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on for Supersonic Flow
Luigi Morino.
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of Engineering



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A FINITE-ELEMENT FORMULATION FOR
SUPERSONIC FLOWS
AROUND COMPLEX CONFIGURATIONS

by

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ABSTRACT

The problem of small perturbation potential supersonic flow around complex configurations is considered. This problem requires the solution of an integral equation relating the values of the potential on the surface of the body to the values of the normal derivative, which is known from the small perturbation boundary conditions. The surface of the body is divided into small (hyperboloidal quadrilateral) surface elements, Σ_i , which are described in terms of the Cartesian components of the four corner points. The values of the potential (and its normal derivative) within each element is assumed to be constant and equal to its value at the centroid of the element. This yields a set of linear algebraic equations. The coefficients of the equation are given by source and doublet integrals over the surface elements, Σ_i . Closed form evaluations of the integrals are presented.

FOREWORD

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SECTION I

FORMULATION OF THE PROBLEM

1.1 Introduction

A general theory for compressible unsteady potential aerodynamic flow around lifting bodies having arbitrary shapes and motions is given in Refs. 1 and 2. Applications to wings in subsonic flows are given in Refs. 3, 4 and 5. A finite element formulation for complex configurations in subsonic flows is given in Ref. 6. Here the formulation is extended to steady supersonic flows.

The equation of the aerodynamic potential is given by

$$(1-M^2) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (1.1)$$

with boundary conditions (\bar{n}_b is the outwardly directed normal to the surface of the body, σ_b)

$$\frac{\partial \varphi}{\partial \bar{n}_b} = -n_x \quad (\text{on } \sigma_b) \quad (1.2)$$

The Green function for Eq. (1.1) is , for $M > 1$

$$G = -\frac{1}{2\pi} \frac{H \left[x_0 - x - B \sqrt{(y-y_0)^2 + (z-z_0)^2} \right]}{\gamma} \quad (1.3)$$

where H is the Heaviside function and

$$\gamma = \sqrt{(x-x_0)^2 - B^2[(y-y_0)^2 + (z-z_0)^2]} \quad (1.4)$$

$$B = \sqrt{M^2 - 1} \quad (1.5)$$

The Green theorem for Eq. (1.1) is (Ref. 1)

$$E(x_0, y_0, z_0) \varphi(x_0, y_0, z_0) = \iint_{\sigma} \left[(1-M^2) \frac{\partial S}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial S}{\partial y} \frac{\partial \varphi}{\partial y} + \frac{\partial S}{\partial z} \frac{\partial \varphi}{\partial z} \right] G \frac{d\sigma}{|\nabla_{ps} S|} - \iint_{\sigma} \left[(1-M^2) \frac{\partial S}{\partial x} \frac{\partial G}{\partial x} + \frac{\partial S}{\partial y} \frac{\partial G}{\partial y} + \frac{\partial S}{\partial z} \frac{\partial G}{\partial z} \right] \varphi \frac{d\sigma}{|\nabla_{ps} S|} \quad (1.6)$$

where ∇_{ps} is the gradient in the Physical Space variables (x, y, z) , σ is a suitable surface (see Section 2), defined by the equation

$$S(x, y, z) = 0 \quad (1.7)$$

and the function E is given by

$$\begin{aligned} E &= 0 && \text{inside } \sigma \\ E &= 1 && \text{outside } \sigma \end{aligned} \quad (1.8)$$

Introducing the supersonic Prandtl-Glauert nondimensional variables (l is a characteristic length of the body)

$$X = \frac{x}{Bl}, \quad Y = \frac{y}{l}, \quad Z = \frac{z}{l}, \quad \phi = \frac{\varphi}{U_0 l} \quad (1.9)$$

yields

$$\frac{1}{r} \frac{d\sigma}{|\nabla_{ps} S|} = \frac{1}{r} \frac{dx dy dz}{|\partial S / \partial z|} = \frac{1}{R} \frac{dX dY}{|\partial S / \partial Z|} = \frac{1}{R} \frac{d\Sigma}{|\nabla S|} \quad (1.10)$$

where ∇ is the gradient in the Prandtl-Glauert variables, Σ is the surface of the space (X, Y, Z) defined by the equation

$$S(BX, Y, Z) = 0 \quad (1.11)$$

and

$$R = \sqrt{(X-X_0)^2 + (Y-Y_0)^2 + (Z-Z_0)^2} \quad (1.12)$$

Combining Eqs. (1.6) and (1.10) yields

$$\begin{aligned} 2\pi E \phi_0 = & - \oint_{\Sigma} \left(-\frac{\partial S}{\partial X} \frac{\partial \phi}{\partial X} + \frac{\partial S}{\partial Y} \frac{\partial \phi}{\partial Y} + \frac{\partial S}{\partial Z} \frac{\partial \phi}{\partial Z} \right) \frac{H}{R} \frac{d\Sigma}{|\nabla S|} \\ & + \oint_{\Sigma} \left(-\frac{\partial S}{\partial X} \frac{\partial}{\partial X} \left(\frac{H}{R} \right) + \frac{\partial S}{\partial Y} \frac{\partial}{\partial Y} \left(\frac{H}{R} \right) + \frac{\partial S}{\partial Z} \frac{\partial}{\partial Z} \left(\frac{H}{R} \right) \right) \phi \frac{d\Sigma}{|\nabla S|} \end{aligned} \quad (1.13)$$

Note that

$$\bar{N} = \frac{\nabla S}{|\nabla S|} \quad (1.14)$$

is the normal to the deformed surface Σ . In order to use compact vector notations, it is usual (see, for instance, Ref. 7) to introduce the concept of the conormal

$$\bar{N}^c = N_x \bar{i} - N_y \bar{j} - N_z \bar{k} \quad (1.15)$$

and the cogradient

$$\bar{\nabla}^c = \frac{\partial}{\partial X} \bar{i} - \frac{\partial}{\partial Y} \bar{j} - \frac{\partial}{\partial Z} \bar{k} \quad (1.16)$$

With this notation, Eq. (1.13) reduces to

$$2\pi E \phi_0 = \oint_{\Sigma} \left[\frac{H}{R} \frac{\partial \phi}{\partial \bar{N}^c} - \phi \frac{\partial}{\partial \bar{N}^c} \left(\frac{H}{R} \right) \right] d\Sigma \quad (1.17)$$

where the conormal derivative is given by

$$\frac{\partial}{\partial \bar{N}^c} = \bar{N}^c \cdot \nabla = \bar{N} \cdot \nabla^c \quad (1.18)$$

Equation (1.16) is in agreement with Eq. (6.89) of Ref. 7,

where the opposite convention on the direction of the normal is used (here the normal \vec{n} is directed from the region $E = 0$ to the region $E = 1$).

On the other hand, the boundary conditions on the surface of the body, Σ_b , can be written as

$$\begin{aligned} \vec{\nabla}S \cdot \vec{\nabla}\phi &= \left(\frac{\partial S_b}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial S_b}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial S_b}{\partial z} \frac{\partial \phi}{\partial z} \right) + U_\infty \frac{\partial S_b}{\partial x} \\ &= \frac{U_\infty}{\beta} \left(\frac{1}{\beta^2} \frac{\partial S_b}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial S_b}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial S_b}{\partial z} \frac{\partial \phi}{\partial z} \right) + \frac{1}{\beta} \frac{\partial S_b}{\partial x} \\ &= \frac{U_\infty}{\beta} \left(-\frac{\partial S_b}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial S_b}{\partial y} \frac{\partial \phi}{\partial y} + \frac{\partial S_b}{\partial z} \frac{\partial \phi}{\partial z} + \left(\frac{1}{\beta^2} + 1 \right) \frac{\partial S_b}{\partial x} \frac{\partial \phi}{\partial x} + \frac{1}{\beta} \frac{\partial S_b}{\partial x} \right) \\ &= 0 \end{aligned} \tag{1.19}$$

or

$$\frac{\partial \phi}{\partial N^c} = -\frac{1}{\beta} N_x \left(1 + \frac{M^2}{\sqrt{M^2 - 1}} \frac{\partial \phi}{\partial x} \right) \tag{1.20}$$

Neglecting terms of the same order of magnitude as those neglected in linearizing the equation of the aerodynamic potential yields

$$\frac{\partial \phi}{\partial N^c} = -\frac{1}{\beta} N_x \quad \text{on the body} \tag{1.21}$$

Finally, note that the linearized Bernoulli theorem for potential flow yields

$$C_p^{(L)} = -\frac{2}{U_\infty} \frac{\partial \phi}{\partial x} \tag{1.22}$$

or, in Prandtl Glauert variables,

$$C_p^{(L)} = -2 \frac{1}{\sqrt{M^2 - 1}} \frac{\partial \phi}{\partial x} \quad (1.23)$$

It may be worth noting that the exact boundary conditions may be rewritten as

$$\frac{\partial \phi}{\partial N^c} = -\frac{1}{B} N_x \left(1 - \frac{1}{2} M^2 C_p^{(L)} \right) \quad (1.24)$$

1.2 Supersonic Vector Algebra

The use of the conormal and the cogradient was found to be quite cumbersome for the extension of the subsonic finite-element formulation to supersonic flow. The algebraic manipulations become much simplified if a special vector algebra is introduced. This algebra is called here, supersonic vector algebra or super-algebra. The sum, dot product and cross product are defined in the usual way. In addition, it is convenient to introduce the supersonic dot product or super-product as

$$\bar{a} \circ \bar{b} = a_x b_x - a_y b_y - a_z b_z \quad (1.25)$$

With this notation,

$$R = \sqrt{(x-x_0)^2 - (y-y_0)^2 - (z-z_0)^2} = \sqrt{\bar{f} \circ \bar{g}} \quad (1.26)$$

where

$$\bar{f} = \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} - \left\{ \begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array} \right\} = \bar{P} - \bar{P}_0 \quad (1.27)$$

Similarly

$$\frac{\partial}{\partial N^c} \equiv \bar{N} \circ \bar{\nabla} \quad (1.28)$$

Furthermore, in addition to the usual norm of a vector

$$|\bar{a}| = \sqrt{\bar{a} \cdot \bar{a}} \quad (1.29)$$

it is convenient to use the supersonic norm or supernorm

$$\|\bar{a}\| = \sqrt{|\bar{a}_1 \circ \bar{a}_1|} \quad (1.30)$$

Note that

$$\bar{a} \circ \bar{a} \geq 0 \quad a_x^2 \geq a_y^2 + a_z^2 \quad (1.31)$$

that is, $\bar{a} \circ \bar{a} > 0$ ($\bar{a} \circ \bar{a} < 0$; $\bar{a} \circ \bar{a} = 0$) if the angle between the vector \bar{a} and the x-axis is less than (greater than; equal to) 45° , which implies that the vector \bar{a} is pointed inside (outside; on) the Mach cone (see Fig. 1). Further development of the superalgebra is given in Appendix A, where the first super-rule

$$(\bar{a} \times \bar{b}) \circ (\bar{c} \times \bar{d}) = (\bar{a} \circ \bar{c})(\bar{b} \circ \bar{d}) - (\bar{a} \circ \bar{d})(\bar{b} \circ \bar{c}) \quad (1.32)$$

and the second super-rule

$$\begin{aligned} & (\bar{a} \circ \bar{a}) \left[(\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) \right] - (\bar{a} \cdot \bar{b} \times \bar{c})^2 \\ & = (\bar{a} \circ \bar{c}) \left[(\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{a}) \right] + (\bar{a} \circ \bar{b}) \left[(\bar{c} \times \bar{d}) \circ (\bar{c} \times \bar{a}) \right] \end{aligned} \quad (1.33)$$

and the third super-rule

$$(\bar{a} \times \bar{d} \circ \bar{b} \times \bar{c})(\bar{a} \cdot \bar{f} \times \bar{g}) + (\bar{a} \cdot \bar{f} \circ \bar{b} \times \bar{c})(\bar{a} \cdot \bar{g} \times \bar{d}) + (\bar{a} \times \bar{g} \circ \bar{b} \times \bar{c})(\bar{a} \times \bar{d} \cdot \bar{f}) = 0 \quad (1.34)$$

are derived.

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With this notation, Eq. (1.13) may be rewritten as*

$$2\pi E.\phi = \oint_{\Sigma} \left[\bar{N} \circ \bar{\nabla} \phi \frac{H}{\sqrt{\bar{g} \circ \bar{g}}} - \bar{N} \circ \bar{\nabla} \left(\frac{H}{\sqrt{\bar{g} \circ \bar{g}}} \right) \phi \right] d\Sigma \quad (1.35)$$

1.3 Finite Element Formulation

Assume that the surface Σ is divided into N small finite elements Σ_i and consider the simplest finite element representation, that is, assume ϕ and

$$\phi^{(N)} = \frac{\partial \phi}{\partial N^c} = -\bar{N} \circ \bar{\nabla} \phi \quad (1.36)$$

constant within each element. Equation (1.35) then reduces to

$$\begin{aligned} E.\phi &= -\sum_{i=1}^N \phi_i^{(N)} \frac{1}{2\pi} \iint_{\Sigma_i} \frac{H}{\sqrt{\bar{g} \circ \bar{g}}} d\Sigma_i - \sum_{i=1}^N \phi_i \frac{1}{2\pi} \iint_{\Sigma_i} \bar{N} \circ \bar{\nabla} \frac{H}{\sqrt{\bar{g} \circ \bar{g}}} d\Sigma_i \\ &= -\sum_{i=1}^N \phi_i^{(N)} S_i - \sum_{i=1}^N \phi_i D_i \end{aligned} \quad (1.37)$$

where S_i and D_i are the source integral

$$S = \frac{1}{2\pi} \iint_{\Sigma} \frac{H}{\sqrt{\bar{g} \circ \bar{g}}} d\Sigma \quad (1.38)$$

and the doublet integral

$$D = \frac{1}{2\pi} \iint_{\Sigma} \bar{N} \circ \bar{\nabla} \frac{H}{\sqrt{\bar{g} \circ \bar{g}}} d\Sigma \quad (1.39)$$

*Note the analogy with the subsonic Green Theorem

$$-4\pi E.\phi = \oint_{\Sigma} \left[\bar{N} \cdot \bar{\nabla} \phi \frac{1}{\sqrt{\bar{g} \cdot \bar{g}}} - \bar{N} \cdot \bar{\nabla} \left(\frac{1}{\sqrt{\bar{g} \cdot \bar{g}}} \right) \phi \right] d\Sigma$$

evaluated on the i th element, Z_i .

In Sections 2 and 3, the evaluation of S and D is derived for a hyperboloidal element (see next subsection). Note that for a planar element*

$$D = -\frac{1}{2\pi} \iint \bar{N}_0 \bar{v}_0 \frac{H}{\sqrt{f_0 g_0}} dz = -\frac{1}{2\pi} \bar{N}_0 \bar{v}_0 \iint \frac{H}{\sqrt{f_0 g_0}} dz = -\bar{N}_0 \bar{v}_0 \bar{S} \quad (1.40)$$

This relation is extremely useful since it is possible to evaluate D (which in general would involve the use of the finite part of the integral) as the conormal derivative of S (which does not contain infinite part).

1.4 Hyperboloidal Element

Consider the equations

$$\begin{aligned} x &= x_c + x_1 \xi + x_2 \eta + x_3 \xi \eta \\ y &= y_c + y_1 \xi + y_2 \eta + y_3 \xi \eta \\ z &= z_c + z_1 \xi + z_2 \eta + z_3 \xi \eta \end{aligned} \quad (1.41)$$

or, in vector notations

$$\bar{p} = \bar{p}_c + \bar{p}_1 \xi + \bar{p}_2 \eta + \bar{p}_3 \xi \eta \quad (1.42)$$

This represents a hyperboloid. The lines $\eta = \text{const}$ and $\xi = \text{const}$ are clearly straight lines. Consider the hyperboloidal element (Fig. 2) defined by the above equation with

$$\begin{aligned} -1 &\leq \xi \leq 1 \\ -1 &\leq \eta \leq 1 \end{aligned} \quad (1.43)$$

*Note that $\bar{v} f(\bar{p}-\bar{p}_0) = -\bar{v}_0 f(\bar{p}-\bar{p}_0)$ where \bar{v}_0 is the gradient for the variables (x_0, y_0, z_0) .

The centroid of the element is $\bar{p}_c (\xi = \eta = 0)$. The corner points of this element are

$$\begin{aligned}
 \bar{p}_{++} &= \bar{p}_c + \bar{p}_1 + \bar{p}_2 + \bar{p}_3 & (\xi = +1, \eta = +1) \\
 \bar{p}_{+-} &= \bar{p}_c + \bar{p}_1 - \bar{p}_2 - \bar{p}_3 & (\xi = +1, \eta = -1) \\
 \bar{p}_{-+} &= \bar{p}_c - \bar{p}_1 + \bar{p}_2 - \bar{p}_3 & (\xi = -1, \eta = +1) \\
 \bar{p}_{--} &= \bar{p}_c - \bar{p}_1 - \bar{p}_2 + \bar{p}_3 & (\xi = -1, \eta = -1)
 \end{aligned} \tag{1.44}$$

The inverse relation is

$$\begin{aligned}
 \bar{p}_c &= \frac{1}{4} (\bar{p}_{++} + \bar{p}_{--} + \bar{p}_{+-} + \bar{p}_{-+}) \\
 \bar{p}_1 &= \frac{1}{4} (\bar{p}_{++} + \bar{p}_{--} - \bar{p}_{+-} - \bar{p}_{-+}) \\
 \bar{p}_2 &= \frac{1}{4} (\bar{p}_{++} - \bar{p}_{--} + \bar{p}_{-+} - \bar{p}_{+-}) \\
 \bar{p}_3 &= \frac{1}{4} (\bar{p}_{+-} - \bar{p}_{-+} - \bar{p}_{++} + \bar{p}_{--})
 \end{aligned} \tag{1.45}$$

Note that the four boundaries of the element ($\xi = \pm 1, \eta = \pm 1$) are straight lines given by

$$\begin{aligned}
 \bar{p} &= (\bar{p}_c + \bar{p}_1) + (\bar{p}_2 + \bar{p}_3) \eta \\
 \bar{p} &= (\bar{p}_c - \bar{p}_1) + (\bar{p}_2 - \bar{p}_3) \eta \\
 \bar{p} &= (\bar{p}_c + \bar{p}_2) + (\bar{p}_1 + \bar{p}_3) \xi \\
 \bar{p} &= (\bar{p}_c - \bar{p}_2) + (\bar{p}_1 - \bar{p}_3) \xi
 \end{aligned} \tag{1.46}$$

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Next, assume that the surface of the aircraft is divided into curved quadrilateral elements with four corner points $\bar{p}_{++}, \bar{p}_{+-}, \bar{p}_{-+}, \bar{p}_{--}$. Then, as already mentioned, these elements can be replaced by the hyperboloidal element (described above) determined by the four corner points $\bar{p}_{++}, \bar{p}_{+-}, \bar{p}_{-+}, \bar{p}_{--}$ (see Fig. 2). It may be noted that the surface is continuous since adjacent elements have in common the straight line connecting the two common corner points. It may be noted also that the

\bar{P}_C is the centroid of the hyperboloidal element Σ , and hence it will be indicated as

$$\bar{P}_C = \bar{P} \quad (1.47)$$

1.5 Surface Geometry for Hyperboloidal Elements

Next note that the geometry of the hyperboloidal element is a particular case of the general equation for a surface in a three-dimensional Euclidean space, which is given by

$$\bar{P} = \bar{P}(\xi, \eta) \quad (1.48)$$

where ξ and η are the generalized curvilinear coordinates.

Then the two base vectors \bar{a}_1, \bar{a}_2 are given by (Fig. 3)

$$\begin{aligned} \bar{a}_1 &= \frac{\partial \bar{P}}{\partial \xi} = \bar{P}_1 + \eta \bar{P}_3 \\ \bar{a}_2 &= \frac{\partial \bar{P}}{\partial \eta} = \bar{P}_2 + \xi \bar{P}_3 \end{aligned} \quad (1.49)$$

The unit normal to the surface is given by

$$\bar{n} = \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} \quad (1.50)$$

and is directed according to the right-hand rule (Fig. 3).

The surface element $d\Sigma$ is given by (Fig. 3)

$$d\Sigma = |\bar{a}_1 d\xi \times \bar{a}_2 d\eta| = |\bar{a}_1 \times \bar{a}_2| d\xi d\eta \quad (1.51)$$

1.6 Expressions for b_{hk} and c_{hk} for Hyperboloidal Element

Combining Eqs. (1.10), (1.50) and (1.51) yields

$$S = \frac{1}{2\pi} \iint_{-1}^{+1} \frac{H}{|\bar{P}|} |\bar{a}_1 \times \bar{a}_2| d\xi d\eta \quad (1.52)$$

Similarly, combining Eqs. (1.8), (1.13) and (1.14) yields

$$D = \frac{1}{2\pi} \iint_{\Sigma_k} \bar{a}_1 \times \bar{a}_2 \cdot \bar{\nabla} \left(\frac{H}{\|\bar{q}\|} \right) d\xi d\eta \quad (1.53)$$

These expressions are evaluated under the hypothesis that the surface element is a portion of a hyperboloid, in Sections 2 and 3, respectively.

1.7 Trapezoidal Planar Element

In order to facilitate the evaluation of the coefficients, the integrals are first evaluated for trapezoidal planar element (the results are then verified to be valid for a general hyperboloidal element). The trapezoidal planar element is a particular use of the hyperboloidal element and is obtained from this last one by assuming that the two edges $\xi = \pm 1$ are parallel. This implies that

$$\bar{p}_1 = \chi \bar{u} \quad (1.54)$$

$$\bar{p}_3 = \psi \bar{u} \quad (1.55)$$

where \bar{u} is the unit vector in the direction of \bar{p}_1 , i.e.

$$\bar{u} = \bar{p}_1 / |\bar{p}_1| \quad (1.56)$$

Note that this implies that

$$\chi = |\bar{p}_1| > 0 \quad (1.57)$$

Furthermore in order to avoid crossing of the element boundaries, one must have

$$|\psi| \leq \chi \quad (1.58)$$

where the equality sign corresponds to triangular element.

It may be worth noting that

$$\bar{a}_1 \equiv \bar{p}_1 + \eta \bar{p}_3 = (\chi + \eta \psi) \bar{u} \quad (1.59)$$

which implies

$$|a_1| = \chi + \eta \psi \geq 0 \quad (1.60)$$

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SECTION 2

SOURCE ELEMENT

2.1 The Case $u \circ u = 1$

Consider the source integral, Eq. (1.38), for a trapezoidal planar element (Subsection 1.7),

$$\begin{aligned} 2\pi S &= \iint_{\Sigma} \frac{H}{\sqrt{\bar{q} \circ \bar{q}}} d\Sigma = \iint_{-1}^1 \frac{H}{\sqrt{\bar{q} \circ \bar{q}}} |\bar{a}_1 \times \bar{a}_2| d\xi d\eta \\ &= |\bar{u} \times \bar{p}_2| \iint_{-1}^1 \|\bar{a}_1\| \frac{H}{\sqrt{\bar{q} \circ \bar{q}}} d\xi d\eta \end{aligned} \tag{2.1}$$

For simplicity, it is assumed that $\bar{u} = \bar{a}_1 / \|\bar{a}_1\|$ is such that

$$\bar{u} \circ \bar{u} = 1 \tag{2.2}$$

The case $u \circ u = -1$ is discussed in Subsections 2.4 and 2.5.

Note that

$$\begin{aligned} &\frac{\partial}{\partial \xi} \ln \frac{|\sqrt{\bar{q} \circ \bar{q}} + \bar{q} \circ \bar{u}|}{|\bar{q} \times \bar{u} \circ \bar{q} \times \bar{u}|} \\ &= \frac{\partial}{\partial \xi} \left[\ln |\sqrt{\bar{q} \circ \bar{q}} + \bar{q} \circ \bar{u}| - \frac{1}{2} \ln (-\bar{q} \times \bar{u} \circ \bar{q} \times \bar{u}) \right] \\ &= \frac{1}{\sqrt{\bar{q} \circ \bar{q}} + \bar{q} \circ \bar{u}} \left(\frac{\bar{q} \circ \bar{a}_1}{\sqrt{\bar{q} \circ \bar{q}}} + \bar{a}_1 \circ \bar{u} \right) + 0 \\ &= \frac{1}{\sqrt{\bar{q} \circ \bar{q}} + \bar{q} \circ \bar{u}} \frac{\bar{q} \circ \bar{a}_1 + \|\bar{a}_1\| \sqrt{\bar{q} \circ \bar{q}}}{\sqrt{\bar{q} \circ \bar{q}}} = \|\bar{a}_1\| \frac{1}{\|\bar{q}\|} \end{aligned} \tag{2.3}$$

Note also that, according to the first superrule, if $\sqrt{\bar{q} \circ \bar{q}} = 0$ (that is, on the Mach cone),

$$\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_1 = -(\bar{q} \circ \bar{a}_1)^2 \quad (2.4)$$

Hence,

$$\ln \frac{|\bar{q} \circ \bar{q} + \bar{q} \circ \bar{u}|}{-\bar{q} \times \bar{u} \circ \bar{q} \times \bar{u}} = \ln 1 = 0 \quad (2.5)$$

along the portion of the contour on the Mach cone (curves M_1 and M_2 in Fig. 4). Thus, no contribution comes from the portions M_i , hence

$$\int_{-1}^1 \|\bar{a}_1\| \frac{H}{\sqrt{\bar{q} \circ \bar{q}}} d\xi = \left[H \ln \left| \frac{\sqrt{\bar{q} \circ \bar{q} + \bar{q} \cdot \bar{u}}}{\|\bar{q} \times \bar{u}\|} \right| \right]_{\xi=-1}^{\xi=1} \quad (2.6)$$

since $H = 0$ on the portion N_i of the lines $\xi = \pm 1$. Hence,

$$S = \frac{1}{2\pi} \left[\dot{\lambda}_s(1) - \dot{\lambda}_s(-1) \right] \quad (2.7)$$

with

$$\dot{\lambda}_s(\xi) = |\bar{u} \times \bar{p}_2| \int_{-1}^1 H \ln \frac{|\sqrt{\bar{q} \circ \bar{q} + \bar{q} \circ \bar{u}}|}{\|\bar{q} \times \bar{u}\|} d\eta \quad (2.8)$$

where $H = 0$ outside the Mach forecone. It may be worth noting that

$$(\bar{q} \circ \bar{q})(\bar{u} \circ \bar{u}) - (\bar{q} \circ \bar{u})^2 = \bar{q} \times \bar{u} \circ \bar{q} \times \bar{u} < 0 \quad (2.9)$$

For, $\bar{q} \times \bar{u}$ is pointed outside the Mach cone, since \bar{q} is pointed inside it. Hence, $\sqrt{\bar{q} \circ \bar{q}} < |\bar{q} \circ \bar{u}|$ and thus,

$$\sqrt{\bar{f} \circ \bar{f}} + \bar{f} \circ \bar{u} \geq 0 \quad \text{for} \quad \bar{f} \circ \bar{u} \geq 0 \quad (2.10)$$

2.2 Indefinite Integration Along η

Consider

$$I = \int \ln \frac{|\sqrt{\bar{f} \circ \bar{f}} + \bar{f} \circ \bar{u}|}{\|\bar{f} \times \bar{u}\|} d\eta \quad (2.11)$$

Integrating by parts yields

$$I = (\eta - \eta_*) \ln \left| \frac{\sqrt{\bar{f} \circ \bar{f}} + \bar{f} \circ \bar{u}}{\sqrt{-\bar{f} \times \bar{u} \circ \bar{f} \times \bar{u}}} \right| - \int (\eta - \eta_*) \left[\frac{1}{\sqrt{\bar{f} \circ \bar{f}} + \bar{f} \circ \bar{u}} \left(\frac{\bar{f} \circ \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}}} + \bar{a}_2 \circ \bar{u} \right) - \frac{1}{2} \frac{1}{\|\bar{f} \times \bar{u}\|^2} (-2 \bar{f} \times \bar{u} \circ \bar{a}_2 \times \bar{u}) \right] d\eta \quad (2.12)$$

