BOSTON UNIVERSITY
COLLEGE OF ENGINEERING

STEADY AND OSCILLATORY,
SUBSONIC AND SUPersonic, AERODYNAMIC
PRESSURE AND GENERALIZED FORCES FOR
COMPLEX AIRCRAFT CONFIGURATIONS
AND APPLICATION TO FLUTTER

by

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ABSTRACT

A general method for steady and oscillatory, subsonic and supersonic, potential linearized aerodynamic flow around complex configurations is introduced. By applying the Green function method, a linear integral equation relating the unknown, small perturbation potential on the surface of the body, to the known downwash is obtained. The surfaces of the aircraft, wake and diaphragm (if necessary) are divided into small quadrilateral elements which are approximated with hyperboloidal surfaces. The potential and its normal derivative are assumed to be constant within each element. This yields a set of linear algebraic equations. The coefficients are evaluated analytically. By using Gaussian elimination method, equations are solved for the potentials at the centroids of elements. The pressure coefficient is evaluated by the finite different method. The lift and moment coefficients are evaluated by numerical integration. Numerical results are presented. Applications to flutter are also included.
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NOMENCLATURE

\( \bar{\alpha}_i \)  
Equations (3.13) and (3.14)

\( A \)  
The area of the wing planform, or \( \omega_d^2/\omega^* \) in Equation (7.35)

\( \text{Arg} \)  
Phase angle

\( AR \)  
Aspect ratio

\( a \)  
Free-stream speed of sound

\( b \)  
Span

\( b_{hk} \)  
Equations (3.4) and (5.4)

\( \hat{b}_{hk} \)  
Equations (4.6) and (6.4)

\( C_k \)  
Equation (7.44)

\( c_p \)  
Pressure coefficient, \( 2 (p - P_\infty) / \rho_\infty U_\infty^2 \)

\( c_\alpha \)  
Lift distribution coefficient, \( C_{p,\alpha} - C_{p_\infty} \)

\( C_L \)  
Sectional lift coefficient

\( C_L \)  
Total lift coefficient

\( \bar{C}_{L_d}, \bar{C}_{M_d} \)  
Total lift coefficient and total moment coefficient about axis \( X = X_{EA} \) for the wing in oscillation in pitch about axis \( X = X_{EA} \) per unit angle amplitude

\( \bar{C}_{L_h}, \bar{C}_{M_h} \)  
Total lift coefficient and total moment coefficient about axis \( X = X_{EA} \) for the wing in oscillation in plunge

\( C \)  
Chord

\( \bar{c}_{ld} \)  
\( \left( \partial C_{ld} / \partial \alpha \right)_d = 0 \)

\( C_{hk} \)  
Equations (3.5) and (5.5)

\( \hat{C}_{hk} \)  
Equations (4.7) and (6.5)

\( E \)  
Equation (2.29)

\( F_j (\bar{z}, \eta) \)  
Equation (5.13)
\( G \)  
Green function, Equations (2.18), (2.20), (2.22) and (2.24)

\( H \)  
Heaviside function, Equation (2.34)

\( I_D(Z, \gamma) \)  
Doublet integral, Equations (3.10), (5.8), (B.36), (B.38) and (B.40)

\( I_S(Z, \gamma) \)  
Source integral, Equations (3.11) and (5.9)

\( I_{s1}(Z, \gamma) \)  
Equations (5.10), (B.26), (B.27) and (B.28)

\( I_{s2}(Z, \gamma) \)  
Equations (5.11), (B.30), (B.31) and (B.32)

\( I_{s3}(Z, \gamma) \)  
Equations (5.12), (B.35), (B.37) and (B.39)

\( I_{w1}(\gamma) \)  
Equation (3.17)

\( I_{w2}(\gamma) \)  
Equation (3.18)

\( \kappa \)  
Reduced frequency, \( \omega \sqrt{\frac{a}{U_\infty}} \)

\( l \)  
Reference length

\( LW \)  
Truncated wake length

\( M \)  
Mach number, \( \frac{U_\infty}{a_\infty} \)

\( \bar{n} \)  
Equation (3.15)

\( \bar{N} \)  
Outward unit normal to surface \( \Sigma_a \)

\( \bar{N}_u \)  
Upper unit normal to surface \( \Sigma_w \)

\( \bar{N}_c \)  
Conormal, defined in Equation (2.37)

\( NX, NY \)  
Number of body elements in X, Y direction

\( ND \)  
Number of diaphragm elements

\( NW \)  
Number of wake elements in \( \Sigma_k \)

\( \mathbf{p} \)  
Point on the surface \( \Sigma \), Equation (2.40)

\( \mathbf{p}_* \)  
Control point, \( (X_*, Y_*, Z_*) \)

\( \bar{q} \)  
Equation (3.12)
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</table>
\( \alpha \) Angle of attack

\( \beta \) \( 1 - \frac{M^2}{1} \)

\( \Sigma \) Surface surrounding body and wake

\( \Sigma_k \) Surface element

\( \Sigma_k' \) Wake strip emanating from the trailing element

\( \Sigma_a \) Surface of aircraft

\( z_w \) Surface of wake

\( \tau \) Thickness ratio

\( \Phi \) Velocity potential, \( U_x + \Phi \)

\( \varphi \) Perturbation velocity potential

\( \phi \) \( \varphi / U_0 \)

\( \phi_p \) \( \phi \) on the diaphragm

\( \psi \) Normal derivative of \( \phi \)

\( \psi_p \) \( \psi \) on the diaphragm

\( \hat{\phi} \) Equations (2.31) and (2.39)

\( \hat{\phi} \) Equations (2.31) and (2.39)

\( \phi_k \), \( \hat{\phi}_k \) Values of \( \phi \) and \( \hat{\phi} \) at the centroid of surface element \( \Sigma_k \)

\( \psi_k \) Values of \( \psi \) at the centroid of surface element \( \Sigma_k \)

\( \Lambda \) Sweep angle

\( \omega \) Frequency of oscillation

\( \omega_h \) Equation (7.12)

\( \omega_c \) Equation (7.13)

\( \Omega \) Compressible reduced frequency, \( \omega / \alpha / \beta = \kappa M / \rho \)
\( (\cdot) \quad (\cdot)_u - (\cdot)_l \)

\( \nabla \)  Gradient in \( X, Y, Z \) space

\( \Theta \)  Super product, Equation (2.36)

\( \| \bar{a} \| \)  Supernorm of vector \( \bar{a} \), Equation (2.35)
SECTION 1
INTRODUCTION

1.1 GENESIS OF THE PROBLEM

The structural design of aircraft requires the evaluation of the aerodynamic loads. Aircraft structures are flexible and this flexibility is responsible for various types of aeroelastic phenomena. For instance, the static aerodynamic loads depend upon the geometry of the surface of the aircraft, while the geometry depends upon the aerodynamic loads. Therefore, it is necessary to evaluate these loads by an iterative process. In addition, the structural design is subject to aeroelastic constraints such as divergence and flutter. These constraints require accurate evaluations of steady and oscillatory loads. Again, the optimal design requires an iterative process, since the resizing of the structure changes the stability boundary. These iterative processes are best performed by computer analysis. This implies that considerable advantages are obtained if the mathematical modeling of the problem is general, flexible and efficient.

The method used in this work is based upon the theoretical formulation developed by Morino (References 1 and 2). A preliminary analysis is described in Reference 3.

It may be worth noting that the usual methods for the evaluation of aerodynamic loads are the computational lifting-surface (zero-thickness wing) theories. These methods are efficient and flexible but not sufficiently general for the purpose of automated design of complex aircraft configurations. In addition, several computational methods around complex aircraft configurations have already been developed. These methods are general, but usually quite cumbersome to use and always require human intervention to define the suitable types of elements to be used. In other words, it may be safely stated that none of the other methods currently available satisfies all the requirements of generality, flexibility and efficiency necessary for automated design. Furthermore, for oscillatory flow around complex configurations, only techniques based upon the doublet-lattice method exist in subsonic range (see References 4 and 5), while no other method is available in the supersonic one.
The method used here is applicable to steady and oscillatory subsonic and supersonic flows (References 6 to 10) and provides an efficient, general, and flexible aerodynamic tool to be used for the design of aircrafts. These advantages of the present method are particularly relevant in the use of automated design.

1.2 OUTLINE OF THE METHOD

The present method is mainly based on the Green function method. By applying the Green function method, the equation of the velocity potential yields a linear equation relating the velocity potential \( \varphi \), at any point, \( \vec{P} * \), in the flow field with the values of \( \varphi \) and its normal derivative on the surface \( \Sigma \), surrounding the body and the wake. The integral equation is obtained by imposing that the value of the potential at \( \vec{P} * \) approaches the value of \( \varphi \) on the surface, if \( \vec{P} * \) approaches a point on the surface.

In order to solve the integral equation, the surface of the aircraft and its wake is divided into small quadrilateral elements. Each element is replaced by a hyperboloidal surface defined by the four corner points of the element. The unknown \( \varphi \) is assumed to be constant within each element. Therefore, the integral equation reduces to a system of algebraic equations with unknown \( \varphi_k \) at the centroids of the elements. The coefficients are evaluated analytically. Once the distribution of the velocity potential is obtained, the pressure distribution, the lift coefficient as well as the generalized forces can be easily evaluated.

In Reference 3, the surface element is replaced by its tangent plane at the element centroids. The improvement obtained by using hyperboloidal element are that the formulation becomes general, flexible, simpler to use and more accurate since the continuity of the surface is maintained (see References 6 to 10).

The work presented here includes further developments with respect to the ones presented in Reference 8. In particular, it includes refinements and applications of the code SOSSA ACTS (Steady and Oscillatory Subsonic and Supersonic Aerodynamics for...
Aerospace Complex Transportation Systems). By using this program, several numerical results have been obtained. These include comparison with ones obtained by lifting surface theory, as well as results for finite thickness wings (in both subsonic and supersonic, steady and oscillatory flow) and results with wing-body configuration (in both subsonic and supersonic steady flow). Also the evaluation of generalized forces in oscillatory subsonic and supersonic flow is introduced. The numerical results presented here include the evaluation of lift and moment coefficient. Also included are a study of the role of diaphragms in supersonic flow, a study of convergence in both subsonic and supersonic, steady and oscillatory flow and a study of the effect of the truncation of the wake in oscillatory subsonic flow. Finally, a preliminary application to flutter problem is presented.

1.3 OUTLINE OF WORK

In Section 2 the theoretical formulation is introduced. In Section 3, numerical formulation as well as numerical results for steady subsonic flow are considered. In Section 4, numerical formulation and results for oscillatory subsonic flow are considered. In Section 5, numerical formulation and results for steady supersonic flow are considered. In Section 6, numerical formulations and results for oscillatory supersonic flow are considered. Preliminary flutter analysis for a rigid model with two degrees of freedom is considered in Section 7. Finally, conclusions are given in Section 8. Details of the proof of the formulation used and a list of the computer program are given in the Appendices.
2.1 INTRODUCTION

In this section, the theoretical formulation is briefly presented. Basic flow equations are considered in Subsection 2.2, while the boundary condition is considered in Subsection 2.3. Green theorems for subsonic and supersonic, steady and oscillatory flow, derived in References 1 and 2, are summarized in Subsection 2.4. The numerical formulation is considered in Subsection 2.5. The evaluation of pressure coefficient and generalized forces are considered in Subsections 2.6 and 2.7, respectively.

2.2 BASIC FLOW EQUATION

The aerodynamic forces are derived from the equations which relate the flow properties to the geometry of the lifting bodies. For many practical applications, the flow may be assumed to be inviscid, isentropic, and initially irrotational. In this case, the flow is described by the velocity potential \( \Phi \). In addition, the perturbation of the flow is assumed to be small, so that the linearized equation may be used.

The equation of the unsteady aerodynamic potential, is

\[
\alpha^2 \nabla^2 \Phi = \frac{D_c \Phi}{D \tau^2}
\]

(2.1)

where \( \alpha \) is the speed of sound, \( \nabla^2 \) is the Laplacian operator, and

\[
\frac{D_c}{D \tau} = \frac{\partial}{\partial t} + \nabla \Phi \cdot \nabla
\]

(2.2)
is the total time derivative. The subscript "c" reminds that $\nabla \Phi$ should be treated as a constant in order to obtain the second total time derivative.

Consider a frame of reference such that the undisturbed flow has velocity, $U_\omega$, in the direction of the positive x-axis. By introducing the perturbation potential $\psi$, such that

$$\Phi = U_\omega x + \psi$$

(where $\psi$ represents perturbation velocity which is assumed to be small with respect to $U_\omega$) and neglecting higher order terms, Equation (2.1) reduces to the linearized equation for the potential flow

$$\nabla^2 \psi - \frac{1}{a_\omega^2} \frac{d^2 \psi}{dt^2} = 0$$

(2.4)

where $a_\omega$ is the free-stream speed of sound, while

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_\omega \frac{\partial}{\partial x}$$

(2.5)

is the linearized total time derivative. For simplicity, the following nondimensional Prandtl-Glauert coordinates are introduced

$$x = x/\beta \lambda, \quad y = y/\lambda, \quad z = z/\lambda, \quad \phi = \psi/\alpha U_\omega \lambda$$

(2.6)

with

$$\beta = \sqrt{1 - M^2}$$

(2.7)

for subsonic flow and

$$\beta = \sqrt{M^2 - 1} = B$$

(2.8)

for supersonic flow.

2.3 BOUNDARY CONDITION

The lifting body considered here has arbitrary shape and its motion consists of
small vibrations of arbitrary nature. Thus, the surface of the body is represented in
general form by
\[ S(x, y, z, t) = 0 \] (2.9)

Then, the boundary condition on the body is given by
\[ \frac{D_{\Sigma} S}{Dt} = \frac{\partial S}{\partial t} + \nabla \Phi \cdot \nabla S = 0 \] (2.10)

By using Equation (2.3), Equation (2.10) yields the boundary condition for unsteady
flow,
\[ \nabla \Phi \cdot \nabla S = - \frac{1}{u_\infty} \left( \frac{\partial S}{\partial t} + u_\infty \frac{\partial S}{\partial x} \right) \] (2.11)

Furthermore, the boundary condition at infinity is given by \( \Phi = 0 \), because the
flow is assumed to be uniform at infinity. In the following, the analysis is limited
to steady state flow. In this case, Equation (2.11) yields the steady-state boundary
condition
\[ \nabla \Phi \cdot \eta_s = - \eta_x \] (2.12a)

Consider the subsonic case first. By using the subsonic nondimensional Prandtl-
Glauert coordinates given by Equations (2.6) and (2.7), Equation (2.12a) yields
\[ \left( \frac{\partial S}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{\partial S}{\partial y} \frac{\partial \Phi}{\partial y} + \frac{\partial S}{\partial z} \frac{\partial \Phi}{\partial z} \right) + \frac{\partial S}{\partial x} = \left( \frac{1}{\beta^2} \frac{\partial \Phi}{\partial x} + \frac{\partial S}{\partial y} \frac{\partial \Phi}{\partial y} + \frac{\partial S}{\partial z} \frac{\partial \Phi}{\partial z} \right) + \frac{1}{\beta} \frac{\partial S}{\partial x} \] (2.12b)

where \( N_x \) is the component of \( \eta \) along \( x \)-axis (direction of the undisturbed flow).
Therefore,
\[
\frac{\partial \phi}{\partial N} = -N_x \left( \frac{1}{\beta} + \frac{M^2}{1-M^2} \frac{\partial \phi}{\partial x} \right) \quad (2.13a)
\]

The second term on the right hand side is of higher order (see Subsection C.1)*. Therefore, Equation (2.13a) may be approximated as

\[
\frac{\partial \phi}{\partial N} = -\frac{N_x}{\beta} \quad (2.13b)
\]

Next, consider the steady supersonic flow. In this case, the boundary condition is still given by Equation (2.12a). By using the supersonic nondimensional Prandtl-Glauert coordinates, given by Equations (2.6) and (2.8), and following the same procedure to obtain Equation (2.12b), one obtains

\[
\frac{\partial \phi}{\partial \bar{N}^c} = -N_x \left( \frac{1}{\beta} + \frac{M^2}{M^2-1} \frac{\partial \phi}{\partial x} \right) \quad (2.14a)
\]

where \( \bar{N}^c = (-N_x', N_y', N_z') \) is the conormal derivative.

Similarly, by neglecting the second term on the right-hand side, Equation (12.14a) may be approximated as

\[
\frac{\partial \phi}{\partial \bar{N}^c} = -\frac{N_x}{\beta} \quad (2.14b)
\]

For clarity and conciseness, the boundary condition as well as the evaluation of the pressure coefficient for oscillatory flow are not derived here, but are presented in Appendix C.

2.4 GREEN FUNCTION METHOD

Observe that Equation (2.4) is a second order linear partial differential

*This is true everywhere on the surface of the aircraft except at the stagnation point.
equation with boundary conditions given by Equation (2.11) on the surface of the body and \( \varphi = 0 \) at infinity.

