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OPTIMUM THREE-DIMENSIONAL ATMOSPHERIC ENTRY FROM THE ANALYTICAL SOLUTION OF CHAPMAN'S EXACT EQUATIONS

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FOREWARD

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ABSTRACT

The general solution for the optimum three-dimensional aerodynamic control of a lifting vehicle entering a planetary atmosphere is developed. A set of dimensionless variables, modified Chapman variables, is introduced. The resulting exact equations of motion, referred to as Chapman's exact equations, have the advantage that they are completely free of the physical characteristics of the vehicle. Furthermore, a completely general lift-drag relationship is used in the derivation. Hence, the results obtained apply to any type of vehicle of arbitrary weight, dimensions and shape, having an arbitrary drag polar, and entering any planetary atmosphere.

The aerodynamic controls chosen are the lift coefficient and the bank angle. General optimum control laws for these controls are developed.

Several earlier particular solutions are shown to be special cases of this general result. This demonstrates a certain universality of this solution. The results are general and apply to any given end conditions. Of particular interest is the fact that the results are valid for both free and constrained terminal position.

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1. INTRODUCTION

With the advancement of space flight technology and the increased frequency of missions requiring atmospheric passage, basic changes have been introduced in the new generation of aerospace vehicles. The new generation of vehicles is distinguished by the ability to use aerodynamic forces in a beneficial sense by controlling the lift and, to a lesser extent, the drag. Such a controllable lifting body can operate first as an integral part of the launch system, using the maneuverability given by the controlled lift to help reach the desired orbit. The vehicle can then function as a satellite for an extended period outside the planetary atmosphere. Upon accomplishing its mission, the vehicle can enter the atmosphere and again use active aerodynamic controls to tailor its trajectory. The aerodynamic maneuverability of the vehicle can be used to reach a prescribed region before performing an approach and landing on an airfield much as an ordinary aircraft. In addition, the lifting capability can be used to limit the peak deceleration and heating, or to achieve a particular deceleration profile. The tailoring of the trajectory can be quite detailed. Such abilities lead naturally to the question of trajectory optimization.

Another area of similar interest comes from the requirements of the vehicle's mission as a satellite. During the period of its mission in extra-atmospheric space, the particular mission requirements may dictate that the vehicle perform one or more orbital changes. An orbital change in space can only be effected at the expense of fuel consumption. The extent of missions which can be accomplished is severely constrained by the fuel requirements of such orbital maneuvers. In some cases the aerodynamic maneuverability of the vehicle can be used to advantage if, instead of a brute-force pure thrusting maneuver in the vacuum of space, a combined thrusting and aerodynamically active maneuver can be performed at lower fuel cost. In such a case the prescribed final orbit is achieved, and the savings in fuel consumption obviously can extend the useful life of the vehicle. Hence, again, aerodynamic controllability leads directly to the question of trajectory optimization.

A schematic of a combined thrusting and lifting maneuver for the orbital change of an aerodynamically active vehicle is illustrated in Figure 1. The state of the vehicle is represented by a point in an n-dimensional Euclidean space E^n . The desired maneuver is to bring the vehicle from an initial state S_1 to a prescribed final state S_2 with minimum fuel consumption. For purpose of illustration, assume that the optimal trajectory without aerodynamic maneuvering is the trajectory C , a pure thrusting maneuver remaining outside the atmosphere. Let the fuel consumption of this maneuver be $J(C)$. It is assumed that the planet, about which the maneuver is centered, is surrounded by an atmosphere inside which a purely aerodynamic maneuver can be achieved without fuel consumption. The only penalty of the aerodynamic maneuver is a loss in the total energy. Of course, the trajectory inside the atmosphere is subject to physical constraints such as upper limits on deceleration and heating rate. The physical description of the atmosphere about the planet, and the inequalities describing the constraints on the vehicle and its trajectory, considered as a whole, limit the region of the state space, E^n, which is open to aerodynamic maneuvers to a subspace, E^m immersed in E^n, Figure 1.

If it is assumed that the analysis of the optimum combined thrust and aerodynamic maneuver yields the optimal trajectory $C_1 + C_* + C_2$, where C_1 and C_2 are thrusting maneuvers in space and C_* is the subarc along which a purely aerodynamic maneuver is used, then the global cost of this combined maneuver must be compared with that of the purely thrusting maneuver, C. Let $J(C_1)$ and $J(C_2)$ be the fuel costs of trajectory subarcs C_1 and C_2. The fuel cost of C_* is zero, since it is purely aerodynamic. Thus, the combined maneuver is optimum if the following condition is satisfied:

$$J(C_1) + J(C_2) < J(C) \tag{1.1}$$

Intuitively, it is felt that, as indicated in the figure, this condition is met if the states S_1 and S_2 are, in some sense, far apart, and if S_1 and S_2 are close enough to E^m. In fact, several numerical studies have shown just that.

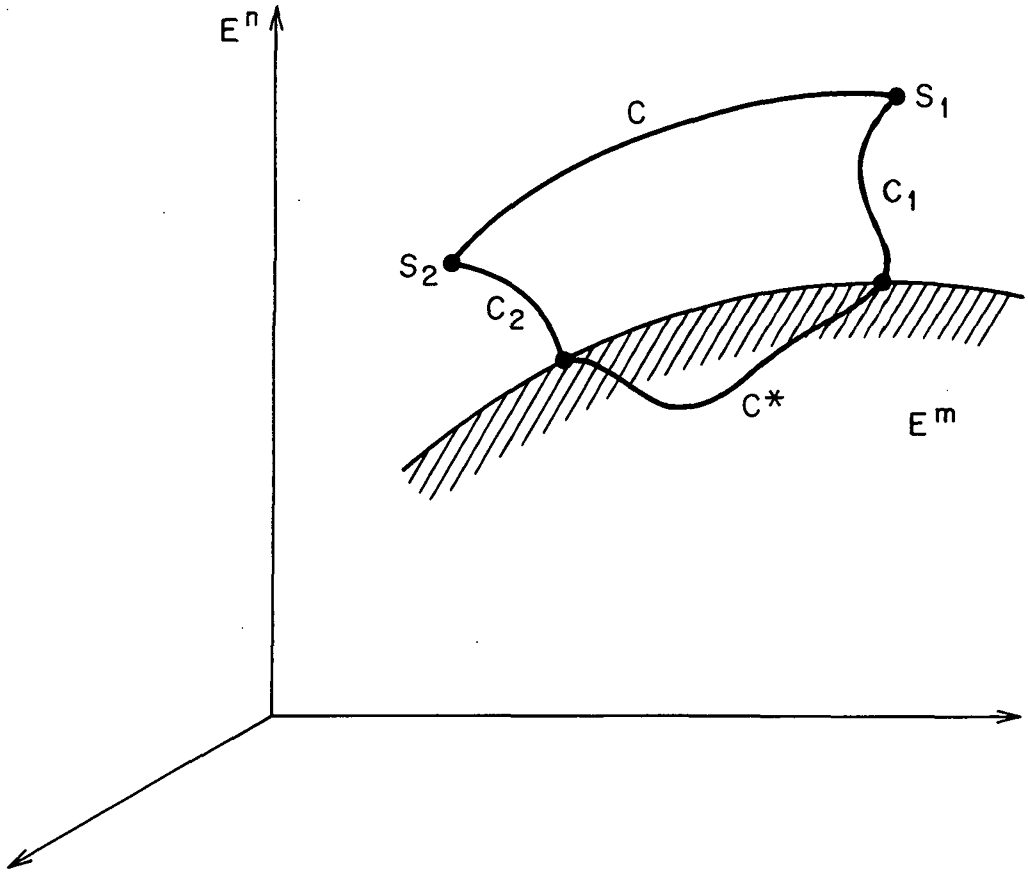

Figure 1. Schematic of Optimal Maneuver

In order to study the problem of optimizing the trajectory for such maneuvers, it is necessary to develop a general theory of optimal aerodynamically controlled maneuvers at orbital entry conditions. To obtain qualitative understanding of the possibilities of these combined trajectories, it is desirable to carry out the investigation analytically as far as is possible. Numerical computations are invaluable, but should be sought only after the analysis has been carried to its greatest result. The results of this investigation of optimal aerodynamic trajectories will serve as a valuable tool both for mission analysis and design, and for operational studies.

It is the purpose of this paper to give a completely general analysis of the optimal three dimensional aerodynamic maneuver of a lifting aerospace vehicle entering and maneuvering within a planetary atmosphere. The results obtained are independent of the physical characteristics of the vehicle such as the weight, dimensions, and shape. The results are applicable to any planetary atmosphere, and are not dependent upon a particular atmosphere model.

A general synthesis of the optimal trajectory problem is possible if a suitable set of dimensionless variables can be constructed. In this respect, a set of modified Chapman variables, (Refs. 1, 2), are used. In addition, the control variables through which the aerodynamic maneuver is directed are selected such that the results are valid for a completely general lift-drag polar.

The analysis is made using the maximum principle, (Ref. 3), with completely free variation of the controls on lift force and bank angle, including the possibility of maximum bounds on these controls. The maneuvers are purely aerodynamic. Thrust is not included at this point.

The particular optimal trajectories which have been previously obtained are presented again here. It is shown how these classical solutions can be obtained directly from the general theory of this report, thus displaying a certain universality for this theory.

2. EXACT EQUATIONS FOR ATMOSPHERIC ENTRY USING VARIABLE AERODYNAMIC CONTROLS

The equations of motion of a nonthrusting, lifting vehicle entering a planetary atmosphere are, Figure 2,

$$
\begin{aligned}
\frac{dr}{dt} &= V \sin \gamma \\
\frac{d\theta}{dt} &= \frac{V \cos \gamma \cos \psi}{r \cos \phi} \\
\frac{d\phi}{dt} &= \frac{V \cos \gamma \sin \psi}{r} \\
\frac{dV}{dt} &= -\frac{\rho S C_D V^2}{2m} - g \sin \gamma \\
\frac{V d\gamma}{dt} &= \frac{\rho S C_L V^2}{2m} \cos \sigma - \left(g - \frac{V^2}{r}\right) \cos \gamma \\
\frac{V d\psi}{dt} &= \frac{\rho S C_L V^2}{2m \cos \gamma} \sin \sigma - \frac{V^2}{r} \cos \gamma \cos \psi \tan \phi
\end{aligned}
\tag{2.1}
$$

The first three equations are simply the kinematic relations. The last three equations are the momentum equations. The planet and its atmosphere are assumed to be spherical and nonrotating. The speed of rotation of the atmosphere is not insignificant. It can reach, for the Earth, six percent of the circular orbit velocity in the altitudes of consequence. However, it greatly simplifies the analysis to ignore it, and account for it approximately by altering the coefficients of lift and drag (Ref. 4).