Noting that

$$(\eta - \eta_*) \bar{a}_2 = \bar{f} - \bar{f}_* \quad (2.13)$$

with

$$\bar{f}_* = (\bar{p}_0 + \bar{\varepsilon} \bar{p}_1) + \eta_* (\bar{p}_2 + \bar{\varepsilon} \bar{p}_3) \quad (2.14)$$

yields, applying the first super-rule,

$$I_1 \equiv I - (\eta - \eta_*) \ln \frac{(\sqrt{\bar{f} \circ \bar{f}} + \bar{f} \circ \bar{u})}{\|\bar{f} \times \bar{u}\|} =$$

$$\begin{aligned}
 &= \int \left\{ \frac{1}{\sqrt{\bar{f}_0 \bar{f}_0 + \bar{f}_0 \bar{u}}} \left(\frac{\bar{f}_0 (\bar{f}_0 - \bar{f}_*)}{\sqrt{\bar{f}_0 \bar{f}_0}} + (\bar{f}_0 - \bar{f}_*) \bar{u} \right) + \frac{1}{\|\bar{f}_* \bar{u}\|} (\bar{f}_* \bar{u} \circ (\bar{f}_0 - \bar{f}_*) \bar{u}) \right\} d\eta \\
 &= - \int \left[\frac{1}{\sqrt{\bar{f}_0 \bar{f}_0 + \bar{f}_0 \bar{u}}} \left(\frac{\bar{f}_0 \bar{f}_0}{\sqrt{\bar{f}_0 \bar{f}_0}} + \bar{f}_0 \bar{u} \right) + \frac{\bar{f}_* \bar{u} \circ \bar{f}_* \bar{u}}{\|\bar{f}_* \bar{u}\|} \right] d\eta \\
 &\quad + \int \left[\frac{1}{\sqrt{\bar{f}_0 \bar{f}_0 + \bar{f}_0 \bar{u}}} \left(\frac{\bar{f}_0 \bar{f}_*}{\sqrt{\bar{f}_0 \bar{f}_0}} + \bar{f}_* \bar{u} \right) + \frac{\bar{f}_* \bar{u} \circ \bar{f}_* \bar{u}}{\|\bar{f}_* \bar{u}\|} \right] d\eta \\
 &= \int \left\{ \frac{\sqrt{\bar{f}_0 \bar{f}_0} - \bar{f}_0 \bar{u}}{\bar{f}_0 \bar{f}_0 - (\bar{f}_0 \bar{u})^2} \left(\frac{\bar{f}_0 \bar{f}_*}{\sqrt{\bar{f}_0 \bar{f}_0}} + \bar{f}_* \bar{u} \right) + \frac{\bar{f}_* \bar{u} \circ \bar{f}_* \bar{u}}{\|\bar{f}_* \bar{u}\|} \right\} d\eta \\
 &= \int \frac{\bar{f}_0 \bar{f}_* - \bar{f}_0 \bar{u} \bar{f}_* \bar{u}}{-\|\bar{f}_* \bar{u}\|^2} d\eta + \int \frac{\bar{f}_0 \bar{f}_* \bar{f}_* \bar{u} - \bar{f}_0 \bar{u} \bar{f}_0 \bar{f}_*}{-\|\bar{f}_* \bar{u}\|^2} \frac{d\eta}{\sqrt{\bar{f}_0 \bar{f}_0}} \\
 &\quad + \int \frac{\bar{f}_* \bar{u} \circ \bar{f}_* \bar{u}}{\|\bar{f}_* \bar{u}\|^2} d\eta \\
 &= - \int \frac{\bar{f}_* \bar{f}_* \circ \bar{f}_* \bar{u}}{\|\bar{f}_* \bar{u}\|} \frac{d\eta}{\sqrt{\bar{f}_0 \bar{f}_0}}
 \end{aligned}$$

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(2.15)

Next set

$$\bar{f}_* = \bar{f}_*^N + \bar{f}_*^T \bar{u} \quad (2.16)$$

with \bar{f}_*^N "super-orthogonal" to \bar{u} and \bar{a}_2 , that is,

$$\bar{f}_*^N \circ \bar{u} = 0$$

$$\bar{f}_*^N \circ \bar{a}_2 = 0 \quad (2.17)$$

This yields

$$(\bar{a}_0 + \eta_* \bar{a}_2 - \xi_*^T u) \circ \bar{u} = 0$$

$$(\bar{a}_0 + \eta_* \bar{a}_2 - \xi_*^T u) \circ \bar{a}_1 = 0 \quad (2.18)$$

or

$$\xi_*^T \bar{u} \circ \bar{u} - \eta_* \bar{a}_2 \circ \bar{u} = \bar{a}_0 \circ \bar{u}$$

$$\xi_*^T \bar{u} \circ \bar{a}_2 - \eta_* \bar{a}_2 \circ \bar{a}_2 = \bar{a}_0 \circ \bar{a}_2 \quad (2.19)$$

which is a system of two equations with two unknowns, with solution

$$\xi_*^T = \frac{\bar{a}_0 \circ \bar{u} \bar{a}_2 \circ \bar{a}_2 - \bar{a}_0 \circ \bar{a}_2 \bar{a}_2 \circ \bar{u}}{\bar{u} \circ \bar{u} \bar{a}_2 \circ \bar{a}_2 - \bar{u} \circ \bar{a}_2 \bar{a}_2 \circ \bar{u}} = \frac{\bar{a}_0 \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2}{\bar{u} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2} = \frac{\bar{f} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2}{-\|\bar{u} \times \bar{a}_2\|^2}$$

$$\eta_* = \frac{\bar{a}_0 \circ \bar{u} \bar{u} \circ \bar{a}_2 - \bar{a}_0 \circ \bar{a}_2 \bar{u} \circ \bar{u}}{-\bar{a}_2 \circ \bar{u} \bar{u} \circ \bar{a}_2 + \bar{a}_2 \circ \bar{a}_2 \bar{u} \circ \bar{u}} = -\frac{\bar{a}_0 \times \bar{u} \circ \bar{a}_2 \times \bar{u}}{\bar{u} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2}$$

(2.20)

Note that

$$\eta - \eta_* = \frac{[\eta \bar{a}_2 - (-\bar{a}_0)] \times \bar{u} \circ \bar{a}_2 \times \bar{u}}{\bar{u} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2} = \frac{\bar{f} \times \bar{u} \circ \bar{a}_2 \times \bar{u}}{-\|\bar{u} \times \bar{a}_2\|^2} \quad (2.21)$$

Combining Eqs. (2.15) and (2.16) yields

$$I_1 = \int \frac{\bar{f} \times (\bar{f}_0^N + \xi_*^T \bar{u}) \circ \bar{f} \times \bar{u}}{-\|\bar{f} \times \bar{u}\|^2} \frac{d\eta}{\sqrt{\bar{f} \circ \bar{f}}} =$$

$$\begin{aligned}
 &= - \int \frac{\bar{f} \times \bar{f}_*^N \circ \bar{f} \times \bar{u}}{\|\bar{f} \times \bar{u}\|^2} \frac{d\eta}{\sqrt{\bar{f} \circ \bar{f}}} + \bar{f}_*^T \int \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} d\eta \\
 &= - \int \frac{\bar{f} \circ \bar{f} \bar{f}_*^N \circ \bar{u} - \bar{f} \circ \bar{u} \bar{f}_*^N \circ \bar{f}}{\|\bar{f} \times \bar{u}\| \sqrt{\bar{f} \circ \bar{f}}} + \bar{f}_*^T \int \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} d\eta \\
 &= \bar{f}_*^N \circ \bar{f}_*^N \int \frac{\bar{f} \circ \bar{u}}{\|\bar{f} \times \bar{u}\|^2} \frac{d\eta}{\sqrt{\bar{f} \circ \bar{f}}} + \bar{f}_*^T \int \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} d\eta
 \end{aligned} \tag{2.22}$$

since (see Eq. C.4)

$$\bar{f}_*^N \circ \bar{f} = \bar{f}_*^N \circ (\bar{f}_*^N + (\eta - \eta_*) \bar{a}_2 + \bar{f}_*^T \bar{u}) = \bar{f}_*^N \circ \bar{f}_*^N \tag{2.23}$$

As shown in Appendix B, by integrating Eq. (2.22), one obtains (Eqs. B.14, B.15, B.16 and B.29)

$$I_1 = \bar{f}_*^T F(\eta) - \frac{\bar{f}_*^N \circ \bar{f}_*^N}{|\bar{f} \cdot \bar{u} \times \bar{a}_2|} \tan^{-1} \left(\frac{-\bar{f} \times \bar{u} \circ \bar{f} \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}} |\bar{f} \cdot \bar{u} \times \bar{a}_2|} \right) \tag{2.24}$$

where

$$\begin{aligned}
 F(\eta) &= \frac{1}{\sqrt{\bar{a}_2 \circ \bar{a}_2}} \ln \frac{|\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|} & \bar{a}_2 \circ \bar{a}_2 > 0 \\
 &= \frac{\sqrt{\bar{f} \circ \bar{f}}}{\bar{f} \circ \bar{a}_2} & \bar{a}_2 \circ \bar{a}_2 = 0 \\
 &= - \frac{1}{\sqrt{\bar{a}_2 \circ \bar{a}_2}} \sin^{-1} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} & \bar{a}_2 \circ \bar{a}_2 < 0
 \end{aligned} \tag{2.25}$$

On the other hand, as shown in Appendix C

$$\bar{f}_2^N \circ \bar{f}_2^N = \frac{|\bar{f} \cdot \bar{a}_1 \times \bar{a}_2|^2}{-\|\bar{a}_1 \times \bar{a}_2\|^2} \quad (2.26)$$

Hence, combining Eqs. (2.11), (2.15), (2.20), (2.21), (2.24), and (2.25) yields

$$\begin{aligned} I = & (\eta - \eta_0) \ln \frac{\|\bar{f}\| + \bar{f} \circ \bar{u}}{\|\bar{f} \times \bar{u}\|} \\ & + \bar{f}_2^T F(\eta) \\ & - \frac{\bar{f}_2^N \circ \bar{f}_2^N}{|\bar{f} \cdot \bar{a}_1 \times \bar{a}_2|} \tan^{-1} \frac{-\bar{f} \times \bar{u} \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| |\bar{f} \cdot \bar{a}_1 \times \bar{a}_2|} \\ & - \frac{1}{-\|\bar{u} \times \bar{a}_2\|^2} \left\{ (\bar{f} \times \bar{u} \circ \bar{a}_2 \times \bar{u}) \ln \left| \frac{\sqrt{\bar{f} \circ \bar{f}} + \bar{f} \circ \bar{u}}{\|\bar{f} \times \bar{u}\|} \right| \right. \\ & \left. + \bar{f} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2 F(\eta) \right. \\ & \left. - \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \tan_p^{-1} \left(\frac{-\bar{f} \times \bar{u} \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{u} \times \bar{a}_2} \right) \right\} \quad (2.27) \end{aligned}$$

where the principal value, \tan_p^{-1} , of the function χ , is defined as

$$-\frac{\pi}{2} \leq \tan_p^{-1}(\chi) \leq \frac{\pi}{2} \quad (2.27a)$$

2.3 Source Integral

Combining Eqs. (2.8) and (2.9) yields

$$i_s(\xi) = I_s(\xi, 1) - I_s(\xi, -1) \quad (2.28)$$

where, (note that $|\bar{u} \times \bar{p}_2| = |\bar{a}_1 \times \bar{a}_2| / \|a, \bar{u}\|$) if $H = 1$ (that is if the corner point is inside the Mach forecone)

$$I_s(\xi, \eta) = \frac{|\bar{a}_1 \times \bar{a}_2|}{-\|\bar{a}_1 \times \bar{a}_2\|^2} \times \left\{ (\bar{f} \times \bar{a}_1 \circ \bar{a}_1 \times \bar{a}_2) \frac{1}{\sqrt{\bar{a}_1 \circ \bar{a}_1}} \ln \left| \frac{\sqrt{\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_1} + \bar{f} \circ \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|} \right| \right. \\ \left. + (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) F(\eta) - \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \tan_p^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \right\} \quad (2.29)$$

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with $F(\eta)$ given by Eq. (2.25).

On the other hand, if $H = 0$, this means that the corner point is outside the Mach cone and there are two possibilities: the whole segment $-1 \leq \eta \leq 1$ is outside the Mach forecone or a portion of the segment is inside it. In the first case, $i(\xi) = 0$ and hence it is legitimate to set

$$I_s = 0$$

In the second case, the integration must be restricted to the portion of the segment with $H = 1$. Hence, the value of I_s must

be evaluated at this extreme of integration, which corresponds to the point where $\bar{q} \circ \bar{q} = 0$. As mentioned above, for $\bar{q} \circ \bar{q} = 0$, the first logarithmic term is equal to zero. In the second term, $F(\eta)$ assumes the value

$$\begin{aligned}
 F(\eta) &= 0 & \bar{a}_2 \circ \bar{a}_2 &\geq 0 \\
 &= -\frac{1}{\|\bar{a}_2\|} \operatorname{sign}(\bar{f} \circ \bar{a}_2) \cdot \frac{\pi}{2} & \bar{a}_2 \circ \bar{a}_2 &< 0
 \end{aligned}
 \tag{2.30}$$

It may be noted that for $\bar{q} \circ \bar{q} = 0$

$$\bar{f} \circ \bar{a}_2 = \bar{f}^c \cdot \bar{a}_2 = |\bar{f}^c| \bar{N}_c \cdot \bar{a}_2$$

where \bar{N}_c is the inward unit normal $\frac{\bar{f}^c}{|\bar{f}^c|}$ to the Mach forecone (see Fig. 5). Hence, noting that \bar{a}_2 is in any case directed from $\eta = -1$ to $\eta = 1$, it turns out that

$$\bar{f} \circ \bar{a}_2 > 0 \tag{2.31}$$

if \bar{a}_2 is directed inside the Mach cone, i.e., for lower extreme of integration and viceversa; hence,

$$\operatorname{sign} \bar{a}_2 \circ \bar{f} = -\operatorname{sign}(\eta) = -\eta \tag{2.32}$$

with $\eta = -1$ ($\eta = +1$) for the lower (upper) limit of integration.

Finally, the last term assumes the value

$$-|\bar{f} \cdot \bar{a}_1 \times \bar{a}_2| \frac{\pi}{2} \operatorname{sign}[-(\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)] \tag{2.33}$$

However, according to the first super-rule for $\bar{q} \circ \bar{q} = 0$,

$$-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2 = \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2 \tag{2.34}$$

On the other hand

$$\bar{f} \odot \bar{a}_1 > 0 \quad (2.35)$$

if \bar{a}_1 is directed inside the Mach cone (and viceversa). This is true if the x-component of \bar{u} , u_x , is negative (positive) (Fig. 5) since $\bar{u} \odot \bar{u} > 0$ by assumption. Hence,

$$\text{sign}(\bar{f} \odot \bar{a}_1) = - \text{sign}(u_x) \quad (2.36)$$

Finally, combining Eqs. (2.30) through (2.36), one obtains, for the case in which portion of the segment is inside the Mach forecone, but the corner is outside it,

$$I_s(\xi, \eta) = \frac{|\bar{a}_1 \times \bar{a}_2|}{-\|\bar{a}_1 \times \bar{a}_2\|^2} \left\{ \bar{f} \times \bar{a}_2 \odot \bar{a}_1 \times \bar{a}_2 F_0(\eta) - |\bar{f} \cdot \bar{a}_1 \times \bar{a}_2| \text{sign}(\eta u_x) \frac{\pi}{2} \right\} \quad (2.37a)$$

where

$$F_0(\eta) = \begin{cases} 0 & \bar{a}_2 \odot \bar{a}_2 \geq 0 \\ \frac{1}{\|\bar{a}_2\|} \frac{\pi}{2} \eta & \bar{a}_2 \odot \bar{a}_2 < 0 \end{cases} \quad (2.37b)$$

In summary, according to Eq. (2.7)

$$S = \frac{1}{2\pi} [\dot{I}_s(1) - \dot{I}_s(-1)] \quad (2.38a)$$

where, according to Eq. (2.28)

$$\dot{I}_s(\xi) = I_s(\xi, 1) - I_s(\xi, -1) \quad (2.38b)$$

In Eq. (2.38b)

$$I_s(\xi, \eta) = 0 \quad (2.39)$$

if the segment $-1 \leq \eta \leq 1$ is completely outside the Mach cone. On the other hand, if a portion of the segment is inside the Mach cone, then two cases are possible: a) the corner $\eta = \pm 1$ is inside the Mach forecone ($H = 1$) or, b) the corner $\eta = \pm 1$ is not inside the Mach forecone ($H = 0$). In the first case $\bar{q} \odot \bar{q} > 0$ and according to Eq. (2.29),

$$\begin{aligned}
 I_s(\xi, \eta) = & \frac{|\bar{a}_1 \times \bar{a}_2|}{-\|\bar{a}_1 \times \bar{a}_2\|^2} \left\{ \bar{f} \times \bar{a}_1 \odot \bar{a}_2 \times \bar{a}_1 \frac{1}{\|\bar{a}_1\|} \ln \left| \frac{\|\bar{f}\| \|\bar{a}_1\| + \bar{f} \odot \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|} \right| \right. \\
 & + (\bar{f} \times \bar{a}_2 \odot \bar{a}_1 \times \bar{a}_2) F(\eta) \\
 & \left. - \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \tan_p^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \odot \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \right\} \quad (2.40)
 \end{aligned}$$

with (see Eq. 2.25)

$$\begin{aligned}
 F(\eta) = & \frac{1}{\|\bar{a}_2\|} \ln \left| \frac{\|\bar{f}\| \|\bar{a}_2\| + \bar{f} \odot \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right| & \bar{a}_2 \odot \bar{a}_2 > 0 \\
 = & \frac{\|\bar{f}\|}{\bar{f} \odot \bar{a}_2} & \bar{a}_2 \odot \bar{a}_2 = 0 \\
 = & -\frac{1}{\|\bar{a}_2\|} \sin^{-1} \left(\frac{\bar{f} \odot \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right) & \bar{a}_2 \odot \bar{a}_2 < 0
 \end{aligned}$$

(2.41)

On the other hand, in the second case, which corresponds to $\bar{q} \odot \bar{q} \neq 0$, according to Eq. (2.37a)

$$I_s(\xi, \eta) = \frac{|\bar{a}_1 \times \bar{a}_2|}{-\|\bar{a}_1 \times \bar{a}_2\|^2} \left\{ \bar{f} \times \bar{a}_2 \odot \bar{a}_1 \times \bar{a}_2 F_0(\eta) - |\bar{f} \cdot \bar{a}_1 \times \bar{a}_2| \operatorname{sign}(\eta u_x) \frac{\pi}{2} \right\} \quad (2.42)$$

where

$$F_0(\eta) = \begin{cases} 0 & \bar{a}_2 \odot \bar{a}_2 \geq 0 \\ \frac{1}{\|\bar{a}_2\|} \frac{\pi}{2} \eta & \bar{a}_2 \odot \bar{a}_2 < 0 \end{cases} \quad (2.43)$$

2.4 Planar Quadrilateral Element Internal to Mach Cone

In this Subsection, it is shown how the results obtained thus far can be extended to quadrilateral planar elements. In order to do this it will be shown that the second mixed derivative of the function

$$I_s(\xi, \eta) = \frac{\bar{a}_1 \times \bar{a}_2}{-\|\bar{a}_1 \times \bar{a}_2\|} \left\{ \bar{f} \times \bar{a}_1 \odot \bar{a}_2 \times \bar{a}_1 F_1(\xi, \eta) + \bar{f} \times \bar{a}_2 \odot \bar{a}_1 \times \bar{a}_2 F_2(\xi, \eta) - \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \odot \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \right\}$$

(2.44)

is proportional to the integrand of the source integral.

In Eq. (2.44),

$$F_1(\xi, \eta) = \frac{1}{\|\bar{a}_1\|} \ln \left| \frac{\|\bar{f}\| \|\bar{a}_1\| + \bar{f} \circ \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|} \right| \quad \bar{a}_1 \circ \bar{a}_1 > 0$$

$$= \frac{\|\bar{f}\|}{\bar{f} \circ \bar{a}_1} \quad \bar{a}_1 \circ \bar{a}_1 = 0$$

$$= -\frac{1}{\|\bar{a}_1\|} \sin^{-1} \left(\frac{\bar{f} \circ \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|} \right) \quad \bar{a}_1 \circ \bar{a}_1 < 0$$

(2.45)

$$F_2(\xi, \eta) = \frac{1}{\|\bar{a}_2\|} \ln \left| \frac{\|\bar{f}\| \|\bar{a}_2\| + \bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right| \quad \bar{a}_2 \circ \bar{a}_2 > 0$$

$$= \frac{\|\bar{f}\|}{\bar{f} \circ \bar{a}_2} \quad \bar{a}_2 \circ \bar{a}_2 = 0$$

$$= -\frac{1}{\|\bar{a}_2\|} \sin^{-1} \left(\frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right) \quad \bar{a}_2 \circ \bar{a}_2 < 0$$

(2.46)

Note that Eqs. (2.44), (2.45) and (2.46) reduce to the formulation derived in the preceding Subsection, if $\bar{a}_1 \circ \bar{a}_1 > 0$. Equation (2.44) may be rewritten as

$$\begin{aligned}
 I_3(\xi, \eta) &= \frac{|\bar{a}_1 \times \bar{a}_2|^2}{-\|\bar{a}_1 \times \bar{a}_2\|^2} \left\{ \bar{f} \times \bar{a}_1 \circ \frac{\bar{a}_2 \times \bar{a}_1}{|\bar{a}_1 \times \bar{a}_2|} F_1(\xi, \eta) \right. \\
 &\quad + \bar{f} \times \bar{a}_2 \circ \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} F_2(\xi, \eta) \\
 &\quad \left. - \bar{f} \cdot \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \right\} \\
 &= \frac{1}{-\|n\|^2} \left\{ -\bar{f} \times \bar{a}_1 \circ \bar{n} F_1(\xi, \eta) + \bar{f} \times \bar{a}_2 \circ \bar{n} F_2(\xi, \eta) \right. \\
 &\quad \left. - \bar{f} \cdot \bar{n} \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \right\}
 \end{aligned}
 \tag{2.47}$$

since
$$\bar{n} \equiv \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|}$$

(2.48)

Next consider the second mixed derivative of Eq. (2.47).

Note that

$$\frac{\partial \bar{f}}{\partial \xi} = \bar{a}_1, \quad \frac{\partial \bar{f}}{\partial \eta} = \bar{a}_2 \quad (2.49)$$

$$\frac{\partial \bar{a}_1}{\partial \xi} = \frac{\partial \bar{a}_2}{\partial \eta} = 0 \quad (2.50)$$

and

$$\frac{\partial \bar{a}_1}{\partial \eta} = \frac{\partial \bar{a}_2}{\partial \xi} = \bar{p}_3 \quad (2.51)$$

Note also that, for a planar element, the unit normal \bar{n} is independent of ξ and η

$$\frac{\partial \bar{n}}{\partial \xi} = \frac{\partial \bar{n}}{\partial \eta} = 0 \quad (2.52)$$

Hence

$$\frac{\partial}{\partial \xi} (\bar{f} \times \bar{a}_1 \circ \bar{n}) = (\bar{a}_1 \times \bar{a}_2 \circ \bar{n}) \equiv 0 \quad (2.53)$$

and

$$\frac{\partial}{\partial \eta} (\bar{f} \times \bar{a}_2 \circ \bar{n}) = (\bar{a}_2 \times \bar{a}_2 \circ \bar{n}) \equiv 0 \quad (2.54)$$

Furthermore, as shown in Appendix B (see Eq. B.5, B.10 and B.12)

$$\begin{aligned} & \frac{\partial}{\partial \eta} \left\{ \frac{1}{\|\bar{a}_2\|} \ln \left| \frac{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right| \right\} \quad (\bar{a}_2 \circ \bar{a}_2 > 0) \\ & \equiv \frac{\partial}{\partial \eta} \frac{\sqrt{\bar{f} \circ \bar{f}}}{\bar{f} \circ \bar{a}_2} \quad (\bar{a}_2 \circ \bar{a}_2 = 0) \end{aligned}$$

$$= \frac{\partial}{\partial \eta} \left\{ \frac{-1}{\|\bar{a}_2\|} \sin^{-1} \left(\frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right) \right\} \quad (\bar{a}_2 \circ \bar{a}_2 < 0)$$

$$= \frac{1}{\sqrt{\bar{f} \circ \bar{f}}}$$

(2.55)

or

$$\frac{\partial F_2}{\partial \eta} = \frac{1}{\|\bar{f}\|} \quad (\text{for } \bar{a}_2 \circ \bar{a}_2 \geq 0) \quad (2.56)$$

Similarly, interchanging indices

$$\frac{\partial F_1}{\partial \xi} = \frac{1}{\|\bar{f}\|} \quad (\text{for } \bar{a}_1 \circ \bar{a}_1 \geq 0) \quad (2.57)$$

Hence

$$\begin{aligned} & \frac{\partial^2}{\partial \xi \partial \eta} \left\{ \bar{f} \times \bar{a}_1 \circ \bar{n} F_1(\xi, \eta) \right\} \\ &= \frac{\partial}{\partial \eta} \left(\bar{f} \times \bar{a}_1 \circ \bar{n} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} \right) \\ &= \frac{\bar{a}_2 \times \bar{a}_1 \circ \bar{n}}{\|\bar{f}\|} + \frac{\bar{f} \times \bar{f}_3 \circ \bar{n}}{\|\bar{f}\|} - \bar{f} \times \bar{a}_1 \circ \bar{n} \frac{\bar{f} \circ \bar{a}_2}{(\bar{f} \circ \bar{f})^{3/2}} \end{aligned}$$

(2.58)

and similarly, interchanging indices

$$\begin{aligned} \frac{\partial^2}{\partial \xi \partial \eta} \left\{ \bar{f} \times \bar{a}_2 \circ \bar{n} F_2(\xi, \eta) \right\} &= \frac{\partial}{\partial \xi} \left(\bar{f} \times \bar{a}_2 \circ \bar{n} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} \right) \\ &= \frac{\bar{a}_1 \times \bar{a}_2 \circ \bar{n}}{\|\bar{f}\|} + \frac{\bar{f} \times \bar{p}_3 \circ \bar{n}}{\|\bar{f}\|} - \bar{f} \times \bar{a}_2 \circ \bar{n} \frac{\bar{f} \circ \bar{a}_2}{(\bar{f} \circ \bar{f})^{3/2}} \end{aligned} \quad (2.59)$$

Finally, as shown in Appendix D,

$$\begin{aligned} \frac{\partial^2}{\partial \xi \partial \eta} \left\{ \bar{f} \cdot \bar{n} \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \right\} \\ = \bar{f} \cdot \bar{n} \frac{\bar{f} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{f}\|^3} \end{aligned} \quad (2.60)$$

Finally, combining Eqs. (2.47), (2.58), (2.59) and (2.60) yields (note that $\|\bar{n}\|^2 = -\bar{n} \circ \bar{n}$)

$$\begin{aligned} \frac{\partial^2 I_5}{\partial \xi \partial \eta} &= \frac{1}{\bar{n} \circ \bar{n}} \times \\ &\left\{ - \left(\frac{\bar{a}_2 \times \bar{a}_1 \circ \bar{n}}{\|\bar{f}\|} + \frac{\bar{f} \times \bar{p}_3 \circ \bar{n}}{\|\bar{f}\|} - \bar{f} \times \bar{a}_1 \circ \bar{n} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f}\|^3} \right) \right. \\ &\left. + \left(\frac{\bar{a}_1 \times \bar{a}_2 \circ \bar{n}}{\|\bar{f}\|} + \frac{\bar{f} \times \bar{p}_3 \circ \bar{n}}{\|\bar{f}\|} - \bar{f} \times \bar{a}_2 \circ \bar{n} \frac{\bar{f} \circ \bar{a}_1}{\|\bar{f}\|^3} \right) \right\} \end{aligned}$$

$$\begin{aligned}
 & \left. - \bar{f} \cdot \bar{n} \frac{\bar{f} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{f}\|^3} \right\} \\
 = & \frac{1}{\bar{n} \circ \bar{n}} \left\{ 2 |\bar{a}_1 \times \bar{a}_2| \bar{n} \circ \bar{n} \frac{1}{\|\bar{f}\|} + \right. \\
 & \left. + \frac{1}{\|\bar{f}\|} \left[\bar{f} \times \bar{a}_1 \circ \bar{n} \bar{f} \circ \bar{a}_2 - \bar{f} \times \bar{a}_2 \circ \bar{n} \bar{f} \circ \bar{a}_1 - \bar{f} \cdot \bar{n} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \right] \right\}
 \end{aligned}
 \tag{2.61}$$

On the other hand, according to Eq. (A.16),

$$\begin{aligned}
 & - |\bar{a}_1 \times \bar{a}_2| (\bar{f} \times \bar{a}_1 \circ \bar{n} \bar{f} \circ \bar{a}_2 - \bar{f} \times \bar{a}_2 \circ \bar{n} \bar{f} \circ \bar{a}_1 - \bar{f} \cdot \bar{n} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2) \\
 = & \bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1 \bar{f} \circ \bar{a}_2 + \bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2 \bar{f} \circ \bar{a}_1 + (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2 \\
 = & \bar{f} \circ \bar{f} \bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2 \\
 = & \bar{f} \circ \bar{f} \bar{n} \circ \bar{n} |\bar{a}_1 \times \bar{a}_2|^2
 \end{aligned}
 \tag{2.62}$$

Hence

$$\frac{\partial^2 I_s}{\partial \zeta \partial \eta} = \frac{1}{\bar{n} \circ \bar{n}} \left(2 \bar{n} \circ \bar{n} \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{f}\|} - \bar{f} \circ \bar{f} \bar{n} \circ \bar{n} \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{f}\|^3} \right)
 \tag{2.63}$$

or

$$\frac{\partial^2 I_s}{\partial \zeta \partial \eta} = \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{f}\|}
 \tag{2.64}$$

Hence, according to Eq. (2.1), the source integral is given by (note that for elements internal to the Mach cone, $H = 1$)

$$2\pi S = \iint_{-1}^1 \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{g}\|} d\zeta d\eta = \iint_{-1}^1 \frac{\partial^2 I_s}{\partial \zeta \partial \eta} d\zeta d\eta \quad (2.65)$$

or

$$2\pi S = I_s(1, 1) - I_s(1, -1) - I_s(-1, 1) + I_s(-1, -1) \quad (2.66)$$

with I_s given by Eq. (2.44).

2.5 General Quadrilateral Planar Element

In order to extend the results to a planar quadrilateral element intersected by the Mach forecone, it is convenient to use the theory of distribution by Schwartz (Reference 8).