The definition of the Green function of Equation (2.4) is given as

\[
\nabla^2 G - \frac{1}{a_{\infty}^2} \frac{d^2 G}{d t^2} = \delta (x-x_i, y-y_i, z-z_i, t-t_i) \tag{2.15}
\]

with

\[
G = 0 \text{ at infinity} \tag{2.16}
\]

where \( \delta \) is the Dirac delta function defined by

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y, z, t) \delta (x-x_i, y-y_i, z-z_i, t-t_i) dx dy dz dt = F(x_i, y_i, z_i, t_i) \tag{2.17}
\]

The solution of \( G \) are derived for instance in Reference 1. For steady subsonic flow, \( G \) is given by

\[
G = \frac{1}{4 \pi R} \tag{2.18}
\]

with

\[
R = \left\{ (x-x_i)^2 + \beta^2 [ (y-y_i)^2 + (z-z_i)^2] \right\}^{\frac{1}{2}} \tag{2.19}
\]

For oscillatory subsonic flow, \( G \) is given by

\[
G = -\frac{1}{4 \pi R} \int (t_1 - t + T) \tag{2.20}
\]

where \( \int (t_1 - t + T) \) is the usual Dirac delta function and

\[
T = \frac{1}{a_{\infty} \beta^2} \left( R - M (x-x_i) \right) \tag{2.21}
\]

with \( R \) given by Equation (2.19) and \( \beta = \sqrt{1-M^2} \)

For steady supersonic flow, \( G \) is given by

\[
G = \frac{1}{2} \tag{2.22}
\]
For oscillatory supersonic flow, \( G \) is given by

\[
G = - \frac{1}{2 \pi R}
\]

with

\[
R = \left\{ (x-x_i)^2 - \beta^2 [(y-y_i)^2 + (z-z_i)^2] \right\}^{1/2}
\]

(2.23)

For oscillatory supersonic flow, \( G \) is given by

\[
G = - \frac{1}{4 \pi R} \left( \delta(t_i - t + T) + \delta(t_i - t + t_i) \right)
\]

(2.24)

where

\[
T^2 = \frac{1}{\alpha^2 \beta^2} \left( M (x-x_i) \pm R \right)
\]

(2.25)

with \( R \) given by Equation (2.23) and \( \beta = \sqrt{M^2 - 1} \)

2.4.1 Green Theorem for Steady Subsonic Flow

By applying the Green function method and using Equation (2.18) the linearized equation of the steady subsonic aerodynamic potential flow can be rewritten as

\[
4 \pi E \mathbf{G} \mathbf{F} = - \iint \frac{\partial \phi}{\partial \mathbf{N}} \frac{1}{R} d \Sigma_A + \iint \phi \frac{\partial}{\partial \mathbf{N}} \left( \frac{1}{R} \right) d \Sigma_A
\]

\[
+ \iint \Delta \phi \frac{\partial}{\partial N_u} \left( \frac{1}{R} \right) d \Sigma_W
\]

(2.26)

with nondimensional Prandtl-Glauert coordinates. Where \( \Sigma_A \) and \( \Sigma_W \) are the surfaces of the aircraft and the wake in \( X, Y, Z \) coordinate system, while \( \mathbf{N} \) is the outward unit normal to \( \Sigma_A \), \( \mathbf{N}_u \) is the upper unit normal to \( \Sigma_W \) and \( R \) is the vector pointing from any point, \( \mathbf{P}_* = (X, Y, Z) \), in the flow field to the dummy point on surface \( \Sigma_A \) or \( \Sigma_W \),
Moreover

\[ R = |\vec{R}| = \left( (X - X^*)^2 + (Y - Y^*)^2 + (Z - Z^*)^2 \right)^{1/2} \]  \(2.28\)

while \( E \) is the domain function, which is defined as (see Reference 1)

\[
E(\vec{r}_A) = \begin{cases} 
1 & \text{outside } \Sigma_A \\
1/2 & \text{on } \Sigma_A \\
0 & \text{inside } \Sigma_A
\end{cases}
\]  \(2.29\)

2.4.2 Green Theorem for Oscillatory Subsonic Flow

By applying Green function method and using Equation (2.20), the linearized equation of the oscillatory subsonic aerodynamic potential flow yields the Green theorem for oscillatory subsonic flow as

\[
4\pi E(\vec{r}_*) \hat{\Phi}(\vec{r}_*) = -\iint_{\Sigma_A} \phi N \left( \frac{e^{-i\alpha R}}{R} \right) d\Sigma_A + \iiint_{\Sigma_w} \phi N_u \left( \frac{e^{-i\alpha R}}{R} \right) d\Sigma_w
\]  \(2.30\)

where \( X, Y, Z, \) and \( \Sigma_A, \Sigma_w, N, N_u, R, E \) are defined in Subsection 2.4.1. while \( \hat{\Phi} \) is such that

\[
\phi(X, Y, Z, T) = \hat{\phi}(X, Y, Z) e^{i\alpha T}
\]  \(2.31\)

with

\[
T = a_0 \beta t / \mu \\
\Omega = \omega / a_0 \beta = KM / \beta
\]  \(2.32\)

where \( \omega \) is natural frequency and \( K \) is the dimensionless reduced frequency, \( K = \omega \delta / U_\infty \).
2.4.3 Green Theorem for Steady Supersonic Flow

By applying the Green function method and using Equation (2.22), the linearized equation of the steady supersonic aerodynamic potential flow can be rewritten as

\[
2\pi E(\tilde{r}_k)\Phi(\tilde{r}_k) = -\oint_{\Sigma A} \frac{\partial \Phi}{\partial N^c} \frac{H}{||R||} d \Sigma_A + \oint_{\Sigma A} \frac{\partial \Phi}{\partial N^c} \left( \frac{H}{||R||} \right) d \Sigma_A
\]

(2.33)

where

\[
H = 1 \quad \text{for} \quad X - X > \left( (Y - Y)^2 + (Z - Z)^2 \right)^{1/2}
\]

= 0 \quad \text{for} \quad X - X \leq \left( (Y - Y)^2 + (Z - Z)^2 \right)^{1/2}

(2.34)

II R II is the "supernorm" of vector R and is defined by

\[
||R|| = \sqrt{\bar{R} \circ \bar{R}}
\]

(2.35)

with the "super-product," \( \circ \), defined by

\[
\bar{a} \circ \bar{b} = a_x b_x - a_y b_y - a_z b_z
\]

(2.36)

and the conormal derivative \( \partial/\partial N^c \) is given by

\[
\frac{\partial}{\partial N^c} \equiv -N_x \frac{\partial}{\partial x} + N_y \frac{\partial}{\partial y} + N_z \frac{\partial}{\partial z} = -N \circ \nabla
\]

(2.37)

while \( \bar{r} \) and \( E \) are defined in Subsection 2.4.1.

2.4.4 Green Theorem for Oscillatory Supersonic Flow

By applying the Green function method and using Equation (2.24) the linearized equation for the oscillatory supersonic aerodynamic potential flow yields the Green Theorem for oscillatory supersonic flow.
\[ 2 \pi \int \overline{p} \cdot \Phi (\overline{p}) = - \frac{\hat{F}}{2} \frac{\partial \Phi}{\partial N} \frac{H}{\| R \|} \cos \alpha \ \| R \| \ d \Sigma \]

\[ + \frac{\hat{F}}{2} \frac{\partial \Phi}{\partial N} (\frac{H}{R} \cos \alpha \ \| R \|) \ d \Sigma \quad (2.38) \]

where \( \hat{F} \) is such that

\[ \Phi (x, y, z, T) = \widehat{\Phi} (x, y, z) e^{i \omega T} = \widehat{\Phi} (x, y, z) e^{i \omega (cT - Mx)} \quad (2.39) \]

while the other parameters are defined in Subsections 2.4.1 and 2.4.3.

### 2.5 NUMERICAL FORMULATION

The integral equations given in Equations (2.26), (2.30), (2.33) and (2.38) can be solved by replacing the surface of the aircraft and wake by a finite number of quadrilateral hyperboloidal elements. Thus, the integral over \( \Sigma \) becomes a summation of a finite number of integrals over hyperboloidal surfaces which represent the surface of each element on \( \Sigma \).

The general expression for the hyperboloidal surface is

\[ \overline{p}(\xi, \eta) = \overline{p}_C + \xi \overline{p}_1 + \eta \overline{p}_2 + \xi \eta \overline{p}_3 \quad (2.40) \]

where \( \overline{p}_C, \overline{p}_1, \overline{p}_2 \) and \( \overline{p}_3 \) are defined in terms of the four corner position vectors, \( \overline{p} (1,1), \overline{p} (1,-1), \overline{p} (-1,1) \) and \( \overline{p} (-1,-1) \).

\[ \overline{p}_C = \frac{1}{4} \left( \overline{p} (1,1) + \overline{p} (1,-1) + \overline{p} (-1,1) + \overline{p} (-1,-1) \right) \]

\[ \overline{p}_1 = \frac{1}{4} \left( \overline{p} (1,1) + \overline{p} (1,-1) - \overline{p} (-1,1) - \overline{p} (-1,-1) \right) \]

\[ \overline{p}_2 = \frac{1}{4} \left( \overline{p} (1,1) - \overline{p} (1,-1) + \overline{p} (-1,1) - \overline{p} (-1,-1) \right) \]

\[ \overline{p}_3 = \frac{1}{4} \left( \overline{p} (1,1) - \overline{p} (1,-1) - \overline{p} (-1,1) + \overline{p} (-1,-1) \right) \quad (2.41) \]
Assume that the values of the velocity potential and its normal derivative within each element are constant. Then the integrand of each surface integral become purely a function of the vector, $\mathbf{p}_e$ and the geometry of the surface element on $\Sigma$.

By using Equation (2.40), the analytical expression for each element-integral can be obtained. Finally, a system of linear equation relate computed coefficients to the unknown, $\phi_k$, at the centroid of element, $Z_k$. The system is solved for $\phi_k$ by using the Gaussian elimination method.

2.6 PRESSURE AND PRESSURE COEFFICIENT

The pressure on the aircraft is evaluated from the linearized Bernoulli

Theorem

\[ p - p_\infty = -p_\infty \left( \frac{\partial \phi}{\partial t} + U_\infty \frac{\partial \phi}{\partial \chi} \right) \quad (2.42a) \]

Therefore, the pressure coefficient is

\[ C_p = (p - p_\infty) / \frac{1}{2} \rho_\infty U_\infty^2 = - \frac{2}{U_\infty^2} \left( \frac{\partial \phi}{\partial t} + U_\infty \frac{\partial \phi}{\partial \chi} \right) \]

\[ = - \frac{2}{U_\infty^2} \frac{d \phi}{dt} \quad (2.42b) \]

For steady flow, by using nondimensional Prandtl-Glauert coordinates given in Equation (2.6), Equation (2.42b) is reduced to

\[ C_p = - 2 \frac{\partial \phi}{\partial \chi} \quad (2.43) \]

For oscillatory flow, details on the calculation of $C_p$ are given in Appendix C.

For all the results presented here, the pressure coefficient is evaluated by finite difference method.
2.7 GENERALIZED FORCES

The evaluation of the generalized forces, used for the flutter analysis, is presented in this section. The generalized forces are defined as

$$Q_h = - \iiint \mathbf{p} \, \mathbf{n} \cdot \mathbf{U}_h \, d\Sigma$$  \hspace{1cm} (2.44)

where $\mathbf{p} \, \mathbf{n}$ is the force acting on the surface of the body, and $\mathbf{U}_h$ is a prescribed shape-function (mode). Two rigid body modes are considered in this section: they correspond to the lift force and the pitch moment. For the lift force,

$$\mathbf{U}_h = \mathbf{R}$$

therefore,

$$L = - \iiint \mathbf{p} \, n_z \, d\Sigma$$

$$= \int \int (p_x - p_y) \, dx \, dy$$  \hspace{1cm} (2.46)

where $n_z$ is the $z$ component of the unit surface normal. The subscript "X" and "n" refer to the lower and upper elements. For a thin wing, $\overline{C_L}$ is evaluated as follows (see Equation C.31)

$$\overline{C_L} = - \frac{1}{A} \int \int_A \bar{C}_p \, dx \, dy$$

$$= + 2 \frac{L}{A} \int_{b_{1/2}}^{b_2} dy \int_{X_{LE}}^{X_{TE}} \frac{\partial}{\partial X} (\bar{\phi}) \, dx + \frac{2K_i}{A} \int \int_{\Sigma_A} (\bar{\phi}) \, dx \, dy$$

$$= \frac{2L}{A} \int_{b_{1/2}}^{b_2} (\bar{\phi})_{X_{TE}} \, dy + \frac{2K_i}{A} \int \int_{\Sigma_A} (\bar{\phi}) \, dx \, dy$$

$$= \frac{2L}{A} \sum_{j=1}^{N_f} (\bar{\phi})_{X_{TE}} J_j (\Delta Y_j) + \frac{2K_i}{A} \sum_{h=1}^{N_f} \sum_{j=1}^{N_f} (\phi_{hj}) A_{hj}$$  \hspace{1cm} (2.47)
where
\[ \Delta \mathcal{C} = \mathcal{C}_M - \mathcal{C}_L \]  \hspace{1cm} (2.48)

and \( A \) is the total projected area of the wing on \( x-y \) plane, \( L \) is the reference length, \( X_{LE} \) and \( X_{TE} \) are the \( x \)-component of vectors pointing to the leading edge and the trailing edge.

For the pitch moment, the mode is given by
\[ \bar{U}_h = - (\bar{z} - \bar{z}_M) \mathbf{I} + (\mathbf{x} - \mathbf{x}_M) \mathbf{K} \]  \hspace{1cm} (2.49)

and corresponds to a rigid body rotation (around the axis \( x = x_M, z = z_M \)) which is positive if the leading edge moves downward. Therefore,
\[
\bar{m} = \iint \rho \bar{n} \cdot \left[ -(\bar{z} - \bar{z}_M) \mathbf{I} + (\mathbf{x} - \mathbf{x}_M) \mathbf{K} \right] d\Sigma
\]
\[
= - \iint \rho \left[ -(\bar{z} - \bar{z}_M) n_x + (\mathbf{x} - \mathbf{x}_M) n_y \right] d\Sigma
\]
\[ \equiv \iint (C_{pL} - C_{pu}) (\mathbf{x} - \mathbf{x}_M) d\Sigma \]  \hspace{1cm} (2.50)

Note that the term with \( p(z - z_M)n_x \) is negligible because both \( z - z_M \) and \( n_x \) are small. For a thin wing the pitch moment coefficient is evaluated as follows (see Equation C.31):

\[
\bar{C}_M = + \frac{1}{\Delta L} \iint_{\Sigma_A} \bar{C}_p (\mathbf{x} - \mathbf{x}_M) d\Sigma
\]
\[ = - \frac{2}{\Delta L} \int_{B} \int_{B} \left[ \frac{\partial}{\partial x} \left( \Delta \phi \right) + \Delta K (\Delta \phi) (\mathbf{x} - \mathbf{x}_M) \right] d\Sigma
\]
\[ = - \frac{2}{\Delta L} \int_{B} \int_{B} \left\{ \frac{\partial}{\partial x} \left( \Delta \phi \right) X_{LE} (\mathbf{x} - \mathbf{x}_M) + \frac{X_{TE}}{X_{LE}} \right\} \]
\[
2 - 13 \\
= +2 \frac{1}{A} \sum_{j=1}^{NY} (\Delta \phi)_j (x_{te} - x_m)_j (\Delta y)_j \\
+ 2 \frac{1}{A} \sum_{h=1}^{nx} \sum_{j=1}^{ny} (\Delta \phi)_{hj} \left[ 1 - K i \frac{x_c - x_m}{2} \right] A_{hj} 
\]
SECTION 3
STEADY SUBSONIC FLOW

3.1 INTRODUCTION

In Subsection 3.2, the numerical formulation for steady subsonic flow, derived in Reference 6, is outlined. For completeness, the proof of this formulation is given in Appendix A. Numerical results are presented in Subsection 3.4.

3.2 NUMERICAL FORMULATION

Consider the Green theorem for steady subsonic flow. According to Equation (2.26)

\[ 4 \pi E \left( \phi \right) \left( \phi \right) = - \oint_{\Sigma} \frac{\partial \phi}{\partial N} \left( \frac{1}{R} \right) d \Sigma + \oint_{\Sigma} \phi \left( \frac{\partial}{\partial N} \left( \frac{1}{R} \right) \right) d \Sigma 
\]

with boundary condition, given by Equation (2.13b) as

\[ \frac{\partial \phi}{\partial N} = - N/X \beta \]

By imposing that the value of the potential at \( P^* \) approaches that at a point \( P \) on the surface \( \Sigma \), if \( P^* \) approaches \( P \), the value of \( E \left( P^* \right) \) on the surface is found to be 1/2 (Reference 1), and an integral equation relating the potential on the surface \( \Sigma \) to its normal derivative is obtained.

Divide the surface \( \Sigma \) into \( N \) small quadrilateral elements, \( \Sigma_k \), assume that the values of \( \phi \) and \( \frac{\partial \phi}{\partial N} \) are constant within each element, and impose that
Equation (3.1) is satisfied at the centroid, $\overline{P_h}$, of each element $\Sigma_h$. This yields $N$ simultaneous linear equations, which can be expressed in the following matrix form:

$$\begin{align*}
\delta_{hK} - C_{hK} - W_{hk} \left\{ \phi_k \right\} &= \left\{ b_{hK} \right\} \left\{ \frac{\partial \phi}{\partial N} \right\}_k \\
(3.3)
\end{align*}$$

where $\delta_{hK}$ is the Kronecker delta

$$C_{hK} = \left[ \frac{1}{2\pi} \int_{\Sigma_k} \frac{\partial}{\partial N} \left( \frac{1}{R} \right) d \Sigma_k \right]_{\overline{P}_h} = \overline{P}_h (3.4)$$

$$b_{hK} = \left[ -\frac{1}{2\pi} \int_{\Sigma_k} \frac{1}{R} d \Sigma_k \right]_{\overline{P}_h} = \overline{P}_h (3.4)$$

The coefficients $W_{hk}$ represent the influence of the wake. These are obtained as follows:\nthe wake is assumed to be composed of straight vortex lines emanating from the trailing edge; the wake surface is divided into strips. The coefficient $W_{hk} = 0$, if the element $\Sigma_k$ is not in contact with the trailing edge, while for the elements $\Sigma_k$ in contact with the trailing edge

$$W_{hk} = W_{hk}^{(TE)} (3.6)$$

with

$$W_{hk}^{(TE)} = \left[ \pm \frac{1}{2\pi} \int_{\Sigma_k} \frac{\partial}{\partial N} \left( \frac{1}{R} \right) d \Sigma_k \right]_{\overline{P}_h} = \overline{P}_h (3.7)$$

3-2
where $\Sigma'_k$ is the strip of the wake, bounded by two streamlines emanating from the element $\Sigma_k$. The upper (lower) sign holds for point $P_h$ on the upper (lower) surface of the aircraft.

Since no pressure difference can exist on the wake, $\Delta \phi$ of the wake is constant along a streamline emanating from the trailing edge, and therefore, equal to $\Delta \phi$ at the trailing edge, or approximately, $\Delta \phi$ at the centroids of the elements adjacent to the trailing edge. Therefore, in deriving Equation (3.3), the values of $\Delta \phi$ on the wake are approximated with the values of $\Delta \phi$ at the centroids of the elements adjacent to the trailing edge.

