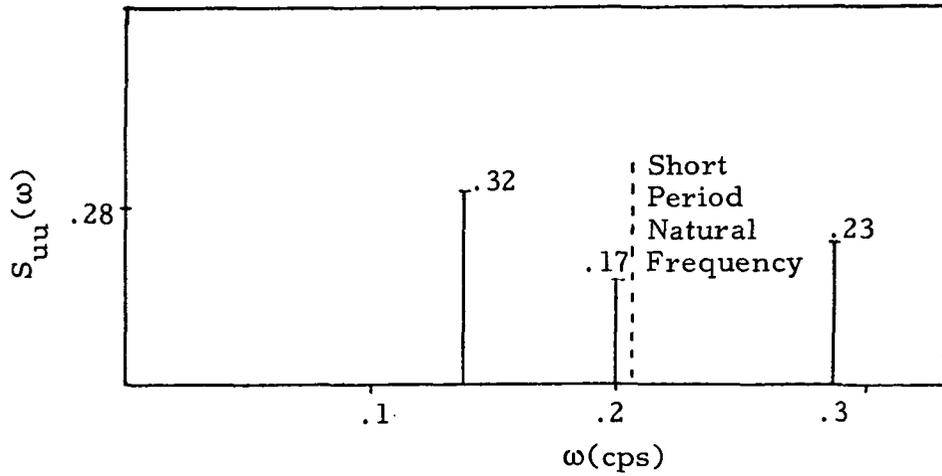


Figure 5.2 Simplified Input Spectrum to Minimize $|D|$

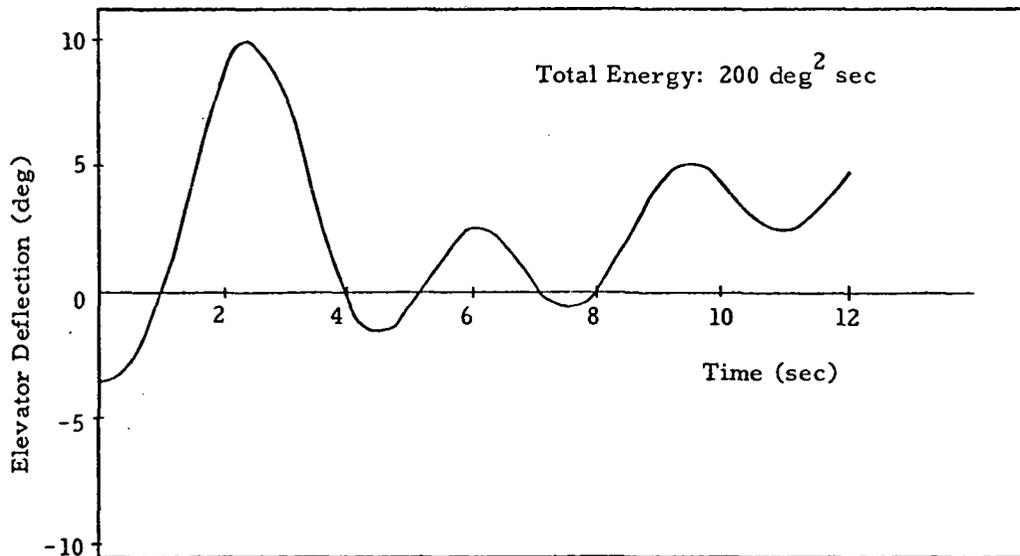


Figure 5.3 Elevator Deflection Time History Based on Spectrum of Figure 5.2 (min $|D|$)

waves is arbitrary. Table 5.1 gives the parameter standard deviations and trace and determinant of information and dispersion matrices when initial phases are chosen at random. The parameter standard deviations deteriorate by 5% to 15%. A better result could be obtained by optimizing the initial phases.

The basic and simplified elevator deflection spectra for the $\text{Tr}(D)$ criterion are shown in Figure 5.4. The frequencies in this input are in the same range as in the input for the $|D|$ criterion. Table 5.2 shows standard deviations in parameter estimates for an average 12 sec period in the steady state when the total input energy is $200 \text{ deg}^2 \text{ sec}$. The simplified design is a little poorer than the basic design. Some parameters have a higher standard deviation than in the $|D|$ case, while others have lower standard deviations. Again, a 12 sec long time domain input (shown in Figure 5.5) is generated and the standard deviations for this input with the system starting from zero initial condition are given in Table 5.2. There is only a moderate deterioration from steady state value.

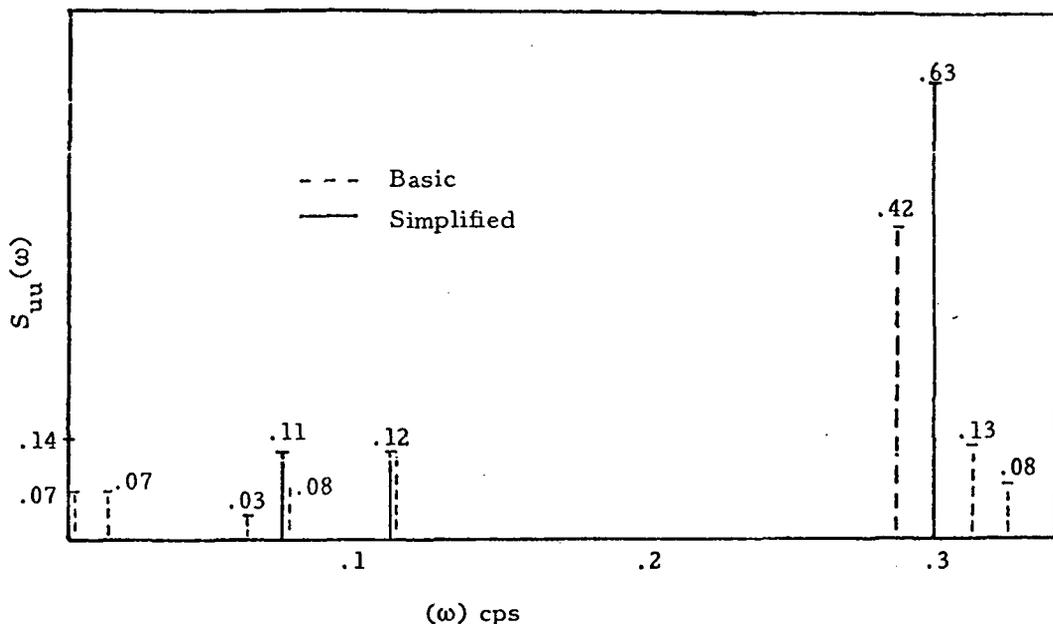


Figure 5.4 Basic and Simplified Elevator Input Spectra to Minimize $\text{Tr}(D)$

Table 5.2 Errors in Parameter Estimates Using Tr(D) as the Optimality Criterion

Duration of Simulated Flight Test = 12 sec
 Energy in $u = 200 \text{ deg}^2 \text{ sec}$

Input	Standard Deviations					Tr(M)	M	Tr(D)
	C_{m_q}	C_{m_α}	C_{Z_α}	$C_{m_{\delta_e}}$	$C_{Z_{\delta_e}}$			
Frequency Domain Input (Steady State Value)	.0709	.0422	.0354	.0437	.0259	$.942 \times 10^5$	1.12×10^{15}	.0106
Simplified Frequency Domain Input (Steady State Value)	.0715	.0412	.0346	.0446	.0257	$.989 \times 10^5$	1.31×10^{15}	.0107
Time Domain Input from Freq. Spectrum and Zero I.C.	.0774	.0416	.0346	.0482	.0257	1.01×10^5	1.12×10^{15}	.0119
Parameter Values	-1.588	-.562	-.737	-1.66	.005			

[All in units of deg, sec]

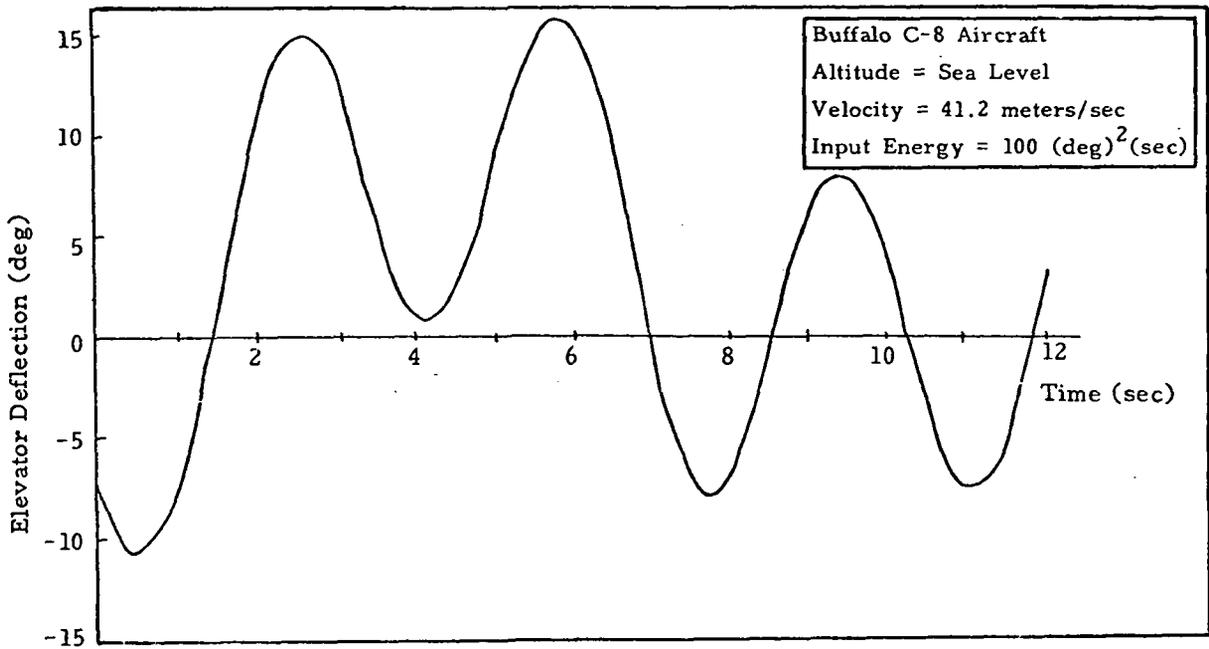


Figure 5.5 Elevator Deflection Sequence to Minimize Tr(D)

The root-mean-square (RMS) state deviations are computed for the frequency domain inputs. For an average input power of 16.67 deg², the RMS states for the inputs of Figures 5.2 and 5.4 are shown in Table 5.3. The values look reasonable.

Table 5.3 RMS State Deviations for Frequency Domain Inputs (Simulation)

Criterion	Pitch Rate	Angle-of-Attack
D	3.19 deg sec ⁻¹	2.95 deg
Tr(D)	3.05 deg sec ⁻¹	2.52 deg

5.2.2 Effect of Short Data Length on Performance of Frequency Domain Inputs

If the system starts from zero initial conditions, the performance of an input of finite length is poorer than predicted on the assumption of steady state. To determine the duration of the experiment when this approximation becomes serious, time traces of elevator deflection 4 to 12 sec long are obtained based on the simplified spectrum of the $\text{Tr}(D)$ criterion (Figure 5.4). Each of these inputs is used with the state and measurement equations and the resulting information and dispersion matrices are determined. Table 5.4 gives the ratio of standard deviations on parameter estimates for these finite inputs (with the system starting from zero initial conditions) to the standard deviations in steady state (for the same average input power and experiment duration). Trace and determinant of the information and dispersion matrices are also compared using simulated data. The asymptotic value of these ratios for long experiments is one. For experiments shorter than 8 sec, the deterioration is serious. The inputs are good for experiments longer than 10 sec. This corresponds to about two cycles of the natural short period mode.

5.2.3 Comparison With Conventional and Optimal Time Domain Inputs

The inputs, which minimize the sum of variances of five parameters in the short period mode of a C-8 aircraft for a 6 sec long experiment are given in Chapter IV. There, the conventional doublet input is also given.

Table 5.5 shows a comparison of standard deviations on parameter estimates for the frequency domain input, the optimal time domain input, and the doublet. The steady state frequency domain value is a lower limit on $\text{Tr}(D)$ for an input with $100 \text{ deg}^2 \text{ sec}$ input energy in a 6 sec long flight test. The time domain input is optimized for a 6 sec long experiment and gives a much better result than the time trace from the frequency domain input. Nevertheless, the input resulting from the frequency domain approach is superior to a doublet. As mentioned before, this would be an excellent first pass at the optimal input and is useful for starting the time domain program.

Table 5.5 Comparison of Time Domain and Frequency Domain Approach
(Tr(D) Criterion)

Duration of Simulated Flight Test = 6 sec
Total Input Energy = 100 deg² sec

Input	Standard Deviations					Tr(D)
	C^{m_q}	C^{m_α}	C^{Z_α}	$C^{m\delta_s}$	$C^{Z\delta_s}$	
Steady State Frequency Domain [cannot achieve; lower limit or Tr(D)]	.102	.0581	.0490	.0628	.0363	.0213
Time Trace from Frequency Spectrum	.140	.0819	.0704	.0801	.0409	.0394
Optimal Time Domain (Figure 4.2)	.113	.0676	.0561	.0672	.0400	.0264
Doublet (Figure 4.1)	.219	.362	.326	.0978	.0957	.304
Parameter Value	-1.588	-.562	-.737	-1.66	.005	

Figures 5.6 and 5.7 show the simulated pitch rate and angle-of-attack variation during the 6 sec long experiment for both inputs. The peak values are comparable.