The oblateness of the planet contributes an important effect also. It has more impact, however, on long term effects such as decay

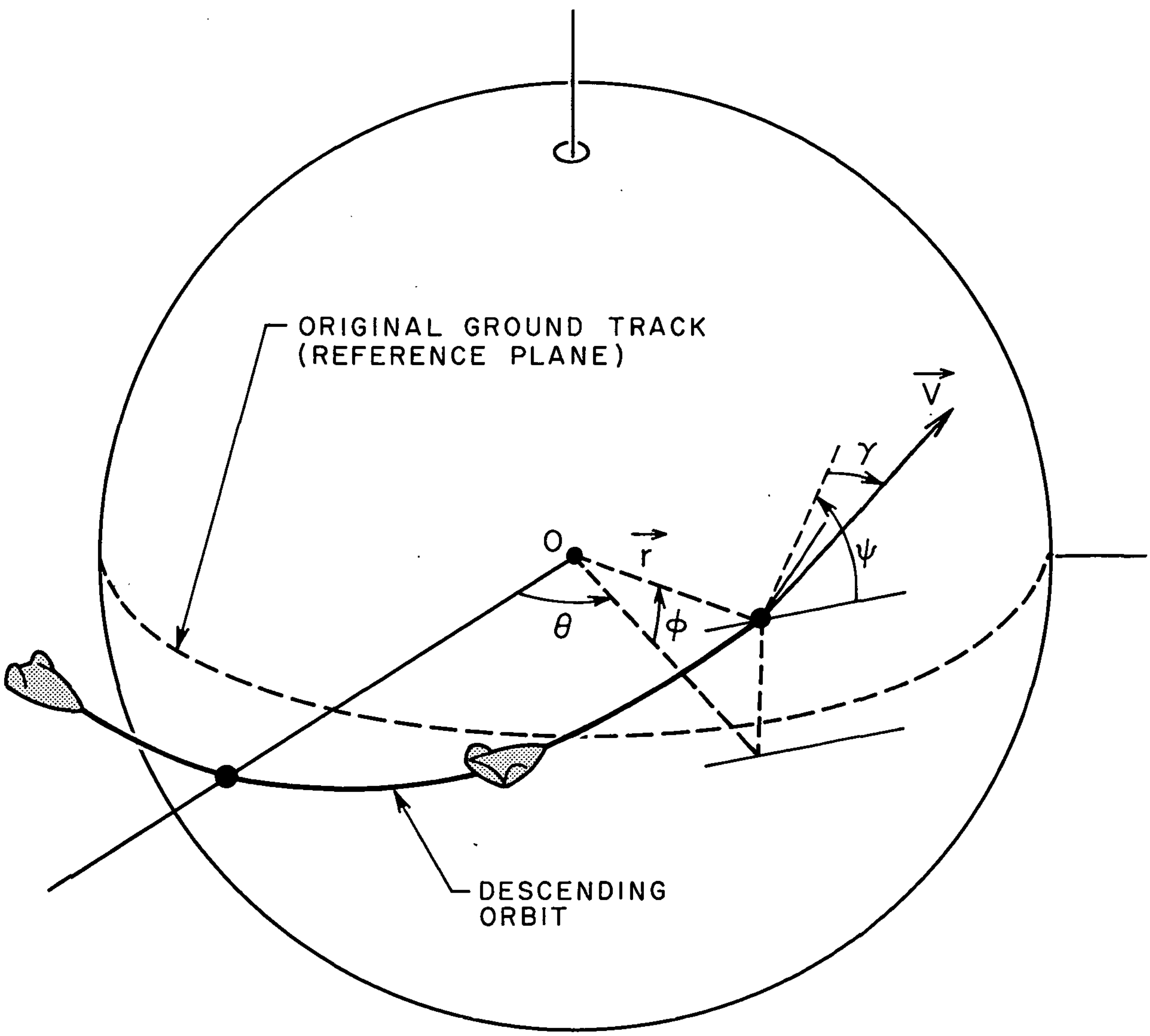

Figure 2. Coordinate System

of a satellite. In the altitudes where major aerodynamic forces are encountered, the atmosphere maintains approximately the same oblateness as the planet's surface. If the maneuver does not cover a large range angle over the planet's surface, it is adequate to treat the planet as spherical and use the local mean value of the planet's radius in the solution.

The six variables r, θ, ϕ, V, γ, and ψ define the state of the vehicle, considered as a point with constant mass m. The six equations of (2.1) are the state differential equations governing the changes in the state of the vehicle.

The initial plane is taken as the reference plane which shall be referred to as the equatorial plane without loss of generality. The velocity vector is defined by the state variables V, the speed, γ, the flight path angle defined positive up from the local horizontal plane, and the heading angle ψ defined positive to the left (in the direction of the North pole) of the initial trajectory.

The bank angle, σ, is taken such that for positive σ the vehicle is turning to the left. This bank angle is defined as the angle between the local vertical plane containing the velocity (that is, the $\vec{r}$, $\vec{V}$ plane), and the plane containing the aerodynamic force $\vec{A}$ and the velocity $\vec{V}$, the $\vec{A}$, $\vec{V}$ plane (or, what is the same, the plane of the lift and the velocity, the $\vec{L}$, $\vec{V}$ plane).

The gravitational field is taken to be a central, inverse square field, with the acceleration $g(r)$ given by

$$g(r) = \frac{\mu}{r^2} \qquad (2.2)$$

but within the consequential atmosphere the acceleration of gravity varies so little that analyses are usually still considered exact when $g(r)$ is replaced by a constant value. In order to be able to continue the trajectories far beyond the sensible atmosphere, however, the Newtonian gravitational field (2.2) will be kept available.

The atmospheric mass density, ρ, is a strong function of the altitude. It shall be assumed to be locally exponential in that it varies according to the differential law

$$\frac{d\rho}{\rho} = -\beta\, dr \qquad (2.3)$$

The local scale height, $1/\beta$, for any specified planetary atmosphere, is a function of the radial distance also. For small altitude intervals β can be taken as constant, though for the Earth, in the altitudes of interest, from sea level to 150 kilometers, $1/\beta$ oscillates from about 5 kilometers to about 8 kilometers. Rather than assume β to be strictly constant in this report, the approach of Chapman (Ref. 1) will be used. This is made possible by the choice of variables, and will be discussed in detail later.

The aerodynamic force is controlled through the bank angle, σ, and the lift coefficient, C_L. It is assumed that there exists a lift-drag relationship characteristic of the vehicle and the particular flight conditions. Thus, the bank angle can be used to control the direction of the lift, and either the lift coefficient, C_L, or the drag coefficient, C_D , can be used to control the magnitude of the lift. To maintain the greatest possible generality, the lift-drag relationship is not explicitly specified but is kept arbitrary throughout the development.

As control for the lift magnitude, a rescaled lift coefficient, λ , defined as

$$\lambda \equiv C_L/C_L^* \tag{2.4}$$

is chosen. Here C_L^* is the lift coefficient corresponding to the maximum lift-to-drag ratio. Thus, maximum lift-to-drag ratio means λ equal to unity.

Similarly, the drag coefficient is replaced by

$$f(\lambda) \equiv C_D/C_D^* \tag{2.5}$$

where, as above, C_D^* means the drag coefficient at maximum L/D . The function $f(\lambda)$ is the specifying function for the lift-drag relationship. Note that $f(1) = 1$. Again it should be emphasized that, in the general formulation of this report, $f(\lambda)$ need not be given explicitly.

In general, the lift coefficient has an upper bound. Also, it is conceivable that the bank angle is bounded. Hence, the control variables λ and σ are bounded by

$$\begin{aligned} |\lambda| &\leq \lambda_{max} \\ |\sigma| &\leq \sigma_{max} \end{aligned} \tag{2.6}$$

For some solutions it is of interest to take λ_{max} and σ_{max} very large so as to determine the unbounded optimal controls. As long as such a solution remains reasonable (that is, within achievable values of the controls), then the unbounded optimal control is also the bounded optimal control. The behavior of a control on the boundary and proper application of the maximum principle in such case will be discussed later in this report.

Finally, the exact equations for atmospheric entry can usefully be written with r as the independent variable instead of t. The first of equations (2.1) is used to achieve the substitution. The remaining five state differential equations (where r is no longer a state variable) are

$$\begin{aligned}
\frac{d\theta}{dr} &= \frac{\cos\psi}{r\cos\phi\tan\gamma} \\
\frac{d\phi}{dr} &= \frac{\sin\psi}{r\tan\gamma} \\
\frac{d\psi}{dr} &= \frac{\rho S C_L \sin\sigma}{2m\sin\gamma\cos\gamma} - \frac{\cos\psi\tan\phi}{r\tan\gamma} \qquad (2.7) \\
\frac{dV}{dr} &= \frac{-\rho S C_D V}{2m\sin\gamma} - \frac{g}{V} \\
\frac{d\gamma}{dr} &= \frac{\rho S C_L\cos\sigma}{2m\sin\gamma} - \left(\frac{g}{V^2} - \frac{1}{r}\right)\frac{\cos\gamma}{\sin\gamma}
\end{aligned}$$

2.1 The Modified Chapman Variables

At this point it becomes convenient to introduce two new variables in the same spirit as did Chapman (Ref. 1). The variables used here differ in several respects from those of Chapman. Hence, these new variables shall be referred to as the modified Chapman variables.

$$Z \equiv \frac{\rho S C_L^*}{2m} \sqrt{\frac{r}{\beta}}$$
$$u \equiv \frac{V^2 \cos^2 \gamma}{gr} \tag{2.8}$$

The minor differences from the original Chapman variables are as follows. Chapman's Z is proportional to C_D. In his case this was feasible since his analysis was for constant lift and drag coefficients. In the present report C_D has been replaced by C_L^*, which is constant for a given vehicle under given flight conditions. The results will not change no matter what constant is used. Hence, even a vehicle and flight conditions such that the actual maximum L/D changes drastically (and, therefore, so does C_L^*) can be considered if a convenient constant is defined as C_L^* in these equations.

Also, in Chapman's report, Z is proportional to $\sqrt{u}$ of this report. However, it is found that the resulting exact equations have a simpler form as defined here. This is the reason, too, for taking as the second variable u of equation (2.8) in this report, rather than the Chapman variable $\bar{u}$, which is simply $\sqrt{u}$.

It is impressive, and of great benefit to following researchers, that Chapman, after many trials, discovered his transformation which now permits the general optimal solution of this report.

Chapman derived his two equations, excluding the range from consideration, for planar motion with constant lift-to-drag ratio.

Here, the full set of five equations for the three-dimensional maneuver is considered, including the range, and with completely variable lift coefficient (and related drag coefficient) and bank angle.

