Almost Periodic Solutions to Difference Equations

Alvin Bayliss

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Abstract

The theory of Massera and Schäffer relating the existence of unique almost periodic solutions of an inhomogeneous linear equation to an exponential dichotomy for the homogeneous equation has been completely extended to discretizations by a strongly stable difference scheme. In addition it has been shown that the almost periodic sequence solution will converge to the differential equation solution at a rate $O(k^p)$ where $p$ is the accuracy of the scheme, uniformly in $t$, if the coefficients are sufficiently smooth.

The preceding theory has also been applied to a class of exponentially stable partial differential equations to which one can apply the Hille-Yoshida Theorem. It is possible to prove the existence of unique almost periodic solutions of the inhomogeneous equation which can be approximated by almost periodic sequences which are the solutions to appropriate discretizations. Two methods of discretizations are discussed; the strongly stable scheme described above and the Lax-Wendroff scheme.
Introduction

This work extends certain facets of the theory of Massera and Schaffer [1] (in the future referred to as M&S) relating properties of the solution of an inhomogeneous ordinary differential equation (ODE) to properties of the solutions of the homogeneous system. In Part I we extend this theory to difference approximations of the ODE. In Part II we use the ODE theory, exclusively in the stable case, together with the Hille-Yoshida Theorem to obtain results for partial differential equations.

The study of admissibility theory and dichotomy theory for difference equations was first done by Coffman and Schäffer [2]. This work differs from their work in that we are concerned with the preservation of admissibility and dichotomy properties by a difference equation which is used to approximate an ODE system which has certain of these properties.

First let us review the theory of M&S. Consider the equations

\[ \dot{y} = A(t)y \]  \hspace{1cm} (0.1)

\[ \dot{y} = A(t)y + f(t) \]  \hspace{1cm} (0.2)

Here the independent variable \( t \) ranges over the whole real line and for each \( t \) the vector \( y(t) \) lies in a Banach space \( E \) (which may be infinite dimensional). \( A(t) \), for each fixed \( t \), belongs to the space of bounded operators on \( E \).
which we will denote by $L(E,E)$. Let $B$ and $D$ be two Banach spaces of a function from the real line into $E$. We say the pair $(B,D)$ is admissible for (0.2) if

a) $\forall f \in B \exists$ a unique solution $y$ (to (0.2)) $\in D$, and

b) $\|y\|_D \leq K\|f\|_B$.

This definition is more restrictive than that of M&S. We are not going to deal with all the subtleties of their theory, but just with certain important parts. It is pointed out, however, that (0.3b) is actually superfluous (see M&S, Chapter 5) but we include it in the definition of admissibility for simplicity.

Observe that (0.3b) states that if we write the $y$ given in (0.3a) as

(0.4) $\quad y = C(f)$

then $C$, which is obviously a linear operator from $B$ into $D$, is also bounded. We will be concerned almost exclusively with the case that $D = B = \tilde{A}(E)$ where $\tilde{A}$ is the space of almost periodic (AP) functions with range in $E$. The range in this notation will generally be omitted and we will simply write $\tilde{A}$, etc., if no confusion might arise. At times we will also be concerned with the cases $B = D = L_\infty$ or $B = D = C_\infty$ which we define as the space of bounded continuous functions with range in $E$. Observe that $A$ and $C_\infty$
are closed subspaces of $L_\infty$ and we have the inclusions

\[ A \subset C_\infty \subset L_\infty \]

$A(t)$ will always be an AP or $L_\infty$ operator function unless stated otherwise.

Associated with the concept of admissibility we define the concept of an exponential dichotomy for (0.1). Specifically (0.1) has an exponential dichotomy if there exist projections $P_1, P_2 = I - P_1$ such that if $Y(t)$ is the fundamental operator solution to (0.1) ($Y(0) = I$, see M&S, Chapter 3), the following estimates hold for some $\alpha > 0$.

\[\begin{align*}
    & a) \quad \|Y(t)P_1Y^{-1}(s)\| \leq Ke^{-\alpha(t-s)}, \quad t \leq s \\
    & b) \quad \|Y(t)P_2Y^{-1}(s)\| \leq Ke^{-\alpha(s-t)}, \quad s \geq t.
\end{align*}\]

Note that this is not the general definition of an exponential dichotomy given in M&S (Chapter 8) but it is equivalent to their definition when $E$ is finite dimensional. In the case that $E$ is infinite dimensional we will in general only be concerned with the stable case; that is, $P_1 = I, P_2 = 0$.

Observe that $P_1$ is merely the projection on $E_+$, the space of initial data of solutions to (0.1) which are bounded for $t \in [0, \infty)$. To see this note that if $y_0$ were in the range of $P_2$ and if the corresponding solution $y(s)$ were bounded for $s \geq 0$ then, by setting $t = 0$ in (0.6b), we obtain
and \( y_0 = 0 \) follows from letting \( s \to \infty \). Similarly \( P_2 \) is just the projection on \( E_- \), the space of initial data which are bounded for \( t \in (\infty, 0] \). An exponential dichotomy merely states that \( E = E_+ \oplus E_- \) and that the solutions, in addition to being bounded, decay exponentially.

Now it is a fundamental result of M&S that if \( A(t) \in \tilde{A} \) then \((\tilde{A}, \tilde{A})\) is admissible for (0.2) iff (0.1) has an exponential dichotomy. The preceding statement is also true if \( \tilde{A} \) is replaced everywhere by \( L_\infty \). We prove here the easy part of the result; namely if we have an exponential dichotomy, then we have admissibility. If \( f \in L_\infty \) define

\[
y(t) = C(f) = \int_{-\infty}^{\infty} G(t,s) f(s) \, ds
\]

where

\[
G(t,s) = Y(t)P_1 Y^{-1}(s), \quad t \geq s
\]

\[
= -Y(t)P_2 Y^{-1}(s), \quad s \geq t.
\]

Observe that the estimate

\[
\|G(t,s)\| \leq Ke^{-a|t-s|}
\]

holds by the definition of an exponential dichotomy.

Using (0.10) we see that the integral in (0.8) exists, that \( y \) satisfies (0.2), and that \( y \) is the unique \( L_\infty \) solution to (0.2) and in fact
So that \( C \), defined in (0.8), is a bounded operator from \( L^\infty(E) \to L^\infty(E) \). To see that \( C : \tilde{A} \to \tilde{A} \) we merely let \( T \) be a common \( \varepsilon \)-almost period for \( f \) and \( A \). Then \( w(t) = y(t+T) - y(t) \) is the unique \( L^\infty \) solution to (0.2) with inhomogeneous term

\[
(A(t+T)-A(t))y(t+T) + f(t+T) - f(t) = O(\varepsilon)
\]

We then use (0.11) to obtain

\[
\|y(t+T) - y(t)\|_\infty = O(\varepsilon).
\]

The proof of the converse is more difficult and can be found in M&S, Chapter 10. A simplified proof valid only in the finite dimensional case is given in the Appendix.

In Section 1 we will define almost periodic sequences, which will be the type of solution we will be searching for. Section 2 is the most important of this work. Here, after discussing the properties of the strongly stable difference schemes we will be using, we will introduce a transformation which will separate out the roots of the scheme inside the unit circle and permit us to work with the Euler 1-step scheme. This technique was originally developed by Engquist [3] although the author was not aware of his work when the formulation given in Section 2 was developed. Engquist's results will be discussed more thoroughly at the end of Section 3.
In Section 3 we settle the question of admissibility when the homogeneous system is exponentially stable. Using an inequality which is an exact discrete analogue to the Gronwall inequality, we will show that the homogeneous difference equation is also exponentially stable. From there it will be a simple matter to obtain, for any sufficiently small time step $k$, the existence of an almost periodic sequence as a solution to the inhomogeneous difference equation. Furthermore the sequences converge uniformly to the unique AP solution to the ODE with a uniform error $O(k^p)$ where $p$ is the order of accuracy of the scheme. We will also show that this solution can in fact be calculated, i.e. it is stable under roundoff errors and errors in initial data.

In Sections 4 and 5 we deal with the case that the homogeneous system has a general exponential dichotomy. We will show that the corresponding inhomogeneous difference equation also has an exponential dichotomy. This is only of theoretical interest as the solution will no longer be stable under roundoff errors or errors in initial data. In Section 6 we will deal with some miscellaneous topics, especially the convergence of the mean value of the AP sequence to the mean value of the AP solution, and also the weakly nonlinear case.

In Part II we extend this theory to a simple class of partial differential equations which can be written as an evolution equation.
\begin{equation}
\dot{y} = (B - \delta)y + f
\end{equation}

where $B$ is an unbounded operator which satisfies the conditions of the Hille-Yoshida Theorem and $\delta(t)$ is an AP function such that the homogeneous system is exponentially stable. We can obtain a unique AP solution to (0.14) by using the formula which would be valid if $B$ were bounded and then showing that under mild restrictions on $f$ the resultant function does in fact satisfy (0.14).

In Section 8 we introduce a family of bounded operators $B_h$ which are spatial discretizations to $B$. We construct functions $y_h$ which are the unique AP solutions to

\begin{equation}
\dot{y}_h = (B_h - \delta)y_h + f.
\end{equation}

We will give conditions to insure that

\begin{equation}
\| (y- y_h) \|_\infty = O(h^j),
\end{equation}

where $j$ is the order of the approximation of $B_h$ to $B$. In Section 9 we apply the theory of Part I to (0.15) to obtain an AP sequence $y_{n;h}$ which approximates $y_h$. We have however the unfortunate restriction

\begin{equation}
\frac{k}{h^{2m}} = O(1)
\end{equation}

where $\|B_h\| = O(1/h^{m})$. A more favorable result

\begin{equation}
\frac{k}{h^{m}} = O(1)
\end{equation}

is obtained in Section 10 when using the Lax-Wendroff scheme.
In Section 11 we will consider the extension of these results in the case that the operator $B$ is perturbed by some bounded AP perturbation $D(t)$.

Finally in the Appendix we will give a proof of the basic ODE theorem that admissibility is equivalent to the existence of an exponential dichotomy. The proof is valid only in the finite dimensional case but is simpler than the proof given in M&S and is also simpler than a finite dimensional proof to be found in Coppel [4].
1. Almost Periodic Sequences

Our first task is to introduce the discrete analogue of an AP function. Following Corduneanu [5] we define an almost periodic (AP) sequence \(a_n\) with range in \(E\) as follows:

\[a_n\text{ is AP iff given } \epsilon > 0 \text{ there exists a length } L(\epsilon) \text{ (a positive integer) such that in any sequence of } L \text{ consecutive integers there exists an } N \text{ such that}
\]

\[\|a_{n+N} - a_n\|_\infty < \epsilon\]  

where the sup in (1.1) is taken over \(n\). As shown in Corduneanu (page 45) this is equivalent to normality i.e. given any sequence of integers \(N_i\) the sequence \(b_{n;i} = a_{n+N_i}\) will have a uniformly convergent subsequence. Although the proof given in Corduneanu is stated for scalar valued sequences this proof is obviously valid if the range is any Banach space \(E\).

If we define the space \(L^1_\infty(E)\) as the Banach space of bounded sequences with range in \(E\) then the AP sequences \(\tilde{A}^n(E)\) form a closed subspace of \(L^1_\infty(E)\). As usual the argument \(E\) will be omitted when no confusion can arise.

We point out that normality can be used, exactly as in the continuous case, to show that for any finite set of AP sequences \(a^1_n, \ldots, a^r_n\) with range in possibly different spaces \(E^1, \ldots, E^r\) and for any \(\epsilon > 0\), there is always a length \(L(\epsilon)\)
such that in any interval of length $L$ we can find a common $\varepsilon$-almost period.

Finally we note that if $f(t)$ is an AP function then the sequence $f_n = f(nk)$ is an AP sequence for any real $k$. The converse is also true as shown by Corduneanu (page 47) but we shall not use that.
2. Properties of the Difference Scheme

We consider linear $\ell$-step, strongly stable schemes described as follows.

\begin{equation}
(2.1) \quad \sum_{j=0}^{\ell} \alpha_j y_{n+j} = k \sum_{j=0}^{\ell} \beta_j y_{n+j} = k \sum_{j=0}^{\ell} \beta_j [A_{n+j} y_{n+j} + f_{n+j}]
\end{equation}

Here we assume that we are discretizing the ODE

\begin{equation}
(2.2) \quad \dot{y} = A(t)y + f(t)
\end{equation}

and $y_n = y(nk)$, $A_n = A(nk)$, $f_n = f(nk)$ where $k$ is the time step.

Associated with (2.1) we have the polynomials

\begin{equation}
(2.3) \quad p(x) = \sum_{j=0}^{\ell} \alpha_j x^j, \quad \sigma(x) = \sum_{j=0}^{\ell} \beta_j x^j.
\end{equation}

It is well known (see D"ahlquist [6] or Heinrici [7]) that consistency implies that $x = 1$ is a simple root of $p(x) = 0$ and that

\begin{equation}
(2.4) \quad p'(1) = \sigma(1) = 1
\end{equation}

where we have normalized the coefficients so that the common value in (2.4) is one.

Strong stability of the scheme is achieved by restricting the size of the other $\ell-1$ roots of $p(x) = 0$. Specifically, if we number these roots $x_u$, $u = 1, \ldots, \ell$, then we require

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that there be a positive number $\theta < 1$, such that

(2.5) \[ x_1 = 1, \quad |x_u| < \theta < 1, \quad u = 2, \ldots, \ell. \]

Finally, we make two further assumptions.

(2.6) \[ x_u \text{ distinct}, \quad u = 2, \ldots, \ell, \]

(2.7) \[ x_u \neq 0, \quad u = 2, \ldots, \ell. \]

(2.7) is necessary because to get an AP solution, the difference equation should be solved backwards and forwards. This assumption can be removed in the important case that the homogeneous ODE system is exponentially stable. Condition (2.6) can be removed in all cases and is included here only to simplify the following proofs. The removal of these conditions will be discussed in Section 6.

Now in working with a multistep scheme, the standard procedure is to convert it into a one-step scheme. To do this we define the space $E_{\ell} = E \times E \times \cdots \times E$. (We will usually write vectors in $E_{\ell}$ in column vector form.) We give $E_{\ell}$ the norm inherited from this definition, namely if $w \in E_{\ell}$ and

(2.8) \[ w = \begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix} \]

then

(2.9) \[ \|w\|_E = \max_{i} \|y_i\|_E. \]

Here we have explicitly indicated the $E$ norm in (2.9).
We now consider the discretization of the linear
inhomogeneous system (2.2). If $w_n = \begin{pmatrix} y_{n+1} \\ \vdots \\ y_n \end{pmatrix}$ then we get
\[(2.10) \quad w_{n+1} = C_n w_n + k\tilde{f}_n.\]

Here if we define $z_n = k A_n$,
\[(2.11) \quad C_n = \tilde{C}(z_n, \ldots, z_{n+\ell}) = \frac{(\alpha_\ell I - \beta_\ell z_{n+\ell})^{-1} (\beta_\ell z_{n+\ell} - \alpha_\ell I)}{(\alpha_\ell I - \beta_\ell z_{n+\ell})^{-1} (\beta_\ell z_n - \alpha_\ell I)}.
\]

Observe that $C_n \in \mathbb{L}(E, E)$ and will be just an $m \times m$ matrix in the case that $E$ is an $m$-dimensional space (in which case it is called the companion matrix). Note also that since
A is uniformly bounded in $t$, the implicit term $(\alpha I - \beta z_{n+1})$ can be inverted, for $k$ sufficiently small, uniformly in $n$.

We finally point out that if $A$ and $f$ are AP ($L_\infty$) then $C_n$ and $\tilde{f}_n$ are AP ($L_\infty$) and also that $\tilde{C}$ given in (2.11) is a smooth function (in the Frechet sense) of its $k+1$ arguments.

Now with $A$ a constant and $z = kA \in L(E,E)$ we consider for small $\|z\|$ the homogeneous difference scheme

(2.12) \[ w_{n+1} = U(z)w_n \]

\[
U(z) = (\alpha I - \beta z)^{-1}(\beta_1 z - \alpha_{k-1} I) \ldots (\alpha_{k-1} I - \beta I)^{-1}(\beta_0 z - \alpha_0 I)
\]

\[
\begin{array}{ccccccc}
  I & 0 & \cdots & \cdots & \cdots & 0 \\
  0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \cdots & \cdots & I & 0 & \cdots \\
  \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & \cdots & \cdots & \cdots & 0 & I & 0
\end{array}
\]

$U(z)$ is a mapping, defined for small $\|z\|$, from $L(E,E)$ into $L(E,L(E))$. Observe the following properties of $U(z)$,

(2.13) (a) $U(z)$ is a smooth (in the Frechet sense) function of $z$ 

(b) $U(0)$ has eigenvalues exactly $x_u$ with eigenspaces $E_u$, where

\[
E_u = \begin{bmatrix} x_u \cdot y \\ \vdots \\ x_u \cdot y \\ \vdots \\ x_u \cdot y \end{bmatrix}, \quad y \in E
\]

(c) $E = \bigoplus \sum_{u=1}^{k} E_u$.  

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Now (a) and (b) should be clear. To see (c), note that the spaces $E_u$ are closed and also that any two have only zero in common. Let $w = \sum x_1^{k-1} y_{1u}$ and suppose $w$ has an expansion

$$w = \sum_{u=1}^{\ell} \begin{pmatrix} x_1^{k-1} y_{1u} \\ \vdots \\ x_1^{k-1} y_{1u} \\ y_{1u} \end{pmatrix}$$

Then if we define the vector $\bar{w}$ to be $\left( \begin{array}{c} y_1 \\ \vdots \\ y_{\ell} \end{array} \right)$ we can write

$$w = V \bar{w}$$

$$V = \begin{pmatrix} x_1^{k-1} & \cdots & x_1^{k-1} \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_\ell \\ 1 & \cdots & 1 \end{pmatrix}$$

i.e. $V$ is just the Vandermonde matrix associated with the distinct numbers $x_1, \ldots, x_\ell$. (2.13c) now follows directly from the invertibility of $V$.

The representations (2.14) and (2.15) enable us to define an equivalent norm on the space $E_\ell$. Specifically if $w$ is expressed as in (2.14) we define

$$\|w\|_u = \max_u \|y_u\|_E$$

That $\|\|_u$ is equivalent to $\|\|_\ell$ follows immediately from the representation (2.15) and the invertibility of $V$. Note that the equivalence of these norms implies the equivalence of the operator norms they induce on $L(E_\ell, E_\ell)$. Of course
this paragraph is superfluous in the case that E is finite dimensional.

For future use we point out that (2.13b,c) imply that \( U^{-1}(z) \) exists for small \( \|z\| \) and is smooth in \( z \).

Now define the space \( E^1 = \bigoplus_{u \geq 1} E_u \). We have \( E_\ell = E_1 \oplus E^1 \) and we note that \( E_1 \) is canonically isomorphic to \( E \). At times we will identify \( E_1 \) with \( E \) but this should not cause any confusion. With respect to the decomposition \( E = E_1 \oplus E^1 \) we see that \( U(0) \) is in block diagonal form; symbolically,

\[
U(0) = \begin{pmatrix}
D(0) & 0 \\
0 & B(0)
\end{pmatrix}
\]

here \( D(0) : E_1 \to E_1 \) and is the identity, while \( B(0) : E^1 \to E^1 \) and \( \|B(0)\|_u^u < \theta \). This is the operator norm induced on \( L(E_\ell, E_\ell) \) by the \( \|\cdot\|_u \) norm on \( E_\ell \) and follows from the fact that \( B(0) \) is just multiplication by \( x_u \) on the space \( E_u \).