For, note that, according to Eqs. (2.61) and (2.64)

$$\begin{aligned} \frac{|\bar{a}_1 \times \bar{a}_2|}{\|\bar{g}\|} = \frac{1}{\bar{n} \circ \bar{n}} \times \left\{ - \left(\frac{\bar{a}_2 \times \bar{a}_1 \circ \bar{n}}{\|\bar{g}\|} - \frac{\bar{g} \times \bar{p}_3 \circ \bar{n}}{\|\bar{g}\|} - \bar{g} \times \bar{a}_1 \circ \bar{n} \frac{\bar{g} \circ \bar{a}_2}{\|\bar{g}\|^3} \right) \right. \\ + \left(\frac{\bar{a}_1 \times \bar{a}_2 \circ \bar{n}}{\|\bar{g}\|} - \frac{\bar{g} \times \bar{p}_3 \circ \bar{n}}{\|\bar{g}\|} - \bar{g} \times \bar{a}_2 \circ \bar{n} \frac{\bar{g} \circ \bar{a}_1}{\|\bar{g}\|^3} \right) \\ \left. - \left[\bar{g} \cdot \bar{n} \frac{\bar{g} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{g}\|} \right] \right\} \end{aligned} \quad (2.67)$$

But while the left-hand side has an integrable singularity, each one of the three terms in brackets on the right-hand side is nonintegrable within the framework of the theory of real function. However, if the right-hand side is treated as generalized

function (or distribution), then the integral of the right-hand side (in the sense of the theory of distributions) equals the desired integral of the left-hand side (in the sense of the theory of real functions).

It is obvious that if the element is fully within the Mach forecone, then the results described here reduce to the one given in the preceding subsection. Also, if the element is completely outside the Mach forecone, the value of the integral is zero. Therefore, the results presented here complete the formulation for a general quadrilateral planar element.

In order to simplify the derivation of the results, consider each of the three terms in the brackets in Eq. (2.67) independently. This yields, according to Eq. (2.1)

$$2\pi S \equiv \iint_{-1}^1 \frac{H}{\|\bar{r}\|} |\bar{a}_1 \times \bar{a}_2| d\xi d\eta = (S_1 + S_2 + S_3) \quad (2.68)$$

where, according to Eq. (2.67)

$$S_1 = - Pf \iint_{-1}^1 \frac{H}{\bar{n} \cdot \bar{n}} \left[\frac{\bar{a}_1 \times \bar{a}_2 \cdot \bar{n}}{\|\bar{r}\|} + \frac{\bar{r} \times \bar{p}_3 \cdot \bar{n}}{\|\bar{r}\|} - \bar{r} \times \bar{a}_1 \cdot \bar{n} \frac{\bar{r} \cdot \bar{a}_2}{\|\bar{r}\|^3} \right] d\xi d\eta \quad (2.69)$$

$$S_2 = + Pf \iint_{-1}^1 \frac{H}{\bar{n} \cdot \bar{n}} \left[\frac{\bar{a}_1 \times \bar{a}_2 \cdot \bar{n}}{\|\bar{r}\|} + \frac{\bar{r} \times \bar{p}_3 \cdot \bar{n}}{\|\bar{r}\|} - \bar{r} \times \bar{a}_2 \cdot \bar{n} \frac{\bar{r} \cdot \bar{a}_1}{\|\bar{r}\|^3} \right] d\xi d\eta \quad (2.70)$$

$$S_3 = - Pf \iint_{-1}^1 \frac{H}{\bar{n} \cdot \bar{n}} \left[\bar{r} \cdot \bar{n} \frac{\bar{r} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{r}\|^3} \right] d\xi d\eta \quad (2.71)$$

where $Pf \int$ indicates the finite part (partie finie de Hadamard), of the integral in agreement with the theory of distributions (Ref. 7, p. 38-43). Note in particular that (Ref. 7, Eq. II, 2; 26 with $m = -3/2$ and $\varphi \equiv 0$ for $x > 0$)

$$Pf \int_0^a x^{-\frac{3}{2}} \varphi(x) dx = \lim_{\epsilon \rightarrow 0} \left[\int_{\epsilon}^a x^{-\frac{3}{2}} \varphi(x) dx + \varphi(0) \frac{\epsilon^{-\frac{1}{2}}}{-\frac{1}{2}} \right] \quad (2.72)$$

Hence, one obtains for $\varphi \equiv x g' - \frac{1}{2} g$

$$\begin{aligned} Pf \int_0^a \frac{d}{dx} \left(\frac{g(x)}{\sqrt{x}} \right) dx &= Pf \int_0^a \left(\frac{x g' - \frac{1}{2} g}{x^{3/2}} \right) dx \\ &= \lim_{\epsilon \rightarrow 0} \left\{ \int_0^a \frac{x g' - \frac{1}{2} g}{x^{3/2}} dx - 2 \left[\frac{x g' - \frac{1}{2} g}{\sqrt{x}} \right]_{x=0} \right\} \\ &= \lim_{\epsilon \rightarrow 0} \left(\left[\frac{g}{\sqrt{x}} \right]_{\epsilon}^a + \frac{g(0)}{\sqrt{\epsilon}} \right) = \lim_{\epsilon \rightarrow 0} \left(\frac{g(a)}{\sqrt{a}} - \frac{g(\epsilon) - g(0)}{\sqrt{\epsilon}} \right) \\ &= \frac{g(a)}{\sqrt{a}} \end{aligned} \quad (2.73)$$

In other words, the singular contribution disappears and should not be taken into account. It may be worth noting that the use of the distribution theory, in particular the finite part of the integrals, is fully legitimate, because the sum of the three integrals considered in the following is a regular integral. Hence, the three singular contributions (which are not taken into account in the theory of distribution) would cancel each

other in the theory of regular function, if the integral is evaluated in the limit (starting from a region of integration without singularity in it and letting the region approach the one under consideration here). The use of theory of distribution however is preferable since it yields the same results with much less complication.

Hence, in order to evaluate S, consider first S_1 . Integrating with respect to η over the portion of the element with $H = 1$ (Mach precone) one obtains (see Eq. 2.58)

$$\begin{aligned}
 S_1 &= \frac{-1}{\bar{n} \cdot \bar{n}} p_v \int_{-1}^1 d\xi \int_{-1}^1 \left[(\bar{a}_2 \times \bar{a}_1 \cdot \bar{o}\bar{n} + \bar{q}_2 \times \bar{p}_1 \cdot \bar{o}\bar{n}) \bar{q}_1 \cdot \bar{q}_2 - \bar{q}_2 \times \bar{a}_1 \cdot \bar{o}\bar{n} \bar{q}_1 \cdot \bar{a}_2 \right] \frac{H}{\|\bar{q}\|^3} d\eta \\
 &= -\frac{1}{\bar{n} \cdot \bar{n}} p_v \int_{-1}^1 d\xi \int_{-1}^1 H \frac{\partial}{\partial \eta} \left(\bar{q}_2 \times \bar{a}_1 \cdot \bar{o}\bar{n} \frac{1}{\|\bar{q}\|^2} \right) d\eta \\
 &= -\frac{1}{\bar{n} \cdot \bar{n}} \int_{-1}^1 \left[\bar{q}_2 \times \bar{a}_1 \cdot \bar{o}\bar{n} \frac{H_0}{\|\bar{q}\|^2} \right]_{\eta=-1}^{\eta=1} d\xi
 \end{aligned}
 \tag{2.74}$$

where, according to Eq. (2.73),

$$\begin{aligned}
 H &= 0 && \text{outside the Mach forecone} \\
 &= 1 && \text{inside the Mach forecone}
 \end{aligned}
 \tag{2.75}$$

In other words, the portions along the intersection of the element with the Mach forecone (lines M_i in Fig. 3) yield no contributions to the integral in agreement with Eq. (2.73). It may be worth noting that

$$\|\bar{q}\|^2 = \mathcal{K} (\eta - \eta_0)^2
 \tag{2.76}$$

where η_0 is the value of η for which $\|\bar{q}\| = 0$, and χ is a suitable constant. This justifies the use of Eq. (2.73) in Eq. (2.74).

In order to discuss the integration along ξ , it is convenient to rewrite Eq. (2.74) as

$$S_1 = S_1^+ - S_1^- \quad (2.77)$$

with

$$S_1^+ = -\frac{1}{\bar{n} \circ \bar{n}} \int_{-1}^1 \left[(\bar{q}, \bar{a}_1 \circ \bar{n}) \frac{H_0}{\|\bar{q}\|} \right]_{\eta=+1} d\xi \quad (2.78)$$

$$S_1^- = -\frac{1}{\bar{n} \circ \bar{n}} \int_{-1}^1 \left[(\bar{q}, \bar{a}_1 \circ \bar{n}) \frac{H_0}{\|\bar{q}\|} \right]_{\eta=-1} d\xi \quad (2.79)$$

Next, consider the two following possibilities. First

$$H_0(\xi) \Big|_{\eta=1} = 0 \quad -1 \leq \xi \leq 1 \quad (2.80)$$

is completely outside the Mach forecone. In this case,

$$S_1^+ = 0 \quad (2.81)$$

In the second case, introduce the abscissas ξ_1^+ and ξ_2^+ such that

$$\begin{aligned} H_0(\xi, 1) &= 0 & -1 \leq \xi \leq \xi_1^+ \\ &= 1 & \xi_1^+ \leq \xi \leq \xi_2^+ \\ &= 0 & \xi_2^+ \leq \xi \leq 1 \end{aligned} \quad (2.82)$$

In particular, $\xi_1^+ = -1$, $\xi_2^+ = 1$ if the corner $\xi = -1, \eta = 1$; ($\xi = 1, \eta = 1$) is inside the Mach forecone. Using Eq. (2.82), Eq. (2.78) may be written as (see also Eq. 2.53).

$$S_1^+ = - \frac{1}{\bar{n} \circ \bar{n}} \bar{f} \times \bar{a}_1 \circ \bar{n} \Big|_{\xi_1}^{\xi_2^+} \int_{\xi_1}^{\xi_2^+} \frac{1}{\|\bar{f}\|} d\xi \quad (2.83)$$

or, according to Eq. (2.58)

$$S_1^+ = - \frac{1}{\bar{n} \circ \bar{n}} \left\{ \bar{f} \times \bar{a}_1 \circ \bar{n} [F_1(\xi_2^+, \eta) - F_1(\xi_1^+, \eta)] \right\}_{\eta=1} \quad (2.84)$$

$$= I_{S_1}(1, 1) - I_{S_1}(-1, 1)$$

where

$$I_{S_1}(1, 1) = - \frac{1}{\bar{n} \circ \bar{n}} \left(\bar{f} \times \bar{a}_1 \circ \bar{n} \right)_{\eta=1} F_1(\xi_2^+, 1) \quad (2.85)$$

$$I_{S_1}(-1, 1) = - \frac{1}{\bar{n} \circ \bar{n}} \left(\bar{f} \times \bar{a}_1 \circ \bar{n} \right)_{\eta=1} F_1(\xi_1^+, 1) \quad (2.86)$$

Note that if $\xi_2^+ \neq 1$ (i.e., if the corner (1,1) is inside the Mach forecone), then ξ_2^+ is the value of ξ at which H_1 becomes equal to zero, i.e., the edge $\eta = 1$ crosses the surface of the Mach forecone. Hence

$$\|\bar{f}\| = 0 \quad \text{for } \xi = \xi_2^+, \eta = 1 \quad (2.87)$$

This implies, according to the first superrule, $|\bar{f} \circ \bar{a}_1| = \|\bar{f} \times \bar{a}_1\|$ and therefore, according to Eq. (2.45)

$$F_1(\xi_2^+, 1) = 0 \quad \bar{a}_1 \circ \bar{a}_1 \geq 0$$

$$= - \frac{1}{\|\bar{a}_1\|} \frac{\pi}{2} \text{sign}(\bar{f} \circ \bar{a}_1) \quad \bar{a}_1 \circ \bar{a}_1 < 0 \quad (2.88)$$

* Here, it is assumed that $\bar{q} \times \bar{a}_1 \neq 0$, that is the centroid of any element is not contained in the line $\eta = 1$ ($-\infty < \xi < \infty$).

Similarly

$$F_1(\xi_1^+, 1) = 0 \quad \bar{a}_1 \circ \bar{a}_1 \geq 0$$

$$= -\frac{1}{\|\bar{a}_1\|} \frac{\pi}{2} \text{sign}(\bar{q} \circ \bar{a}_1) \quad \bar{a}_1 \circ \bar{a}_1 < 0 \quad (2.89)$$

if $\xi_1^+ = -1$. In order to obtain the expression for $\text{sign}(\bar{q} \circ \bar{a}_1)$ the same reasoning used in deriving Eq. (2.32) may be employed. For, note that, if $\bar{q} \circ \bar{q} = 0$, the vector \bar{q} is tangent to the Mach forecone (Figs. 1 and 5). Therefore \bar{q}^c (see Eq. A.5) is directed like the inward normal to the surface of the Mach forecone at P. Hence

$$\bar{q} \circ \bar{a}_1 = \bar{q}^c \cdot \bar{a}_1 > 0 \quad (2.90)$$

is inwardly directed and vice versa. Note that the case $\bar{q} \circ \bar{a}_1 = 0$ implies that \bar{a}_1 is not tangent to the Mach forecone (see Fig. 1). On the other hand, \bar{a}_1 is necessarily directed from $\xi = -1$ to $\xi = +1$ since along $\eta = 1$

$$\bar{a}_1 = \bar{p}_1 + \bar{p}_3 = \frac{1}{2} (\bar{p}_{++} - \bar{p}_{+-}) \quad (2.91)$$

Therefore, if $\xi_1^+ \neq -1$, \bar{a}_1 is inwardly directed at ξ_1^+ (lower limit of integration, see Fig. 6), while, if $\xi_2^+ \neq -1$, \bar{a}_1 is outwardly directed at ξ_2^+ (upper limit of integration). In other words

$$\text{Sign}(\bar{q} \circ \bar{a}_1) = 1 \quad \text{at } \xi = \xi_1^+ \neq -1$$

$$= -1 \quad \text{at } \xi = \xi_2^+ \neq +1 \quad (2.92)$$

Similar results may be obtained for S_1^- . In summary, it is possible to rewrite the results in a more compact form, as follows:

$$S_1 = I_{S_1}(1,1) - I_{S_1}(1,-1) - I_{S_1}(-1,1) + I_{S_1}(-1,-1) \quad (2.93)$$

where

$$I_{S_1}(1, 1) - I_{S_1}(-1, 1) \equiv 0 \quad (2.94)$$

or

$$I_{S_1}(1, -1) - I_{S_1}(-1, -1) \equiv 0 \quad (2.95)$$

if the edge $\eta = 1$, or $\eta = -1$, respectively, is completely outside the Mach forecone. Otherwise

$$I_{S_1}(s_1, s_2) = -\frac{1}{\bar{n} \circ \bar{n}} \left[\bar{\xi} \times \bar{a}_1 \circ \bar{n} \right]_{\eta=s_2} F_1(s_1, s_2) \quad (s_1 = \pm 1, s_2 = \pm 1) \quad (2.96)$$

if the corner $\xi = s_1$, $\eta = s_2$ is inside the Mach forecone, while (see Eqs. 2.88, 2.89, and 2.92)

$$\begin{aligned} I_{S_1}(s_1, s_2) &= 0 & \bar{a}_1 \circ \bar{a}_1 &\geq 0 \\ &= \frac{1}{\bar{n} \circ \bar{n}} \left[\bar{\xi} \times \bar{a}_1 \circ \bar{n} \right]_{\eta=s_2} \frac{1}{\|\bar{a}_1\|} \frac{\pi}{2} s_1 & \bar{a}_1 \circ \bar{a}_1 < 0 \end{aligned} \quad (2.97)$$

if the corner is outside the Mach forecone, i.e., if

$$\bar{\xi} \circ \bar{\xi} \Big|_{\xi=s_1, \eta=s_2} < 0 \quad (2.98)$$

In writing Eq. (2.97), Eq. (2.92) has been modified to read

$$\begin{aligned} \text{Sign}(\bar{\xi} \circ \bar{a}_1) &= +1 & \text{for } s_1 &= -1 \\ &= -1 & \text{for } s_1 &= +1 \end{aligned} \quad (2.99)$$

Next, consider the second contribution, S_2 , to the source integral. This is given by Eq. (2.70). Interchanging the order of integration and repeating the procedure used for S_1 , one obtains (see Eq. 2.59)

$$\begin{aligned} S_2 &= \frac{1}{\bar{n} \circ \bar{n}} \text{Pr} \int_{-1}^1 d\eta \int_{-1}^1 \left[(\bar{a}_1 \times \bar{a}_2 \circ \bar{n} + \bar{\xi} \times \bar{\rho}_2 \circ \bar{n}) \bar{\xi} \circ \bar{\xi} - \bar{\xi} \times \bar{a}_2 \circ \bar{n} \bar{\xi} \circ \bar{a}_2 \right] \frac{H}{\|\bar{\xi}\|} d\eta \\ &= \frac{1}{\bar{n} \circ \bar{n}} \text{Pr} \int_{-1}^1 d\eta \int_{-1}^1 H \frac{\partial}{\partial \xi} \left(\bar{\xi} \times \bar{a}_2 \circ \bar{n} \frac{1}{\|\bar{\xi} \circ \bar{\xi}\|} \right) d\xi \\ &= \frac{1}{\bar{n} \circ \bar{n}} \text{Pr} \int_{-1}^1 \left[\bar{\xi} \times \bar{a}_2 \circ \bar{n} \frac{H}{\|\bar{\xi}\|} \right]_{\xi=-1}^{\xi=1} d\eta \end{aligned} \quad (2.100)$$

where

$$\begin{aligned} H_0 &= 0 && \text{outside the Mach forecone} \\ &= 1 && \text{inside the Mach forecone} \end{aligned}$$

(2.101)

or

$$\begin{aligned} H_0(\pm 1, \eta) &= 0 && -1 \leq \eta \leq \eta_1^\pm \\ &= 1 && \eta_1^\pm < \eta < \eta_2^\pm \\ &= 0 && \eta_2^\pm \leq \eta \leq 1 \end{aligned}$$

(2.102)

Hence, Eq. (2.100) may be rewritten as

$$S_2 = S_2^+ - S_2^- \quad (2.103)$$

where

$$S_2^\pm = \frac{1}{\bar{n} \circ \bar{n}} \left[\bar{f} \times \bar{a}_2 \circ \bar{n} \int_{\eta_1^\pm}^{\eta_2^\pm} \frac{1}{|\bar{f}|} d\eta \right]_{\bar{\zeta} = \pm 1} \quad (2.104)$$

Note that

$$S_2^\pm \equiv 0 \quad (2.105)$$

if $H_0 = 0$ along the edge $\eta = \pm 1$, i.e., if the edge $\eta = \pm 1$ is completely outside the Mach forecone. Otherwise, (see Eq. 2.59)

$$S_2^\pm = I_{S_1}(\pm 1, 1) - I_{S_1}(\pm 1, -1) \quad (2.106)$$

where

$$I_{S_2}(\pm 1, 1) = \frac{1}{\bar{n} \circ \bar{n}} \left[\bar{f} \times \bar{a}_2 \circ \bar{n} \right]_{\bar{\zeta} = \pm 1} F_2(\pm 1, \eta_2^\pm)$$

$$I_{S_2}(\pm 1, -1) = \frac{1}{\bar{n} \circ \bar{n}} \left[\bar{f} \times \bar{a}_2 \circ \bar{n} \right]_{\bar{\zeta} = \pm 1} F_2(\pm 1, \eta_1^\pm)$$

(2.107)

Note that if $\eta_2^{\pm} \neq 1$ (or $\eta_1^{\pm} \neq -1$),

$$I_{S_2}(S_1, S_2) = 0 \quad \bar{a}_2 \circ \bar{a}_2 \geq 0$$

$$= \frac{1}{\bar{n} \circ \bar{n}} \bar{f} \cdot \bar{a}_2 \circ \bar{n} \frac{1}{\|\bar{a}_2\|} \frac{\pi}{2} \text{sign}(\bar{f} \circ \bar{a}_2) \quad \bar{a}_2 \circ \bar{a}_2 < 0 \quad (2.108)$$

where

$$\text{Sign}(\bar{f} \circ \bar{a}_2) = -S_2 = +1 \quad S_2 = -1$$

$$= -1 \quad S_2 = +1 \quad (2.109)$$

In summary, it is possible to rewrite the results in a more compact form as follows

$$S_2 = I_{S_1}(1, 1) - I_{S_2}(1, -1) - I_{S_2}(-1, 1) + I_{S_2}(-1, -1) \quad (2.110)$$

where

$$I_{S_2}(1, 1) - I_{S_2}(1, -1) \equiv 0$$

or (2.111)

$$I_{S_2}(-1, 1) - I_{S_2}(-1, -1) \equiv 0$$

if the edge $\zeta = 1$ or $\zeta = -1$, respectively, is completely outside the Mach forecone. Otherwise,

$$I_{S_2}(1, S_2) = \frac{1}{\bar{n} \circ \bar{n}} \bar{f} \cdot \bar{a}_1 \circ \bar{n} \Big|_{\zeta=S_1} F_2(S_1, S_2) \begin{pmatrix} S_1 = \pm 1 \\ S_2 = \pm 1 \end{pmatrix} \quad (2.112)$$

if the corner $\zeta = S_1, \eta = S_2$ is inside the Mach forecone, while

$$I_{S_2}(S_1, S_2) = 0 \quad \bar{a}_2 \circ \bar{a}_2 \geq 0$$

$$= \frac{-1}{\bar{n} \circ \bar{n}} \left[\bar{f} \cdot \bar{a}_1 \circ \bar{n} \right]_{\zeta=S_1} \frac{1}{\|\bar{a}_1\|} \frac{\pi}{2} S_2 \quad \bar{a}_2 \circ \bar{a}_2 < 0 \quad (2.113)$$

if the corner is outside the Mach forecone.

Next consider the last contribution, S_3 , to the source integral. This is given by Eq. (2.70). Note that for planar elements

$$\bar{f} \cdot \bar{n} = \text{const} \quad (2.114)$$

Therefore

$$S_3 = - \frac{\bar{g} \cdot \bar{n}}{\bar{n} \cdot \bar{n}} \text{pf} \int_{-1}^1 \int_{-1}^1 H \frac{\bar{g} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{g}\|} d\xi d\eta$$

$$\equiv \frac{\bar{g} \cdot \bar{n}}{\bar{n} \cdot \bar{n}} 2\pi D \quad (2.115)$$

with D as the doublet integral given by Eq. (3.28). The evaluation of D is discussed in Subsection 3.4 under the less restrictive hypothesis of hyperboloidal (nonnecessarily planar) element. The results obtained there are applicable here and this, according to Eq. (3.39)

$$S_3 = - \frac{\bar{g} \cdot \bar{n}}{\bar{n} \cdot \bar{n}} \left[I_D(1,1) - I_D(1,-1) - I_D(-1,1) + I_D(-1,-1) \right] \quad (2.116)$$

where $I_D(S_1, S_2)$ is given by Eqs. (3.40) to (3.43).

In conclusion, by combining Eqs. (2.68), (2.93), (2.110), and (2.116), one obtains

$$2\pi S = I_S(1,1) - I_S(1,-1) - I_S(-1,1) + I_S(-1,-1) \quad (2.117)$$

with

$$I_S(S_1, S_2) = I_{S_1}(S_1, S_2) + I_{S_2}(S_1, S_2) - \frac{\bar{g} \cdot \bar{n}}{\bar{n} \cdot \bar{n}} I_D(S_1, S_2) \quad (2.118)$$

where $I_{S_1}(S_1, S_2)$ is given by Eqs. (2.94), to (2.97), $I_{S_2}(S_1, S_2)$ is given by Eqs. (2.111) to (2.113) and $I_D(S_1, S_2)$ is given by Eqs. (3.40) to (3.43).

SECTION 3

DOUBLET ELEMENT

3.1 Introduction

Consider the doublet integral, Eq. (1.39)

$$D = \frac{1}{2\pi} \iint \bar{N} \circ \nabla \left(\frac{H}{\sqrt{\xi \circ \bar{\xi}}} \right) d\Sigma \quad (3.1)$$

As mentioned in Section 1, in order to avoid evaluation of finite parts of integrals, it is convenient to use Eq. (1.40)

$$D = - \frac{1}{2\pi} \bar{N} \circ \nabla \cdot \iint \frac{H}{\sqrt{\xi \circ \bar{\xi}}} d\Sigma = - \bar{N} \circ \nabla \cdot S \quad (3.2)$$

Hence, according to Eq. (2.38)

$$D = + \frac{1}{2\pi} [\dot{\lambda}_D(1) - \dot{\lambda}_D(-1)] \quad (3.3)$$

with

$$\dot{\lambda}_D(\xi) = + \bar{N} \circ \nabla \cdot \dot{\lambda}_S(\xi) \quad (3.4)$$

or, according to Eq. (2.38)

$$\dot{\lambda}_D(\xi) = I_D(\xi, 1) - I_D(\xi, -1) \quad (3.5)$$

where

$$I_D(\xi, \eta) = + (\bar{N} \circ \nabla \cdot) I_S(\xi, \eta) \quad (3.6)$$

with I_S given by Eqs. (2.40) to (2.43).

3.2 Evaluation of I_D

Consider first Eq. (2.40). In order to evaluate I_D , it is convenient to use the definition of the unit normal \bar{N}

$$\bar{N} = \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} \quad (3.7)$$

and recast Eq. (2.40) as

$$\begin{aligned} I_s(\xi, \eta) = & \frac{1}{\bar{N} \circ \bar{N}} \left\{ -\bar{f} \times \bar{a}_1 \circ \bar{N} \frac{1}{\|\bar{a}_1\|} \ln \left| \frac{\|\bar{f}\| \|\bar{a}_1\| + \bar{f} \circ \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|} \right| \right. \\ & + \bar{f} \times \bar{a}_2 \cdot \bar{N} F(\eta) \\ & \left. - \bar{f} \cdot \bar{N} \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \right\} \end{aligned} \quad (3.8)$$

Note that \bar{P}_0 appears only in $\bar{q} = \bar{P} - \bar{P}_0$, (Eq. 1.27). Hence

$$(\bar{N} \circ \bar{v}_0) \bar{a}_1 = 0$$

$$(\bar{N} \circ \bar{v}_0) \bar{a}_2 = 0$$

$$(\bar{N} \circ \bar{v}_0) \bar{N} = 0$$

$$(\bar{N} \circ \bar{v}_0) \bar{f} = -\left(N_x \frac{\partial}{\partial x_0} - N_y \frac{\partial}{\partial y_0} - N_z \frac{\partial}{\partial z_0}\right) \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} = \begin{Bmatrix} N_x \\ -N_y \\ -N_z \end{Bmatrix} = -\bar{N}^c \quad (3.9)$$

where \bar{N}^c is the conormal, Eq. (1.15). Note that \bar{N} is orthogonal to \bar{a}_1 and \bar{a}_2 (Eq. 3.7). Thus

$$\bar{N}^c \circ \bar{a}_1 \equiv \bar{N} \cdot \bar{a}_1 = 0$$

$$\bar{N}^c \circ \bar{a}_2 \equiv \bar{N} \cdot \bar{a}_2 = 0$$

(3.10)

Thus

$$\begin{aligned} \bar{N} \circ \nabla_{\cdot} (\bar{f} \times \bar{a}_1 \circ \bar{N}) &= -\bar{N}^c \times \bar{a}_1 \circ \bar{N} = -\bar{N}^c \times \bar{a}_1 \circ \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} \\ &= -(\bar{N}^c \circ \bar{a}_1, \bar{a}_1 \circ \bar{a}_2 - \bar{N}^c \circ \bar{a}_2, \bar{a}_1 \circ \bar{a}_1) / |\bar{a}_1 \times \bar{a}_2| \equiv 0 \\ \bar{N} \circ \nabla_{\cdot} (\bar{f} \times \bar{a}_2 \circ \bar{N}) &= -\bar{N}^c \times \bar{a}_2 \circ \bar{N} = -\bar{N}^c \times \bar{a}_2 \circ \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} \\ &= -(\bar{N}^c \circ \bar{a}_1, \bar{a}_2 \circ \bar{a}_2 - \bar{N}^c \circ \bar{a}_2, \bar{a}_1 \circ \bar{a}_2) / |\bar{a}_1 \times \bar{a}_2| \equiv 0 \end{aligned} \tag{3.11}$$

and

$$\bar{N} \circ \nabla_{\cdot} (\bar{f} \cdot \bar{N}) = -\bar{N}^c \cdot \bar{N} = -\bar{N} \circ \bar{N} \tag{3.12}$$

On the other hand, for $\bar{a}_1 \circ \bar{a}_1 > 0$,

$$\begin{aligned} &\bar{N} \circ \nabla_{\cdot} \left(\frac{1}{\|\bar{a}_1\|} \ln \left(\frac{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_1 \circ \bar{a}_1} + \bar{f} \circ \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|} \right) \right) \\ &= \frac{1}{\|\bar{a}_1\|} \bar{N} \circ \nabla_{\cdot} \left\{ \ln (\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_1 \circ \bar{a}_1} + \bar{f} \circ \bar{a}_1) - \frac{1}{2} \ln (-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_1) \right\} \\ &= \frac{1}{\|\bar{a}_1\|} \left\{ \frac{1}{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_1 \circ \bar{a}_1} + \bar{f} \circ \bar{a}_1} \left(\frac{-\bar{N}^c \circ \bar{f}}{\sqrt{\bar{f} \circ \bar{f}}} \sqrt{\bar{a}_1 \circ \bar{a}_1} + \bar{N}^c \circ \bar{a}_1 \right) - \frac{1}{2} \frac{-2(-\bar{N}^c) \times \bar{a}_1 \circ \bar{f} \times \bar{a}_1}{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_1} \right\} \\ &= \frac{1}{\|\bar{a}_1\|} \left\{ \frac{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_1 \circ \bar{a}_1} - \bar{f} \circ \bar{a}_1}{\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_1 - (\bar{f} \circ \bar{a}_1)^2} \frac{(-\bar{N}^c \circ \bar{f})}{\sqrt{\bar{f} \circ \bar{f}}} \sqrt{\bar{a}_1 \circ \bar{a}_1} + \frac{\bar{N}^c \circ \bar{f} \bar{a}_1 \circ \bar{a}_1 - \bar{N}^c \circ \bar{a}_1 \bar{f} \circ \bar{a}_1}{-\|\bar{f} \times \bar{a}_1\|^2} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\|\bar{a}_1\|} \left\{ \frac{-\bar{N} \cdot \bar{f} \bar{a}_1 \circ \bar{a}_1}{-\|\bar{f} \times \bar{a}_1\|^2} + \frac{\bar{N} \cdot \bar{f} \bar{f} \circ \bar{a}_1 \sqrt{\bar{a}_1 \circ \bar{a}_1}}{-\|\bar{f} \times \bar{a}_1\|^2 \sqrt{\bar{f} \circ \bar{f}}} + \frac{\bar{N} \cdot \bar{f} \bar{a}_1 \circ \bar{a}_1}{-\|\bar{f} \times \bar{a}_1\|^2} \right\} \\
 &= \bar{f} \cdot \bar{N} \frac{\bar{f} \circ \bar{a}_1}{-\|\bar{f} \times \bar{a}_1\|^2} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}}
 \end{aligned}$$