Chapman used two simplifying assumptions that limited his solutions. The first is that the fractional change in distance from the planet center is small compared to the fractional change in velocity, $|dr/r| << |dV/V|$. The second is, for lifting vehicles, the flight path angle γ relative to the local horizontal plane is sufficiently small that the component of lift in the horizontal direction is small compared to the component of drag, $|(L/D) \tan \gamma)| << 1$.

Because of these assumptions, his resulting equations are approximate. Here, these assumptions are not used. The resulting equations are exact. Even g and r are kept as varying, though for many trajectories they may be taken as constant without loss of accuracy.

2.2 Chapman's Exact Equations

Because of their origin, in recognition of Chapman's contributions, the exact equations for three-dimensional atmospheric entry with variable lift coefficient and bank angle obtained in this report will be referred to as Chapman's exact equations.

By taking the derivative of u, as defined in equations (2.8), with respect to r,

$$\frac{du}{dr} = \frac{\cos^2 \gamma}{gr} \frac{dV^2}{dr} - \frac{2V^2 \sin \gamma \cos \gamma}{gr} \frac{d\gamma}{dr} - \frac{V^2 \cos^2 \gamma}{gr^2} \tag{2.9}$$

is obtained. Using equations (2.7) and the definitions (2.4), (2.5), and (2.8), one obtains

$$\frac{du}{dr} = - \frac{2Zu}{\sin\gamma}\sqrt{\frac{\beta}{r}}\left[\frac{f(\lambda)}{E^*} + \lambda\cos\sigma\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\, Z}\right] \tag{2.10}$$

where E^* is the maximum lift-to-drag ratio, and is equal to C_L^*/C_D^*.

In this notation the differential equations for the flight path angle, γ, and the heading angle, ψ, are

$$\frac{d\gamma}{dr} = \frac{Z}{\sin\gamma}\sqrt{\frac{\beta}{r}}\left[\lambda\cos\sigma + \frac{\cos\gamma}{\sqrt{\beta r}\, Z}\left(1 - \frac{\cos^2\gamma}{u}\right)\right] \tag{2.11}$$

and

$$\frac{d\psi}{dr} = \frac{Z}{\sin\gamma\cos\gamma}\sqrt{\frac{\beta}{r}}\left[\lambda\sin\sigma - \frac{\cos^2\gamma\cos\psi\tan\phi}{\sqrt{\beta r}\, Z}\right] \tag{2.12}$$

where the effect of the controls, λ and σ, is obvious.

From equation (2.8) for Z comes

$$\frac{dZ}{dr} = -\beta Z\left(-\frac{1}{\rho\beta}\frac{d\rho}{dr} - \frac{1}{2\beta r} + \frac{1}{2\beta^2}\frac{d\beta}{dr}\right) = -\beta Z a \tag{2.13}$$

in which the coefficient "a"

$$a \equiv -\frac{1}{\rho\beta}\frac{d\rho}{dr} - \frac{1}{2\beta r} + \frac{1}{2\beta^2}\frac{d\beta}{dr} \tag{2.14}$$

becomes, for the locally exponential atmosphere of equation (2.3)

$$a = 1 - \frac{1}{2\beta r} + \frac{1}{2\beta^2}\frac{d\beta}{dr} \tag{2.15a}$$

If the atmosphere is taken to be strictly exponential, then β is constant and $d\beta/dr$ is zero. Thus,

$$a = 1 - \frac{1}{2\beta r} \tag{2.15b}$$

If, on the other hand, an isothermal atmosphere is assumed, then βr^2 is constant and $d\beta/dr$ is $-2\beta/r$. Then

$$a = -\frac{1}{\rho\beta}\frac{d\rho}{dr} - \frac{3}{2\beta r} \tag{2.15c}$$

Finally, if βr is assumed constant, $d\beta/dr$ is $-\beta/r$ and

$$a = -\frac{1}{\rho\beta}\frac{d\rho}{dr} - \frac{1}{\beta r} \tag{2.15d}$$

In any of these cases, if "a" is set equal to unity, the required density relationship will differ very little from equation (2.3).

$$\frac{d\rho}{\rho} = -\beta dr + \text{small term} \tag{2.16}$$

It shall now be determined which assumption is most convenient.

If the equations of motion are rewritten using the state vector $\vec{X}$ defined as

$$\vec{X} = (x_1, x_2, x_3, x_4, x_5) \equiv (\theta , \phi , \psi , u , \gamma) \tag{2.17}$$

with the variable Z as the independent variable, the resulting differential equation for the state vector will be of the form

$$\frac{d\vec{X}}{dZ} = \vec{F}(\vec{X} , \lambda , \sigma , \beta r , Z) \tag{2.18}$$

in which now β and r appear only as the product βr.

For Earth, and as well as is known for the other planets with atmospheres, the quantity βr oscillates about a mean value throughout the altitude band of primary concern for atmospheric entry. Chapman (Ref. 1) pointed out this and made the assumption that βr was constant and equal to its mean value in this lower region of the atmosphere. For Earth, for altitudes below 120 kilometers, the mean value of βr is about 900. The deviation of βr from its mean value is large. In this same region βr varies from a low of about 750 to a high of about 1300.

It is, however, a better assumption than simply putting β constant and using the simple exponential atmosphere at this point. This development will follow Chapman's lead and put βr constant.

In any case, the coefficient "a" defined in equation (2.14) must be very nearly one. It will be henceforth put equal to unity.

Explicitly, the equations (2.18) are

$$\begin{aligned}
\frac{d\theta}{dZ} &= -\frac{\cos\psi}{\beta r\ Z \cos\phi \tan\gamma} \\
\frac{d\phi}{dZ} &= -\frac{\sin\psi}{\beta r\ Z \tan\gamma} \\
\frac{d\psi}{dZ} &= -\frac{1}{\sqrt{\beta r}\sin\gamma\cos\gamma}\left[\lambda\sin\sigma - \frac{\cos^2\gamma\cos\psi\tan\phi}{\sqrt{\beta r}\ Z}\right] \\
\frac{du}{dZ} &= \frac{2u}{\sqrt{\beta r}\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\cos\sigma\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\ Z}\right] \\
\frac{d\gamma}{dZ} &= -\frac{(\lambda\cos\sigma + G)}{\sqrt{\beta r}\sin\gamma}
\end{aligned} \tag{2.19}$$

in which the new variable

$$G \equiv \frac{\cos\gamma}{\sqrt{\beta r}\ Z}\left(1 - \frac{\cos^2\gamma}{u}\right) \tag{2.20}$$

has been introducted for convenience. The physical significance of G will be discussed later.

The equations (2.19) are the equations for three-dimensional flight inside a planetary atmosphere with a variable lift coefficient and variable bank angle. They are referred to as Chapman's exact equations.

It should be noticed that, rather than Z as the independent variable, any one of the state variables, for example, the flight path angle, γ, can be chosen as the independent variable. Sometimes it will prove advantageous to change from one independent variable to another to obtain monotone behavior. The quantity βr is to be treated as a constant, thus eliminating the problem of r appearing on the right-hand side of the equations.

The equations (2.19) can be considered as the exact equations for planetary entry since the two restrictive assumptions of Chapman have been removed. The only restriction in these equations is that concerning βr . This assumption has proven to be valuable and not damaging to the results. This assumption concerns only the given planetary atmosphere. The equations properly reduce to the equations for Keplerian motion in the vacuum of space. Thus, in this further sense they are exact.

These equations are completely free of the physical characteristics of the vehicle. The results obtained apply to any type of vehicle, regardless of weight, dimensions, and shape. The equations apply to any atmosphere, specified by its mean value of βr at this point, and later by the specification of its density-altitude relation. There are no restrictions on flight path angle, lift-to-drag ratio, or type of trajectory.

3. APPLICATION OF THE MAXIMUM PRINCIPLE

The problem is now formulated as an optimal control problem. The state of the vehicle is defined by the five-vector, $\vec{X}$, as given by equation (2.17). The motion of the vehicle is governed by the state equations (2.19). The normal problem would be posed by prescribing an initial state, $\vec{X}_i$, and partially prescribing a terminal state, $\vec{X}_f$. The problem is to select the control functions, $\lambda(Z)$ and $\sigma(Z)$, subject to the constraints (2.6), such that some function, $J(\vec{X}_f, Z_f)$, is minimized. Such a solution is an optimal trajectory, and is determined by the control functions, $\lambda(Z)$ and $\sigma(Z)$, which are the optimal controls.

Using the maximum principle (Ref. 3), a five-vector, $\vec{p}$, associated with the state vector, $\vec{X}$, to form the Hamiltonian, H, one obtains

$$H \equiv \vec{p} \cdot \dot{\vec{X}} \tag{3.1}$$

which, in this case, is

$$H = -\frac{p_1 \cos\psi}{\beta r\, Z \cos\phi \tan\gamma} - \frac{p_2 \sin\psi}{\beta r\, Z \tan\gamma} - \frac{p_3}{\sqrt{\beta r}\sin\gamma\cos\gamma}\left[\lambda \sin\sigma - \frac{\cos^2\gamma\cos\psi\tan\phi}{\sqrt{\beta r}\, Z}\right] + \frac{2p_4 u}{\sqrt{\beta r}\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\cos\sigma\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\, Z}\right] - \frac{p_5(\lambda\cos\sigma + G)}{\sqrt{\beta r}\sin\gamma} \tag{3.2}$$

The canonical equations of Hamilton

$$\frac{dx_i}{dZ} = \frac{\partial H}{\partial p_i} \qquad i = 1 , \ldots , 5 \tag{3.3}$$

and

$$\frac{dp_i}{dZ} = - \frac{\partial H}{\partial x_i} \qquad i = 1 , \ldots , 5 \tag{3.4}$$

govern the propagation of the state variables, x_i , and the adjoint variables, p_i . Equations (3.3) are nothing more than equations (2.19). The adjoint equations, (3.4), provide the additional equations required for solving the optimal control problem.

A necessary condition for optimality is that, at every point, the Hamiltonian, (3.2), considered as a function of the control variables λ and σ , be an absolute maximum (Ref. 3). This may be illustrated in the cylindrical space of (λ , σ , H) , Figure 3. The control variables λ and σ may be either an interior point or a boundary point of the set of controls, λ and σ . This leads to the following possible types of optimal solution.

$$\begin{array}{llll} \text{a)}\ \lambda = \lambda(Z) & \text{, variable} & \sigma = \sigma(Z) & \text{, variable} \\ \text{b)}\ \lambda = \lambda(Z) & \text{, variable} & \sigma = \sigma_{max} & \text{, constant} \\ \text{c)}\ \lambda = \lambda_{max} & \text{, constant} & \sigma = \sigma(Z) & \text{, variable} \\ \text{d)}\ \lambda = \lambda_{max} & \text{, constant} & \sigma = \sigma_{max} & \text{, constant} \end{array} \tag{3.5}$$

The optimal trajectory may consist of any one or a combination of these optimal subarcs, depending on the given end conditions.