We can now state the fundamental theorem of this section.

**Theorem 1.** For small \( \|z\| \) there exists an operator \( T(z) : L(E, E) \to L(E_\ell, E_\ell) \) such that

(a) \( T(0) = I \) (Identity on \( E_\ell \))

(b) \( T(z), T^{-1}(z) \) are smooth in \( z \)

(c) \( L(z) = T^{-1}(z)U(z)T(z) \) is in block diagonal form with respect to the decomposition \( E_\ell = E_1 \oplus E^1 \).

(d) Writing \( L(z) \) symbolically as

\[
L(z) = \begin{pmatrix}
D(z) & 0 \\
0 & B(z)
\end{pmatrix}
\]

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then $B(z) |_{z=0} = B(0)$ (from (2.17)) and $\|B(z)\| u < \theta$ while $D(z)$ has an expansion

\[(2.18) \quad D(z) = I + z + O(\|z\|^2)\]

where $E_1$ is identified with $E$.

Theorem 1 is basically trivial and the proof involves familiar arguments. First we note that $U(z)$ is smoothly invertible for small $\|z\|$. (The restriction "for small $\|z\|$" will not be stated explicitly in the future.)

Next we observe that $(wI - U(z))^{-1}$ exists for $w$ in a small annulus around the circle $|w| = \theta$ and this holds uniformly in $z$. This follows from (2.13b) and the geometric series.

Now define the projections

\[(2.19) \quad P(z) = \frac{1}{2\pi i} \int_{|w| = \theta} (wI - U(z))^{-1} \, dw, \quad Q(z) = I - P(z) .\]

That $P$ and $Q$ are projections is a familiar result which follows from the resolvent identity,

\[(2.20) \quad (w_1 I - U(z))^{-1} - (w_2 I - U(z))^{-1} = (w_2 - w_1)(w_1 I - U(z))^{-1} (w_2 I - U(z))^{-1},\]

calculating $P^2$ by integrating around two slightly different circles, and interchanging the order of integration. It follows from the construction that $P(z)$ and $Q(z)$ commute with $U(z)$ and that they are smooth functions of $z$. 
It is clear by applying $P(0)$ to an arbitrary vector $w \in E_\perp$ and using the expansion given in (2.15) that $P(0)$ is exactly the projection onto $E^1$ along $E_1$ and $Q(0)$ is the projection onto $E_1$ along $E^1$.

Now define

$$T(z) = P(z)P(0) + Q(z)Q(0).$$

Clearly $T(0) = I$ and $T(z)$ is smooth whence $T^{-1}(z)$ exists and is also smooth. Note that this holds in either of the norms $\| \cdot \|_u$ or $\| \cdot \|_\ell$ on $L(E_\perp, E_k)$. Let

$$L(z) = T^{-1}(z)U(z)T(z).$$

We claim $L$ is in block diagonal form with respect to the decomposition $E_\perp = E_1 \oplus E^1$. This follows directly from the fact that $P(z)$ and $Q(z)$ commute with $U(z)$. In fact let $x \in E_1$; then

$$U(z)T(z)x = U(z)Q(z)x = Q(z)U(z)x = w \text{ (say)}.$$ 

Now if $y = T^{-1}(z)w$ and $y = y_1 + y^1$ with $y_1 \in E_1$ and $y^1 \in E^1$ we have

$$w = Q(z)U(z)x = T(z)y = Q(z)y_1 + P(z)y^1$$

whence

$$P(z)y^1 = T(z)y^1 = 0$$

and so $y^1 = 0$ by the invertibility of $T(z)$. Thus $L(z)$ maps $E_1 \to E_1$ and a similar argument shows $L(z): E^1 \to E^1$. 

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If we write (using the notation of (2.17)) \( L(z) \) as

\[
L(z) = \begin{pmatrix} D(z) & 0 \\ 0 & B(z) \end{pmatrix}
\]

(2.26)

then we see that \( D(z) \) and \( B(z) \) are smooth in \( z \); \( D(0) = I \) and \( \|B(0)\|^u < \theta \). It follows immediately by continuity that

\[
\|B(z)\|^u < \theta
\]

(2.27)

for small \( \|z\| \). Now (2.27) will imply that given \( z_1, ..., z_N \) with \( \|z_1\| \) small then

\[
\| \sum_{i=1}^{N} B(z_i) \| < K \theta^N
\]

(2.28)

for a constant \( K \) independent of the \( z_i \) and \( N \).

It only remains to study the first order structure of \( D(z) \) in order to obtain (2.18). Expanding \( D(z) \) about \( z = 0 \) we can write

\[
D(z) = I + C(z) + O(\|z\|^2)
\]

(2.29)

where \( C \) is the Frechet derivative of \( D(z) \) at \( z = 0 \) and is a bounded linear map from \( L(E,E) \) into \( L(E_1,E_1) \) which we identify with \( L(E,E) \). It is necessary to show that \( C \) is the identity, and as one might expect this will follow very easily from consistency.

Let \( A \) be an arbitrary element in \( L(E,E) \) and let \( z = kA \), \( A' = C(A) \). For \( y \in E \) and identifying \( E_1 \) with \( E \), we
calculate the following limit

\[(2.30)\quad \lim_{k \to 0} D(kA)^n y = (I + kA' + O(k^2))^n y\]

for any fixed \( t > 0 \).

As one would expect this limit is \( e^{tA'} y \) because the difference scheme expressed in (2.30) is consistent with the ODE \( \dot{y} = A' y \).

More precisely since

\[(2.31)\quad \|D(kA)\| \leq 1 + kr\]

for small \( k \), where \( R \) is some fixed constant, it follows that

\[(2.32)\quad \|D(kA)^N\| \leq e^{RT} , \quad Nk \leq T , \quad 0 \leq t \leq T .\]

Let \( x_n \) solve the difference equation

\[(2.33)\quad x_{n+1} = D(kA)x_n , \quad x_0 = y ;\]

i.e. \( x_n = D(kA)^n y \). Now if \( \tilde{x}_n = e^{A'kn} y \) then \( \tilde{x}_n \) satisfies the perturbed difference equation

\[(2.34)\quad \tilde{x}_{n+1} = [D(kA) + O(k^2)]\tilde{x}_n \]

where the term \( O(k^2) \) is uniform for \( nk \leq T \). Equation (2.34) follows directly from the fact that \( \tilde{x}(t) \) satisfies the ODE

\[(2.35)\quad \dot{\tilde{x}} = A' \tilde{x} .\]
Letting \( w_n = x_n' - x_n \) we can show easily enough that

\[
(2.36) \quad \|w_n\| = O(k), \quad nk \leq T.
\]

In fact from (2.33) and (2.34) we obtain

\[
(2.37) \quad w_n = k^2 \sum_{j=1}^{n} D(kA)^{n-j} O(1) w_{j-1}.
\]

If we use (2.32) to bound the powers of \( D(kA) \) and thus \( \|x_n\| \), we then obtain (2.36) and letting \( k \to 0 \) we obtain

\[
(2.38) \quad \lim_{n \to \infty} D(kA)^n \bar{y} = e^{tA'} \bar{y}
\]

Recalling the identification of \( E_1 \) with \( E \) we define the vector \( \bar{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \) whence \( \bar{y} \in E_1 \).

Thus (2.38) can be expressed

\[
(2.39) \quad \lim_{k \to 0} L(kA)^n \bar{y} = \begin{cases}
    e^{A't}y \\
    \vdots \\
    e^{A't}y
\end{cases}
\]

Now we are going to show that \( A' = A \) by calculating this limit directly from the definition of \( L(kA) \). In fact since

\( \|U(0)\| = 1 \) it follows that \( \|U(kA)^n\| \) is uniformly bounded for \( nk \leq T \) (and of course this is also true if we replace \( \|U\| \) by \( \|U^k\| \)). Then writing

\[
(2.40) \quad L(kA)^n \bar{y} = T^{-1}(kA) U(kA)^n T(kA) \bar{y},
\]

and letting \( k \to 0 \) such that \( nk = t \), using the continuity of \( T(z) \) and the fact that \( T(0) = I \) together with the
uniform boundedness of the powers $U(kA)^n$, we are left with

$$\lim_{k \to 0} L(kA)^n \tilde{y} = \lim_{k \to 0} U(kA)^n \tilde{y} = e^{At} \tilde{y}$$

(2.41)\[ \begin{array}{ccc} nk = t & nk = t & e^{At} \tilde{y} \\ k \to 0 & k \to 0 & \end{array} \]

the last equality follows from the fact that a consistent
and stable difference scheme is convergent.

Since $A$ was arbitrary, (2.41) together with (2.39) shows
that $A = A'$, which proves that the map $C$ is the identity
and this completes the proof of Theorem 1.
3. The Exponentially Stable Case

We now assume that the homogeneous system (0.1) is exponentially stable. Thus, if \( Y(t) \) is the fundamental operator solution to (0.1) \((Y(0) = I)\), then the estimate

\[
\|Y(t)Y^{-1}(s)\| \leq Ke^{-\alpha(t-s)}, \quad t \geq s,
\]

will hold with positive constants \( K \) and \( \alpha \). In the sequel we will use \( K \) as a generic positive constant so it will appear in contexts other than (3.1).

We wish to prove a similar estimate for the homogeneous difference equation (see (2.10), (2.11))

\[
w_{n+1} = C_n w_n.
\]

Now \( C_n \) is invertible for all \( n \) by (2.7). Then (3.2) has a fundamental solution \( W_n \) \((W_0 = I)\) which is simply

\[
W_n = \prod_{i=0}^{n-1} C_i, \quad n > 0,
\]

\[
W_{-n} = \prod_{i=-1}^{-n} C_i^{-1}, \quad n > 0.
\]

It can easily be verified that the unique solution to (3.2), \( w_n \), given initial data \( w_j \) is simply

\[
w_n = W_n w_j.
\]

Note that for \( n > j \), \( W_n W_j^{-1} \) is simply

-23-
(3.5) \[ W_n \cdot W_n^{-1} = \prod_{i=j}^{n-1} C_i, \quad n > j. \]

In the stable case we will only use \( W_n \cdot W_n^{-1} \) for \( n \geq j \) and we can see that the invertibility of \( C_i \) is not required in this case and assumption (2.7) is therefore not required here.

We intend to show that for small \( k \) (this will not be stated explicitly in the future) (3.1) implies

(3.6) \[ \| W_n \cdot W_n^{-1} \| \leq Ke^{-\alpha k(n-j)}, \quad n \geq j, \]

where \( K \) is used as a generic constant and can be taken independently of \( k \), while \( \alpha_1 = \alpha + O(k) \). Thus since \( \alpha > 0 \), \( \alpha_1 \) can also be taken independent of \( k \) for small \( k \).

To attain (3.6) we suppose \( A(t) \) is \( C^1 \). All derivatives that are assumed will always be required to be \( AP \) or \( L_\infty \). This will not be stated explicitly in the future. The smoothness condition on \( A \) will be removed in Section 6.

Expanding \( A_{n+1} \) about \( A_n \) we can write (3.2) as

(3.7) \[ w_{n+1} = U(kA_n)w_n + k^2O(1)w_n. \]

Here the \( O(1) \) term is uniform in \( n \) and the operator \( U \) is given by (2.12). We are going to show that perturbations of the type \( k^2O(1) \) preserve exponential stability. We thus consider the unperturbed system

(3.8) \[ \tilde{w}_{n+1} = U(kA_n)\tilde{w}_n \]
If we let \( \bar{v}_n = T(kA_n)\bar{w}_n \), where the operator \( T \) is introduced from Theorem 1, and note that by the smoothness of \( T \) and \( A \) we have \( T(kA_{n+1}) = T(kA_n) + k^2O(1) \), we see that we should consider the difference equation

\[
(3.9) \quad v_{n+1} = L(kA_n)v_n
\]

where the operator \( L \) has been studied in Theorem 1.

If \( V_n \) denotes the fundamental solution to (3.9) then since \( L \) is block diagonal we see that \( V_n \) itself is also block diagonal. If we write \( V_n \) in the form

\[
(3.10) \quad V_n = \begin{pmatrix} V^1_n & 0 \\ 0 & V^2_n \end{pmatrix}
\]

we can see that the second part, the contribution from the roots inside the unit circle, causes no difficulty. In fact for \( n > j \) we have

\[
(3.11) \quad \| V^j_n V^{2-1}_n \| \leq K \theta^{n-j},
\]

where we have used (2.28) and \( K \) as a generic constant. Since \( \theta \) is a fixed number less than 1 we see immediately that (3.6) holds for \( V^2_n \) where \( \alpha_1 \) can in fact be taken equal to \( \alpha \) for small \( k \).

We must now deal with \( V^1_n \). Neglecting perturbations of \( k^2O(1) \) and identifying the space \( E_1 \) with \( E \) we
consider the difference equation

\[ X_{n+1} = (I + kA_n)X_n \]  

i.e. we have reduced the problem to the study of the Euler 1-step scheme.

We will compare \( X_n \), the fundamental solution to (3.12), with \( Y_n \) (= \( Y(nk) \)) where \( Y \) is the fundamental solution to the homogeneous equation (0.1).

Let \( H(t,s) = Y(t)Y^{-1}(s) \). As a function of \( t \), \( H \) satisfies the equations

\[
\begin{align*}
\dot{H} &= A(t)H \\
\ddot{H} &= A^2(t)H + \dot{A}(t)H, \\
H(t,t) &= I
\end{align*}
\]

Integrating (3.13) we obtain

\[
H((n+1)k,nk) = I + k \int_0^1 \dot{H}(nk+\theta k,nk) d\theta
\]

(3.14)

\[
= I + kA(nk) + k^2 \int_0^1 \int_0^1 \theta d\theta d\phi \ddot{H}(nk + \phi k, nk).
\]

Now since \( A \) is bounded it is a standard result (M&S, Theorem 3.1 C), that

\[
\|H(t,s)\| \leq K_1(R) \quad \text{for} \quad |t-s| \leq R,
\]

(3.15)

where \( R \) is any positive number. This does not require exponential stability and in fact is an immediate consequence of Gronwall's inequality. It now follows from the second
equation in (3.13) that the coefficient of $k^2$ in (3.14) is bounded uniformly in $n$ and $k$. This implies that $Y_n$ is the fundamental solution operator to an equation

\begin{equation}
Y_{n+1} = [I + kD_n(k)]Y_n = [I + kA_n + k^2 O(1)]Y_n
\end{equation}

where the $O(1)$ term is uniform in $n$. This means that we can regard the $x_n$ equation (3.12) as a perturbation of the $Y_n$ equation. We point out that the reduction we have obtained does not use exponential stability and so is valid in the case that (0.1) has a general exponential dichotomy. If we use the stability, however, we see that $Y_n$ satisfies the estimate

\begin{equation}
\|Y_n^{-1}\| \leq Ke^{-\alpha k(n-j)}, \quad n \geq j.
\end{equation}

We must now prove the proposition that exponential stability is preserved under $k^2$ perturbations.

**Theorem 2.** Consider two difference equations

\begin{align*}
(3.18) & \quad \tilde{Y}_{n+1} = R(n,k)\tilde{Y}_n \\
(3.19) & \quad \tilde{x}_{n+1} = R(n,k)\tilde{x}_n + kS(n,k)\tilde{x}_n
\end{align*}

defined for small $k$. Suppose that (3.18) is exponentially stable, that is there exists constants $K$ and $\alpha$ independent of $k$ such that

\begin{equation}
\|\tilde{Y}_n^{-1}\| \leq Ke^{-\alpha k(n-j)}, \quad n \geq j,
\end{equation}
where \( \tilde{y}_n \) is the fundamental solution to (3.18). We point out that we do not require \( R(n,k) \) to be invertible since we only study the solution \( \tilde{y}_n \tilde{y}_{n-j}^{-1} \) for \( n \geq j \). In that case \( \tilde{y}_n \) is not invertible and we must use (3.5) in place of \( \tilde{y}_n \tilde{y}_{n-j}^{-1} \), but we will retain this notation for simplicity.

Under this hypothesis there exists an \( \varepsilon_0 \) such that if for small \( k \),

\[
\|S(n,k)\|_\infty \leq \varepsilon \leq \varepsilon_0
\]

where the sup in (3.21) is over \( n \), then (3.19) is exponentially stable and if \( \tilde{x}_n \) is the fundamental solution to (3.19) (the dependence on \( k \) has been suppressed) the estimate

\[
\|\tilde{x}_n \tilde{x}_{n-j}^{-1}\| \leq K_1 e^{-\alpha_1 (n-j)}
\]

where \( K_1 \) can be taken independent of \( k \) and \( \alpha_1 = \alpha + O(\varepsilon) \).

Before proving this we note that the case where \( S(n,k) = O(k) \) is automatically covered. The more general formulation will be used in Section 6 and more importantly in Part II. Note also that it is a discrete analogue to the Gronwall inequality and the proof is in fact immediately suggested by the proof of the Gronwall inequality. This theorem is equivalent to a lemma of Engquist itself based on a theorem of Strang [8], but the proof given here is simpler than Strang's proof and much more suggestive of the Gronwall inequality.

To prove Theorem 2 we first note that for \( n > j \) we have

\[
\tilde{x}_n \tilde{x}_{n-j}^{-1} = \tilde{y}_n \tilde{y}_{n-j}^{-1} + k \sum_{k=j+1}^{n} \tilde{y}_n \tilde{y}_{n-j}^{-1} S_{k-1} \tilde{x}_{k-1} \tilde{x}_{j}^{-1}
\]

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This analogue of the variation of constants formula can be verified immediately. Using (3.20) we get

\[
\|X_n^{-1}X_j\| \leq K e^{-\alpha k(n-j)} + kK e^{\alpha k} \sum_{\ell=j}^{n-1} e^{-\alpha k(n-\ell)} \|X_\ell^{-1}X_j\|
\]

Here \( K \) is the constant of (3.20). If we redefine \( K \) as \( K e^{\alpha k} \) (for small \( k \)) and define \( v_n,j = e^{\alpha kn} \|X_n^{-1}X_j\| \) we can write

\[
v_{n,j} \leq K + kK e^{\sum_{\ell=j}^{n-1} v_{\ell,j}} \equiv s_{n-1,j}, \quad n > j.
\]

From (3.25) we obtain

\[
v_{n,j} = \frac{s_{n,j}-s_{n-1,j}}{kK e^{\sum_{\ell=j}^{n-1} v_{\ell,j}}} \leq s_{n-1,j}
\]

whence

\[
s_{n,j} \leq (1+eK) e^{K(n-j)}
\]

where \( K_1 \) is defined so that \( s_{j,j} \leq K_1 \) independent of \( k \) and \( j \) (from (3.25) specialized to \( n = j+1 \) and the fact that \( v_{j,j} = 1 \)).