5.2.4 Frequency Domain Lateral Inputs

The lateral system of Section 4.3 with unknown $C_{y\beta}$, $C_{\ell\beta}$, $C_{n\beta}$, C_{n_r} and $C_{n\delta_r}$ is used as the example to determine the optimal rudder input spectrum to minimize the trace of the dispersion matrix. The optimum input spectrum has two frequencies: at 0 cps with 12% of total input power and at .285 cps with 88% of input power. The second frequency is very close to the natural frequency of the Dutch-roll mode. Since the Dutch-roll mode for this aircraft has low damping, this input would produce large state excursions in steady state. This occurs because there are no constraints on the state variables.

Table 5.6 shows a comparison of standard deviations for this frequency domain input and the time domain input designed in Chapter IV for an 8 sec long simulated flight test. Because of low damping, the system is far from steady state for the duration of the experiment. Standard deviations on parameter estimates predicted on the assumption of steady state are too optimistic.

5.3 CONCLUSIONS

Though the frequency domain inputs are optimal only if the flight test is long, they are useful even for short experiments where steady state conditions cannot be reached. These inputs could be used as they are for flight tests which are longer than two cycle times of the mode of interest. Since frequency domain inputs can be obtained in a much shorter computation time, this technique may have advantages in real-time and on-line applications. These inputs are excellent for starting other more complicated algorithms, for example, the time domain input design.

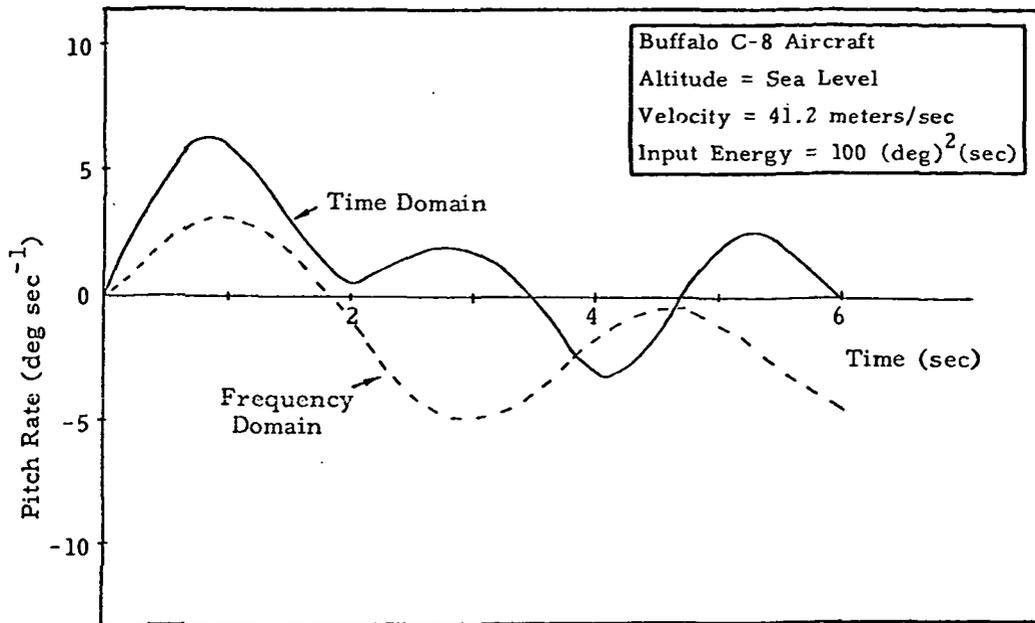


Figure 5.6 Pitch Rate Time Histories for Time Domain and Frequency Domain Optimal Inputs (Simulation)

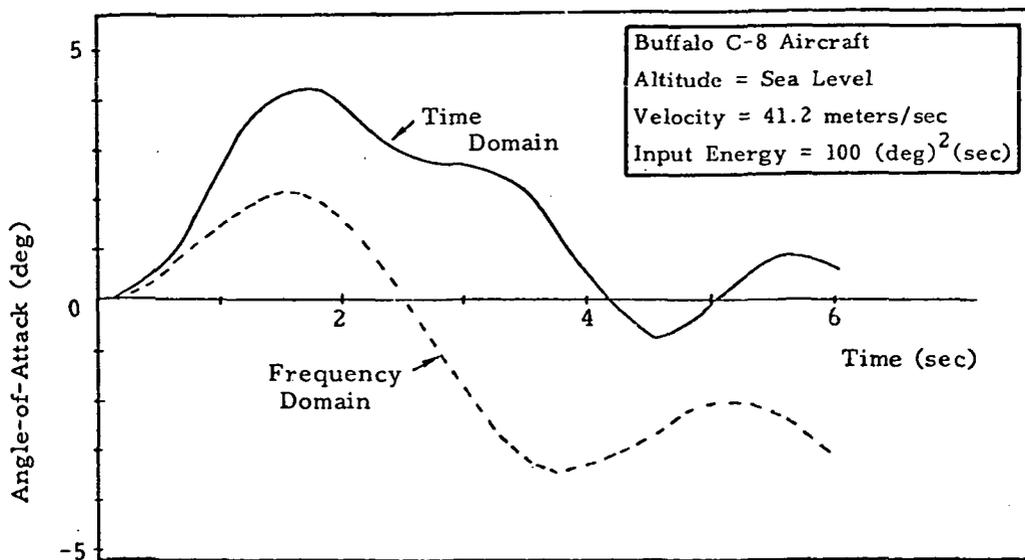


Figure 5.7 Angle-of-Attack Time Histories for Time Domain and Frequency Domain Optimal Inputs (Simulation)

Table 5.6 Comparison of Frequency Domain and Time Domain Optimal Lateral Inputs

Duration of Simulated Flight Test: 8 sec
 Total Input Energy: 100 deg² sec

Input	Standard Deviations						Tr(D)
	$C_{y\beta}$	$C_{\ell\beta}$	$C_{n\beta}$	C_{n_r}	$C_{n\delta_r}$		
Frequency Domain (Steady State)	.00119	.00281	.000140	.000948	.00666		.0000546*
Time Trace from Frequency Spectrum	.00988	.0242	.00366	.0103	.0102		.000904
Optimal Time Domain	.00880	.0204	.00277	.00880	.00860		.000648
Parameter Value	-.119	-4.43	2.99	-.178	1.55		

[All in units of deg, sec]

* Lower limit on Tr(D); cannot be achieved starting from zero initial conditions.

A small input energy near the natural frequency of a lightly damped mode can cause large state excursions. Since the input design places a constraint on the input energy, there would be a concentration of power near the natural frequency, because this would increase the signal-to-noise ratio. If this is undesirable, it can be avoided by putting state constraints or limiting the allowable frequencies in the input spectrum.

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VI. EVALUATION OF TIME AND FREQUENCY DOMAIN OPTIMAL INPUTS

6.1 BASIS OF EVALUATION

The optimal inputs of Chapter III are evaluated for performance sensitivity when design conditions are not satisfied. There are two possible sources of error. The control surface deflection may be different from the optimum because of the pilot's inability to follow the input exactly. Secondly, the input design may be based on incorrect parameter values (based on incorrect a priori estimates) or overly simplified models.

The optimal inputs are approximated by a series of four steps. A simulation of these step inputs is used on the system to evaluate parameter error covariances resulting from the measurements. This evaluates the degradation in performance of optimal inputs from errors in implementation.

Next, it is assumed that the parameters of the system are 50% off from the design values. The optimal inputs based on design conditions are used under off-design conditions. A comparison of these inputs with conventional inputs determines the loss in input efficiency resulting from inaccurate knowledge of parameters.

In Chapter IV, the optimal elevator inputs were computed based on the two state approximation of the short period mode in the longitudinal motions. Those inputs are used in this chapter on the full four state model to study the effects of model approximation on performance of the input resulting from it.

6.2 LONGITUDINAL SYSTEM

6.2.1 Approximation to Optimal Input

Figure 6.1 shows the optimal input of Chapter IV to identify five short period parameters of a C-8 Buffalo aircraft. This input is approximated by a

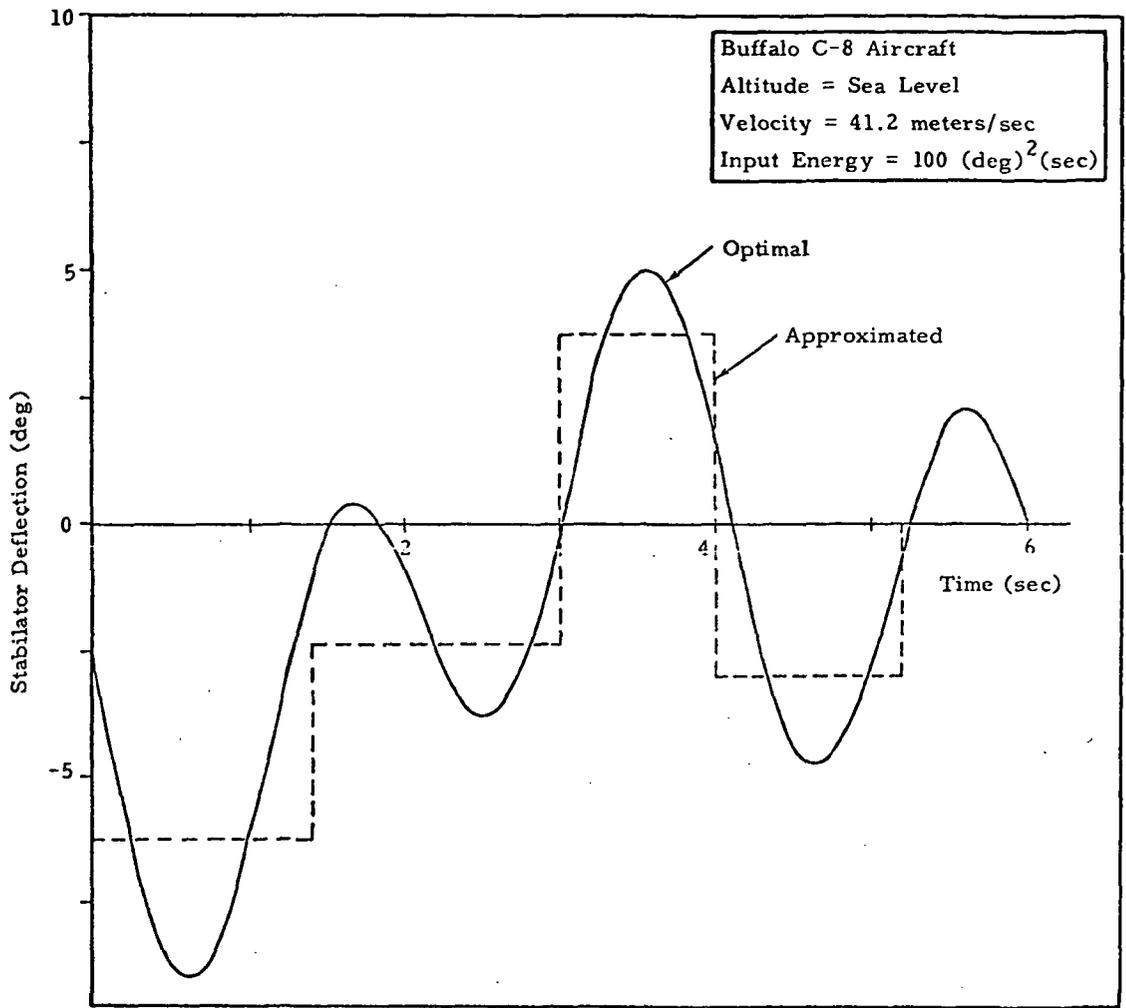


Figure 6.1 Optimal and Approximated Elevator Inputs

sequence of four steps with the same total energy. The approximated input is shown by the broken line in Figure 6.1. This input is used on the two state, short period approximation of the longitudinal equations of motion (Equation (4.2)). Table 6.1 compares parameter dispersions resulting from the optimal and the approximated (suboptimal) inputs. Some parameters have better estimates while others have poorer estimates.

Table 6.1 Comparison of Optimal and Approximated Inputs

Duration of Simulated Flight Test = 6 sec.

Total Input Energy = 100 deg² sec

	Standard Deviations					Tr(D)
	C_{m_q}	C_{m_α}	C_{Z_α}	$C_{m_{\delta_s}}$	$C_{Z_{\delta_s}}$	
Optimal Input (Solid line, Figure 6.1)	.113	.0676	.0561	.0672	.0400	.0264
Approximated Input (Broken line, Figure 6.1)	.126	.0590	.0497	.0807	.0381	.0299
Parameter Value	-1.588	-.562	-.737	-1.66	0.0	

[All in units of deg, sec.]

The pitch rate and angle-of-attack for the optimal and the approximated input are compared in Figure 6.2. The maximum values are about the same in the two cases.