### 3.3 ANALYTICAL EXPRESSIONS OF $C_{hk}$, $b_{hk}$, and $W_{hk}$

The integrals in Equations (3.4), (3.5) and (3.7) are evaluated analytically by approximating the surface of each element with a hyperboloidal surface described in Subsection 2.5. The derivations are given in Reference 6 and are outlined here in Appendix A. The expressions for $C_{hk}$ and $b_{hk}$ are given by

\[ C_{hk} = I_p(1, 1) - I_p(1, -1) - I_p(-1, 1) + I_p(-1, -1) \]  
\[ b_{hk} = I_s(1, 1) - I_s(1, -1) - I_s(-1, 1) + I_s(-1, -1) \]  

with

\[ I_p(\xi, \eta) = \frac{1}{2\pi} \tan^{-1} \left( \frac{-\bar{g} \cdot \bar{a}_4 \cdot \bar{g} \cdot \bar{a}_2}{|\bar{g}| |\bar{g} \cdot \bar{a}_x \bar{a}_z|} \right) \]  
\[ \text{(3.10)} \]  

\[ \text{3-3} \]
(The subscript \( p \) reminds that the principal value of \( \tan^{-1} \) from \(-\frac{\pi}{2} \) to \( \frac{\pi}{2} \) should be considered) and

\[
I_3 (\xi, \eta) = -\frac{1}{2\pi} \left\{ -\frac{q \times \overline{a}_1 \cdot \eta}{|\overline{a}_1|} \ln \left| \frac{q \times \overline{a}_1 + q \cdot \overline{a}_2}{|q \times \overline{a}_1 + q \cdot \overline{a}_2|} \right| 
+ \frac{q \times \overline{a}_2 \cdot n}{|\overline{a}_2|} \ln \left| \frac{q \times \overline{a}_1 + q \cdot \overline{a}_2}{|q \times \overline{a}_1 + q \cdot \overline{a}_2|} \right| \right. 
+ \left. \frac{q \cdot n \tan^{-1} \left( \frac{-q \times \overline{a}_1 \cdot \overline{q} \times \overline{a}_2}{|q \times \overline{a}_1 + q \cdot \overline{a}_2|} \right)}{q \times \overline{a}_1 \times \overline{a}_2} \right\} \tag{3.11}
\]

where

\[
\overline{q} (\xi, \eta) = \overline{p}_0 + \xi \overline{p}_1 + \eta \overline{p}_2 + \xi \eta \overline{p}_3 - \overline{p}_n \tag{3.12}
\]

\[
\overline{a}_1 = \overline{p}_1 + \eta \overline{p}_2 \tag{3.13}
\]

\[
\overline{a}_2 = \overline{p}_2 + \xi \overline{p}_3 \tag{3.14}
\]

\[
\overline{n} = \frac{\overline{a}_1 \times \overline{a}_2}{|\overline{a}_1 \times \overline{a}_2|} \tag{3.15}
\]

Furthermore, the expression for \( \mathcal{W}_{hk} \) may be obtained by taking the limit of \( I_D \) in Equation (3.10) by letting \(|\overline{a}_1|\) go to infinity. The derivation is given in Appendix A.4. The expression of \( \mathcal{W}_{hk}^{(\text{cr})} \) is then given as

\[
\mathcal{W}_{hk}^{(\text{cr})} = I_{\omega_1} (1) - I_{\omega_2} (1) - I_{\omega_1} (-1) + I_{\omega_2} (-1) \tag{3.16}
\]
In the above equations, \( P^+ \) and \( P^- \) are the two corner points of the element, \( \Sigma_h \), which are on the trailing edge. (See Figure 2b). Equation (3.3) can be solved numerically to yield the values of \( \phi_K \). Once the \( \phi_K \) are obtained, the pressure can be evaluated from the linearized Bernoulli Theorem, given by Equation (2.43).

### 3.4 NUMERICAL RESULTS

Numerical results obtained for steady subsonic flow are presented here. In Subsection 3.4.1, the results are compared with those obtained by the lifting surface theory. Comparison with experimental results are considered in Subsection 3.4.2. Analysis of convergence is considered in Subsection 3.4.3. The evaluation of the lift coefficient is given in Subsection 3.4.4. Finally, the pressure distribution coefficient of a wing-body configuration is presented in Subsection 3.4.5.
3.4.1 Comparison With Lifting Surface Theories

Comparison with various lifting surface theories are presented in Figures 4–7. Figure 4 shows the lift distribution per unit angle of attack, $C_\alpha$, for a rectangular wing with $AR = 1$ and $M = .2$. The results, obtained with $NX = NY = 7$ are compared with the ones of Cunningham and Kulakowski and Haskell. Figure 5 shows the distribution of $C_\alpha$ for a tapered swept wing with aspect ratio $AR = 3$, taper ratio $TR = .5$, $\alpha = 5^\circ$, sweep angle along quarter chord $\alpha_{1/4} = 45^\circ$ and Mach number $M = .8$. The results obtained with $NX = NY = 7$ are compared with ones of Cunningham and Kolbe and Boltz. Figure 6 shows the distribution of the section lift coefficient per unit angle of attack, $C_{L\alpha}$, for a rectangular wing with aspect ratio $AR = 4$ and Mach number $M = .507$. The result, obtained with $NX = NY = 7$ and 10, are compared with the ones by Yates.

All these results are in good agreement with the existing one. Thickness ratio used in evaluating the above results is chosen to be 0.1%. The reasons to choose this value is that, as shown in Reference 3, the solution with $T = 0.1\%$ is a good approximation for the zero thickness solution and that values of thickness ratio smaller than 0.01% may cause elimination of significant figures.

3.4.2 Comparison With Experimental Results

Figures 7, 8a and 8b present a comparison with the experimental results of Lessing, Troutman and Menees. The results are relative to a rectangular wing with aspect ratio $AR = 3$. The airfoil consists of a biconvex circular arc section with 5 percent thickness and with sharp leading and trailing edges. Figure 7 shows the pressure
distribution on the upper and the lower surfaces, $C_{pu}$ and $C_{pl}$, versus $x/c$ at $2y/b = 0.0, 0.5, 0.7$ and $0.9$ for the above mentioned wing with $\alpha = 0^\circ$ and $M = 0.24$.

Figure 8a shows the lift distribution, $C_l$, versus $x/c$ at $2y/b = 0, 0.5, 0.7$ and $0.9$ for the above mentioned wing with $\alpha = 5^\circ$ and $M = 0.24$, while Figure 8b shows the pressure distribution on the upper and the lower surface, $C_{pu}$ and $C_{pl}$, versus $x/c$ at $2y/b = 0, 0.5, 0.7$ and $0.9$ for the above mentioned wing with $\alpha = 5^\circ$ and $M = 0.24$.

It should be mentioned that the results for Figure 8a are obtained by taking the advantage of antisymmetry. If the velocity potential, $\phi$, is separated into two parts, symmetric part, $\phi_s$, and antisymmetric part, $\phi_{aS}$, then the difference of the velocity potential on the upper and lower surfaces of the wing is twice the value of $\phi_{aS}$ while the velocity potential on the upper surface of the wing is $\phi_u = \phi_s + \phi_{aS}$ and the velocity potential on the lower surface is $\phi_l = \phi_s - \phi_{aS}$. The value of $\phi_{aS}$ can be obtained by taking advantage of antisymmetry while the value of $\phi_s$ can be obtained by taking advantage of symmetry. This is mainly because the doublet, source and wake integrals are nearly independent of angle of attack, $\alpha$. Therefore $C_{hk}$, $b_{hk}$ and $\omega_{hk}$ can be evaluated at $\alpha = 0$, while the only $\alpha$-dependent term in Equation (3.3) is the linearized boundary term, $\frac{2\phi}{\sigma N}$, which can be separated into symmetric and antisymmetric as follows: Set

$$\frac{2\phi}{\sigma N} = -\eta_x = -(n_k)_{\alpha=0} \cos \alpha + (n_2)_{\alpha=0} \sin \alpha$$

$$(3.22)$$

and as mentioned before,

$$\phi = \phi_s \pm \phi_{aS}$$

$(3.23)$
where \( n_1 \) and \( n_2 \) are the x-component and z-component of the surface unit normal.
The upper (or lower) sign holds for the upper (or lower) surface. A similar example of the symmetric and antisymmetric problem is given in Section B.5. It may be worth noting that the results for Figure 8b are obtained by directly solving \( \phi \), or with \( C_{hk} \), \( b_{hk} \) and \( \omega_{hk} \) evaluated at \( \alpha = 5^\circ \). However, the lift coefficient obtained by subtracting \( C_{pl} \) from \( C_{pu} \) is consistent with results obtained for Figure 8a.

### 3.4.3 Analysis of Convergence

Figure 9 presents a study of convergence of the values of \( \phi_k \) in terms of the number of elements. The wing is the same as that of Figure 7 with \( \alpha = 0^\circ \) and \( M = 0.24 \). In the results shown in Figure 9, there is a sharp change of the values near the leading edge and the wing tip, therefore, in order to increase the rate of convergence, smaller elements along the leading edge and the wing tip are used. For this purpose, the coordinates of the nodes \((x, y)\) are defined by

\[
X_i = \left( \frac{1}{NX} \right)^2 \quad i = 0, 1, 2 \ldots \quad NX
\]

\[
Y_j = \left( 1 - \frac{1}{NY} \right)^2 \quad j = 0, 1, 2 \ldots \quad NY
\]

Result shown is the distribution of \( \phi \) along \( y = 0 \). It may be noted that 100 elements on the whole wing (i.e., \( NX = NY = 5 \)) are sufficient for convergence.

### 3.4.4 Total Lift Coefficient

The total lift coefficient for steady flow is evaluated from Equation (2.47) without the imaginary terms, i.e.,
\[ C_L = 2 \frac{L}{A} \sum_{j=1}^{NY} \left( \left( \phi_L - \phi_U \right) \chi_{TE} \right) \left( \Delta y \right) \]  \hspace{1cm} (3.25)

where \( \phi_L \) and \( \phi_U \) are the extrapolated values of the velocity potential along the lower and upper trailing edges. Figure 10 shows the values of \( C_L \) of a triangular wing with different aspect ratios in incompressible flow. The result obtained at \( \tau = 0.1\% \) are compared with the analytical ones.\(^{16}\)

3.4.5 Wing-Body Configurations

In Figure 11, results for the pressure distribution coefficient of a wing-body configuration in steady subsonic flow are compared, with the results presented by Labruijere, Loeve and Slooff.\(^{17}\) The results were obtained for \( M = 0 \) and a rectangular midpositioned wing with chord \( c = 1 \), span \( b = 6 \), thickness ratio \( \tau = 0.9 \), and angle of attack \( \alpha = 6^\circ \). The body is at zero angle of attack and is composed of a forebody with length \( L_A = 2 \) and radius

\[ \gamma = 0.5 - 0.125 (\chi - \chi_{LE}) \]  \hspace{1cm} (3.26)

a midsection of length \( L_M = 1 \) and radius \( r = 0.5 \) and an aftbody of length \( L_A = 9 \), and radius \( r = 0.5 \). It is worth note that the extra long aftbody is used to simulate the wake emanating from the circular fuselage. The number of elements is 200 on the whole configuration (\( NX = 5, NY = 5 \) on the wing, \( NX = 2, NY = 3 \) on the forebody, \( NX = 5, NY = 3 \) on the midsection, and \( NX = NY = 3 \) on the aftbody). Figure 11b presents the distribution of \( \phi_L - \phi_U \) along three circumferential stations (see Figure 11b) for the same wing-body configuration.
4.1 INTRODUCTION

In subsection 4.2, the numerical formulations for oscillatory subsonic flow are considered. The boundary condition and the pressure coefficient are considered in Appendix C. In subsection 4.3, numerical results obtained are presented.

4.2 NUMERICAL FORMULATION

Consider the Green Theorem for oscillatory subsonic flow, given by Eq. (2.30)

$$4\pi \mathbf{E}(\mathbf{P}_*) \cdot \mathbf{P}(\mathbf{P}_*) = - \iint_{\Sigma_H} \frac{\partial}{\partial n} \left( \frac{e^{-j\alpha R}}{R} \right) d\Sigma_H + \iint_{\Sigma_H} \frac{\partial}{\partial n} \left( \frac{e^{-j\alpha R}}{R} \right) d\Sigma_H$$

with boundary condition given by Equation (C.24)

$$\frac{\partial \mathbf{P}}{\partial n} = N_{\Sigma} (i k \mathbf{z} + \frac{1}{\rho} \frac{\partial \mathbf{z}}{\partial X}) e^{-j\alpha M_X}$$

where \( \mathbf{z}(X,Y) \) is the vibration mode defined by Equation (C.18)

By imposing that the value of the velocity potential at \( \mathbf{P}_* \) approaches that at a point, \( \mathbf{P} \), on the surface \( \Sigma_H \), if \( \mathbf{P}_* \) approaches \( \mathbf{P} \), the value of \( \mathbf{E}(\mathbf{P}_*) \) is found to be 1/2 (Reference 1), and an integral equation relating the potential on the surface \( \Sigma_H \) to its normal derivative is obtained. Noting that no pressure difference can exist on the wake, and using Equation (C.32)

$$\mathbf{C}_p = - \frac{2}{\rho} \varepsilon^{-ik\lambda_x} \frac{\partial}{\partial X} \left( \mathbf{P} e^{ik\lambda_y} \right)$$
one can conclude that \((\Delta \hat{\phi}) e^{ikY_R}\) is constant along a streamline emanating from the trailing edge and equal to the value at the trailing edge

\[
(\Delta \hat{\phi}) e^{ikY_R} = (\Delta \hat{\phi}_{\text{TE}}) e^{ikY_R} \approx (\Delta \hat{\phi}_K) e^{ikY_R}
\]

(4.4)

In Equation 4.4, the value of \(\hat{\phi}_{\text{TE}} e^{ikY_R}\) is approximated with this value \(\Delta \phi e^{ikY_R}\) at the centroid of the element, \(\Sigma_K\), adjacent to the trailing edge. In Equation 4.4, \(X_K\) represents the \(X\) component of the position vector of the centroid of elements, \(\Sigma_K\), adjacent to the trailing edge where the wake starts to develop. Using Equation (4.4) and applying the same procedures used for the steady flow, one obtains

\[
\left\{ \delta_{hk} - \hat{C}_{hk} - \hat{W}_{hk} \right\} \left\{ \hat{\phi}_K \right\} = \left\{ \hat{B}_{hk} \right\} \left\{ \left( \frac{\partial \phi}{\partial N} \right)_K \right\}
\]

(4.5)

where

\[
\hat{C}_{hk} = \left[ \frac{-1}{2\pi} \oint_{\Sigma_K} \frac{\partial}{\partial N} \left( \frac{e^{iAR}}{R} \right) d\Sigma_K \right] \hat{P}_{h} - \hat{P}_h
\]

(4.6)

\[
\hat{B}_{hk} = \left[ \frac{-1}{2\pi} \oint_{\Sigma_K} \frac{e^{-iAR}}{R} d\Sigma_K \right] \hat{P}_{h} - \hat{P}_h
\]

(4.7)

and for elements not adjacent to the trailing edge, \(\hat{W}_{hk}=0\), otherwise,

\[
\hat{W}_{hk} = \left[ \frac{1}{2\pi} \oint_{\Sigma_K'} \frac{e^{-iAR}}{R} \frac{\partial}{\partial N} \left( \frac{e^{iAR}}{R} \right) d\Sigma_K' \right] \hat{P}_{h} - \hat{P}_h
\]

(4.8)

where \(X_K\) is defined previously. \(\Sigma_K'\) is the strip of the wake, bounded by two streamlines emanating from the element \(\Sigma_K\). The upper (or lower) sign holds for point, \(\hat{P}_h\), on the upper (or lower) surface of the aircraft. Using the following equation

\[
\frac{\partial}{\partial N} \left( \frac{e^{iAR}}{R} \right) = \frac{1}{R^2} \left( 1 + i \omega R \right) \frac{\partial R}{\partial N} = \left( 1 + i \omega R \right) e^{-iAR} \frac{\partial}{\partial N} \left( \frac{1}{R} \right)
\]

(4.9)
the approximate evaluations of \( \hat{C}_{hK} \) and \( \hat{b}_{hK} \) are obtained by replacing the exponential term with its value at the centroid of the element. This yields

\[
\hat{C}_{hK} = e^{-i\alpha R_c} (1 + i\alpha R_c) C_{hK}
\] (4.10)

\[
\hat{b}_{hK} = e^{-i\alpha R_c} b_{hK}
\] (4.11)

where \( C_{hK} \) and \( b_{hK} \) are given by Equations (3.8) and (3.9) whereas \( R_c \) is the distance between the centroids of elements \( \Sigma_h \) and \( \Sigma_K \). For the wake contribution one obtains

\[
\hat{W}_{hK} = 0 \quad \text{if the element } \Sigma_k \text{ is not in contact with the trailing edge, while, for the element } \Sigma_k \text{ in contact with the trailing edge, dividing the strip } \Sigma_k \text{ into small elements, one obtains}
\]

\[
\hat{W}_{hK} = \sum_{j=1}^{NW} e^{-i(k(x_j-x_K)+NR_c)\beta} (1 + i\alpha R_c) C_{hj}
\] (4.12)

where, \( j \) refers to the \( j \)th wake element emanating from the element \( \Sigma_k \), \( C_{hj} \) is given by Equation (3.8) \( R_c \) is the distance between the centroids of element, \( \Sigma_h \), and the \( i \)th wake element; \( NW \) is the number of elements of the truncated wake emanating from \( \Sigma_K \). (A more refined analysis is obtained by taking the limit of the present one when \( NW \) tends to infinity).

4.3 NUMERICAL RESULT

Results for oscillatory supersonic flow are presented here. In Subsection 4.3.1 the results are compared with the experimental ones. In Subsection 4.3.2 the analysis of convergence is considered. In oscillatory flow, instead of using semi-infinite wake, truncated wake is used. Therefore a study of the wake length \( LW \) as well as the
analysis of the convergence due to the number of wake elements are considered in Subsection 4.3.3.

4.3.1. Comparison with Experimental Results

Figure 12 presents a comparison with the experimental results of Lessing, Troutman and Menees. The results are relating to a rectangular wing with aspect ratio AR = 3. The airfoil consists of a biconvex circular-arc section, 5% thickness with sharp leading and trailing edges. The wing considered has the same geometry as the wing considered for Figures 7 and 8 and is oscillating in bending mode described by

\[ Z = 1.8043 \left| \frac{2y}{b} \right| + 1.70255 \left( \frac{2y}{b} \right)^2 - 1.13688 \left( \frac{2y}{b} \right)^3 + 2.12538 \left( \frac{2y}{b} \right)^4 \]  

(4.13)

with

\[ \psi = \frac{\omega c}{2 \ U_\infty} = 0.47 \quad \text{and} \quad M = 0.24. \]

The values showed are the real and imaginary part of the distribution of \( C_L \) along \( \frac{2y}{b} = 0 \) and \( \frac{2y}{b} = 0.5 \). They are obtained with \( NX = NY = 7, NW = 20 \), and \( LW = 2 \). The node coordinates are given by Equation (3.22).

4.3.2. Analysis of Convergence

A convergence study of the problem considered in above subsection are presented in Figures 13a and 13b. The values showed are the real and imaginary parts of the distribution of \( C_L \) along \( \frac{2y}{b} = 0.5 \) and \( 2y/b = .1328 \)
respectively. They are obtained with different numbers of wing elements,
$\text{NX} = \text{NY} = 5, 6, 7$, while the number of wake element emenating from the
trailing edge and the wake length, $LW$, are fixed such that $\text{NW}=30, LW=3.5C$.
Using 64 elements on the whole wing (i.e. $\text{NX} = \text{NY} = 4$) is sufficient for convergence.