For types a) and b) of (3.5), the lift coefficient varies along the trajectory. The maximization of H with respect to λ requires that

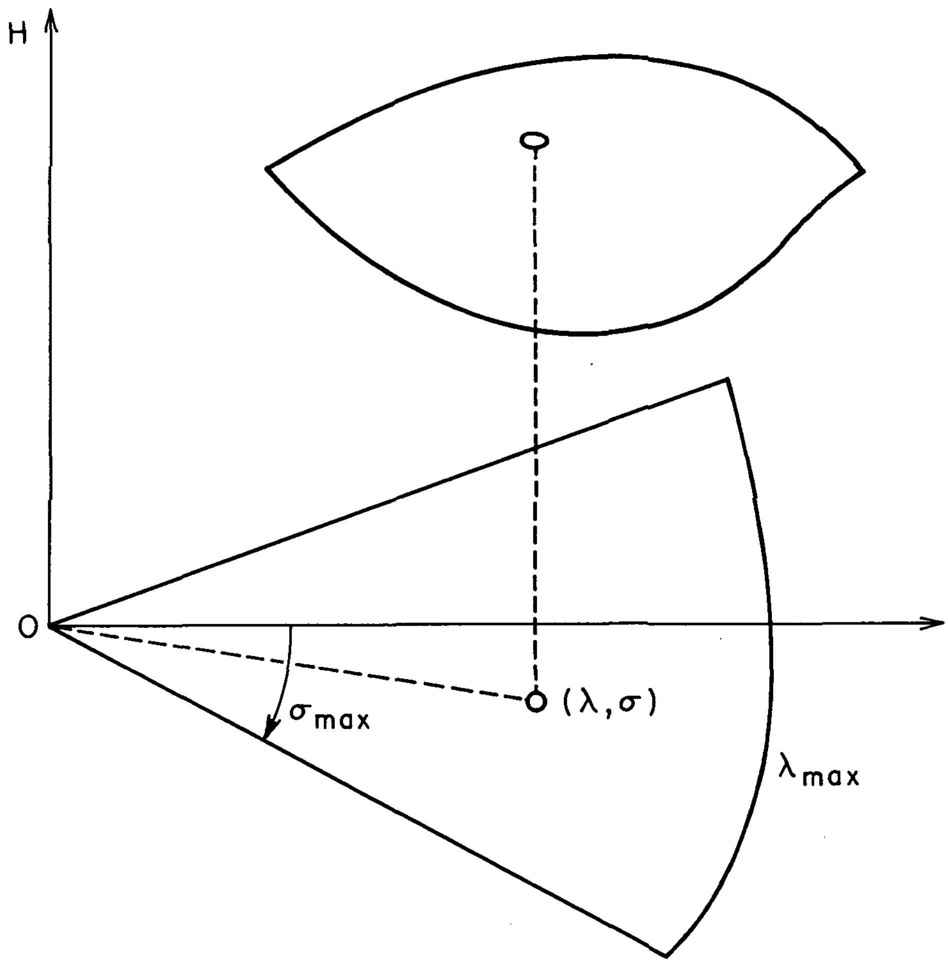

Figure 3. Maximization of the Hamiltonian

$$\frac{\partial H}{\partial \lambda} = 0 \tag{3.6}$$

or, for the Hamiltonian of (3.2), that

$$\frac{df}{d\lambda} + E^* \cos\sigma \tan\gamma = \frac{E^*}{2p_4 u}\left(\frac{p_3 \sin\sigma}{\cos\gamma} + p_5 \cos\sigma\right) \tag{3.7}$$

which, when solved, yields the optimal control law, $\lambda(Z)$.

For types a) and c) of (3.5), the bank angle varies along the trajectory. As above, maximization of the Hamiltonian with respect to σ requires that

$$\frac{\partial H}{\partial \sigma} = 0 \tag{3.8}$$

or, for the Hamiltonian of this problem, that

$$E^* \sin\sigma \tan\gamma = -\frac{E^*}{2p_4 u}\left(\frac{p_3 \cos\sigma}{\cos\gamma} - p_5 \sin\sigma\right) \tag{3.9}$$

which, when solved, yields the optimal control law, $\sigma(Z)$.

More concisely, along an arc of type a), with both λ and σ varying to maximize the Hamiltonian, the following equations hold.

$$\frac{df}{d\lambda} \cos\sigma + E^* \tan\gamma = \frac{E^* p_5}{2p_4 u} \tag{3.10}$$

$$\frac{df}{d\lambda} \sin\sigma = \frac{E^* p_3}{2p_4 u \cos\gamma} \tag{3.11}$$

Obviously, except for type d), the optimal controls are functions of the adjoint variables p_3 , p_4 , and p_5 , components of the adjoint vector $\vec{p}$. Since the equations (3.4) for the p_i are coupled with the state equations, (3.3) or (2.19), the complete solution requires the simultaneous integration of these two sets of five differential equations.

Analytically, these tasks would appear to be impossible without some simplifying assumptions. Fortunately, the present formulation not only provides a completely dimensionless set of equations which can be used for a purely numerical analysis for a general type of aerospace vehicle with completely arbitrary drag polar, but also eases the way to obtaining approximate optimal control laws independent of the physical characteristics of the vehicle.

As an indication of the universality of this presentation, it is possible to show the different known approximate solutions and how they may be derived from this theory. After this, the general solution from this theory will be presented.

4. SOME OPTIMAL SOLUTIONS BY APPROXIMATE MEANS

The two sets of equations, (3.3) and (3.4), for the state variables and adjoint variables can be integrated if certain special cases are considered and realistic assumptions are made. In the past, these particular solutions were obtained by different authors through various ad hoc coordinate transformations and simplifying devices. It will now be shown how these special solutions of other authors can be obtained directly from the theory developed here. In this way, a certain universality of the theory is established.

It should be pointed out that, although the optimal control laws obtained are approximations, the exact differential equations for the state variables have been presented, equations (2.19). Thus, once the control laws are known, these exact equations can be integrated, using the approximate optimal control, to yield the precise trajectory flown by the vehicle.

4.1 Contensou's Formulation

Contensou considered the problem of a skip trajectory in the vertical plane with the range unconstrained (Ref. 5). The range, θ , does not appear in the Hamiltonian, (3.2). Thus, from (3.4) the adjoint variable conjugate to θ is constant. Since the range is unconstrained, the transversality condition (Ref. 3) gives p_1 as zero. The range, θ , is therefore an ignorable coordinate in every sense of

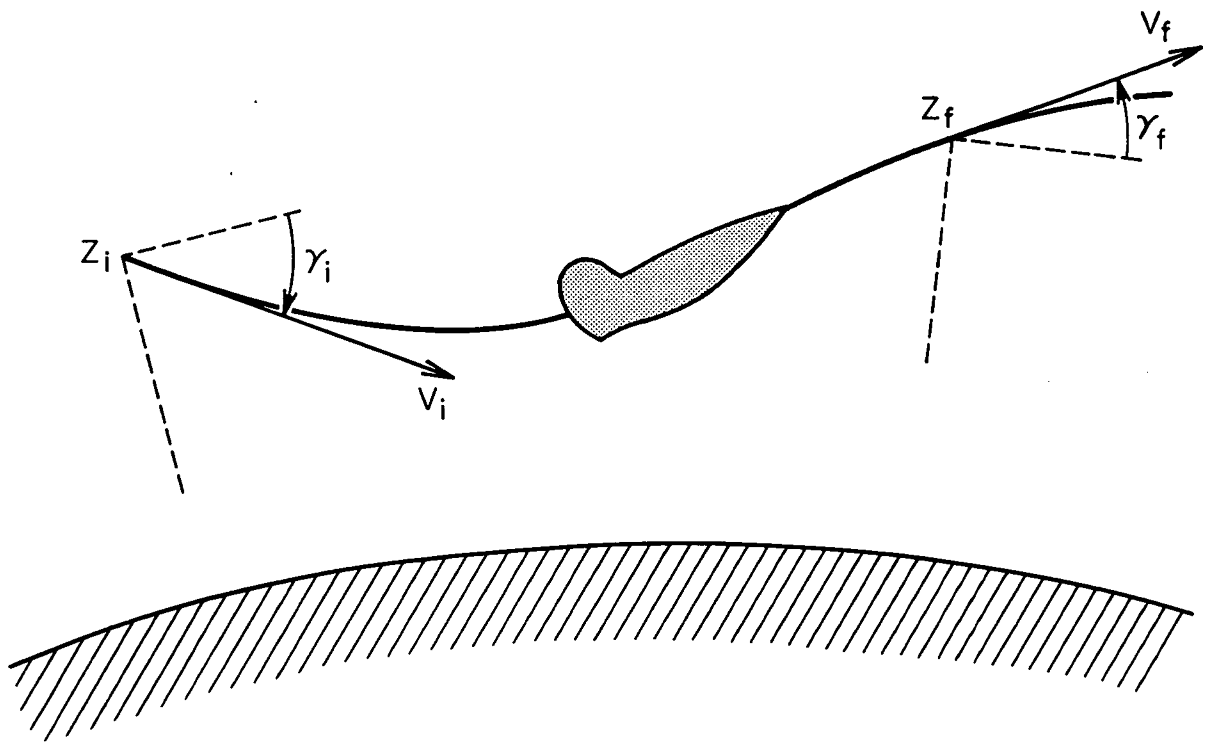

Figure 4. Skip Trajectory

the word (Ref. 6). The assumption of flight in a vertical plane eliminates ϕ and ψ as state variables. The remaining state equations are the last two of (2.19) for u and γ, in which the bank angle σ is set identically equal to zero.

The governing force is primarily the aerodynamic force. The Allen and Eggers assumption is justified (Ref. 7). This allows the last term in the equation for u to be dropped. This is the term arising from the gravity component along the tangent to the flight path. Similarly, in the equation for γ, the G term, expressing the combined gravitation and centrifugal force normal to the flight path, is neglected. The equations now have the following form.

$$\begin{aligned} \frac{du}{dZ} &= \frac{2u}{\sqrt{\beta r}\,\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\tan\gamma\right] \\ \frac{d\gamma}{dZ} &= -\frac{\lambda}{\sqrt{\beta r}\,\sin\gamma} \end{aligned} \tag{4.1}$$

Since the right-hand sides of equations (4.1) are free of the independent variable Z, the Hamiltonian, (3.2), is a constant of the motion, giving

$$H = \frac{2p_4 u}{\sqrt{\beta r}\,\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\tan\gamma\right] - \frac{p_5\lambda}{\sqrt{\beta r}\,\sin\gamma} = \frac{a_5}{\sqrt{\beta r}} \tag{4.2}$$

where a_5 is a constant of integration.