Inequality (3.27) together with (3.25) and the definition of \( v_{n,j} \) yields (3.22) immediately and thus completes the proof of Theorem 2.

Returning to our specific case, it should be clear that after several applications of Theorem 2 we can prove exponential stability for \( \tilde{v}_n \) defined as \( T(kA_n)\tilde{w}_n \) (see (3.8)
and the following paragraph). Since in terms of fundamental solution operators we have

\[
(3.28) \quad \bar{W}_n \bar{W}_j^{-1} = T^{-1}(kA_n) \bar{V}_n \bar{V}_j^{-1} T(kA_j)
\]

we obtain exponential stability for (3.8) and another application of Theorem 2 yields exponential stability for the full homogeneous system (3.2). It is a simple matter to go from exponential stability to admissibility for the inhomogeneous difference equation

\[
(3.29) \quad w_{n+1} = C_n w_n + k g_n
\]

We first define admissibility in the obvious way.

The pair \((L^n_\infty, \tilde{L}^n_\infty)\) will be admissible for (3.29), for small \(k\), iff for any sequence \(g_n\) in \(L^n_\infty\) (3.29) has a unique solution \(w_n\) in \(L^n_\infty\) and this assignment is a bounded mapping, i.e. there exists a \(K\) (independent of \(k\)) such that

\[
(3.30) \quad \|w_n\|_\infty \leq K \|g_n\|_\infty.
\]

An entirely analogous definition holds for the admissibility of the pair \((\tilde{A}^n_\infty, \tilde{A}^n_\infty)\). As in the ODE case we will show \((L^n_\infty, \tilde{L}^n_\infty)\) is admissible and then show that the solution is AP if the coefficients are.

To show \((L^n_\infty, \tilde{L}^n_\infty)\) admissibility we merely write the solution

\[
(3.31) \quad w_n = k \sum_{j=-\infty}^{n} \bar{W}_n \bar{W}_j^{-1} g_{j-1}.
\]

Exponential stability implies the convergence of this series,
which can be immediately verified to be a solution of (3.29).

To show \( w_n \in L_\infty \) we estimate, using (3.6)

\[
\|w_n\|_\infty \leq \|g_n\|_\infty K \sum_{j=0}^{\infty} e^{-\alpha jk} = \|g_n\|_\infty K \frac{k}{1-e^{-\alpha k}} \leq \|g_n\|_\infty \frac{1}{\bar{k}}
\]

(using \( x/(1-e^{-x}) + 1 \) as \( x \to 0 \) and this yields (3.30) after a redefinition of \( k \).

To show uniqueness is trivial, if \( w_n \) were a bounded solution to the homogeneous equation (3.2) we would have for \( n > j \),

\[
w_n = W_n^{-1} w_j
\]

and \( w_n = 0 \) follows immediately upon letting \( j \to -\infty \).

It remains to show that \((\tilde{A}^n, \tilde{A}^n)\) is admissible. In fact if \( N \) is a common \( \varepsilon \)-almost period for \( C_n \) and \( g_n \), we would have \( w_n w_n+\bar{N} - w_n \) to be the unique \( L_\infty \) solution to an inhomogeneous equation with inhomogeneous term

\[
\frac{1}{k} [C_{n+N} - C_n] w_n w_n+\bar{N} + g_{n+N} - g_n
\]

and for fixed \( k > 0 \) the almost periodicity of \( w_n \) is a consequence of (3.30).

We can remove the factor \( 1/k \) in the denominator by using the expression (2.11) for \( C_n \),

\[
C_n = \bar{C}(kA_n, \ldots, kA_{n+k})
\]

where \( \bar{C} \) is a smooth function of its arguments. Expanding \( C_{n+N} - C_n \) using (3.35) we see that if \( N \) is a common \( \varepsilon \)-almost
period for $A_n$ and $f_n$, then $N$ will be an $O(\varepsilon)$-almost period of $w_n$, with the bound in the $O(\varepsilon)$ term independent of $k$.

Note that these two paragraphs follow only from $(L^n, L^n)$ admissibility of (3.29) and do not depend on the stability of (0.1).

The convergence of $w_n$ to the solution $y_n$ also follows very easily. If $y$ is the unique AP solution to (0.2) we have

\begin{equation}
\sum_j \alpha_j \gamma_{n+j} - k \sum_j \beta_j \gamma_{n+j} = \kappa \left[ \sum_j \alpha_j \int_0^1 (\dot{y}(nk+\theta kj) - \dot{y}(nk)) \, d\theta \right. \\
- \left. \sum_j \beta_j [\dot{y}(nk+kj) - \dot{y}(nk)] \right] = k \, g(k,nk) .
\end{equation}

(The summation is from 0 to $k$ in the above.) Here $g(k,t)$ is AP in $t$ for each $k$ and

\begin{equation}
\|g(k,t)\|_\infty = o(1) , \quad k \to 0 ;
\end{equation}

(3.37) follows from the uniform continuity of AP functions on the whole real axis.

Now if $\bar{w}_n = \left( \begin{array}{c} Y_{n+k-1} \\ \vdots \\ Y_n \end{array} \right)$ then (3.36) implies that $\bar{w}_n$ is the unique AP solution to the same equation as $w_n$ except for an error,

\begin{equation}
\begin{pmatrix}
\begin{array}{c}
g(k,nk) \\
0 \\
\vdots \\
0
\end{array}
\end{pmatrix}
\end{equation}

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and (3.37) together with (3.30) yield

\[ \| w_n - \bar{w}_n \| _\infty = o(1) \quad k \rightarrow 0. \]

We can obtain finer convergence results by imposing some smoothness conditions on \( A(t) \) and \( f(t) \). In fact if the scheme has order of accuracy \( p \), and \( A \) and \( f \) have \( p+1 \) derivatives, then \( y \) also has \( p+1 \) derivatives. It is shown in Henrici (p. 247) that

\[ \sum_j \alpha_j y_{n+j-k} \sum_j \beta_j y_{n+j} \leq k^{p+1} G y^{p+1} \| \_\infty, \]

where the constant \( G \) depends only on the scheme. (3.40) together with (3.30) yield

\[ \| w_n - \bar{w}_n \| _\infty = O(k^p). \]

We point out that these convergence arguments are valid whenever we have admissibility and do not require stability.

Before leaving the stable case we would like to discuss problems relating to the computability of the solution \( w_n \).

Consider first errors in initial data. Suppose we solve the exact difference equation (3.29) but use as initial data \( w_0^*(k) = w_0(k) + e(k) \) where \( w_0(k) \) is the exact initial data for the AP sequence solution \( w_n \) and \( e(k) \) is bounded for small \( k \), i.e. \( \| e(k) \| \leq e_0 \). Now the solution \( w_n^* \), which we solve for, will be

\[ w_n^* = w_n + W_n e(k), \]

and we have
Now in general we cannot expect $e(k) \to 0$ as $k \to 0$. This would certainly occur if we could obtain initial data consistent with the initial data of the solution $y$ to the ODE. However, this solution is known only by an improper integral involving not only the given forcing function $f(s)$ but also the fundamental solution $Y(t)$ to the homogeneous system which we could not expect to know explicitly unless $A$ were a constant. Thus the most we can assert is that, using fixed initial data (say $w^*_0(k) = 0$), if we integrate over a sufficiently long interval we will get the solution $w_n$ up to some exponentially decaying error.

Now let us consider round-off error. If we could postulate that we solved a perturbed equation

\begin{equation}
\label{eq:3.44}
\begin{aligned}
w^*_{n+1} &= C_n w^*_n + k f_n + O(k^{r+1}), \\
n &\geq 0, \quad r > 0
\end{aligned}
\end{equation}

where the error is uniform in $n$ then we could relate this solution to $w_n^*$ by

\begin{equation}
\label{eq:3.45}
\begin{aligned}
w^*_n &= w_n + k \sum_{j=1}^{n} W_n^{-1} O(k^r), \\
n &> 0
\end{aligned}
\end{equation}

and we can estimate the error simply,

\begin{equation}
\label{eq:3.46}
\begin{aligned}
\| k \sum_{j=1}^{n} W_n^{-1} O(k^r) \| &\leq k^r K k \sum_{r=0}^{\infty} e^{-akr} = O(k^r).
\end{aligned}
\end{equation}

Let us further examine the validity of (3.44). If we include rounding errors in forming the companion matrix (or operator), together with the error of taking only a finite
expansion of the term \((\alpha I - \beta kA_{n+1})^{-1}\) in the implicit case (see (2.11)), then we could assume that the linear part of (3.44) would be replaced by

\[(3.47) \quad C_{n}^{**} = C_{n} + O(k^{r+1}) .\]

If we also assume an error \(O(k^{r+1} \| w_{n} \|)\) in forming the product \(C_{n} w_{n}\) we see we should postulate a system

\[(3.48) w_{n+1}^{**} = C_{n}^{**} w_{n} + k g_{n} + O(k^{r+1}) + O(k^{r+1} \| w_{n}^{**} \|)\]

and in order to justify (3.44) we must show that the solution to (3.48) will be bounded for \(n \geq 0\).

We can write this solution as

\[(3.49) \quad w_{n}^{**} = W_{n} w_{0}^{**} + k \sum_{j=1}^{n} W_{n} W_{j}^{-1} g_{j-1} + k \left( \sum_{j=1}^{n} W_{n} W_{j}^{-1} (O(k^{r}) + O(k^{r} \| w_{j-1}^{**} \|)) \right)\]

Now if we use the fact that

\[(3.50) \quad k \sum_{r=0}^{\infty} e^{-\alpha kr} = O(1)\]

and define \(h_{n} = \max_{j=0, \ldots, n} \| w_{j}^{**} \|\) we can estimate \(h_{n}\) by

\[(3.51) \quad h_{n} = O(1) + k^{r} h_{n} O(1)\]

where the \(O(1)\) terms are independent of \(n\). This implies, for small \(k\), the boundedness of \(\| w_{n}^{**} \|\) and so justifies (3.44).

The techniques used in Section 2 of block diagonalizing the companion matrix were first developed by Engquist.
although I was not aware of his work when my formulation was developed.

Engquist's procedure may appear to be more complicated than the procedure presented here, but this is because he proves a more general result; namely that the companion matrix can be block diagonalized to within an error of arbitrary order (say $O(k^{p_1})$). (He also does not restrict himself to strongly stable schemes, but this imposes stability requirements on several additional homogeneous systems.) If one were to consider his procedure restricted to a block diagonalization up to $O(k^2)$ then, while the two formulations differ in the lines of approach, they are equally simple.

Engquist also shows that the upper block will agree with the Taylor series expansion of (0.1), up to terms of order $p$, if $p < p_1$ is the accuracy of the scheme. The proof of the first order structure of the upper block (the term $D(z)$ in Section 2) is, I believe, somewhat simpler than his proof restricted to the first order term.

I would like to point out that in the case we are considering (an exponentially stable ODE and a strongly stable scheme) the extra fineness of the block diagonalization being carried to order $p+1$ will only give the advantage of the homogeneous system having a stability exponent $-\alpha + O(k^p)$ instead of $-\alpha + O(k)$. In particular in Engquist's study of the uniform convergence (for $t > 0$) of the solutions of the homogeneous equation to the solution to the ODE (see his
Theorem 3, p. 24-27 and note that what he uses as $\alpha$ is what we call $-\alpha$). One would still obtain uniform convergence of $O(k^p)$ (neglecting round-off and errors in initial data), however the decay exponent would be $-\alpha+O(k)$ instead of $-\alpha+O(k^p)$. 
4. Admissibility in the Case of a General Exponential Dichotomy

Here we are concerned with admissibility properties in the case that the homogeneous equation (0.1) has a general exponential dichotomy as described by (0.6) with $p_2 \neq 0$. The results will be valid in the infinite dimensional case, although in this case one must bear in mind that the situation described by equations (0.6) is not the most general form of an exponential dichotomy.

First of all consider arbitrary homogeneous and inhomogeneous difference equations

\begin{align*}
(4.1) & \quad w_{n+1} = C_n(k)w_n \\
(4.2) & \quad w_{n+1} = C_n(k)w_n + kg_n .
\end{align*}

Here $C_n$ is defined for $k \in (0,k_0]$ and is AP in $n$ for fixed $k$, and is now assumed invertible so that a fundamental solution operator $W_n$, as described by (3.3), exists and is invertible for all $n$.

Associated with (4.2) we have variation of constants formulas in both the forward and backward directions,

\begin{align*}
(a) & \quad w_n = W_nw_0 + k \sum_{j=1}^{n} W_nW_j^{-1}g_{j-1} , \quad n > 0 \\
(4.5) & \quad w_n = W_nw_0 - k \sum_{j=n+1}^{0} W_nW_j^{-1}g_{j-1} , \quad n < 0 \\
(b) & \quad \text{as can be easily verified.}
\end{align*}
Now we will define an exponential dichotomy for (4.1) as follows. Suppose there exist projections \( \bar{P}_1(k) \), \( \bar{P}_2(k) = I - \bar{P}_1(k) \), such that the estimates

\[
\| W_n \bar{P}_1 W_j^{-1} \| \leq K e^{-\alpha k(n-j)}, \quad n \geq j,
\]

(4.6)

\[
\| W_n \bar{P}_2 W_j^{-1} \| \leq K e^{-\alpha k(j-n)}, \quad j \geq n,
\]

hold with \( K \) and \( \alpha \) independent of \( k \). The only difference between this and the ODE case is that the projections may depend on \( k \).

If we have an exponential dichotomy then \( (L_n^\infty, L_n^\infty) \) is admissible for (4.2). In fact the unique bounded solution \( w_n \) is

\[
w_n(k) = k \sum_{j=-\infty}^{\infty} G_j^n g_{j-1}
\]

(4.7)

where

\[
G_j^n = W_n \bar{P}_1 W_j^{-1}, \quad n \geq j
\]

(4.8)

\[
= -W_n \bar{P}_2 W_j^{-1}, \quad j \geq n+1.
\]

Equation (4.7) is of course suggested by the ODE case and is easily verified using (4.6). The admissibility bound (3.30) can be derived exactly as in the stable case as can \((\bar{A}^n, \bar{A}^n)\) admissibility.

We next observe that \((L_n^\infty, L_n^\infty)\) (or \((\bar{A}^n, \bar{A}^n)\)) admissibility is preserved under perturbations of the linear term. In fact if we had a system
\[(4.9) \quad W_{n+1} = C_n(k)W_n + kS_n(k)W_n + k\tilde{g}_n\]

where, for small \(k\),

\[(4.10) \quad \|S_n(k)\|_\infty \leq \epsilon_0\]

and \(\epsilon_0\) is some number to be determined, then the unique \(L^n\) solution can be found by defining \(W_{n;0} = 0\) and \(W_{n;i+1}\) to be the unique \(L^n\) solution to

\[(4.11) \quad W_{n+1;i+1} = C_nW_{n,i+1} + kS_nW_{n,i} + k\tilde{g}_n\]

(for simplicity the dependence of \(C_n\) and \(S_n\) on \(k\) has been suppressed). If \(\epsilon_0 K_1 \leq 1\), where \(K_1\) is the admissibility bound for (4.2), the contracting mapping principle establishes \((L^n, L^n)\) admissibility with admissibility bound

\[(4.12) \quad K_2 = \frac{K_1}{1 - \epsilon_0 K_1}\]

Of course we will get AP solutions if \(D_n + kS_n\) and \(g_n\) are AP.

Now these two principles certainly settle the question of admissibility when the homogeneous system has a general exponential dichotomy. Referring to the reduction obtained in Section 3 we see that after a nonsingular change of dependent variable, \(V_n = T(kA_n)w_n\), \(V_n\) satisfies the equation (see (3.8) ff.);

\[(4.13) \quad V_{n+1} = \begin{pmatrix} I + D_n(k) & 0 \\ 0 & B_n \end{pmatrix} V_n + k^2 O(1)V_n + k\tilde{g}_n\]

Here \(\tilde{g}_n = T(kA_{n+1})g_n\). By the boundedness of \(T\) and \(T^{-1}\),

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admissibility for (4.13) is equivalent to admissibility for (4.2). But (4.13) expresses the homogeneous term as a perturbation of the system

\[ X_{n+1} = \begin{pmatrix} I+D_n & 0 \\ 0 & B_n \end{pmatrix} X_n \]

with fundamental solution

\[ X_n = \begin{pmatrix} Y_n \\ 0 \end{pmatrix} \prod_{i=1}^{n-1} B_i, \quad n > 0, \]

(4.15)

\[ X_{-n} = \begin{pmatrix} Y_{-n} \\ 0 \end{pmatrix} \prod_{i=0}^{-n+1} B_i, \quad n > 0, \]

and this has an exponential dichotomy with projections independent of \( k \) and in fact (after identifying \( E_1 \) with \( E \))

\[ \overline{P}_1 = P_1 \otimes P(0), \]

(4.16)

\[ \overline{P}_2 = P_2 \]

Here \( P_1 \) and \( P_2 \) are defined in (0.6) while \( P(0) \) is the projection onto \( E_1 \) along \( E \), (see (2.19) ff).

We thus obtain admissibility in this case and we point out that the convergence proofs given in Section 3 did not require stability and remain valid here.
5. **Existence of an Exponential Dichotomy for the Homogeneous Linear Equation**

We will prove here that admissibility for the inhomogeneous difference equation

\[(5.1) \quad w_{n+1} = C_n w_n + k g_n \]

implies an exponential dichotomy for the homogeneous equation

\[(5.2) \quad w_{n+1} = C_n w_n \]

The proof is very similar to a proof of the analogous property for the ODE case given in the Appendix. It is a strictly finite dimensional proof and it may then be simpler to think in terms of matrices rather than linear operators. We can then regard \(T(z)\) as a similarity transformation which follows an initial transformation putting \(U(0)\) into block diagonal form. The matrix \(L(z)\) is strictly in block diagonal form and it is no longer necessary for us to continually make the qualification "identifying \(E_1\) with \(E\)."

If we let \(v_n = T(k A_n) w_n\), (5.1) is transformed into (see (4.13))

\[(5.3) \quad v_{n+1} = \begin{pmatrix} I + k D_n & 0 \\ 0 & B_n \end{pmatrix} v_n + k^2 O(1) v_n + k g_n \]

(For simplicity we will write the inhomogeneous term in (5.13) as \(g_n\) rather than \(\tilde{g}_n\)).
If we call the leading matrix in (5.3) $R_n(k)$, then the unperturbed linear system

\[(5.4) \quad x_{n+1} = R_n x_n\]

has an exponential dichotomy with projections $p_1$ and $p_2$, independent of $k$ (see (4.15) and (4.16)), and we must exhibit an exponential dichotomy for the homogeneous version of (5.3).

\[(5.5) \quad v_{n+1} = R_n v_n + k^2 O(1)v_n.\]

By admissibility (5.5) can have no $L^\infty$ solution. Let $S_+(k)$ and $S_-(k)$ be the subspaces of initial data which give rise to solutions bounded for positive and negative $n$ respectively. We have $S_+ \cap S_- = \{0\}$ (suppressing the $k$ dependence). If we let $S$ be any complementary space we have

\[(5.6) \quad E_k = S_+ \oplus S_- \oplus S\]

with associated projections

\[(5.7) \quad I = P_1 + P_2 + P_3\]

Our first task is to show $S = \{0\}$ i.e. $P_3 = 0$.