4.3.3 Analysis of Wake Effect

The wake effect of the oscillatory subsonic flow is evaluated from
Equation (4.12). The values of $\hat{W}_{k}$ depend on the number of wake elements and
the wake length. Note that, $C_{hi}$ in Equation (4.12) is a doublet integral which
represents the solid angle. The value of the solid angle decreases as the distance
between the control point and wake element increases. Therefore, the wake effect
is dominated by the wake elements closest to the trailing edge. For this reason,
it is necessary to determine the effective wake length. Also after the effective
wake length is determined, it is necessary to study the convergence due to the
number of wake elements. Figure 14 shows the distribution of $C_L$, along
$\gamma_{lb} = 0.13265$, of the same problem considered for Figure 12a. The results
are obtained with different number of $\text{NW}=5, 10, 20, 30, 40$, while using 196
elements on the whole wing, i.e., $\text{NX} = \text{NY} = 7$, and using four times chord length
for the wake length, or $LW=4.0C$. The length of the wake elements varies from
$0.8C$ to $0.1C$ from the figure shown. At least $\text{NW}=20$ should be used for convergence. Figure 15 shows the distribution of $C_L$, along $\gamma_{lb} = 0.13265$, of the same
problem considered for Figure 12a. The results are obtained with different lengths
of wake \( LW = 0.5C, 2C, 3C \) and \( 4C \). The length of the wake element is the same for each run, or \((\Delta X)_W = LW/NW = 0.1C\). The numbers of \( NW \) for each run are 5, 10, 20 and 40 respectively. From the figure shown, it can be concluded that at least, \( LW = 2C \), should be used as effective truncated wake length.

4.3.4 Results of Generalized Forces

Figures 16a and 16b show the absolute values and the phase angles of \( \overline{C}_L \) and \( \overline{C}_M \) for a delta wing in incompressible oscillatory flow, as functions of the reduced frequency, \( K = \frac{\omega C}{U_\infty} \). The wing with \( AR = 4 \) and \( \tau = 0.5\% \) is oscillating in pitch about the axis \( X = X_M \). The vibration mode is described by \( \frac{(X-X_M)}{L} \) with \( X_M = 0.5 (X_{LE} + X_{TE}) \). The results are compared with the experimental, as well as, the numerical results by Laschka \(^{18}\).

Figures 17a and 17b show the absolute values and the phase angles of \( \overline{C}_L \) and \( \overline{C}_M \) as functions of the Mach number which varies from 0.0 to 2.5. (Section 5 presents the formulation for the supersonic range) for a rectangular wing with \( AR = 2 \) and \( \tau = 0.001 \). The vibration mode is the same as that of the above problem, and the reduced frequency is set at \( K = 1 \). The results are compared with the ones obtained by Laschka \(^{18}\). Figures 18a and 18b show the thickness effect on the problem of Figure 17. The results are obtained with two thickness ratios \( \tau = 0.001 \) and \( \tau = 0.05 \). The Mach number varies from 0.0 to 2.5.

Figures 19a and 19b show the absolute values and the phase angles of \( \overline{C}_L \) and \( \overline{C}_M \) as functions of the reduced frequencies for a rectangular wing.
with $AR=2$ and $\zeta=0.001$. The wing in incompressible flow is oscillating in plunge. The vibration mode is described by $e^{i\omega t}$. The results are compared with the ones by Lawrance and Gerber \(^{19}\) as well as Laschka \(^{18}\).

Figures 20a and 20b show the absolute values and the phase angles of $C_L$ and $C_M$ of the same problem as that given for Figures 19a and 19b, while the reduced frequency is set at $K=1.0$ and the Mach number varies from 0.0 to 2.5. Results are compared with the ones of Laschka \(^{18}\).
SECTION 5
STEADY SUPERSONIC FLOW

5.1 INTRODUCTION

In Subsection 5.2 the numerical formulations for steady supersonic flow, derived in Reference 7, is outlined. For completeness, the derivation of these formulations and more detailed formulations are summarized in Appendix B. In Subsection 5.3 wings with supersonic leading edges and diaphragms for wings with subsonic leading edges are discussed. In Subsection 5.4 numerical results are presented.

5.2 NUMERICAL FORMULATIONS

Consider the Green theorem for steady supersonic flow. According to Equation (2.33)

\[ 2 \oint E(\vec{p}_e)\phi(\vec{p}_e) = - \iint \frac{\partial \phi}{\partial \vec{H}} \cdot \frac{H}{||\vec{r}||} d \Sigma_A + \iint \phi \frac{\partial}{\partial N^c} \left( \frac{H}{||\vec{r}||} \right) d \Sigma_A \]  

(5.1)

with the boundary condition given by Equation (2.14a)

\[ \frac{\partial \phi}{\partial N^c} = - N_x \left( \frac{1}{\beta} + \frac{M^2}{M^2 - 1} \frac{\partial \phi}{\partial x} \right) \]  

(5.2)

Applying the same procedure for subsonic flow, one obtains the following equations, expressed in matrix form,

\[ \begin{bmatrix} \Delta_{hk} - C_{hk} - \mathcal{W}_{hk} \end{bmatrix} \begin{bmatrix} \phi_h \end{bmatrix} = \begin{bmatrix} b_{hk} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial N^c} \end{bmatrix} \]  

(5.3)
where

\[ C_{hK} = \left( \frac{1}{\Pi} \int_{\Sigma_k} \frac{2}{N} \left( \frac{H}{\| \vec{q} \|} \right) d\Sigma \right) \vec{p}_x - \vec{p}_h \]  

(5.4)

and

\[ b_{hK} = \left( \frac{1}{\Pi} \int_{\Sigma_k} \frac{H}{\| \vec{q} \|} d\Sigma \right) \vec{p}_x - \vec{p}_h \]  

(5.5)

For hyperboloidal elements completely inside the Mach forecone, \( C_{hK} \) and \( b_{hK} \) are still given by

\[ C_{hK} = I_D(1, 1) - I_D(-1, -1) - I_S(-1, 1) + I_S(-1, -1) \]  

(5.6)

\[ b_{hK} = I_S(1, 1) - I_S(1, -1) - I_S(-1, 1) + I_S(-1, -1) \]  

(5.7)

For elements completely inside the Mach forecone \( I_D \) and \( I_S \) are given by

\[ I_D(\vec{q}, \eta) = \frac{1}{\Pi} \tan^{-1} \left( \frac{-\vec{q} \cdot \vec{a}_0}{\sqrt{\| \vec{q} \|^2 \vec{a}_0 \cdot \vec{a}_2}} \right) \]  

(5.8)

and

\[ I_S(\vec{q}, \eta) = \frac{1}{\Pi} \frac{1}{n \circ \bar{n}} \left( I_{S1}(\vec{q}, \eta) + I_{S2}(\vec{q}, \eta) + I_{S3}(\vec{q}, \eta) \right) \]  

(5.9)
with

\[ I_{s_1}(\xi, \eta) = -\left( \frac{\mathbf{q} \times \mathbf{a}_1}{\mathbf{n}} \right) F_1(\xi, \eta) \] (5.10)

\[ I_{s_2}(\xi, \eta) = \left( \frac{\mathbf{q} \times \mathbf{a}_2}{\mathbf{n}} \right) F_2(\xi, \eta) \] (5.11)

\[ I_{s_3}(\xi, \eta) = \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{n}} \tan^{-1} \left( \frac{-\frac{\mathbf{q} \times \mathbf{a}_1}{\mathbf{n}} \cdot \mathbf{a}_2}{\frac{\mathbf{q} \cdot \mathbf{a}_1}{\mathbf{n}} \cdot \mathbf{a}_2} \right) \] (5.12)

where \( \bar{q}, \bar{a}_1, \bar{a}_2, \) and \( \mathbf{n} \) are given by Equations (3.12) through (3.15) while

\[ F_j(\xi, \eta) = \frac{1}{||\mathbf{a}_j||} \ln \left| \frac{||\mathbf{q}|| + \mathbf{a}_j}{||\mathbf{q}|| + \mathbf{a}_j} \right| \quad \bar{a}_j \circ \bar{a}_j > 0 \]

\[ = \frac{||\mathbf{q}||}{\mathbf{q} \circ \bar{a}_j} \quad \bar{a}_j \circ \bar{a}_j = 0 \] (5.13)

\[ = -\frac{1}{||\mathbf{a}_j||} \sin^{-1} \left( \frac{\mathbf{q} \circ \bar{a}_j}{||\mathbf{q}|| \cdot ||\mathbf{a}_j||} \right) \quad \bar{a}_j \circ \bar{a}_j < 0 \]

For elements partially inside the Mach forecone \( I_D \) and \( I_{S1}, I_{S2}, I_{S3} \) are still given by Equations (5.8), (5.10), (5.11) and (5.12) if the corner point is inside the Mach forecone. However, if the corner point is outside the Mach forecone, integrals in Equations (5.4) and (5.5) involve singularity. Details of these kinds of integrals are discussed in Subsection B.4. For elements completely outside the Mach forecone,

\[ I \equiv 0 \] (5.14a)
Therefore

\[ C_{hK} = b_{hK} = 0 \]  \hspace{1cm} (5.14b)

Note that the wake effect has not been considered for the results presented here, because the geometry of the lifting body under consideration includes only wing, nose, or middle-section fuselage. Therefore, for each element, the wake is always outside the Mach forecone, i.e., \( W_{hk} = 0 \).

In addition, by neglecting the higher order terms, the boundary condition is given by (see Equation 2.14b)

\[ \frac{2 \phi}{\phi_N} = -\frac{N_x}{\beta} \]  \hspace{1cm} (5.15)

5.3 DIAPHRAGMS

There are three categories of wing geometry in supersonic flow: 1) supersonic leading edges (the elements \( \Sigma_k \) on the upper or the lower sides are influenced only by the elements on the same sides, therefore, the integration in Equations (5.4) and (5.5) are around the upper or lower surface only for the upper or lower elements); 2) mixed supersonic-subsonic leading edge (for this kind of wing, diaphragms have to be used to separate upper and lower sides of the aircraft; then, the problem can be handled as the case of supersonic leading edge); 3) subsonic leading edge (for this kind of wing, results obtained with or without diaphragms are the same [see Subsection 5.4.5]; however, for small thickness ratio, diaphragms are still needed to avoid the elimination of significant figures).

If diaphragms are used, both values of the velocity potential and its normal derivative, \( \phi_0 \) and \( \phi_N \), of the diaphragm element are unknown. However, two
equations are obtained for each element of the diaphragm: one relates $\varphi_p$ and $\gamma_p$ of the diaphragm element, $\Sigma_p$, to the upper geometry of the aircraft and the upper side diaphragm, the other one relates the same quantities to the lower geometry. Therefore, the total number of equations is equal to the total number of unknowns. Details of the numerical procedures are given in Subsection B.5. As mentioned before, for wing with subsonic leading edge, procedure with or without diaphragm can be used. If the procedure without diaphragm is employed, then the integral equations are solved the same way as for the subsonic case (solutions are obtained by solving Equation (5.3)). However, if the procedure with diaphragm is employed, solutions are obtained by solving Equations (B.41) or (B.46).

5.4 NUMERICAL RESULTS

Results for steady supersonic flow are presented here. In Subsection 5.4.1, results are compared with the experimental ones. In Subsection 5.4.2, analysis of convergence is studied. Problems of airfoils in conical flow are studied in Subsection 5.4.3. The role of diaphragms has already been discussed in Subsection 5.3. In order to show that diaphragms are not necessary for geometries with subsonic leading edge, two results obtained with and without diaphragms for a circular cone in conical flow are compared in Subsection 5.4.4. Finally, in order to study body interference and also to show the generality of the present method, result of wing-body configuration is considered in Subsection 5.4.5.

5.4.1 Comparison With Experimental Results

Figure 21 shows the distribution of the pressure coefficient $C_p$ on the lower and upper surfaces for a rectangular wing with aspect ratio $AR = 3$. The airfoil consists of a biconvex circular-arc section, 5% thick, with sharp leading and trailing edges. The results are obtained for $\alpha = 0^\circ$, $M = 1.3$ and $NX = NY = 7$. Figure 22a shows the
distribution of the lift coefficient for the same wing considered in Figure 21. The results are obtained for \( \alpha = 5^\circ \), \( M = 1.3 \) and \( NX = NY = 7 \). These results are compared with the ones obtained by Lessing, et al. In Figure 22b, the pressure distributions on the upper and the lower sides of the wing are shown. It should be mentioned that results in Figure 22a are obtained by taking the advantage of anti-symmetry as in the case described in Subsection 3.4.2.

5.4.2 Analysis of Convergence

Convergence study of the problem considered in Figure 21 is presented in Figure 23 which shows the distribution of the velocity potential along \( 2y/b = 0 \) for different numbers of elements. The curves are obtained with \( NX = NY = 5, 6 \) and 7. It is shown that 144 elements on the whole wing, i.e., \( NX = NY = 6 \), are sufficient for convergence.

5.4.3 Delta Wings

Figure 24 shows the distribution of life coefficient per unit angle of attack for a delta wing with supersonic leading edge and

\[
M = \frac{\beta}{\tan \Lambda} = 1.2
\]

where \( \Lambda \) is the sweep angle of the leading edge. The results obtained with \( NX = 8 \), \( NY = 12 \) and \( M = 1.2 \) are compared with the exact solution which is given by (Reference 20)

\[
\Delta C_p = \frac{4\alpha}{\pi \beta} \frac{m}{\sqrt{m^2 - 1}} \Re \left[ \cos^{-1} \left( \frac{1-\frac{m^2}{m^2 + \gamma^2}}{m^2} \right) + \cos^{-1} \left( \frac{1+\frac{m^2}{m^2 + \gamma^2}}{m^2} \right) \right]
\]

(5.16)

where

\[
\gamma = \frac{\beta y}{\alpha}
\]
Figure 25 shows the distribution of the lift coefficient per unit angle of attack for a delta wing with subsonic leading edges and \( m = 0.8333 \). The results obtained with \( NX = NY = 7 \) and \( M = \sqrt{2} \) are compared with the exact solution, which is given by (Reference 20)

\[
\Delta C_\rho = \frac{4m^2 \Delta \alpha}{\beta E(K)} \frac{1}{\sqrt{m^2 - \gamma^2}} \tag{5.17}
\]

where \( K = \sqrt{1 - m^2} \) and \( E \) is the elliptic integral of the second kind. It may be worth noting that there are two types of elements — grids which can be used for delta wings (see Figure 2c). For supersonic delta wings, the flow is conical, therefore, the element-grid shown in Figure 2c (b) appears to be the more natural one. In addition, the use of this grid implies the use of a completely general quadrilateral element. Let \( \tilde{x} \) and \( \tilde{\eta} \) be the generalized coordinates such that \( \tilde{\eta} \) has same direction as \( j \) while the direction of \( \tilde{x} \) passes the origin and pointed downward. The derivative of \( \phi \) with respect to \( X \) is obtained as follows: consider the directional derivatives \( \frac{d\phi}{d\xi} \) and \( \frac{d\phi}{d\eta} \)

\[
\frac{d\phi}{d\xi} = \left( \frac{\partial \phi}{\partial x} \bar{x} + \frac{\partial \phi}{\partial y} \bar{j} \right) \cdot \bar{\xi} = \frac{\partial \phi}{\partial x} \frac{x}{\sqrt{x^2 + y^2}} + \frac{\partial \phi}{\partial y} \frac{y}{\sqrt{x^2 + y^2}} \tag{5.18}
\]

\[
\frac{d\phi}{d\eta} = \left( \frac{\partial \phi}{\partial x} \bar{x} + \frac{\partial \phi}{\partial y} \bar{j} \right) \cdot \bar{\eta} = \frac{\partial \phi}{\partial y} \tag{5.19}
\]

where

\[
\bar{\xi} = X \bar{x} + Y \bar{j} \tag{5.20a}
\]

\[
\bar{\eta} = Y \bar{j} \tag{5.20b}
\]
Therefore

\[
\frac{\partial \phi}{\partial x} = \left( \frac{d \phi}{d \xi} \right) \frac{\sqrt{x^2+y^2}}{x} \left( \frac{d \phi}{d \eta} \right) \frac{y}{x} \tag{5.21}
\]

The values of \( \frac{d \phi}{d \xi} \) and \( \frac{d \phi}{d \eta} \) can be evaluated by the finite difference method. By using Equation (5.21), the values of \( C_p \) are obtained. The results obtained above by applying this scheme are very accurate. The results obtained for Figure 24 are especially accurate (up to the fifth significant figure).

5.4.4 Circular Cone (Analysis of Diaphragm Existence)

For most cases of supersonic flow considered here, diaphragms seem to be necessary to avoid determinant singularity. However, in this Subsection, an example is given to show that diaphragms are unnecessary, for example, for bodies with subsonic leading edge. This example is related to a circular cone with base radius \( r = 1 \), length \( l = 5 \), and \( M = 2.0 \). One is obtained with diaphragm by solving Equation (B.46), while the other one is obtained without diaphragm by solving Equation (5.3). The results are in good agreement. The pressure coefficient of the body is evaluated from the slope of \( \phi - \chi \) curve. This value is evaluated as 0.112 which is consistent with the exact solution given in Reference 21 (Figure 17, p. 187, \( C_p \approx 0.12 \) for \( M = 2 \), \( \theta_c = \tan^{-1} \left( \frac{1}{5} \right) \approx 11^\circ \)).

5.4.5 Wing-Body Configuration

The present method is sufficiently general to be applied for any complex configuration. Presented here is an example of this application. A wing-body combination in supersonic flow with \( M = 1.48 \) is considered in Figures 27a, 27b, and 27c. The geometry of the configuration is shown in Figure 27c.

The combination is composed of a wing with chord \( c = 3 \), span \( s = 9 \), thickness \( t = 5\% \), a forebody of length \( L_A = 6.0 \), and radius varying from 0 to 0.75 linearly, and
a midsection of length $L_M = 3.0$ and radius $r = 0.75$. Wake and aftbody are not considered. The angle of attack of the wing is $\alpha_w = 1.92$, while the angle of attack of the body is zero, $\alpha_\beta = 0$. The results are obtained with 580 elements on the whole configuration ($NX = NY = 10$ on the wing, $NX = 5$, $NY = 3$ on the nose, $NX = 10$, $NY = 3$ on the middle section). In Figure 27a the distribution of $\beta C_\rho / \omega_w$ on the wing section is presented, and the results are compared with the ones by Nielsen$^{22}$ and Woodward, Tinoco and Larsen$^{23}$, while in Figure 27b the distribution of $\beta C_\rho / \omega_w$ on the fuselage is shown.
SECTION 6
OSCILLATORY SUPERSONIC FLOW

6.1 INTRODUCTION

In Subsection 6.2, the numerical formulation for oscillatory supersonic flow is considered. The boundary conditions and the pressure coefficient are considered in Appendix C. The role of the diaphragm in oscillatory supersonic flow is the same as in steady supersonic flow. It has been already discussed in Subsection 5.3, therefore it is unnecessary to repeat it here. In Subsection 6.3, numerical results obtained are presented.