The equation for the adjoint variable, p_4, is, from (3.4),

$$\frac{dp_4}{dZ} = -\frac{2p_4}{\sqrt{\beta r}\,\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\tan\gamma\right] \tag{4.3}$$

which, combined with the first of equations (4.1) gives

$$d(p_4 u) = 0 \quad . \tag{4.4}$$

Thus,

$$p_4 u = a_4 \tag{4.5}$$

with a_4 another constant of integration.

It should be noted that the terms neglected do not change the general optimal control law for the lift coefficient, equation (3.7). Therefore, using the integrals (4.2) and (4.5) in equation (3.7) gives the optimal control law for the lift coefficient in the form

$$f(\lambda) - \lambda\frac{df}{d\lambda} = \frac{\alpha_5}{2} \sin \gamma \tag{4.6}$$

where

$$\alpha_5 \equiv \frac{E^* a_5}{a_4} \tag{4.7}$$

is the only constant appearing as a parameter in the optimal control law. The optimal control is either λ_{max} or λ variable, given by equation (4.6).

The control is valid for any drag polar. For the special case of a parabolic polar

$$f(\lambda) = \frac{1}{2}(1 + \lambda^2) \tag{4.8}$$

which yields Contensou's law (Ref. 5),

$$\lambda^2 = 1 - \alpha_5 \sin \gamma \quad . \tag{4.9}$$

This solution applies to the problem of maximizing the final speed, V_f, with a prescribed final altitude, Z_f, or to the problem of maximizing the final altitude with a prescribed final speed.

A detailed discussion of these two problems, and, in particular, the evaluation of the constant α_5 and the switching between λ_{max} arcs and variable λ arcs is available (Refs. 8, 9).

The control law of Contensou, equation (4.9), provides a close approximation to the exact optimal lift control. It can be improved by considering the exact equation for the state variable u . That is, the first of equations (4.1) is replaced by

$$\frac{du}{dZ} = \frac{2u}{\sqrt{\beta r}\,\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\,Z}\right] \quad . \tag{4.10}$$

$$H = \frac{2p_4 u}{\sqrt{\beta r}\,\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\,Z}\right] - \frac{p_5\lambda}{\sqrt{\beta r}\,\sin\gamma} \tag{4.11}$$

is the new Hamiltonian.

The equation for the adjoint variable p_4 is now

$$\frac{dp_4}{dZ} = -\frac{2p_4}{\sqrt{\beta r}\,\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\,Z}\right] \tag{4.12}$$

from which it follows that integral (4.5) still holds.

In this case the Hamiltonian is no longer constant (Ref. 3).

$$\frac{dH}{dZ} = \frac{\partial H}{\partial Z} = -\frac{p_4 u}{\beta r\,Z^2} \tag{4.13}$$

With p_4u from equation (4.5), this can be integrated.

$$H = \frac{a_4}{\beta r\,Z} + \frac{a_5}{\sqrt{\beta r}} \tag{4.14}$$

Substitution of the integrals (4.5) and (4.14) into the general optimal control law, (3.7), yields exactly the same optimal lift control as before, equation (4.6). This demonstrates the accuracy of Contensou's law. The improvement comes in the behavior of the state variable u, now obtained from the exact equation (4.10).

A second, and more important, improvement can be made by considering also the exact equation for the flight path angle, γ. This will provide a correction to the control law, (4.6).

The state equations for u and γ are now

$$\frac{du}{dZ} = \frac{2u}{\sqrt{\beta r}\ Z}\left[\frac{f(\lambda)}{E^*} + \lambda \tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\ Z}\right]$$
$$\frac{d\gamma}{dZ} = -\frac{(\lambda + G)}{\sqrt{\beta r}\sin\gamma} \tag{4.15}$$

The Hamiltonian is

$$H = \frac{2p_4 u}{\sqrt{\beta r}\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\ Z}\right] - \frac{p_5(\lambda + G)}{\sqrt{\beta r}\sin\gamma} \tag{4.16}$$

The additional term from G, defined by equation (2.20), contains u, γ, and Z. It will be impossible to handle without some simplifying assumption. For the skip trajectory, the assumption used, (Ref. 7), neglected G compared with λ. For a glide trajectory, the correct assumption, (Ref. 10), is that $(\lambda + G)$ is approximately zero. For many trajectories, G remains nearly constant, as was noted by Loh (Ref. 11). This assumption of constant G has been referred to as Loh's conjecture. An effective use of this approach is to assume G constant for the sole purpose of obtaining the optimal lift control law. Once the control law has been obtained, the state equations, (4.15), can be used with a varying G, equation (2.20), to generate the exact trajectory of the vehicle. This method of handling G has been used by several authors (Refs. 12, 13).

The equation for p_4, equation (4.12), still holds, as do the two integrals, (4.5) and (4.14), if G is treated as constant as described in the preceding paragraph. Substitution of these integrals into the general optimal control law, (3.7), gives the optimal control law in the following form.

$$f(\lambda) - (\lambda + G)\frac{df}{d\lambda} = \frac{\alpha_5}{2} \sin \gamma + E^{*}G \tan \gamma \tag{4.17}$$

For a parabolic drag polar, (4.8), the optimal control law is

$$(\lambda + G)^2 = 1 - \alpha_5 \sin \gamma + G^2 - 2E^{*}G \tan \gamma \tag{4.18}$$

The inclusion of the G term provides a higher order effect for, or correction to, the optimal control law of Contensou, (4.9). In particular, the last term, involving E^{*}, shows the influence of the maximum lift-to-drag ratio, which is a design parameter, on the optimal lift control. This term is important in another sense because it provides a correct limiting form for the optimal control law at low speeds. Without this term, for a maximum range problem the lift coefficient would tend to zero at the terminal point. With this term, the lift coefficient will correctly tend to the maximum lift-to-drag ratio condition at the terminal point.

4.2 Fave's Formulation

Fave studied the optimal bank control to obtain the maximum footprint for an orbital glider entering the atmosphere, (Refs. 14, 15). He analyzed a three-dimensional problem using a reduced set of equations. Fave followed the approach and used the assumptions of Ref. 16, but allowed a variable lift coefficient and a variable bank angle.

The basic simplifying assumptions are that the lateral range is small, and that the conditions of equilibrium glide hold, the assumption first put forward in Ref. 10.

With the lateral range small, ϕ is approximately zero. The equilibrium glide trajectory assumes that the lift, gravitational force, and centrifugal force are balanced in the vertical direction. Thus, the equilibrium term is zero.

$$\lambda \cos \sigma + G = 0 \tag{4.19}$$

The glide angle is small, and small angle approximations are applied to γ. Thus, $\sin \gamma \simeq \tan \gamma$ when in the denominator, and $\cos \gamma \simeq 1$. Using this and the definition of G, (2.20), equation (4.19) is

$$\frac{1}{\sqrt{\beta r}\, Z} = \frac{u\lambda \cos \sigma}{1 - u} \tag{4.20}$$

which serves as a constraining relation.

All these assumptions applied to the state equations, (2.19), yields

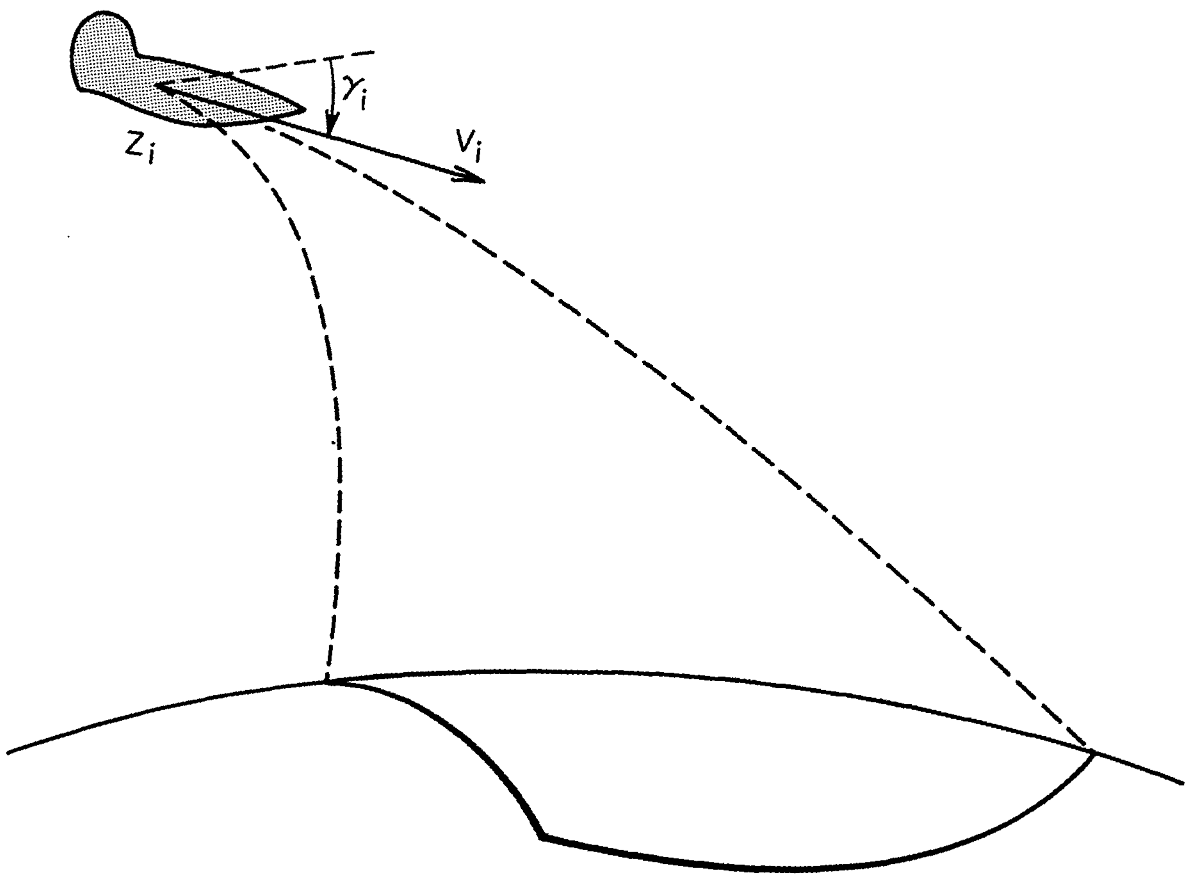

Figure 5. Footprint of an Entry Vehicle

$$\frac{d\theta}{dZ} = \frac{-\cos\psi}{\beta r\ Z\ \tan\gamma}$$

$$\frac{d\phi}{dZ} = -\frac{\sin\psi}{\beta r\ Z\ \tan\gamma}$$

$$\frac{d\psi}{dZ} = -\frac{\lambda \sin\sigma}{\sqrt{\beta r}\ \sin\gamma} \tag{4.21}$$

$$\frac{du}{dZ} = \frac{2uf(\lambda)}{E^* \sqrt{\beta r}\ \sin\gamma}$$

With equation (4.20) giving Z in terms of u, the state equations (4.21) can be written with u as the independent variable.

$$\frac{d\theta}{du} = -\frac{E^*\lambda}{f(\lambda)} \frac{\cos\sigma \cos\psi}{2(1-u)}$$

$$\frac{d\phi}{du} = -\frac{E^*\lambda}{f(\lambda)} \frac{\cos\sigma \sin\psi}{2(1-u)} \tag{4.22}$$

$$\frac{d\psi}{du} = -\frac{E^*\lambda}{f(\lambda)} \frac{\sin\sigma}{2u}$$

Note that

$$\frac{E^*\lambda}{f(\lambda)} = \frac{C_L}{C_D} = E \quad , \tag{4.23}$$

the lift-to-drag ratio. In this case E can be used as one of the control variables.