Define $\mathcal{L}_n^\infty$ to be the submanifold of $L_n^\infty$ of sequences with only finitely many nonzero components. Let $g_n \in \mathcal{L}_n^\infty$ and let $v_n$ be the corresponding unique $L_n^\infty$ solution to (5.3). Then using (4.5a), we see that for large positive $n$ we have

\[(5.8) \quad v_n = v_0 + k \sum_{j=1}^{\infty} v_j^{-1} g_{j-1}\]
and since \( v_n \) is bounded the vector in the brackets lies in \( S_+ \) i.e.

\[
\begin{align*}
(a) \quad & P_2 v_0 = -k \sum_{j=1}^{\infty} P_2 v_j^{-1} g_{j-1} \\
(5.9) & \\
(b) \quad & P_3 v_0 = -k \sum_{j=1}^{\infty} P_3 v_j^{-1} g_{j-1}
\end{align*}
\]

Similarly looking at \( v_n \) for \( n \to -\infty \) and using (4.5b) we obtain

\[
\begin{align*}
(a) \quad & P_1 v_0 = k \sum_{j=-\infty}^{0} P_1 v_j^{-1} g_{j-1} \\
(5.10) & \\
(b) \quad & P_3 v_0 = k \sum_{j=-\infty}^{0} P_3 v_j^{-1} g_{j-1}
\end{align*}
\]

Now if we simply set

\[
g_0 = V_1 P_3 z
\]

\[
(5.11)
\]

\[
g_i = 0 \quad , \quad i \neq 0 ,
\]

where \( z \) is an arbitrary vector in \( E \) we see immediately that \( P_3 = 0 \). We can also see that for any given \( g_n \in L_\infty^n \) the unique \( L_\infty^n \) solution \( v_n \) is given by

\[
v_n = k \sum_{j=-\infty}^{\infty} G^n_j g_{j-1}
\]

where \( G^n_j \) is given by (see (4.8))

\[
G^n_j = V_n P_1 v_j^{-1} , \quad n \geq j
\]

\[
= -V_n P_2 v_j^{-1} , \quad j \geq n+1 .
\]
It is of course $G^n_j$ which we wish to prove is exponentially damped.

Let $\tilde{G}^n_j$ be the Green's function associated with (5.4). We know

\[(5.14) \quad \|\tilde{G}^n_j\| \leq K e^{-\alpha k|n-j|}\]

Now define $H^n_j = G^n_j - \tilde{G}^n_j$. If $g_j \in L^\infty$ and $v_n$ the corresponding solution to (5.3), while $x_n$ is the $L^\infty$ solution to the inhomogeneous version of (5.4),

\[(5.15) \quad x_{n+1} = R_n x_n + k g_n .\]

Then $u_n = v_n - x_n$ is the unique bounded solution to

\[(5.16) \quad u_{n+1} = R_n u_n + k^2 o(1) v_n .\]

We thus have

\[(5.17) \quad u_n = k \sum_{j=-\infty}^{\infty} H^n_j g_{j-1} = k^2 \sum_{j=-\infty}^{\infty} \tilde{G}^n_j o(1) v_{j-1} .\]

If we fix $i$ and let $g_{i-1} = z$, $g_j = 0$ for $j \neq i-1$ where $z$ is an arbitrary vector in $E_k$, we then obtain, using (5.14) and the admissibility bound for (5.3),

\[(5.18) \quad \|H^n_j\| \leq K_1 , \quad \text{for all } n, i\]

where $K_1$ is some constant independent of $k$. (The restriction "for small $k" is always understood.) The use of (5.14) then implies the boundedness of $G^n_j$ i.e.

\[(5.19) \quad \|G^n_j\| \leq K_2 ,\]

where $K_2$ is independent of $k$. 

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We will now show that

\[(5.20) \quad \| \mathcal{P}_1(k) - \mathcal{P}_1 \| = O(k) \, . \]

Observe that \( \mathcal{P}_1 \) is independent of \( k \). To obtain (5.20) we note that \( P_1 - \mathcal{P}_1 = H^0 \). If we define \( g_{i-1} = z, g_j = 0 \) (\( j \neq i \)) where \( z \) is an arbitrary vector we then have from (5.12)

\[(5.21) \quad v_{j-1} = k G_{i-1}^j G_{i-1} = k G_{i-1}^j z \, . \]

Substituting into (5.17) we obtain

\[(5.22) \quad u_n = k H_i^n z = k 3 \left[ \sum_{j=-\infty}^{\infty} G^n_j O(1) G_{j-1}^j \right] z \]

Now using (5.19) and (5.14) we see the sum in brackets in (5.22) is \( O(1/k) \) whence we conclude that

\[(5.23) \quad \| H_{-1}^n \| = O(k) \]

and setting \( n = i = 0 \) we obtain (5.20).

Note that this proves that the ranks of the stable and unstable manifolds are unchanged for small \( k \).

We can now show quite easily that (5.19) can be replaced by an exponential decay factor.

For a certain small positive \( \varepsilon \) consider new systems with the linear part of (5.3) and (5.4) multiplied by \( e^{-\varepsilon k} \) i.e.

\[(5.24) \quad v_{n+1}^\varepsilon = e^{-\varepsilon k} [ R_n + k^2 O(1) ] v_n^\varepsilon \]
\[(5.25) \quad x_{n+1}^\varepsilon = e^{-\varepsilon k} R_n x_n^\varepsilon . \]

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Now by Section 4 (see (4.9) ff.) the inhomogeneous versions of (5.24) and (5.25) will both have \((L^n_\omega, L^n_\omega)\) admissible, if \(\varepsilon\) is small enough. But the new fundamental solutions are

\[
V_n^\varepsilon = e^{-\varepsilon nk} V_n
\]

\[
X_n^\varepsilon = e^{-\varepsilon nk} X_n
\]

and clearly (5.25) will have an exponential dichotomy with projections \(\overline{P}_1\) and \(\overline{P}_2\) if \(\varepsilon\) is small enough.

Now if \(P_1(k)\) and \(P_2(k)\) are the corresponding stable and unstable projections for (5.24) while \(G^n_{j;\varepsilon}\) is the Green's function for (5.24), the same analysis that led to (5.19) will yield for some constant \(K_3\),

\[
\|G^n_{j;\varepsilon}\| \leq K_3
\]

In particular if \(j \geq n+1\), (5.28) reduces to

\[
\|V_n P^c_2 V^{-1}_j\| \leq K_3 e^{-\varepsilon k(j-n)}
\]

and one part of the definition of an exponential dichotomy for (5.5) will have been shown if we can show \(P^c_1 = P_1\), \(P^c_2 = P_2\). But this is a simple consequence of (5.20). In fact if \(S^c_+, S^c_-\) are the new stable and unstable manifolds for (5.24) then (5.26) yields

\[
(a) \quad S^c_+ \subset S^c_+
\]

\[
(b) \quad S^c_- \subset S_-
\]

But (5.20) shows that the ranks of \(P^c_1\) and \(P_1\) are equal.
(being equal to the rank of $P_1$) and similarly for $P_2^c$ and $P_2$. This shows that the inclusions in (5.30) are equalities and that

\begin{align*}
(a) & \quad P_1^c = P_1 \\
(b) & \quad P_2^c = P_2
\end{align*}

Thus one part of the requirements of an exponential dichotomy for (5.5) has been shown and the other part will follow on replacing $-\varepsilon$ by $+\varepsilon$.

Lastly we point out that an exponential dichotomy for (5.5) implies one for (5.1) since in terms of fundamental solutions we have

\begin{equation}
W_n = T^{-1}(kA_n)V_n T(kA_0)
\end{equation}

whence the projections $P_1'(k)$, $P_2'(k)$ for (5.2) are related to $P_1$ and $P_2$ by a similarity transformation

\begin{align*}
(a) & \quad P_1'(k) = T^{-1}(kA_0)P_1(k)T(kA_0) \\
(b) & \quad P_2'(k) = T^{-1}(kA_0)P_2(k)T(kA_0)
\end{align*}
6. **Miscellaneous Results**

This section is concerned with certain generalizations of the theory developed in the preceding section. We first remove certain restrictions which had been imposed previously in order to make the exposition clearer. In Section 6A we will remove the restriction that $A$ be $C^1$ while in Section 6B we remove the restrictions (2.6) and (2.7) on the roots of the polynomial $p(x) = 0$ which lie inside the unit circle.

We then consider certain trivial extensions of the theory. In Section 6C we remark on the general $L_\infty$ (non-AP) case. In Section 6D we consider the convergence of the mean value of our sequence solution to the mean value of the solution of the ODE and finally in Section 6E we consider the weakly nonlinear case.
6A. **Removal of the Differentiability Condition on \( A \)**

First consider the case when the homogeneous equation

\[
(6.1) \quad \dot{y} = Ay
\]

is exponentially stable; i.e.

\[
(6.2) \quad \|Y(t)Y^{-1}(s)\| \leq Ke^{-\alpha(t-s)}, \quad t > s,
\]

where \( Y(t) \) is the fundamental solution to (6.1). We assume the \( A \) is \( AP \) but is not \( C^1 \). Now Theorem 2 does not require a perturbation \( O(k^2) \) but is certainly valid for a perturbation \( k o(1) (k \to 0) \). This leads us to expect that the requirement that \( A \) be \( C^1 \) can be replaced by the uniform continuity of \( AP \) functions on the whole real axis. This is indeed the case as one can verify with little difficulty.

In this subsection, we will merely trace through the proofs in Section 3 and indicate what changes must be made if \( A \) is not \( C^1 \).

Discretizing (6.1) we obtain the homogeneous difference equation

\[
(6.3) \quad w_{n+1} = C_n w_n
\]

where \( C_n \) is given in (2.11). Using the uniform continuity of \( A \) we see that (3.7) can be replaced by

\[
(6.4) \quad w_{n+1} = U(k A_n)w_n + k o(1)w_n
\]

where the term \( o(1) \) is uniform in \( n \) (this will not be stated

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explicitly in the future). If we also use the uniform continuity of \( A \) to obtain

\[(6.5) \quad T(k A_{n+1}) = T(k A_n) + k o(1)\]

we see that Theorem 2 will yield exponential stability for \( (6.3) \) provided we can show exponential stability for the system

\[(6.6) \quad v_{n+1} = \begin{pmatrix} I + kA_n & 0 \\ 0 & B_n \end{pmatrix} v_n\]

The lower block causes no difficulty and we are left with the system

\[(6.7) \quad x_{n+1} = [I + k A_n] x_n.\]

Finally we can show that \( (6.7) \) is exponentially stable by using Theorem 2 to compare the fundamental solution \( X_n \) with \( Y_n (= Y(nk)) \). This requires some modification, as the proof leading to \( (3.16) \) used the differentiability of \( A \) (see \( (3.13) \) ff).

Following the notation of Section 3, we define

\[H(t,s) = Y(t)Y^{-1}(s).\]

We can then write (compare with \( (3.14) \))

\[(6.8) \quad H((n+1)k, nk) = I + kH(nk,nk) + k \int_0^1 d\theta [H(nk+\theta k,nk) - H(nk,nk)]\]

\[\quad = I + kA_n + k \int_0^1 d\theta [A(nk+\theta k) - A_n] H(nk+\theta k,nk)\]

\[\quad + k \int_0^1 d\theta A_n [H(nk+\theta k,nk) - H(nk,nk)].\]
Finally we see that if we use (6.2) together with the uniform continuity of $A$ on the whole real axis and the equation

\[(6.9) \quad H(t,s) = A(t)H(t)\]
	hen (6.8) can be rewritten as

\[(6.10) \quad Y_{n+1} = [I + kA_n + k o(1)] Y_n\]

and Theorem 2 is immediately applicable.

We thus have exponential stability for (6.3) and hence $(\tilde{A}^n, \tilde{A}^n)$ admissibility for the inhomogeneous version of (6.3).

Finally we point out that the results of Section 4, regarding admissibility in the case that (6.1) has a general exponential dichotomy is equally valid if $A$ is not $C^1$, as the contracting mapping principle (see (4.9) ff) would certainly be applicable if the perturbation is $k o(1)$. The results in Section 5, however, use crucially the differentiability of $A$ (in the argument involving the deduction of (5.18) from (5.17)) and it has not been possible to extend this result when $A$ is not $C^1$. 

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6B. **Removal of Restrictions on the Roots of** \( p(x) = 0 \)

In Section 2 we prescribed two conditions on the roots of \( p(x) = 0 \) which lie inside the unit circle; namely

\[
(6.11) \quad x_u \text{ distinct, } \quad u = 2, \ldots, \ell, \\
(6.12) \quad x_u \neq 0 \quad \text{, } \quad u = 2, \ldots, \ell.
\]

(Recall that \( x_1 = 1 \) and \( |x_u| < \theta < 1 \) for \( u > 1 \)). We will now remove these restrictions.

It should be clear by now that in the exponentially stable case (6.12) is unnecessary. In fact the fundamental solution to the homogeneous equation is used only for \( n \geq j \), and the equation (3.5) shows that the invertibility of the linear term \( C_i \) is not required for \( n \geq j \). Of course the notation \( W_n W_j^{-1} \) is no longer accurate, but except for this detail the results of Sections 2 and 3 are valid without assuming (6.12).

We deal next with the restriction (6.11). (Note that the root \( x_1 = 1 \) is always simple.) The removal of (6.11) in the finite dimensional case is trivial. In fact suppose the root \( x_r \) has multiplicity \( q > 1 \); i.e.

\[
(6.13) \quad p^1(x_r) = \ldots = p^{q-1}(x_r) = 0 \\
p^q(x_r) \neq 0.
\]

It is shown in Henrici (p. 214) that the effect of (6.13) is that the eigenvalue \( x_r \) of \( U(0) \) (see (2.12)) will now have
nontrivial Jordan blocks. Thus $U(0)$ can still be put in block diagonal form (see (2.17))

$$U(0) = \begin{pmatrix} I & 0 \\ 0 & B(0) \end{pmatrix}$$

where $B(0)$ is no longer diagonalizable, but has all of its eigenvalues bounded by $\theta$ in the norm. Now since $B(0)$ can be put in Jordan normal form with $\varepsilon$ instead of 1 on the superdiagonal, for any $\varepsilon > 0$, we can certainly choose a matrix norm such that

$$\|B(0)\| < \theta$$

Since all norms are equivalent on a finite dimensional space, the proof of Theorem 1 can now proceed exactly as in Section 2.

The case when $E$ is infinite dimensional can be handled in exactly the same manner; however since we can no longer appeal to the theorems of linear algebra, we will have to carry out the proof in more detail.

It is shown in Henrici (p. 214), that in the scalar case a basis for the generalized eigenspace $E_{x_r}$ corresponding to the eigenvalue $x_r$ is $\{e_{r,j}\}_{j=1,\ldots,q}$ where

$$\bar{e}_{r,1} = \begin{pmatrix} x_r^{l-1} \\ \vdots \\ 1 \end{pmatrix},$$

$$e_{r,j} = \begin{pmatrix} x_r^{j-1} \\ \vdots \\ 1 \end{pmatrix}.$$
\[
\begin{align*}
\bar{e}_{r,j} &= \\
&= \begin{cases} \\
&\frac{j-1}{r=1} (l-r)x \frac{j-1}{r=1} (j-1) \\
&\vdots \\
&\frac{j-1}{r=1} (n-r)x (n-(j-1)) \\
&\vdots \\
&\frac{j-1}{r=1} (j-r)x \\
&0 \\
&\vdots \\
&0 \\
\end{cases}, \quad j > 1.
\end{align*}
\]

One can verify easily that in fact

(a) \( U(0) \bar{e}_{r,1} = x_r \bar{e}_{r,1} \) 

(6.17) 

(b) \( U(0) \bar{e}_{r,j} = x_r \bar{e}_{r,j} + \bar{e}_{r,j-1}, \quad j > 1. \) 

Also since the vectors \( \bar{e}_{r,j} \) form a new basis the \( l \times l \) matrix with columns \( \bar{e}_{r,j} \) is nonsingular. Now in order to have \( \varepsilon \) instead of 1 on the superdiagonal, we replace \( \bar{e}_{r,j} \) by \( e_{r,j} \) where

(6.18) 

\[
e_{r,j} = \varepsilon^{j-1} \bar{e}_{r,j}
\]

(see Bellman [9], p. 198). Here \( \varepsilon > 0 \) is to be specified.

The matrix with columns \( e_{r,j} \) is still nonsingular but (6.17b) will now be replaced by

(6.19) 

\[
U(0) e_{r,j} = x_r e_{r,j} + \varepsilon e_{r,j-1}.
\]
Now the extension of this to infinite dimensional space is very simple and follows closely the procedure of Section 2. For any vector \( y \in E \) define the vector \( e_{r,j}(y) \in E_{\ell} \) by

\[
e_{r,j}(y) = \varepsilon^{j-1} \begin{bmatrix}
\frac{j}{r=1} (l-r)x^l_j y \\
\vdots \\
\frac{j}{r=1} (j+1-r)x^l_j y \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Equation (6.19) clearly generalizes to

\[
U(0) e_{r,j}(y) = x_r e_{r,j}(y) + \varepsilon e_{r,j-1}(y), \quad j > 1
\]

(6.20)

\[
U(0) e_{r,1}(y) = x_r e_{r,1}(y).
\]

We now define the space \( E_{r,j} = \{ w \in E_{\ell} \text{ such that } w = e_{r,j}(y) \text{ for some } y \in E \} \}. Clearly the space \( E_{r,j} \) is closed for all \( r \) and \( j \). Furthermore we have

\[
E_{\ell} = \bigoplus_{r,j} E_{r,j}
\]

This follows by the same argument as that given in Section 2 (see (2.14) ff). In fact if \( w = \begin{bmatrix} x_1 \\ \vdots \\ x_{\ell} \end{bmatrix} \) is a vector in \( E_{\ell} \) and we had an expansion

\[
\begin{bmatrix} x_1 \\ \vdots \\ x_{\ell} \end{bmatrix} = \sum_{r,j} e_{r,j}(y_{r,j})
\]

(6.23)

then the \( \{ y_{r,j} \} \) would be related to the \( \{ x_i \} \) by multiplication
by a nonsingular $\mathcal{L} \times \mathcal{L}$ matrix and this establishes (6.22). It also follows from this that if we define a new norm on $E_\mathcal{L}$ by $\|w\|_u = \max_{r,j} \|y_{r,j}\|_E$ then $\|w\|_u$ is equivalent to $\|w\|_\mathcal{L}$ (max $\|x_i\|_E$).

