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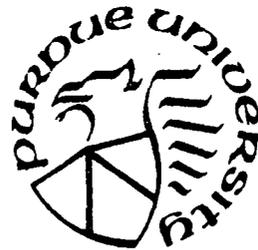
AMPLIFICATION, ATTENUATION, AND DISPERSION OF SOUND
IN INHOMOGENEOUS FLOWS

BY

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ABSTRACT

First order effects of gradients in nonuniform potential flows of a compressible gas are included in a dispersion relation for sound waves. Three nondimensional numbers, the ratio of the change in the kinetic energy in one wavelength to the thermal energy of the gas, the ratio of the change in the total energy in one wavelength to the thermal energy, and the ratio of the dilatation frequency (the rate of expansion per unit volume) to the acoustic frequency, play the dominant rôle permitting the separation of the effects of flow gradients into isotropic and anisotropic effects. Dispersion and attenuation (or amplification) of sound are proportional to the wavelength for small wavelength and depend on the direction of wave propagation relative to flow gradients. Modification of ray acoustics for the effects of flow gradients is suggested and conditions for amplification and attenuation of sound are discussed.

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INTRODUCTION

Recent interest in propagation of noise in high speed flows through variable area ducts, e.g., in turbomachinery, calls for reexamination of existing methods of treatment of acoustics of inhomogeneous flowing gas. The existing methods⁽¹⁻³⁾ are based on the zero-wavelength approximation and use wave propagation properties corresponding to a locally homogeneous (uniform) flow. The objective of this investigation is to display the phenomena which manifest themselves when otherwise small inhomogeneities are not neglected but are retained in the analysis.

The present work is applicable to cases where the boundary conditions of the mean flow induce large gradients of the the mean flow so that changes in the properties of the medium over the distance of a wavelength of sound are no longer negligible while still small enough to permit a classical (deterministic) treatment. As an example consider a plane wave propagating in the upstream direction in a flow with Mach number $M = U/a$ through a duct with a sudden constriction. The phase velocity is $a-U = a(1-M)$ and the frequency is $\nu = a(1-M)/\lambda$. If the characteristic length of the flow inhomogeneities at the constriction is $L = 1$ m, and we take the wavelength $\lambda < L/10 = 10$ cm, then we are limited to $\nu > 3(1-M) \times 10^3$ Hz. Thus the assumption of small but not negligible wavelength would limit the frequencies considered to those above about 3 kHz at low speeds. However, at, e.g., $M = 0.9$ all frequencies of upstream-propagating waves above 300 Hz could be considered in the small wavelength approximation and only those above 6 kHz for waves propagating downstream.

ANALYSIS OF THE POTENTIAL EQUATION

The potential equation for an irrotational flow of an inviscid barotropic gas, as given, e.g., by von Mises⁽⁴⁾, is

$$c^2 \nabla^2 \Phi + \frac{DP}{Dt} = 0 \quad (1)$$

where $c^2 = (\gamma-1)P =$ adiabatic speed of sound, $P = \int [\rho(p)]^{-1} dp = -(\frac{\partial \Phi}{\partial t} + \frac{1}{2} \bar{u}^2)$, $\bar{u} = \nabla \Phi$, $D/Dt = \partial/\partial t + \bar{u} \cdot \nabla$, $\gamma =$ ratio of specific heats, $p =$ hydrostatic pressure, $\rho =$ density.

The term DP/Dt may be expanded and interpreted as follows,

$$\begin{aligned} -\frac{DP}{Dt} &= \left(\frac{\partial}{\partial t} + \bar{u} \cdot \nabla\right) \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} \bar{u}^2\right) = \frac{\partial^2 \Phi}{\partial t^2} + \bar{u} \cdot \nabla \frac{\partial \Phi}{\partial t} + \bar{u} \cdot \frac{\partial}{\partial t} \nabla \Phi + \bar{u} \cdot (\bar{u} \cdot \nabla \nabla \Phi) \\ &= \frac{\partial^2 \Phi}{\partial t^2} + 2\bar{u}_o \cdot \nabla \frac{\partial \Phi}{\partial t} + \bar{u}_o \cdot \nabla (\bar{u}_o \cdot \nabla) \Phi = \left(\frac{\partial}{\partial t} + \bar{u}_o \cdot \nabla\right) \left(\frac{\partial}{\partial t} + \bar{u}_o \cdot \nabla\right) \Phi \\ &= \frac{D^2 \Phi}{Dt^2} \end{aligned}$$

where the subscript ()_o indicates the convention that the convective velocity \bar{u}_o is to be regarded as constant when operated upon with the operator D/Dt . With this convention Eq.(1) may be written as

$$c^2 \nabla^2 \Phi - \left[\frac{\partial^2 \Phi}{\partial t^2} + 2\bar{u}_o \cdot \nabla \frac{\partial \Phi}{\partial t} + \bar{u}_o \cdot \nabla (\bar{u}_o \cdot \nabla) \Phi\right] = c^2 \nabla^2 \Phi - \frac{D^2 \Phi}{Dt^2} \quad (2)$$