The Hamiltonian, with E a control variable, is

$$H = -\frac{E}{2}\left[\frac{p_1 \cos\sigma \cos\psi}{1-u} + \frac{p_2 \cos\sigma \sin\psi}{1-u} + \frac{p_3 \sin\sigma}{u}\right] \tag{4.24}$$

Since the independent variable u is decreasing, so that du will be taken negative, the optimal control is that which minimizes the Hamiltonian. H is minimized with respect to E when E is equal to E^* . The flight is always at maximum lift-to-drag ratio. The optimal bank angle σ will either be at its maximum value, or will vary as a solution of equation (3.8). Explicitly, this is

$$\tan \sigma = \frac{(1 - u)\, p_3}{u(p_1 \cos \psi + p_2 \sin \psi)} \tag{4.25}$$

The differential equations for the adjoint variables are

$$\frac{dp_1}{du} = 0$$

$$\frac{dp_2}{du} = 0$$

$$\frac{dp_3}{du} = - \frac{E^* \cos \sigma}{2(1 - u)} (p_1 \sin \psi - p_2 \cos \psi) \tag{4.26}$$

The integration is immediate.

$$\begin{aligned} p_1 &= a_1 \\ p_2 &= a_2 \\ p_3 &= a_1\phi - a_2\theta + a_3 \end{aligned} \tag{4.27}$$

In this, a_1 , a_2 , and a_3 are constants of integration.

The optimal bank angle varies according to equation (4.25), which is now

$$\tan \sigma = \frac{(1 - u)(a_1\phi - a_2\theta + a_3)}{u(a_1 \cos \psi + a_2 \sin \psi)} \tag{4.28}$$

This, along with E equal to E^* , is the optimal control law for the problem of maximum footprint of a hypervelocity vehicle on gliding entry to the atmosphere, using Fave's formulation.

It should be noted that in this solution the flight path angle is assumed to be negligibly small, and the optimal lift control is λ equal to unity. This same solution comes from Contensou's formulation, as can be seen from equation (4.9) with γ taken as zero. It is shown in Section 4.4 that, in fact, Contensou's law, equation (4.9), is also applicable to general three-dimensional flight. It will also be shown, in Section 5, that the equations for the adjoint variables p_1 , p_2 , and p_3 can be integrated in the general three-dimensional case with no restrictive assumptions.

4.3 Busemann's Formulation

The exact equations for horizontal coasting flight of a hypervelocity vehicle, following a great circle course, were integrated in Ref. 17. The problem is that of the flight of a vehicle along a great circle, starting from an initial speed u_i (Figure 6). As the speed decreases due to atmospheric drag, a constant altitude is maintained by continuously increasing the lift coefficient until the maximum lift coefficient is obtained. In Ref. 17 it is shown that the resulting range can be maximized by a proper selection of the optimal constant altitude.

This approach and problem formulation can be extended to lateral flight at constant altitude. The optimal lift and bank controls can be found such that the maximum reachable domain, at any given altitude, is obtained. Then the constant altitude can be used as a parameter to find the optimum flight level for maximum longitudinal range, maximum

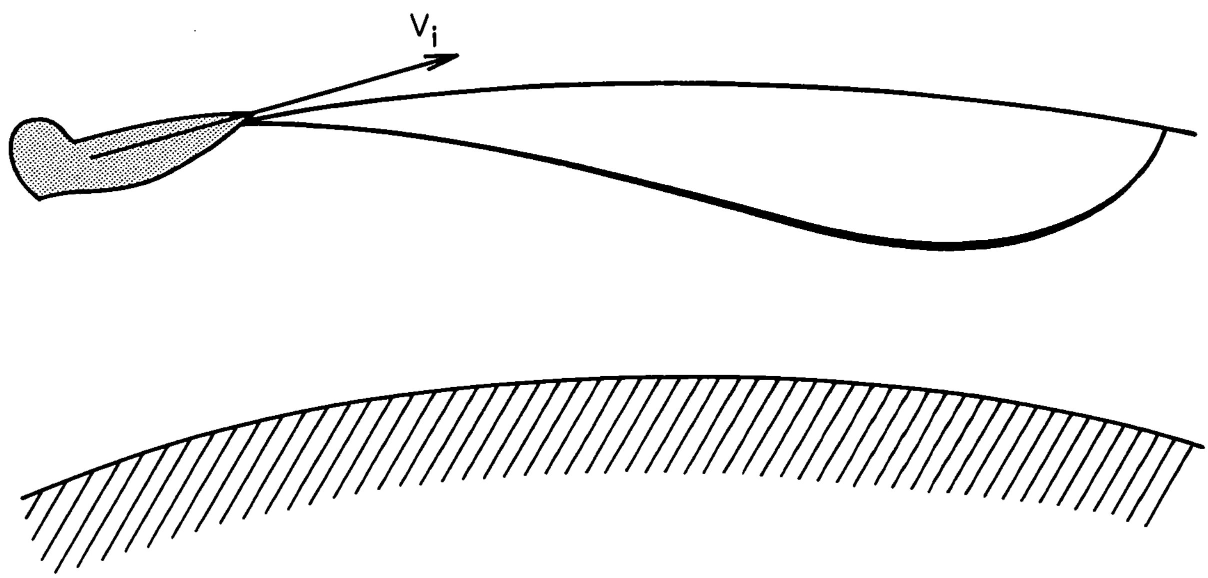

Figure 6. Reachable Domain in Constant Altitude Coast

lateral range, or maximum area of the reachable domain.

Unlike Fave's formulation, the exact solution to this problem can be obtained since the adjoint equations for p_1 , p_2 , and p_3 , which are the only adjoint variables involved in this problem, can be integrated exactly.

First, for horizontal flight the flight path angle, γ , is identically zero. Thus, the last of the state variable equations, (2.19), is zero and becomes a simple constraining relation.

$$\lambda \cos \sigma = \frac{1 - u}{wu} \tag{4.29}$$

$$w \equiv \sqrt{\beta r}\ Z \tag{4.30}$$

Since the altitude is constant, w is a parameter which serves to define the flight altitude.

With Z no longer a variable, the first three state variable equations, (2.19), can be rewritten using u as the independent variable, and setting γ equal to zero.

$$\begin{aligned}
\frac{d\theta}{du} &= - \frac{E^* \cos \psi}{2f(\lambda)\ wu \cos \phi} \\
\frac{d\phi}{du} &= - \frac{E^* \sin \psi}{2f(\lambda)\ wu} \\
\frac{d\psi}{du} &= - \frac{E^*}{2f(\lambda)\ u}[\lambda \sin \sigma - \frac{\cos \psi \tan \phi}{w}]
\end{aligned} \tag{4.31}$$

As shown by the constraining relation for horizontal flight, (4.29), the lift and the bank are no longer independent controls. One immediately noticed effect is that, for $u > 1$, negative lift is required to maintain horizontal flight. The consequences of this are interesting and are discussed in detail in Ref. 17. Here, the discussion will be limited to the case where $u < 1$. That is, the horizontal velocity is taken to be less than the local circular orbital velocity. The independent control is chosen to be λ , subject to the inequality, (2.6), and, with u less than unity, λ is never negative.

The flight at constant altitude is terminated by the constraining relation (4.29) when λ reaches λ_{max} with the bank angle, σ , equal to zero. Hence, for a prescribed flight altitude w , the final speed is

$$u_f = \frac{1}{1 + w\lambda_{max}} \tag{4.32}$$

The state variable equation for ψ , the last of equations (4.31), is rewritten by expressing $\lambda \sin \sigma$ in terms of λ , u , and w by using equation (4.29).

$$\frac{d\psi}{du} = - \frac{E^*}{2f(\lambda)\ wu}\left[\pm \frac{\sqrt{\lambda^2 w^2 u^2 - (1 - u)^2}}{u} - u \cos \psi \tan \phi\right] \tag{4.33}$$

The Hamiltonian for this problem,

$$\begin{aligned} H = - \frac{E^*}{2f(\lambda)\ wu}\Big[p_1 \frac{\cos \psi}{\cos \phi} + p_2 \sin \psi &+ \\ + p_3\Big(\pm \frac{\sqrt{\lambda^2 w^2 u^2 - (1 - u)^2}}{u} &- \cos \psi \tan \phi\Big)\Big] \end{aligned} \tag{4.34}$$

which must be maximized with respect to λ . This occurs for either λ equal to λ_{max} , or for a variable λ given by the solution of $\partial H/\partial \lambda = 0$. The latter condition is, explicitly,

$$\pm \frac{p_3}{u\sqrt{\lambda^2 w^2 u^2 - (1 - u)^2}}[w^2 u^2 (\frac{\lambda f}{f'} - \lambda^2) + (1 - u)^2]$$

$$= p_1 \frac{\cos\psi}{\cos\phi} + p_2 \sin\psi - p_3 \cos\psi \tan\phi \tag{4.35}$$

where f' is $df/d\lambda$.