Now if $E_1$ is defined as $E_{1,1}$ (the root $x_1$ is simple) while $E^\bot$ is defined as $E_{r,j} \sum_{r>1,j}$, then, exactly as in Section 2, we see that $U(0)$ is in block diagonal form with respect to the decomposition $E = E_1 \oplus E^\bot$, and we can write (see (2.17))

$$U(0) = \begin{pmatrix} D(0) & 0 \\ 0 & B(0) \end{pmatrix}$$

where $D(0): E_1 \rightarrow E_1$ and is the identity, while $B(0): E^\bot \rightarrow E^\bot$ and for $\varepsilon$ sufficiently small we have from (6.19)

$$\|B(0)\|_u < \theta$$

The proof of Theorem 1 can now be carried out exactly as in Section 2 if we make the final observation that the spectrum of $U(0)$ is exactly $\{x_u\}$. This should be obvious and follows from the fact that on each "generalized eigenspace" $E_r = \sum_j E_{r,j}$, $U(0) - \lambda I$ (for $\lambda \not\in \{x_u\}$) acts as the matrix

$$x_r - \lambda \quad \varepsilon \quad 0 \quad \ldots \quad 0$$
$$0 \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$0 \quad \ldots \quad \ldots \quad 0 \quad x_r - \lambda$$

(6.26)

and the inverse is obtained merely by inverting this matrix.
6C. **Extension to the $L_\infty$ Case**

Here we would like to make the simple observation that the difference equation theory, like the ODE theory, is essentially an $L_\infty$ theory which produces AP solutions when the coefficients are AP.

In fact, if we now agree that derivatives are to be understood as $L_\infty$ instead of AP, the theory developed in Sections 3, 4 and 5 is entirely valid in the $L_\infty$ case, with only one unimportant exception. The single exception is the proof given in (3.36) ff of the convergence of the $L_\infty^n$ solution $w_n$ to the vector $\bar{w}_n = \begin{pmatrix} y_{n+\lambda-1} \\ \vdots \\ y_n \end{pmatrix}$ in the case that the coefficients are not smooth. In fact if we refer to (3.36) and (3.37) we see that (3.37) need not be valid because the function $y$ which is $C_\infty$ (we take $A$ and $f$ continuous) need not be uniformly continuous on the whole real axis. In this case we have not been able to show that $w_n \to \bar{w}_n$ uniformly for all $n$ but only uniformly for $nk$ lying in compact intervals.

To see this let us write $w_n(k), \bar{w}_n(k)$ to indicate explicitly the dependence on $k$. Now if the initial data for $w_n(k)$ is consistent, that is

\begin{equation}
(6.27) \quad w_0(k) \xrightarrow[k\to0]{} \begin{pmatrix} Y_1 \\ \vdots \\ Y_1 \end{pmatrix}
\end{equation}

for some vector $y_1 \in E$, then it is well known (see Henrici, p. 244) that $w_n(k)$ must converge uniformly on
compact subsets, to some solution to the equation
\[ y = A(t)y + f. \]

But since every solution to (6.28) but one is unbounded, this limiting solution can only be the unique \( L_\infty \) solution \( y(t) \). Thus we must show (6.27) and the proof of this will in fact show that the vector \( y_1 \) in (6.27) is \( y(0) \).

We assume at least that \( A \) is \( C^1 \) (with bounded derivative) so that the results of Section 5 are applicable. We can then write
\[ y = A(t)y + f. \]

Observe that \( g(k,t) \to 0 \) uniformly for \( t \) in compact intervals. Let \( T \) be unspecified for the moment and rewrite the sum in (6.29) as
\[ w_0 - \bar{w}_0 = k \sum_{j=\infty}^\infty G^0_j g(k,(j-1)k) \]
where \( g \) is given in (3.37), and \( G^0_j \) satisfies the estimate
\[ \|G^0_j\| \leq K_1 e^{-\alpha_1 k|n-j|}. \]

The first term \( \to 0 \) because \( g(k,t) \to 0 \) uniformly on the compact interval \([-T,T]\). The second term is \( O(e^{-T}) \) as we can easily see from (6.30). Thus by first choosing \( T \) sufficiently large to make the second term small, and then choosing \( k \) small enough to make the first term small we see that \( \|w_0 - \bar{w}_0\| \to 0 \) (\( k \to 0 \)) and this establishes the convergence of \( w_n \) to \( \bar{w}_n \) uniformly on compact intervals.
6D. Mean Value Properties

Let \( a_n \in \mathbb{A}^n \). It is shown in Corduneanu (p. 48) that the limit

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=n}^{n+N-1} a_j = m(a_n)
\]

exists and that this limit is uniform and independent of \( n \), which we henceforth set equal to zero. This is of course the exact analogy to the mean value of an AP function

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt = \bar{m}(f).
\]

Now if \( w_n \) is our AP sequence solution, we would like to study the behavior of \( m(w_n) \) as \( k \to 0 \). Define \( w_n \) as

\[
(w_n) = \left( \begin{array}{c}
  y_{n+1} - 1 \\
  \vdots \\
  y_n 
\end{array} \right)
\]

where \( y \) is the AP solution to the ODE

\[
\dot{y} = Ay + f
\]

Clearly

\[
m(w_n) = \left( \begin{array}{c}
  m(y_{n+1}) \\
  \vdots \\
  m(y_n) 
\end{array} \right)
\]

Now since (see \( 3.39 \))

\[
\|w_n - w_n\|_{\infty} = o(1) \quad k \to 1
\]
we must compare \( m(y_n) \) with \( m(y) \). We first claim for any AP function \( g \) we will have

\[
(6.38) \quad m(g_n) \xrightarrow[k \to 0]{} \bar{m}(g)
\]

where \( g_n = g(nk) \). This is obvious because (6.38) holds if \( g \) is a trigonometric polynomial and for arbitrary \( g \), we simply approximate by a sequence of trigonometric polynomials.

Now (6.38), together with (6.37) and (6.34), certainly yields

\[
(6.39) \quad m(w_n) \xrightarrow[k \to 0]{} \begin{pmatrix} \bar{m}(y) \\ \vdots \\ \bar{m}(y) \end{pmatrix}
\]

We now concern ourselves with the rates of convergence, assuming smooth coefficients. If \( A \) and \( f \) are \( C^{p+1} \) (so that \( y \) is) we can replace (6.37) by (see (3.41))

\[
(6.40) \quad \| w_n - \bar{w}_n \|_\infty = O(k^p)
\]

where \( p \) is the order of accuracy of the scheme. It therefore follows that

\[
(6.41) \quad m(w_n) = \begin{pmatrix} m(y_n) \\ \vdots \\ m(y_n) \end{pmatrix} + O(k^p)
\]

and it is only necessary to study the convergence of \( m(y_n) \) to \( \bar{m}(y) \). In fact for any AP function \( y \in C^{r+1} \) we have
Equation (6.42) is an almost periodic analogue of a theorem of Isaacson and Keller [10] for periodic functions (p. 340) and in fact the proof is very similar, and is a simple consequence of Taylor's theorem. Letting $t_n = nk$ and $s \in [0,1]$ we can write

\begin{equation}
(6.43) \quad y(t_n + sk) = y_n + (sk)y_n + \ldots + \frac{(sk)_r}{r!} y_n^r + O(k^{r+1})
\end{equation}

where the remainder is uniform in $t$ since $y^{r+1}$ is AP.

Integrating (6.43) from $s = 0$ to $s = 1$ we obtain

\begin{equation}
(6.44) \quad \int_{t_n}^{t_n+1} y(t) \, dt = k \int_0^1 y(t_n + sk) \, ds \\
= k[y_n + \frac{k}{2} y_n' + \ldots \frac{k^r}{(r+1)!} y_n^r + O(k^{r+1})].
\end{equation}

If we sum (6.44) from $n = 0$ to $N$, divide by $Nk$ and let $N \to \infty$ we obtain

\begin{equation}
(6.45) \quad \tilde{m}(y) = m(y_n) + \frac{k}{2} m(y_n^1) + \ldots \frac{k^r}{(r+1)!} m(y_n^r) + O(k^{r+1})
\end{equation}

Now if $r = 0$, i.e. $y$ is $C^1$ the terms involving $m(y_n^1)$ through $m(y_n^r)$ are missing and we obtain (6.42). For $r > 0$ we assume (6.42) holds for $j = 0, \ldots, r-1$. We can then write, as $y^1$ is $C^r$,

\begin{equation}
(6.46) \quad 0 = \tilde{m}(y^1) = m(y_n^1) + O(k^r)
\end{equation}
and similar expressions for $\bar{m}(y^i)$ for $i = 2, \ldots, r$ since $y^i \in C^{(r-i)+1}$ (we have used the obvious fact that the mean value of a derivative is zero), to obtain (6.42). If we now apply this to (6.41) we obtain

$$m(w_n) = \begin{bmatrix} \bar{m}(y) \\ \vdots \\ \bar{m}(y) \end{bmatrix} + O(k^p).$$
6E. Weakly Nonlinear Equations

Consider the equations

\begin{align}
\dot{y} &= Ay \\
\dot{y} &= Ay + f(t) \\
\dot{y} &= Ay + h(t,y)
\end{align}

If any pair of function spaces (B,D) is admissible for (6.49) the contracting mapping principle provides a technique to obtain D-solutions to (6.50) under certain conditions on the nonlinear term \( h(t,y) \), the most important being that \( h \) has a small Lipschitz constant (see Hartman [11], Chapter 12).

The case of \((\bar{A},\bar{A})\) admissibility is particularly simple. We require that \( h \) be almost periodic in \( t \), uniformly for \( y \) in compact subsets (see Hale [12] pp. 113 ff). This simply insures that \( h(t,x(t)) \) will be AP for any AP function \( x \). We also require that

\begin{align}
\|h(t,x_1) - h(t,x_2)\|_\infty &\leq \varepsilon_1 \|x_1 - x_2\|
\end{align}

for any \( x_1, x_2 \in E \). The restriction on \( \varepsilon_1 \) is simply

\begin{align}
\varepsilon_1 K_1 &< 1
\end{align}

where \( K_1 \) is the admissibility bound for (6.49) (see (0.3b)).

The unique AP solution to (6.50) will simply be the limit of the iterates.
(6.53) \[ y^0 = 0 \]
\[ y_{i+1} \] is the unique AP solution to
\[ \dot{y}_{i+1} = A y_{i+1} + h(t, y_i) . \]

This is of course an immediate consequence of the contracting mapping principle.

The same proof will yield AP sequence solutions to the discretized version of (6.50).

(6.54) \[ w_{n+1} = C_n w_n + k g(nk, k, w_{n+1}, w_n) \]

Here the nonlinear term is
\[ (a I - k \beta_1 A(t+\ell k))^{-1} \left[ \beta_1 h(t, \bar{x}_{\ell-1}) + \sum_{j=0}^{\ell-1} \beta_j h(t, x_j) \right] \]
\[ g(t, k, \bar{w}, w) = \left\{ \begin{array}{c}
    (a I - k \beta_1 A(t+\ell k))^{-1} [\beta_1 h(t, \bar{x}_{\ell-1})] \\
    + \sum_{j=0}^{\ell-1} \beta_j h(t, x_j) \\
    0 \\
    \vdots \\
    0
  \end{array} \right. \]

where \( w = \begin{bmatrix} x_{\ell-1} \\
  \vdots \\
  x_0 \end{bmatrix} \) and \( \bar{w} = \begin{bmatrix} \bar{x}_{\ell-1} \\
  \vdots \\
  \bar{x}_0 \end{bmatrix} \). Observe that \( g \) is, for fixed \( k \), AP in \( t \) uniformly for \( w \) and \( \bar{w} \) in compact subsets, and that \( g \) will have an \( O(\varepsilon_1) \) Lipschitz constant which we call \( \varepsilon_1 \), i.e.

(6.56) \[ \| g(t, k, \bar{w}_2, w_2) - g(t, k, \bar{w}_1, w_1) \|_\infty \]
\[ \leq \max \left[ \| \bar{w}_2 - \bar{w}_1 \|, \| w_2 - w_1 \| \right]. \]

Now if \( K \) is the admissibility bound for the linear inhomogeneous version of (6.54) (see (3.30)) and \( \mu = \varepsilon K < 1 \) then (6.54)
will have a unique AP solution \( w_n \), which in fact will be the limit of the iterates \( w_{n;i} \) defined by

\[
(6.57) \quad w_{n;0} = 0
\]

\[
 w_{n;i+1} \text{ is the unique AP solution to }
 w_{n+1;i+1} = C_n w_{n;i+1} + kg(nk,k,w_{n+1;i},w_{n;i})
\]

This is an immediate consequence of the contracting mapping principle and we note for future reference that

\[
(6.58) \quad \| w_n - w_{n;m} \| = O(\mu^m).
\]

Now the first thing to consider is the convergence of \( w_n \) to \( \tilde{w}_n \) defined as

\[
(6.50) \quad \tilde{w}_n \text{ satisfies (6.54) up to an error } k o(1) \ (O(k^{p+1}) \text{ if everything is smooth}). \]

Hence the difference \( z_n = \tilde{w}_n - w_n \) is the unique AP solution to

\[
(6.59) \quad z_{n+1} = C_n z_n + k[g(nk,k,w_{n+1}+z_{n+1},w_n+z_n) - g(nk,k,w_{n+1},w_n)] + k o(1) \ (O(k^{p+1}))
\]

and we have

\[
(6.60) \quad \| z_n \|_\infty \leq \| z_n \|_\infty + o(1) \ (O(k^p))
\]

and since \( \mu < 1 \) this settles the question of convergence.

We now examine the question of the computability of the solution in the exponentially stable case; that is when we have constants \( \bar{k} \) and \( \alpha \) such that
(6.61) \[ \| w_n W_j^{-1} \| \leq K e^{-\alpha (n-j)} , \quad n \geq j, \]

where \( W_n \) is the fundamental solution to the homogeneous version of (6.54).

To do this we use (6.58) to reduce the problem to a study of the effect of round-off errors and errors in initial data on the solution \( w_{n;m} \) for a fixed \( m \). This will be a simple consequence of the stability results of Section 3 (see (3.42) ff) for the inhomogeneous equation.

In fact the results of Section 3 show that at the first iteration we would solve (for \( n > 0 \)) for a solution \( w^*_n \) which is related to \( w_{n;l} \) by

(6.62) \[ w^*_n = w_{n;l} + O(k^r) + O(e^{-\alpha kn}) . \]

We now consider the effect of this error on the second iterate. Assuming, for the moment, exact calculations and exact initial data, we obtain a solution \( w^*_n \) which solves the equation

(6.63) \[ w_{n+1;l}^* = C_n w^*_{n;l} + k g(nk,k, w^*_{n+1;l}, w^*_{n;l}) . \]

If we let \( z_{n;i} = w^*_{n;i} - w_{n;i} \) for \( i = 1,2 \) then we obtain

(6.64) \[ z_{n+1;l} = C_n z_{n;l} + k [ g(nk,k, w_{n+1;l}^*; z_{n+1;l}^*; w_{n+l}; z_{n;l}) \\
- g(nk,k, w_{n+1;l}^*; w_{n;l}) ] \]

The last term can be written (using (6.62)) as

\[ k O(k^r) + k O(e^{-\alpha kn}) \]

and we have

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where $\alpha_1$ can be arbitrarily close to $\alpha$. We have used (6.62), (6.61) and the trivial fact that $t = O(e^\delta t)$ for any positive $\delta$.

It is now obvious that (6.65) is preserved if we include in the equation for $w_{n;2}^*$ (see (6.63)) the effect of roundoff and errors in the initial data of $w_{n;2}^*$ (compared with the initial data for $w_{n;2}$) and it is also obvious that this process can be repeated for $m$ iterations, for any fixed finite $m$, and this establishes the computability of the solution $w_n$. 

\begin{equation}
(6.65) \quad z_{n;2} = k \sum_{j=1}^{n} W_n W_j^{-1} O(k^r) + k \sum_{j=1}^{n} W_n W_j^{-1} O(e^{-\alpha_1 k j})
\end{equation}

\begin{equation}
= O(k^r) + O(e^{-\alpha_1 kn}) = O(k^r) + O(e^{-\alpha_1 kn})
\end{equation}
Part II. Extensions to Partial Differential Equations

We will now use the techniques of Massera and Schäffer, in the simple exponentially stable case, to obtain AP solutions to a certain class of inhomogeneous partial differential equations (PDE's) to which one can apply ODE formalism by use of the Hille Yoshida Theorem. The homogeneous equation has been extensively studied by Krein [13], to which we will refer often. A more succinct study of the Hille-Yoshida Theorem can be found in the appendix of Lax and Phillips [14].

Throughout Part II E will denote an infinite dimensional Banach space.
7. Existence Theorems

Consider the linear equation

\begin{equation}
\dot{y} = By
\end{equation}

where $B$ is an unbounded operator. The study of the solutions to (7.1) has been answered by the Hille-Yoshida Theorem in the case that the resolvent $R_\lambda(B) = (\lambda I - B)^{-1}$ satisfies the inequality

\begin{equation}
\|R_\lambda(B)\| \leq \frac{1}{\lambda}, \quad \lambda > 0.
\end{equation}

A more general formulation is given in Krein (in particular Section 2, Chapter 1). Krein shows that if:

\begin{equation}
\text{(a) } B \text{ is closed} \\
\text{(b) } B \text{ is densely defined} \\
\text{(c) } \|R_\lambda^n(B)\| \leq \frac{M}{(\text{Re } \lambda - \omega)^n}, \quad \text{R.P. } \lambda > \omega,
\end{equation}

then there exists a semigroup $Z_t, t \geq 0$, such that:

\begin{equation}
\text{(a) } Z_t \text{ is strongly continuous} \\
\text{(b) } Z_0 = I \\
\text{(c) } \|Z_t\| \leq M e^{\omega t} \\
\text{(d) } \lim_{\Delta \to 0} [Z_{\Delta} - I]e/\Delta \text{ converges iff } e \in D_B \\
\text{and in that case it converges to } Be \\
\text{(e) if } e \in D_B \text{ then for } t \geq t_0 \text{ the unique solution to (7.1) such that } y(t_0) = e \\
\text{is simply } y(t) = Z_{t-t_0}e \text{ (uniqueness is}
\end{equation}
shown in Theorem 2.7, page 47)

\( Z_t : D_B \to D_B \) and commutes with \( B \) for \( t \geq 0 \).

For a well posed problem we obviously require \( \omega \leq 0 \) in (7.4c).
To insure exponential stability we are going to proceed as if \( \omega = 0 \) and introduce our own damping term which will be allowed to depend on \( t \). The reader should note that if \( \omega < 0 \) this term is not necessary. It is well known (see Krein, page 43) that one can always introduce an equivalent norm so that the factor \( M \) in (7.4c) can be replaced by 1, however we will eventually approximate \( B \) by a family of bounded operators \( B_h \) where

\[
(7.5) \quad \| Z^h_t \| \leq e^{h^t} \leq M
\]

and we may not be able to introduce a norm so that \( M \) can be taken as 1 for all \( h \), and thus we will leave \( M \) unspecified.

Now to make the homogeneous equation exponentially stable we introduce a damping term \(-\delta(t)\) and consider the equation

\[
(7.6) \quad \dot{y} = [B - \delta(t)]y
\]

For conditions on \( \delta \) we first let it be a scalar \( \text{AP} \) function such that

\[
(7.7) \quad \text{R.P. } (\tilde{m}(\delta))' > 0.
\]

This implies that if \( g(t,s) = \exp\left\{-\int_s^t \delta(r) \, dr\right\} \) then

\[
(7.8) \quad |g(t,s)| \leq Ke^{-\alpha(t-s)}, \quad t \geq s,
\]
as can easily be seen.