(3.13)

Furthermore, by setting

$$S_n = \text{sign}(\bar{f} \cdot \bar{N})$$

(3.14)

one obtains

$$\begin{aligned}
 &N \circ \nabla \cdot \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| |\bar{f} \cdot \bar{a}_1 \times \bar{a}_2|} \right) \\
 &= S_n \frac{1}{1 + \frac{(\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)^2}{\bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2}} \times \\
 &\times (-1) \times \left\{ \frac{-\bar{N} \cdot \bar{a}_1 \circ \bar{f} \times \bar{a}_2 + \bar{f} \times \bar{a}_1 \circ (-\bar{N}^c) \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}} S_n (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)} + \right. \\
 &\left. + (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2) \left[-\frac{-\bar{N}^c \circ \bar{f}}{(\bar{f} \circ \bar{f})^{3/2}} \frac{1}{S_n (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)} + \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} S_n \frac{-\bar{N}^c \cdot \bar{a}_1 \times \bar{a}_2}{-(\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2} \right] \right\} \\
 &= S_n \frac{\|\bar{f}\|^2 (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2}{\bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) + (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)^2} \frac{1}{\|\bar{f}\|^3 (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2} \times
 \end{aligned}$$

$$\times \left\{ (\bar{N}^c \cdot \bar{a}_1 \circ \bar{f} \times \bar{a}_2 + \bar{f} \times \bar{a}_1 \circ \bar{N}^c \cdot \bar{a}_2) \bar{f} \circ \bar{f} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \right. \\ \left. - \bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2 \left[\bar{N}^c \circ \bar{f} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 + \bar{f} \circ \bar{f} \bar{N}^c \cdot \bar{a}_1 \times \bar{a}_2 \right] \right\}$$

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(3.15)

In order to obtain a simpler expression, note that according to the first and second super-rules and Eq. (3.10),

$$\begin{aligned} & (\bar{N}^c \cdot \bar{a}_1 \circ \bar{f} \times \bar{a}_2 + \bar{f} \times \bar{a}_1 \circ \bar{N}^c \cdot \bar{a}_2) \bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) \\ & - (\bar{N}^c \circ \bar{f} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 + \bar{f} \circ \bar{f} \bar{N}^c \cdot \bar{a}_1 \times \bar{a}_2) (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2) \\ = & (\bar{N}^c \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 - \bar{N}^c \circ \bar{a}_2 \bar{a}_1 \circ \bar{f} + \bar{f} \circ \bar{N}^c \bar{a}_1 \circ \bar{a}_2 - \bar{f} \circ \bar{a}_2 \bar{a}_1 \circ \bar{N}^c) \\ & \times (\bar{f} \circ \bar{f}) (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) \\ & - (\bar{f} \circ \bar{N}^c \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 + \bar{f} \circ \bar{f} \bar{N}^c \cdot \bar{a}_1 \times \bar{a}_2) (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2) \\ & + (\bar{f} \circ \bar{N}^c \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 + \bar{f} \circ \bar{f} \bar{N}^c \cdot \bar{a}_1 \times \bar{a}_2) (\bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \\ = & (\bar{f} \cdot \bar{N} \bar{a}_1 \circ \bar{a}_2) \bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) - (\bar{f} \circ \bar{f}) (\bar{a}_1 \circ \bar{a}_2) (\bar{N}^c \cdot \bar{a}_1 \times \bar{a}_2) \\ & + \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2 \bar{f} \cdot \bar{N} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 + \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2 \bar{f} \circ \bar{f} \bar{N}^c \cdot \bar{a}_1 \times \bar{a}_2 \\ = & \bar{f} \cdot \bar{N} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 + \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \\ & - \bar{f} \circ \bar{f} \bar{N} \circ \bar{a}_1 \times \bar{a}_2 (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 - \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \\ = & \frac{1}{|\bar{a}_1 \times \bar{a}_2|} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 + \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \\ & - \frac{1}{|\bar{a}_1 \times \bar{a}_2|} \bar{f} \circ \bar{f} (\bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 - \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{|\bar{a}_1 \times \bar{a}_2|} \left\{ \bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 \circ \bar{a}_1 \circ \bar{a}_2 (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 + \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \right. \\
 &\quad - \bar{f} \circ \bar{a}_2 (\bar{a}_1 \circ \bar{a}_1 \bar{f} \circ \bar{a}_2 - \bar{f} \circ \bar{a}_1 \bar{a}_1 \circ \bar{a}_2) (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 + \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \\
 &\quad - \bar{f} \circ \bar{a}_1 (\bar{a}_2 \circ \bar{a}_2 \bar{f} \circ \bar{a}_1 - \bar{f} \circ \bar{a}_2 \bar{a}_2 \circ \bar{a}_1) (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 + \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \\
 &\quad \left. - \bar{f} \circ \bar{f} (\bar{a}_1 \circ \bar{a}_2 \circ \bar{a}_1 \circ \bar{a}_2) (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 - \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2) \right\} \\
 &= \frac{1}{|\bar{a}_1 \times \bar{a}_2|} \left\{ \bar{f} \circ \bar{f} (\bar{a}_1 \circ \bar{a}_1 \bar{a}_2 \circ \bar{a}_2 - (\bar{a}_1 \circ \bar{a}_2)^{\wedge}) \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2 + \right. \\
 &\quad - (\bar{f} \circ \bar{a}_2)^{\wedge} \bar{a}_1 \circ \bar{a}_1 \bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 - (\bar{f} \circ \bar{a}_2)^{\wedge} \bar{a}_1 \circ \bar{a}_1 \bar{f} \circ \bar{a}_1 + \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2 \bar{f} \circ \bar{f} (\bar{a}_1 \circ \bar{a}_2)^{\wedge} \\
 &\quad + (\bar{f} \circ \bar{a}_1)^{\wedge} (\bar{f} \circ \bar{a}_2)^{\wedge} \bar{a}_1 \circ \bar{a}_2 - (\bar{f} \circ \bar{a}_1)^{\wedge} \bar{a}_2 \circ \bar{a}_2 \bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 - (\bar{f} \circ \bar{a}_2)^{\wedge} \bar{a}_2 \circ \bar{a}_2 \bar{f} \circ \bar{a}_2 \\
 &\quad \left. + \bar{f} \circ \bar{a}_2 \bar{f} \circ \bar{a}_2 \bar{f} \circ \bar{f} (\bar{a}_1 \circ \bar{a}_2)^{\wedge} + (\bar{f} \circ \bar{a}_2)^{\wedge} (\bar{f} \circ \bar{a}_1)^{\wedge} \bar{a}_1 \circ \bar{a}_2 \right\} \\
 &= \frac{1}{|\bar{a}_1 \times \bar{a}_2|} \left\{ \bar{f} \circ \bar{a}_1 (- (\bar{f} \circ \bar{a}_2 \bar{a}_1 \circ \bar{a}_1 - \bar{f} \circ \bar{a}_1 \bar{a}_1 \circ \bar{a}_2) (\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2 - (\bar{f} \circ \bar{a}_2)^{\wedge}) \right. \\
 &\quad \left. + \bar{f} \circ \bar{a}_2 ((\bar{f} \circ \bar{a}_1 \bar{a}_2 \circ \bar{a}_2 - \bar{f} \circ \bar{a}_2 \bar{a}_2 \circ \bar{a}_1) (\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2 - (\bar{f} \circ \bar{a}_1)^{\wedge}) \right\} \\
 &= \frac{1}{|\bar{a}_1 \times \bar{a}_2|} \left\{ \bar{f} \circ \bar{a}_1 (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) (\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2) + \bar{f} \circ \bar{a}_2 (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_1) \right\} \\
 &= (\bar{f} \times \bar{a}_1 \circ \bar{N}) (\bar{f} \circ \bar{a}_1) \|\bar{f} \times \bar{a}_2\|^{\wedge} - (\bar{f} \times \bar{a}_2 \circ \bar{N}) (\bar{f} \circ \bar{a}_2) \|\bar{f} \times \bar{a}_1\|^{\wedge} \tag{3.16}
 \end{aligned}$$

Furthermore, as shown in Appendix C (Eq. C.14)

$$\bar{f} \circ \bar{f} (\bar{f} \times \bar{a}_1 \circ \bar{a}_2)^{\wedge} + (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)^{\wedge} = \|\bar{f} \times \bar{a}_1\|^{\wedge} \|\bar{f} \times \bar{a}_2\|^{\wedge} \tag{3.17}$$

Hence, combining Eqs. (3.15), (3.16) and (3.17), yields

$$N \circ \nabla \cdot \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| |\bar{f} \times \bar{a}_1 \times \bar{a}_2|} \right) =$$

$$\begin{aligned}
 &= S_n \frac{\bar{f} \times \bar{a}_1 \circ \bar{N} \bar{f} \circ \bar{a}_1 \|\bar{f} \times \bar{a}_2\|^2 - \bar{f} \times \bar{a}_2 \circ \bar{N} \bar{f} \circ \bar{a}_2 \|\bar{f} \times \bar{a}_1\|^2}{\|\bar{f} \times \bar{a}_1\|^2 \|\bar{f} \times \bar{a}_2\|^2} \frac{1}{\|\bar{f}\|^2} \\
 &= \frac{S_n}{\|\bar{f}\|^2} \left(\bar{f} \times \bar{a}_1 \circ \bar{N} \frac{\bar{f} \circ \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|^2} - \bar{f} \times \bar{a}_2 \circ \bar{N} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|^2} \right)
 \end{aligned} \tag{3.18}$$

Next, it is shown that for any value of $\bar{a}_2 \circ \bar{a}_2$

$$\bar{N} \circ \nabla \cdot F(\eta) = \bar{N} \cdot \bar{f} \frac{\bar{f} \circ \bar{a}_2}{-\|\bar{f} \times \bar{a}_2\|^2} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} \tag{3.19}$$

where F is given by Eq. (2.41). For, if $\bar{a}_2 \circ \bar{a}_2 > 0$, repeating the same derivation as shown in Eq. (3.13),

$$\begin{aligned}
 \bar{N} \circ \nabla \cdot F &= \bar{N} \circ \nabla \cdot \left(\frac{1}{\|\bar{a}_2\|^2} \ln \left(\frac{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right) \right) \\
 &= \bar{f} \cdot \bar{N} \frac{\bar{f} \circ \bar{a}_2}{-\|\bar{f} \times \bar{a}_2\|^2} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}}
 \end{aligned} \tag{3.20}$$

Similarly, if $\bar{a}_2 \circ \bar{a}_2 = 0$, then $-(\bar{f} \circ \bar{a}_2)^2 = \bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2 = -\|\bar{f} \times \bar{a}_2\|^2$

and

$$\begin{aligned}
 \bar{N} \circ \nabla \cdot F &= \bar{N} \circ \nabla \cdot \left(\frac{\sqrt{\bar{f} \circ \bar{f}}}{\bar{f} \circ \bar{a}_2} \right) \\
 &= -\frac{\bar{N} \circ \bar{f}}{\sqrt{\bar{f} \circ \bar{f}}} \frac{1}{\bar{f} \circ \bar{a}_2} - \sqrt{\bar{f} \circ \bar{f}} \frac{1}{-\|\bar{f} \times \bar{a}_2\|^2} (-\bar{N} \circ \bar{a}_2) \\
 &= \bar{f} \cdot \bar{N} \frac{\bar{f} \circ \bar{a}_2}{-\|\bar{f} \times \bar{a}_2\|^2} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}}
 \end{aligned} \tag{3.21}$$

Finally, if $\bar{a}_2 \circ \bar{a}_2 < 0$

$$\begin{aligned}
 \bar{N} \circ \bar{\nabla}_s F &= \bar{N} \circ \bar{\nabla}_s \left(\frac{1}{\|\bar{a}_2\|} \sin^{-1} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right) \\
 &= \frac{-1}{\|\bar{a}_2\|} \frac{1}{\sqrt{1 - \left(\frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right)^2}} \left(-\frac{\bar{N} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} + \bar{f} \circ \bar{a}_2 \left(-\frac{1}{2} \right) \frac{(-2)(-\bar{N} \circ \bar{a}_2 \circ \bar{f} \times \bar{a}_2)}{\|\bar{f} \times \bar{a}_2\|^3} \right) \\
 &= \frac{-1}{\|\bar{a}_2\|} \frac{\|\bar{f} \times \bar{a}_2\|}{\sqrt{-\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2 - (\bar{f} \circ \bar{a}_2)^2}} \left(-\frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} (\bar{N} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2 - \bar{N} \circ \bar{a}_2 \bar{f} \circ \bar{a}_2) \right) \\
 &= \frac{1}{\|\bar{a}_2\|} \frac{1}{\sqrt{-\bar{a}_2 \circ \bar{a}_2 \bar{f} \circ \bar{f}}} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|^2} \bar{f} \cdot \bar{N} \bar{a}_2 \circ \bar{a}_2 \\
 &= -\bar{f} \cdot \bar{N} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|^2} \frac{1}{\|\bar{f}\|}
 \end{aligned}$$

(3.22)

Finally, combining Eqs. (3.8), (3.11), (3.12), (3.13), (3.18), and (3.19) yields

$$\begin{aligned}
 I_s = \bar{N} \circ \bar{\nabla}_s I_s &= \frac{1}{\bar{N} \circ \bar{N}} \left\{ -(\bar{f} \times \bar{a}_2 \circ \bar{N}) \bar{N} \circ \bar{\nabla}_s \left(\frac{1}{\|\bar{a}_2\|} \ln \left| \frac{\|\bar{f}\| \|\bar{a}_2\| + \bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right| \right) \right. \\
 &\quad \left. + (\bar{f} \times \bar{a}_2 \circ \bar{N}) \bar{N} \circ \bar{\nabla}_s F - S_n (\bar{f} \cdot \bar{N}) \bar{N} \circ \bar{\nabla}_s \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \|\bar{f} \times \bar{a}_2\|} \right) \right. \\
 &\quad \left. - S_n [\bar{N} \circ \bar{\nabla}_s (\bar{f} \cdot \bar{N})] \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \|\bar{f} \times \bar{a}_2\|} \right) \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\bar{N} \circ \bar{N}} \left\{ + \bar{f} \times \bar{a}_1 \circ \bar{N} (\bar{f} \cdot \bar{N}) \frac{\bar{f} \circ \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|^2} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} \right. \\
 &\quad \left. - \bar{f} \times \bar{a}_2 \circ \bar{N} (\bar{f} \cdot \bar{N}) \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|^2} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} \right. \\
 &\quad \left. - S_n \bar{f} \cdot \bar{N} \frac{S_n}{\sqrt{\bar{f} \circ \bar{f}}} \left(\bar{f} \times \bar{a}_1 \circ \bar{N} \frac{\bar{f} \circ \bar{a}_1}{\|\bar{f} \times \bar{a}_1\|^2} - \bar{f} \times \bar{a}_2 \circ \bar{N} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|^2} \right) \right. \\
 &\quad \left. - S_n [-\bar{N} \circ \bar{N}] \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \|\bar{f} \cdot \bar{a}_1 \times \bar{a}_2\|} \right) \right\} \\
 &= \text{Sign}(\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}} \|\bar{f} \cdot \bar{a}_1 \times \bar{a}_2\|} \right) \\
 &= \tan_p^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}} \|\bar{f} \cdot \bar{a}_1 \times \bar{a}_2\|} \right) \tag{3.23}
 \end{aligned}$$

3.3 Direct Integration (for elements completely inside the Mach cone)

3.3 Quadrilateral Element Internal to Mach Cone

Consider Eq. (3.1) for elements internal to the Mach forecone. In this case, $H \equiv 1$ and Eq. (3.1) reduces to

$$\begin{aligned}
 D &= \frac{1}{2\pi} \int_{-1}^1 \int_{-1}^1 \bar{\nabla} \left(\frac{1}{\|\bar{f}\|} \right) \circ \bar{N} \, d\xi \, d\eta \\
 &= \frac{1}{2\pi} \int_{-1}^1 \int_{-1}^1 \left(-\frac{1}{2} \frac{1}{\|\bar{f}\|^3} \right) \bar{\nabla}(\bar{f} \circ \bar{f}) \circ \bar{N} \|\bar{a}_1 \times \bar{a}_2\| \, d\xi \, d\eta
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{-1}{2\pi} \int_{-1}^1 \int_{-1}^1 \frac{1}{\|\bar{f}\|^3} \bar{f}^c \circ \bar{a}_1 \times \bar{a}_2 \, d\xi \, d\eta \\
 &= -\frac{1}{2\pi} \int_{-1}^1 \int_{-1}^1 \frac{\bar{f} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{f}\|^3} \, d\xi \, d\eta
 \end{aligned}$$

(3.24)

since $\nabla(\bar{f} \circ \bar{f}) = 2\bar{f}^c$. Consider the function

$$I_0(\xi, \eta) = \tan^{-1}_P \frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2}$$

(3.25)

As shown in Appendix D (see Eq. D.17)

$$\frac{\partial^2 I_0}{\partial \xi \partial \eta} = \frac{\bar{f} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{f}\|^3}$$

(3.26)

Therefore, combining Eqs. (3.24) and (3.26), one obtains

$$\begin{aligned}
 -2\pi D &= \int_{-1}^1 \int_{-1}^1 \frac{\partial^2 I_0}{\partial \xi \partial \eta} \, d\xi \, d\eta \\
 &= I_0(1, 1) - I_0(-1, 1) - I_0(1, -1) + I_0(-1, -1)
 \end{aligned}$$

(3.27)

3.4 General Quadrilateral Element

In order to extend the above results to elements partially inside the Mach forecone, it is necessary to use the methods of theory of distributions, which were introduced in Section 2. For, the conormal derivative of $\frac{H}{\|\bar{f}\|}$ is not integrable in the theory of regular functions. The interpretation of this fact is given

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in Ref. 1, Appendix H, where the doublet integral is obtained in the limit (by replacing $\frac{H}{\|\bar{r}\|}$ with a suitable continuous function and then taking the limit as the continuous function approaches the original function).

In the case of a planar element, the problem can be circumvented in the way used in Subsection 3.2, that is, by replacing the integral of the normal derivative with the normal derivative of the integral. On the other hand, in the case of general hyperboloidal quadrilateral element, it is still possible to evaluate the integral, however only by using the methods of the theory of distributions. Consider therefore, Eq. (3.1) which for an element inside the Mach forecone, may be written as (see Eq. 3.24)

$$\begin{aligned}
 2\pi D &= \int_{-1}^1 \int_{-1}^1 \bar{N} \circ \bar{V} \left(\frac{H}{\|\bar{r}\|} \right) d\Sigma \\
 &= \rho_f \int_{-1}^1 \int_{-1}^1 H \bar{N} \circ \bar{V} \left(\frac{1}{\|\bar{r}\|} \right) d\Sigma \\
 &= -\rho_f \int_{-1}^1 \int_{-1}^1 H \frac{\bar{r} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{r}\|^3} d\xi d\eta
 \end{aligned}$$

(3.28)

Using Eq. (D.17), one obtains

$$-2\pi D = \int_{-1}^1 \frac{1}{\|\bar{r} \times \bar{a}_1\|^2 \|\bar{r}\|} \frac{H_0}{\|\bar{r}\|} \left(\bar{r} \circ \bar{r} \bar{r} \cdot \bar{a}_1 \times \bar{a}_2 - \bar{r} \circ \bar{a}_1 \bar{r} \cdot \bar{a}_1 \times \bar{a}_2 \right) \Big|_{\xi=-1}^{\xi=1} d\eta \quad (3.29)$$

where H_0 , defined by Eq. (2.75), is used in order to take into account the fact that the contributions of the intersection of the element with the surface of the Mach forecone is equal to zero (in agreement with Eq. 2.73). This is the same procedure used in Eq. (2.74).

Next, introduce the abscissas η_1^+ , η_2^+ , η_1^- , and η_2^- such that

$$\begin{aligned} H_0(\pm \eta) &= 0 & -1 \leq \eta \leq \eta_1^+ \\ &= 1 & \eta_1^+ \leq \eta \leq \eta_2^+ \\ &= 0 & \eta_2^+ \leq \eta \leq 1 \end{aligned} \quad (3.30)$$

Note that $\eta_2^+ \equiv +1$ if the point (1,1) is inside the Mach forecone.

Similar possibilities hold for η_2^- , η_1^- , η_1^- . Using Eq. (3.30),

Eq. (3.29) may be rewritten as

$$\begin{aligned} -2\pi D &= \int_{\eta_1^+}^{\eta_2^+} \left[\frac{1}{\|\bar{\mathbf{r}} \times \bar{\mathbf{a}}_1\|^2} \frac{1}{\|\bar{\mathbf{r}}\|} (\bar{\mathbf{r}} \cdot \bar{\mathbf{r}} \bar{\mathbf{r}} \cdot \bar{\mathbf{a}}_1 \times \bar{\mathbf{p}}_3 - \bar{\mathbf{r}} \cdot \bar{\mathbf{a}}_1 \bar{\mathbf{r}} \cdot \bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2) \right]_{\xi=1} d\eta \\ &- \int_{\eta_1^-}^{\eta_2^-} \left[\frac{1}{\|\bar{\mathbf{r}} \times \bar{\mathbf{a}}_1\|^2} \frac{1}{\|\bar{\mathbf{r}}\|} (\bar{\mathbf{r}} \cdot \bar{\mathbf{r}} \bar{\mathbf{r}} \cdot \bar{\mathbf{a}}_1 \times \bar{\mathbf{p}}_3 - \bar{\mathbf{r}} \cdot \bar{\mathbf{a}}_1 \bar{\mathbf{r}} \cdot \bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2) \right]_{\xi=-1} d\eta \end{aligned} \quad (3.31)$$

or, by using Eq. (D.15)

$$\begin{aligned} -2\pi D &= \int_{\eta_1^+}^{\eta_2^+} \frac{\partial I_D}{\partial \eta} \Big|_{\xi=1} d\eta - \int_{\eta_1^-}^{\eta_2^-} \frac{\partial I_D}{\partial \eta} \Big|_{\xi=-1} d\eta \\ &= I_D(1, \eta_2^+) - I_D(1, \eta_1^+) - I_D(-1, \eta_2^-) + I_D(-1, \eta_1^-) \end{aligned} \quad (3.32)$$

It should be emphasized that if $H_0(\pm 1, \eta) \equiv 0$, then $\eta_2^\pm \equiv \eta_1^\pm$, and thus

$$I_D(\pm 1, \eta_2^\pm) - I_D(\pm 1, \eta_1^\pm) \equiv 0 \quad (3.33)$$

Note that, if $\xi_2^+ \neq 0$, then $\|\bar{f}\| = 0$ and hence, $-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2 = \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2$ and therefore

$$\begin{aligned} I_D(1, \eta_2^+) &= \text{sign} \left(\frac{\bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2}{\bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \\ &= \text{sign}(\bar{f} \circ \bar{a}_1) \text{sign}(\bar{f} \circ \bar{a}_2) \text{sign}(\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) \end{aligned} \quad (3.34)$$

with $\text{sign}(\bar{f} \circ \bar{a}_2)$ given by Eq. (2.109). In order to find the values of the other terms, note that, for $\xi = 1$, η_1^+ and η_1^- are the roots of

$$\begin{aligned} \bar{f} \circ \bar{f} &= (\bar{p}_0 + \bar{p}_1) + \eta(\bar{p}_2 + \bar{p}_3) \\ &= (\bar{p}_0 + \bar{p}_1) \circ (\bar{p}_0 + \bar{p}_1) + 2\eta(\bar{p}_0 + \bar{p}_1) \circ (\bar{p}_2 + \bar{p}_3) + \eta^2(\bar{p}_2 + \bar{p}_3) \circ (\bar{p}_2 + \bar{p}_3) = 0 \end{aligned} \quad (3.35)$$

and thus, at $\xi = 1$ and $\eta = \eta_2^+$

$$\bar{f} \circ \bar{a}_1 = [(\bar{p}_0 + \bar{p}_1) + \eta_2^+(\bar{p}_2 + \bar{p}_3)] \circ (\bar{p}_1 + \eta_2^+ \bar{p}_3) \quad (3.36)$$

while

$$\bar{f} \cdot \bar{a}_1 \times \bar{a}_2 = [(\bar{p}_0 + \bar{p}_1) + \eta_2^+(\bar{p}_2 + \bar{p}_3)] \cdot (\bar{p}_1 + \eta_2^+ \bar{p}_3) \times (\bar{p}_2 + \bar{p}_3) \quad (3.37)$$

Finally, it may be noted that $\bar{f} \circ \bar{a}_1 = \bar{f}^c \cdot \bar{a}_1 = 0$ (or, $\bar{f} \circ \bar{a}_2 = \bar{f}^c \cdot \bar{a}_2 = 0$) at $\eta = \eta_2^+$ implies that \bar{a}_1 (\bar{a}_2 respectively) is tangent to the surface of the Mach forecone. Hence, one can conclude that $\bar{f} \circ \bar{a}_2 = 0$ and $\bar{f} \circ \bar{a}_1 = 0$ cannot occur simultaneously, otherwise the element is tangent to the Mach forecone, contrary to the hypothesis of small perturbation. Therefore, if $\bar{f} \circ \bar{a}_1 = 0$, the configuration is as shown in Fig. 6 and it is evident that a more convenient way to evaluate D is by interchanging the order of integration, which yields

$$-2\pi D = I_D(\xi_2^+, 1) - I_D(\xi_1^+, 1) - I_D(\xi_2^-, -1) + I_D(\xi_1^-, -1) \quad (3.38)$$

Obviously Eq. (3.38) is equivalent to Eq. (3.32).

In summary, it is possible to rewrite the results in a more compact form, as follows,

$$-2\pi\mathcal{D} = I_p(1, 1) - I_p(1, -1) - I_p(-1, 1) + I_p(-1, -1) \quad (3.39)$$

where

$$I_p(1, 1) - I_p(-1, +1) \equiv 0 \quad (3.40)$$

or

$$I_p(1, -1) - I_p(-1, -1) \equiv 0 \quad (3.41)$$

if the edge $\eta = 1$ or $\eta = -1$, respectively, is completely outside the Mach forecone. Otherwise

$$I_p(s_1, s_2) = \tan_p^{-1} \frac{-\bar{\eta} \times \bar{a}_1 \circ \bar{\eta} \times \bar{a}_2}{\|\bar{\eta}\| \bar{\eta} \cdot \bar{a}_1 \times \bar{a}_2} \quad (3.42)$$

if the corner $\xi = s_1$, $\eta = s_2$, is inside the Mach forecone, otherwise

$$I_p(s_1, s_2) = \frac{\pi}{2} \text{sign} [(\bar{\eta} \circ \bar{a}_1)(\bar{\eta} \circ \bar{a}_2)(\bar{\eta} \cdot \bar{a}_1 \times \bar{a}_2)] \quad (3.43)$$

where the term in brackets is evaluated as indicated by Eqs. (3.34) to (3.38).

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SECTION 4

METHOD OF SOLUTION

4.1 Introduction

In the preceding Sections, the problem has been formulated and the expressions for the source and doublet integrals obtained. The method of solution is outlined in this Section. This requires a discussion of the value of E on the surface Σ and an analysis of the role of the diaphragms. These are presented in the following Subsections. Then, the general method of solution is presented. For simplicity, supersonic trailing edges are considered so that the contribution of the wake need not be considered. For subsonic leading edges the wake may be included by following the same procedure used in Ref. 6.