6.2 NUMERICAL FORMULATIONS

Consider the Green Theorem for oscillatory supersonic flow. According to Equation (2.38)

\[ 2 \pi \tilde{E}(\tilde{p}_0) \phi(\tilde{p}_0) = \iint_{\Sigma} \frac{\partial \phi}{\partial N^c} \frac{H}{||\tilde{R}||} \cos (\omega \frac{\alpha}{||\tilde{R}||}) \, d\Sigma \]

(6.1)

\[ + \iint_{\Sigma} \phi \frac{\partial}{\partial N^c} \left( \frac{H}{||\tilde{R}||} \cos \frac{\alpha}{||\tilde{R}||} \right) \, d\Sigma \]

with the boundary condition, given by Equation (C.28)

\[ \left. \frac{\partial \phi}{\partial N^c} \right|_{\Sigma} = N_Z \left( ik \tilde{Z} + \frac{1}{\tilde{\rho}} \frac{\partial \tilde{Z}}{\partial X} \right) e^{i\alpha M X} \]

(6.2)

where \( \tilde{Z} (X,Y) \) is the vibration mode defined by Equation (C.18). Applying the same procedure used for subsonic flow, one obtains the following equations, expressed in matrix form.
The wake effect for supersonic flow is not considered here since only supersonic-trailing-edge wings are presented here. Therefore \( \hat{W}_{hk} = 0 \).

Noting that

\[
\frac{\partial}{\partial N_c} \left[ \frac{1}{n R_1} \cos \left( \frac{1}{n R_1} \right) \right]
\]

\[
= - \frac{1}{n R_1^2} \left[ \cos \left( \frac{1}{n R_1} \right) + \frac{1}{n R_1} \sin \left( \frac{1}{n R_1} \right) \right] \frac{1}{n R_1} \frac{1}{N_c}
\]

the approximating evaluations of \( \hat{C}_{hk} \) and \( \hat{\beta}_{hk} \) are obtained by replacing the sine and cosine terms with their values at the centroid of the element. This yields

\[
\hat{C}_{hk} = \left[ \cos \left( \frac{1}{n R_{ec}} \right) + \frac{1}{n R_{ec}} \sin \left( \frac{1}{n R_{ec}} \right) \right] C_{hk}
\]
and

\[ \hat{b}_{hk} = \cos (\omega \, \| \vec{r} \|) \, b_{hk} \]  \tag{6.8} \]

where \( C_{hk} \) and \( b_{hk} \) are given by Equations (5.6) and (5.7) with \( I_D \) and \( I_S \) given by Equations (5.8) through (5.12) for elements completely inside the Mach forecone. For elements partially inside the Mach forecone, \( I_D \) and \( I_S \) are still given by Equations (5.8) and (5.9) if the corner point is inside the Mach forecone, otherwise, they are given by Equations (B.26) through (B.28) for \( I_{S1} \), Equations (B.30) through (B.32) for \( I_{S2} \), and Equations (B.35) through (B.40) for \( I_{S3} \) and \( I_D \). For elements completely outside the Mach forecone \( \hat{C}_{hk} = \hat{b}_{hk} = 0 \).

6.3 NUMERICAL RESULTS

Results for oscillatory supersonic flow are presented here.

In Subsection 6.3.1 a wing oscillating in bending mode is considered. In Subsection 6.3.2, the convergence problem is considered. In Subsection 6.3.3, the evaluation of generalized forces is considered.

6.3.1 Comparison With Experimental Results

Figure 28 shows the distribution of absolute value and phase angle of \( \tilde{C}_d \) for a rectangular wing oscillating in bending mode

\[ Z = 1.18043 \left| \frac{2y}{b} \right| + 1.70255 \left| \frac{2y}{b} \right|^2 - 1.13688 \left| \frac{2y}{b} \right|^3 + 2.5397 \left| \frac{2y}{b} \right|^4 \]  \tag{6.9} \]

with \( K = \frac{\omega_c}{2} \, U_\infty = 0.1 \)

The airfoil, with aspect ratio \( AR = 3 \), consists of a biconvex circular-arc section, 5% thick with sharp leading and trailing edges. Results obtained for \( M = 1.3 \)
and \( NX = NY = 7 \) are compared with the ones by Lessing, et al.\(^{15} \) It should be mentioned that the sign of phase angle of \( \phi \) obtained by the present method is opposite to that of Ref. 15. The author believes that this is due to the different choice of the sign convention.

### 6.3.2 Analysis of Convergence

A convergence study of the above problem is presented here. In Figure 29, the distributions of the imaginary and real parts of the velocity potential along \( 2y/b = 0.5 \) for the problem given for Figure 28 are shown. The curves are obtained with \( NX = NY = 5, 6, 7 \) and 8. It is shown that 100 elements (i.e., \( NX = NY = 5 \)) are sufficient for convergence.

### 6.3.3 Results of Generalized Forces

The typical results for the evaluation of generalized forces are presented in Figures 17a, 17b, 18a, 18b, 20a and 20b. In Figures 17a and 17b, a rectangular wing with oscillation in pitch is considered. In Figures 18a and 18b the thickness effect of the problem given for Figures 17a and 17b is considered. In Figure 20a and 20b a rectangular wing with oscillation in plunge is considered. These results are obtained for Mach number varying from 0.0 to 2.0 and \( K = \frac{2\omega C}{U_p} = 1.0 \) and compared with the ones obtained by Laschka. Details of the problems have already been given in Subsection 4.3.4, therefore, they are not repeated here.
7.1 INTRODUCTION

The previous sections present an efficient aerodynamic tool to analyze flutter phenomenon. In this section, a preliminary flutter analysis is presented. In Subsection 7.2, the mathematical model is presented. In Subsection 7.3, the Lagrange's equation forces are discussed. In Subsection 7.5, the evaluation of aerodynamic forces is outlined. In Subsection 7.6, the flutter analysis is described. In Subsection 7.7, numerical results are presented.

7.2 MATHEMATICAL MODEL

As usual, the mathematical model is established by describing the aerodynamic loads due to simple harmonic motion and then studying the stability of infinitesimal vibration of this kind of motion. The model is based on the fact that the motion at the flutter boundary is simple harmonic and that the flutter boundary is by definition the one relative to an infinitesimal oscillation around the next configuration.
The most typical aircraft flutter results from a coupling between the bending and torsion modes of a wing. A typical section (airfoil) analysis is considered for simplicity. This kind of flutter is considered in this section. The airfoil under consideration is permitted freedom to execute small vertical and angular displacements, \( h \) and \( \alpha \), and is constrained by two springs representing bending and torsion respectively. Thus, the displacement in \( Z \) direction is

\[
W = Z - Z (x, y) = h(t) + (x - X_{EA}) \alpha(t)
\]  
(7.1)

where \( X_{EA} \) denotes the position of elastic axis and \( h \) is the vertical displacement of the elastic axis (Figure 3).

Mathematically, it is easier to describe displacement with exponential time dependent \( e^{i\omega t} \) (\( \omega \) real) and assume that all dependent variables are proportional to \( e^{i\omega t} \). Therefore, Equation (7.1) can be rewritten as

\[
W = \tilde{h} e^{i\omega t} + (x - X_{EA}) \tilde{\alpha} e^{i\omega t} = \tilde{W} e^{i\omega t}
\]  
(7.2)

with

\[
\tilde{W} = \tilde{h} + \tilde{\alpha} (x - X_{EA})
\]  
(7.3)

\[
h = \tilde{h} e^{i\omega t}
\]  
(7.4)

\[
\alpha = \tilde{\alpha} e^{i\omega t}
\]  
(7.5)

The actual quantities of \( W \), \( h \) and \( \alpha \) are found by taking the real part of the complex counterparts.
7.3 LAGRANGE'S EQUATION OF MOTION

Equation (7.1) describes the displacement of airfoil having two degree of freedom. The variables $h$ and $\alpha$ are two generalized coordinates. The motion of the unrestrained airfoil is defined by the Lagrange's Equation which states that

$$\frac{d}{dt} \left( \frac{\partial \mathcal{T}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{T}}{\partial q_k} + \frac{\partial U}{\partial q_k} = Q_k$$

(7.6)

where $\dot{q}_k$ is the $k$-th generalized coordinates, $\mathcal{T}$ is the kinetic energy of the wing segment, $U$ is the internal strain energy of the wing segment, $Q_k$ is the $k$-th generalized force corresponding to the $k$-th generalized coordinate. The kinetic energy of the wing segment is

$$\mathcal{T} = \frac{1}{2} \int_{\text{chord}} \dot{\mathbf{W}}^2 \, dm$$

$$= \frac{1}{2} \int_{\text{chord}} \left( \dot{h}^2 + \left( X - X_{EA} \right) \dot{\alpha}^2 \right) \, dm$$

$$= \frac{1}{2} m \dot{h}^2 + \frac{1}{2} m J \dot{\alpha}^2 + m J \dot{X}_{\alpha} \ddot{h}$$

(7.7)

where $J$ is a reference length, while $r_\alpha$ and $X_{\alpha}$ are the dimensionless radius of gyration and static balance about the elastic axis,

$$Y_\alpha = \frac{1}{m J^2} \int_{\text{chord}} (X - X_{EA})^2 \, dm \quad \text{and} \quad X_{\alpha} = \frac{1}{m J} \int_{\text{chord}} (X - X_{EA}) \, dm$$

(7.8)

If the stiffness of bending and torsion spring are represented by $K_h$ and $K_\alpha$ respectively, the internal strain energy is given by

$$U_e = \frac{1}{2} K_h h^2 + \frac{1}{2} K_\alpha \alpha^2$$

(7.10)
or

\[ U_E = \frac{1}{2} m \omega_h^2 h^2 + \frac{1}{2} m k^2 \omega_d^2 \omega_c^2 \] (7.11)

where \( \omega_d \) and \( \omega_h \) are the natural frequencies without mass coupling, or

\[ \omega_h = \sqrt{\frac{K_h}{m}} \] (7.12)

\[ \omega_d = \sqrt{\frac{K_d}{m k^2 \omega_c}} \] (7.13)

Combining Equations (7.7) and (7.11) with the Lagrange Equation, Equation (7.6), with \( q_1 = h, q_2 = d, Q_1 = Q_h \) and \( Q_2 = Q_d \), the following two equations of motion are obtained

\[ m h'' + m l X_d \dot{\omega}_d + m \omega_d^2 h = \ddot{Q}_h \] (7.14)

\[ m l X_d h'' + m l^2 Y_d \dot{\omega}_d + m \omega_d^2 l^2 Y_d \dot{\omega}_d = \ddot{Q}_d \] (7.15)

Combining Equations (7.4) and (7.5) with Equations (7.14) and (7.15) yields

\[ -m \omega^2 h - m \omega^2 l X_d \ddot{\omega}_d + m \omega h \dddot{h} = \dddot{\tilde{Q}}_h \] (7.16)

\[ -m \omega^2 l X_d \dddot{h} - m \omega^2 l Y_d \ddot{\omega}_d + m \omega_d^2 l^2 Y_d \ddot{\omega}_d = \dddot{\tilde{Q}}_d \] (7.17)

As mentioned above, the symbol tilde above each quantity represents the complex time-independent part.

7.4. GENERALIZED FORCES

The generalized forces are found from the virtual work which is given by

\[ \delta \tilde{\mathbf{W}} = \int \int C_\rho \delta \omega \, dx \, dy \]

\[ = \delta h \int \int C_\rho \, dx \, dy + \delta d \int \int C_\rho (X - X_{EA}) \, dx \, dy \]

\[ = \tilde{\omega}_h \delta h + \tilde{\omega}_d \delta d \] (7.18)
where

\[ \text{\( \tilde{\Omega}_h \)} = \mathcal{L} = \frac{CPU_0^2}{2} \iint \Delta \tilde{C}_p \, dx \, dy \]  

(7.19)

and

\[ \text{\( \tilde{\Omega}_d \)} = \tilde{M} = \frac{CPU_0^2}{2} \iint \Delta \tilde{C}_p (X-X_{EA}) \, dx \, dy \]  

(7.20)

Combining Equations (7.19) and (7.20) with Equations (7.14) and (7.15) yields

\[ \left[ -m \omega^2 + \omega \right] \hat{h} - m \xi \omega^2 \hat{x} = \mathcal{L} \]  

(7.21)

\[ -m \omega^2 \xi \omega^2 \hat{x} + \left[ -m \omega^2 \xi \omega^2 \hat{x} + m \omega^2 \xi \omega^2 \right] \mathcal{L} = \tilde{M} \]  

(7.22)

### 7.5 AERODYNAMIC FORCES

Following is the procedure to evaluate \( \mathcal{L} \) and \( \tilde{M} \). Because of the assumption of small vibration, the doublet integral and source integral (see Equations (4.4), (4.5), (6.4), and (6.5)) are not influenced by the displacement, \( W \), while the following boundary condition is a function of \( W \)

\[ \frac{\partial \phi}{\partial N} = N_2 \left( i \xi \omega + \frac{1}{\beta} \frac{\partial W}{\partial X} \right) \]  

(7.23)

Therefore, it is legitimate to write

\[ C_p = \mathcal{L} \left( \frac{\partial \phi}{\partial N} \right) = \mathcal{L} (W) \]

\[ = \mathcal{L} \left( h e^{i\omega t} + (X-X_{EA}) \tilde{z} e^{i\omega t} \right) \]

\[ = \tilde{h} \mathcal{L} (e^{i\omega t}) + \mathcal{L} [(X-X_{EA}) e^{i\omega t}] \]  

(7.24)
where $\mathcal{L}$ denotes the linear operator.

Therefore, by applying Equation (7.24) to Equation (7.19) and (7.20) the lift force and pitch moment are found.

\[ \mathcal{L} = \frac{\rho U_\infty^2 c}{2} \iint \mathcal{C}_p \, dx \, dy \]

\[ = \frac{\rho U_\infty^2 c}{2} \left\{ \frac{\mathcal{K}}{k} \iint \mathcal{L} \left[ \mathcal{L} (x e^{i\omega t}) \right] dx \, dy + \mathcal{L} \iint \mathcal{L} \left[ (x - x_E) e^{i\omega t} \right] dx \, dy \right\} \]

\[ = \frac{\rho U_\infty^2 c}{2} \left\{ \left( \frac{h}{x} \right) \mathcal{C}_{Lh} + \mathcal{L} \mathcal{C}_{Lh} \right\} \quad (7.25) \]

\[ \mathcal{M} = \frac{\rho U_\infty^2 c^2}{2} \iint \mathcal{C}_p \left( x - x_{EA} \right) dx \, dy \]

\[ = \frac{\rho U_\infty^2 c^2}{2} \left\{ \left( \frac{h}{x} \right) \iint \mathcal{L} \left[ \mathcal{L} (x e^{i\omega t}) \right] (x - x_{EA}) dx \, dy \right. \]

\[ + \mathcal{L} \iint \mathcal{L} \left[ (x - x_{EA}) e^{i\omega t} \right] (x - x_{EA}) dx \, dy \left\} \right. \]

\[ = \frac{\rho U_\infty^2 c^2}{2} \left( \left( \frac{h}{x} \right) \mathcal{C}_{Mh} + \mathcal{L} \mathcal{C}_{Mh} \right) \quad (7.26) \]

In Equations (7.25) and (7.26) $\mathcal{C}_{Lh}$ and $\mathcal{C}_{Mh}$ are lift force and pitch moment coefficients about elastic axis of the airfoil in the motion with oscillation in plunge.

The displacement for plunge motion with $\frac{h}{x} = 1$ is described by

\[ W = x e^{i\omega t} \quad (7.27) \]

The boundary condition is then given by

\[ \frac{\partial \phi}{\partial N} = N_x (iK e^{i\omega t}) \quad (7.28) \]

or

\[ \frac{\partial \phi}{\partial N} = N_x \quad (7.29) \]
Similarly, $C_{Ld}$ and $C_{Md}$ are lift force and pitch moment about elastic axis of the airfoil in the motion with oscillation in pitch about the elastic axis with unit angle amplitude. The displacement is described by

$$W = (\chi - \chi_{Eh}) e^{i\omega t} \quad (7.30)$$

the boundary condition is then given by

$$\frac{\partial \phi}{\partial N} = N_z \left[ 1 + iK \frac{(\chi - \chi_{EA})}{\ell} \right] e^{i\omega t} \quad (7.31)$$

i.e.

$$\frac{\partial \phi}{\partial N} = N_z \left[ 1 + iK \frac{(\chi - \chi_{EA})}{\ell} \right] \quad (7.32)$$

7.6 FLUTTER ANALYSIS

Combining Equations (7.25) and (7.26) with Equations (7.21) and (7.22) yields

$$\left\{ m \omega^2 \xi (-1 + \frac{\omega_k}{\omega}) - \frac{p U_o^2 c}{2} C_{Lh} \right\} \frac{\ddot{\xi}}{\ell} - \left\{ m \omega^2 \chi_k + \frac{p U_o^2 c}{2} C_{Ld} \right\} \ddot{\chi} = 0 \quad (7.33)$$

and

$$\left\{ m \omega^2 \varepsilon (\ell \chi_k) - \frac{p U_o^2 c}{2} C_{Mh} \right\} \frac{\ddot{\varepsilon}}{\ell} - \left\{ m \omega^2 \chi_k^2 \left( 1 - \frac{\omega_k}{\omega^2} + \frac{p U_o^2 c}{2} \right) \right\} \varepsilon = 0 \quad (7.34)$$

By introducing

$$A = \frac{\omega_k^2}{\omega^2} \quad (7.35)$$

and

$$K = \frac{\omega}{U_{bo}} \quad (7.36)$$
Equation (7.33) and (7.34) may be rewritten in the following matrix form

\[
\begin{bmatrix}
1 - \left(\frac{\omega}{\omega_d}\right)^2 A + \frac{1}{\pi K^2 \mu} C_{Lh} & X_d + \frac{1}{\pi K^2 \mu} C_{Ld} \\
X_d + \frac{2}{\pi K^2 \mu} C_{Mh} & 2 h - \frac{2}{\pi K^2 \mu} C_{Mh}
\end{bmatrix}
\begin{bmatrix}
\frac{\dot{h}}{h} \\
\frac{\dot{\alpha}}{\alpha}
\end{bmatrix}
= 0
\]

(7.39)

In order to yield a non-trivial solution, the determinant of the above left-hand side matrix must be zero. The value of \( A \) can, therefore, be evaluated. By plotting the values of the imaginary part of \( A \) versus different reduced frequencies, the flutter is found at a specific reduced frequency where the imaginary part of \( A \) is zero.