This equation can be solved for λ in terms of the state variables, θ , ϕ , ψ , if the adjoint components, p_1 , p_2 , and p_3 , are obtained in terms of the state variables. The adjoint equations are

$$\begin{aligned}
\frac{dp_1}{du} &= 0 \\
\frac{dp_2}{du} &= \frac{E^*}{2f(\lambda)\ wu}[p_1 \frac{\cos\psi \tan\phi}{\cos\phi} - p_3 \frac{\cos\psi}{\cos^2\phi}] \\
\frac{dp_3}{du} &= -\frac{E^*}{2f(\lambda)\ wu}[p_1 \frac{\sin\psi}{\cos\phi} - p_2 \cos\psi - p_3 \sin\psi \tan\phi]
\end{aligned} \tag{4.36}$$

Immediately,

$$p_1 = a_1 \tag{4.37}$$

To solve the other two adjoint equations, first change the independent variable from u to θ .

$$\begin{aligned}
\frac{dp_2}{d\theta} &= -p_1 \tan\phi + \frac{p_3}{\cos\phi} \\
\frac{dp_3}{d\theta} &= p_1 \tan\psi - p_2 \cos\phi - p_3 \sin\phi \tan\psi
\end{aligned} \tag{4.38}$$

Differentiating the first of equations (4.38) with respect to θ , gives

$$\frac{d^2 p_2}{d\theta^2} = \left(\frac{-p_1}{\cos^2 \phi} + p_3 \frac{\sin \phi}{\cos^2 \phi}\right) \frac{d\phi}{d\theta} + \frac{1}{\cos \phi} \frac{dp_3}{d\theta} \qquad (4.39)$$

Using the second of equations (4.38) and, from equations (4.31), the relation

$$\frac{d\phi}{d\theta} = \cos \phi \tan \psi \qquad (4.40)$$

puts equation (4.39) into the simple form

$$\frac{d^2 p_2}{d\theta^2} + p_2 = 0 \qquad (4.41)$$

The general solution is

$$p_2 = a_2 \cos \theta + a_3 \sin \theta \qquad (4.42)$$

where a_2 and a_3 are the last two constants of integration. The adjoint variable p_3 is now obtained from the first of equations (4.38).

$$p_3 = a_1 \sin \phi - (a_2 \sin \theta - a_3 \cos \theta) \cos \phi \qquad (4.43)$$

The expressions for the adjoint variables, p_i , equations (4.37), (4.42), and (4.43), are substituted into equation (4.35). This yields, upon specifying the lift-drag relationship $f(\lambda)$, the optimal variable lift coefficient, λ , and, through equation (4.29), the optimal variable bank angle, σ .

One particular case of interest, that of the parabolic drag polar, deserves to be written out specifically. The function $f(\lambda)$ is given by (4.8), and the optimal control law is

$$\pm \frac{p_3[w^2u^2(1-\lambda^2) + 2(1-u)^2]}{2u\sqrt{\lambda^2 w^2 u^2 - (1-u)^2}} \tag{4.44}$$

$$= p_1 \frac{\cos\psi}{\cos\phi} + p_2 \sin\psi - p_3 \cos\psi \tan\phi$$

This equation is a quadratic in λ^2 , and can be solved explicitly for λ^2 as a function of the state variables, the three constants of integration, a_i , and the independent variable u , when the solutions for the adjoint variables, p_i , are inserted.

Thus, the generalized Busemann problem is completely solved. As far as can be determined, this problem is the only one of this category for which the set of exact adjoint equations is completely integrable. Thus, it is the only such problem for which the exact optimal aerodynamic controls are obtained explicitly.

4.4 The Three-Dimensional Free Range Problem

For this problem, the assumption of small lateral range is used, $\phi \simeq 0$. The motion is described by the state variable equations, with Z as independent variable.

$$\begin{aligned}
\frac{d\theta}{dZ} &= -\frac{\cos\psi}{\beta r\, Z \tan\gamma} \\
\frac{d\phi}{dZ} &= -\frac{\sin\psi}{\beta r\, Z \tan\gamma} \\
\frac{d\psi}{dZ} &= -\frac{\lambda \sin\sigma}{\sqrt{\beta r}\,\sin\gamma\cos\gamma} \\
\frac{du}{dZ} &= \frac{2u}{\sqrt{\beta r}\,\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\cos\sigma\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\,Z}\right]
\end{aligned} \tag{4.45}$$

$$\frac{d\gamma}{dZ} = - \frac{(\lambda \cos \sigma + G)}{\sqrt{\beta r} \sin \gamma} \qquad (4.45, \text{ continued})$$

The general optimal control laws for the lift coefficient and bank angle, equations (3.7) and (3.9), are still valid.

The Hamiltonian is now

$$H = - \frac{p_1 \cos \psi}{\beta r \ Z \tan \gamma} - \frac{p_2 \sin \psi}{\beta r \ Z \tan \gamma} - \frac{p_3 \lambda \sin \sigma}{\sqrt{\beta r} \sin \gamma \cos \gamma}$$
$$+ \frac{2 p_4 u}{\sqrt{\beta r} \sin \gamma} [\frac{f(\lambda)}{E^*} + \lambda \cos \sigma \tan \gamma + \frac{\sin \gamma}{2\sqrt{\beta r} \ Z}] - p_5 \frac{(\lambda \cos \sigma + G)}{\sqrt{\beta r} \sin \gamma} \qquad (4.46)$$

The equations for the first three adjoint variables, p_1 , p_2 , and p_3 , are

$$\frac{dp_1}{dZ} = 0$$
$$\frac{dp_2}{dZ} = 0 \qquad (4.47)$$
$$\frac{dp_3}{dZ} = - \frac{p_1 \sin \psi}{\beta r \ Z \tan \gamma} + \frac{p_2 \cos \psi}{\beta r \ Z \tan \gamma}$$

Integration gives the adjoint variables as

$$p_1 = a_1$$
$$p_2 = a_2 \qquad (4.48)$$
$$p_3 = a_1 \phi - a_2 \theta + a_3$$

with a_1 , a_2 , and a_3 constants of integration.

If the term G is treated as a constant in the differentiation, as explained in detail in Section 4.1, the equation for p_4 is

$$\frac{dp_4}{dZ} = - \frac{2p_4}{\sqrt{\beta r}\ \sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\cos\sigma\tan\gamma + \frac{\sin\gamma}{2\sqrt{\beta r}\ Z}\right] \tag{4.49}$$

which is precisely equation (4.12). Thus, again

$$p_4 u = a_4 \tag{4.50}$$

is the integral for p_4. For the final integration, instead of integrating the adjoint differential equation for p_5, the variation of H along an optimal trajectory is considered. The maximum principle requires that

$$\frac{dH}{dZ} = \frac{\partial H}{\partial Z} \tag{4.51}$$

which, in this case, is

$$\frac{dH}{dZ} = \frac{p_1 \cos\psi}{\beta r\ Z^2 \tan\gamma} + p_2 \frac{p_2 \sin\psi}{\beta r\ Z^2 \tan\gamma} - \frac{p_4 u}{\beta r\ Z^2} \tag{4.52}$$

If the longitudinal and lateral ranges are not constrained, the transversality condition (Ref. 3) gives a_1 and a_2 as zero. Then, with $p_4 u$ from (4.50), H is the same as in equation (4.14),

$$H = \frac{a_4}{\beta r\ Z} + \frac{a_5}{\sqrt{\beta r}} \tag{4.53}$$

in which a_5 is the last constant of integration. In this case p_3, from the last of equations (4.48), is just a_3, the integral (4.53) can be written as

$$\frac{E^* p_5}{\alpha_4}(\lambda\cos\sigma + G) = 2f(\lambda) + 2E^*\lambda\cos\sigma\tan\gamma - \frac{\alpha_3\lambda\sin\sigma}{\cos\gamma} - \alpha_5\sin\gamma \tag{4.54}$$

where

$$\alpha_3 \equiv \frac{E^* a_3}{a_4}$$

and (4.55)

$$\alpha_5 \equiv \frac{E^* a_5}{a_4}$$

are the two constants of integration required in this problem.

Substitution into the general optimal control laws, equations (3.7) and (3.9), gives

i) the equation for the optimal control law for varying lift coefficient:

$$2[\lambda\frac{df}{d\lambda} - f(\lambda)] + \frac{2G}{\cos\sigma}\frac{df}{d\lambda} = \frac{\alpha_3 G \tan\sigma}{\cos\gamma} - \alpha_5 \sin\gamma - 2E^* G \tan\gamma \tag{4.56}$$

ii) the equation for the optimal control law for varying bank angle:

$$2f(\lambda) = \frac{\alpha_3(\lambda + G\cos\sigma)}{\sin\sigma\cos\gamma} + \alpha_5 \sin\gamma + 2E^* G\tan\gamma \tag{4.57}$$

Hence, referring to the four possible types of optimal trajectories, equations (3.5), for type a) both equations (4.56) and (4.57) hold. For type b), equation (4.56) gives $\lambda(Z)$ while $|\sigma| = \sigma_{max}$. For type c), equation (4.57) gives $\sigma(Z)$ while $|\lambda| = \lambda_{max}$. For type d), both λ and σ are held constant at the limit of permissable values.

Solution gives, for type a),

$$\frac{df}{d\lambda}\sin\sigma = \frac{\alpha_3}{2\cos\gamma} \tag{4.58}$$

and

$$2(\lambda + G\cos\sigma)\frac{df}{d\lambda} - 2f(\lambda) = -\alpha_5 \sin\gamma - 2E^* G \tan\gamma \qquad (4.59)$$

For the parabolic drag polar, $f(\lambda)$ is given by equation (4.8), and the optimal lift coefficient control law of equation (4.58) becomes

$$\lambda \sin\sigma = \frac{\alpha_3}{2\cos^2\gamma} \qquad (4.60)$$

while the optimal bank angle control law of equation (4.59) becomes

$$(\lambda \cos\sigma + G)^2 = 1 + G^2 - 2E^* G \tan\gamma - \alpha_5 \sin\gamma - \frac{\alpha_3^2}{4\cos^2\gamma} \qquad (4.61)$$

The constants of integration α_3 and α_5 are evaluated at the prescribed end conditions.

5. THE GENERAL SOLUTION

The discussion of the particular solutions in Section 4 indicated that the integration of the last adjoint equation is singularly difficult, especially in the case where the final position is prescribed. Nevertheless, these particular solutions and the discussions accompanying them have shed some light on the behavior of the optimal variable lift control.

For the skip trajectory, the Contensou control law, equation (4.9), shows that the optimal lift control oscillates about the maximum lift-to-drag ratio. This is also true for the case of gliding flight to achieve a maximum footprint, as shown by Fave's analysis, Refs. 14 and 15. It is to be expected that this is a characteristic of a large class of entry trajectories.

In this section an approximate lift control law for the general case is developed. It is shown that this approximation affects primarily the vertical component of the lift. The lateral component can be obtained in essentially exact form.

The equations used in this section are the exact equations for the state variables, equations (2.19). The Hamiltonian is given by equation (3.2).