Small Amplitude Approximation

We shall assume that the scales of the rate of change of the medium and that of the acoustic oscillations are vastly different permitting us to separate the scales by averaging over times large as compared to the acoustic scale but small on the scale of the

slowly varying mean motion of the gas.. Let $\Phi = \bar{\Phi} + \phi$ where $\bar{\Phi}$ is the potential of the mean flow and ϕ is the perturbation potential of the sound field with zero time average. Assuming low intensity sound and neglecting squares of the perturbations, we have

$$c^2 = -(\gamma-1) \left[\frac{\partial \bar{\Phi}}{\partial t} + \frac{1}{2}(\nabla \bar{\Phi})^2 + \frac{\partial \phi}{\partial t} + \nabla \bar{\Phi} \cdot \nabla \phi \right] = a^2 - (\gamma-1) \left(\frac{\partial}{\partial t} + \nabla \bar{\Phi} \cdot \nabla \right) \phi$$

where a^2 is the adiabatic speed of sound of the mean flow. Substituting the above expression in Eq.(2) yields

$$\begin{aligned} & \left[a^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} - 2 \nabla \bar{\Phi} \cdot \frac{\partial}{\partial t} \nabla \phi - \nabla \bar{\Phi} \cdot (\nabla \bar{\Phi} \cdot \nabla \nabla \phi) \right] \\ & + \left[a^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} - 2 \nabla \bar{\Phi} \cdot \frac{\partial}{\partial t} \nabla \phi - \nabla \bar{\Phi} \cdot (\nabla \bar{\Phi} \cdot \nabla \nabla \phi) \right] \\ & = 2 \left(\frac{\partial}{\partial t} \nabla \bar{\Phi} \right) \cdot \nabla \phi + \nabla \bar{\Phi} \cdot (\nabla \bar{\Phi} \cdot \nabla \nabla \phi) + (\nabla \bar{\Phi} \cdot \nabla \nabla \phi) \cdot \nabla \phi \\ & \quad + (\gamma-1) \nabla^2 \phi \left(\frac{\partial \bar{\Phi}}{\partial t} + \nabla \bar{\Phi} \cdot \nabla \bar{\Phi} \right). \end{aligned} \quad (3)$$

If we assume that time averages of the perturbation quantities vanish, then, due to the fact that squares of the perturbations were neglected, averaging Eq.(3) leaves only the first bracketed term. Thus the potential of the mean flow, $\bar{\Phi}$, must satisfy the original potential equation, Eq.(1). Subtracting Eq.(1) from Eq.(3) we obtain

$$\begin{aligned} & a^2 \nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} - 2 \nabla \bar{\Phi} \cdot \frac{\partial}{\partial t} \nabla \phi - \nabla \bar{\Phi} \cdot (\nabla \bar{\Phi} \cdot \nabla \nabla \phi) \\ & = 2 \left(\frac{\partial}{\partial t} \nabla \bar{\Phi} \right) \cdot \nabla \phi + \nabla \bar{\Phi} \cdot (\nabla \bar{\Phi} \cdot \nabla \nabla \phi) + (\nabla \bar{\Phi} \cdot \nabla \nabla \phi) \cdot \nabla \phi \\ & \quad + (\gamma-1) \nabla^2 \phi \left(\frac{\partial \bar{\Phi}}{\partial t} + \nabla \bar{\Phi} \cdot \nabla \bar{\Phi} \right). \end{aligned} \quad (4)$$

Equation (4) is valid when squares of the perturbations are negligible. Further, Eq.(4) is a linear equation with variable coefficients, the latter being known function of the mean flow which

is independent of the perturbations. With the coefficients set equal to their instantaneous local values we may deduce valuable information concerning instantaneous and local propagation properties of the sound field and its dependence on the gradients of the mean flow.