4.2 Value of E on Σ

Consider Eqs. (1.6) and (1.8). In order to evaluate the value of E on the surface, Σ , it is convenient to obtain the limit value, as the control point, P_* , approaches the surface, Σ , of the body. Following the same procedure used in Appendix C of Ref. 1, consider a small neighborhood Σ_ϵ of the point P_* on the body (Fig. 7). Equation (1.6) may be rewritten as

$$2\pi\epsilon_0\dot{\phi}_0 = - \iint_{\Sigma-\Sigma_\epsilon} \left(\frac{\partial\phi}{\partial N^c} \frac{H}{\|\mathbf{q}\|} - \phi \frac{\partial}{\partial N^c} \left(\frac{H}{\|\mathbf{q}\|} \right) \right) d\Sigma + 2\pi\dot{\phi}_\epsilon \quad (4.1)$$

where

$$2\pi\delta_\varepsilon = - \oint_{\Sigma_\varepsilon} \left(\frac{\partial\phi}{\partial N^c} \frac{H}{\|\bar{q}\|} - \phi \frac{\partial}{\partial N^c} \left(\frac{H}{\|\bar{q}\|} \right) \right) d\Sigma \quad (4.2)$$

Assume that the surface Σ_ε is a small quadrilateral element, as shown in Fig. 8. If ε is sufficiently small, the surface may be replaced by a quadrilateral planar element. Then, if P_0 approaches the point P_* , the value of $\bar{q} \cdot \bar{a}, x\bar{a}_2$ tends to zero; and, neglecting higher order terms in ε (which includes, in particular, the first integral in Eq. 4.2) in analogy with the results of Appendix C of Ref. 1, one obtains (see Eq. 3.29)

$$\begin{aligned} \lim_{P_0 \rightarrow P_*} 2\pi\delta_\varepsilon &= \phi_* \lim_{P_0 \rightarrow P_*} \oint_{\Sigma_\varepsilon} \frac{\partial}{\partial N^c} \left(\frac{H}{\|\bar{q}\|} \right) d\Sigma \\ &= -\phi_* \lim_{P_0 \rightarrow P_*} \int_{\eta^-}^{\eta^+} \left[\frac{H_0}{\|\bar{q}\|} \frac{\bar{q}_0 \bar{q} \cdot \bar{a}, x\bar{p}_3 - \bar{q} \partial \bar{a}, \bar{q} \cdot \bar{a}, x\bar{a}_2}{\|\bar{q} \times \bar{a}\|^2} \right]_{\xi=-1}^{\xi=1} d\eta \\ &= \phi_* \lim_{P_0 \rightarrow P_*} \left[\tan^{-1} \left(\frac{-\bar{q} \times \bar{a}, \cdot \bar{q} \times \bar{a}_2}{\|\bar{q}\| \bar{q} \cdot \bar{a}, x\bar{a}_2} \right) \right]_{\eta=\eta^-}^{\eta=\eta^+} \\ &= \pm \phi_* \pi \end{aligned} \quad (4.3)$$

where the upper sign hold if P is outside the surface Σ (region $E = 1$) and vice versa. Hence, for infinitesimal values of ε , combining Eqs. (4.1) and (4.3), yields, for P_* on Σ

$$2\pi E_* \phi_* = - \oint_{\Sigma} \left(\frac{\partial\phi}{\partial N^c} \frac{H}{\|\bar{q}\|} - \phi \frac{\partial}{\partial N^c} \frac{H}{\|\bar{q}\|} \right) d\Sigma \quad (4.4)$$

where

$$\begin{aligned} E_* &= E_0 + 1 = 1 - \frac{1}{2} = \frac{1}{2} && (P_0 \text{ outside } \Sigma) \\ &= 0 + \frac{1}{2} = \frac{1}{2} && (P_0 \text{ inside } \Sigma) \end{aligned} \quad (4.5)$$

in both cases (P_0 inside or outside Σ), in correspondence with Eq. (C.13) of Ref. 1. Note that Eq. (4.4) may be included as a particular case of Eq. (1.17) by extending the definition of E as

$$\begin{aligned} E &= 1 && \text{Outside } \Sigma \\ &= 1/2 && \text{On } \Sigma \\ &= 0 && \text{Inside } \Sigma. \end{aligned} \quad (4.6)$$

Finally, it may be noted that Eq. (4.4) is an integral expression relating the value of ϕ at one point P of the surface Σ to the values of ϕ and $\partial\phi/\partial N^c$ on the surface Σ . The values of ϕ are unknown, while the values of $\partial\phi/\partial N^c$ are known from the boundary condition, Eq. (1.21). (except for terms of the same order of magnitude as the ones neglected in the process of linearizing the differential equation for the supersonic potential flow).

4.3 Numerical Procedure

As mentioned above only wings with a supersonic trailing edge are considered here. In this case, the wake does not affect the wing and, thus, it can be

ignored.* Consider first a wing with subsonic leading edge. In this case Eq. (4.4) is the desired integral equation and can be solved as follows: divide Σ into small quadrilateral elements, assume ϕ and $\partial\phi/\partial N^c$ constant within the elements. Then, Eq. (4.4) (written at the centroid, P_h , of the element Σ_h) yields

$$[\delta_{hk} - c_{hk}] \{\phi_k\} = [b_{hk}] \{\psi_k\} \quad (4.7)$$

where

$$\psi_k = \left[\frac{\partial\phi}{\partial N^c} \right]_{P=P_k} \quad (4.8)$$

is given by the boundary conditions Eq. (1.21), while, approximating the element with a quadrilateral hyperboloidal element, c_{hk} and b_{hk} are given by

$$c_{hk} = \left[\frac{1}{\pi} \iint_{\Sigma_k} \frac{\partial}{\partial N^c} \left(\frac{H}{|\vec{r}|} \right) d\Sigma_k \right]_{P=P_h} = \frac{1}{2} [D]_{\Sigma=\Sigma_k; P_0=P_h} \quad (4.9)$$

$$b_{hk} = \left[\frac{1}{\pi} \iint_{\Sigma_k} \frac{H}{|\vec{r}|} d\Sigma_k \right]_{P=P_h} = \frac{1}{2} [S]_{\Sigma=\Sigma_k; P_0=P_h} \quad (4.10)$$

with D and S given by Eqs. (3.38) and (2.117) respectively.

Next, consider a wing with supersonic leading edge. In this case, Eq. (4.7) should not be used since the system may have a determinant equal to zero. In order to show this, consider the case shown in Fig. 9. For simplicity assume that the elements are such that the Mach forecones,

* The contribution of the wake may be included in the same way used in Ref. 6 for subsonic flows.

C_1 , and C_2 , with vertices in P_1 and P_2 respectively, intersect only the elements Σ_1 , and Σ_2 . Assuming that the element Σ_1 is planar, then

$$c_{hh} = 0 \quad (h=1, 2) \quad (4.11)$$

since $\vec{a}_1 \times \vec{a}_2 \equiv 0$ on Σ_h . Furthermore note that according to Eq. (F.7) (or Eq. (F.13) with $E = 1/2$),

$$\sum_h c_{hh} = \frac{1}{\pi} \oint_{\Sigma} \frac{\vec{e}}{\partial N^c} \left(\frac{\#}{\|y\|} \right) d\Sigma = -\frac{1}{\pi} \oint_{\Sigma} d\Omega' = -1 \quad (4.12)$$

On the other hand,

$$\begin{aligned} c_{1h} &\neq 0 & h=2 \\ &= 0 & h=3, 4, \dots \end{aligned} \quad (4.13)$$

since all the elements except Σ_1 and Σ_2 are outside the Mach forecone C_1 . Equations (4.11, 4.12 and 4.13) are equivalent to

$$\begin{aligned} c_{1h} &= -1 & h=2 \\ &= 0 & h \neq 2 \end{aligned} \quad (4.14)$$

Similarly

$$\begin{aligned} c_{2h} &= -1 & h=1 \\ &= 0 & h \neq 1 \end{aligned} \quad (4.15)$$

Therefore the first two equations in Eq. (4.7) are

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} b_{1k} \\ 0_{1k} \end{bmatrix} \begin{Bmatrix} \psi_k \end{Bmatrix} \quad (4.16)$$

which shows clearly that the determinant of the system in Eq. (4.7) is equal to zero. Therefore in order to solve the problem a different procedure is used.

Consider the Green theorem for the function E as defined in Fig. 10. This yields a systems of equations similar to Eq. 4.7 where the summation however is limited to the upper surface of the wing. Similarly for points on the lower surface the summation is limited to the elements on the lower side of the wing.

For wings with subsonic leading edges the determinant is close to zero if the edges are near sonic. In this case as well as for wings with leading edges partially subsonic and partially supersonic, the solution may be obtained by using a diaphragm to separate upper and lower sides of the aircraft. For the elements on the diaphragm, both ϕ and $\partial\phi / \partial N^c$ are unknown, while two different equations are obtained by writing Eq. (4.7) for the upper and lower side, respectively. The solution of the problem is obtained from the system derived by writing Eq. (4.7) for the upper and lower sides of the body and the diaphragm.

The method described in this section was used to obtain the numerical results presented in Ref. 9.

SECTION 5

CONCLUDING REMARKS

A general method for solving steady supersonic flows around complex aircraft configurations has been presented. The extension to oscillatory flows is presented in Appendix E. Numerical results for steady flows around wing body configurations and for oscillatory flows around finite thickness wings are presented in Ref. 9, and indicate that the method, besides being intrinsically general and flexible is also accurate and fast.

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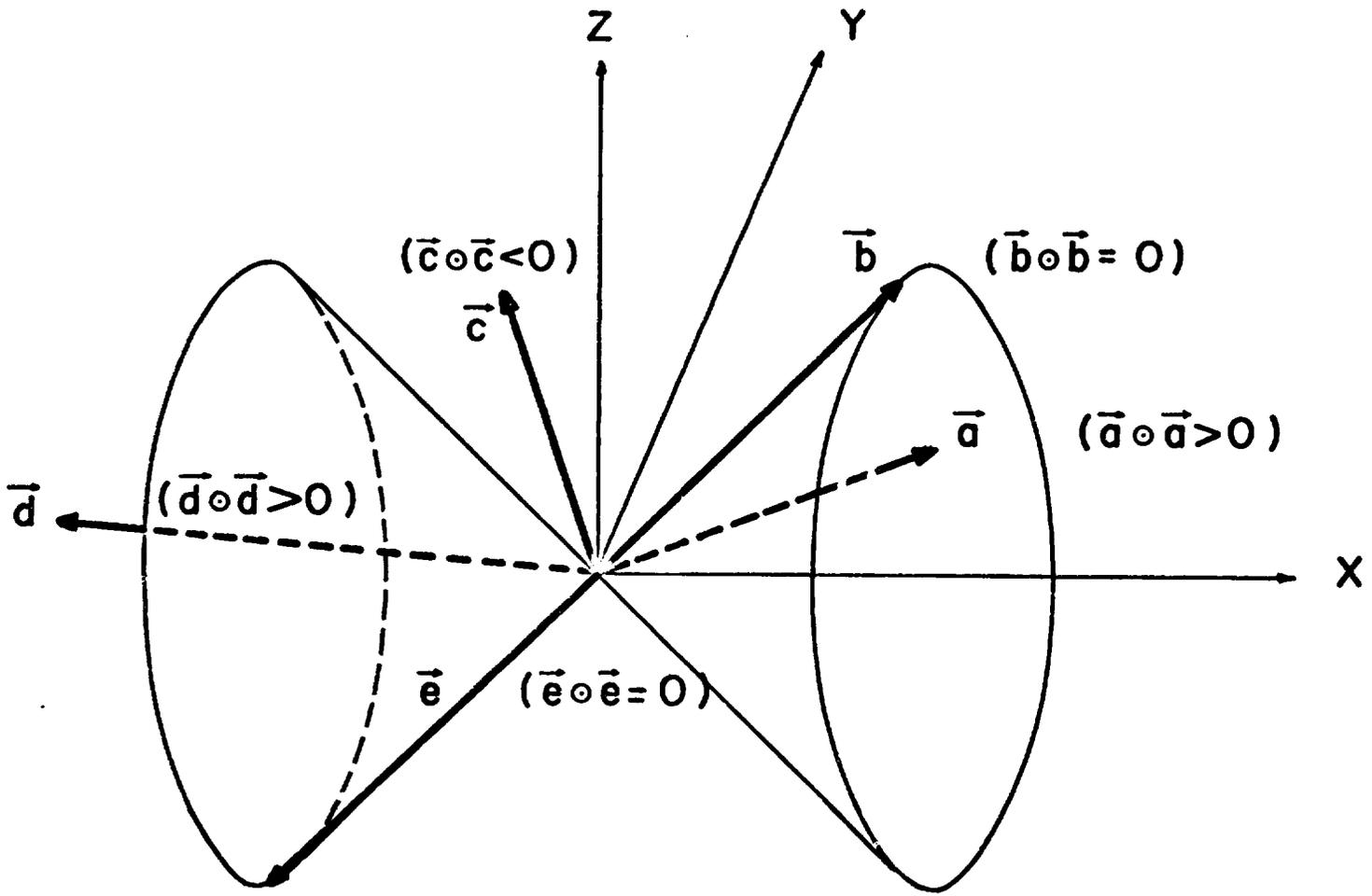


Fig. 1. Signs of $\vec{v} \circ \vec{v}$.

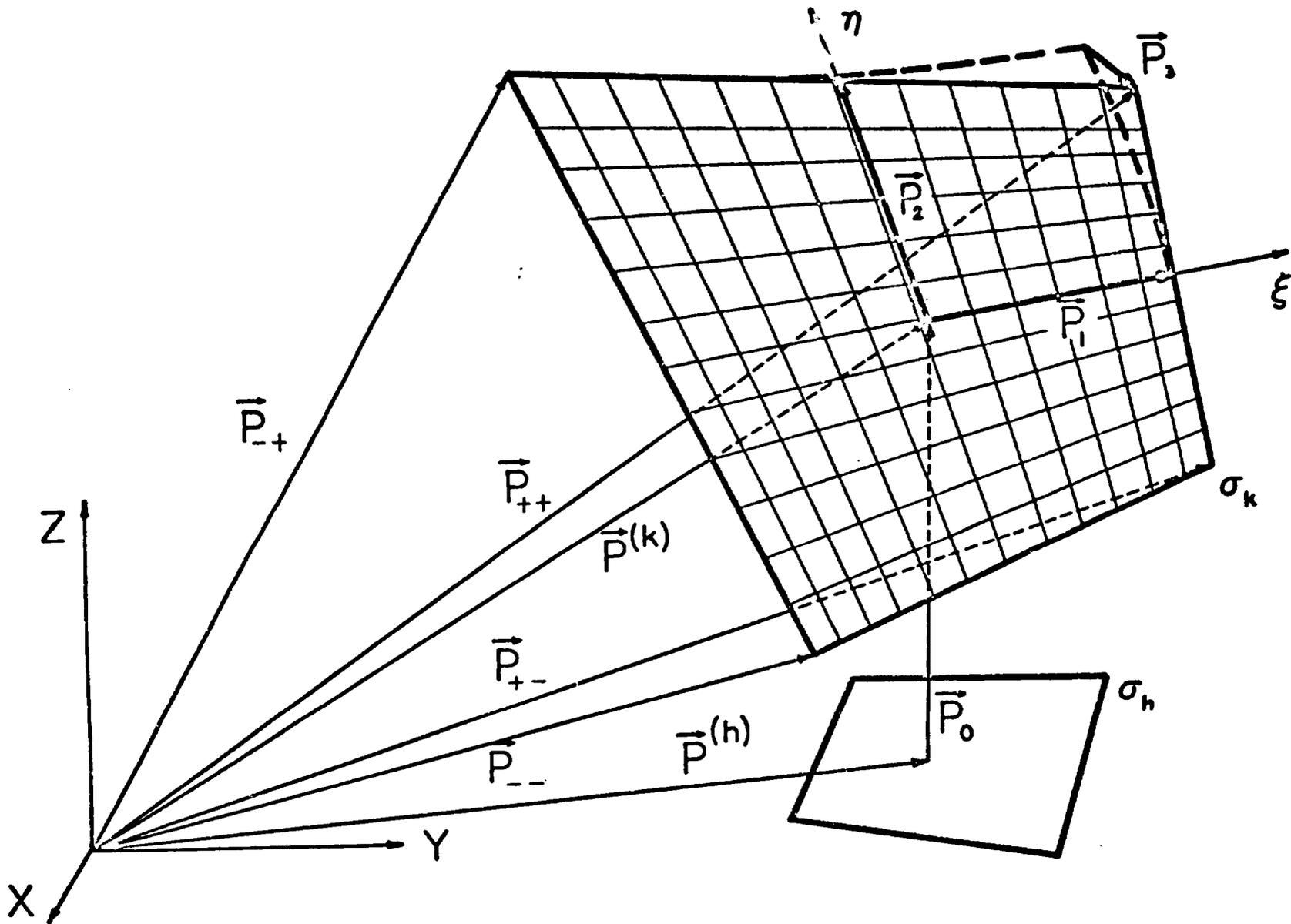
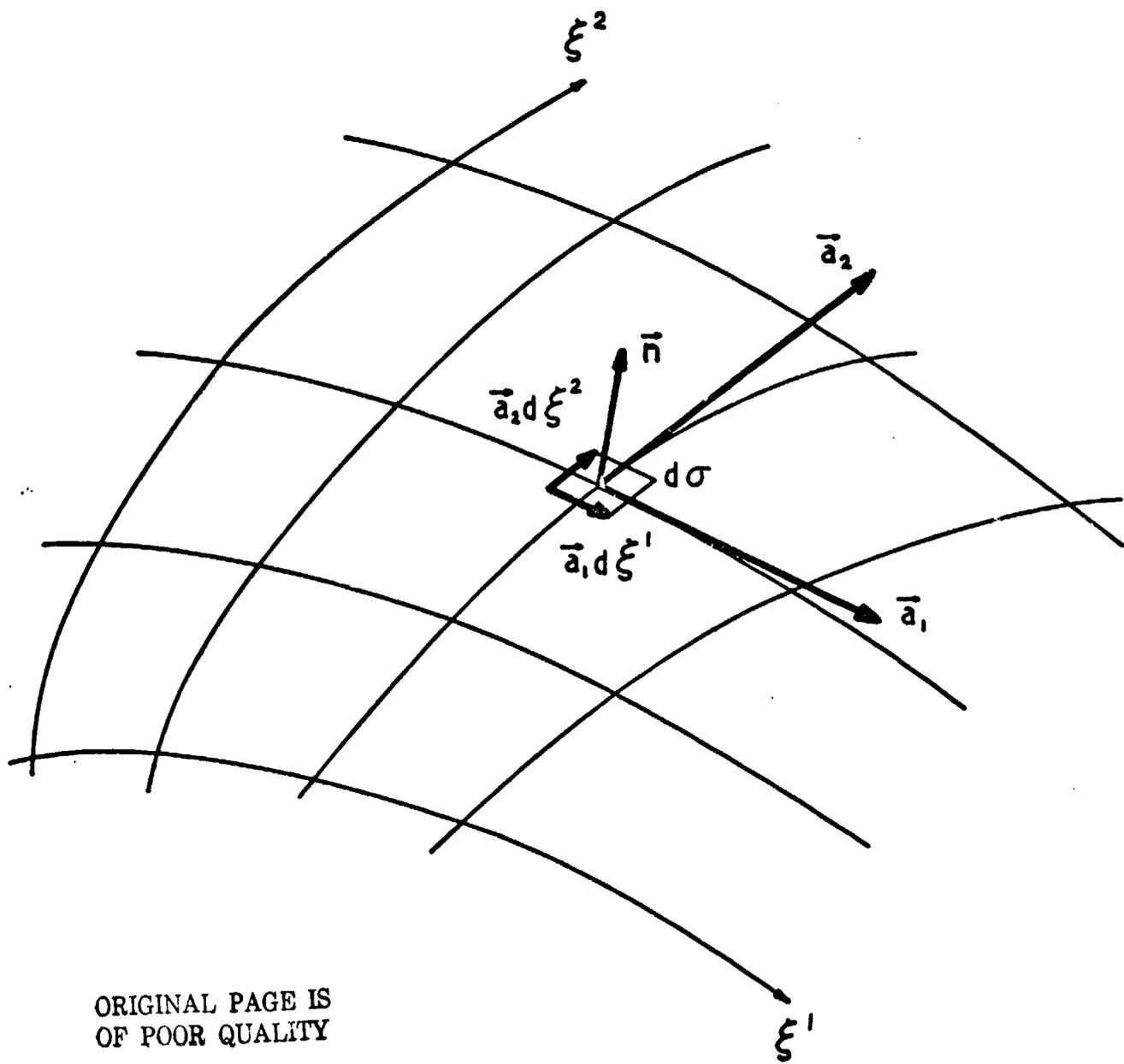


Fig. 2 Geometry of hyperboloidal element



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Fig. 3 Surface geometry

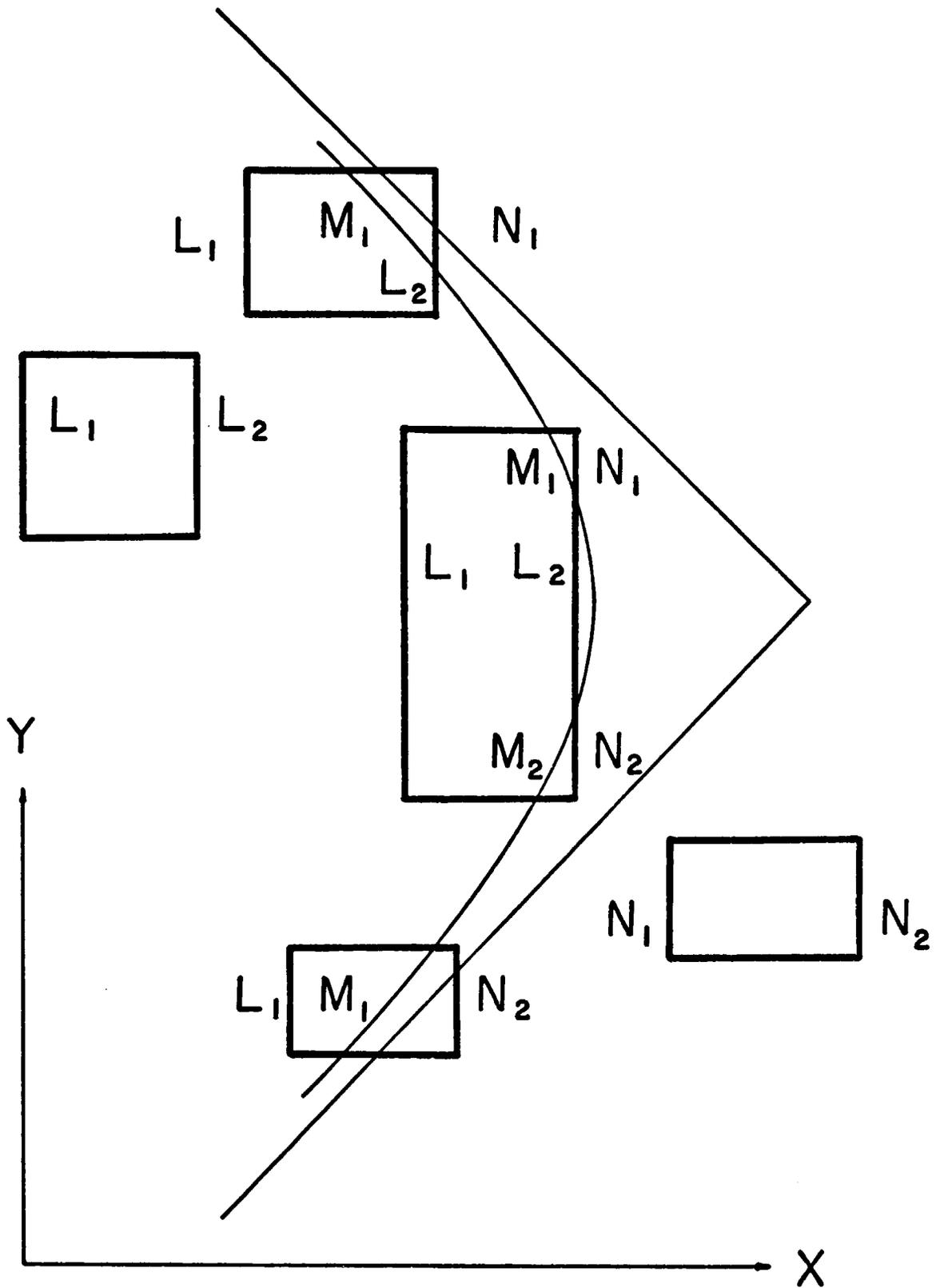


Fig. 4. Several types of supersonic elements.

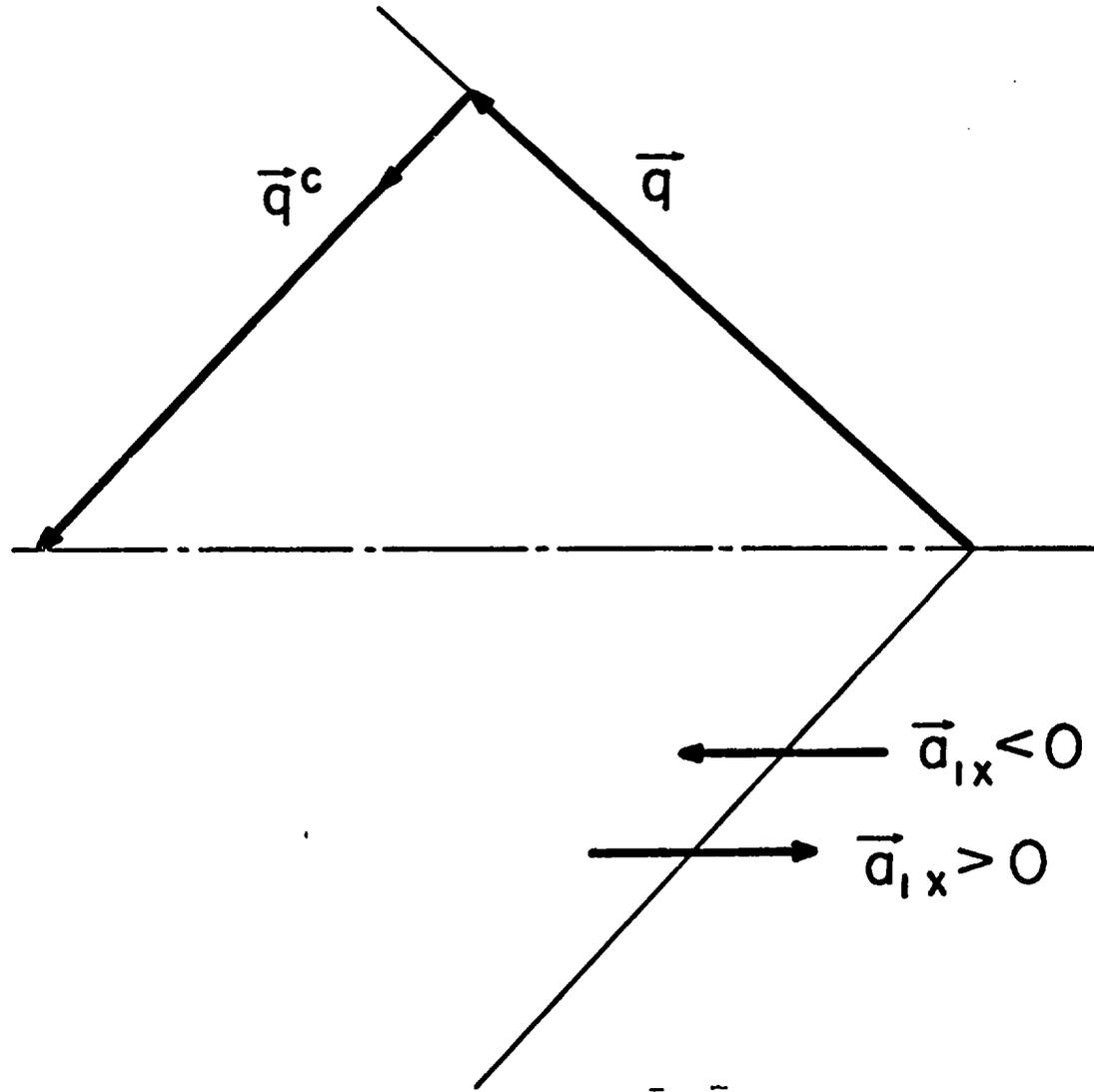


Fig. 5. Sign of $\vec{a}_1 \cdot \vec{a}$

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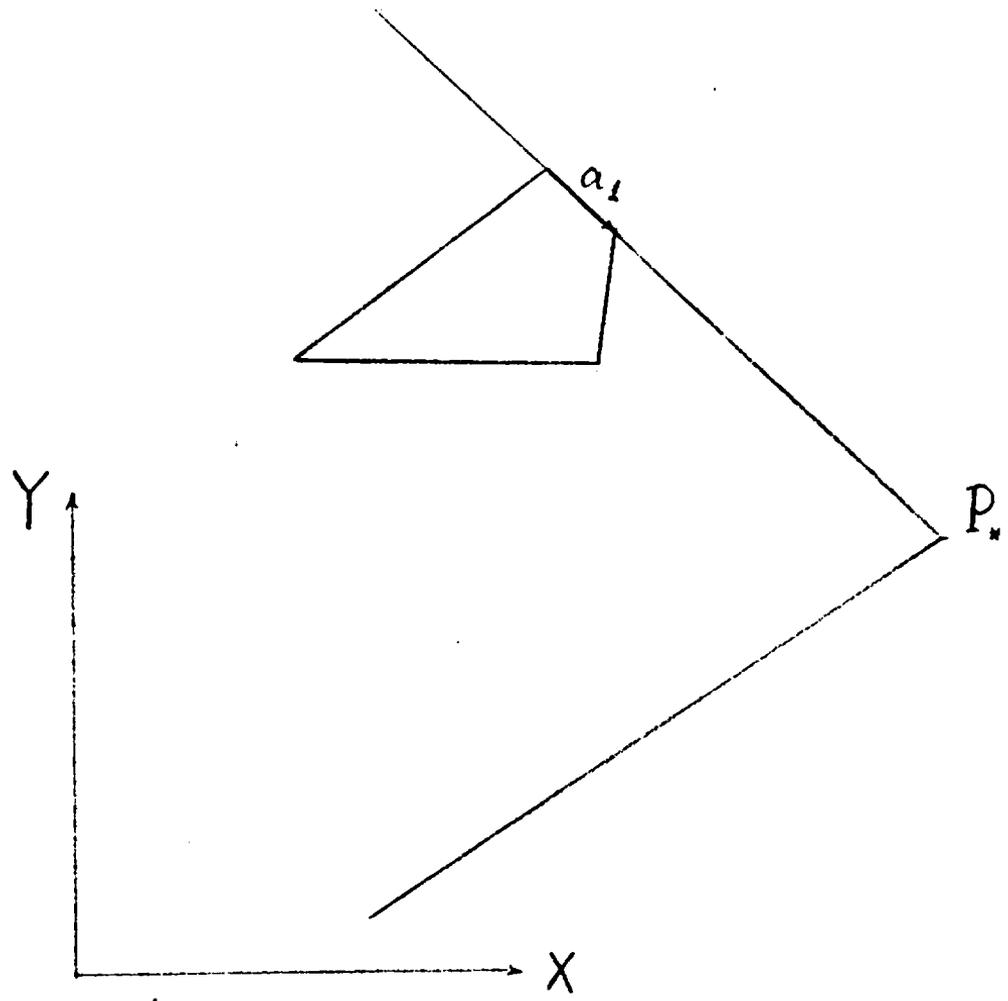


Fig. 6. Configuration with $q \circ a_1 = 0$ and $q \circ q = 0$.