It is also possible to permit $\delta$ to be an AP operator function satisfying certain commutivity properties. Specifically if $Y(t)$ is the fundamental solution to

\begin{equation}
\dot{Y} = -\delta Y
\end{equation}

then

\begin{equation}
\begin{align}
(a) & \quad \|Y(t)Y^{-1}(s)\| \leq Ke^{-\alpha(t-s)}, \quad t \geq s \\
(b) & \quad \delta(t), Y(t), Y^{-1}(t) \text{ commute with } Z_r \\
& \quad \text{for all } t \text{ and all } r > 0 \\
(c) & \quad \delta(t), Y(t), Y^{-1}(t) \text{ map } D_B \text{ into itself and commute with } B.
\end{align}
\end{equation}

These hypotheses permit us to essentially treat $\delta$ as a scalar in the following analysis. If we define

\begin{equation}
g(t,s) = Y(t)Y^{-1}(s)
\end{equation}

then the "fundamental solution" to (7.6) is

\begin{equation}
g(t,s)Z_{t-s}, \quad t \geq s,
\end{equation}

and

\begin{equation}
\|g(t,s)Z_{t-s}\| \leq KM e^{-\alpha(t-s)} = K e^{-\alpha(t-s)}
\end{equation}

after a redefinition of $K$.

Now consider the inhomogeneous equation

\begin{equation}
\dot{y} = [B-\delta]y + f(t)
\end{equation}

where $f \in \tilde{A}(E)$ or $C_\infty(E)$. We want to study bounded (for all $t$)
solutions to (7.14). First we see that any such solution is unique. In fact if \( w(t) \) were a bounded solution to (7.6) we would have (for \( t \geq s \))

\[
(7.15) \quad w(t) = Z_{t-s} g(t,s) \, w(s)
\]

and \( w = 0 \) follows immediately on letting \( s \to -\infty \).

It is equally simple to see that any \( C_\infty \) solution must be given by

\[
(7.16) \quad y(t) = \int_{-\infty}^{t} g(t,s) \, Z_{t-s} f(s) \, ds \equiv C(f) .
\]

This is in fact an immediate consequence of letting \( t_0 \to -\infty \) in the variation of constants formula

\[
(7.17) \quad y(t) = g(t,t_0) Z_{t-t_0} y(t_0) + \int_{t_0}^{t} ds \, g(t,s) Z_{t-s} f(s)
\]

which is proved in Krein (Theorem 6.1, page 129) for the autonomous case, and it is a trivial matter to see that the same proof will work if \( \delta \) is time dependent. Note that (7.17) need not be a solution to (7.14), but any such solution must be given by (7.17).

We now consider the operator \( C \) defined by (7.16). \( C \) is a bounded operator mapping \( L_\infty \to L_\infty \). Boundedness is an immediate consequence of (7.13) and in fact

\[
(7.18) \quad \|y\|_\infty \leq \overline{K} \left( = \frac{K}{\alpha} \right) \|f\|_\infty
\]

The integrand is obviously continuous if \( f \in C_\infty \) and the only point in question is its measurability for a general
f ∈ L∞ (for measurability of functions in a Banach space see Hille and Phillips [15]). However it is shown by Krein, and will be used crucially very shortly, that Zr is the strong limit, pointwise, of e^n where Br is a bounded operator (Theorem 2.9, page 48). Since e^n will be norm continuous this establishes the integrand as the pointwise limit of measurable functions, hence measurable.

We will now show that y as defined in (7.16) is a solution to (7.14). We will make use of the above mentioned approximation theorem of Krein (Theorem 2.9). It is shown that there exists a sequence of bounded operators Bn such that

\[(7.19) \quad \begin{align*}
(a) & \quad \| (B-B_n)e \| \xrightarrow{n \to \infty} 0 \quad (\forall e \in D_B) \\
(b) & \quad \| (Z_r-Z_r^n)e \| \xrightarrow{n \to \infty} 0 \quad (\forall e, r \geq 0, \text{ where } Z_r^n = e_n^r ) \\
(c) & \quad \| Z_r^n \| \leq M, \quad r \geq 0 \\
(d) & \quad B_n, Z_r^n (r \geq 0) \text{ satisfy the same commutativity relations with } \delta(t) \text{ (7.10b,c) as does } B \text{ and } Z_r. 
\end{align*} \]

We only have to point out that (7.19d) follows from the explicit definition of Bn as \( -\lambda_n I - \lambda_n^2 R \) where \( \lambda_n \to \infty \) (see Krein, page 49). We point out that the operators Bn cannot in general be considered as spatial discretizations. A family of such approximations will be assumed in the following section.

Now consider the operators Cn(f) defined by
\( (7.20) \quad y_n = \int_{-\infty}^{t} g(t,s) z_{t-s}^n f(s) \, ds = C_n(f) \).

It follows immediately that the operators \( C_n \) are uniformly bounded operators mapping \( L_{\infty} \) into \( C_\infty \) and in fact

\( (7.21) \quad \|y_n\|_\infty \leq \bar{K} \|f\|_\infty \)

where we take the same bound as in (7.18). It is also immediate that \( y_n \) is the unique \( C_\infty \) solution to

\( (7.22) \quad y_n = [B_n-\delta]y_n + f \)

and \( y_n \) is AP if \( f \) is.

We are going to show that \( C_n \to C \) strongly in a certain subspace of \( L_{\infty} \). Specifically define \( S \) to be the subspace spanned by the functions whose range has compact closure. It is well known (see Amerio [16]) that \( A \subseteq S \). For \( f \in S \) we will have

\( (7.23) \quad C_n(f) \to C(f) \)

To show (7.23) note that by the uniform boundedness of \( C_n \) it is sufficient to show it for a dense subset of \( S \) and by the definition of \( S \) it is sufficient to take \( f \) as

\( (7.24) \quad f = \sum_{j=1}^{n} \chi_j(t)e_j \)

where \( e_j \in E \) and \( \chi_j \) is the characteristic function of a measurable set. By linearity we only have to work with one term in the sum in (7.24). Thus if \( I \) is some measurable
set let \( f = \chi e \). Then \( w_n(t) = y(t) - y_n(t) \) is explicitly

\[
(7.25) \quad w_n(t) = \int_0^\infty \chi(t-r) g(t,t-r) [Z_r - Z_r^n] e^{-r} dr .
\]

Estimating (7.25) we obtain

\[
(7.26) \quad \|w_n\|_\infty \leq K \int_0^\infty e^{-\alpha r} \|Z_r - Z_r^n\| e^{-r} dr.
\]

and (7.23) is an immediate consequence of the Lebesgue dominated convergence theorem.

Observe that (7.23) implies that \( C(f) \) is continuous if \( f \in S \) and it is AP if \( f \) is.

Now consider the almost periodic case. First we point out that if \( \delta \) and \( f \) are \( C^1 \) (AP derivatives) then \( y \) is also and in fact

\[
(7.27) \quad \dot{y} = C(-\dot{\delta}y + \dot{f})
\]

This is trivial if we observe that

\[
(7.28) \quad \dot{y}_n = C_n(-\dot{\delta}y_n + \dot{f})
\]

and if \( \tilde{y} \) is defined to be the right-hand side of (7.27) we have

\[
(7.29) \quad \tilde{y} - \dot{y}_n = [C-C_n](-\dot{\delta}y) + C_n(-\dot{\delta}(y-y_n)) + [C-C_n](\dot{f}) ,
\]

and clearly

\[
(7.30) \quad \|Y - \dot{y}_n\|_\infty \xrightarrow{n \to \infty} 0
\]

Equation (7.27) is now an immediate consequence of letting
(7.31) \[ y_n(t_1) - y_n(t_2) = \int_{t_1}^{t_2} \dot{y}_n(t) \, dt \]

We next show that \( y \) satisfies an integrated version of (7.14). Specifically

(7.32) \[ y(t_1) - y(t_2) = B \int_{t_1}^{t_2} dt \, y(t) - \int_{t_1}^{t_2} dt \, \delta(t) y(t) + \int_{t_1}^{t_2} dt \, f(t) \]

Since \( D_B \) is dense and \( B \) is closed it is sufficient to show (7.32) when \( f \) is of the form

(7.33) \[ f = e^{i\lambda t} b, \quad b \in D_B. \]

Now observe that \( Bf \) is AP. It certainly follows that

(7.34) \[ By = C(Bf). \]

In fact this holds whenever \( Bf \in C_\infty \) and is a simple consequence of the closure of \( B \), approximating the integral in (7.16) by Riemann sums over finite intervals and then noting that \( B \) applied to each sum is the Riemann sum approximating \( C(Bf) \).

Now the same property certainly holds for \( B_n \) and \( C_n \) i.e.

(7.35) \[ B_n y_n = C_n (B_n f). \]

Finally, since by its construction \( B_n \) commutes with \( Z_r \).
(Krein, Theorem 2.9) we can write

\[ (7.36) \quad B_n y = C(B_n f) . \]

Since \( y_n \) satisfies (7.22) we have

\[ (7.37) \quad y_n(t_1) - y_n(t_2) \]
\[ = B_n \int_{t_1}^{t_2} \delta(t) y_n(t) \, dt + \int_{t_1}^{t_2} f(t) \, dt \]
\[ = B n \int_{t_1}^{t_2} y(t) \, dt \quad [B_n - B] \int_{t_1}^{t_2} y(t) \, dt + B_n \int_{t_1}^{t_2} \delta(t) y_n(t) \, dt \]
\[ - \int_{t_1}^{t_2} \delta(t) y_n(t) \, dt + \int_{t_1}^{t_2} f(t) \, dt . \]

where the fact that \( \int_{t_1}^{t_2} \, dt y(t) \in D_B \) is a consequence of
(7.34) and the same type of closure argument as followed
that equation.

If we now let \( n \to \infty \) in the last equation in (7.37) we see that the term \( B_n \int_{t_1}^{t_2} \, dt (y_n-y) \xrightarrow{n \to \infty} 0 \) because

\[ (7.38) \quad B_n(y_n-y) = [C_n - C](B_n f) = [C_n - C](B f) + [C_n - C]((B_n - B)f) \]

and the right-hand side of (7.38) clearly \( \to 0 \) in the sup norm
as \( n \to \infty \). It therefore follows, on letting \( n \to \infty \) in (7.37),
that (7.32) does in fact hold for functions of the form (7.33)
and thus for all \( f \in \tilde{\mathcal{A}} \).

Now if \( \delta \) and \( f \) are \( C^1 \) (hence \( y \) is) we can show very easily
that (7.32) implies (7.14). In fact since $y$ is $\mathcal{C}^1$ we can let $h \to 0$ in the equation
\[
\frac{y(t+h) - y(t)}{h} = \frac{\int_0^t ds \, y(s) - \int_t^{t+h} ds \, \delta(s) y(s) + \int_t^{t+h} f(s)}{h}
\]
where $h$ can be positive or negative, using the closure of $B$.

Similarly (7.14) will hold if we only require $Bf$ is AP.

In fact since in that case $By$ is AP, $B$ can be brought inside the integral in (7.32), which can then be differentiated directly.

We point out that in both these cases both $y$ and $By$ are also AP. For future use we point out, using the integral equation (7.32) applied to (7.27), that if $\delta$ and $f$ are $\mathcal{C}^2$ then $y$ is $\mathcal{C}^2$ and $\dot{y}$ satisfies
\[
\dot{y} = [B-\delta]y - \delta y + f
\]
i.e. (7.14) can be differentiated formally. (Note in particular that if $Bf$ is AP, $B^2y$ is AP.) This argument can obviously be extended; namely if $f$ and $\delta$ are $\mathcal{C}^p$ then $y$ is $\mathcal{C}^p$ and we have
\[
y^r = [B-\delta]y^{r-1} + f^{r-1} - (\delta y)^{r-1} + \delta y^{r-1}, \quad r=1, \ldots, p.
\]

We will now examine briefly the case that $f \in S$.

We still have $y_n \to y$ but (7.30) need not hold since $\dot{y}$ and $f$ need not belong to $S$. However we see very easily that if $f \in L^\infty$ and
\[ x = C(g) \]
\[ x_n = C_n(g) \]
then we will have \( x_n(t) \to x(t) \) pointwise and boundedly. In fact if \( w_n(t) = x(t) - x_n(t) \), then
\[ \|w_n(t)\| \leq K \int_{-\infty}^{t} e^{-\alpha r} \|Z_r - Z_r^n\| g(t-r) \, dr \]
and the Lebesgue dominated convergence theorem is still applicable and hence (7.31) will still yield (7.27), if we use the bounded convergence theorem and the fact that
\[ \lim_{n \to \infty} (C-C_n)(\delta y+f)(t) + C_n(\delta(y-y_n))(t) = 0 \]
Thus \( y \) has a bounded derivative. Now if we assume that \( f \) is of the form
\[ f = \chi^b, \quad b \in D_B, \]
then the derivation of (7.32) from (7.37) and (7.38) still follows, if we use the bounded convergence theorem for integrals over finite intervals. Since finite sums of such \( f \)'s are dense in \( S \), (7.32) holds for all \( f \in S \) and if \( Bf \in S \) or \( \delta \) and \( f \) are \( C^1 \), \( y \) will still be the unique \( C^\infty \) solution to (7.14). Thus the theory will hold for a certain class of bounded, non AP, inhomogeneous terms, but is not as simple as the AP case and in the future we will restrict ourselves to this case.

Finally we point out that (7.32) can be derived by brute force differentiation of the formula (7.16) but use of the approximating operators \( B_n \) permits a cleaner and more straightforward development.
8. Discretization in Space

The operators $B_n$ which proved so useful in the preceding section need not correspond to a discretization in space. We therefore postulate the existence of a family of bounded operators $B_h$, where $h$ will generally play the role of a spatial grid size, such that:

\begin{equation}
(8.1) \quad \begin{aligned}
(a) & \quad B_h \text{ are defined and bounded for } h \in (0, h_0] \\
(b) & \quad \|B_h\| = O\left(\frac{1}{h^m}\right), \ m > 0 \\
(c) & \quad B_h \text{ generate bounded semi-groups (i.e.} \\
& \quad \|e^{B_h t}\| \leq M \text{ for } t \geq 0) \\
(d) & \quad e \in D_B \text{ implies } \|e^{(B-B_h)_h}\|_{h \to 0} \to 0 \\
(e) & \quad \text{If } \delta \text{ is not a scalar function, then the commutativity relations (7.10)b,c hold with } \\
& \quad B \text{ and } Z_r \text{ replaced by } B_h \text{ and } Z_r^h \\
(f) & \quad \text{If } e \in D_B \text{ (or any dense manifold in } D_B \text{ invariant under } Z_r \text{ for } r \geq 0) \text{ we have} \\
& \quad \|B_h Z_r e\| \leq K(I,e) \text{ for } r \in I, \text{ where } I \text{ is any compact interval on the nonnegative real axis.}
\end{aligned}
\end{equation}

We point out that (8.1f) is designed to insure that $Z_r^h \to Z_r$ strongly, for all $r \geq 0$. This is certainly reasonable since $B_h Z_r e$ converges to $B Z_r e$. It is of course trivial if $B_h$ commutes with $Z_r$. Also note that the convergence in (8.1d) does not deal with the order of the approximation of $B_h$ to $B$. 
This will be dealt with by a subsequent assumption.

Before proceeding further, we indicate a simple example of the case we are considering.

(8.2) (a) \[ E = L_2(0,2\pi) \]

(b) \[ B = \frac{\partial}{\partial x} \] (with periodic boundary conditions)

(c) \[ B_h = \frac{u(x+h)-u(x-h)}{2h} \]

In this example (8.1f) is unnecessary as \( B_h \) in fact commutes with \( Z_r \).

Now associated with \( B_h \) we have the equations

(8.3) \[ y = [B_h - \delta]y \]

(8.4) \[ y = [B_h - \delta]y + f \]

The homogeneous equation (8.4) has as its fundamental solution \( Y_h(t) = g(t,s) e^{h t} \), and is exponentially stable, i.e. there exist constants \( K \) and \( \alpha \) independent of \( h \) such that

(8.5) \[ ||Y_h(t)Y_h^{-1}(s)|| \leq K e^{-\alpha(t-s)} , \quad t \geq s \]

Associated with (8.4) we form the operator \( C_h \) where

(8.6) \[ Y_h = C_h(f) = \int_{-\infty}^{t} g(t,s) Z_h^{h} f(s) \, ds \]

for any AP \( f \), \( Y_h \) is the unique AP solution to (8.4).

Equation (8.5) shows that the operators \( C_h \) are bounded uniformly in \( h \), i.e.

(8.7) \[ ||Y_h||_{\infty} \leq K ||f||_{\infty} \]

where \( K \) is independent of \( h \).
Now our first task is to show that $C_h \to C$ strongly. Using the results of Section 7 (see (7.23) ff) it is sufficient to show that $Z_r^h \to Z_r$ strongly, for $r \geq 0$. This will be an easy consequence of (8.1f). In fact if $e \in D_B$ (or the manifold described in (8.1f)) and if $y = Z_t e$, $y_h = Z_t^h e$, then if $w_h = y - y_h$ we will have

$w_h = B_h w_h + [B - B_h] y$

(8.8)

$w_h(0) = 0$.

From (8.8) we can write

(8.9) $w_h(t) = \int_0^t ds Z_h^t [B - B_h] Z_s e$

and the fact that $\|w_h(t)\| \to 0$, in fact uniformly on compact intervals, is an immediate consequence of (8.1f) and the Lebesgue bounded convergence theorem. The result for any $e \in E$ follows from the uniform boundedness of the operators $Z_r^h$ and the denseness of the manifold of (8.1f).

Now the fact that

(8.10) $\|(y - y_h)\|_\infty \xrightarrow{h \to 0} 0$

for all $f \in \tilde{A}$ (in fact for all $f \in S$) is not any improvement of the corresponding result for the operators $B_n$ of Section 7. We want to show that the difference in (8.10) approaches zero as fast as some power of $h$, and to do this we must require that $B_h$ approximate $B$ up to a certain degree of accuracy. Specifically we suppose that there exists an operator $L$
(unbounded) with \( D_L \subseteq D_B \) and such that if \( e \in D_L \) we will have
\[
\| (B-B_h)e \| \leq C h^j \| e \| \quad \text{where} \quad j > 0
\]
and \( C \) and \( j \) are independent of \( h \) and \( e \).

If we have \( Lf(t) \) AP we would like to show that (8.11) implies
\[
\| (y-y_h) \|_{\infty} = O(h^j).
\]
Unfortunately (8.12) requires a further assumption; namely

1. \( Z_r: D_L \rightarrow D_L \)
2. If \( Lf \) is AP then \( \| L Z_r f(t) \| \) will be uniformly bounded for \( r \geq 0 \) and all \( t \).

Assumption (8.13) is of course trivial if \( L \) commutes with \( Z_r \).