Small Wavelength Approximation

For the purpose of a discussion we shall nondimensionalize Eq.(4) by referring $\nabla\phi$ to $U =$ local speed of the mean flow, spatial gradients of ϕ to the length scale L and those of ϕ to $\lambda =$ wavelength, and time derivatives of ϕ to U/L and time derivatives of ϕ to a/λ . Dividing by a^2/λ^2 , the nondimensional form of Eq.(4) becomes

$$\begin{aligned} \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} - 2M \nabla\phi \cdot \frac{\partial}{\partial t} \nabla\phi - M^2 \nabla\phi \cdot (\nabla\phi \cdot \nabla\nabla\phi) \\ = M^2 \frac{\lambda}{L} \left\{ 2 \left(\frac{\partial}{\partial t} \nabla\phi \right) \cdot \nabla\phi + \nabla\phi \cdot (\nabla\phi \cdot \nabla\nabla\phi) + (\nabla\phi \cdot \nabla\nabla\phi) \cdot \nabla\phi + (\gamma-1) \left(\frac{1}{M} \frac{\partial\phi}{\partial t} + \nabla\phi \cdot \nabla\phi \right) \right\}. \end{aligned}$$

It is apparent that the inhomogeneities of the mean flow, i.e., the derivatives of $\nabla\phi$, will have a negligible effect on the sound propagation if the coefficient of the terms on the right side tends to zero, that is, when

$$M^2 \frac{\lambda}{L} \rightarrow 0.$$

The above condition will be satisfied if $M \rightarrow 0$, $\lambda \rightarrow 0$, or $L \rightarrow \infty$. In the case of vanishingly small Mach number we obtain the wave equation governing the classical acoustics of stationary media,

$$\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = 0.$$

The more important case is that of flows with $M > 0$ and with the length L characterizing the spatial scale of the inhomogeneities of

the flowing medium. In the case of high speed flow the inhomogeneities of the medium will have no effects on the sound propagation if $\lambda/L \rightarrow 0$. Thus, in the limit of zero wavelength we obtain the convective wave equation,

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} - 2M \nabla \phi \cdot \frac{\partial}{\partial t} \nabla \phi - M^2 \nabla \phi \cdot (\nabla \phi \cdot \nabla \nabla \phi) = 0$$

which is the nondimensional form of Eq.(2) with the coefficients evaluated at the local values of the mean flow. Consequently, the convective wave equation should be regarded as the zero wavelength limit of the acoustics of inhomogeneous media or as a locally homogeneous ($L = \infty$) approximation for small wavelengths.

The problem of sound propagation in an inhomogeneous medium is not unlike that confronting the physicist at the turn of the century. The realization that the classical mechanics is equivalent to the zero wavelength limit of the geometrical mechanics lead to the development of the quantum mechanics. Quantum-like (probabilistic) formulation of acoustics may be necessary for the wavelengths comparable to the scale of fluid inhomogeneities $\lambda = O(L)$, e.g., for the study of noise generated by fluid turbulence, cf. Kentzer⁽⁵⁾. In the limit of infinite wavelength or infinitesimal inhomogeneities, $\lambda/L \rightarrow \infty$, various scattering theories are applicable. In this work we shall be concerned with the extension of the classical (deterministic) methods for the case of small wavelengths,

$$0 < \lambda/L \ll 1,$$

in the hope of displaying at least the first order effects of dispersion and attenuation or amplification of sound by an inhomogeneous high speed flow. Consequently, we shall retain all the terms in Eq.(4) keeping in mind the fact that the equation is valid for small wavelengths only.

DISPERSION RELATION FOR NONUNIFORM FLOWS

We observe first that Eq.(4), as it stands, could be solved numerically by the method of characteristics with the right-hand-side playing the rôle of a forcing function. However, extensive numerical calculations would have to be performed in order to study the effects of the inhomogeneities, to identify the causes, and to separate the varied effects. Further, the method of characteristics would not permit the evaluation of the changes in the noise spectrum. An insight into the effects of flow nonuniformities on the sound spectrum may be gained only by a spectral analysis of Eq.(4).