### 7.7 NUMERICAL RESULTS

In order to compare the numerical results with the analytical ones, a two-dimensional thin airfoil in incompressible, oscillatory flow was studied. For this airfoil, the analytical expressions of \( \bar{C}_{Lh} \), \( \bar{C}_{Ld} \), \( \bar{C}_{Mh} \) and \( \bar{C}_{Md} \) are given in several standard text books (see e.g. Reference 24) as

\[
\bar{C}_{Lh} = 2\pi K^2 \left(2 - \frac{C_E}{K}\right) - 1
\]

(7.40)

\[
\bar{C}_{Ld} = \pi K^2 \left\{X_{EA} + 2 \frac{C_E}{K^2} + \left(1 - 2X_{EA}\right)\frac{C_k}{K}\right\}
\]

(7.41)

\[
\bar{C}_{Mh} = \pi K^2 \left((2X_{EA} + 1) - \frac{C_k}{K}\right)
\]

(7.42)
\[
C_{Mh} = \frac{1}{2} \pi K^2 \left\{ (0.125 + \chi_{EA}^2) + (2\chi_{EA} + 1) \frac{C_k}{K^2} + (0.5 - \chi_{EA})(2\chi_{EA} + 1) \right\} (7.43)
\]

where
\[
C_k = \left( -J_1(k) + iY_1(k) \right) \left[ \frac{-\left( J_0(k) + Y_0(k) \right) + i \left( J_1(k) - Y_1(k) \right)}{J_0(k) - Y_0(k)} \right] (7.44)
\]

where \( J_1 \) and \( J_0 \) and \( J \)-Bessel functions of the first and zero orders and \( Y_1 \) and \( Y_0 \)
are \( Y \)-Bessel functions of the first and zero orders and reduced frequency, \( K \), is defined as
\[
K = \frac{\omega}{2} \frac{c}{U_{\infty}} (7.45)
\]

It should be noted that the purpose for this two dimensional (airfoil) flutter analysis is to assess the validity of the three dimensional aerodynamic theory presented in the proceeding sections. For this reason the two dimensional airfoil is approximated by a thin rectangular wing with high aspect ratios. For the current problem rectangular wing with \( AR=16 \) can be considered as a two dimensional wing as being verified in Figure 30. In Figures 30a, 30b, 30c and 30d, the curves of \( \tilde{C}_{Lh}, \tilde{C}_{Ld}, \tilde{C}_{Mh}, \) and \( \tilde{C}_{Mh} \) as functions of the aspect ratio for a thin rectangular wing with \( K=0.25 \) are presented. It is shown that most curves converge at \( AR=16 \). Values of \( \tilde{C}_{Lh}, \tilde{C}_{Ld}, \tilde{C}_{Mh}, \) and \( \tilde{C}_{Mh} \) versus \( K \) varying from 0.1 to 1.5 are obtained with \( X_{EA}=2c, N_X=8, N_Y=10, N_W=20, LW=3.0c, \tau=0.1\% \) and \( AR=16 \) are compared with the exact ones in Figures 34a, 34b, 34c and 34d. The numerical results are in good agreement with the exact ones for reduced frequencies higher than 0.15, while the results for lower reduced frequencies are not completely satisfactory.

As mentioned before, by plotting the value of imaginary \( \Lambda \) versus reduced frequency \( K \), the flutter speed can be located at a specific \( K \) where the imaginary \( \Lambda \) is zero. An example of this work is presented in Figure 31 which is related to a
rectangular wing with \( AR = 16, \ T = 0.1\% \), \( M = 5 \), \( X_{EA} = -0.2c \), \( \omega_h/\omega_a = 0.5 \), \( \chi_c = 0.2 \) and \( r_c = 0.5 \). The flutter boundary occurs at \( K = 0.53 \) and \( A = \omega_a^2/\omega_c^2 \) = 1.775. Combining Equations (4.35) (4.36) and (4.37) gives the nondimensional flutter speed, \( 2U_F/C \omega_a = 1.42 \) which is in good agreement with the exact solution (Reference 24)*. The flutter speed as a function of \( \omega_h/\omega_a \) for the same wing with \( M = 5 \), \( \chi_c = 0.2 \), \( r_c = 0.5 \) and \( X_{EA} = -0.2c \) is presented in Figure 32a and compared with the solution given by two dimensional airfoil theory (Figure 9-5(A), Reference 24). Additional results for the same wing with \( M = 10 \), \( \chi_c = 0.2 \), \( r_c = 0.5 \) and \( X_{EA} = -0.2c \) are presented in Figure 32b.

*See Figure 9-5(A)(i) of Reference 24 (p. 540), which yields \( 2U_F/C\omega_a \approx 1.47 \).
8.1 INTRODUCTION

The general comments of the present method are made in Subsection 8.2. The outline of the computer program is introduced in Subsection 8.3, while the CPU time and the memory space required for the computer program are listed in Subsection 8.4.

8.2 GENERAL COMMENTS

The results obtained in the previous sections are in very good agreement with the existing ones. Especially, compared with the exact solutions, several results are remarkably accurate. For example, values of $C_{dd}$, evaluated for a supersonic delta wing in Subsection 5.4.3, is accurate up to fifth significant figure. In addition to the accuracy, the method is very efficient, extremely general and very easy to use. The results obtained include a variety of problems for either steady or oscillatory, subsonic or supersonic flow. Two results for wing-body configurations in subsonic and supersonic flow are included, while the method can be further applied to wing-body-tail combinations, or more complex configurations. A preliminary result of flutter analysis is presented in Section 7. The pressure is evaluated by finite difference method. However, a hyperboloidal distribution of the velocity potential, $\phi$, within each element in terms of $\phi$'s at corners can be obtained by evaluating $\phi$ at each corner by taking average of $\phi$ at the centroids of the surrounding elements. Therefore, an increase of generality and efficiency can be attained. In addition, the present method for the oscillatory flow can be generalized for the study of the unsteady flow without additional increase in computational complexity (Reference 25). Also it should be mentioned that the computer code presented here is believed to be the only one available for supersonic oscillatory flows around non-zero-thickness configuration.
8.3 COMPUTER PROGRAM

The formulations in the previous sections are imbedded into a general computer program called SOSSA ACTS (Steady and Oscillatory Subsonic and Supersonic Aerodynamics for Complex Transportation System). The listing of this program is given in Appendix E. In general, the program is separated into three parts. The first part of the program is to define the geometry of the aircraft, wake and diaphragm (if necessary) and divide their surface surfaces into a number of elements. Each element is then replaced by a hyperboloidal surface defined by four corner points. This part of the program includes Subroutines GEOMET and VEC123. The second part of the program is to evaluate the coefficients \( C_{hk} \), \( b_{hk} \), and \( W_{hk} \) and to solve a system of linear equations by using the Gaussian elimination method for unknowns \( \phi \) or \( \hat{\phi} \). This part of the program includes Subroutines CEOFF and CCGELG. The third part of the program is to evaluate the pressure coefficient and the generalized forces from the computed velocity potential \( \phi \) or \( \hat{\phi} \). This part of the program includes Subroutine COEFPR.

8.4 COMPUTER TIME AND MEMORY SPACE

The following table gives the computer time used by the program SOSSA ACTS in terms of the total number of elements on the wing, \( N_{\text{ELEM}} = 4 \times N_x \times N_y \).

<table>
<thead>
<tr>
<th>( N_{\text{ELEM}} )</th>
<th>Subsonic</th>
<th>Supersonic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Steady</td>
<td>Unsteady</td>
</tr>
<tr>
<td>4 \times 4 \times 4</td>
<td>29</td>
<td>161</td>
</tr>
<tr>
<td>4 \times 5 \times 5</td>
<td>70</td>
<td>324</td>
</tr>
<tr>
<td>4 \times 6 \times 6</td>
<td>143</td>
<td>543</td>
</tr>
<tr>
<td>4 \times 7 \times 7</td>
<td>268</td>
<td>928</td>
</tr>
</tbody>
</table>
All the results are obtained by taking advantage of the symmetry (or anti-symmetry) in y and z directions. The wake for subsonic flow has \( N_W = 1 \) for steady and \( N_W = 30 \) for unsteady flow. The diaphragm for supersonic flow has \( N_D = 9 \). The computer times are expressed in seconds and were obtained on an IBM 370/145 computer of the Boston University Computer center.

Finally, the memory space required for steady flow is approximately given by

\[
N_{\text{words}} = 9,500 + N_{\text{EQN}} + N_{\text{EQN}}^2
\]  

(8.1)

where \( N_{\text{words}} \) is the number of words, while \( N_{\text{EQN}} \) is the number of equations. Similarly for unsteady flow, the memory space requirement is approximately given by

\[
N_{\text{words}} = 12,500 + 22 N_{\text{EQN}} + 2N_{\text{EQN}}^2
\]  

(8.2)
REFERENCES


Figure 1. Hyperboloidal Element
Figure 2a. Wing Planform
Figure 2b. Wake Element
Figure 2c. Two Types of Element-Grids for Delta Wings.
Figure 3. Model for Flutter Analysis
Figure 4. The distribution of $c_{ld}$ along $2y/b = .7$ for a rectangular wing with $AR = 1.0$, $M = .2$, and $NX = NY = 7$ for comparison with results of Ref. 11.
Figure 5. The distribution of $c_L$ along $2y/b = .707$ for a tapered swept wing with $AR = 3$, $TR = .5$, $\Lambda_{1/4} = 45^\circ$, $M = .8$, $\alpha = 5^\circ$ and $NX = NY = 7$ for comparison with results of Ref. 13.
Figure 6. The distribution of the section lift coefficient per unit angle of attack for a rectangular wing with AR = 4, $M = .507$ and $NX=NY=7$ and 10 for comparison with results of Ref. 14.
Figure 7. The distribution of $C_p$ on the upper and lower surfaces of a symmetric rectangular wing with $AR = 3$, $\tau = 5\%$, $\alpha = 0^\circ$, $M = 0.24$ and $NX=NY=10$ for comparison with results of Ref. 15.
Figure 8a. The distribution of the lift coefficient, $c_L$, on a symmetric rectangular wing with $AR = 3, \tau = 0.05, \alpha = 5^\circ, M = 0.24$ and $NX = NY = 7$ for comparison with results of Ref. 15.
Figure 8b. The distribution of \( c_{pu} \) on the upper and lower surfaces of a symmetric rectangular wing with \( AR = 3, \tau = 0.05, \alpha = 5^\circ, M = .24 \) and \( NX=NY=6 \) for comparison with results of Ref. 15.
Figure 9. Analysis of Convergence: Potential Distribution, $\xi$, Versus $x/c$, at $y = 0$, for Rectangular Wing With Biconvex Section, in Steady Subsonic Flow, for $AR = 3$, $\tau = 0.05$, $M = 0.24$, $\alpha = 0^\circ$. $NX = NY = 5, 6, 7$. 
Figure 10. Lift Distribution Coefficient, $C_{L\alpha}$, Versus Aspect Ratio AR, for Delta Wing in Steady Subsonic Flow, With $\alpha = 0.001$, $M = 0$, $N_x = N_y = 7$. Comparison With Lifting Surface Theory of Reference 16.
Figure 11a. The distribution of section lift coefficient, $C_l$, for a wing-body configuration with $\alpha_w = 6^\circ$, $\alpha_b = 0^\circ$, $\tau = 9\%$, $M=0$, $b=6c$, $r = 0.5c$ and 200 elements on the whole wing for the comparison with results of Ref. 17.
Figure 11b. The distribution of $\phi_u - \phi_i$ along three circumferential stations for a wing-body configuration with $\alpha_w = 6^\circ$

$\alpha_B = 0^\circ$, $\tau = 9\%$, $M=0$, $b=6c$, $r=0.5c$
Figure 12. The distribution of lift coefficient, $c_l$, for a rectangular wing oscillating in bending mode with $k = \omega c/2U_e = .47$, $M = .24$, $AR = 3, \tau = 0.05$, $NW = 20$, $L_w = 2.5c$ and $NX = NY = 7$ for comparison with results of Ref. 15.
Figure 13a. Analysis of Convergence: The distribution of $\tilde{C}_f$ versus $x/c$ at $2\gamma/b = .05$ for a rectangular wing oscillating in bending mode with $k = \omega c/2 U_m = .47$, $M = .24$, $AR=3$
$\zeta = 0.01$, $\alpha = 0^\circ$, $N_w = 30$, $L_w = 3.5c$, $NX = NY = 5, 6, 7$. 

- $7 \times 7$
- $6 \times 6$
- $5 \times 5$
Figure 13b. Analysis of Convergence: Lift Distribution Coefficient, $\tilde{c}_L$, Versus $x/c$, at $2\gamma/b = 0.1328$, for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Subsonic Flow; for $AR = 3$, $\tau = 0.01$, $M = 0.24$, $\alpha = 0^\circ$, $K = 0.47$, $N_W = 30$, $L_W = 3.5c$, $NX = NY = 5,6,7$. 
Figure 14. Analysis of Convergence: Lift Distribution Coefficient, $c_l$, Versus $x/c$, at $2y/b = 0.1328$ for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Subsonic Flow, for $AR = 3$, $\tau = 0.01$, $M = 0.24$, $\alpha = 0^\circ$, $K = 0.47$, $L_W = 4c$, $N_x = N_y = 7$. 
Figure 15. Analysis of Convergence: Lift Distribution Coefficient, $\bar{c}_L$, Versus $x/c$ at $2y/b = 0.1328$, for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Subsonic Flow, for AR = 3, $\tau = 0.01$, $M = 0.24$, $\alpha = 0^\circ$, $K = 0.47$, $\Delta x_W = 0.1$, $N_x = N_y = 7$. 
Figure 160. Lift Coefficient, $\tilde{C}_L$, Versus $k$, for Delta Wing Oscillating in Pitch, With $AR = 4$, $\tau = 0.005$, $M = 0$, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W/c = 2$. Comparison With Results of Reference 18.
Figure 16b. Moment Coefficient, $\tilde{C}_M$, Versus $k$, for Delta Wing Oscillating in Pitch With $AR = 4$, $\tau = 0.005$, $M = 0$, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W = 2c$. Comparison With Results of Reference 18.
Figure 17a. Lift Coefficient, $\tilde{C}_L$, Versus $M$, for Rectangular Wing Oscillating in Pitch, With $AR = 2$, $\tau = 0.001$, $k = 1$, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W/c = 2$, $N_D = 30$. Comparison With Results of Reference 18.
Figure 17b. Moment Coefficient, $\tilde{C}_M$, Versus $M$, for Rectangular Wing Oscillating in Pitch, for $AR = 2$, $\tau = 0.001$, $k = 1$, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W/c = 2$, $N_D = 30$. Comparison With Results of Reference 18.
Figure 18a. Thickness effects in oscillatory subsonic and supersonic flows. Results are total lift coefficient, $\bar{C}_L$, versus $M$, for a rectangular wing with $AR = 2$ and $K = ac/2U_w = 1.0$ oscillating in pitch about axis $X = X_m$. 
Figure 18b. Thickness Effect in Oscillatory Subsonic and Supersonic Flows. Results are total moment coefficient, $\overline{C_M}$, versus $M$, for a rectangular wing with $AR = 2$ and $k = \omega c / 2U_\infty = 1.0$ oscillating in pitch about axis $x = x_m$. 
Figure 19a. Lift Coefficient, $|\tilde{C}_L|$, Versus $k$, for Rectangular Wing Oscillating in Plunge, With $AR = 2$, $\tau = 0.001$, $M = 0$, $N_x = 10$, $N_y = 6$, $N_W = 20$, $L_W = 2c$. Comparison With Results of Reference 18.
Figure 19b. Moment Coefficient, $\tilde{C}_M$, Versus $k$ for a Rectangular Wing Oscillating in Plunge, With $AR = 2$, $\tau = 0.001$, $M = 0$, $N_x = 10$, $N_y = 6$, $N_{WY} = 20$, $L_W = 2c$. Comparison With Results of Reference 18.
Figure 20a. Total lift coefficient, $C_L$, versus $M$, for a rectangular wing with $AR = 2$, $\tau = 0.18$ and $k = \omega \alpha / 2U_\infty = 1.0$ oscillating in plunge. Results are related to both subsonic and supersonic flows.
Figure 20b. Total moment coefficient, $C_M$, versus $M$, for a rectangular wing with AR = 2, $\tau = 0.1\%$ and $k = \omega c/2U_\infty$ oscillating in plunge. Results are related to both subsonic and supersonic flows (same problem as Figure 20a).
Figure 21. The pressure distribution on a symmetric rectangular wing with AR = 3, τ = 5%, α = 0°, M = 1.3 and NX = NY = 7 for the comparison with results of Ref. 15.
Figure 22a. The lift distribution on symmetric rectangular wing with \( AR = 3 \), \( \tau = 5\% \), \( \alpha = 5^\circ \), \( M = 1.3 \) and \( NX = NY = 7 \) for the comparison with results of Ref. 15.
Figure 22b. The distribution of $C_p$ on the upper and lower surfaces of a symmetric rectangular wing with $AR = 3$, $\tau = 5\%$, $\alpha = 5^\circ$, $M = 1.3$ and $NX = NY = 6$ for comparison with results of Ref. 15.
Figure 23. Analysis of Convergence: Potential Distribution, $\psi$, Versus $x/c$, at $y = 0$, for Rectangular Wing With Biconvex Section, in Steady Supersonic Flow, for $AR = 3$, $r = 0.05$, $M = 1.3$, $\alpha = 0^\circ$, $N_D = 3N_x$, $NX = NY = 5, 6, 7$. 
Figure 24. Lift Distribution Coefficient, $C_{L\alpha}$, for Delta Wing With Supersonic Leading Edge, in Steady Supersonic Flow, With $B/\tan \Lambda = 1.2$, $\tau = 0$, $N_x = 8$, $N_y = 12$. Comparison With Exact, Conical-Flow Solution, Reference 20.
Figure 25. Lift Distribution Coefficient, $C_{g2\alpha}$, for Delta Wing with Subsonic Leading Edge, in Steady Supersonic Flow, with $B/\tan \Lambda = 0.833$, $\tau = 0$, $N_x = N_y = 7$. Comparison with Exact Conical-Flow Solution, Reference 20.
Figure 26. Study of existence of diaphragm; the distribution of velocity potential, $\phi$, for a circular cone with subsonic leading edge, $r = 1$, $\beta = 5$, $M = 2.0$ and $NX = 12$, $NY = 5$. Results obtained with and without diaphragm are compared.
Figure 27a. The distribution of $\frac{\beta C_l}{\alpha_w}$ on the wing section for a wing-body configuration (shown in Figure 27c) with $M = 1.48$, $\alpha_w = 1.92^\circ$ and $\alpha_B = 0^\circ$, for the comparison with results of Ref. 21.
Figure 27b. The distribution of $\beta C_\ell/\alpha_w$ on the fuselage at three circumferential stations for the same problem of Figure 27a.
Figure 27c. Wing-body Configuration in Supersonic Flow

\[ \tan^{-1} \beta \]

\[ M = 1.48 \]

\[ 8r \]

\[ 5r \]

\[ \alpha_w = 1.92^\circ \]

\[ \alpha_B = 0^\circ \]

\[ \tau = 5\% \]
Figure 28. Lift coefficient, $\tilde{C}_l$, for a rectangular wing oscillating in bending mode with $k = \omega c/2U_\infty = 0.1$, $M = 1.3$, $AR = 3$ and $NX = NY = 10$. 
Figure 29 Analysis of Convergence: Distribution of $\delta = \sum_{\Omega} \Delta M^x \text{ Versus } x/c$, at $2y/b = 0.5$, for Rectangular Wing With Biconvex Section, Oscillating in Bending Mode in Supersonic Flow, for $AR = 3$, $\tau = 0.01$, $M = 1.3$, $K = 0.1$, $\alpha = 0^\circ$, $N_D = 3N_x$, $NX = NY = 5,6,7$. 
Figure 30a. $C_{L_H}$ as a function of aspect ratio for a rectangular wing with $\tau = 0.1\%$, $M = 0$ and $NX = 8$, $NY = 10$ ($X_{EA} = -0.2C$).
Figure 30b. $C_{L\alpha}$ as a function of aspect ratio for a rectangular wing with $	au = .1\%$, $M = 0$ and $NX = 8$, $NY = 10$ ($X_{EA} = -0.2C$).
Figure 30c. $\tilde{C}_{Mh}$ as a function of aspect ratio for a rectangular wing with $\tau = 0.1\%, M = 0$ and $NX = 8$, $NY = 10$ ($X_{EA} = -0.2C$).
Figure 30d. $\bar{C}_{M\alpha}$ as a function of aspect ratio for a rectangular wing with $\tau = 0.1\%$, $M = 0$ and $NX = 8$, $NY = 10$ ($X_{EA} = -0.2C$).
Figure 31. Parameter $A = \omega / \omega_\alpha^2$ as a function of reduced frequency, $K = \omega c / 2U_\infty$, for a rectangular airfoil with $AR = 16$, $M = 0$, $\omega_h / \omega_\alpha = 0.5$, $X_{\alpha} = 0.2$, $r_\alpha = 0.5$, $M = 5$ and $NX = 8$, $NY = 10$ ($X = -0.2C$).
Figure 32a. Flutter speed as a function of \( \frac{\omega_f}{\omega_\alpha} \) for a rectangular wing with \( AR = 16, M = 0, \tau = 0.1\%, \mu = 5, X_\alpha = 0.2, R_\alpha = 0.5 \), and \( NX = 8, NY = 10 \). Results are compared with exact solution given by two dimensional airfoil theory (Ref. 24) \( (X_{EA} = -0.2C) \).