6.2.2 Off-Design Parameter Values

All five parameters in the short period model are increased by 50% of their initial values. This results in a system with a natural frequency of 1.86 rad/sec and a damping ratio of .94. It is more difficult to identify parameters of this system with

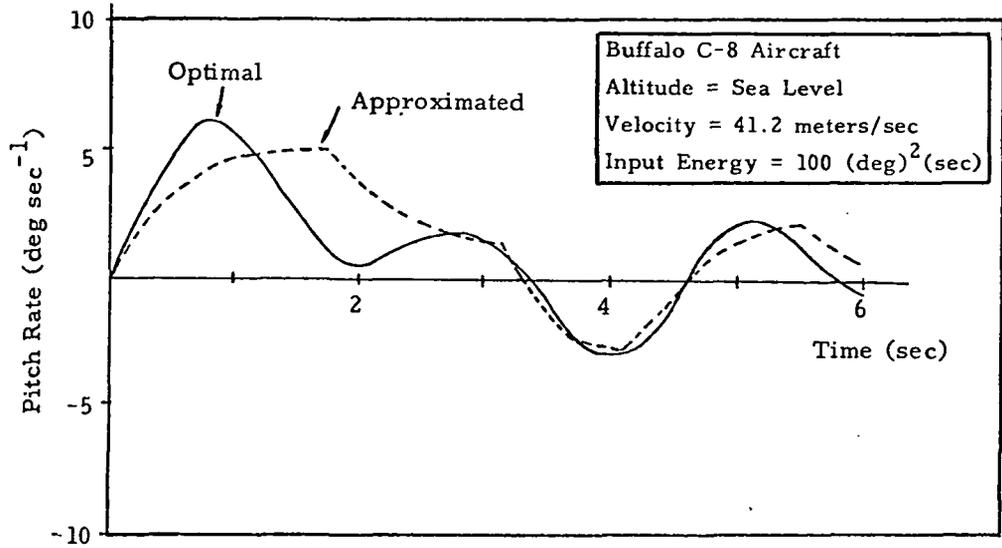


Figure 6.2a Comparison of Pitch Rates for Optimal and Approximated Inputs (Simulation)

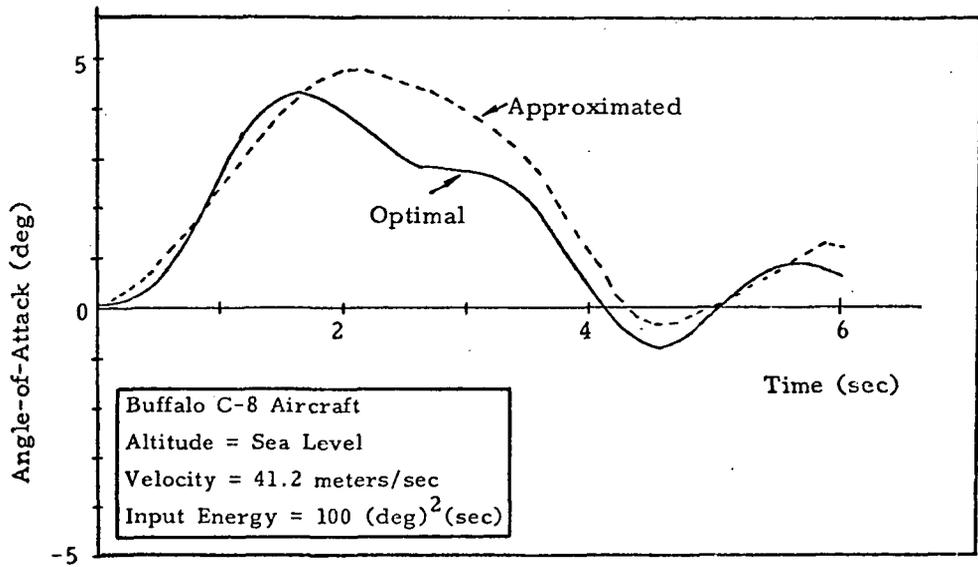


Figure 6.2b Angle-of-Attack for Optimal and Approximated Inputs (Simulation)

the same input energy because of increased damping. Table 6.2 shows standard deviations of parameter estimates when the approximated input is used on this system with off-design parameter values. There is a sharp increase in estimation errors from design conditions, partly because of the higher damping ratio. Table 6.2 also shows parameter error covariances with a doublet input. Next, the parameters of the system are halved, reducing the natural frequency to .76 rad/sec and the damping ratio to .77. The parameters in this system are easier to identify. The performances of the approximated input (broken line of Figure 6.1) and the doublet are given in Table 6.2. For the approximated input, the parameter error covariances are smaller than under design conditions. In both cases, the approximated input compares favorably with the conventional doublet input. Figure 6.3 compares the simulated pitch rate and angle-of-attack for the approximated input under design and off-design conditions (all parameters halved).

6.2.3 Fourth Order Model

The approximation to optimal inputs, obtained using the two state approximation, is simulated on the four state longitudinal equations of motion (Equation (4.1)). Again, there are measurements of only q and α . The measurement error and the sampling rate are the same as before. Table 6.3 is a comparison of the standard deviations on estimates of C_{m_q} , C_{m_α} , $C_{m_{\delta_s}}$, C_{Z_α} , and $C_{Z_{\delta_s}}$ on the assumption that the remaining parameters in the system are known. The estimates predicted by the four state model are better than the estimates predicted by the two state model. This is because there is an additional excitation of the short period mode from variations in forward speed.

Simulated time histories of the deviations in forward speed, pitch rate, and angle-of-attack are shown in Figure 6.4. They are compared to the output of the two state model. There is an insignificant difference, except in forward speed which is assumed to remain constant in the two state short period approximation of the longitudinal equations of motion.

Table 6.2 Approximated Input Under Design and Off-Design Conditions

Length of Simulation: 6 sec
 Total Input Energy: $100 \text{ deg}^2 \text{ sec}$

Parameters	Natural Frequency	Damping Ratio	Input	Standard Deviations					Tr(D)
				C^m_q	C^m_α	$C^m_{Z_\alpha}$	$C^m_{\delta_s}$	$C^m_{Z_\delta_s}$	
Nominal	1.33	.87	Approximated	.126	.0590	.0497	.0807	.0387	.0299
			Doublet	.219	.362	.326	.0978	.0957	.304
All Parameters 50% Higher	1.86	.94	Approximated	.220	.134	.0845	.143	.0508	.0968
			Doublet	.243	.517	.385	.117	.0949	.498
All Parameters 50% Lower	.76	.77	Approximated	.0656	.0200	.0269	.0370	.0269	.00753
			Doublet	.224	.213	.287	.0719	.0866	.191

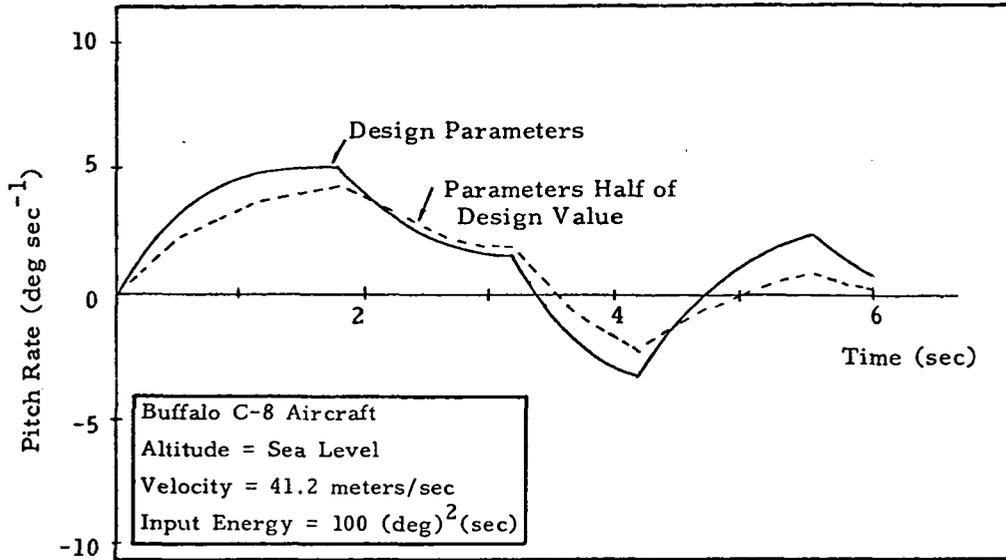


Figure 6.3a Pitch Rate Variation for Systems Under Design and Off-Design Conditions (Approximated Inputs) Based on Simulation

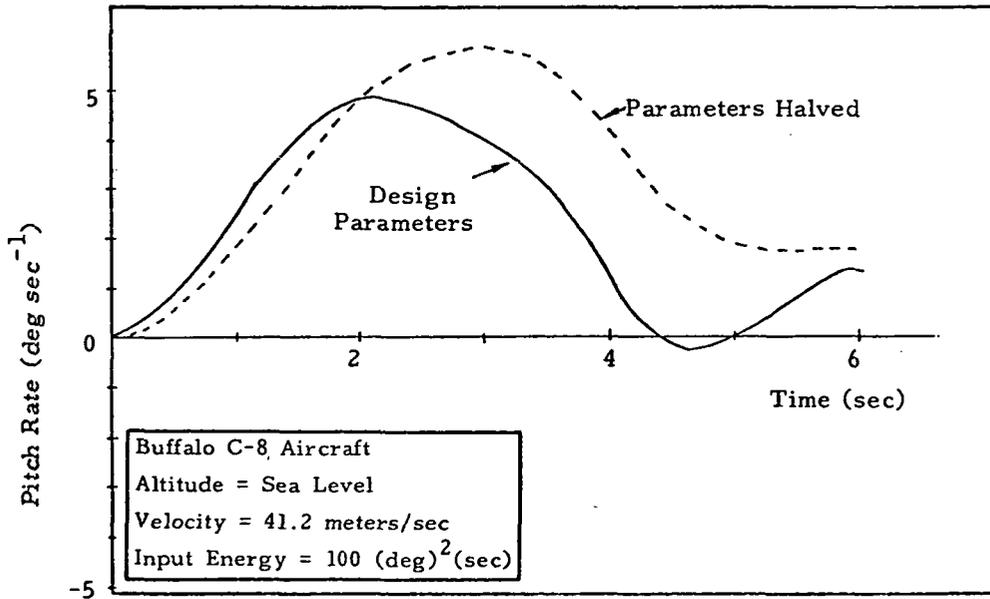


Figure 6.3b Angle-of-Attack Variation for Systems Under Design and Off-Design Conditions (Approximated Input) Based on Simulation

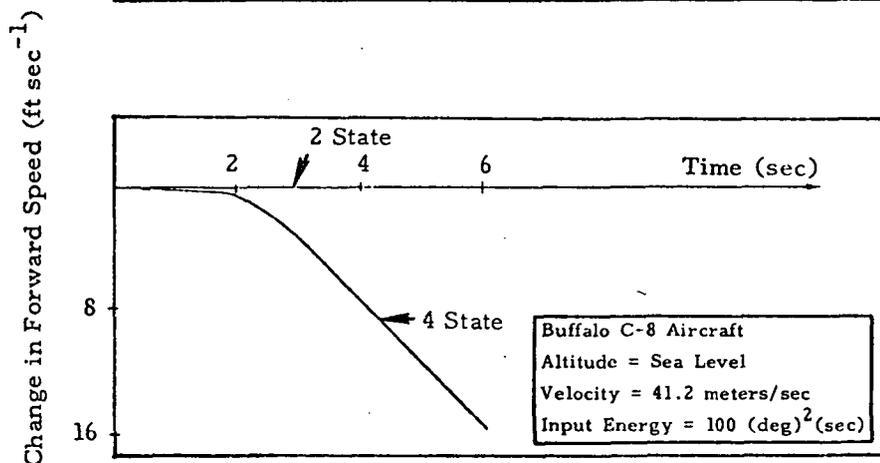
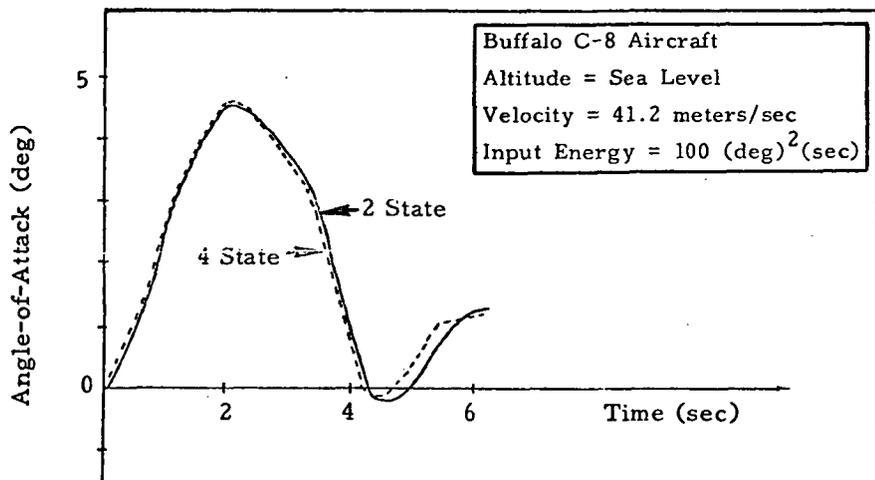
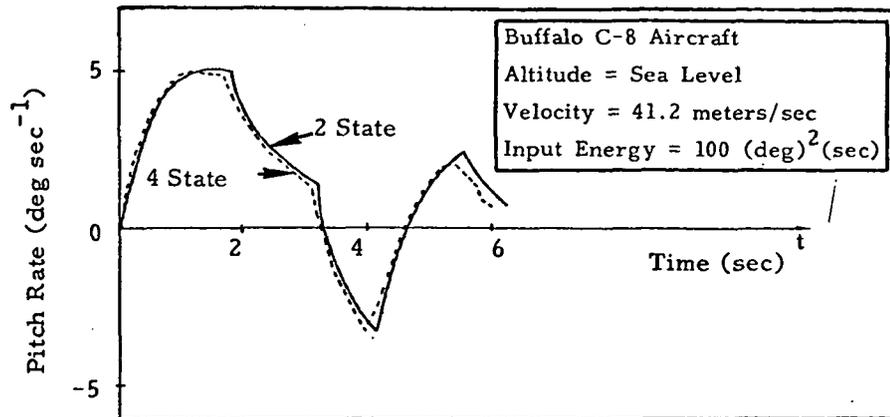