It is now a simple matter to show (8.12). In fact \( w_h = y-y_h \) is the unique AP solution to
\[
w_h = [B_h-\delta]w_h + (B-B_h)y
\]
whence by (8.7) we have
\[
\| w_h \|_{\infty} = O(\| [B-B_h] y \|_{\infty})
\]
and so it is only necessary to study \( (B-B_h)y \). But now if \( Bf \in \tilde{A} \), which will generally follow if \( Lf \in \tilde{A} \) since \( L \) will be a higher order operator than \( B \), we can write
\[ (8.16) \quad [B-B_h]y = \int_0^\infty \, dr \, g(t, t-r) [B-B_h]Z_r f(t-r) \]

and clearly

\[ (8.17) \quad \| [B-B_h]y \|_\infty = O(h^j) \]

is an immediate consequence of (8.13).
9. **Discretization in Time by the Multistep Linear Method**

We will now apply the theory developed in Section 3 to the spatially discretized equation

\[(9.1)\]

\[y_h = [B_h - \delta] + f.\]

Since \(B_h\) is bounded the theory is immediately applicable and we can assert that for \(k\) sufficiently small (but depending on \(h\)) there will be an AP sequence \(w_n(k,h)\) which will converge to \(w_n(k,h)\) defined by

\[(9.2)\]

\[\tilde{w}_n(k,h) = \begin{pmatrix} y_h((n+\ell-1)k) \\ \vdots \\ y_h(nk) \end{pmatrix}\]

and if everything is smooth in \(t\) we will have

\[(9.3)\]

\[\|w_n - \tilde{w}_n\|_\infty = O_h(k^p)\]

where the bound in (9.3) may depend on \(h\). It is only left to show that the bound in (9.3) is independent of \(h\) and to discuss the relationship of \(k\) to \(h\). We will show that what is required is

\[(9.4)\]

\[k + \frac{k}{h^{2m}} = O(1)\]

i.e. there exists a constant \(c\) independent of \(k\) and \(h\), such that if \(k + k/h^{2m} < c\) the difference equation corresponding to (9.1) will be exponentially stable with constants independent of \(k\) and \(h\).
The quadratic dependence on $h$ in (9.4) is restrictive and will be removed in the following section.

Throughout this section we will use the notation of Sections 2 and 3. We will also take $\delta$ to be $C^1$.

We now fix $h$ and consider the homogeneous difference equation on $E_\infty$ (see (3.7)),

$$w_{n+1} = U(kB_h - k\delta_n) + k^2O(1)w_n.$$  \hspace{1cm} (9.5)

It is clear that this will hold uniformly in $k$ and $h$ if $k + k/h^m$ is sufficiently small, i.e. if we are in a region

$$k + \frac{k}{h^m} = O(1).$$  \hspace{1cm} (9.6)

Of course restrictions of the form (9.6) are included in (9.4). Also note from the explicit definition of the full linear term in (9.5) (see (2.11)) the last term in (9.5) will be uniform in $k$ and $h$. For the rest of this section this will be understood for all "O" signs unless stated otherwise.

If $W_n(k,h)$ is the fundamental solution to (9.5) it is clear that all we must show is that in a region of the form (9.4) we will have

$$||W_nW_j^{-1}|| \leq K_1 e^{-\alpha_1 k(n-j)}, \hspace{1cm} n \geq j,$$ \hspace{1cm} (9.7)

where the constants $K_1$ and $\alpha_1$ are independent of $k$ and $h$.

Following the procedure of Section 3 we let

$$v_n = T(kB_h - k\delta_h)w_n$$ \hspace{1cm} (9.8)
and we then have in a region of the form (9.6)

\begin{equation}
(9.9) \quad v_{n+1} = L(kB_n - k\delta_n)v_n + k^2 O(1)v_n
\end{equation}

Using the structure of the block diagonal operator $L(z)$ derived in Theorem 1 we can write this as

\begin{equation}
(9.10) \quad v_{n+1} = \begin{pmatrix}
[I + k(B_n - \delta_n)] & 0 \\
0 & B_n
\end{pmatrix}v_n + kO(k + \frac{k}{h^{2m}})v_n
\end{equation}

Here we have used the notation $B_n$ for the lower block (see (2.26)) so as not to confuse it with the operator $B_h$.

Observe that in (9.10) we have the first appearance of the term $k/h^{2m}$. This is because in order to apply Theorem 2 we must separate out a factor of $k$ in the perturbing term.

Now it certainly follows that in a region of the type (9.6) we have

\begin{equation}
(9.11) \quad \| I - B_i \| \leq K_1 e^{\theta n-j}, \quad n \geq j,
\end{equation}

where $K_1$ and $\theta$ are independent of $h$ and $k$. We therefore only have to consider the upper block in the leading term of (9.10) and in order to apply Theorem 2 we are going to compare this with $Y_{n,h} = Y_h(nk)$ where $Y_h$ is the fundamental solution to

\begin{equation}
(9.12) \quad \dot{Y}_h = [B_h - \delta]Y_h
\end{equation}

Observe that the estimate

\begin{equation}
(9.13) \quad \| Y_{n,h}^{-1} \| \leq Ke^{-\alpha k(n-j)}, \quad n \geq j,
\end{equation}
holds, where $K$ and $\alpha$ are independent of $h$.

We will show that $Y_{n;h}$ satisfies the difference equation

$$Y_{n+1;h} = [I + kD_n(k,h)]Y_{n;h}$$

$$= [I + k(B_h - \delta_n)]Y_{n;h} + kO(k + \frac{k}{h^2m})Y_{n;h}.$$

Observe that the last term in (9.14) is of the same order as the perturbation in (9.10).

Equation (9.14) follows easily from the same argument as in Section 3 (see (3.13) and (3.14)) if we simply observe that

$$B_h - \delta_n = -\delta$$

and that (3.15) holds in this case for $t \geq s$ by (8.5).

We can now apply Theorem 2 comparing the equation

$$V_{n+1;h} = [I + k(B_h - \delta_n)]V_{n;h}$$

with (9.14). We simply note that the proof of Theorem 2 indicates that the terms $\varepsilon_0$, $\alpha_1$ and $K_1$ (see the statement of Theorem 2) depend only on the constants $K$ and $\alpha$ in (9.13) and are thus independent of both $k$ and $h$. Another application of Theorem 2 comparing the equation

$$\tilde{V}_{n+1} = \begin{pmatrix} I + k(B_h - \delta_n) & 0 \\ 0 & \tilde{E}_n \end{pmatrix} \tilde{V}_n$$

with (9.10), together with the boundedness of the operators
T(k(B_n - δ_n)), T^{-1}(k(B_n - δ_n)) in the region described by (9.6) will yield (9.7) with the restriction k + k/h^{2m} sufficiently small (i.e. (9.4)).

If we now refer to the proof of admissibility in the stable case given by (3.31) ff we see immediately that (9.7) implies (L_{\omega}^n, L_{\infty}^n) admissibility for the inhomogeneous version (9.5), with an admissibility bound independent of k and h. It is only necessary to show that the convergence of \tilde{w}_\infty to \tilde{w}_n is independent of h (see (9.3)). This will follow from (3.40) if we can show \|y_h^{p+1}\|_{\infty} will be bounded uniformly in h, assuming that \delta and f are smooth.

In fact if \delta and f have p+1 derivatives the differentiability properties of y_h are the same as those of y_n introduced in Section 7. In addition, the formula given in (7.28) can obviously be extended (for the operators C_h) to yield

(9.18) \quad y_h^{p+1} = C_h f^{p+1} + (-\delta y_h)^{p+1} + \delta y_h^{p+1} \]

(note that the argument of C_h in (9.18) does not involve y_h^{p+1}) and the uniform boundedness of the operators C_h immediately yields the uniform boundedness of the derivatives of y_h.

If we now define \bar{w}_n = \{y((n+\ell-1)k) \}

(9.19) \quad \|w_n - \bar{w}_n\|_{\infty} = O(k^{p+h}j^4) = O(h^{2mp+h}j^4)

where we have used (9.4) for small k.
10. **The Lax-Wendroff Scheme**

The term \( k/h^2 \) in (9.4) can be improved for operators such that the Lax-Wendroff scheme is stable for the homogeneous equation

\[
(10.1) \quad \dot{w} = Bw
\]

We will then be able to obtain AP sequence solutions under the more favorable restriction

\[
(10.2) \quad k + \frac{k}{h^m} = O(1).
\]

First of all we require that \( \delta \) and \( f \) are \( C^3 \). As shown in Section 7 this implies that \( y \) is \( C^3 \). It implies further that \( \dot{y} \) satisfies

\[
(10.3) \quad \ddot{y} = [B-\delta] \dot{y} - \delta y + \dot{f}
\]

i.e. that the basic equation

\[
(10.4) \quad \dot{y} = [B-\delta]y + f
\]

can be differentiated formally.

We can now apply the Lax-Wendroff scheme to (10.4). Proceeding formally we write, assuming \( Bf \) is AP,

\[
(10.5) \quad y(t+k) = y(t) + k \dot{y}(t) + \frac{k^2}{2} \ddot{y}(t)
\]

\[
= y(t) + k[B-\delta]y(t) + \frac{k^2}{2} [B-\delta]^2 y(t)
\]

\[
+ \frac{k^2}{2} [-\delta y + (B-\delta)f + f] + kf(t) =
\]
\[
= [I + kB + \frac{k^2}{2} B^2]y(t) - k\delta y + k[\frac{k}{2} \delta^2 - \frac{k}{2} \delta + k\delta B]y + k[f + \frac{k}{2}(B-\delta) + \frac{k}{2} \dot{f}]
\]

We have written this as an inhomogeneous difference equation in order to apply techniques already developed.

Now in the expression for \( \dot{y} \) we approximate \( B \) by \( B_h \) and in the expression for \( \ddot{y} \) approximate \( B^2 \) by \( B_{ch}^2 \), for a certain constant \( c > 0 \) to be determined (\( c = 1/2 \) for the example described in (8.2)). Of course we will have to assume that \( B_h^2 \) approximates \( B^2 \) in the same sense that \( B_h \) approximates \( B \), but we will leave the precise hypothesis for later.

Applying these approximations to (10.5) and letting \( w_n(k, h) \) stand for the dependent variable we obtain the difference equation

\[
(10.6) \quad w_{n+1}(k, h) = U(k, h)w_n - k\delta w_n + kR(n, k, h)w_n + k\tilde{f}_n
\]

where

\[
(10.7) \quad (a) \quad U(k, h) = I + kB_h + \frac{k^2}{2} B_{ch}^2
\]

\[
(b) \quad \tilde{f}_n = f_n + \frac{k}{2} [B_{ch} - \delta]f_n + \frac{k}{2} \dot{f}_n
\]

\[
(c) \quad R(n, k, h) = \frac{k}{2} \delta^2 - \frac{k}{2} \dot{\delta} - k\delta B_{ch} = O(k^2 h^{-n}).
\]

Now \( U(k, h) \) represents an approximation to (10.1). We choose \( c \) so that for

\[
(10.8) \quad \frac{k}{h^m} = O(1)
\]
this discretization is stable, i.e.

\[(10.9) \quad \|U(k,h)^r\| \leq H, \quad r \geq 0.\]

We are going to use Theorem 2 to show that if \(\|R(n,k,h)\| \) (i.e. \(k + k/h^m\)) is sufficiently small, then the homogeneous version of (10.6) is exponentially stable; that is if \(W_n(k,h)\) is the fundamental solution to the homogeneous equation, then there exists constants \(K_1\) and \(\alpha_1\) independent of \(k\) and \(h\) such that

\[(10.10) \quad \|W_nW_j^{-1}\| \leq K_1 e^{-\alpha_1 k(n-j)}, \quad n \geq j,\]

where for simplicity we have suppressed the dependence of \(W_n\) on \(k\) and \(h\). Once we have obtained (10.10), \((L_n^r, L_n^s)\) (and \((\tilde{A}_n, \tilde{A}_n)\)) admissibility will follow in the usual manner.

To obtain (10.10) we observe that since we can allow \(k + k/h^m\) to be as small as required, Theorem 2 permits us to neglect the perturbing term \(R\). Also by neglecting a perturbation of the same order as \(kR\) we can work with the simpler equation

\[(10.11) \quad w_{n+1}^1 = [I-k\delta_n] U(kh)w_n^1.\]

Now consider the function \(g(t,s)\) defined in (7.11). Using the same argument given in Section 3 (see (3.16)), we see that \(g_n = g(nk,0)\) will satisfy the equation

\[(10.12) \quad g_{n+1} = [I+kD_n]g_n = [I-k\delta_n + k^2O(1)]g_n\]

where the "0" sign in (10.12) is independent of \(n\). Thus by
again neglecting a perturbation of the form \( k_0(k+k/h^m) \)
we obtain finally the equation

\[
(10.13) \quad \tilde{w}_{n+1} = [1+kD_n] U(k,h) \tilde{w}_n.
\]

But the fundamental solution to (10.13) is simply

\[
(10.14) \quad \tilde{w}_n^{-1} \tilde{w}_j^{-1} = g(nk,jk) U(k,h)^{n-j}
\]
as one can verify immediately, using the commutivity
properties of \( g_n \) with \( B_h \) (see (8.1e)) in the case that
\( \delta \) is an operator. Now the estimate

\[
(10.15) \quad \| \tilde{w}_n^{-1} \tilde{w}_j^{-1} \| \leq \frac{\bar{k}}{n-j}, \quad n \geq j,
\]
follows from (7.10a) and (10.9) and we can now apply
Theorem 2, noting, again that the terms \( k_1, \alpha_1, \varepsilon_0 \) depend
only on the terms \( \bar{k} \) and \( \bar{\alpha} \) (see the hypothesis of Theorem 2),
to obtain (10.10).

We now have \((L_n^\infty, L_n^\infty)\) (or \((\tilde{A}_n, \tilde{A}_n)\)) admissibility for
(10.6) and it is only necessary to study the convergence in
the case that the inhomogeneous term is given by (10.7b).
To do this we must now assume that \( B_h^2 \) approximates \( B^2 \); namely
that there exists an operator \( \tilde{L} \) such that if \( e \in D_\tilde{L} \cap D_{B^2} \)
then

\[
(10.16) \quad \| (B^2 - B_h^2) e \| \leq C_2 h^j \| \tilde{L} e \|
\]
We also assume that \( \tilde{L} \) satisfies the same assumption as \( L \)
(see (8.13)) so that if \( Lf \) and \( \tilde{L}f \) are AP we can conclude
If \( w_n(k,h) \) is the unique AP solution to (10.6), we can now show that \( w_n \) converges to \( y_n \) uniformly in \( n \).

In fact since \( y \) is \( C^3 \), \( y_n \) satisfies (10.5) up to an error \( O(k^3) \). If we then make use of (10.17) we see that \( y_n \) will satisfy (10.6) with an error \( kO(h^{j+k^2}) \), where this is uniform in \( n \). Then as the admissibility bound of (10.6) is independent of \( h \) (by (10.10)), we can conclude

\[
\| w_n - y_n \|_\infty = O(h^{j+k^2}) = O(h^{j+2m})
\]

where we have used (10.2).

For the example described in (8.2) we will have

\[
L = \frac{\partial^3}{\partial x^3} \quad \text{and} \quad \tilde{L} = \frac{\partial^4}{\partial x^4}, \quad c = \frac{1}{2}, \quad m = 1, \quad j = 2 \quad (\text{periodic boundary conditions})
\]

Finally we would like to point out that in the term \( \tilde{f}_n \) in (10.7b) one can replace \( B_{ch} f_n \) by \( Bf \), without changing the order of the error. This is simply because \( Lf \) is AP, hence

\[
\| (B-B_h) f \|_\infty = O(h^j)
\]
11. Linear Perturbations

We now consider the perturbed system

\[(11.1) \quad \dot{y} = [B-\delta]y + D(t)y + f(t)\]

Here \(D(t)\) is a bounded operator for each \(t\) and is almost periodic as an operator function. We also require

\[(11.2) \quad \|D\|_\infty \leq \varepsilon_0\]

where \(\varepsilon_0\) is to be specified.

The first step in finding AP solutions to (11.1) is to solve the equation

\[(11.3) \quad y = C(Dy) + C(f)\]

This equation can be solved uniquely by the contracting mapping principle provided \(\mu = \varepsilon_0 \overline{K} < 1\), where \(\overline{K}\) is the norm of \(C\) (see (7.18)). The contracting mapping principle also shows that the assignment of \(y\) from \(f\) is a bounded operator and in fact

\[(11.4) \quad \|y\|_\infty \leq \frac{\overline{K}}{1-\mu} \|f\|_\infty \equiv K_1 \|f\|_\infty\]

where \(K_1\) depends only on \(\varepsilon_0\) and is independent of \(D\).

We want to show that \(y\) satisfies (11.1), and if we assume that \(D, \delta\) and \(f\) are \(C^1\), the results of Section 7 show that it is sufficient to show that \(y\) is \(C^1\). This will follow if we observe that \(y\) will be the limit of iterates
\( y_i \) defined as

\begin{align}
& (a) \quad Y_0 = 0 \\
& (b) \quad Y_{i+1} = C(Dy_i) + C(f) .
\end{align}

We will then have for each \( i \)

\begin{equation}
\dot{Y}_{i+1} = C(D\dot{y}_i) + C(-\delta y_i + \dot{D}y_i + f)
\end{equation}

If we then define \( \ddot{y} \) as the unique solution to

\begin{equation}
\ddot{y} = C(D\dot{y}) + C(-\delta y + \dot{D}y + f)
\end{equation}

we will have

\begin{equation}
\dot{Y}_i \to \ddot{y}.
\end{equation}

To see (11.8) note that \( \|\dot{y}_i\|_\infty \) is bounded uniformly in \( i \) (from (11.6) and \( \mu < 1 \)) and if \( g_i \) is defined as

\begin{equation}
g_i = \sup_{r \geq i} \|\ddot{y} - \dot{y}_r\|
\end{equation}

Then \( \{g_i\} \) form a nonincreasing sequence satisfying

\begin{equation}
g_{i+1} \leq \mu g_i + o(1)_{i \to \infty}
\end{equation}

and (11.10) implies \( g_i \to 0 \) which, in turn, implies (11.8).

The fact that \( \ddot{y} = \dot{y} \) follows exactly as in Section 7 (see (7.27) ff). Observe that since \( \dot{y} \) is AP, \( By \) is also AP (from (11.1)).

Now consider the operators \( C_h \) defined by (8.6). The equation

\begin{equation}
Y_h = C_h(Dy_h) + C_h(f)
\end{equation}
can be solved exactly as (11.3) can. (For simplicity we suppose that \( \| C_h \| = \| C \| \) so that we have the same restriction on \( \varepsilon_0 \).

Our first task is to show that \( y_h \to y \). This will be a simple consequence of the strong convergence of \( C_h \) to \( C \). In fact if \( w_h = y - y_h \) we have

\[
(11.12) \quad w_h = [C-C_h](Dy) + C_h(Dw_h) + [C-C_h](f)
\]

whence

\[
(11.13) \quad \| w_h \|_\infty \leq \mu \| w_h \|_\infty + O(1)_{h \to 0}
\]

and this certainly implies \( \| w_h \| \to 0 \) as \( h \to 0 \).