Figure 32b. Flutter speed as a function of \( \frac{\omega_f}{\omega_\alpha} \) for a rectangular wing with \( AR = 16, M = 0, \tau = 0.1\%, \mu = 10, X_\alpha = 0.2, R_\alpha = 0.5 \) and \( NX = 8, NY = 10 \). Results are compared with exact solution given by two dimensional airfoil theory (Ref. 24) \( (X_{EA} = -0.2C) \).
Figure 33a. $\tilde{C}_{L_h}$ as a function of the reduced frequency, $k = \omega c/2U_\infty$, for a rectangular wing with $AR = 16$, $\tau = 0.1\%$, $M = 0$ and $NX = 8$, $NY = 10$. Comparison with the exact solution given by 2-D airfoil theory (Ref. 24) ($X_{EA} = -0.2C$).
Figure 33b. \( \tilde{C}_{L\alpha} \) as a function of the reduced frequency, \( k = \omega c/2U_\infty \) for a rectangular wing with \( AR = 16 \), \( \tau = 0.1\% \), \( M = 0 \) and \( NX = 8 \), \( NY = 10 \). Comparison with the exact solution given by 2-D airfoil theory (ref. 24) \( (X_{EA} = -0.2C) \).
Figure 33c. $\tilde{C}_{mh}$ as a function of reduced frequency, $k = \omega c/2U_\infty$ for a rectangular wing with $AR = 16$, $\tau = 0.1\%$, $M = 0$ and $NX = 8$, $NY = 10$. Comparison with the exact solution given by 2-D airfoil theory (Ref. 24) ($X_{EA} = -0.2C$).
Figure 33d. $\tilde{C}_{M\alpha}$ as a function of reduced frequency, $k = \omega c / 2U_\infty$ for a rectangular wing with $AR = 16$, $\tau = 0.1\%$, $M = 0$ and $NX = 8$, $NY = 10$. Comparison with the exact solution given by 2-D airfoil theory (Ref. 24) ($X_{EA} = -0.2C$).
APPENDIX A
FORMULATIONS FOR STEADY SUBSONIC FLOW

A.1 INTRODUCTION

The derivation of Equations (3.10) and (3.11) is presented in Reference 6.

In this appendix it will be shown by differentiation that doublet and source integrals in Equations (3.10) and (3.11) are valid for any planar quadrilateral element. In Subsection A.2, basic equations for hyperboloidal surface are introduced. In Subsection A.3, the doublet integral in Equation (3.10) is shown to be valid for any planar quadrilateral element. In Subsection A.4, the source integral in Equation (3.11) is shown to be valid for any planar quadrilateral element. In addition, the wake term in Equation (3.16) is derived in Subsection A.5.

A.2 BASIC EQUATION

According to Equation (3.15)
\[ \mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 / | \mathbf{a}_1 \times \mathbf{a}_2 | \]  
(A.1)
is independent of \( \xi \) and \( \eta \), therefore,
\[ \frac{\partial \mathbf{n}}{\partial \xi} = \frac{\partial \mathbf{n}}{\partial \eta} = 0 \]  
(A.2)
Also, it is noted that
\[ \frac{\partial \mathbf{a}_1}{\partial \xi} = 0 \]  
(A.3)
\[ \frac{\partial \mathbf{a}_2}{\partial \eta} = 0 \]  
(A.4)
\[ \frac{\partial}{\partial \xi} (\mathbf{q} \cdot \mathbf{n}) = \mathbf{a}_1 \cdot \mathbf{n} = 0 \]  
(A.5)
\[ \frac{\partial}{\partial \eta} (\mathbf{q} \cdot \mathbf{n}) = \mathbf{a}_2 \cdot \mathbf{n} = 0 \]  
(A.6)
\[ \frac{\partial \mathbf{a}_1}{\partial \eta} = \frac{\partial \mathbf{a}_2}{\partial \xi} = \mathbf{p}_3 \]  
(A.7)
For hyperboloidal elements with \( \bar{\alpha}_1 \) and \( \bar{\alpha}_2 \) defined by Equations (3.12), (3.13), and (3.14) with

\[
d \Sigma = 1 \frac{\bar{\alpha}_1 \times \bar{\alpha}_2}{|\bar{\alpha}_1 \times \bar{\alpha}_2|} \ d\xi \ d\eta
\]

the doublet and source integrals in Equations (3.4) and (3.5) can be rewritten as

\[
C_{hk} = \frac{1}{2\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{\sqrt{\bar{q} \cdot \bar{q}}} \left( \frac{1}{\sqrt{\bar{q} \cdot \bar{q}}} \right) |\bar{\alpha}_1 \times \bar{\alpha}_2| \ d\xi \ d\eta
\]

\[
= -\frac{1}{2\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{\bar{\alpha}_1 \times \bar{\alpha}_2 \cdot \bar{q}}{(\bar{q} \cdot \bar{q})^{3/2}} \ d\xi \ d\eta
\]

and

\[
b_{hk} = \frac{1}{2\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{\sqrt{\bar{q} \cdot \bar{q}}} \left( 1 \frac{\bar{\alpha}_1 \times \bar{\alpha}_2}{|\bar{\alpha}_1 \times \bar{\alpha}_2|} \right) \ d\xi \ d\eta
\]

since

\[
\frac{\bar{q}}{\sqrt{\bar{q} \cdot \bar{q}}} \left( \frac{1}{\sqrt{\bar{q} \cdot \bar{q}}} \right) |\bar{\alpha}_1 \times \bar{\alpha}_2|
\]

\[
= -\frac{|\bar{\alpha}_1 \times \bar{\alpha}_2|}{(\bar{q} \cdot \bar{q})^{3/2}} \left( \frac{\bar{\alpha}_1 \times \bar{\alpha}_2}{\sqrt{\bar{q} \cdot \bar{q}}} \right) \frac{1}{\sqrt{\bar{q} \cdot \bar{q}}}
\]

\[
= -\frac{|\bar{\alpha}_1 \times \bar{\alpha}_2|}{(\bar{q} \cdot \bar{q})^{3/2}} \left( \frac{\bar{\alpha}_1 \times \bar{\alpha}_2}{\sqrt{\bar{q} \cdot \bar{q}}} \right) \frac{1}{\sqrt{\bar{q} \cdot \bar{q}}}
\]

A.3 DOUBLET INTEGRAL

The following proves that Equation (3.10) is valid for any quadrilateral element. By using Equations (A.3) to (A.7) one obtains:

\[
\frac{\partial}{\partial \eta} \tan^{-1} \left[ \frac{(\bar{q} \cdot \bar{\alpha}_i) \cdot (\bar{q} \cdot \bar{\alpha}_2)}{\sqrt{\bar{q} \cdot \bar{q}} \cdot (\bar{q} \cdot \bar{\alpha}_1 \times \bar{\alpha}_2)} \right] = \frac{1}{1 + \left( \frac{(\bar{q} \cdot \bar{\alpha}_i) \cdot (\bar{q} \cdot \bar{\alpha}_2)}{\sqrt{\bar{q} \cdot \bar{q}} \cdot (\bar{q} \cdot \bar{\alpha}_1 \times \bar{\alpha}_2)} \right)^2}
\]
Considering terms inside the bracket yields

\[
\begin{align*}
\{(\bar{a}_2 \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2) + (\bar{q} \times \bar{p}_3) \cdot (\bar{q} \times \bar{a}_2) + (\bar{q} \times \bar{a}_1) \cdot (\bar{a}_2 \times \bar{a}_1)\} = \frac{1}{(\bar{q} \cdot \bar{a}_2)^2} \times \frac{1}{(\bar{q} \cdot \bar{a}_1, \bar{a}_2)^2} \times \\
\frac{\gamma^2 (\bar{q} \cdot \bar{a}_2 \times \bar{a}_1)^2}{\gamma^2 (\bar{q} \cdot \bar{a}_2 \times \bar{a}_1)^2 + (\bar{q} \times \bar{a}_1, \bar{a}_2)^2} \times \left\{\left((\bar{a}_2 \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2) \cdot (\bar{q} \times \bar{a}_1) \cdot (\bar{a}_2 \times \bar{a}_1)\right) \cdot (\bar{q} \cdot \bar{a}_2)\right\}.
\end{align*}
\]

(A.12)

\[
\begin{align*}
\{(\bar{q} \cdot \bar{a}_1) \times (\bar{a}_2) + (\bar{q} \times \bar{p}_3) \cdot (\bar{q} \times \bar{a}_2) \cdot (\bar{q} \cdot \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2) - (\bar{q} \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2) \cdot (\bar{q} \cdot \bar{p}_3) \cdot (\bar{q} \cdot \bar{a}_2)\} = \frac{1}{(\bar{q} \cdot \bar{a}_2)^2} \times \frac{1}{(\bar{q} \cdot \bar{a}_1, \bar{a}_2)^2} \times \\
\frac{\gamma^2 (\bar{q} \cdot \bar{a}_2 \times \bar{a}_1)^2}{\gamma^2 (\bar{q} \cdot \bar{a}_2 \times \bar{a}_1)^2 + (\bar{q} \times \bar{a}_1, \bar{a}_2)^2} \times \left\{\left((\bar{a}_2 \times \bar{a}_1) \cdot (\bar{q} \times \bar{a}_2) \cdot (\bar{q} \times \bar{a}_1) \cdot (\bar{a}_2 \times \bar{a}_1)\right) \cdot (\bar{q} \cdot \bar{a}_2)\right\}.
\end{align*}
\]
\[-(q - a_1) \cdot (q - a_2) \cdot a_3] \cdot [(q - a_2) - (q - a_1) \cdot (q - a_3)] - (q \cdot a_1) \cdot (q \cdot a_2) \cdot (q \cdot a_3) = 1 \| q \times a_2 \| ^2 q \cdot p_3 \cdot a_1 \text{ (A.13)}

Since (note the change of the order in the triple product)
\[
(q \cdot q) \cdot [(q \cdot p_3) \cdot (q \cdot a_2) - (q \cdot a_2) \cdot (q \cdot p_3)] = -(q \cdot q) \cdot [(q \cdot p_3) \cdot (q \cdot a_2) - (q \cdot a_2) \cdot (q \cdot p_3)]
\]
\[
= (q \cdot q) \cdot [(q \cdot a_2) \cdot (q \cdot p_3) - (q \cdot p_3) \cdot (q \cdot a_2)]
\]
\[
= (q \cdot q) \cdot [(q \cdot a_2) \cdot (q \cdot p_3) - (q \cdot p_3) \cdot (q \cdot a_2)]
\]
\[
= \frac{1}{2} \| q \times a_2 \| ^2 q \cdot p_3 \cdot a_1 \text{ (A.14)}
\]

Finally, by using Equation (D.2) and combining Equations (A.12) and (A.13) one obtains
\[
\frac{2}{\tan \rho} \cdot \frac{\tan \rho}{\sqrt{q \cdot q} \cdot (q \cdot a_2)} = \frac{1}{\sqrt{q \cdot q} \cdot \| q \times a_2 \| ^2} \| q \times a_2 \| ^2 \cdot (q \cdot a_1) \cdot (q \cdot a_2) - (q \cdot q) \cdot (q \cdot a_1) \cdot (q \cdot p_3) \text{ (A.15)}
\]
Finally, by using Equations (A.3), (A.7) and
\[ \frac{\partial}{\partial \xi} (\bar{q} \times \bar{a}_1) = \bar{a}_1 \times \bar{a}_1 = 0 \]  
(A.16)

one obtains,
\[ 2\pi \frac{\partial^2 I_{ij}}{\partial x_{ij}} = \frac{\partial^2}{\partial x_{ij}} \tan^{-1} \left( \frac{-\bar{q} \cdot \bar{a}_1}{\sqrt{\bar{q} \cdot \bar{q}} \sqrt{\bar{a}_1 \cdot \bar{a}_1}} \right) \]
\[ = \frac{\partial}{\partial x_{ij}} \left( \frac{(\bar{q} \cdot \bar{q})(\bar{q} \cdot \bar{a}_1 \times \bar{p}_3) - (\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2)}{\sqrt{\bar{q} \cdot \bar{q}} \sqrt{\bar{a}_1 \cdot \bar{a}_1}} \right) \]
\[ + \frac{\partial}{\sqrt{\bar{q} \cdot \bar{q}} \sqrt{\bar{a}_1 \cdot \bar{a}_1}} \left[ 2(\bar{a}_1 \cdot \bar{q})(\bar{q} \cdot \bar{a}_1 \times \bar{p}_3) + (\bar{q} \cdot \bar{a}_1)(\bar{a}_1 \cdot \bar{a}_1 \times \bar{a}_2) \right] \]
\[ - (\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2) - (\bar{q} \cdot \bar{a}_1)(\bar{a}_1 \cdot \bar{a}_1 \times \bar{a}_2) - (\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \times \bar{p}_3) \]
\[ = \frac{1}{(\bar{q} \cdot \bar{q})^{\frac{3}{2}} \sqrt{\bar{a}_1 \cdot \bar{a}_1}} \left[ (\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1) - (\bar{q} \cdot \bar{a}_1)(\bar{a}_1 \cdot \bar{a}_1) \right] (\bar{q} \cdot \bar{a}_1 \times \bar{a}_2) \]
\[ = \frac{\bar{q} \cdot \bar{a}_1 \times \bar{a}_2}{(\bar{q} \cdot \bar{q})^{\frac{3}{2}}} \]  
(A.18)

This proves that Equation (3.10) provides an analytical solutions of the doublet integral, given by Equation (A.9), for any hyperboloidal element.

A.4 SOURCE INTEGRAL

Use Equation (A.3) and note that
\[ \frac{\partial}{\partial \xi} (\bar{q} \times \bar{a}_1 \cdot \bar{n}) = \bar{a}_1 \times \bar{q} \cdot \bar{n} = 0 \]  
(A.19)
and

\[
\frac{\sigma^2}{\sigma^2_0} \ln \left( \frac{\bar{a}_i - \bar{q}}{\bar{q} \cdot \bar{a}_i} \right)
\]

\[
= \frac{1}{\bar{a}_i} \ln \left( \frac{\bar{q} \cdot \bar{a}_i}{\bar{q} \cdot \bar{q}} \right) + \frac{\bar{q} \cdot \bar{a}_i}{\sqrt{\bar{q} \cdot \bar{q}}}
\]

Hence,

\[
\frac{\sigma^2}{\sigma^2_0} \ln \left( \frac{\bar{q} \cdot \bar{a}_i \cdot \bar{n}}{\bar{a}_i} \right) \ln \left( \frac{\bar{q} \cdot \bar{q}}{\bar{q} \cdot \bar{q}} \right) = \frac{\sigma}{\sigma^2_0} \left( \frac{\bar{q} \cdot \bar{a}_i \cdot \bar{n}}{\sqrt{\bar{q} \cdot \bar{q}}} \right)
\]

\[
\frac{\bar{a}_x \cdot \bar{a}_i \cdot \bar{n}}{\sqrt{\bar{q} \cdot \bar{q}}} + \frac{\bar{q} \cdot \bar{a}_i \cdot \bar{n}}{\sqrt{\bar{q} \cdot \bar{q}}} - \frac{\bar{q} \cdot \bar{a}_i}{1 \bar{q}^2}
\]  

(A.20)

Similarly

\[
\frac{\sigma^2}{\sigma^2_0} \ln \left( \frac{\bar{q} \cdot \bar{a}_i \cdot \bar{n}}{\bar{a}_i} \right) \ln \left( \frac{\bar{q} \cdot \bar{q}}{\bar{q} \cdot \bar{q}} \right) = \frac{\sigma}{\sigma^2_0} \left( \frac{\bar{q} \cdot \bar{a}_i \cdot \bar{n}}{\sqrt{\bar{q} \cdot \bar{q}}} \right)
\]

\[
= \frac{\bar{a}_x \cdot \bar{a}_i \cdot \bar{n}}{\sqrt{\bar{q} \cdot \bar{q}}} + \frac{\bar{q} \cdot \bar{a}_i \cdot \bar{n}}{\sqrt{\bar{q} \cdot \bar{q}}} - \frac{\bar{q} \cdot \bar{a}_i}{1 \bar{q}^2}
\]  

(A.22)

Furthermore, noting that

\[
\frac{\sigma}{\sigma^2_0} = \frac{\bar{q}}{\bar{q}^2}
\]

and combining Equation (A.14) with Equations (A.5) and (A.6) yields

\[
\frac{\sigma^2}{\sigma^2_0} \ln \left( \frac{\bar{q} \cdot \bar{n}}{\bar{a}_i} \right) \tan^{-1} \left( \frac{\bar{q} \cdot \bar{n}}{\bar{a}_i} \right)
\]

\[
= - \frac{\bar{q} \cdot \bar{n}}{\bar{q} \cdot \bar{a}_i \bar{a}_x} \left( \frac{\bar{q} \cdot \bar{q}}{\bar{q} \cdot \bar{q}} \right)^{3/2}
\]

(A.24)

Finally combining Equations (A.21), (A.22) and (A.24) yields
This proves that Equation (3.11) provides an analytical solution of source integral, given by Equation (A.10), for any quadrilateral planar element.