Figure 6.4 Comparison of State Excursions Predicted by Two State and Four State Models (Approximated Input) Based on Simulation

Table 6.3 Comparison of Standard Deviations on Parameter Estimates Predicted by Two State and Four State Models

Length of Simulation = 6 sec
 Total Input Energy = 100 deg² sec

Model	Standard Deviations					Tr(D)
	C_{m_q}	C_{m_α}	C_{Z_α}	$C_{m_{\delta_s}}$	$C_{Z_{\delta_s}}$	
Four State [Equation (4.1)]	.115	.0560	.0499	.0752	.0378	.0260
Two State [Equation (4.2)]	.113	.0676	.0561	.0672	.0400	.0264
Parameter Value	-1.588	-.562	-.737	-1.66	0.0	

[All in units of deg, sec]

6.3 LATERAL MODES

The lateral system equations and the optimal input to obtain good estimates of C_{y_β} , C_{l_β} , C_{n_β} , C_{n_r} , and $C_{n_{\delta_r}}$ are presented in Section 4.2. As for the lateral system, this input is approximated by a series of four steps and is shown in the broken line in Figure 6.5. Table 6.4 compares the parameter error dispersions for the optimal and the approximated inputs. There is less than a 20% increase in the trace of the dispersion matrix (this corresponds to an average increase of 10% in standard deviation on parameter estimates).

The parameters are next increased by 50%. This increases the damping and natural frequency of the Dutch-roll mode significantly. Table 6.5 compares the standard deviations on estimates with optimal input and doublet under off-design conditions. Though the standard deviation on parameter estimates rises considerably, the optimal input is still much better than a doublet. It is important to note that a part of the degradation in estimation accuracy is due to increased

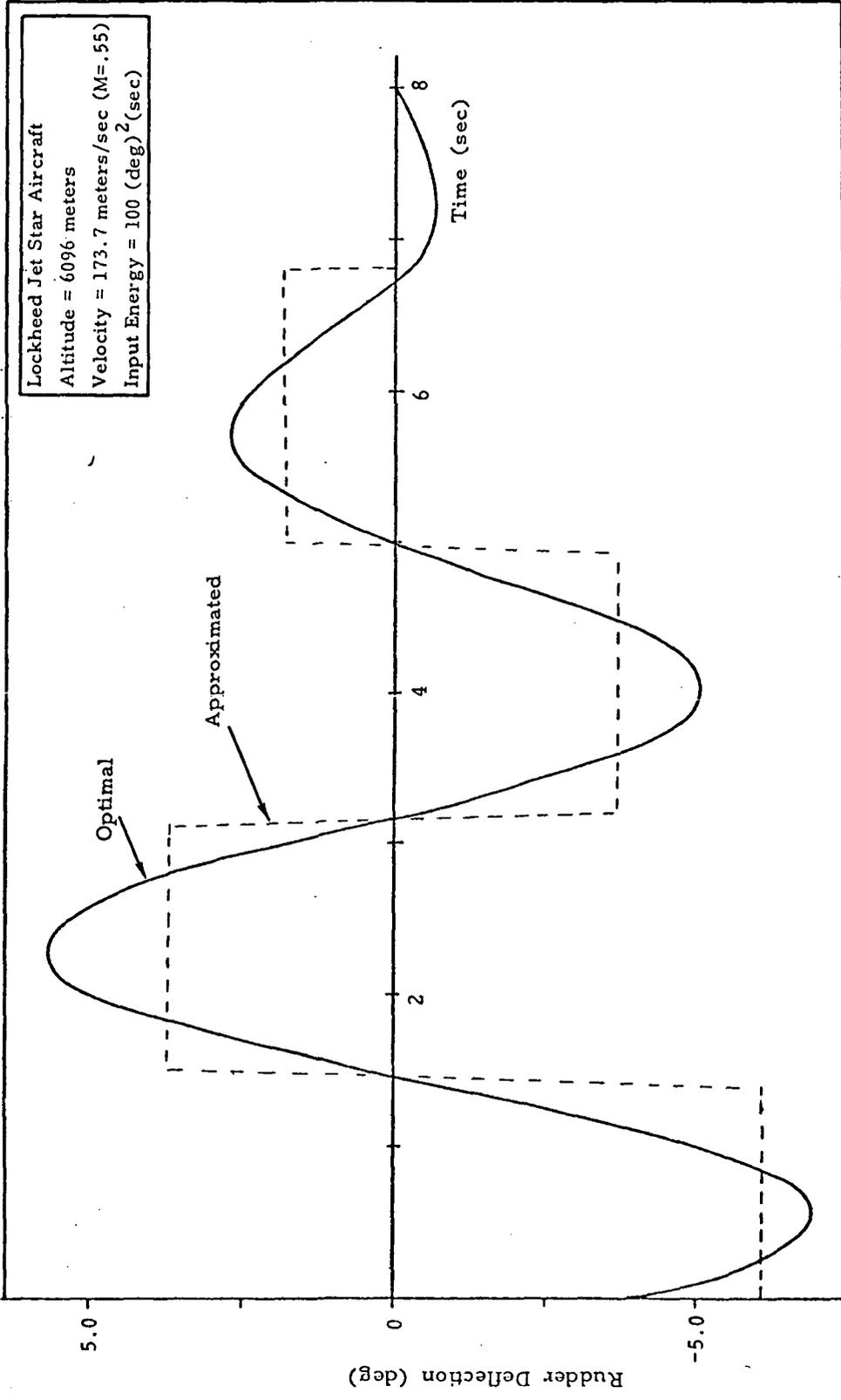


Figure 6.5 Optimal and Approximated Rudder Input to Identify Five Lateral Parameters

Table 6.4 Comparison of Optimal and Approximated Rudder Inputs

Duration of Simulated Flight Test = 8 sec

Total Input Energy = 100 deg² sec

Input	Standard Deviations					Tr(D)
	$C_{y\beta}$	$C_{\ell\beta}$	$C_{n\beta}$	C_{n_r}	$C_{n_{\delta_r}}$	
Optimal	.00880	.0204	.00277	.00880	.00860	.000648
Approximated	.00933	.0221	.00310	.00977	.00908	.000764
Parameter Value	-.119	-4.43	2.99	-.178	1.55	

[All in units of deg, sec]

damping ratio under off-design conditions which make parameters less identifiable from a given input energy.

6.4 CONCLUSIONS

Large errors in the implementation of optimal inputs result in small deterioration in parameter estimation accuracy. In particular, it is possible to approximate the optimal inputs by a series of steps at a small cost in increased standard deviations on parameter estimates. This is true both of a system with low damping and of a system with high damping in the mode of interest.

Changes in parameter values from design conditions investigated here changed the damping and natural frequency of the mode of interest considerably. For the same input energy, the parameter of the resulting system can be estimated with either higher or lower accuracy, depending mainly on the increase or decrease in damping and to some extent on changes in natural frequency. The optimal input is sensitive to off-design parameters; however, with a 50% change in all

Table 6.5 Comparison of Lateral Rudder Inputs Under Design and Off-Design Conditions

Duration of Simulated Flight Test = 8 sec
 Total Input Energy = $100 \text{ deg}^2 \text{ sec}$

Parameters	Input	Standard Deviations					Tr(D)
		$C_{y\beta}$	$C_{l\beta}$	$C_{n\beta}$	C_{nr}	$C_{n\delta_r}$	
Design	Optimal (Fig. 4.8)	.00880	.0204	.00277	.00880	.00860	.000648
	Doublet	.0238	.0576	.00635	.0240	.0207	.00492
Off-Design (All Parameters 50% Up)	Optimal (Fig. 4.8)	.0133	.0452	.00582	.0143	.0115	.00260
	Doublet	.0280	.0844	.00794	.0277	.0203	.00916
Parameter Value		-.119	-4.43	2.99	-.178	1.55	

[All in units of deg, sec]

parameters from their design values, the optimal inputs are at least as good as conventional doublet inputs. The deterioration in performance of the optimal input is more severe when the damping is low with a priori estimates of parameter values but is higher for true parameter values.

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VII. SUMMARY AND CONCLUSIONS

A method for designing inputs for linear time-invariant systems is developed for application to the identification of aircraft stability and control coefficients.

The method consists of two complementary techniques:

1. The time domain technique can be used for any linear dynamic system under various conditions and all durations of flight test time. It is general, but computationally complicated.
2. The frequency domain technique gives an input spectrum, the time traces of which are good inputs for parameter identification for flight tests which have long allowable data lengths compared to the time constants of the mode of interest. This method could also be used to determine a set of specific frequencies where the flight test may be carried out, obviating the need for lengthy in-flight frequency sweeps.

Both of these techniques can be used for input designs when test constraints allow for relatively long time aircraft response to such inputs. The time domain method gives the best results and should be used whenever possible in spite of its computational complexity, because computer time is usually a less important constraint than flight test time. The frequency domain method has application in obtaining fairly good inputs quickly for long experiments, and in finding starting inputs for the time domain program.

As the limitation on allowable aircraft response time becomes stricter, the time domain method is the desirable approach. For example, a test at an extreme flight condition (e.g., high angle-of-attack, sonic transition, etc.) may preclude extensive times for response data acquisition. At such points, the time domain capabilities are best used to generate shorter inputs which supply maximum parameter estimate accuracy. It is also noted that considerations of cost and/or fuel restrictions may be achieved by using efficient inputs requiring less data acquisition time.

The improved applicability of these optimal input design techniques to flight test is achieved because of the resolution of several computational problems which have previously hindered such application. The solution to these problems has been attained by the following accomplishments:

- (a) Design of inputs based on a wider class of optimizing criteria which address more directly the requirements for which the parameters are identified. The criteria are based on the variances of parameter estimates and include determinant and trace of the dispersion matrix. Other criteria may also be used (cf., Appendix A and Section 3.4.2).
- (b) Increase in the number of parameters to be identified by design of inputs for a larger number of parameters than previously computationally feasible. This benefit is obtained by implementation of two new algorithms. These are:
 - (i) Sensitivity Function Reduction: The technique of sensitivity function reduction lowers the order of the computational problem which must be solved in the time domain input design method. For example, in a system with 4 states, 2 inputs, 4 measurements, and 15 unknown parameters, the number of differential equations is reduced from 64 to a maximum of 12. Since the computation time varies as the cube of the order of the problem, this sensitivity reduction method would reduce computation time by a factor of more than 100 (Appendix D and Section 3.4.3). By using sensitivity functions reduction, it has become numerically feasible to compute inputs for high order systems with many unknown parameters.
 - (ii) Eigenvalue-Eigenvector Decomposition Method to Solve the Two Point Boundary Value Problem: Properties of the transition matrix occurring in the time domain method are used to develop an eigenvalue-eigenvector decomposition solution

technique. This has been found to be quite efficient and general (Appendix C and Section 3.4.4).

- (c) Frequency domain input design specification is desired specifically for tests where sinusoidal input generators are available. By making the steady state assumption, the input design procedure can be simplified considerably. In this method, the frequency domain representation of a system is used to determine the optimal input spectrum. This technique is useful for designing long inputs (Appendix B and Section 3.5) and obtaining first estimates of inputs for short experiments.
- (d) Specially designed inputs for primary/secondary derivative consideration allow flexibility in tailoring inputs to identify specific important parameters. Alternate formulations where a varying amount of confidence in "known" parameters exists can be easily incorporated into the methods (cf., Section 3.6.1).

The evaluation and verification of these benefits have been performed by the design of optimal longitudinal and lateral inputs and subsequent testing on simulated data. These evaluations have led to the following conclusions:

- (a) Inputs based on the dispersion matrix criteria $\text{Tr}(D)$ and $|D|$ give lower parameter error covariances than inputs based on the information matrix criterion $\text{Tr}(M)$ (Section 4.2.4). For example, a longitudinal input based on $\text{Tr}(D)$ gave C_{m_q} and $C_{m\delta_s}$ estimate error deviations which were approximately one-third of that for the input given by the $\text{Tr}(M)$ criterion (cf., Table 4.3). This conclusion is significant because it confirms the advantages which were anticipated by reformulating the problem to be more applicable to flight test requirements. This reformulation involved using the dispersion matrix directly, a previously intractable computational procedure for a large number of parameters.