Now of course this would not be useful unless we could prove

\[
(11.14) \quad \| w_h \|_\infty = O(h^j)
\]

This estimate can be shown provided we make certain assumptions on the interaction of \( D \) and the operator \( L \) defined in (8.11). We first observe that \( w_h \) satisfies the equation

\[
(11.15) \quad w_h = C_h(Dw_h) + C_h([B-B_h]y)
\]

and using the analogue of (11.4) for the equation (11.11), together with the uniform boundedness of the operators \( C_h \), we see

\[
(11.16) \quad \| w_h \|_\infty = O(\| [B-B_h]y \|_\infty)
\]
and it is sufficient to show

\begin{equation}
(11.17) \\
\| [B - B_h] y \|_{\infty} = O(h^j) .
\end{equation}

To obtain (11.17) we make an assumption on the operator L. We will call it Assumption I as we will later have to make the same type of assumption for other operators.

**Assumption I**: Suppose there exists a sequence of operators \( L_0 = I, L_1, \ldots, L_n = L \) such that

\begin{equation}
(11.18) \\
(\text{a}) \quad L_i \text{ are closed} \\
(\text{b}) \quad E = D_{L_0} \supseteq D_{L_1} \supseteq \ldots \supseteq D_{L_n} \\
(\text{c}) \quad L_i \text{ commutes with } Z_t \text{ for } t \geq 0 \quad (\text{i.e. } Z_t : D_{L_i} \rightarrow D_{L_i} \text{ and } L_i Z_t e = Z_t L_i e \text{ for } e \in D_{L_i}) \\
(\text{d}) \quad \text{If } e \in D_{L_L} \text{ then } D(t)e \in D_{L_L} \text{ and } \\
L_{L_L} D(t)e = \sum_{j=0}^{n} E_{L_L}^j(t)L_L^j e \\
\text{where the operators } E_{L_L}^j(t) \text{ are bounded and AP, and } E_{L_L}^j = D.
\end{equation}

Assumption I is clearly motivated by the example where

\( D(t) \) is multiplication by some function \( d(t,x) \) (\( x \) is the spatial variable).

Now Assumption I implies that if \( L_{L_L} f \) is AP then \( L_{L_L} y \) is AP, for \( \ell = 1, \ldots, n \). First note that if \( z = C(f) \) then

\begin{equation}
(11.19) \\
L_{L_L} z = C(L_{L_L} f) .
\end{equation}

This is a consequence of (11.18a,c) using the same argument as was used for B (see (7.34) ff). Now if we define \( z_{L_L,j} = L_{L_L} y_j \) where the \( y_j \) are defined in (11.5); then it certainly follows
from (11.18d) that \( z_{k;i} \) is AP for \( i \) and for \( k = 0, \ldots, n \).

We will now show that the sequence \( \{z_{k;i}\} \) converges as \( i \to \infty \).

To see this first set \( k = 1 \). We then have

\[
(11.20) \quad z_{1;i+1} = C(Dz_{1;i}) + C(E_0(t)y_i) + C(f)
\]

and \( z_{1,i} \to z_1 \), where \( z_1 \) is the solution to

\[
(11.21) \quad z_1 = C(Dz_1) + C(E_0y) + C(f) .
\]

In fact \( w_{1,i} = z_1 - z_{1,i} \) satisfies the equation

\[
(11.21) \quad w_{1,i+1} = C(Dw_{1,i}) + C(E_0(y-y_i))
\]

and

\[
(11.22) \quad \|w_{1,i}\|_\infty = \|z_1 - z_{1,i}\|_\infty \to 0 \quad i \to \infty
\]

follows from the same argument used in proving \( y \) is \( C^1 \) (see (11.9) ff). Finally by the closure of \( L_1 \) we must have \( z_1 = L_1y \). It is clear that a simple induction, using the same proof, will establish that

\[
(11.23) \quad L_\ell y_i \to \ell L_\ell y \quad \ell = 1, \ldots, n.
\]

In particular \( L (= L_n)y \) is AP and (11.17), hence (11.14) is valid.

Finally we point out that if we assume a chain of operators, as in Assumption I, for the operator \( B \), we will then have \( BDy \) AP. We will also have to make such an assumption for the operator \( \tilde{L} \) (see (10.16)), when applying the Lax-Wendroff scheme to (11.1), as will be discussed shortly.
We will now apply to (11.1) the two methods of time discretization that have already been described. We are interested in the existence of AP sequence solutions, or equivalently exponential stability for the homogeneous difference equation, with \( k \) and \( h \) restricted according to relations (9.4) and (10.2). As one might expect this will be a simple consequence of Theorem 2, provided \( \epsilon_0 \) is sufficiently small.

We consider first the linear multistep scheme as discussed in Section 9. Since the linear term \( C_n \) of the difference scheme (see (2.11)) is a smooth function of its \( \lambda+1 \) arguments, it follows that if \( k + k/h^m \) is sufficiently small (i.e. \( \|kB_h\| \) is small) then the difference equation for (11.1) can be written

\[
(11.24) \quad w_{n+1} = C_n^* w_n + k O(\epsilon_0) w_n + k \tilde{f}_n
\]

where \( C_n^* \) is the linear term corresponding to the unperturbed version of (11.1), i.e.

\[
(11.25) \quad y = [B_h-\delta]y + f
\]

Since we have shown in Section 9 that in a region of the form

\[
(11.26) \quad k + \frac{k}{h^{2m}} = O(1)
\]

the unperturbed homogeneous difference equation

\[
(11.27) \quad w_{n+1} = C_n^* w_n
\]

is exponentially stable, with constants \( K_1 \) and \( a_1 \) independent
of \( k \) and \( h \) (see (9.7) ff) it will follow from Theorem 2 (see the remark following (9.16)) that in the same region the homogeneous version of (11.24) will be exponentially stable if \( \epsilon_0 \) is small enough. It then follows easily that the inhomogeneous equation (11.24) will have \((L^n, L^n)\) (or \((\tilde{A}^n, \tilde{A}^n)\)) admissible, with an admissibility bound independent of \( k \) and \( h \).

We can thus obtain AP sequence solutions to (11.24) and, as shown in Section 9 (see (9.18) ff), we will obtain convergence of \( O(k^p) \) provided \( \|y_h^{p+1}\|_{\infty} \) is bounded in \( h \). However, as one can trivially see, if \( D, \delta, \) and \( f \) are \( C^{p+1} \) then the same argument used to show \( y \) is \( C^1 \) (see (11.6) ff) can be repeated \( p+1 \) times and applied to the operators \( C_h \) as well as \( C \) to show that \( y \) and \( y_h \) are \( C^{p+1} \) and in fact (11.7) can be generalized to (compare with (9.18))

\[
(11.28) \quad y^i = C(Dy^i) + C([(D-\delta)y]^i - (D - \delta)y^i + f^i), \quad i = 1, \ldots, p+1,
\]

with a similar formula for \( y_h \) and \( C_h \). This certainly shows that \( \|y_h^i\| \) will be bounded in \( h \) for \( i = 1, \ldots, p+1 \) and thus that we can obtain a complete extension of the theory of Section 9. We note in passing, that the same proof which led to (11.13) will show that \( y_h^i - y^i \).

The extension of the Lax-Wendroff scheme will be just as simple. In fact (under the assumption that \( D, \delta \) and \( f \) are \( C^3 \)) the difference equation for (11.1) will have the
same linear term as the equation for (11.25), (10.6) except for a perturbation

\[ (11.29) \quad k[D_n + \frac{k}{2} (B_{ch} - \delta_n)D_n + D_n(B_{ch} - \delta_n) + D_n^2 + \delta_n) ] \]

as one can easily see by carrying out the expansion of Section 10 for the equation (11.1). It follows that if \( k/h = O(1) \) (so that \( k\|B_{ch}\| \) is bounded), Theorem 2 is applicable, provided \( k \) and \( \varepsilon_0 \) are sufficiently small. One can then obtain AP sequence solutions and carry out the convergence argument of Section 10 (see (10.17) ff), with the only difficulty being the verification that

\[ (11.30) \quad \| (B^2 - B^2_h) y \|_\infty = O(h^j) \]

and this will follow immediately if one assumes a chain as in Assumption I for the operator \( \tilde{L} \) introduced in (10.16).
Appendix. Proof that \((L_{\infty}, L_{\infty})\) Admissibility Implies an Exponential Dichotomy.

Here we would like to give a simplified proof of the proposition that \((\tilde{A}, \tilde{A})\) admissibility for the inhomogeneous equation

\[
\dot{y} = Ay + f
\]

implies an exponential dichotomy for the homogeneous equation

\[
\dot{y} = Ay
\]

It is assumed of course that \(A(t)\) is AP although we will actually do the \(L_{\infty}\) case. The proof is valid only if the underlying space \(E\) is finite dimensional and we can then regard \(A\) as a matrix and \(y\) and \(f\) as \(m\) vectors, where \(m\) is the dimension of \(E\). The reader is referred to Section 5 where an entirely similar proof for the difference equation case is given.

The general proof is given in M & S (p. 344, Theorem 103.A), but this proof relies strongly on preceding material. A finite dimensional proof is given in Coppel (p. 134 ff) for an equation on \([0, \infty)\). His proof can be extended to an equation on the whole real axis, but the proof given here is simpler and more in keeping with the ideas of Massera and Schäffer.

It is shown in M&S (Theorem 103.A) that \((\tilde{A}, \tilde{A})\) admissibility implies \((L_{\infty}, L_{\infty})\) admissibility and this will be our
starting point. We thus assume that for every \( f \) in \( L_\infty \), (A.1) has a unique solution \( y \) in \( L_\infty \) and the estimate

\[
(A.3) \quad \|y\|_\infty \leq \tilde{K}\|f\|_\infty
\]

will hold for a certain constant \( \tilde{K} \).

Now let \( S_1 \) be the subspace of initial data which gives rise to solutions to (A.2) that are bounded for \( t \in [0, \infty) \). Let \( S_2 \) be the analogous subspace giving rise to solutions that are bounded for \( t \in (-\infty, 0] \). We must have \( S_1 \cap S_2 = \{0\} \) because any nontrivial, bounded solution to (A.2) would violate the uniqueness requirement of our definition of admissibility. Let \( S_3 \) be any subspace complementary to \( S_1 \oplus S_2 \) so that

\[
(A.4) \quad E = S_1 \oplus S_2 \oplus S_3
\]

Let \( P_1, P_2, P_3 \) be the associated projections, i.e. \( P_1 \) is the projection onto \( S_1 \) along \( S_2 \oplus S_3 \), etc. We have

\[
(A.5) \quad I = P_1 + P_2 + P_3
\]

\[
P_i P_j = 0, \quad i = 1, 2, 3; \quad j = 1, 2, 3; \quad i \neq j.
\]

We will first show \( S_3 = \{0\} \), i.e. \( P_3 = 0 \). This will follow from the variation of constants formula, just as in the difference equation case. If \( f \) has compact support and \( y(t) \) is the unique \( L_\infty \) solution to (A.1), then since \( y \) is a bounded solution to (A.2) for large \( |t| \), we must have
\begin{align}
\text{(A.6)} & \quad (a) \quad P_2 Y_0 = - \int_0^\infty P_2 Y^{-1}(s) f(s) \, ds \\
& \quad (b) \quad P_3 Y_0 = - \int_0^\infty P_3 Y^{-1}(s) f(s) \, ds \\
& \quad (c) \quad P_1 Y_0 = \int_{-\infty}^0 P_1 Y^{-1}(s) f(s) \, ds \\
& \quad (d) \quad P_3 Y_0 = \int_{-\infty}^0 P_3 Y^{-1}(s) f(s) \, ds 
\end{align}

Here \( Y(t) \) is the fundamental solution to (A.2).

Now for any vector \( z \in E \), if we set \( f = \chi_{[0,1]} Y(t) z \) then (A.6b,d) yield immediately \( P_3 z = 0 \) and we can conclude \( P_3 = 0 \), i.e.

\begin{align}
E = S_1 \oplus S_2 \\
I = P_1 + P_2
\end{align}

Using (A.6a,c) we also see that for any \( f \) with compact support we have

\begin{align}
\text{(A.8)} & \quad y(t) = \int_{-\infty}^\infty G(t,s) f(s) \, ds \\
\text{where} & \quad G(t,s) = Y(t) P_1 Y^{-1}(s), \quad t \geq s \\
& \quad \quad -Y(t) P_2 Y^{-1}(s), \quad s \geq t
\end{align}

We will next use (A.3) to show that for every fixed \( t \), \( G(t,s) \) is in \( L_1(-\infty,\infty) \) as a function of \( s \), and in fact
where $\bar{K}$ is given in (A.3). This is a simple consequence of using (A.8) as an operator on $L_\infty(I)$, where $I$ is any compact interval to conclude that

$$\int_{-\infty}^{\infty} ds \|G(t,s)\| \leq \bar{K}$$

and then letting $I \rightarrow (-\infty, \infty)$.

We now will show that (A.10) implies

$$\|G(t,s)\| = O(1) .$$

To see this we simply observe that $Y^{-1}(s)$ satisfies the adjoint equation

$$Z' = - Z A(s)$$

where we have used "'" to stand for $d/ds$. Hence fixing $t$ and considering the region $s < t$, we have

$$G(t,s)' = - G(t,s) A(s)$$

Now since $A$ is bounded it follows that $G'(t,s)$ is in $L_1$, as a function of $s$, on the interval $(-\infty, t]$ with an $L_1$ norm which can be bounded uniformly in $t$. This of course means that $G(t,s)$ must approach a limit as $s \rightarrow -\infty$ and since $G(t,s)$ is in $L_1$ (as a function of $s$), this limit can only be zero. We thus have

$$G(t,s) = \int_{-\infty}^{s} dr G'(t,r) = O(1) , \quad s \leq t,$$
and a similar argument establishes (A.12) for \( s \geq t \).

We must now show that (A.12) can be replaced by an exponentially decaying factor, and to do this we consider perturbations of (A.1) and (A.2); namely for small \( \varepsilon > 0 \) consider the equations

\begin{align}
(A.16) \quad \dot{y} &= [A-\varepsilon I]y + f \\
(A.17) \quad \dot{y} &= [A-\varepsilon I]y
\end{align}

If \( \varepsilon \) is sufficiently small we will have \((I_{\infty} I_{\infty})\) admissible for (A.16). This is a simple consequence of the contracting mapping principle analogous to the argument of Section 4C (see (4.9) ff). The contracting mapping principle will also show that the admissibility bound for (A.16) will be given by (see A.3)

\begin{equation}
(A.18) \quad \bar{K}_\varepsilon = \frac{\bar{K}}{1 - \varepsilon \bar{K}} \leq K, \quad |\varepsilon| \leq \varepsilon_0,
\end{equation}

if \( \varepsilon_0 \) is sufficiently small. Thus the admissibility bound for (A.16) can be taken independent of \( \varepsilon \), if \( \varepsilon \) is small enough.

Now the fundamental solution to (A.18) is

\begin{equation}
(A.19) \quad Y_\varepsilon(t) = e^{-\varepsilon t} Y(t)
\end{equation}

and if \( P_1^\varepsilon, P_2^\varepsilon \) denote the projections replacing \( P_1 \) and \( P_2 \), while \( G_\varepsilon(t,s) \) is the new Green's function, then the analysis which led to (A.12), and which depended only on admissibility, is equally valid here and we can conclude
(A.20) \[ \| G^\varepsilon(t,s) \| = O(1) ; \| G^\varepsilon(t,\cdot) \|_1 = O(1) , \]

where the bound in (A.20) can be taken independent of \( \varepsilon \) (because of (A.18)).

The estimate (A.20) means in particular that for small \( \varepsilon \) there is a constant \( K_2 \) independent of \( \varepsilon \) such that if \( s \geq t \),

(A.21) \[ \| Y(t)P_2 Y^{-1}(s) \| \leq K_2 e^{-\varepsilon(s-t)} \]

and it is only necessary to show that

(A.22) \[
\begin{align*}
P_1^\varepsilon &= P_1 \\
P_2^\varepsilon &= P_2
\end{align*}
\]

To prove (A.22) we let \( S_1^\varepsilon \) and \( S_2^\varepsilon \) replace \( S_1 \) and \( S_2 \).

It then follows from (A.19) and \( \varepsilon > 0 \)

(A.23) \[
\begin{align*}
S_1 &\subseteq S_1^\varepsilon \\
S_2^\varepsilon &\subseteq S_2
\end{align*}
\]

and (A.22) will be established if we can show that the ranks of the stable and unstable projections are unchanged for small \( \varepsilon \), and this, in turn, will follow from

(A.24) \[ \| P_1^\varepsilon - P_1 \| = O(\varepsilon) . \]

To obtain (A.22) we set \( H^\varepsilon(t,s) = G^\varepsilon(t,s) - G(t,s) \).

Now for any \( f \) with compact support we let \( y \) and \( y_\varepsilon \) be the unique bounded solutions to (A.1) and (A.16) respectively. If \( w_\varepsilon = y_\varepsilon - y \) we see that \( w_\varepsilon \) is the unique bounded solution to

(A.25) \[ \dot{w}_\varepsilon = Aw_\varepsilon - \varepsilon y_\varepsilon , \]
and we can conclude from (A.3) together with its analogue for (A.16) that

(A.26) \[ \|w_\varepsilon\|_\infty = O(\varepsilon)\|f\|_\infty \]

Now \( w_\varepsilon \) can also be written as

(A.27) \[ w_\varepsilon(t) = \int_{-\infty}^\infty ds \, H_\varepsilon(t,s) \, \delta(s) \]

and using the same argument that led to (A.10) we see that

(A.28) \[ \int_{-\infty}^\infty ds \, \|H_\varepsilon(t,s)\| = O(\varepsilon) \]

(Note that the bound in (A.28) will be independent of \( t \), although we will only use it for \( t = 0 \).)

Now if \( s \leq t \) we see that as a function of \( s \), \( H_\varepsilon(t,s) \)

satisfies the equation

(A.29) \[ H'_\varepsilon = -H_\varepsilon A + \varepsilon \, G_\varepsilon(t,s) \]

This equation, together with (A.20) and (A.28), shows that

(A.30) \[ \int_{-\infty}^\infty ds \, \|H'_\varepsilon(t,s)\| = O(\varepsilon) \]

and since \( H_\varepsilon(t,s) \) is in \( L_1 \) as a function of \( s \) we can write for \( s \leq t \)

(A.31) \[ H_\varepsilon(t,s) = \int_{-\infty}^t ds \, H'_\varepsilon(t,s) = O(\varepsilon) \]

and if we set \( t = 0 \), \( s = 0_+ \), (A.31) yields (A.24), which proves one part of the requirement of an exponential dichotomy and the other part follows on replacing \(-\varepsilon\) by \(+\varepsilon\).
References


The theory of Massera and Schäffer relating the existence of unique almost periodic solutions of an inhomogeneous linear equation to an exponential dichotomy for the homogeneous equation has been completely extended to discretizations by a strongly stable difference scheme. In addition it has been shown that the
almost periodic sequence solution will converge to the differential equation solution at a rate $O(k^p)$, where $p$ is the accuracy of the scheme, uniformly in $t$, if the coefficients are sufficiently smooth.

The preceding theory has also been applied to a class of exponentially stable partial differential equations to which one can apply the Hille-Yoshida Theorem. It is possible to prove the existence of unique almost periodic solutions of the inhomogeneous equation which can be approximated by almost periodic sequences which are the solutions to appropriate discretizations. Two methods of discretizations are discussed; the strongly stable scheme described above and the Lax-Wendroff scheme.