Thus, the equations for the first three adjoint variables are

$$\frac{dp_1}{dZ} = 0$$

$$\frac{dp_2}{dZ} = \frac{p_1 \cos\psi \sin\phi}{\beta r\, Z \cos^2\phi \tan\gamma} - \frac{p_3 \cos\psi}{\beta r\, Z \cos^2\phi \tan\gamma} \tag{5.1}$$

$$\frac{dp_3}{dZ} = - \frac{p_1 \sin \psi}{\beta r\ Z \cos \phi \tan \gamma} + \frac{p_2 \cos \psi}{\beta r\ Z \tan \gamma} + \frac{p_3 \sin \psi \tan \phi}{\beta r\ Z \tan \gamma}$$

(5.1, continued)

The integration of these three equations is precisely the same as in Section 4.3. The adjoint variables p_1 , p_2 , and p_3 are obtained exactly, introducing a_1 , a_2 , and a_3 as constants of integration.

$$p_1 = a_1$$

$$p_2 = a_2 \cos \theta + a_3 \sin \theta \qquad (5.2)$$

$$p_3 = a_1 \sin \phi - (a_2 \sin \theta - a_3 \cos \theta) \cos \phi$$

The adjoint variable p_4 has as its associated state variable, u , which appears in the last two terms of the Hamiltonian. If, in obtaining the differential equation for p_4 from the Hamiltonian, the quantity G in the last term is treated as a constant, an assumption which is valid for both strong aerodynamic lifting maneuvers and smooth equilibrium glide, the integral for p_4 is obtained as before.

$$p_4 u = a_4 \qquad (5.3)$$

This integral, for the broad range of cases for which the one assumption used, Loh's conjecture (Ref. 11), is very accurately satisfied, can be recognized as extremely accurate. For the case of variable lift coefficient and bank angle control, equation (3.11) shows that the lateral component of the lift has been obtained very accurately.

Explicitly, this optimal control law is

$$\frac{df}{d\lambda} \sin \sigma = \frac{\alpha_1 \sin \phi - (\alpha_2 \sin \theta - \alpha_3 \cos \theta) \cos \phi}{2 \cos \gamma} \tag{5.4}$$

where

$$\alpha_1 \equiv \frac{E^* a_1}{a_4}$$

$$\alpha_2 \equiv \frac{E^* a_2}{a_4} \tag{5.5}$$

$$\alpha_3 \equiv \frac{E^* a_3}{a_4}$$

For the vertical component of the lift, equation (3.10) applies. The control law depends on the adjoint variable, p_5 . The state variable associated with p_5 is γ , which enters explicitly every term of the Hamiltonian. This makes a direct integration of the differential equation for p_5 highly unlikely. Therefore, as was done before in Section 4, the differential equation for the Hamiltonian itself, equation (4.51), is used to determine p_5 .

$$\frac{dH}{dZ} = \frac{p_1 \cos \psi}{\beta r\ Z^2 \cos \phi \tan \gamma} + \frac{p_2 \sin \psi}{\beta r\ Z^2 \tan \gamma} - \frac{p_3 \cos \psi \tan \phi}{\beta r\ Z^2 \tan \gamma}$$

$$- \frac{p_4 u}{\beta r\ Z^2} + \frac{p_5}{\beta r\ Z^2 \tan \gamma}\left(1 - \frac{\cos^2 \gamma}{u}\right) \tag{5.6}$$

With the right-hand side written in terms of H , this becomes

$$Z\frac{dH}{dZ} + H = -\frac{\lambda}{\sqrt{\beta r}\sin\gamma}\left(\frac{p_3 \sin\sigma}{\cos\gamma} + p_5 \cos\sigma\right)$$
$$+ \frac{2p_4 u}{\sqrt{\beta r}\sin\gamma}\left[\frac{f(\lambda)}{E^*} + \lambda\cos\sigma\tan\gamma\right] \tag{5.7}$$

The optimal control law for variable lift coefficient, equation (3.7), is used to eliminate p_3 and p_5 from this equation.

$$\frac{E^*\sqrt{\beta r}\sin\gamma}{2p_4 u}\left(Z\frac{dH}{dZ} + H\right) = f(\lambda) - \lambda\frac{df}{d\lambda} \tag{5.8}$$

This equation for H is exact since no restrictive assumptions have been made in deriving it.

There is one case for which equation (5.8) can be integrated exactly, even without using the expression for p_4 in equation (5.3). If the vehicle has a linear drag polar

$$f(\lambda) \sim \lambda \tag{5.9}$$

so that, taking the proportionality constant equal to unity for convenience,

$$C_D = C_D^*\lambda \tag{5.10}$$

then the right-hand side of equation (5.8) is zero. The integration is immediate.

This hypothetical case can still be realized, as is discussed by Contensou, Ref. 5. Even if it is not the case, for an arbitrary polar it can be assumed that near the point of maximum lift-to-drag ratio, which, as discussed earlier, is the range of optimal lift control for many general problems, the drag polar is linear (Figure 7). Thus, the right-hand side of equation (5.8) is approximately zero.

In the same spirit, an approximate law for the lift coefficient, λ , can be substituted in the right-hand side of equation (5.8) to integrate the equation. With the Contensou control, equation (4.9), equation (5.8) becomes

$$Z \frac{dH}{dZ} + H = \text{constant} \tag{5.11}$$

where the constant should be near zero since the optimal modulation of the lift should remain near the maximum lift-to-drag ratio.

If the maximum lift-to-drag ratio program is used as a first approximation to enable one to perform the integration to obtain a better approximation for the optimal lift control, equation (5.8) becomes

$$\frac{d(HZ)}{dZ} = 0 \tag{5.12}$$

which can be integrated to obtain

$$H = \frac{a_5}{\beta r\ Z} \tag{5.13}$$

where a_5 is the last constant of integration. Substitution of this H into equation (3.2), and then that into the optimal control law, equation (3.10), gives

$$\begin{aligned} f(\lambda) - (\lambda + G \cos \sigma) \frac{df}{d\lambda} &= \frac{\alpha_5 \sin \gamma}{2\sqrt{\beta r}\ Z} + E^{*}G \tan \gamma \\ &+ \frac{\cos \gamma}{2\sqrt{\beta r}\ Z}[\alpha_1 \cos \psi \cos \phi + \alpha_2(\cos \theta \sin \psi + \sin \theta \cos \psi \sin \phi) \\ &+ \alpha_3(\sin \theta \sin \psi - \cos \theta \cos \psi \sin \phi)] \end{aligned} \tag{5.14}$$

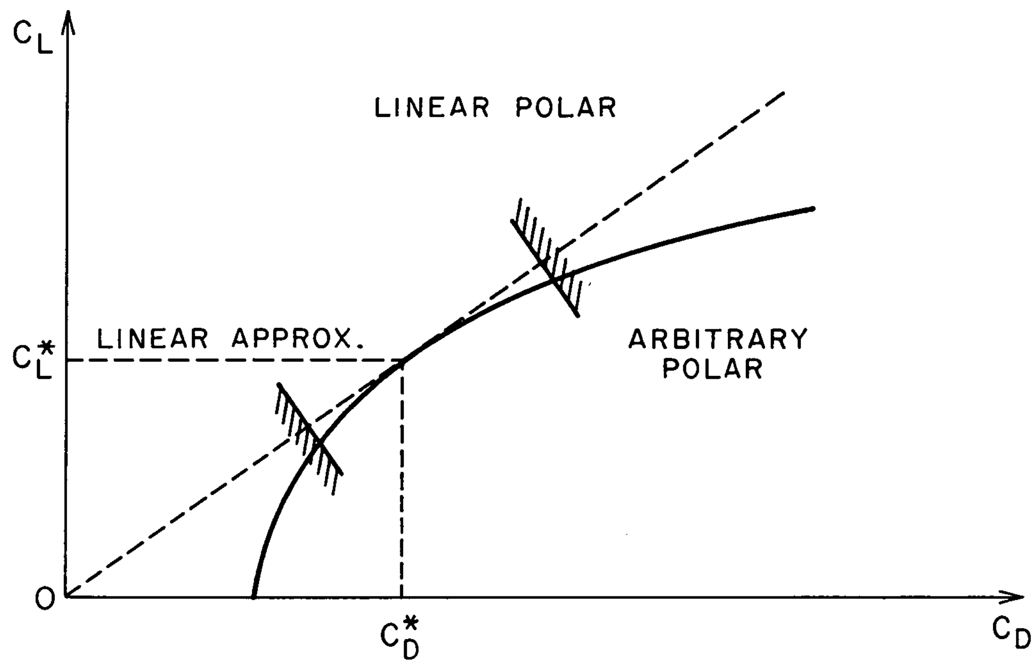

Figure 7. Drag Polar

where

$$\alpha_5 \equiv \frac{E^* a_5}{a_4} - E^* \tag{5.15}$$

Thus, equations (5.4) and (5.14) provide the optimal control laws, for variable lift coefficient control and variable bank angle control, in the general case where the final position is specified. In contrast to the simpler cases of Section 4, these show the stabilizing effect of the altitude in the Z appearing in the denominator, which tends to drive the optimal control to the maximum lift-to-drag ratio condition.

6. CONCLUSION

The analytical solution presented in the previous section is the general solution for the optimum three-dimensional aerodynamic control of a lifting vehicle entering a planetary atmosphere. Ordinarily any particular such optimal trajectory will be flown with lift and bank controls well within the capabilities of the vehicle. However, for completeness the possibilities of all four types of subarcs, equations (3.5), must be considered. Such an optimal trajectory, composed of more than one of the possible subarcs, must be pieced together using the proper corner conditions. These corner conditions, or switching laws are not discussed in this report. The simplest approach is to calculate the optimal trajectory as if there were no limits on the lift coefficient or the bank angle. The resulting optimal controls from equations (5.4) and (5.14) are compared with the realistic bounds for the vehicle, equations (2.6). If the bounds are not exceeded, then the optimal trajectory is wholly of type a) and no switching is involved.

Numerical determination of the optimal trajectory is straightforward, though in individual cases can be quite difficult. First, a functional form for the general drag polar must be chosen. After that, the problem is just the numerical solution of a boundary value problem. It involves numerical integration of the differential equations combined with a search of the four-space, $(\alpha_1 , \alpha_2 , \alpha_3 , \alpha_5)$.

Depending upon the type of problem, the approach will vary slightly because of differing sets of free or fixed end conditions. In general, the constant vector $\vec{\alpha} = (\alpha_1 , \alpha_2 , \alpha_3 , \alpha_5)$ is selected, the exact equations of motion, equations (2.19), are integrated numerically, the values of α_i are updated and the process repeated until the constrained final relations and the transversality conditions of the given problem are satisfied.

In Section 4 of this report, several earlier particular solutions employing various simplifying assumptions are shown to be special cases of this general result. This demonstrates a certain universality of the present development.

These results are comprehensive in that they apply to any problem with any given end conditions. Of particular interest is the fact that the results are valid for both constrained terminal position, and the hitherto unchallenged problem of free terminal position.

The special set of exact equations of motion are given their particularly simple and useful form through the use of the dimensionless variables referred to as modified Chapman variables. Hence, the equations are named Chapman's exact equations. They are completely free of the physical characteristics of the vehicle, and permit the use of a completely general lift-drag relationship.

The primary job remaining is to use these results to generate families of optimal three-dimensional atmospheric entry trajectories from which general qualities may be deduced.

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