- (b) Optimal inputs give lower parameter error covariances than conventional control doublets (for the same input energy). The doublet was chosen as the representative input because of its wide use as a basic flight test maneuver. In all cases, the optimal input gave derivative estimate accuracies almost 50% or better than the doublet (c.f., Table 4.1 for longitudinal and Table 6.5 for lateral examples). The greatest improvement in accuracy through optimal inputs is obtained in derivatives affecting highly damped modes, and this is one singular benefit of such optimal inputs (cf., Section 4.2 and 4.3). Even under off-design conditions, where the assumed derivatives for calculating the optimal input were as much as 50% off from the "true" values, the doublet gave higher error covariances (cf., Sections 6.2 and 6.3).
- (c) Optimal inputs may often be approximated by a series of steps without significant increase in derivative estimation errors (Sections 6.2 and 6.3). For example, the increase in the sum of parameter error covariances was less than 15% when the optimal input was approximated by a series of four steps for both longitudinal and lateral simulated responses.

Several useful guidelines have been established for use of the time and frequency domain input design techniques. Such guidelines are useful for reducing the computation time required for designing optimal inputs. These include:

- (a) Based on the computed results, it is suggested that frequency domain inputs be designed when the experiment duration is longer than two cycle times of the mode of interest (Section 5.2.2).
- (b) There are several practical approximations possible in the computation of optimal inputs. These simplifications reduce design time with only a slight loss in accuracy.

- (i) Inputs can be designed based on simplified models. A two state representation of the short period mode is adequate in the design of inputs to identify the five parameters which affect this mode (Section 6.2.3).

- (ii) Sensitivity equations can, sometimes, be reduced below the minimum number required to obtain the exact value of all sensitivity functions. This approximation was made for the lateral system in Section 4.3 and compared to the exact case in Table 4.6.

- (c) Direct state constraints in the design may be desired, particularly in weakly damped modes to avoid excessive aircraft responses (Section 4.3). As long as the pilot can control the aircraft, however, such motions will not limit the use of the inputs studied in this report.

These results are preliminary, but do indicate that the approaches developed are suitable for flight test applications.

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APPENDIX A
INPUT DESIGN IN TIME DOMAIN

A.1 PROBLEM STATEMENT

Consider a continuous time varying system

$$\begin{aligned} \dot{x} &= F(t, \theta)x + G(t, \theta)u \\ x(0) &= 0 \quad 0 \leq t \leq T \end{aligned} \tag{A.1}$$

where

x is a $n \times 1$ state vector

u is a $q \times 1$ control vector, and

$F(t, \theta)$ and $G(t, \theta)$ are $n \times n$ and $n \times q$ matrices, which depend on m unknown parameter θ .

There are noisy measurements of some linear combinations of state

$$y = H(t, \theta)x + v \quad \underline{\Delta} \quad z + v \tag{A.2}$$

y is a $p \times 1$ output vector and v is $p \times 1$ white noise vector with zero mean and known covariance $R(t)$.

The problem is to choose u from a class of second order processes independent of v to obtain good estimates of parameters from the measurements of u and y . The class of inputs is such that

$$\int_0^T u^T(t) u(t) dt = 1 \tag{A.3}$$

A.2 INFORMATION MATRIX

The information matrix for parameters θ is

$$M = \int_0^T \left(\frac{\partial z}{\partial \theta} \right)^T R^{-1}(t) \frac{\partial z}{\partial \theta} dt \quad (\text{A.4})$$

(A.1) can be solved for state vector x ,

$$x(t) = \int_0^t \phi(t, \tau) G(\tau, \theta) u(\tau) d\tau \quad (\text{A.5})$$

where $\phi(t, \tau)$ is the transition matrix and follows the differential equation

$$\frac{\partial \phi(t, \tau)}{\partial t} = F(t, \theta) \phi(t, \tau) \quad (\text{A.6})$$

$$\phi(\tau, \tau) = I$$

Vector z can be written as

$$\begin{aligned} z(t) &= \int_0^t H(t, \theta) \phi(t, \tau) G(\tau, \theta) u(\tau) d\tau \\ &= \sum_{i=1}^q \int_0^t H(t, \theta) \phi(t, \tau) G_i(\tau, \theta) u_i(\tau) d\tau \end{aligned} \quad (\text{A.7})$$

Therefore,

$$\frac{\partial z(t)}{\partial \theta} = \sum_{i=1}^q \int_0^t \frac{\partial}{\partial \theta} \{ H(t, \theta) \phi(t, \tau) G_i(\tau, \theta) \} u_i(\tau) d\tau$$

$$\underline{\Delta} = \sum_{i=1}^q \int_0^t A_i(t, \tau) u_i(\tau) d\tau \quad (A.8)$$

Using (A.7) in (A.4),

$$\begin{aligned} M &= \int_0^T \sum_{i=1}^q \int_0^t A_i^T(t, \tau) u_i(\tau) d\tau R^{-1}(t) \\ &\quad \sum_{j=1}^q \int_0^t A_j(t, s) u_j(s) ds dt \\ &= \int_0^T \int_0^T \sum_{i=1}^q \sum_{j=1}^q \int_{\sup(\tau, s)}^T \left\{ A_i^T(t, \tau) R^{-1}(t) A_j(t, s) dt \right\} u_i(\tau) u_j(s) d\tau ds \end{aligned}$$

where $\sup(\tau, s)$ is the higher of τ and s . This can be written as

$$M = \int_0^T \int_0^T \sum_{i,j=1}^q M_{ij}(\tau, s) u_i(\tau) u_j(s) d\tau ds \quad (A.9)$$

where

$$\begin{aligned} M_{ij}(\tau, s) &= 1/2 \int_{\sup(\tau, s)}^T \left\{ A_i^T(t, \tau) R^{-1}(t) A_j(t, s) \right. \\ &\quad \left. + A_j^T(t, \tau) R^{-1}(t) A_i(t, s) \right\} dt \end{aligned} \quad (A.10)$$

It is clear that

$$M_{ij}(\tau, s) = M_{ji}(\tau, s) = M_{ij}^T(s, \tau) \quad (A.11)$$

The next sections show how one can work with the three criteria of optimality outlined in Chapter III.

A.3 MAXIMIZATION OF A LINEAR FUNCTION OF THE INFORMATION MATRIX

From (A.9), a linear functional \mathcal{L} (see Section 3.3 for definition of \mathcal{L}) of the information matrix is

$$\begin{aligned} J_1 \triangleq \mathcal{L}(M) &= \int_0^T \int_0^T \sum_{i,j=1}^q \mathcal{L}(M_{ij}(\tau, s)) u_i(\tau) u_j(s) d\tau ds \\ &= \int_0^T \int_0^T u^T(\tau) P(\tau, s) u(s) d\tau ds \end{aligned} \quad (\text{A.12})$$

where

$$P_{ij}(\tau, s) = \mathcal{L}(M_{ij}(\tau, s)) \quad (\text{A.13})$$

$P(\tau, s)$ is a symmetric positive definite matrix. To maximize a linear function of the information matrix for the given class of inputs, it is necessary to maximize (A.12) under the constraint (A.3). It is straightforward to show that this is achieved by solving the following eigenvalue problem:

$$\begin{aligned} \int_0^T P(\tau, s) u(s) ds &= \lambda u(\tau) \\ 0 \leq \tau \leq T \end{aligned} \quad (\text{A.14})$$

The maximum eigenvalue λ is the maximum value of the trace of the information matrix and the corresponding eigenvector is the desired optimal input. Mehra^[22] shows this eigenvalue problem can be recast into a linear two-point boundary value problem. The solution technique to this problem is given in Chapter III.

A.4 DETERMINANT OF THE DISPERSION MATRIX

In this and the next sections, we prove some important theorems. These lead to a computation algorithm presented in Section 3.4.3.

Theorem A.1: A necessary condition that an input $u^*(t)$ minimizes $|D^*|$ is that

- (i) u^* is an eigenvector of P^* with eigenvalue m .
- (ii) Any other eigenvalue, λ_k with eigenvector u^k , of P^* follows the inequality

$$\lambda_k \leq m + 2 \operatorname{Tr}\{(M^{*-1} M^{*k})^2\} \quad (\text{A.15})$$

where

$$P_{ij}^*(\tau, s) = \operatorname{Tr}(M^{*-1} M_{ij}(\tau, s)) \quad (\text{A.16})$$

$$M^{*k} = \int_0^T \int_0^T \sum_{i,j=1}^q M_{ij}(\tau, s) u_i^*(\tau) u_j^k(s) d\tau ds \quad (\text{A.17})$$

and

$$\int_0^T P^*(\tau, s) u^k(s) ds = \lambda_k u^k(\tau) \quad (\text{A.18})$$

Proof: (a) From (A.11)

$$M^* = \int_0^T \int_0^T \sum_{i,j=1}^q M_{ij}(\tau, s) u_i^*(\tau) u_j^*(s) d\tau ds$$

$$\text{Tr}(I) = \int_0^T \int_0^T \sum_{i,j=1}^q \text{Tr}(M^{*-1} M_{ij}(\tau, s)) u_i^*(\tau) u_j^*(s) d\tau ds$$

$$m = \int_0^T \int_0^T u^{*T}(\tau) P^*(\tau, s) u^*(s) d\tau ds \quad (\text{A.19})$$

Therefore, $\lambda_{\max} \geq m$ (A.20)

Equality holds if $u^*(t)$ is an eigenvector of P^* corresponding to its maximum eigenvalue m .

- (b) Consider an input $u^*(t)$ which minimizes $|D|$. Since logarithm is a monotonic function, $u^*(t)$ must also minimize $\log |D|$. Let λ_k be an eigenvalue of P^* and $u^k(t)$ the corresponding eigenvector. Consider an input u ,

$$u(t) = \alpha u^*(t) + \beta u^k(t) \quad (\text{A.21})$$

The energy constraint (A.3) requires

$$\alpha^2 + \beta^2 + 2\alpha\beta\gamma = 1 \quad (\text{A.22})$$

where

$$\gamma = \int_0^T u^{*T}(t) u^k(t) dt \quad (\text{A.23})$$

giving

$$\frac{\partial \alpha}{\partial \beta} = - \frac{(\beta + \alpha\gamma)}{(\alpha + \beta\gamma)} \quad (\text{A.24})$$

The information matrix for input $u(t)$ is

$$M = \alpha^2 M^* + \beta^2 M^k + 2\alpha\beta M^{*k} \quad (\text{A.25})$$

where

$$M^k = \int_0^T \int_0^T \sum_{i,j=1}^q M_{ij}(\tau, s) u_i^k(\tau) u_j^k(s) d\tau ds \quad (\text{A.26})$$

$$\begin{aligned} \left. \frac{\partial \log |D|}{\partial \beta} \right|_{\beta=0} &= -\text{Tr}\{M^{*-1}(-2\gamma M^* + 2M^{*k})\} \\ &= 2\gamma(m - \lambda_k) \end{aligned} \quad (\text{A.27})$$

Since u^* minimizes $|D|$ and γ can be positive or negative, for all eigenvalues of P^*

$$\lambda_k = m \quad \text{or} \quad \gamma = 0 \quad (\text{A.28})$$

If $\gamma \neq 0$ for one particular eigenvector of P^* (except possibly u^*), it cannot be zero for any other eigenvector of P^* , since the eigenvectors of a positive self adjoint kernel are orthogonal. Then all eigenvalues of P^* must equal m . In the light of Equation (A.19), this implies that u^* is also an eigenvector of P^* with eigenvalue m . Then $\gamma = 0$ for all other eigenvectors.

If $\gamma = 0$, then u^* is an eigenvector and the corresponding eigenvalue is m . Thus, we show that u^* is an eigenvector of P^* with eigenvalue m .

Since u^* minimizes $|D|$

$$\begin{aligned}
\left. \frac{\partial^2 \log |D|}{\partial \beta^2} \right|_{\beta=0} &= -\text{Tr}\{-M^{*-1}(M^{*k} M^{*-1} M^{*k} + 2M^* - 2M^k)\} \\
&= 4\text{Tr}\{(M^{*-1} M^{*k})^2\} + 2m - 2\lambda_k \\
&\geq 0
\end{aligned} \tag{A.29}$$

or

$$\lambda_k \leq m + 2 \text{Tr}\{(M^{*-1} M^{*k})^2\} \tag{A.30}$$

Theorem A.2: A sufficient condition that an input $u^*(t)$ is a stationary point for $|D|$ is that u^* be an eigenvector of P^* with eigenvalue m .

Proof: Assume u^* is an eigenvector of P^* . Consider any input $u^m(t)$ satisfying the energy constraint and define

$$u(t) = \alpha u^*(t) + \beta u^m(t) \tag{A.31}$$

Since the eigenvectors of P^* form a complete set and are orthogonal,

$$u^m(t) = \sum_{i=1}^{\infty} \alpha_i u^i(t) \tag{A.32}$$

$$\sum_{i=1}^{\infty} \alpha_i^2 = 1 \tag{A.33}$$

Using (A.27),

$$\begin{aligned}
\left. \frac{\partial \log |D|}{\partial \beta} \right|_{\beta=0} &= -\text{Tr}\{M^{*-1}(-2\alpha^* M^* + \sum_{i=1}^{\infty} \alpha_i M^{*i})\} \\
&= 2\alpha^* m - 2\alpha^* m = 0
\end{aligned} \tag{A.34}$$

where α^* is the component of u^* in u^m . Since u^* is a stationary point of $\log |D|$ and logarithm is a monotonic function, it must be a stationary point for $|D|$.