We consider now Eq.(4) with its coefficients kept constant at their local values. This permits us to treat Eq.(4) as locally linear and to obtain local propagation properties of the wave solutions with the derivatives of the mean flow as parameters. The fundamental solutions of constant coefficient linear equation are of the form

$$\phi = A \exp[i(\bar{\kappa} \cdot \bar{k} - \omega t)]$$

where we take the frequency ω as being complex in general, and where \bar{k} is the wavenumber vector. Then we may substitute

$$\nabla \phi = i\bar{k}\phi, \quad \nabla^2 \phi = -k^2 \phi, \quad \frac{\partial \phi}{\partial t} = -i\omega \phi, \quad \frac{\partial^2 \phi}{\partial t^2} = -\omega^2 \phi$$

in Eq.(4) to obtain the secular equation

$$\begin{aligned} & -a^2 k^2 + \omega^2 - 2(\bar{U} \cdot \bar{k})\omega + (\bar{U} \cdot \bar{k})^2 \\ & = .1[-2\left(\frac{\partial \bar{U}}{\partial t}\right) \cdot \bar{k} + \nabla U^2 \cdot \bar{k} + (\gamma-1)\nabla \cdot \bar{U}(-\omega + \bar{U} \cdot \bar{k})]. \end{aligned} \quad (5)$$

We introduce now the following nondimensional ratios

$$K_1 \equiv \frac{\bar{k} \cdot \nabla U^2}{a^2 k^2} = \frac{\text{change in the kinetic energy over one wavelength}}{\text{thermal energy}},$$

$$K_2 \equiv \frac{\gamma-1}{2} \frac{\nabla \cdot \bar{U}}{ak} = \frac{\text{dilatation frequency}}{\text{acoustic frequency}},$$

$$K_3 \equiv -\frac{\frac{\partial \bar{U}}{\partial t} \cdot \bar{k}}{a^2 k^2} = \frac{\nabla \left(\frac{1}{2} U^2 + P \right) \cdot \bar{k}}{a^2 k^2}$$

$$= \frac{\text{change in total energy over one wavelength}}{\text{thermal energy}}.$$

The ratios K_1 and K_3 are measures of, respectively, the convective and nonstationary rates of stretching of linear elements normal to the wave front, while K_2 measures the importance of the rate of change of material volume elements. The ratios K_1 and K_3 introduce anisotropy or directional dependence of the propagation of sound.

We may introduce three characteristic length scales,

$$L_1 = 2\pi U^2 / |\nabla U^2|, \quad L_2 = \left[\frac{4\pi U}{\gamma-1} \right] / |\nabla \cdot \bar{U}|, \quad L_3 = 2\pi U^2 / |\nabla \left(\frac{1}{2} U^2 + P \right)|.$$

Then, with $\lambda = 2\pi/k$, the assumption of small wavelength,

$$|K_1|, |K_2|, |K_3| \ll 1,$$

takes the form

$$\lambda \ll \min \left\{ \frac{L_1}{M^2}, \frac{L_2}{M}, \frac{L_3}{M^2} \right\}.$$

In terms of the nondimensional ratios the secular equation (5) becomes

$$(\omega - \bar{U} \cdot \bar{k})^2 + 2iakK_2(\omega - \bar{U} \cdot \bar{k}) - a^2 k^2 [1 + i(K_1 + 2K_3)] = 0.$$

With $\omega = \omega_r + i\omega_i$ we may separate real and imaginary parts and solve for ω_r and ω_i ,

$$\omega_r = \bar{U} \cdot \bar{k} + ak \left\{ \frac{1}{2}(1-K_2^2) \left[1 + \left\{ 1 + \frac{(K_1+2K_3)^2}{(1-K_2^2)^2} \right\}^{1/2} \right] \right\}^{1/2} = \bar{U} \cdot \bar{k} + a^*k, \quad (6)$$

$$\omega_i = -akK_2 + \frac{a^2k^2 \left(\frac{1}{2}K_1 + K_3 \right)}{a^*k}. \quad (7)$$

Since $\phi = A \exp[\omega_i t + i(\bar{x} \cdot \bar{k} - \omega_r t)]$, the initial amplitude A will be amplified if $\omega_i > 0$ and attenuated if $\omega_i < 0$. Whether sound waves are amplified or attenuated depends on the signs of K_1 , K_2 , and K_3 . We may observe at this point that in a steady flow K_1 is positive if the wave propagates in the direction of acceleration, and K_2 is positive in an expanding flow, that is, in an accelerating flow regardless of the direction of acceleration. Thus maximum amplification will occur in compression regions for waves propagating in the direction of lower pressures and higher kinetic energies of the mean flow. Such conditions exist, e.g., for sound waves propagating in an upstream direction in a diverging subsonic duct flow.