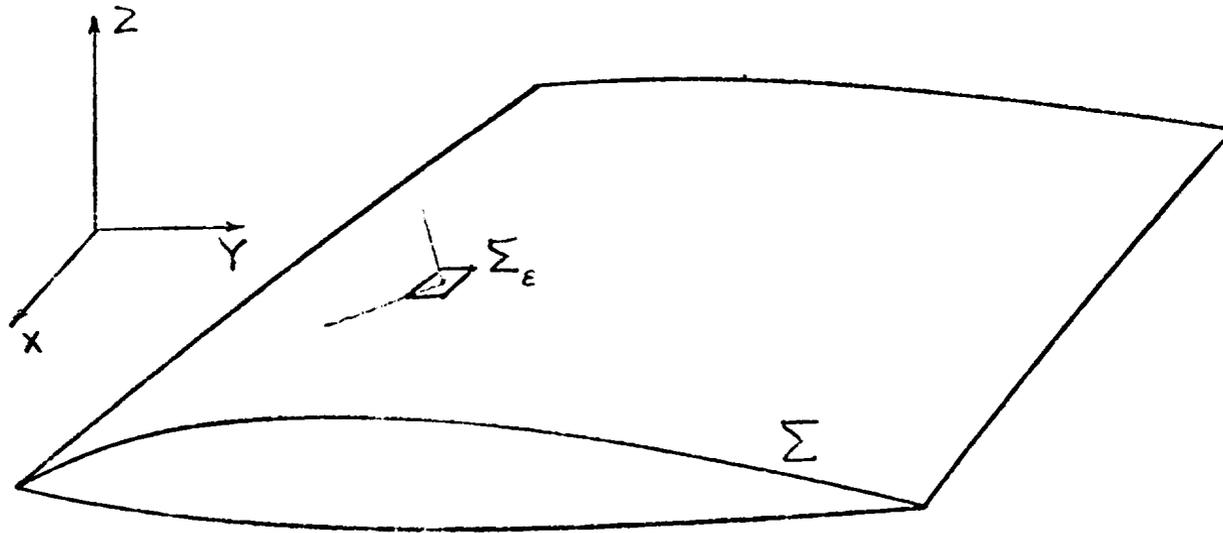


Fig. 7. Geometry for evaluating the value of E on Σ .

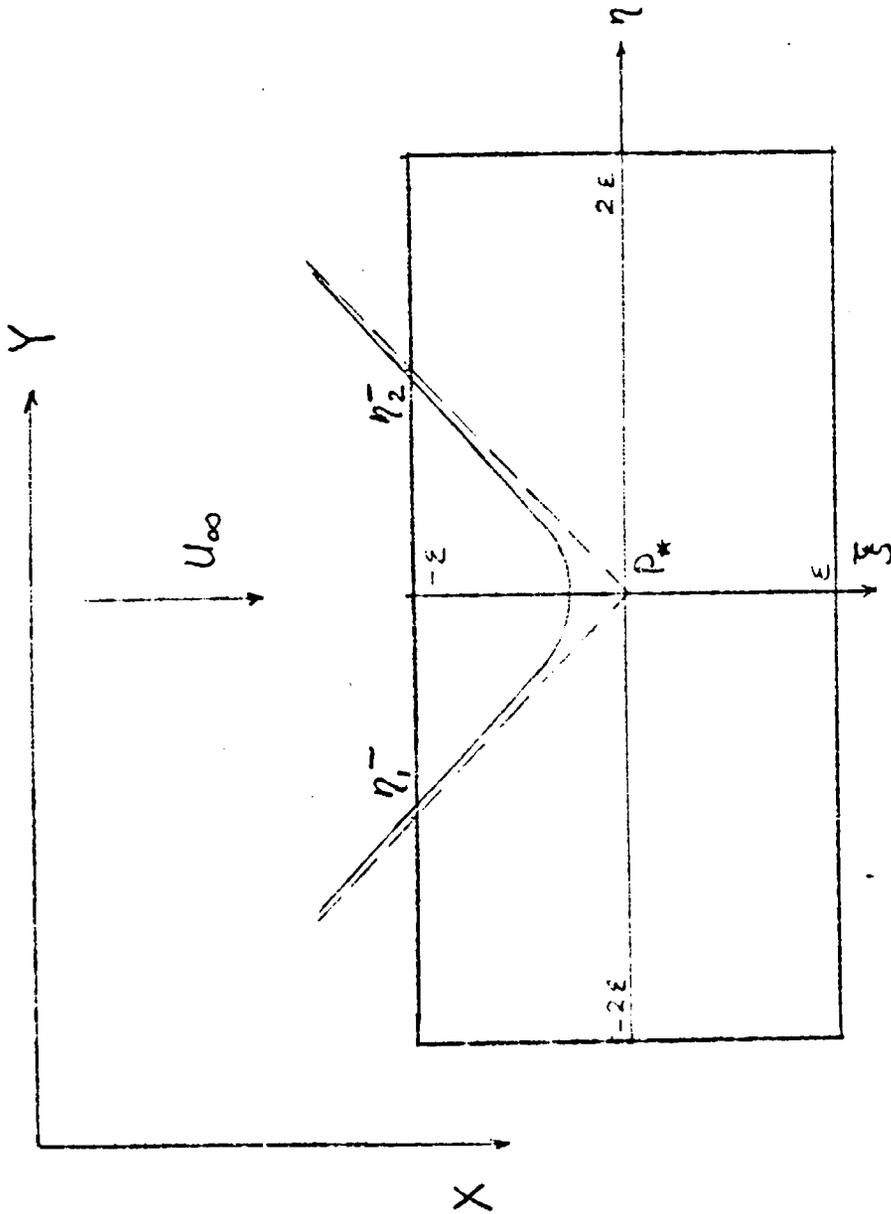


Fig. 8. The surface Σ_ϵ .

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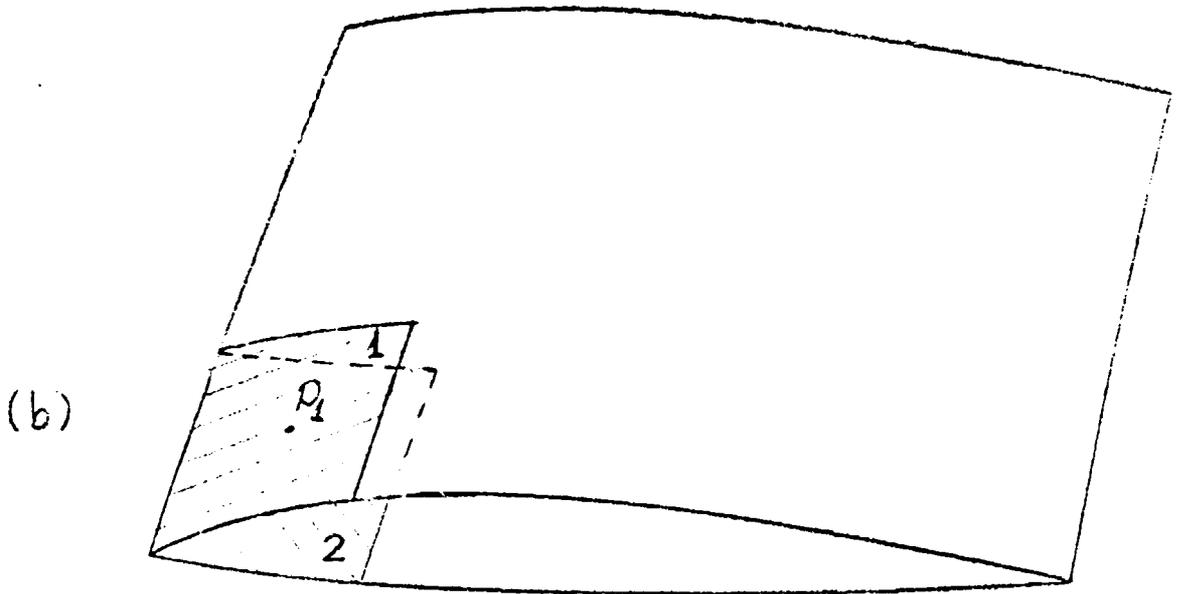
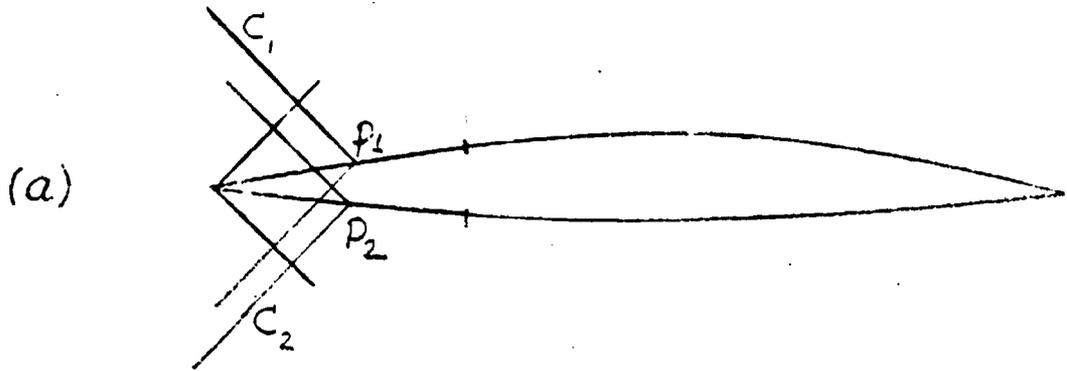


Fig. 9. Geometry for determinant equal to zero for supersonic leading edge wings.

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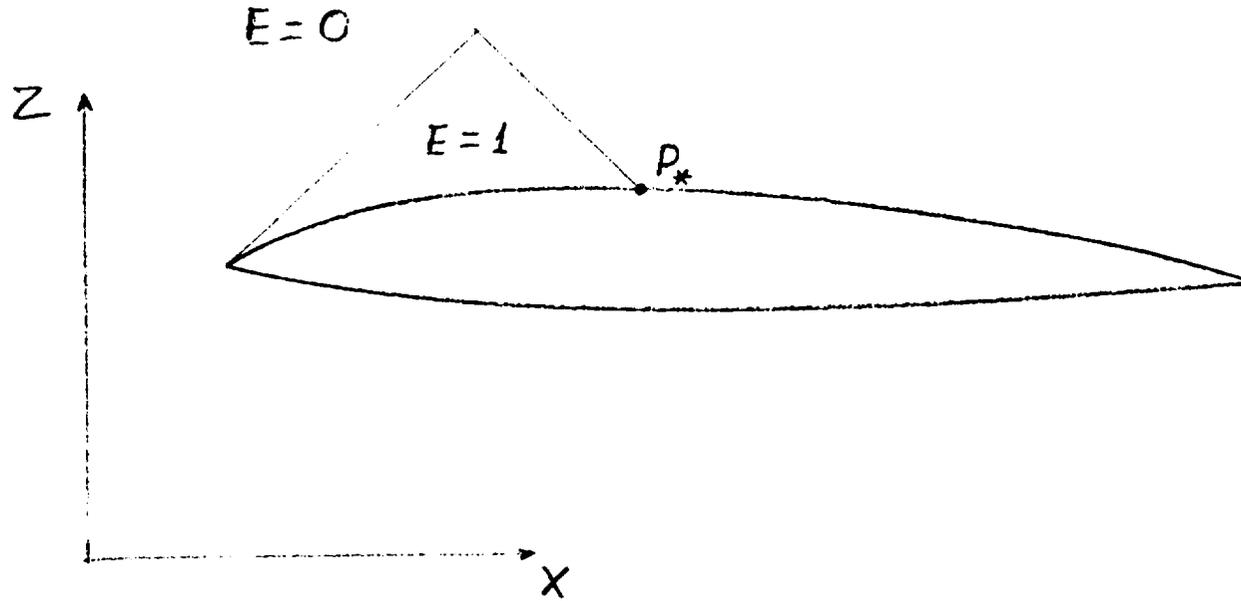


Fig. 10. Function E for supersonic leading edge configurations.

APPENDIX A

SUPERALGEBRA

A.1 Super-product

As mentioned in Section 1, in order to simplify the algebraic manipulation for the supersonic flow theory, it is convenient to introduce a special algebra, called supersonic vector algebra or super-algebra. In addition to the rules of the ordinary vector algebra, the super-algebra includes a supersonic dot product or super-product

$$\bar{a} \circ \bar{b} = a_x b_x - a_y b_y - a_z b_z \quad (\text{A.1})$$

The additive and distributive rules are obviously valid for the super-product. Note that $\bar{a} \circ \bar{a}$ is

$$\bar{a} \circ \bar{a} \begin{matrix} \geq \\ \leq \end{matrix} 0 \quad \text{for} \quad a_x \begin{matrix} \geq \\ \leq \end{matrix} \sqrt{a_y^2 + a_z^2} \quad (\text{A.2})$$

that is for \bar{a} pointed, respectively, inside, on, outside the Mach cone (Fig. 1). Hence, in addition to the ordinary norm of a vector (or dot-norm)

$$|a| = \sqrt{\bar{a} \cdot \bar{a}} \quad (\text{A.3})$$

it is convenient to introduce the supersonic norm (or super-norm)

$$\|a\| = \sqrt{|\bar{a} \circ \bar{a}|} \quad (\text{A.4})$$

Finally, it is convenient to introduce the concept of covector

$$\bar{a}^c = \begin{pmatrix} a_x \\ -a_y \\ -a_z \end{pmatrix} \quad (\text{A.5})$$

With these notations, it is immediately verified that

$$\bar{a} \circ \bar{b} = \bar{a}^c \cdot \bar{b} = \bar{a} \cdot \bar{b}^c \quad (\text{A.6})$$

It may be worth noting that

$$\bar{a} \cdot \bar{b} \times \bar{c} = \bar{a}^c \circ \bar{b} \times \bar{c} = \bar{a} \circ (\bar{b} \times \bar{c})^c = \bar{a} \circ \bar{b}^c \times \bar{c}^c \quad (\text{A.6a})$$

A.2 First Super-rule

Throughout the subsonic finite-element formulation (Ref. 6) the following rule is used

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c}) \quad (\text{A.7})$$

The corresponding supersonic rule, called for convenience, first super-rule, is also valid

$$(\bar{a} \times \bar{b}) \circ (\bar{c} \times \bar{d}) = (\bar{a} \circ \bar{c})(\bar{b} \circ \bar{d}) - (\bar{a} \circ \bar{d})(\bar{b} \circ \bar{c}) \quad (\text{A.8})$$

For

$$\begin{aligned} & (\bar{a} \times \bar{b}) \circ (\bar{c} \times \bar{d}) \\ &= (a_y b_z - a_z b_y)(c_y d_z - c_z d_y) \\ & \quad - (a_z b_x - a_x b_z)(c_z d_x - c_x d_z) \\ & \quad - (a_x b_y - a_y b_x)(c_x d_y - c_y d_x) \\ &= a_y b_z c_y d_z + a_z b_y c_z d_y - a_y b_z c_z d_y - a_z b_y c_y d_z \\ & \quad - a_z b_x c_z d_x - a_x b_z c_x d_z + a_z b_x c_x d_z + a_x b_z c_z d_x \\ & \quad - a_x b_y c_x d_y - a_y b_x c_y d_x + a_x b_y c_y d_x + a_y b_x c_x d_y \quad (\text{A.9}) \end{aligned}$$

while

$$\begin{aligned}
& (\bar{a} \circ \bar{c})(\bar{b} \circ \bar{d}) - (\bar{a} \circ \bar{d})(\bar{b} \circ \bar{c}) \\
&= (a_x c_x - a_y c_y - a_z c_z)(b_x d_x - b_y d_y - b_z d_z) \\
&\quad - (a_x d_x - a_y d_y - a_z d_z)(b_x c_x - b_y c_y - b_z c_z) \\
&= a_x c_x (\cancel{b_x d_x} - b_y d_y - b_z d_z) \\
&\quad - a_y c_y (b_x d_x - \cancel{b_y d_y} - b_z d_z) \\
&\quad - a_z c_z (b_x d_x - b_y d_y - \cancel{b_z d_z}) \\
&\quad - a_x d_x (\cancel{b_x c_x} - b_y c_y - b_z c_z) \\
&\quad + a_y d_y (b_x c_x - \cancel{b_y c_y} - b_z c_z) \\
&\quad + a_z d_z (b_x c_x - b_y c_y - \cancel{b_z c_z}) \\
&= a_y b_z c_y d_z + a_z b_y c_z d_y - a_y b_z c_z d_y - a_z b_y c_y d_z \\
&\quad - a_z b_x c_z d_x - a_x b_z c_z d_z + a_z b_x c_x d_z + a_x b_z c_z d_x \\
&\quad - a_x b_y c_y d_y - a_y b_x c_y d_x + a_x b_y c_y d_x + a_y b_x c_x d_y
\end{aligned} \tag{A.10}$$

A.3 Second Super-rule

A second rule of the super-algebra is

$$\begin{aligned}
 & (\bar{a} \circ \bar{a})(\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - (\bar{a} \cdot \bar{b} \times \bar{c})^{\wedge} \\
 = & \bar{a} \circ \bar{c} (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{a}) + \\
 & \bar{a} \circ \bar{b} (\bar{c} \times \bar{a}) \circ (\bar{c} \times \bar{a})
 \end{aligned}
 \tag{A.11}$$

Note that the dot product appears in the triple product. In order to prove Eq. (A.11), consider the regular vector algebra rule

$$\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b} (\bar{a} \cdot \bar{c}) - \bar{c} (\bar{a} \cdot \bar{b})
 \tag{A.12}$$

This yields, for the covector \bar{a}^c , (see Eq. A.6)

$$\begin{aligned}
 \bar{a}^c \times (\bar{b} \times \bar{c}) &= \bar{b} (\bar{a}^c \cdot \bar{c}) - \bar{c} (\bar{a}^c \cdot \bar{b}) \\
 &= \bar{b} (\bar{a} \circ \bar{c}) - \bar{c} (\bar{a} \circ \bar{b})
 \end{aligned}
 \tag{A.13}$$

On the other hand, according to Eqs. (A.6) and (A.8)

$$\begin{aligned}
 & \bar{a}^c \times (\bar{b} \times \bar{c}) \circ \bar{a}^c \times (\bar{b} \times \bar{c}) \\
 = & \bar{a}^c \circ \bar{a}^c (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - [\bar{a}^c \circ (\bar{b} \times \bar{c})]^{\wedge} \\
 = & \bar{a} \circ \bar{a} (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - (\bar{a} \cdot \bar{b} \times \bar{c})^{\wedge}
 \end{aligned}$$

while, according to Eqs. (A.6) and (A.8)

$$\begin{aligned}
 & [\bar{b} (\bar{a} \circ \bar{c}) - \bar{c} (\bar{a} \circ \bar{b})] \circ [\bar{b} (\bar{a} \circ \bar{c}) - \bar{c} (\bar{a} \circ \bar{b})] \\
 = & \bar{b} \circ \bar{b} (\bar{a} \circ \bar{c})^{\wedge} - 2 \bar{b} \circ \bar{c} \bar{a} \circ \bar{c} \bar{a} \circ \bar{b} + \bar{c} \circ \bar{c} (\bar{a} \circ \bar{b})^{\wedge} \\
 = & \bar{a} \circ \bar{c} (\bar{b} \times \bar{c} \circ \bar{b} \times \bar{a}) + \bar{a} \circ \bar{b} (\bar{c} \times \bar{b} \circ \bar{c} \times \bar{a})
 \end{aligned}
 \tag{A.14}$$

Combining Eqs. (A.12), (A.13) and (A.14) yields

$$\begin{aligned}
 & \bar{a} \circ \bar{a} (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{c}) - (\bar{a} \cdot \bar{b} \times \bar{c})^2 \\
 &= (\bar{a}^c \times (\bar{b} \times \bar{c})) \circ (\bar{a}^c \times (\bar{b} \times \bar{c})) \\
 &= (\bar{b} (\bar{a} \circ \bar{c}) - \bar{c} (\bar{a} \circ \bar{b})) \circ (\bar{b} (\bar{a} \circ \bar{c}) - \bar{c} (\bar{a} \circ \bar{b})) \\
 &= \bar{a} \circ \bar{c} (\bar{b} \times \bar{c}) \circ (\bar{b} \times \bar{a}) + \bar{a} \circ \bar{b} (\bar{c} \times \bar{b}) \circ (\bar{c} \times \bar{a})
 \end{aligned} \tag{A.15}$$

that is the second super-rule, Eq. (A.11). In particular, for $\bar{a} \equiv \bar{q}$, $\bar{b} \equiv \bar{a}_1$, $\bar{c} \equiv \bar{a}_2$, one obtains

$$\begin{aligned}
 & \bar{q} \circ \bar{q} \bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2 - |\bar{q} \cdot \bar{a}_1 \times \bar{a}_2|^2 \\
 &= \bar{q} \circ \bar{a}_2 \bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{q} + \bar{q} \circ \bar{a}_1 (\bar{a}_2 \times \bar{q}_1 \circ \bar{a}_2 \times \bar{q})
 \end{aligned} \tag{A.16}$$

A.4 Third Supersonic Rule

A third useful formula, called the third supersonic rule, is

$$\begin{aligned}
 & \bar{a} \times \bar{d} \circ \bar{b} \times \bar{c} \bar{a} \cdot \bar{f} \times \bar{g} + \bar{a} \times \bar{f} \circ \bar{b} \times \bar{c} \bar{a} \cdot \bar{g} \times \bar{d} \\
 & + \bar{a} \times \bar{g} \circ \bar{b} \times \bar{c} \bar{a} \times \bar{d} \cdot \bar{f} = 0
 \end{aligned} \tag{A.17}$$

The proof of this rule follows

$$\begin{aligned}
 & (\bar{a} \times \bar{d} \circ \bar{b} \times \bar{c})(\bar{a} \cdot \bar{f} \times \bar{g}) - (\bar{a} \times \bar{g} \circ \bar{b} \times \bar{c})(\bar{a} \cdot \bar{f} \times \bar{d}) \\
 &= (\bar{a} \circ \bar{b} \bar{c} \circ \bar{d} - \bar{a} \circ \bar{c} \bar{b} \circ \bar{d}) \bar{a} \times \bar{f} \cdot \bar{g} \\
 & - (\bar{a} \circ \bar{b} \bar{c} \circ \bar{g} - \bar{a} \circ \bar{c} \bar{b} \circ \bar{g}) \bar{a} \times \bar{f} \cdot \bar{d} \\
 &= \bar{a} \circ \bar{b} (\bar{c} \circ \bar{d} \bar{a} \times \bar{f} \cdot \bar{g} - \bar{c} \circ \bar{g} \bar{a} \times \bar{f} \cdot \bar{d}) \\
 & - \bar{a} \circ \bar{c} (\bar{b} \circ \bar{d} \bar{a} \times \bar{f} \cdot \bar{g} - \bar{b} \circ \bar{g} \bar{a} \times \bar{f} \cdot \bar{d})
 \end{aligned}$$

$$\begin{aligned}
 &= \bar{a} \cdot \bar{b}^c (\bar{c}^c \cdot \bar{d} \bar{a} \times \bar{f} \cdot \bar{g} - \bar{c}^c \cdot \bar{g} \bar{a} \times \bar{f} \cdot \bar{d}) \\
 &\quad - \bar{a} \cdot \bar{c}^c (\bar{b}^c \cdot \bar{d} \bar{a} \times \bar{f} \cdot \bar{g} - \bar{b}^c \cdot \bar{g} \bar{a} \times \bar{f} \cdot \bar{d}) \\
 &= \bar{a} \cdot \bar{b}^c [\bar{c}^c \times (\bar{a} \times \bar{f})] \cdot \bar{d} \times \bar{g} - \bar{a} \cdot \bar{c}^c [\bar{b}^c \times (\bar{a} \times \bar{f})] \cdot \bar{d} \times \bar{g} \\
 &= \bar{a} \cdot \bar{b}^c \{ (\bar{c}^c \cdot \bar{f}) (\bar{a} \cdot \bar{d} \times \bar{g}) - (\bar{c}^c \cdot \bar{a}) (\bar{f} \cdot \bar{d} \times \bar{g}) \} \\
 &\quad - \bar{a} \cdot \bar{c}^c \{ (\bar{b}^c \cdot \bar{f}) (\bar{a} \cdot \bar{d} \times \bar{g}) - (\bar{b}^c \cdot \bar{a}) (\bar{f} \cdot \bar{d} \times \bar{g}) \} \\
 &= (\bar{a} \cdot \bar{b}^c \bar{c}^c \cdot \bar{f} - \bar{a} \cdot \bar{c}^c \bar{b}^c \cdot \bar{a}) \bar{a} \cdot \bar{d} \times \bar{g} \\
 &= (\bar{a} \circ \bar{b} \bar{c} \circ \bar{f} - \bar{a} \circ \bar{c} \bar{b} \circ \bar{f}) = (\bar{a} \times \bar{f} \circ \bar{b} \times \bar{c}) \bar{a} \cdot \bar{d} \times \bar{g}
 \end{aligned} \tag{A.18}$$

In particular, for $\bar{a} = \bar{b} = \bar{q}$, $\bar{c} = \bar{f} = \bar{a}_2$, $\bar{d} = \bar{p}_3$, $\bar{g} = \bar{a}_1$,
the third superrule reduces to

$$\begin{aligned}
 &(\bar{q} \times \bar{p}_3 \circ \bar{q} \times \bar{a}_2) (\bar{q} \cdot \bar{a}_2 \times \bar{a}_2) - (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_2) (\bar{q} \cdot \bar{a}_2 \times \bar{p}_3) \\
 &= (\bar{q} \times \bar{a}_2 \circ \bar{q} \times \bar{a}_2) (\bar{q} \cdot \bar{p}_3 \times \bar{a}_1)
 \end{aligned} \tag{A.19}$$

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APPENDIX B

BASIC INTEGRALS

B.1 Integral I_1

As mentioned in Section 2, in order to obtain Eq. (2.24) from Eq. (2.22) is necessary to evaluate the two integrals

$$I_1 = \int \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} d\eta \quad (\text{B.1})$$

$$I_2 = \int \frac{\bar{f} \circ \bar{u}}{\|\bar{f} \times \bar{u}\|^2 \sqrt{\bar{f} \circ \bar{f}}} d\eta \quad (\text{B.2})$$

The integral I_2 is evaluated in the next subsection.