A.5 LINEAR FUNCTIONAL OF THE DISPERSION MATRIX

Theorem A.3: A necessary condition that an input $u^*(t)$ minimizes $\mathcal{L}(D)$ is that

- (i) $u^*(t)$ is an eigenvector of Q^* with eigenvalue $\mathcal{L}(D^*)$.
- (ii) Any other eigenvalue λ_k of Q^* follows the inequality

$$\lambda_k \leq \mathcal{L}\{M^{*-1}(1 + (M^{*-1}M^{*k})^2)\} \quad (\text{A.35})$$

and

$$Q_{ij}^*(\tau, s) = \mathcal{L}(M^{*-1}M_{ij}(\tau, s)) \quad (\text{A.36})$$

Proof: It is similar to proof of Theorem A.1.

Theorem A.4: A sufficient condition that an input $u^*(t)$ is a stationary point for $\mathcal{L}(D)$ is that u^* be an eigenvector of Q^* with eigenvalue $\mathcal{L}(D^*)$.

Proof: Again the proof follows along the same lines as in Theorem A.2.

A.6 EXTENSIONS

We state one more theorem without proof.

Theorem A.5: A necessary and sufficient condition than an input $u^*(t)$ is a stationary point for $\mathcal{L}(D^l)$ is that $u^*(t)$ be an eigenvector of Q_l^* with eigenvalue $\mathcal{L}(D^{*l})$, where

$$(Q_{\ell}^*(\tau, s))_{ij} = \mathcal{L}(M^{*-\ell} M_{ij})(\tau, s) \quad (\text{A.37})$$

This is an important theorem because by choosing the linear operator as trace and a high value of ℓ , the maximum eigenvalue of D can be minimized. In other words, the maximum error in any direction in parameter space is minimum.

APPENDIX B
INPUT DESIGN IN FREQUENCY DOMAIN

B.1 PROBLEM STATEMENT

Consider the state space representation of a discrete time system

$$x(k+1) = \Phi x(k) + Gu(k) \quad k=1,2,\dots,N \quad (B.1)$$

and the noisy measurements

$$y(k) = Hx(k) + v(k) \quad (B.2)$$

where

$x(\cdot)$ is $n \times 1$ state vector

$u(\cdot)$ is $q \times 1$ control vector

$y(\cdot)$ and $v(\cdot)$ are $p \times 1$ measurement and noise vectors, respectively.

Φ , G , and H are matrices of appropriate dimensions and contain m unknown parameters θ . It is assumed that:

- (a) The system is stable and θ identifiable.
- (b) v is a second order process with known covariance.

Mehra^[22] developed a technique for determining the spectrum of input u which minimizes the trace or determinant of the dispersion matrix of parameter estimates. A summary is presented here.

B.2 FREQUENCY DOMAIN APPROACH

The approach consists of transforming the frequency domain representation of the system into a regression problem. The results of Kiefer and Wolfowitz^[11] and Fedorov,^[14] in the theory of optimal experiments, are subsequently applied to develop schemes for designing optimal inputs.

Fourier transform (1) and (2) to get:

$$e^{-j\omega_0 n} \tilde{x}(n) = \Phi \tilde{x}(n) + \tilde{G}u(n) \quad (\text{B.3})$$

and

$$\tilde{y}(n) = H\tilde{x}(n) + \tilde{v}(n) \quad (\text{B.4})$$

where

$$n = -\frac{N}{2}, -\frac{N}{2} + 1, \dots, \frac{N}{2} - 1$$

$$\omega_0 = 2\pi/N$$

and \sim over a variable denotes its finite Fourier transform.

From (B.3) and (B.4)

$$\begin{aligned} \tilde{y}(n) &= H(e^{-j\omega_0 n} - \Phi)^{-1} \tilde{G}u(n) + \tilde{v}(n) \\ &\stackrel{\Delta}{=} T(n, \theta) \tilde{u}(n) + \tilde{v}(n) \\ &= \left\{ T(n, \theta_0) + \frac{\partial T}{\partial \theta} (\theta - \theta_0) + o(\Delta\theta^2) \right\} \tilde{u}(n) + \tilde{v}(n) \end{aligned} \quad (\text{B.5})$$

where θ_0 is the a priori estimate of θ and u is a scalar. Thus,

$$\begin{aligned}\Delta \tilde{y}(n) &\triangleq \tilde{y}(n) - T(n, \theta_0) \tilde{u}(n) \\ &= \tilde{u}(n) \frac{\partial T}{\partial \theta} (\theta - \theta_0) + \tilde{v}(n)\end{aligned}\quad (\text{B.6})$$

From (B.6) we can estimate $\Delta \theta$ using generalized least-squares

$$\Delta \hat{\theta} = \left[\frac{1}{N} \operatorname{Re} \begin{array}{c} \Sigma \\ -N/2 \end{array} \frac{\partial T^*}{\partial \theta} S_{vv}^{-1}(n) \frac{\partial T}{\partial \theta} S_{uu}(n) \right]^{-1} \frac{1}{N} \operatorname{Re} \begin{array}{c} \Sigma \\ -N/2 \end{array} \frac{\partial T^*}{\partial \theta} S_{vv}^{-1}(n) u^*(n) \Delta \tilde{y}(n) \quad (\text{B.7})$$

and $\operatorname{cov}(\Delta \hat{\theta}) = M^{-1}$ where M is the information matrix given by

$$M = \operatorname{Re} \begin{array}{c} \Sigma \\ -N/2 \end{array} \frac{\partial T^*}{\partial \theta} S_{vv}^{-1}(n) \frac{\partial T}{\partial \theta} S_{uu}(n) \quad (\text{B.8})$$

$S_{uu}(n)$ and $S_{vv}(n)$ are the spectra of u and v and $*$ denotes transpose and complex conjugate.

The average information matrix per sample is

$$M_N = \frac{M}{N} = \frac{1}{N} \operatorname{Re} \begin{array}{c} \Sigma \\ -N/2 \end{array} \frac{\partial T^*}{\partial \theta} S_{vv}^{-1}(n) \frac{\partial T}{\partial \theta} S_{uu}(n) \quad (\text{B.9})$$

As the number of sample points increases, the information matrix approaches infinity. However, the average information matrix per sample reaches a constant value.

We now take limits of equation (B.9) as $N \rightarrow \infty$. It is straightforward to show that $\lim_{N \rightarrow \infty} M_N \rightarrow M_\infty$, where

$$M_{\infty} = \frac{1}{2\pi} \operatorname{Re} \int_{-\pi}^{\pi} \frac{\partial T}{\partial \theta}^* S_{vv}^{-1}(\omega) \frac{\partial T}{\partial \theta} dF_{uu}(\omega) \quad (\text{B.10})$$

where F_{uu} is the spectral distribution function of u . In the input design procedure the total power in u will be constrained and, without loss of generality, we may fix it at unity. Such an input will be called normalized input and the corresponding information matrix is the normalized information matrix. We will only consider normalized inputs in the following development.

The normalized information matrix has many important properties. Some of them are given in Theorem 1 (see [2] for proof).

Theorem 1:

- (a) M is a symmetric positive semi-definite matrix.
- (b) If the spectrum of u contains fewer than $\frac{m}{2p}$ frequencies, the matrix M is singular and not all parameters can be identified.
- (c) For any normalized input with mixed spectrum $f_1(\omega)$, it is always possible to find another normalized input $f_2(\omega)$ which has the same information matrix and has discrete spectrum with no more than $\frac{m(m+1)}{2} + 1$ frequencies. In addition, if $f(\omega)$ is an optimizing normalized input, the number of frequencies cannot exceed $\frac{m(m+1)}{2}$.

This theorem has important implications. Thus, the optimal input is a sum of a finite number of sinusoidal waves. If k_q is the number of independent frequencies in the optimal design,

$$\frac{m}{2p} \leq k_q \leq \frac{m(m+1)}{2} \quad (\text{B.11})$$

There are two other theorems which are important for input design.

Theorem 2: The following are equivalent:

(a) The normalized design f^* maximizes the determinant of the information matrix (or minimizes the determinant of the dispersion matrix).

(b) The normalized design f^* minimizes

$$\max_{\omega \in [-\pi, \pi]} \psi(\omega, f) \quad (\text{B.12})$$

(c) $\max_{\omega \in [-\pi, \pi]} \psi(\omega, f^*) = m$ (B.13)

$$\text{where } \psi(\omega, f) = \text{Re Tr} \left(S_{\mathbf{v}\mathbf{v}}^{-1}(\omega) \frac{\partial \mathbf{T}}{\partial \theta} M^{-1} \frac{\partial \mathbf{T}^*}{\partial \theta} \right) \quad (\text{B.14})$$

Theorem 3: The following are equivalent:

(a) The normalized design f minimizes $\mathcal{L}(D(f^*))$

(b) The normalized design f^* minimizes

$$\max_{\omega \in [-\pi, \pi]} \psi(\omega, f)$$

(c) $\max_{\omega \in [-\pi, \pi]} \psi(\omega, f^*) = \mathcal{L}(D(f^*))$ (B.15)

$$\text{where } \psi(\omega, f) = \mathcal{L} \left[D(f) \frac{\partial \mathbf{T}^*(\omega)}{\partial \theta} S_{\mathbf{v}\mathbf{v}}^{-1}(\omega) \frac{\partial \mathbf{T}(\omega)}{\partial \theta} D(f) \right] \quad (\text{B.16})$$

and \mathcal{L} is a linear operator (see Equation (3.9) for definition of \mathcal{L}).

All input spectra satisfying (a)-(c) of theorem 2 and their linear combinations give the same information matrix. The same holds of Theorem 3.

The above theorems are very important because they convert complex non-linear problems into simpler ones. Instead of minimizing $|D|$ or $\mathcal{L}(D)$, it is only necessary to minimize the $\max_{\omega \in [-\pi, \pi]} \psi(\omega, f)$. This leads to a very powerful computation technique given in Section 3.5.

Example 1: Consider a second order system in which we wish to estimate the frequency $\theta^{1/2}$

$$\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -\theta & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (\text{B.17})$$

$$y = x_1 + v \quad (\text{B.18})$$

Discrete time equivalent to this system is (for small Δ)

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & \Delta \\ -\Delta\theta & -\Delta \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ \Delta \end{bmatrix} u(k) \quad (\text{B.19})$$

$$z(k) = x_1(k) + v(k) \quad (\text{B.20})$$

We assume that $v(k)$ is white with known power spectral density S_{vv} .

$$\begin{aligned} T(\omega, \theta) &= [1 \quad 0] \begin{bmatrix} e^{-j\omega} - 1 & 0 \\ \Delta\theta & e^{-j\omega} + \Delta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \Delta \end{bmatrix} \\ &= \frac{\Delta^2}{(e^{-j\omega} - 1)(e^{-j\omega} - 1 + \Delta) + \Delta^2\theta} \end{aligned} \quad (\text{B.21})$$

$$\frac{\partial T^*}{\partial \theta} = - \frac{\Delta^4}{\{(e^{j\omega} - 1)(e^{j\omega} - 1 + \Delta) + \Delta^2\theta\}^2} \quad (\text{B.22})$$

$$M(\omega) = + \frac{\Delta^8 S_{vv}^{-1}}{\{(e^{j\omega} - 1)(e^{j\omega} - 1 + \Delta) + \Delta^2\theta\}^2 \{(e^{-j\omega} - 1)(e^{-j\omega} - 1 + \Delta) + \Delta^2\theta\}^2} \quad (\text{B.23})$$

In this case, the maximum number of frequencies is one and can be determined by maximizing $M(\omega)$ for $\omega \in [-\pi, \pi]$. It can be shown that for small Δ , M is maximized at

$$\begin{aligned} \omega^* &= \sqrt{\theta - \frac{1}{2}} & \theta > \frac{1}{2} \\ &= 0 & \frac{1}{2} \geq \theta \geq \frac{1}{4} \end{aligned} \quad (\text{B.24})$$

θ is not smaller than $\frac{1}{4}$ from the assumption of a stable system. For high damping, it is best to use a constant input. With low damping, an oscillatory input at the damped natural frequency is the best.