MODIFICATIONS OF GEOMETRICAL ACOUSTICS

We shall consider now corrections which, in high speed flows, should be applied to the nondispersive geometrical acoustics as developed over the years by, e.g., Blokhintsev⁽¹⁾, Kornhauser⁽²⁾, Warren⁽³⁾, and others. The corrections are due to the small terms retained on the right hand side of Eq.(4). They modify the expression for the real part of the frequency, as evidenced by the appearance of the effective speed of sound, a^* , replacing the acoustic speed a of the mean flow in Eq.(6), and introduce an exponential dependence of amplitudes on time.

The time variation of wave amplitudes implies that the acoustic energy flux in a ray tube is not constant. However, an inspection of Eq. (7) reveals that ω_1 is independent of the magnitude of the wavenumber vector, so that all rays in a given ray tube of small cross section are equally attenuated or amplified. The number of rays in a given ray tube is not conserved due to dispersion, and one must consider separately ray tubes of rays with initial wavenumbers in a small range $d\bar{k}$ centered at a given value of \bar{k} . The rays corresponding to a given initial position \bar{x}_0 and initial wavenumber \bar{k}_0 , may be constructed with the help of the group velocity. Differentiating ω_r with respect to \bar{k} we obtain for the group velocity \bar{V}_g

$$\bar{V}_g = (\bar{U} + a\bar{n}) + \frac{1}{2} \frac{a}{a^*} a\bar{n} \left\{ 1 + \sigma - 2 \frac{a^*}{a} - \frac{\bar{k} \cdot (\nabla U^2 - 2 \frac{\partial \bar{U}}{\partial t})}{4 \left[\left(\frac{a^*}{a} \right)^2 - \frac{1}{2} a^2 k^2 + 2 \left(\frac{\gamma-1}{2} \nabla \cdot \bar{U} \right)^2 \right]} \right\} + \left\{ \frac{\nabla U^2 - 2 \frac{\partial \bar{U}}{\partial t}}{8 a^* k \sigma} \right\} \quad (8)$$

where $\bar{n} = \bar{k}/k =$ unit wavenormal, and where

$$\sigma = \left\{ 1 + \frac{\bar{k} \cdot (\nabla U^2 - 2 \frac{\partial \bar{U}}{\partial t})}{a^2 k^2 - \left(\frac{\gamma-1}{2} \nabla \cdot \bar{U} \right)^2} \right\}^{1/2} .$$

The first term on the right hand side of Eq. (8) represents the group velocity in absence of inhomogeneities, the second term represents effects of the change in the apparent speed of sound and gives an anisotropic correction to the group velocity component normal to the wave front, and the last term gives an anisotropic correction in the directions of the local convective and nonstationary accelerations. The last term may be thought of as representing effects of changes in the apparent velocity of the mean flow. A superposition of these corrections is illustrated schematically in Fig. 1.

The local values of the group velocity determine a direction field, \overline{dx}/dt , which is a function of local mean flow and of its gradients as well as a function of the wavenumber.

The phase velocity, \overline{V}_p , is, by definition,

$$\overline{V}_p = \omega \overline{k} / k^2 = (\overline{U} \cdot \overline{n}) \overline{n} + a^* \overline{n} = [\overline{U} \cdot \overline{n} + a^*] \overline{n}.$$

We observe that the phase velocity \overline{V}_p is not equal to the projection of the group velocity \overline{V}_g on the wavenormal \overline{n} . The implications of this fact are discussed by Brillouin⁽⁶⁾.

CONCLUSIONS

Retaining temporal and spatial derivatives of the mean flow in the potential equation for a low intensity sound in high speed flows, and evaluating the coefficients of the potential equation at the instantaneous local values of the mean flow, we obtained a local dispersion relation which is valid for small amplitude sound field and for small, but not necessarily negligible wavelengths. Differentiation of the frequency with respect to the wavenumber defines the local group velocity which furnishes a direction field. The direction field is a function of the mean flow, its gradients and of the direction and magnitude of the wavenumber vector. The so determined direction field may be used to generalize the geometrical acoustics to the cases where the effects of local inhomogeneities of the mean flow are not negligible but small and proportional to the wavelength.