In order to obtain the integral in Eq. (B.1), it is convenient to treat independently the three cases $\bar{a}_2 \circ \bar{a}_2 \gtrless 0$. Consider first, $\bar{a}_2 \circ \bar{a}_2 > 0$. Note that

$$\begin{aligned} & \frac{\partial}{\partial \eta} \ln \left(\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{f} \circ \bar{a}_2 \right) \\ &= \frac{1}{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{f} \circ \bar{a}_2} \left(\frac{\bar{f} \circ \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{a}_2 \circ \bar{a}_2 \right) \\ &= \frac{\sqrt{\bar{a}_2 \circ \bar{a}_2}}{\sqrt{\bar{f} \circ \bar{f}}} \end{aligned} \quad (\text{B.3})$$

Note also that

$$\begin{aligned} \frac{\partial}{\partial \eta} (\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2) &= \frac{\partial}{\partial \eta} [(\bar{p}_0 + \bar{z}\bar{p}_1) \times (\bar{p}_0 + \bar{z}\bar{p}_2)] \circ [(\bar{p}_0 + \bar{z}\bar{p}_1) \times (\bar{p}_0 + \bar{z}\bar{p}_2)] \\ &= 0 \end{aligned} \quad (\text{B.4})$$

Hence

$$\frac{\partial}{\partial \eta} \left(\ln \frac{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right) = \frac{\sqrt{\bar{a}_2 \circ \bar{a}_2}}{\sqrt{\bar{f} \circ \bar{f}}} \quad (\text{B.5})$$

Note that

$$\begin{aligned}
 & \ln \frac{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \ln \left| \frac{(\bar{f} \circ \bar{f})(\bar{a}_2 \circ \bar{a}_2) - (\bar{f} \circ \bar{a}_2)^2}{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} - \bar{f} \circ \bar{a}_2} \right| \frac{1}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \ln \left| \frac{\|\bar{f} \times \bar{a}_2\|}{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} - \bar{f} \circ \bar{a}_2} \right| \\
 &= - \ln \left| \frac{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} - \bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right|
 \end{aligned}$$

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(B.6)

Hence, it is possible to write

$$I_1 = \text{sign}(\bar{f} \circ \bar{a}_2) \ln \left| \frac{\sqrt{\bar{f} \circ \bar{f}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + |\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|} \right| \quad (\bar{a}_2 \circ \bar{a}_2 > 0) \quad (\text{B.7})$$

Note that

$$\text{Sinh}^{-1} x = \ln (x + \sqrt{x^2 + 1})$$

$$\text{Cosh}^{-1} x = \ln (x + \sqrt{x^2 - 1}) \quad , \quad x \geq 1 \quad (\text{B.8})$$

Hence

$$\begin{aligned}
 & \ln \frac{\sqrt{\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2} + |\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \ln \frac{\sqrt{(\bar{f} \circ \bar{a}_2)^2 + \bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2} + |\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \ln \sqrt{\left(\frac{|\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|}\right)^2 - 1} + \frac{|\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \cosh^{-1} \frac{|\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|}
 \end{aligned}$$

and

$$\begin{aligned}
 & \ln \frac{\sqrt{\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2} + |\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \ln \frac{\sqrt{\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2} + \sqrt{\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2 - \bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2}}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \ln \left(\frac{\sqrt{\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2}}{\|\bar{f} \times \bar{a}_2\|} + \sqrt{\left(\frac{\sqrt{\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2}}{\|\bar{f} \times \bar{a}_2\|}\right)^2 + 1} \right) \\
 &= \sinh^{-1} \left(\frac{\sqrt{\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2}}{\|\bar{f} \times \bar{a}_2\|} \right)
 \end{aligned}$$

Next, consider the case $\bar{a}_2 \circ \bar{a}_2 = 0$. In this case

$$\begin{aligned} \frac{\partial}{\partial \eta} (\bar{f} \circ \bar{a}_2) &= \frac{\partial}{\partial \eta} \left((\bar{p}_0 + \varepsilon \bar{p}_1 + \eta \bar{a}_2) \circ \bar{a}_2 \right) \\ &= \frac{\partial}{\partial \eta} \left((\bar{p}_0 + \varepsilon \bar{p}_1) \circ (\bar{p}_2 + \varepsilon \bar{p}_3) \right) = 0 \end{aligned} \tag{B.9}$$

Hence

$$\frac{\partial}{\partial \eta} \frac{\sqrt{\bar{f} \circ \bar{f}}}{\bar{f} \circ \bar{a}_2} = \frac{1}{\bar{f} \circ \bar{a}_2} \frac{\bar{f} \circ \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}}} = \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} \tag{B.10}$$

Thus

$$I_1 = \frac{\sqrt{\bar{f} \circ \bar{f}}}{\bar{f} \circ \bar{a}_2} \tag{B.11}$$

Finally, consider the case $\bar{a}_2 \circ \bar{a}_2 < 0$. Note that in this case

$$\begin{aligned} \frac{\partial}{\partial \eta} \sin^{-1} \left(\frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right) &= \frac{1}{\sqrt{1 - \left(\frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \right)^2}} \frac{\bar{a}_2 \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \\ &= \frac{\|\bar{f} \times \bar{a}_2\|}{\sqrt{-\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2 - (\bar{f} \circ \bar{a}_2)^2}} \frac{1}{\|\bar{f} \times \bar{a}_2\|} \bar{a}_2 \circ \bar{a}_2 \\ &= \frac{\bar{a}_2 \circ \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f} (-\bar{a}_2 \circ \bar{a}_2)}} = \frac{-\sqrt{-\bar{a}_2 \circ \bar{a}_2}}{\sqrt{\bar{f} \circ \bar{f}}} \end{aligned} \tag{B.12}$$

Hence

$$I = - \frac{1}{\sqrt{-\bar{a}_2 \circ \bar{a}_2}} \sin^{-1} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} , \quad \bar{a}_2 \circ \bar{a}_2 < 0 \tag{B.13}$$

In summary

$$\begin{aligned}
 I_1 &= \frac{\text{Sign}(\bar{f} \circ \bar{a}_2)}{\|\bar{a}_2\|} \ln \frac{\|\bar{f}\| \|\bar{a}_2\| + |\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \frac{\text{Sign}(\bar{f} \circ \bar{a}_2)}{\|\bar{a}_2\|} \text{Cosh}^{-1} \frac{|\bar{f} \circ \bar{a}_2|}{\|\bar{f} \times \bar{a}_2\|} \\
 &= \frac{\text{Sign}(\bar{f} \circ \bar{a}_2)}{\|\bar{a}_2\|} \text{Sinh}^{-1} \frac{\|\bar{f}\| \|\bar{a}_2\|}{\|\bar{f} \times \bar{a}_2\|} \quad , \quad \bar{a}_2 \circ \bar{a}_2 > 0
 \end{aligned}$$

(B.14)

$$I_1 = \frac{\|\bar{f}\|}{\bar{f} \circ \bar{a}_2} \quad , \quad \bar{a}_2 \circ \bar{a}_2 = 0$$

(B.15)

$$I_1 = - \frac{1}{\|\bar{a}_2\|} \text{Sin}^{-1} \frac{\bar{f} \circ \bar{a}_2}{\|\bar{f} \times \bar{a}_2\|} \quad , \quad \bar{a}_2 \circ \bar{a}_2 < 0$$

(B.16)

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B.2 Integral I₂

Next consider Eq. (B.2). Setting

$$\bar{f} = \bar{a}_0 + \eta \bar{a}_1$$

(B.17)

one obtains for $\bar{u} \circ \bar{u} = 1$

$$\bar{f} \circ \bar{u} = a + b\eta$$

(B.18)

$$\bar{f} \circ \bar{f} = \alpha + 2\beta\eta + \gamma\eta^2$$

(B.19)

$$\begin{aligned}
 \bar{f} \cdot \bar{u} \circ \bar{f} \cdot \bar{u} &= \bar{f} \circ \bar{f} \bar{u} \circ \bar{u} - (\bar{f} \circ \bar{u})^2 \\
 &= d^2 - a^2 + 2(\beta - ab)\eta + (\gamma - b^2)\eta^2 \\
 &= c + 2d\eta + e\eta^2
 \end{aligned}
 \tag{B.20}$$

with

$$a = \bar{a}_1 \circ \bar{u}$$

$$b = \bar{a}_2 \circ \bar{u}$$

$$d = \bar{a}_1 \circ \bar{a}_1$$

$$\beta = \bar{a}_1 \circ \bar{a}_2$$

$$\gamma = \bar{a}_2 \circ \bar{a}_2$$

$$c = d - a^2 = \bar{a}_1 \circ \bar{a}_1 - (\bar{a}_1 \circ \bar{u})^2 = \bar{a}_1 \cdot \bar{u} \circ \bar{a}_1 \cdot \bar{u}$$

$$d = \beta - ab = \bar{a}_1 \circ \bar{a}_2 - \bar{a}_1 \circ \bar{u} \bar{a}_2 \circ \bar{u} = \bar{a}_1 \cdot \bar{u} \circ \bar{a}_2 \cdot \bar{u}$$

$$e = \gamma - b^2 = \bar{a}_2 \circ \bar{a}_2 - (\bar{a}_2 \circ \bar{u})^2 = \bar{a}_2 \cdot \bar{u} \circ \bar{a}_2 \cdot \bar{u}$$

Combining Eqs. (B.2), (B.18), (B.19) and (B.20) (see Eq. A.1 of Ref. 6)

$$\begin{aligned}
 I_2 &= \int \frac{a+b\eta}{c+2d\eta+e\eta^2} \frac{1}{\sqrt{d+2\beta\eta+\gamma\eta^2}} d\eta \\
 &= \frac{1}{\sqrt{ec-d^2}} \tan^{-1} \left(\frac{(ae-bd)\eta + (ad-bc)}{\sqrt{ec-d^2}} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} \right)
 \end{aligned}
 \tag{B.22}$$

Note that

$$\begin{aligned}
 &(ae-bd)\eta + (ad-bc) \\
 &= a(e\eta + d) - b(d\eta + c)
 \end{aligned}$$

$$\begin{aligned}
 &= a [(\gamma - b^2)\eta + (\beta - ab)] - b [(\beta - ab)\eta + (\lambda - a^2)] \\
 &= a(\gamma\eta + \beta) - b(\beta\eta + \lambda) \\
 &= \bar{a}_0 \circ \bar{u} (\bar{a}_1\eta + \bar{a}_0) \circ \bar{a}_2 - \bar{a}_2 \circ \bar{u} (\bar{a}_1\eta + \bar{a}_0) \circ \bar{u} \\
 &= \bar{a}_0 \circ \bar{u} \bar{f} \circ \bar{a}_2 - \bar{a}_2 \circ \bar{u} \bar{f} \circ \bar{a}_0 \\
 &= -\bar{f} \times \bar{u} \circ \bar{a}_1 \times \bar{a}_2 = -\bar{f} \times \bar{u} \circ \bar{f} \times \bar{a}_2
 \end{aligned} \tag{B.23}$$

Furthermore,

$$\begin{aligned}
 ec - d^2 &= \bar{f} \circ \bar{f} (\bar{u} \times \bar{a}_2) \circ (\bar{u} \times \bar{a}_2) + \bar{f} \circ \bar{a}_2 (\bar{f} \times \bar{u}) \circ (\bar{u} \times \bar{a}_2) \\
 &\quad + \bar{f} \times \bar{u} (\bar{f} \times \bar{a}_2 \circ \bar{a}_2 \times \bar{u})
 \end{aligned} \tag{B.24}$$

For

$$\begin{aligned}
 ec - d^2 &= (\bar{a}_2 \times \bar{u} \circ \bar{a}_2 \times \bar{u}) (\bar{a}_0 \times \bar{u} \circ \bar{a}_0 \times \bar{u}) - (\bar{a}_1 \times \bar{u} \circ \bar{a}_1 \times \bar{u})^2 \\
 &= [(\bar{a}_2 \circ \bar{a}_2)(\bar{u} \circ \bar{u}) - (\bar{a}_2 \circ \bar{u})^2] [(\bar{a}_0 \circ \bar{a}_0)(\bar{u} \circ \bar{u}) - (\bar{a}_0 \circ \bar{u})^2] \\
 &\quad - [(\bar{a}_0 \circ \bar{a}_2)(\bar{u} \circ \bar{u}) - \bar{a}_0 \circ \bar{u} \bar{a}_2 \circ \bar{u}]^2 \\
 &= (\bar{a}_0 \circ \bar{a}_1)(\bar{a}_2 \circ \bar{a}_2) - \bar{a}_2 \circ \bar{a}_2 (\bar{a}_0 \circ \bar{u})^2 - \bar{a}_0 \circ \bar{a}_1 (\bar{a}_2 \circ \bar{u})^2 + (\bar{a}_0 \circ \bar{u})^2 (\bar{a}_2 \circ \bar{u})^2 \\
 &\quad - (\bar{a}_0 \circ \bar{a}_2)^2 + 2\bar{a}_0 \circ \bar{a}_2 \bar{a}_0 \circ \bar{u} \bar{a}_2 \circ \bar{u} - (\bar{a}_0 \circ \bar{u})^2 (\bar{a}_2 \circ \bar{u})^2
 \end{aligned} \tag{B.25}$$

while

$$\begin{aligned}
 &\bar{f} \circ \bar{f} (\bar{u} \times \bar{a}_2) \circ (\bar{u} \times \bar{a}_2) + \bar{f} \circ \bar{a}_2 (\bar{f} \times \bar{u}) \circ (\bar{u} \times \bar{a}_2) \\
 &+ \bar{f} \circ \bar{u} (\bar{f} \times \bar{a}_2) \circ (\bar{a}_2 \times \bar{u}) =
 \end{aligned}$$

$$\begin{aligned}
 &= [(\bar{a}_0 + \eta \bar{a}_2) \circ (\bar{a}_0 + \eta \bar{a}_2)] [\bar{u} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2] \\
 &\quad + [(\bar{a}_0 + \eta \bar{a}_2) \circ \bar{a}_2] [(\bar{a}_0 + \eta \bar{a}_2) \times \bar{u} \circ (\bar{u} \times \bar{a}_2)] \\
 &\quad + [(\bar{a}_0 + \eta \bar{a}_2) \circ \bar{u}] [(\bar{a}_0 + \eta \bar{a}_2) \times \bar{a}_2 \circ (\bar{a}_2 \times \bar{u})] \\
 &= (\bar{a}_0 \circ \bar{a}_2 + 2\eta \bar{a}_0 \circ \bar{a}_2 + \eta^2 \bar{a}_2) (\bar{u} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2) \\
 &\quad + (\bar{a}_0 \circ \bar{a}_2 + \eta \bar{a}_2 \circ \bar{a}_2) (\bar{a}_0 \times \bar{u} \circ \bar{u} \times \bar{a}_2) \\
 &\quad + (\bar{a}_0 \circ \bar{a}_2 + \eta \bar{a}_2 \circ \bar{a}_2) \eta (\bar{a}_2 \times \bar{u} \circ \bar{u} \times \bar{a}_2) \\
 &\quad + (\bar{a}_0 \circ \bar{u} + \eta \bar{a}_2 \circ \bar{u}) (\bar{a}_0 \times \bar{a}_2 \circ \bar{a}_2 \times \bar{u}) \\
 &\quad + (\bar{a}_0 \circ \bar{u} + \eta \bar{a}_2 \circ \bar{u}) \eta (\bar{a}_2 \times \bar{a}_2 \circ \bar{a}_2 \times \bar{u}) \\
 &= \bar{a}_0 \circ \bar{a}_0 [\bar{u} \times \bar{u} \bar{a}_2 \circ \bar{a}_2 - (\bar{u} \circ \bar{a}_2)^2] \\
 &\quad + \bar{a}_0 \circ \bar{a}_2 [\bar{a}_0 \circ \bar{u} \bar{u} \circ \bar{a}_2 - \bar{a}_0 \circ \bar{a}_2 \bar{u} \circ \bar{u}] \\
 &\quad + \bar{a}_0 \circ \bar{u} [\bar{a}_0 \circ \bar{a}_2 \bar{a}_2 \circ \bar{u} - \bar{a}_0 \circ \bar{u} \bar{a}_2 \circ \bar{a}_2] \\
 &\quad + \eta \left\{ \bar{a}_0 \circ \bar{a}_2 [\bar{u} \circ \bar{u} \bar{a}_2 \circ \bar{a}_2 - (\bar{u} \circ \bar{a}_2)^2] \right. \\
 &\quad \quad + \bar{a}_2 \circ \bar{a}_2 [\bar{a}_0 \circ \bar{u} \bar{u} \circ \bar{a}_2 - \bar{a}_0 \circ \bar{a}_2 \bar{u} \circ \bar{u}] \\
 &\quad \quad \left. + \bar{a}_2 \circ \bar{u} [\bar{a}_0 \circ \bar{a}_2 \bar{a}_2 \circ \bar{u} - \bar{a}_0 \circ \bar{u} \bar{a}_2 \circ \bar{a}_2] \right\} \\
 &= \bar{a}_0 \circ \bar{a}_0 \bar{a}_2 \circ \bar{a}_2 - \bar{a}_0 \circ \bar{a}_0 (\bar{u} \circ \bar{a}_2)^2 + 2 \bar{a}_0 \circ \bar{a}_2 \bar{a}_0 \circ \bar{u} \bar{u} \circ \bar{a}_2 \\
 &\quad - (\bar{a}_0 \circ \bar{a}_2)^2 - \bar{a}_2 \circ \bar{a}_0 (\bar{a}_0 \circ \bar{u})^2
 \end{aligned}$$

(B.26)

On the other hand, applying the second super-rule, Eq. (A.11)
 (with $\bar{a} = \bar{q}$, $\bar{b} = \bar{u}$, $\bar{c} = \bar{a}_2$)

$$\begin{aligned}
 (\bar{q} \cdot \bar{u} \times \bar{a}_2)^{\dagger} &= \bar{q} \circ \bar{q} (\bar{u} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2) - \bar{q} \circ \bar{a}_2 (\bar{u} \times \bar{a}_2 \circ \bar{u} \times \bar{q}) \\
 &\quad - \bar{q} \circ \bar{u} (\bar{a}_2 \times \bar{u} \circ \bar{a}_2 \times \bar{q})
 \end{aligned}
 \tag{B.27}$$

Thus, comparing Eqs. (B.24) and (B.27)

$$e c - d^{\dagger} = (\bar{q} \cdot \bar{u} \times \bar{a}_2)^{\dagger}
 \tag{B.28}$$

Hence, finally combining Eqs. (B.22), (B.23) and (B.28)

$$I_2 = \frac{1}{|\bar{q} \cdot \bar{u} \times \bar{a}_2|} \tan^{-1} \frac{-\bar{q} \times \bar{u} \circ \bar{q} \times \bar{a}_2}{\sqrt{\bar{q} \circ \bar{q}} |\bar{q} \cdot \bar{u} \times \bar{a}_2|}
 \tag{B.29}$$

APPENDIX C

TWO USEFUL FORMULAE

C.1 An expression for $\bar{f}_*^N \circ \bar{f}_*^N$

In this Appendix, it is proved that

$$\bar{f}_*^N \circ \bar{f}_*^N \parallel \bar{a}_1 \times \bar{a}_2 \parallel^2 = -(\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2 \quad (C.1)$$

Note that

$$\bar{f} - \bar{f}_* = (\eta - \eta_*) \bar{a}_2 \quad (C.2)$$

and

$$\bar{f}_* = \bar{f}_*^N + f_*^T \bar{u} \quad (C.3)$$

Hence

$$\bar{f}_*^N = \bar{f}_* - f_*^T \bar{u} = \bar{f} - (\eta - \eta_*) \bar{a}_2 - f_*^T \bar{u} \quad (C.4)$$

or, according to Eqs. (2.20) and (2.21)

$$\begin{aligned} \bar{f}_*^N &= \bar{f} - \frac{\bar{f} \times \bar{a}_2 \circ \bar{u} \times \bar{a}_2}{-\parallel \bar{u} \times \bar{a}_2 \parallel^2} \bar{u} - \frac{\bar{f} \cdot \bar{u} \circ \bar{a}_2 \cdot \bar{u}}{-\parallel \bar{u} \times \bar{a}_2 \parallel^2} \bar{a}_2 \\ &= \frac{1}{\parallel \bar{a}_1 \times \bar{a}_2 \parallel^2} \left\{ \parallel \bar{a}_1 \times \bar{a}_2 \parallel^2 \bar{f} + (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) \bar{a}_1 + \right. \\ &\quad \left. + (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \bar{a}_2 \right\} \quad (C.5) \end{aligned}$$

Hence

$$\begin{aligned} &\parallel \bar{a}_1 \times \bar{a}_2 \parallel^4 \bar{f}_*^N \circ \bar{f}_*^N \\ &= \bar{f} \circ \bar{f} \parallel \bar{a}_1 \times \bar{a}_2 \parallel^4 + (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2)^2 \bar{a}_1 \circ \bar{a}_1 + (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1)^2 \bar{a}_2 \circ \bar{a}_2 + \end{aligned}$$

$$\begin{aligned}
 & + 2 \left\{ \|\bar{a}_1 \times \bar{a}_2\|^2 (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) \bar{f} \circ \bar{a}_1 \right. \\
 & \quad + \|\bar{a}_1 \times \bar{a}_2\|^2 (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \bar{f} \circ \bar{a}_2 \\
 & \quad \left. + (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \bar{a}_1 \circ \bar{a}_2 \right\} \quad (C.6)
 \end{aligned}$$

On the other hand, according to the second super-rule

$$\begin{aligned}
 (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2 &= \bar{f} \circ \bar{f} (\bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) \\
 &\quad - \bar{f} \circ \bar{a}_1 (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) \\
 &\quad - \bar{f} \circ \bar{a}_2 (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \quad (C.7)
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \|\bar{a}_1 \times \bar{a}_2\|^2 \left[\bar{f} \cdot \bar{f} \|\bar{a}_1 \times \bar{a}_2\|^2 + (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2 \right] \\
 &= \bar{f} \circ \bar{f} \|\bar{a}_1 \times \bar{a}_2\|^4 + (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2)^2 \bar{a}_1 \circ \bar{a}_1 + (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1)^2 \bar{a}_2 \circ \bar{a}_2 \\
 & \quad + \cancel{\|\bar{a}_1 \times \bar{a}_2\|^2} \left\{ (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) \bar{f} \circ \bar{a}_1 + (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \bar{f} \circ \bar{a}_2 \right\} \\
 & \quad + 2 (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \bar{a}_1 \circ \bar{a}_2 \\
 & \quad + \|\bar{a}_1 \times \bar{a}_2\|^2 \left\{ \bar{f} \circ \bar{f} (\bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) - \bar{f} \circ \bar{a}_1 (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) \right. \\
 & \quad \left. - \bar{f} \circ \bar{a}_2 (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2)^2 \bar{a}_1 \circ \bar{a}_1 + (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1)^2 \bar{a}_2 \circ \bar{a}_2 \\
 &+ \|\bar{a}_1 \times \bar{a}_2\|^2 \left\{ (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) \bar{f} \circ \bar{a}_1 + (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \bar{f} \circ \bar{a}_2 \right\} \\
 &+ 2 (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1) \bar{a}_1 \circ \bar{a}_2 \\
 &= \bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2 (\bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2 \bar{a}_1 \circ \bar{a}_2 + \bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1 \bar{a}_1 \circ \bar{a}_2 \\
 &+ \|\bar{a}_1 \times \bar{a}_2\| \bar{f} \circ \bar{a}_1) + \bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1 (\bar{f} \times \bar{a}_1 \circ \bar{a}_2 \times \bar{a}_1 \bar{a}_2 \circ \bar{a}_2 \\
 &+ \bar{f} \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2 \bar{a}_1 \circ \bar{a}_2 + \|\bar{a}_1 \times \bar{a}_2\| \bar{f} \circ \bar{a}_2) \\
 &= 0
 \end{aligned}$$

(C.8)

since*

$$\begin{aligned}
 \bar{f} \circ \bar{a}_j \bar{a}_l \times \bar{a}_k \circ \bar{a}_l \times \bar{a}_k &= \bar{f} \times \bar{a}_k \circ \bar{a}_l \times \bar{a}_k \bar{a}_j \circ \bar{a}_l \\
 &+ \bar{f} \times \bar{a}_l \circ \bar{a}_k \times \bar{a}_l \bar{a}_j \circ \bar{a}_k \\
 &\quad (\ell = 1, 2; k = 2, 1)
 \end{aligned} \tag{C.9}$$

For

$$\bar{f} \circ \bar{a}_j \bar{a}_l \times \bar{a}_k \circ \bar{a}_l \times \bar{a}_k = \bar{f} \circ a_j (\bar{a}_l \circ \bar{a}_l \bar{a}_k \circ \bar{a}_k - (\bar{a}_j \circ \bar{a}_k)') \tag{C.10}$$

* For ordinary algebra, this corresponds to the well known re-

lation between tensor components $f_{jk} = a_{jlk} f^k$

and

$$\begin{aligned}
 & \bar{f} \times \bar{a}_K \circ \bar{a}_1 \times \bar{a}_K \bar{a}_1 \circ \bar{a}_2 + \bar{f} \times \bar{a}_1 \circ \bar{a}_K \times \bar{a}_1 \bar{a}_2 \circ \bar{a}_K \\
 = & \bar{f} \circ \bar{a}_1 \bar{a}_K \circ \bar{a}_K \bar{a}_2 \circ \bar{a}_2 - \bar{f} \circ \bar{a}_K \bar{a}_K \circ \bar{a}_2 \bar{a}_2 \circ \bar{a}_1 \\
 & + \bar{f} \circ \bar{a}_K \bar{a}_1 \circ \bar{a}_1 \bar{a}_2 \circ \bar{a}_K - \bar{f} \circ \bar{a}_2 (\bar{a}_1 \circ \bar{a}_K)^2 \\
 = & \bar{f} \circ \bar{a}_1 [\bar{a}_K \circ \bar{a}_K \bar{a}_1 \circ \bar{a}_2 - (\bar{a}_1 \circ \bar{a}_K)^2]
 \end{aligned}
 \tag{C.11}$$

Equation (C.8) is the desired proof of Eq. (C.1).

C.2 A Second Useful Formula

In subsonic theory, it was shown that

$$\bar{f} \cdot \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{f} \times \bar{a}_1 \cdot \bar{f} \times \bar{a}_2)^2 \equiv |\bar{f} \times \bar{a}_1|^2 |\bar{f} \times \bar{a}_2|^2
 \tag{C.13}$$

Here it is shown that for supersonic flow

$$\bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)^2 \equiv \|\bar{f} \times \bar{a}_1\|^2 \|\bar{f} \times \bar{a}_2\|^2
 \tag{C.14}$$

For, according to the second super-rule (see also Eq. B.27)

$$\begin{aligned}
 & \bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)^2 \\
 = & \bar{f} \circ \bar{f} \left\{ \bar{f} \circ \bar{f} (\bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{a}_2) - \bar{f} \circ \bar{a}_2 (\bar{a}_1 \times \bar{a}_2 \circ \bar{a}_1 \times \bar{f}) \right. \\
 & \left. - \bar{f} \circ \bar{a}_1 (\bar{a}_2 \times \bar{a}_1 \circ \bar{a}_2 \times \bar{f}) \right\} + (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 - \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2)^2 \\
 = & (\bar{f} \circ \bar{f})^2 [\bar{a}_1 \circ \bar{a}_1 \bar{a}_2 \circ \bar{a}_2 - (\bar{a}_1 \circ \bar{a}_2)^2] - \bar{f} \circ \bar{f} \bar{f} \circ \bar{a}_2 (\bar{a}_1 \circ \bar{a}_1 \bar{a}_2 \circ \bar{f} \\
 & - \bar{a}_1 \circ \bar{f} \bar{a}_2 \circ \bar{a}_1) - \bar{f} \circ \bar{f} \bar{f} \circ \bar{a}_1 (\bar{a}_2 \circ \bar{a}_2 \bar{a}_1 \circ \bar{f} - \bar{a}_2 \circ \bar{f} \bar{a}_1 \circ \bar{a}_2) +
 \end{aligned}$$

$$\begin{aligned}
 & + (\bar{f} \circ \bar{f})^* (\bar{a}_1 \circ \bar{a}_2)^* - 2 \bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_2 \bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2 + (\bar{f} \circ \bar{a}_1 \bar{f} \circ \bar{a}_2)^2 \\
 & = (\bar{f} \circ \bar{f} \bar{a}_1 \circ \bar{a}_1 - (\bar{f} \circ \bar{a}_1)^2) (\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2 - (\bar{f} \circ \bar{a}_2)^2) \\
 & = (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_1) = (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_1) (\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2) \\
 & = \|\bar{f} \times \bar{a}_1\|^2 \|\bar{f} \times \bar{a}_2\|^2
 \end{aligned}
 \tag{C.15}$$

APPENDIX D

DERIVATIVES OF I_D AND I_{S3}

D.1 Introduction

In this Appendix, it will be shown that for any hyperboloidal quadrilateral element, the second mixed derivative of

$$I_D = \frac{\bar{f} \cdot \bar{n}}{|\bar{f} \cdot \bar{n}|} \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| |\bar{f} \cdot \bar{a}_1 \times \bar{a}_2|} \right) = \tan_p^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \quad (D.1)$$

is given by

$$\frac{\partial^2 I_D}{\partial \xi \partial \eta} = \frac{\bar{f} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{f}\|^3} \quad (D.2)$$

while, for any planar quadrilateral element, the second mixed derivative of I_{S3} ,

$$I_{S3} = |\bar{f} \cdot \bar{n}| \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| |\bar{f} \cdot \bar{a}_1 \times \bar{a}_2|} \right) = \bar{f} \cdot \bar{n} \tan_p^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\|\bar{f}\| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right) \quad (D.3)$$

is given by

$$\frac{\partial^2 I_{S3}}{\partial \xi \partial \eta} = \bar{f} \cdot \bar{n} \frac{\bar{f} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{f}\|^3} \quad (D.4)$$

Note that

$$\frac{\partial \bar{a}_1}{\partial \xi} = \frac{\partial \bar{a}_2}{\partial \eta} = 0 \quad (D.5)$$

$$\frac{\partial \bar{a}_1}{\partial \eta} = \frac{\partial \bar{a}_2}{\partial \xi} = \bar{p}_3 \quad (D.6)$$

and

$$\frac{\partial \bar{f}}{\partial \xi} = \bar{a}_1 \quad (D.7)$$

$$\frac{\partial \bar{f}}{\partial \eta} = \bar{a}_2 \quad (D.8)$$

D.2 Derivative of I_D

Consider Eq. (D.1), or

$$I_D = S_n \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}} S_n (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)} \right) \quad (D.9)$$

where

$$S_n = \frac{\bar{f} \cdot \bar{n}}{|\bar{f} \cdot \bar{n}|} = \text{Sign}(\bar{f} \cdot \bar{n}) \quad (D.10)$$

The derivative of I_D with respect to η is given by

$$\begin{aligned} \frac{\partial I_D}{\partial \eta} &= \frac{\partial}{\partial \eta} S_n \tan^{-1} \left(\frac{-\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}} S_n (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)} \right) \\ &= -(S_n)^2 \frac{1}{1 + \left(\frac{\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right)^2} \\ &\times \left\{ \left[\bar{a}_2 \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2 + \bar{f} \times \bar{p}_3 \circ \bar{f} \times \bar{a}_2 + \bar{f} \times \bar{a}_2 \circ \bar{a}_2 \times \bar{a}_2 \right] \frac{1}{|\bar{f}| \bar{f} \cdot \bar{a}_1 \times \bar{a}_2} \right. \\ &\left. - \frac{\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2}{\bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)} \left(\frac{\bar{a}_2 \circ \bar{f}}{\sqrt{\bar{f} \circ \bar{f}}} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 + \sqrt{\bar{f} \circ \bar{f}} (\bar{a}_2 \cdot \bar{a}_1 \times \bar{a}_2 + \bar{f} \cdot \bar{p}_3 \times \bar{a}_2) \right) \right\} \end{aligned}$$

$$\begin{aligned}
 &= - \frac{\bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \cdot \bar{a}_2)^2}{\bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \cdot \bar{a}_2) + (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)^2} \frac{1}{\|\bar{f}\|^3 (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2} \times \\
 &\quad \left\{ (\bar{a}_2 \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2 + \bar{f} \times \bar{p}_3 \circ \bar{f} \times \bar{a}_2) \bar{f} \circ \bar{f} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \right. \\
 &\quad \left. - \bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2 (\bar{a}_2 \circ \bar{f} \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 + \bar{f} \circ \bar{f} \bar{f} \cdot \bar{p}_3 \times \bar{a}_2) \right\} \\
 &= \frac{-1}{\bar{f} \circ \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) + (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)} \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} \times \\
 &\quad \left\{ \left[(\bar{a}_2 \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2) \bar{f} \circ \bar{f} - (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2) \bar{a}_2 \circ \bar{f} \right] \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \right. \\
 &\quad \left. + \left[(\bar{f} \times \bar{p}_3 \circ \bar{f} \times \bar{a}_2) \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 - (\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2) \bar{f} \cdot \bar{p}_3 \times \bar{a}_2 \right] \bar{f} \circ \bar{f} \right\} \\
 &\hspace{20em} \text{(D.11)}
 \end{aligned}$$