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APPENDIX C
COMPUTATION OF TIME DOMAIN INPUTS USING
EIGENVALUE-EIGENVECTOR DECOMPOSITION

In the time domain input design problem, a weighted trace of information matrix has to be maximized in each iteration, whichever criteria of Section 3.3 we may be using. This leads to a two point boundary value problem of the type (see Equation (3.10)):

$$\frac{d}{dt} \begin{bmatrix} \mathbf{x}_\theta \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\theta & -\mu \mathbf{G}_\theta \mathbf{G}_\theta^T \\ \mathbf{H}_\theta^T \mathbf{R}_\theta^{-1} \mathbf{H}_\theta & -\mathbf{F}_\theta^T \end{bmatrix} \begin{bmatrix} \mathbf{x}_\theta \\ \lambda \end{bmatrix}$$

$$\underline{\Delta} \quad \mathcal{H} \begin{bmatrix} \mathbf{x}_\theta \\ \lambda \end{bmatrix}$$

$$\mathbf{x}_\theta(0) = 0 \quad \lambda(T) = 0 \tag{C.1}$$

The smallest value of μ to give a nontrivial solution to the above equation is to be determined. It has been shown by Bryson and Hall^[28] that the Hamiltonian matrix \mathcal{H} is symplectic. Therefore, the eigenvalues of \mathcal{H} occur in pairs $+\lambda$ and $-\lambda$. Let \mathcal{P}_+ and $-\mathcal{P}_+$ be the positive and negative sets of eigenvalues of \mathcal{H} . Then the corresponding eigenvector matrix can be written in the partitioned form as

$$\mathbf{S} = \begin{bmatrix} \mathbf{X}_+ & \mathbf{X}_- \\ \mathbf{\Lambda}_+ & \mathbf{\Lambda}_- \end{bmatrix} \tag{C.2}$$

The eigenvectors are normalized such that

$$\mathbf{\Lambda}_-^T \mathbf{X}_+ - \mathbf{X}_-^T \mathbf{\Lambda}_+ = \mathbf{I} \tag{C.3}$$

It can be shown that (see Bryson and Hall)

$$S^{-1} = \begin{bmatrix} \Lambda_-^T & -X_-^T \\ -\Lambda_+^T & X_+^T \end{bmatrix} \quad (C.4)$$

Premultiply both sides of Equation (C.1) by S^{-1} :

$$\frac{d}{dt} S^{-1} \begin{bmatrix} x_\theta \\ -\lambda \end{bmatrix} = S^{-1} \mathcal{H} S^{-1} \begin{bmatrix} x_\theta \\ -\lambda \end{bmatrix} \quad (C.5)$$

or

$$\frac{d}{dt} \begin{bmatrix} \Lambda_-^T x_\theta - X_-^T \lambda \\ \hline -\Lambda_+^T x_\theta + X_+^T \lambda \end{bmatrix} \begin{bmatrix} \delta_+ & 0 \\ \hline 0 & -\delta_+ \end{bmatrix} \begin{bmatrix} \Lambda_-^T x_\theta - X_-^T \lambda \\ \hline \Lambda_+^T x_\theta + X_+^T \lambda \end{bmatrix} \quad (C.6)$$

Using the boundary conditions and simplifying,

$$\begin{bmatrix} X_-^T & \hline \hline e^{-\delta_+ T} X_+^T & \hline \hline \Lambda_-^T & \hline \hline \Lambda_+^T \end{bmatrix} \begin{bmatrix} \lambda(0) \\ \hline \hline x_\theta(T) \end{bmatrix} = 0 \quad (C.7)$$

For a nontrivial solution,

$$\begin{bmatrix} X_-^T & e^{-s_+ T} \Lambda_-^T \\ \hline \hline e^{-s_+ T} X_+^T & \Lambda_+^T \end{bmatrix} \text{ is singular}$$

$$\Lambda_+^T - e^{-s_+ T} X_+^T X_-^T e^{-s_+ T} \Lambda_-^T \text{ is singular}$$

(C.8)

i.e.,
$$U = \Lambda_+^{-1} \Lambda_- e^{-s_+ T} X_-^{-1} X_+ e^{-s_+ T}$$

has at least one eigenvalue equal to one.

Suppose for a certain μ_0 and T_0 , U has eigenvalues $\lambda^{(1)}$ close to one. Let the corresponding normalized left and right eigenvectors of U be $y_L^{(1)}$ and $y_R^{(1)}$. Then

$$U y_R^{(1)} = \lambda^{(1)} y_R^{(1)} \quad (C.9)$$

Differentiating with respect to T ,

$$\frac{\partial U}{\partial T} y_R^{(1)} + U \frac{\partial y_R^{(1)}}{\partial T} = \frac{\partial \lambda^{(1)}}{\partial T} y_R^{(1)} + \lambda^{(1)} \frac{\partial y_R^{(1)}}{\partial T} \quad (C.10)$$

Premultiply by $y_L^{(1)T}$ and simplify to get

$$\frac{\partial \lambda^{(1)}}{\partial T} = \left(y_L^{(1)T} \frac{\partial U}{\partial T} y_R^{(1)} \right) / \left(y_L^{(1)T} y_R^{(1)} \right)$$

Similarly,

$$\frac{\partial \lambda^{(1)}}{\partial \mu} = \left(y_L^{(1)T} \frac{\partial U}{\partial \mu} y_R^{(1)} \right) / \left(y_L^{(1)T} y_R^{(1)} \right) \quad (C.11)$$

$$\begin{aligned} \frac{\partial U}{\partial T} &= -\Lambda_+^{-1} \Lambda_-^{-1} e^{-s_+ T} X_-^{-1} X_+^{-1} e^{-s_+ T} - \Lambda_+^{-1} \Lambda_-^{-1} e^{-s_+ T} X_-^{-1} X_+^{-1} e^{-s_+ T} \\ \frac{\partial U}{\partial \mu} &= \frac{\partial(\Lambda_+^{-1} \Lambda_-^{-1})}{\partial \mu} e^{-s_+ T} X_-^{-1} X_+^{-1} e^{-s_+ T} - \Lambda_+^{-1} \Lambda_-^{-1} e^{-s_+ T} \frac{\partial s_+}{\partial \mu} X_-^{-1} X_+^{-1} e^{-s_+ T} \\ &\quad + \Lambda_+^{-1} \Lambda_-^{-1} e^{-s_+ T} \frac{\partial X_-^{-1} X_+^{-1}}{\partial \mu} e^{-s_+ T} - \Lambda_+^{-1} \Lambda_-^{-1} e^{-s_+ T} X_-^{-1} X_+^{-1} e^{-s_+ T} \frac{\partial s_+}{\partial \mu} \cdot T \end{aligned} \quad (C.12)$$

Since \mathcal{P}_+ are the first half eigenvalues of \mathcal{H} ,

$$\begin{aligned}
\frac{\partial \phi_{+i}}{\partial \mu} &= \begin{pmatrix} \Lambda_{-i}^T & -x_{-i}^T \\ 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \mu} \begin{pmatrix} x_{+i} \\ \Lambda_{+i} \end{pmatrix} \\
&= \begin{pmatrix} \Lambda_{-i}^T & -x_{-i}^T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -G_\theta G_\theta^T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{+i} \\ \Lambda_{+i} \end{pmatrix} \\
&= -\Lambda_{-i}^T G_\theta G_\theta^T \Lambda_{+i}
\end{aligned}$$

or
$$\frac{\partial \phi_+}{\partial \mu} = \text{diag} [-\Lambda_{-}^T G_\theta G_\theta^T \Lambda_{+}] \quad (\text{C.13})$$

It can also be shown that

$$\frac{\partial}{\partial \mu} \begin{bmatrix} X_+ & X_- \\ \Lambda_+ & \Lambda_- \end{bmatrix} = \begin{bmatrix} X_+ & X_- \\ \Lambda_+ & \Lambda_- \end{bmatrix} \Psi \quad (\text{C.14})$$

where

$$\begin{aligned}
\psi_{ji} &= -\Lambda_{-j}^T G_\theta G_\theta^T \Lambda_{+i} / (\lambda_j - \lambda_i) \quad 1 \leq j \leq n\theta, 1 \leq i \leq n\theta \\
&= \Lambda_{+(j-n\theta)}^T G_\theta G_\theta^T \Lambda_{+i} / (\lambda_j - \lambda_i) \quad n\theta < j \leq 2n\theta, 1 \leq i \leq n\theta \\
&= -\Lambda_{-j}^T G_\theta G_\theta^T \Lambda_{-(j-n\theta)} / (\lambda_j - \lambda_i) \quad 1 \leq j \leq n\theta, n\theta < i \leq 2n\theta \\
&= -\Lambda_{-(j-n\theta)}^T G_\theta G_\theta^T \Lambda_{-(i-n\theta)} / (\lambda_j - \lambda_i) \\
&\quad n\theta < j \leq 2n\theta, n\theta < i \leq 2n\theta
\end{aligned} \quad (\text{C.15})$$

Equation (C.14) can be used to compute $\frac{\partial(\Lambda_+^{-1}\Lambda_-)}{\partial\mu}$ and $\frac{\partial(X_-^{-1}X_+)}{\partial\mu}$.

To find the change in μ for a small change ΔT in T , we use the relation

$$\frac{\partial\lambda^{(1)}}{\partial T} \Delta T + \frac{\partial\lambda^{(1)}}{\partial\mu} \Delta\mu = 0 \quad (C.16)$$

If the desired eigenvalue is close to one, but not exactly equal to one, the approximate change in μ required to bring this eigenvalue closer to one is given by

$$\frac{\partial\lambda^{(1)}}{\partial\mu} \Delta\mu = 1 - \lambda^{(1)} \quad (C.17)$$

For a given μ , there may be several values of T for which the matrix U has at least one eigenvalue equal to unity. We are interested in the smallest T for which this is true. If we start with a correct μ_0, T_0 pair, this iteration technique will always give the correct μ, T pair for small changes, ΔT . Thus, if the desired T is "far" from T_0 , μ is updated in steps, each step involving a small change in T . In each step (C.17) is used to bring $\lambda^{(1)}$ closer to one. Once the optimum μ is found, $\lambda(0)$ can be determined. Then the state and control time histories and information matrix are evaluated.

This technique gives an excellent insight into the structure of the optimal inputs. By studying the eigenvalues and eigenvectors, we can determine if they consist of basically damped or undamped oscillatory functions of rising and falling exponentials. The time constants of these oscillatory functions and exponentials can usually be correlated with model time constants. This approach is excellent for obtaining good approximations to optimal inputs and building input generators. Instead of specifying the input as a function of time, it is possible to give its functional representation. The sensitivity of the nature of optimal inputs to the length of the experiment can also be studied.

Example

Consider the following first order system with an unknown parameter θ , with a priori value one.

$$\dot{x} = -x + \theta u \quad x(0) = 0 \quad (\text{C.18})$$

There is a continuous measurement of the scalar state x :

$$\dot{y} = x + v \quad (\text{C.19})$$

Let

$$E(v(t)) = 0, \quad E(v(t)v(\tau)) = \delta(t - \tau)$$

$$F_{\theta} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$G_{\theta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$H_{\theta} = [0 \quad 1]$$

$$R_{\theta} = [1]$$

(C.20)

The controllability matrix of (F_{θ}, G_{θ}) is

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

The system is uncontrollable. The controllable part can be represented by

$$\begin{aligned}
 F_{\theta} &= [-1] \\
 G_{\theta} &= [1] \\
 H_{\theta} &= [1]
 \end{aligned}
 \tag{C.21}$$

The Hamiltonian is

$$\mathcal{H} = \begin{pmatrix} -1 & -\mu \\ 1 & 1 \end{pmatrix}
 \tag{C.22}$$

with eigenvalues $\pm \sqrt{\mu - 1} j$ and the corresponding normalized eigenvector matrix

$$S = \begin{bmatrix} -1 + \sqrt{\mu - 1} j & -.5 + \frac{.5j}{\sqrt{\mu - 1}} \\ 1 & \frac{-.5j}{\sqrt{\mu - 1}} \end{bmatrix}
 \tag{C.23}$$

$$\Lambda_+^{-1} \Lambda_- = \frac{-.5j}{\sqrt{\mu - 1}}
 \tag{C.24}$$

$$X_-^{-1} X_+ = \frac{2(\mu - 1)}{\mu} \left(2 + \frac{2 - \mu}{\sqrt{\mu - 1}} j \right)
 \tag{C.25}$$

$$U = e^{j \tan^{-1} \left(\frac{2\sqrt{\mu - 1}}{2 - \mu} \right)} e^{-2j \sqrt{\mu - 1} T}
 \tag{C.26}$$

One eigenvalue of U is one if

$$\frac{-2\sqrt{\mu - 1}}{2 - \mu} = \tan 2\sqrt{\mu - 1} T$$

or
$$\tan \sqrt{\mu - 1} T = -\sqrt{\mu - 1}$$

(C.27)

Equation (C.27) has many solutions. The smallest T corresponds to $\sqrt{\mu - 1} T$ falling in the second quadrant. For example, if $T = 2$, $\mu \approx 2.30$.

APPENDIX D

PRACTICAL TECHNIQUES FOR SENSITIVITY FUNCTIONS REDUCTION IN LINEAR TIME-INVARIANT SYSTEMS

D.1 INTRODUCTION

The problem of computing state sensitivities using reduced order models has become very important in parameter estimation involving high order models and many unknown parameters. These techniques allow a considerable saving in computation time which makes the determination of optimal inputs feasible for practical systems. Most efforts to date [29-31] have concentrated on finding bounds on the order of the model which can generate state sensitivities for all system parameters. Very little attention has been given to the formulation of practical techniques leading to these lowest order models. Formulations by Wilkie and Perkins, [29] Denery, [30] and Neuman and Sood [31] lead to fairly complicated transformations and are not capable of exploiting the characteristics of the system in most cases.

A practical method for obtaining lowest order models for sensitivity functions computations is developed. The technique makes full use of special system characteristics and has general application to high order systems with a large number of unknown parameters.

D.2 PROBLEM STATEMENT

Consider a system

$$\dot{x} = Fx + Gu \quad x(0) = x_0 \quad (D.1)$$

where x is a $n \times 1$ state vector, u is a $q \times 1$ control vector. F and G are matrices of appropriate dimensions and are functions of m parameters θ .

A heretofore uncited property of systems, which depends on the parameters θ , is important in sensitivity computation.

Definition 1 - Structural Controllability: A system is said to be structurally controllable if it is controllable for almost all values of parameters. The system may be uncontrollable if certain relations hold among the parameters.

Definition 2 - Structural Linear Dependence: A set of vectors has structural linear dependence if a linear combination of these vectors is zero for almost all values of parameters. The particular linear combination may depend on the values of the parameters.