Due to the dependence of the group velocity on the magnitude and direction of the wavenumber vector, the sound propagation is dispersive in nature and anisotropic. Further, sound waves may be either ampli-

fied or attenuated depending on the direction of propagation relative to the directions of the gradients of the mean flow. As a consequence of the dispersive nature of the propagation of sound, the Huygens' principle does not apply, that is, the solution of Cauchy's problem at any point (\bar{x}, t) does not depend solely on the Cauchy data on the intersection of the initial data surface with a characteristic cone with its vertex at (\bar{x}, t) . As a matter of fact, no unique characteristic cone exists since the rays passing through a given point (\bar{x}, t) and corresponding to different wavenumbers are not confined to a particular surface, that is, the rays do not form a one-parameter family of lines.

Refraction of sound in inhomogeneous moving gas is likewise modified when temporal and spatial derivatives of the mean flow are retained. In particular, refraction of sound becomes wavelength dependent in presence of mean flow gradients and "gas prism" effects are expected to occur. The "gas prism" phenomenon will alter spectral distribution of acoustic energy. Estimation or detailed calculations of energy spectra are currently of great interest in connection with the design of tuned acoustical liners for treating jet engine intakes.

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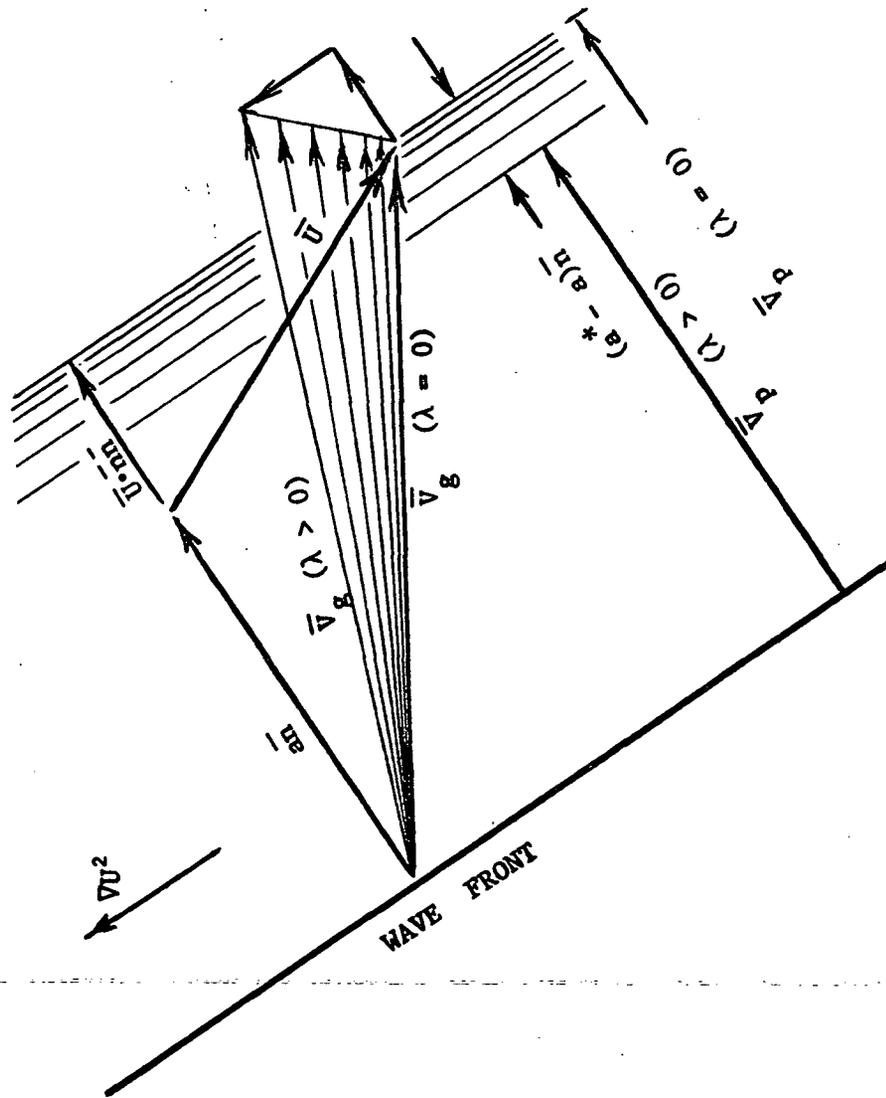


Fig. 1 A sketch showing group and phase velocities for steady potential flow with $\nabla U^2 \cdot \bar{k} = 0$ and $\nabla \cdot \bar{U} \neq 0$.