Next note, as shown in Appendix C, Eq. (C.14),

$$\bar{f} \cdot \bar{f} (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2)^2 + (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2)^2 = \|\bar{f} \times \bar{a}_1\|^2 \|\bar{f} \times \bar{a}_2\|^2 \quad \text{(D.12)}$$

Moreover, note that (see Eq. A.19)

$$\begin{aligned}
 & [(\bar{a}_2 \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2) \bar{f} \circ \bar{f} - (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2) \bar{a}_2 \circ \bar{f}] \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \\
 & + [(\bar{f} \times \bar{p}_3 \circ \bar{f} \times \bar{a}_2) (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) - (\bar{f} \times \bar{a}_1 \circ \bar{f} \times \bar{a}_2) (\bar{f} \cdot \bar{p}_3 \times \bar{a}_2)] \bar{f} \circ \bar{f} \\
 \equiv & [(\bar{a}_2 \circ \bar{f} \times \bar{a}_1 \circ \bar{a}_2 - \bar{a}_2 \circ \bar{a}_2 \bar{a}_1 \circ \bar{f}) \bar{f} \circ \bar{f} \\
 & - (\bar{f} \circ \bar{f} \times \bar{a}_1 \circ \bar{a}_2 - \bar{f} \circ \bar{a}_2 \bar{f} \circ \bar{a}_1) \bar{a}_2 \circ \bar{f}] \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \\
 & - [(\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2) \bar{f} \cdot \bar{a}_1 \times \bar{p}_3] \bar{f} \circ \bar{f} \\
 \equiv & [-\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2 - (\bar{f} \circ \bar{a}_2)^2] (\bar{f} \circ \bar{a}_1) (\bar{f} \cdot \bar{a}_1 \times \bar{a}_2) \\
 & + (\bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2) (\bar{f} \cdot \bar{a}_1 \times \bar{p}_3) \bar{f} \circ \bar{f} \\
 \equiv & \|\bar{f} \times \bar{a}_2\|^2 (\bar{f} \circ \bar{a}_1 \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 - \bar{f} \circ \bar{f} \bar{f} \cdot \bar{a}_1 \times \bar{p}_3)
 \end{aligned} \tag{D.13}$$

since

$$\bar{f} \circ \bar{f} \bar{a}_2 \circ \bar{a}_2 - (\bar{f} \circ \bar{a}_2)^2 \equiv \bar{f} \times \bar{a}_2 \circ \bar{f} \times \bar{a}_2 \equiv -\|\bar{f} \times \bar{a}_2\|^2 \tag{D.14}$$

Finally, combining Eqs. (D.11), (D.12) and (D.13), yields

$$\begin{aligned}
 \frac{\partial I_0}{\partial \eta} &= \frac{-\|\bar{f} \times \bar{a}_2\|^2 (\bar{f} \circ \bar{a}_1 \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 - \bar{f} \circ \bar{f} \bar{f} \cdot \bar{a}_1 \times \bar{p}_3)}{\|\bar{f} \times \bar{a}_1\|^2 \|\bar{f} \times \bar{a}_2\|^2 \|\bar{f}\|} \\
 &= \frac{1}{-\|\bar{f} \times \bar{a}_1\|^2 \|\bar{f}\|} (\bar{f} \circ \bar{a}_1 \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 - \bar{f} \circ \bar{f} \bar{f} \cdot \bar{a}_1 \times \bar{p}_3) \tag{D.15}
 \end{aligned}$$

Next, consider the second mixed derivative, noting that

$$\frac{\partial}{\partial \bar{f}} (\bar{f} \cdot \bar{a}_1) = \frac{\partial}{\partial \bar{f}} \left\{ (\bar{p}_0 + \eta \bar{p}_2) \times (\bar{p}_1 + \eta \bar{p}_3) \right\} \equiv 0 \quad (\text{D.16})$$

one obtains

$$\begin{aligned} \frac{\partial^2 I_p}{\partial \bar{f} \partial \eta} &= \frac{1}{\|\bar{f} \times \bar{a}_1\|^2} \frac{\partial}{\partial \bar{f}} \left(\frac{\bar{f} \circ \bar{f} \cdot \bar{f} \cdot \bar{a}_1 \times \bar{p}_3 - \bar{f} \circ \bar{a}_1 \cdot \bar{f} \cdot \bar{a}_1 \times \bar{a}_2}{\sqrt{\bar{f} \circ \bar{f}}} \right) \\ &= \frac{1}{\|\bar{f} \times \bar{a}_1\|^2} \left\{ -\frac{\bar{f} \circ \bar{a}_1}{(\bar{f} \circ \bar{f})^{3/2}} (\bar{f} \circ \bar{f} \cdot \bar{f} \cdot \bar{a}_1 \times \bar{p}_3 - \bar{f} \cdot \bar{a}_1 \cdot \bar{f} \cdot \bar{a}_1 \times \bar{a}_2) \right. \\ &\quad \left. + \frac{1}{\sqrt{\bar{f} \circ \bar{f}}} (2 \bar{f} \circ \bar{a}_1 \cdot \bar{f} \cdot \bar{a}_1 \times \bar{p}_3 + \bar{f} \circ \bar{f} \cdot \bar{a}_1 \cdot \bar{a}_1 \times \bar{p}_3 - \bar{a}_1 \circ \bar{a}_1 \cdot \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \right. \\ &\quad \left. - \bar{f} \circ \bar{a}_1 \cdot \bar{a}_1 \cdot \bar{a}_1 \times \bar{a}_2 - \bar{f} \circ \bar{a}_1 \cdot \bar{f} \cdot \bar{a}_1 \times \bar{p}_3) \right\} \\ &= \frac{1}{\|\bar{f} \times \bar{a}_1\|^2} \frac{1}{\|\bar{f}\|^3} \left[(\bar{f} \circ \bar{a}_1)^2 - \bar{f} \circ \bar{f} \cdot \bar{a}_1 \circ \bar{a}_1 \right] \bar{f} \cdot \bar{a}_1 \times \bar{a}_2 \\ &= \frac{\bar{f} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{f}\|^3} \quad (\text{D.17}) \end{aligned}$$

D.3 Derivative of I_{s3}

Note that for a quadrilateral planar element $\bar{q} \cdot \bar{n}$ is constant. For

$$\frac{\partial}{\partial \xi} (\bar{q} \cdot \bar{n}) = \bar{a}_1 \cdot \bar{n} = 0 \quad (D.18)$$

$$\frac{\partial}{\partial \eta} (\bar{q} \cdot \bar{n}) = \bar{a}_2 \cdot \bar{n} = 0 \quad (D.19)$$

Therefore, using Eqs. (D.1) and (D.17)

$$\begin{aligned} \frac{\partial^2 I_s}{\partial \xi \partial \eta} &= \bar{q} \cdot \bar{n} \frac{\partial^2}{\partial \xi \partial \eta} \tan^{-1} \left(\frac{-\bar{q} \times \bar{a}_1 \cdot \bar{q} \times \bar{a}_2}{\|\bar{q}\| \bar{q} \cdot \bar{a}_1 \times \bar{a}_2} \right) \\ &= \bar{q} \cdot \bar{n} \frac{\partial^2 I_0}{\partial \xi \partial \eta} = \bar{q} \cdot \bar{n} \frac{\bar{q} \cdot \bar{a}_1 \times \bar{a}_2}{\|\bar{q}\|} \end{aligned}$$

(D.20)

in agreement with Eq. (D.4).

APPENDIX E

SUPERSONIC OSCILLATORY FLOW

E.1 Integral Equation

In this Appendix it is shown how the results obtained in the main body of this report can be extended to supersonic oscillatory flow. Introducing the variables

$$X = \frac{x}{B^2} \quad Y = \frac{y}{l} \quad Z = \frac{z}{l} \quad T = \frac{Ba_\infty t}{l} \quad \Omega = \frac{\omega l}{Ba_\infty} \quad (\text{E.1})$$

and the complex potential $\hat{\phi}$ such that

$$\varphi(x, y, z, t) = U_\infty l \hat{\phi}(X, Y, Z) e^{i\Omega(T - MX)} \quad (\text{E.2})$$

the integral equation for the subsonic oscillatory flow is given by

$$\pi \hat{\phi} = \oint_{\Sigma} \left[\frac{\partial \hat{\phi}}{\partial N^c} \frac{H}{R_{12}^2} \cos(\Omega \parallel \mathbf{q} \parallel) + \hat{\phi} \frac{\partial}{\partial N^c} \left(\frac{H}{R_{12}^2} \cos(\Omega \parallel \mathbf{q} \parallel) \right) \right] d\Sigma \quad (\text{E.3})$$

where Σ surrounds body and wake,

C.2 Boundary Condition

The boundary condition is given by

$$\nabla_{xy2} S \cdot \mathbf{V}_{xy2} \varphi = - \frac{\partial S}{\partial t} - U_\infty \frac{\partial S}{\partial x} \quad (\text{E.4})$$

or

$$-\nabla_{XYZ} S_0 \nabla_{XYZ} \phi + \frac{B}{M} \frac{\partial S}{\partial T} + \frac{1}{B} \frac{\partial S}{\partial X} + \frac{M^2}{B^2} \frac{\partial S}{\partial X} \frac{\partial \phi}{\partial X} = 0 \quad (\text{E.5})$$

where ψ and ϕ are such that

$$\bar{\phi} = U_\omega x + \psi = U_\omega l (\beta X + \phi) \quad (\text{E.6})$$

Next, assume that the motion of the surface consists of small harmonic oscillations around a rest configuration, that is

$$S = S_0(X, Y, Z) + \tilde{S}(X, Y, Z) e^{i\Omega T} \quad (\text{E.7})$$

Then, setting

$$\phi = \phi_0(X, Y, Z) + \tilde{\phi}(X, Y, Z) e^{i\Omega T} \quad (\text{E.8})$$

one obtains

$$\begin{aligned} & -\nabla_{XYZ} S_0 \nabla_{XYZ} \phi_0 - \left(\nabla_{XYZ} S_0 \nabla_{XYZ} \tilde{\phi} + \nabla_{XYZ} \tilde{S} \nabla_{XYZ} \phi_0 \right) e^{i\Omega T} \\ & - \left(\nabla_{XYZ} \tilde{S} \nabla_{XYZ} \tilde{\phi} \right) e^{i2\Omega T} + \frac{B}{M} i\Omega \tilde{S} e^{i\Omega T} \\ & + \frac{1}{B} \left(\frac{\partial S_0}{\partial X} + \frac{\partial \tilde{S}}{\partial X} e^{i\Omega T} \right) \\ & + \frac{M^2}{B^2} \left[\frac{\partial S_0}{\partial X} \frac{\partial \phi_0}{\partial X} + \frac{\partial S_0}{\partial X} \frac{\partial \tilde{\phi}}{\partial X} + \frac{\partial \tilde{S}}{\partial X} \frac{\partial \phi_0}{\partial X} \right] e^{i\Omega T} \\ & + \frac{\partial \tilde{S}}{\partial X} \frac{\partial \tilde{\phi}}{\partial X} e^{i2\Omega T} \Big] = 0 \quad (\text{E.9}) \end{aligned}$$

Assume that the surface is given in the form

$$S = + [Z - Z_u(X, Y) - \tilde{Z}_u(X, Y) e^{i\Omega T}] = 0 \quad \text{(Upper surface)}$$

$$S = - [Z - Z_l(X, Y) - \tilde{Z}_l(X, Y) e^{i\Omega T}] = 0 \quad \text{(Lower surface)}$$

(E.10)

with

$$Z_{u, \epsilon}(X, Y) = O(\epsilon) \quad \text{(E.11)}$$

$$\tilde{Z}_{u, \epsilon}(X, Y) = O(\epsilon^2) \quad \text{(E.12)}$$

or, in general by Eq. (E.7) with

$$S_0 = \pm [Z - Z_{u, \epsilon}(X, Y)] = O(1) \quad \text{(E.13)}$$

$$\frac{\partial S_0}{\partial X} = \mp \frac{\partial Z_{u, \epsilon}}{\partial X} = O(\epsilon) \quad \text{(E.14)}$$

$$\nabla S_0 = O(1) \quad \text{(E.15)}$$

and

$$\tilde{S}_0 = \pm \tilde{Z}_{u, \epsilon}(X, Y) = O(\epsilon^2) \quad \text{(E.16)}$$

$$\frac{\partial \tilde{S}_0}{\partial X} = O(\epsilon^2) \quad \text{(E.17)}$$

Assume also

$$\Omega = O(1) \quad (\text{E.18})$$

This implies (see Eqs. E.25 and E.26) that

$$\phi_0 = O(\epsilon) \quad (\text{E.19})$$

$$\tilde{\phi} = O(\tilde{\epsilon}) \quad (\text{E.20})$$

Neglecting the terms which contain $e^{i2\Omega T}$ (which are of order ϵ^4) and separating the steady from the oscillatory terms, one obtains

$$-\nabla_{xyz} \tilde{S}_0 \otimes \nabla_{xyz} \phi_0 + \frac{1}{B} \frac{\partial \tilde{S}_0}{\partial X} + \frac{M^2}{\beta^2} \frac{\partial \tilde{S}_0}{\partial X} \frac{\partial \phi_0}{\partial X} = 0 \quad (\text{E.21})$$

$$-\nabla_{xyz} \tilde{S}_0 \otimes \nabla_{xyz} \tilde{\phi} - \nabla_{xyz} \tilde{S} \otimes \nabla_{xyz} \phi_0 + \frac{B}{M} i\Omega \tilde{S} + \frac{1}{B} \frac{\partial \tilde{S}}{\partial X} + \frac{M^2}{B^2} \left(\frac{\partial \tilde{S}}{\partial X} \frac{\partial \tilde{\phi}}{\partial X} + \frac{\partial \tilde{S}}{\partial X} \frac{\partial \phi_0}{\partial X} \right) = 0 \quad (\text{E.22})$$

Introducing $\hat{\phi}$ such that

$$\tilde{\phi} = \hat{\phi} e^{-i\Omega M X} \quad (\text{E.23})$$

Equation (E.22) reduces to

$$-\nabla_{xyz} \tilde{S}_0 \otimes \nabla_{xyz} \hat{\phi} e^{-i\Omega M X} + i\Omega M \frac{\partial \tilde{S}}{\partial X} \hat{\phi} e^{-i\Omega M X} - \nabla_{xyz} \tilde{S} \otimes \nabla_{xyz} \phi_0 + \frac{B}{M} i\Omega \tilde{S} + \frac{1}{B} \frac{\partial \tilde{S}}{\partial X} + \frac{M^2}{B^2} \left[\frac{\partial \tilde{S}}{\partial X} \left(\frac{\partial \hat{\phi}}{\partial X} - i\Omega M \hat{\phi} \right) e^{-i\Omega M X} + \frac{\partial \tilde{S}}{\partial X} \frac{\partial \phi_0}{\partial X} \right] = 0 \quad (\text{E.24})$$

Finally, neglecting terms of order ε^2 in Eq. (E.21) and terms of order ε^3 in Eq. (E.24), one obtains

$$-\nabla_{xy2} S_0 \odot \nabla_{xy2} \hat{\phi}_0 = -\frac{1}{B} \frac{\partial S_0}{\partial x} \quad (\text{E.25})$$

$$-\nabla_{xy2} S_0 \odot \nabla_{xy2} \hat{\phi} = -\left(i \frac{\beta \Omega}{M} \tilde{S} + \frac{1}{B} \frac{\partial \tilde{S}}{\partial x} \right) e^{i2MX} \quad (\text{E.26})$$

In particular, for

$$S = \pm \frac{1}{\ell} \left[z - \tilde{z}_{u,l}(x,y) - \tilde{z}_{u,l}(x,y) e^{i\omega t} \right] \quad (\text{E.27})$$

(where the upper[lower] sign holds on the upper[lower] surface), one obtains

$$S_0 = \pm \frac{1}{\ell} \left[z - \tilde{z}_{u,l}(x,y) \right] \quad (\text{E.28})$$

$$\tilde{S} = \mp \frac{1}{\ell} \tilde{z}_{u,l}(x,y) \quad (\text{E.29})$$

$$\frac{1}{|\nabla_{xy2} S_0|} = |N_z| = \pm N_z \quad (\text{E.30})$$

and

$$\frac{\partial \hat{\phi}}{\partial N_0} = \frac{\nabla_{xy2} S_0 \odot \nabla_{xy2} \hat{\phi}}{|\nabla_{xy2} S_0|} = -N_z \left(i k \frac{\tilde{z}}{\ell} + \frac{\partial \tilde{z}}{\partial x} \right) e^{i2MX} \quad (\text{E.31})$$

where

$$k = \frac{\beta \Omega}{M} = \frac{\omega \ell}{U_\infty} \quad (\text{E.32})$$

C.3 Pressure Coefficient

The pressure coefficient can be evaluated by using the linearized Bernoulli theorem, as

$$\begin{aligned}
 c_p &= -\frac{2}{U_\infty^2} \left(\frac{\partial \phi}{\partial t} + U_\infty \frac{\partial \phi}{\partial x} \right) \\
 &= -2 \left(\frac{B}{M} \frac{\partial \phi}{\partial T} + \frac{1}{B} \frac{\partial \phi}{\partial X} \right)
 \end{aligned}
 \tag{E.33}$$

For oscillatory flow, setting

$$\phi = \tilde{\phi} e^{i\Omega T} = \hat{\phi} e^{i\Omega(T-MX)}
 \tag{E.34}$$

$$c_p = \tilde{c}_p e^{i\Omega T}
 \tag{E.35}$$

one obtains

$$\begin{aligned}
 \tilde{c}_p &= -2 \left(\frac{B}{M} i\Omega \tilde{\phi} + \frac{1}{B} \frac{\partial \tilde{\phi}}{\partial X} \right) \\
 &= -2 \left[i\Omega \left(\frac{B}{M} - \frac{M}{B} \right) \hat{\phi} + \frac{1}{B} \frac{\partial \hat{\phi}}{\partial X} \right] e^{-i\Omega MX} \\
 &= -\frac{2}{B} \left[-i\Omega \frac{\hat{\phi}}{M} + \frac{\partial \hat{\phi}}{\partial X} \right] e^{i\Omega MX} \\
 &= -\frac{2}{B} \left[e^{i\Omega X/M} \frac{\partial}{\partial X} \left(\hat{\phi} e^{-i\Omega X/M} \right) \right] e^{-i\Omega MX} \\
 &= -\frac{2}{B} e^{-i\Omega B^2 X/M} \frac{\partial}{\partial X} \left(\hat{\phi} e^{-i\Omega X/M} \right) \\
 &= -\frac{2}{B} e^{-ikBX} \frac{\partial}{\partial X} \left(\hat{\phi} e^{-ikX/B} \right)
 \end{aligned}
 \tag{E.36}$$

APPENDIX F.

SUPER-SOLID ANGLE

In this Appendix the concept of supersonic solid angle or supersolid-angle, $d\Omega'$, is introduced and it is shown that

$$\begin{aligned} \Omega' &= \oint_{\Sigma} d\Omega' = 0 && (P_0 \text{ outside } \Sigma) \\ &= 2\pi && (P_0 \text{ on } \Sigma) \\ &= 4\pi && (P_0 \text{ inside } \Sigma) \end{aligned} \tag{F.1}$$

Note that (see, for instance, Ref. 1, Eq. 6.6)

$$\begin{aligned} - \iint_{\Sigma} \frac{\partial}{\partial N} \left(\frac{H}{\|\bar{q}\|} \right) d\Sigma &= \iint_{\Sigma} \frac{H}{\|\bar{q}\|^3} \bar{q} \cdot \bar{N} d\Sigma \\ &= \iint_{\Sigma} H \frac{|\bar{q}|^3}{\|\bar{q}\|^3} \frac{\bar{q} \cdot \bar{N}}{|\bar{q}|} \frac{d\Sigma}{|\bar{q}|^2} = \iint_{\Sigma} H \frac{|\bar{q}|^3}{\|\bar{q}\|^3} \cos(\bar{q} \cdot \bar{N}) \frac{d\Sigma}{\|\bar{q}\|^2} \\ &= \iint_{\Sigma} H \frac{|\bar{q}|^3}{\|\bar{q}\|^3} \frac{d\Sigma}{|\bar{q}|^2} = \iint_{\Sigma} H \frac{|\bar{q}|^3}{\|\bar{q}\|^3} d\Sigma \\ &= \frac{1}{2} \iint d\Omega' \end{aligned} \tag{F.2}$$

where the finite parts of the integrals are understood, $d\Omega$ is the usual solid angle, while

$$d\Omega' = 2H \frac{|\bar{q}|^3}{\|\bar{q}\|^3} d\Omega \tag{F.3}$$

is the analogous one for the supersonic flow and is called here supersonic solid angle or super-solid-angle. Note that

$$\frac{\|\vec{q}\|}{|\vec{r}|} \equiv \left\| \frac{\vec{q}}{|\vec{r}|} \right\| \quad (\text{F.4})$$

depends upon the direction of \vec{q} but not upon its magnitude. Therefore if P_0 is outside the closed surface Σ , then (following the same reasoning used for the usual solid angles)

$$\oint_{\Sigma} d\Omega' = 0 \quad (P_0 \text{ outside } \Sigma) \quad (\text{F.5})$$

Furthermore note if P_0 approaches P_* on Σ , one obtains, according to Eq. (4.3) and (F.5)

$$\begin{aligned} \lim_{P_0 \rightarrow P_*} \left(\oint_{\Sigma} d\Omega' \right) &= \lim_{\epsilon \rightarrow 0} \left[\lim_{P_0 \rightarrow P_*} \left(\oint_{\Sigma - \Sigma_{\epsilon}} d\Omega' - 2 \oint_{\Sigma_{\epsilon}} \frac{\partial}{\partial N_{\epsilon}} \left(\frac{H}{\|\vec{q}\|} \right) d\bar{\Sigma} \right) \right] \quad (\text{F.6}) \\ &= \left(\oint_{\Sigma} d\Omega' \right)_{P_0 = P_*} - 2\pi \equiv 0 \end{aligned}$$

Therefore

$$\oint_{\Sigma} d\Omega' = 2\pi \quad (P_0 \text{ on } \Sigma) \quad (\text{F.7})$$

Similarly, if P is inside Σ , the supersolid-angle is the one intersected by the Mach forecone on an arbitrary surface, for instance the plane $X - X_0 = -a$ (Fig. F.1).

$$\begin{aligned}
 \frac{1}{2} \oint_{\Sigma} d\Omega' &= - \oint \frac{\partial}{\partial N} \left(\frac{H}{|\vec{q}|} \right) d\Sigma = \frac{\partial}{\partial N_0} \iint \frac{H}{|\vec{q}|} d\Sigma \\
 &= \frac{\partial}{\partial x_0} \iint \frac{H}{|\vec{q}|} d\Sigma = \frac{\partial}{\partial a} \iint \frac{H}{|\vec{q}|} d\Sigma = \frac{\partial}{\partial a} \left(2\pi \int_0^a \frac{1}{\sqrt{a^2 - \rho^2}} \rho d\rho \right) \\
 &= 2\pi \frac{\partial}{\partial a} \left[-\sqrt{a^2 - \rho^2} \right]_{\rho=0}^{\rho=a} = \frac{\partial}{\partial a} (2\pi a) = \\
 &= 2\pi
 \end{aligned}
 \tag{F.8}$$

Equations (F.5, F.6 and F.8) are equivalent to Eq. (F.1).

It may be noted that according to Eqs. (4.6) and (F.1)

$$E = 1 - \frac{1}{4\pi} \oint d\Omega'
 \tag{F.9}$$

It should be noted that Eq. (F.1) and (4.6) are valid only if the surface, Σ , has a unique tangent plane on P_* , while Eq. (F.9) is valid even without this restriction. The proof of this statement is not of interest here, but could be obtained by following the procedure used to derive Eq. (4.6) by using Eq. (F.5) in order to evaluate \oint_{Σ} in Eq. (4.3).

It is worth noting that the analogous of Eq. (F.9) for subsonic flow is

$$E = 1 - \frac{1}{4\pi} \oint d\Omega
 \tag{F.10}$$

The proof of this equation is similar to Eq. (F.9) and is not given here. As for Eq. (F.9), Eq. (F.10) is valid even if Σ does not have a unique tangent plane in P_* .

Equation (F.9) implies that

$$\begin{aligned}
 \sum_K (2E \delta_{nK} - c_{nK}) &= 2E - \frac{1}{\pi} \oint \frac{\partial}{\partial N} \left(\frac{H}{|\vec{q}|} \right) d\Sigma \\
 &= 2E + \frac{1}{2\pi} \oint d\Omega' = 2
 \end{aligned}
 \tag{F.11}$$

that is the sum of the coefficients of Eq. (4.7), where $E = 1/2$, is always equal to two. Note that this result is valid even if the surface does not have a unique tangent plane in $P_v = P_h$ since in this case Eq. (4.7) must be modified as

$$\left[2E \delta_{hk} - c_{hk} \right] \{ \phi_k \} = [c_{hk}] \{ \psi_k \} \quad (\text{F.12})$$

Equation F.11 may be used to evaluate E for points with slopes discontinuities (such as corners of quadrilateral elements or the apex of a cone) as

$$E = \frac{1}{2} \sum_k c_{hk} + 1 \quad (\text{F.13})$$

The above results are valid for subsonic flow as well.

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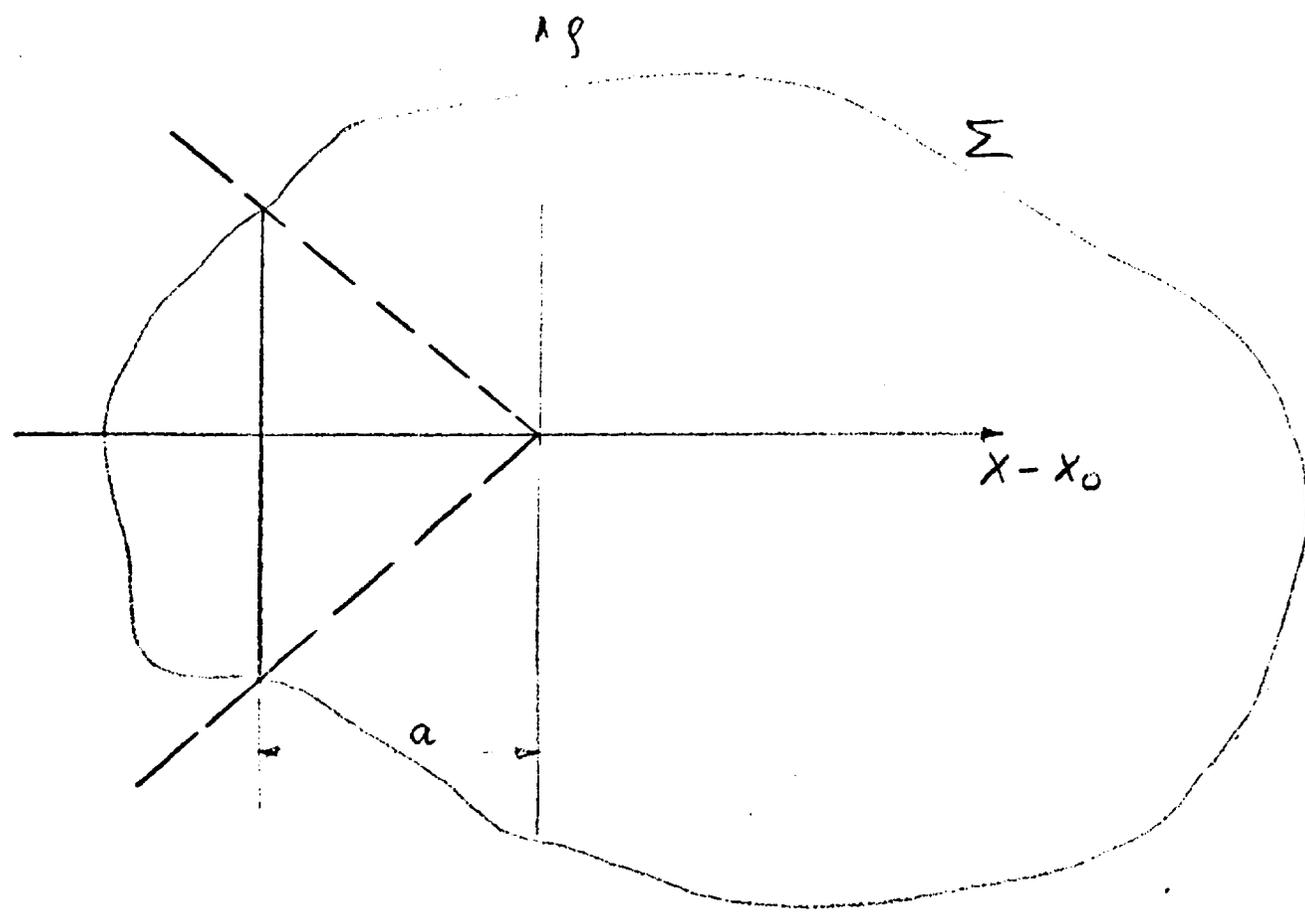


Fig. F.1. Super-solid - angle