Example 1: Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathbf{u} \quad (\text{D.2})$$

The controllability matrix is

$$\begin{pmatrix} 1 & \theta_1 + \theta_2 \\ 1 & \theta_3 + \theta_4 \end{pmatrix}$$

The system is controllable unless $\theta_1 + \theta_2 = \theta_3 + \theta_4$. Thus, if $\theta_1 = \theta_4 = -1$ and $\theta_2 = \theta_3 = -5$, the system is uncontrollable in the classical sense but structurally controllable.

Initially, the following simplifications can be made:

- a. The system is made structurally controllable by dropping uncontrollable states. Since the initial condition is zero, the system never moves into the uncontrollable subspace. This reduces the

order of the system. Note that the states which are uncontrollable only for the given values of the parameters but which are structurally controllable should not be dropped.

- b. All structurally linearly dependent columns of G matrix are lumped with other columns. This reduces the number of effective controls.

The state sensitivities for all parameters θ can be written as [29]

$$\dot{x}_\theta = F_\theta x_\theta + G_\theta u \quad (D.3)$$

$$x_\theta(0) = 0$$

F_θ , G_θ and x_θ are defined in Chapter III. If (F_θ, G_θ) is uncontrollable, the corresponding controllability matrix is of rank less than $(m+1)n$, say r . Let Q_1 be the set of r independent columns in the controllability matrix. If Q_2 is such that Q_1 and Q_2 form a set of $n(m+1)$ linearly independent vectors, then

$$\dot{x}_\theta \triangleq (Q_1 : Q_2)^{-1} x_\theta \triangleq \begin{pmatrix} Q_1^\dagger \\ \dots \\ Q_2^\dagger \end{pmatrix} x_\theta \quad (D.4)$$

follows the differential equation

$$\dot{x}_\theta = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} x_\theta + \begin{pmatrix} G_1 \\ 0 \end{pmatrix} u \quad (D.5)$$

Since the initial condition in (D.5) is zero, the last $(m+1)n-r$ uncontrollable states remain zero throughout. The remaining r states, x_c , follow the differential equation

$$\dot{x}_c = F_c x_c + G_c u \quad (D.6)$$

$$x_c(0) = 0$$

where

$$F_c \triangleq F_{11}' = Q_1^\dagger F_\theta Q_1 \quad (D.7)$$

$$G_c \triangleq G_1' = Q_1^\dagger G_\theta$$

Also,

$$\begin{aligned} x_\theta &= (Q_1' : Q_2') x_\theta' \\ &= Q_1' x_c \end{aligned} \quad (D.8)$$

since other states in x_θ' are zero. Note that Q_1^\dagger is a pseudo-inverse of Q_1 depending on Q_2 . The transformation from F_θ, G_θ to F_c, G_c and from x_c to x_θ does not involve Q_2 explicitly. Therefore, Q_1^\dagger can be chosen to be any pseudo-inverse of Q_1 , for example,

$$Q_1^\dagger = (Q_1^T Q_1)^{-1} Q_1^T \quad (D.9)$$

It is assumed here that the inputs are linearly independent. If this is not so, the number of inputs can be reduced until they are linearly independent. This will usually result in a reduction in the controllable subspace of (F_θ, G_θ) as shown in Section D.4. Under the assumption of linear independence of inputs, it is necessary and sufficient to solve a system of r linear equations (D.6) to determine the state and its sensitivities at all times. The next sections investigate the nature and dimension of the controllable subspace of (F_θ, G_θ) and explore efficient methods for finding Q_1 .

D.3 SINGLE INPUT SYSTEMS

In single input systems, F_θ is a $(m+1)n \times (m+1)n$ matrix and G_θ is a $(m+1)n$ vector. The following theorem holds.

Theorem D.1: For a single input system, the rank of the controllability matrix of (F_θ, G_θ) is less than $2n$.

Proof: The controllability matrix of (F_θ, G_θ) is

$$C_\theta = \left[G_\theta : F_\theta G_\theta : \dots : F_\theta^{(m+1)n-1} G_\theta \right] \quad (D.10)$$

It is easy to show that

$$F_\theta^p G_\theta = \begin{bmatrix} F^p G \\ \dots \\ \frac{\partial}{\partial \theta_1} F^p G \\ \dots \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dots \\ \frac{\partial}{\partial \theta_m} F^p G \end{bmatrix} \triangleq D(F^p G) \triangleq \begin{pmatrix} F^p G \\ 0 \end{pmatrix} + D^*(F^p G) \quad (D.11)$$

If

$$F^n = \alpha_0 I + \alpha_1 F + \dots + \alpha_{n-1} F^{n-1} \quad (D.12)$$

the $(n+k)$ th column of C_θ is

$$D(F^{n+k-1} G) = \sum_{i=0}^{n-1} \{ D^* (\alpha_i I) F^{k+i-1} G + \alpha_i D(F^{k+i-1} G) \} \quad (D.13)$$

The second term is a linear combination of n preceding columns of C_θ . Thus,

$$\begin{aligned} \text{rank } C_\theta = \text{rank} \{ & D(G) \vdots \dots \vdots D(F^{n-1}G) \vdots \sum_{i=0}^{n-1} D^*(\alpha_i I) F^i G \vdots \dots \dots \\ & \dots \dots \vdots \sum_{i=0}^{n-1} D^*(\alpha_i I) F^{i+n-1} G \vdots \sum_{i=0}^{n-1} D^*(\alpha_i I) F^{n+i} G \vdots \dots \dots \} \end{aligned} \quad (D.14)$$

The $(2n+k)$ th column of the right hand side matrix is

$$\begin{aligned} \sum_{i=0}^{n-1} D^*(\alpha_i I) F^{n+i+k-1} G &= \sum_{i=0}^{n-1} D^*(\alpha_i I) \sum_{j=0}^{n-1} \alpha_j F^{i+j+k-1} G \\ &= \sum_{j=0}^{n-1} \alpha_j \sum_{i=0}^{n-1} D^*(\alpha_i I) F^{i+(j+k)-1} G \end{aligned} \quad (D.15)$$

which is a linear combination of n previous columns for $k \geq 0$. Therefore,

$$\text{rank } C = \text{rank} [G_\theta \vdots F_\theta G_\theta \vdots \dots \vdots F_\theta^{2n-1} G_\theta] \leq 2n \quad (D.16)$$

Thus, the order of the system required to compute all state sensitivities for a single input system cannot exceed $2n$. In many practical cases, it is smaller as shown in Example 1.

Corollary 1: If the structurally controllable subspace of (F, G) is of the order p , the maximal order of the controllable subspace of (F_θ, G_θ) is $2p$.

Since $F^p G$ is a linear combination of $G, FG, \dots, F^{p-1}G$, the corollary follows immediately.

Example 2: Consider the following system

$$\dot{\mathbf{x}} = \begin{pmatrix} \theta_1 & \theta_2 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \quad \mathbf{x}(0) = 0 \quad (\text{D.17})$$

The state vector and its sensitivities for θ_1 and θ_2 form a set of six differential equations. Since the number of states is two and the number of controls is one, only the first four columns can be independent in the controllability matrix of (F_θ, G_θ) . These columns are

$$\text{rank}(C_\theta) = \text{rank} \left[D \begin{pmatrix} 0 \\ 1 \end{pmatrix}, D \begin{pmatrix} \theta_2 \\ -1 \end{pmatrix}, D \begin{pmatrix} \theta_1 \theta_2 - \theta_2 \\ 1 \end{pmatrix}, D \begin{pmatrix} \theta_1^2 \theta_2 - \theta_1 \theta_2 + \theta_2 \\ -1 \end{pmatrix} \right] \quad (\text{D.18})$$

$$= \text{rank} \begin{bmatrix} 0 & \theta_2 & \theta_1 \theta_2 - \theta_2 & \theta_1^2 \theta_2 - \theta_1 \theta_2 + \theta_2 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & \theta_2 & 2\theta_1 \theta_2 - \theta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \theta_1 - 1 & \theta_1^2 - \theta_1 + 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{D.19})$$

The first three columns are independent for $\theta_2 \neq 0$. If θ_2 is zero, only the first two columns are independent and the required model is of the second order.

D.4 MULTI-INPUT SYSTEMS

We state and prove the following theorem.

Theorem D.2: The rank of the controllability matrix of (F_θ, G_θ) cannot exceed $(q+1)n$ for a q inputs system.

In multi-input systems, G_θ is a matrix with q columns. The controllability matrix of (F_θ, G_θ) has $(m+1)nq$ columns which can be written thus:

$$C_\theta = [D(G_1, FG_1, \dots, F^{n(m+1)-1}G_1) : D(G_2, FG_2, \dots, F^{n(m+1)-1}G_2) : \dots : D(G_q, FG_q, \dots, F^{n(m+1)-1}G_q)] \quad (D.20)$$

where G_i is the i th column of G . By using Theorem 1, it is easily seen that the last $(m-1)n$ columns involving any of vectors G_i 's are linearly dependent on the first $2n$ columns for that vector. From (D.14) and (D.15)

$$\text{rank}(C_\theta) \stackrel{\Delta}{=} \rho = \text{rank}\{[D(G_1, FG_1, \dots, F^{n-1}G_1), \Sigma D^*(\alpha_i I)F^i G_1, \dots, \Sigma D^*(\alpha_i I)F^{i+n-1}G_1] : \dots : [D(G_q, FG_q, \dots, F^{n-1}G_q), \Sigma D^*(\alpha_i I)F^i G_q, \dots, \Sigma D^*(\alpha_i I)F^{i+n-1}G_q]\} \quad (D.21)$$

where all summations are from 0 to $n-1$. Let the structurally independent columns in the controllability matrix of (F, G) be

$$[G_1, \dots, F^{k_1-1}G_1, G_2, \dots, F^{k_2-1}G_2, \dots, G_q, \dots, F^{k_q-1}G_q] \quad (D.22)$$

$$\sum_{i=1}^q k_i = n \quad (D.23)$$

This set of linearly independent vectors in the controllability matrix spans the complete n -dimensional space. So any vector can be represented as a linear combination of these vectors. In particular,

$$F^{j-1}G_k = \beta_1 G_1 + \dots + \beta_{k_1} F^{k_1-1}G_1 + \dots + \beta_n F^{k_q-1}G_q \quad \begin{matrix} 1 \leq j \leq n \\ 1 \leq k \leq q \end{matrix} \quad (D.24)$$

Therefore,

$$\begin{aligned} \sum_{i=0}^{n-1} D^*(\alpha_i I) F^{i+j-1} G_k &= \beta_1 \sum_{i=0}^{n-1} D^*(\alpha_i I) F^i G_1 + \beta_2 \sum_{i=0}^{n-1} D^*(\alpha_i I) F^{i+1} G_1 \dots \\ &\dots, + \beta_n \sum_{i=0}^{n-1} D^*(\alpha_i I) F^{i+k-1} G_q \end{aligned} \quad (D.25)$$

This is a linear combination of n vectors in the right hand side matrix of (D.24) for all j and k (the values of β_i depend on j and k). Thus,

$$\begin{aligned} \rho &= \text{rank} [D(G_1, FG_1, \dots, F^{n-1} G_1) \vdots D(G_2, FG_2, \dots, F^{n-1} G_2) \vdots \dots \vdots D(G_q, FG_q, \dots, F^{n-1} G_q) \\ &\quad \Sigma D^*(\alpha_i I) F^i G_1, \Sigma D^*(\alpha_i I) F^{i+1} G_1, \dots, \Sigma D^*(\alpha_i I) F^i G_j, \dots, \Sigma D^*(\alpha_i I) F^{i+k-1} G_q] \end{aligned} \quad (D.26)$$

$$\leq (q+1)n$$

Thus, the procedure for finding independent columns of C_θ consists of finding structurally independent columns of the controllability matrix of (F, G) choosing $(q+1)n$ appropriate columns from C_θ and checking to see if there is any further linear dependence.

Another simplification is possible in large order systems in which each input controls only a small number of states. If k_i is the dimension of the controllable subspace of the i th input, no more than $2k_i$ columns involving G_i can be linearly independent in the right hand side matrix of (D.26) as shown in Corollary 1 to Theorem 1.

Corollary 2: If for any single input u_j the system is completely controllable,

$$\begin{aligned} \rho &= \text{rank} [D(G_1, \dots, F^{n-1} G_1), D(G_2, \dots, F^{n-1} G_2), \dots, D(G_q, \dots, F^{n-1} G_q), \\ &\quad \dots, \Sigma D^*(\alpha_i I) F^i G_j, \dots, \Sigma D^*(\alpha_i I) F^{n+i-1} G_j] \end{aligned} \quad (D.27)$$

The proof is obvious since $G_j, \dots, F^{n-1}G_j$ form a set of n linearly independent columns.

D.5 CONCLUSIONS

A systematic method for finding the controllable subspace of the augmented system, in which the state vector is the system state and its sensitivities, is presented. It is necessary to start with no more than $(q+1)n$ columns of the controllability matrix of the augmented system. These columns can be selected quickly by inspection of the controllability matrix of the initial system. If r is the dimension of the controllable subspace of the augmented system, it is necessary to solve r linear equations to evaluate the state vector and its sensitivities.

This method of sensitivity functions reduction fully exploits the characteristics of the system and the cases in which the sensitivity to all parameters in the system is not required. In other words, it leads to the minimal order model under the circumstances.

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