A.5 WAKE COEFFICIENT

Consider Figure 2b, and assume that the wake element is truncated such that

\[ \vec{a}_1 = \vec{p}_i = \chi \vec{u} = \chi \vec{i} \]  

(A.26)

By letting \( \chi \) go to infinity, wake coefficient can be obtained from \( \Gamma_D \) in Equation (3.10). Note that (see Figure 2b)

\[ \vec{p}_0 - \vec{p}_i = (\vec{p}_+ + \vec{p}_-)/2 = \vec{p}_m \]  

(A.27)

\[ \vec{p}_i = \frac{\vec{p}_+ - \vec{p}_-}{2} = \vec{p}_d = \vec{a}_2 \]  

(A.28)

\[ \vec{p}_3 = 0 \]  

(A.29)
It is convenient to separate the contribution from the trailing edge ($\xi = -1$) and the edge that goes to infinity ($\xi = 1$):

$$I_w = \tan^{-1} \left( \frac{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}} \right)$$

where (note that $\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x} + \frac{1}{\xi} \frac{\partial \psi}{\partial x} + \eta \frac{\partial \psi}{\partial x}$)

$$S = \text{sign} \left( \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} \right) = \text{sign} \left( \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} \right)$$

with

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x} + \eta \frac{\partial \psi}{\partial x}$$

while (note that $\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x} + 2 \frac{\partial \psi}{\partial x}$)

$$J_w (1, \eta) = \lim_{x \to \infty} \tan^{-1} \left( \frac{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}} \right)$$

$$= \lim_{x \to \infty} \tan^{-1} \left( \frac{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}} \right)$$

$$= \lim_{x \to \infty} \tan^{-1} \left( \frac{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}} \right)$$

$$= \tan^{-1} \left( \frac{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x}} \right)$$

and similarly (note that $\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x} + \eta \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x}$)

A. 8
\[ J_W(-1, \eta) = \tan^{-1}\left( -\frac{\frac{q \times \bar{a}_1}{|q \times \bar{a}_1|} \cdot \frac{q \times \bar{a}_2}{|q \times \bar{a}_2|}}{1 - \frac{1}{q \times \bar{a}_1 \times \bar{a}_2}} \right) \] \[ \eta = -1 \]

\[ = \tan^{-1}\left( -\frac{\frac{q \times \bar{a}_2}{|q \times \bar{a}_2|} \cdot \frac{q \times \bar{a}_1}{|q \times \bar{a}_1|}}{1 - \frac{1}{q \times \bar{a}_1 \times \bar{a}_2}} \right) \]

\[ = \tan^{-1}\left( -\frac{\frac{P_{md} \times \bar{a}_2}{|P_{md} \times \bar{a}_2|} \cdot \frac{P_{md} \times \bar{a}_1}{|P_{md} \times \bar{a}_1|}}{1 - \frac{1}{P_{md} \times \bar{a}_2 \times \bar{a}_1}} \right) \]

\[ = \tan^{-1}\left( -\frac{(P_{md} \cdot P_{md}) (\bar{a}_1 \cdot \bar{a}_2) + (P_{md} \cdot \bar{a}_2) (P_{md} \cdot \bar{a}_1)}{|P_{md} \cdot |P_{md} \times \bar{a}_1 \times \bar{a}_2|} \right) \]  \[ (A.34) \]
APPENDIX B
FORMULATIONS FOR STEADY SUPERSONIC FLOW

B.1 INTRODUCTION

The derivation of Equations (5.8) and (5.9) is presented in Reference 7. In Subsections B.2 and B.3 it will be shown that these equations are valid for any quadrilateral planar element. In Subsection B.4, formulations of the doublet and source integral of the element intersected with the Mach forecone are considered.

For hyperboloidal elements with \( \overline{a}_r, \overline{a}_1, \overline{a}_2, \) and \( d\xi \) defined by Equations (3.12), (3.13), and (3.14) and (A.8), the doublet and source integral in Equations (4.7a) and (4.7b) can be rewritten as

\[
C_{hk} = \frac{1}{4} \int \int \frac{1}{\eta^2 N^2} \left( \frac{H}{||\overline{q}||} \right) |\overline{a}_1 \times \overline{a}_2| \, d\xi \, d\eta
\]

and

\[
b_{hk} = -\frac{1}{4} \int \int \frac{H}{||\overline{q}||} |\overline{a}_1 \times \overline{a}_2| \, d\xi \, d\eta
\]

Since

\[
\frac{\varphi}{\eta N^2} \left( \frac{H}{||\overline{q}||} \right) |\overline{a}_1 \times \overline{a}_2| = -\frac{|\overline{a}_1 \times \overline{a}_2|}{(\overline{q} \circ \overline{q})^{3/2}} \left( \overline{\varphi} N^2 \circ \overline{q} \right)
\]

\[
= -\frac{|\overline{a}_1 \times \overline{a}_2|}{(\overline{q} \circ \overline{q})^{3/2}} \left( N_x \frac{\varphi}{\eta^2} (x-x_4) \overline{\dot{a}} - N_y \frac{\varphi}{\eta^2} (y-y_4) \overline{\dot{a}} - N_z \frac{\varphi}{\eta^2} (z-z_4) \overline{\dot{a}} \right) \circ \overline{q}
\]

\[
= -\frac{|\overline{a}_1 \times \overline{a}_2|}{(\overline{q} \circ \overline{q})^{3/2}} N^2 \circ \overline{q} = \frac{\overline{a}_1 \times \overline{a}_2 \cdot \overline{q}}{(\overline{q} \circ \overline{q})^{3/2}}
\]
B.2 DOUBLET INTEGRAL

Following is the proof that Equation (5.8) is valid for any quadrilateral element within the Mach forecone. By using Equations (A.3), (A.4) and

\[
\frac{\partial \mathbf{a}_1}{\partial \eta} = \frac{\partial \mathbf{a}_2}{\partial \zeta} = \mathbf{P}_3
\]  

one obtains

\[
\frac{\partial}{\partial \eta} \tan^{-1} \frac{-\mathbf{q} \cdot \mathbf{a}_1 \circ \mathbf{q} \cdot \mathbf{a}_2}{\sqrt{\mathbf{q} \cdot \mathbf{q}}} \left( \frac{\mathbf{q} \cdot \mathbf{a}_1 \circ \mathbf{q} \cdot \mathbf{a}_2}{\mathbf{q} \cdot \mathbf{a}_1 \circ \mathbf{q} \cdot \mathbf{a}_2} \right) = \frac{1}{1 + \left( \frac{\mathbf{q} \cdot \mathbf{a}_1 \circ \mathbf{q} \cdot \mathbf{a}_2}{\mathbf{q} \cdot \mathbf{a}_1 \circ \mathbf{q} \cdot \mathbf{a}_2} \right)^2} \left( \mathbf{a}_2 \circ \mathbf{q} \cdot \mathbf{a}_2 + \mathbf{q} \cdot \mathbf{p}_3 \circ \mathbf{q} \cdot \mathbf{a}_2 - \mathbf{q} \cdot \mathbf{a}_1 \circ \mathbf{q} \cdot \mathbf{a}_2 - \mathbf{q} \cdot \mathbf{p}_3 \circ \mathbf{q} \cdot \mathbf{a}_2 \right)
\]

Furthermore,
\[
\left( (\vec{a}_3 \times \vec{a}_1 \circ \vec{q} \times \vec{a}_2) \vec{q} \circ \vec{g} - (\vec{q} \times \vec{a}_1 \circ \vec{q} \times \vec{a}_3) \vec{a}_2 \circ \vec{g} \right) (\vec{g} \cdot \vec{a}_1 \times \vec{a}_2) + \\
+ \left( (\vec{g} \cdot \vec{p}_3 \circ \vec{g} \times \vec{a}_2) (\vec{g} \cdot \vec{a}_1 \times \vec{a}_2) - (\vec{g} \times \vec{a}_1 \circ \vec{g} \times \vec{a}_3) (\vec{q} \cdot \vec{p}_3 \times \vec{a}_2) \right) \vec{q} \circ \vec{g}
\]

\[
= \left\{ \left( (\vec{q} \circ \vec{q}) (\vec{a}_1 \times \vec{a}_2) - (\vec{q} \circ \vec{a}_2)(\vec{a}_1 \circ \vec{q}) \right) \vec{g} \circ \vec{g} - \left( (\vec{q} \circ \vec{a}_3)(\vec{a}_1 \times \vec{a}_2) - (\vec{q} \circ \vec{a}_2)(\vec{a}_1 \circ \vec{q}) \right) \vec{q} \circ \vec{g} \right\} \vec{q} \cdot \vec{a}_2
\]

\[
= - \left\{ (\vec{q} \circ \vec{q})(\vec{a}_2 \circ \vec{a}_2) - (\vec{q} \circ \vec{a}_2)^2 \right\} (\vec{q} \circ \vec{a}_1)(\vec{q} \cdot \vec{a}_1 \times \vec{a}_2) + \left\{ (\vec{q} \circ \vec{q})(\vec{a}_2 \circ \vec{a}_2) - (\vec{q} \circ \vec{a}_2)^2 \right\} (\vec{q} \circ \vec{a}_1)(\vec{q} \cdot \vec{a}_1 \times \vec{a}_2)
\]

\[
= \| \vec{q} \times \vec{a}_2 \|^2 \left( (\vec{q} \circ \vec{a}_1)(\vec{q} \cdot \vec{a}_1 \times \vec{a}_2) - (\vec{q} \circ \vec{q})(\vec{q} \cdot \vec{a}_1 \times \vec{a}_2) \right)
\]

since

\[
(\vec{q} \circ \vec{q})(\vec{a}_2 \circ \vec{a}_2) - (\vec{q} \circ \vec{a}_2)^2 = (\vec{q} \times \vec{a}_2) \circ (\vec{q} \times \vec{a}_2) = - \| \vec{q} \times \vec{a}_2 \|^2 \quad \text{(B.7)}
\]

Finally, combining Equation (B.5) with (B.6) yields

\[
\frac{\partial^2 \mathcal{I}_p}{\partial \eta \partial \vec{g}} = \frac{\| \vec{q} \times \vec{a}_2 \|^2 \left( (\vec{q} \circ \vec{a}_1)(\vec{q} \cdot \vec{a}_1 \times \vec{a}_2) - (\vec{q} \circ \vec{a}_2)(\vec{q} \cdot \vec{a}_1 \times \vec{a}_2) \right)}{\| \vec{q} \times \vec{a}_2 \|^2 \| \vec{q} \|^2}
\]

\[
= \frac{1}{\| \vec{q} \times \vec{a}_2 \|^2 \| \vec{q} \|^2} \left( (\vec{q} \circ \vec{a}_1)(\vec{q} \cdot \vec{a}_1 \times \vec{a}_2) - (\vec{q} \circ \vec{a}_2)(\vec{q} \cdot \vec{a}_1 \times \vec{a}_2) \right)
\]

(B.8)

Next, consider the second mixed derivative, since

\[
\frac{\partial^2}{\partial \vec{g} \partial \vec{g}} (\vec{q} \times \vec{a}_1) = \frac{\partial}{\partial \vec{g}} \left( (\vec{p}_0 + \eta \vec{p}_2) \times (\vec{p}_1 + \eta \vec{p}_3) \right) = 0
\]

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Therefore,

\[
\frac{\partial^2 \tau_{xy}}{\partial \gamma \partial \eta} = \frac{1}{\bar{\eta} \cdot \bar{\eta} \cdot \bar{a}_1^2} \frac{\partial}{\partial \gamma} \left\{ \frac{\bar{q} \cdot \bar{a}_1}{\bar{q} \cdot \bar{a}_1} \left( \frac{(\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \cdot \bar{F}_3) - (\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \cdot \bar{a}_2)}{\sqrt{\bar{q} \cdot \bar{q}}} \right) \right\}
\]

\[
= \frac{1}{\bar{q} \cdot \bar{a}_1^2} \left\{ \frac{\bar{q} \cdot \bar{a}_1}{\bar{q} \cdot \bar{a}_1} \left( \frac{(\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \cdot \bar{F}_3) - (\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \cdot \bar{a}_2)}{\sqrt{\bar{q} \cdot \bar{q}}} \right) + \frac{1}{\sqrt{\bar{q} \cdot \bar{q}}} \left( \bar{q} \cdot \bar{a}_1 \cdot \bar{a}_3 \right) \right\}
\]

\[
+ \frac{1}{\sqrt{\bar{q} \cdot \bar{q}}} \left( \left( \frac{(\bar{q} \cdot \bar{a}_1)^2}{\bar{q} \cdot \bar{a}_1} - \frac{(\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \cdot \bar{F}_3)}{\sqrt{\bar{q} \cdot \bar{q}}} \right) \bar{q} \cdot \bar{a}_1 \cdot \bar{a}_2 \right)
\]

\[
= \frac{1}{\bar{q} \cdot \bar{a}_1^2} \left\{ \frac{\bar{q} \cdot \bar{a}_1}{\bar{q} \cdot \bar{a}_1} \left( \frac{(\bar{q} \cdot \bar{a}_1)^2}{\bar{q} \cdot \bar{a}_1} - \frac{(\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_1 \cdot \bar{F}_3)}{\sqrt{\bar{q} \cdot \bar{q}}} \right) \right\}
\]

Thus, prove that Equation (5.8) provides an analytical solution of the doublet integral given by Equation (B.1) for any quadrilateral element.

B.3 SOURCE INTEGRAL

Following is the proof that Equation (5.19) is valid for any quadrilateral element within the Mach forecone. Noting that, as shown in Appendix D,

\[
\frac{\partial \bar{F}_3}{\partial \gamma} = \frac{\partial \bar{F}_2}{\partial \eta} = \frac{1}{\bar{q} \cdot \bar{a}_1^2} \quad \text{(B.10)}
\]

and

\[
\frac{\partial}{\partial \gamma} \left( \bar{q} \cdot \bar{a}_1 \cdot \bar{a}_2 \right) = \bar{a}_1 \cdot \bar{a}_2 \cdot \bar{a}_2 = 0 \quad \text{(B.11a)}
\]

\[
\frac{\partial}{\partial \eta} \left( \bar{q} \cdot \bar{a}_1 \cdot \bar{a}_2 \right) = \bar{a}_1 \cdot \bar{a}_2 \cdot \bar{a}_2 = 0 \quad \text{(B.11b)}
\]

yields

\[
\frac{\partial}{\partial \gamma} \left\{ \left( \frac{\bar{a}_1 \cdot \bar{a}_2}{\bar{q} \cdot \bar{a}_1 \cdot \bar{a}_2} \right) \bar{F}_3(\gamma, \eta) \right\} = 0
\]

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\[
\frac{2^*}{\omega_0 \theta} \left( \frac{\partial}{\partial \theta} \left( \frac{1}{q_x a_y} \frac{1}{q_y} \right) \right) = \frac{\partial_x a_y}{\|q_y\|} + \frac{\partial_y a_x}{\|q_x\|} - \frac{q \cdot a}{\|q\|} \left( \frac{\partial \theta}{\partial \theta} \right)_{1/2} \tag{B.12a}
\]

Similarly,
\[
\frac{2^*}{\omega_0 \theta} \left\{ \frac{\partial_x a_y \cdot \partial_y g}{\|q_y\|} \right\} = \frac{\partial_x a_y}{\|q_y\|} + \frac{\partial_y a_x}{\|q_x\|} - \frac{q \cdot a}{\|q\|} \left( \frac{\partial \theta}{\partial \theta} \right)_{1/2} \tag{B.12b}
\]

In addition, from Equation (B.9),
\[
\frac{2^*}{\omega_0 \theta} \left\{ \frac{g \cdot q \cdot \partial \theta}{\|q\|} \right\} = \frac{-q \cdot a}{\|q\|} \tag{B.13}
\]

Finally, combining Equations (4.10), (B.12b), (B.13) and (B.14) and noting that
\[
\|q\| = -n \cdot n
\]
yields
\[
\frac{2^*}{\omega_0 \theta} \left\{ \frac{\partial_x a_y \cdot \partial_y g}{\|q_y\|} \right\} = \frac{\partial_x a_y}{\|q_y\|} + \frac{\partial_y a_x}{\|q_x\|} - \frac{q \cdot a}{\|q\|} \left( \frac{\partial \theta}{\partial \theta} \right)_{1/2} \tag{B.15a}
\]

According to Equation (D.7) for
\[
-\left| \frac{\partial_x a_y}{\|q_x a_y \|} \right| - \left( \frac{\partial_x a_y \cdot \partial_y g}{\|q_y\|} \right) = \frac{\partial \theta}{\partial \theta} \tag{B.15b}
\]

According to Equation (D.7) for
\[
-\left| \frac{\partial_x a_y}{\|q_x a_y \|} \right| - \left( \frac{\partial_x a_y \cdot \partial_y g}{\|q_y\|} \right) = \frac{\partial \theta}{\partial \theta} \tag{B.15b}
\]

According to Equation (D.7) for
\[
-\left| \frac{\partial_x a_y}{\|q_x a_y \|} \right| - \left( \frac{\partial_x a_y \cdot \partial_y g}{\|q_y\|} \right) = \frac{\partial \theta}{\partial \theta} \tag{B.15b}
\]
By using Equation (B.16), Equation (B.15) reduces to

\[
\frac{\partial^2 I_s}{\partial \xi \partial \eta} = \frac{1}{\hat{n} \cdot \overline{n}} \left\{ 2\hat{n} \cdot \overline{n} \frac{\overline{a}_1 \times \overline{a}_2}{\| \overline{q} \|} - \overline{q} \cdot \frac{\overline{a}_1 \times \overline{a}_2}{\| \overline{q} \|^2} \right\}
\]

\[
= \frac{1}{\hat{n} \cdot \overline{n}} \left\{ \frac{\overline{a}_1 \times \overline{a}_2}{\| \overline{q} \|^2} \right\}
\]

(B.17)

This proves that Equation (5.9) provides an analytical solution of the integral in Equation (5.5) for any quadrilateral element.

B.4 FINITE PARTS OF INTEGRALS

In order to extend the results to a planar quadrilateral element intersected with the Mach forecone, the finite part of the following special integral is investigated.

\[
I = \int_a^{\infty} \frac{g(x)}{\sqrt{x}} H(x) \, dx
\]

(B.18)

where \( g \) is a regular function and \( H \) is the Heaviside function, Equation (B.18) can be rewritten as

\[
I = \lim_{\varepsilon \to 0} \int_a^{\infty} \frac{g(x)}{\sqrt{x}} \left( \frac{H(x - \varepsilon)}{\varepsilon} \right) \, dx
\]
\[
\lim_{\varepsilon \to 0} \left( \int_{-\infty}^{\infty} \frac{g}{\sqrt{x}} \left( \frac{g}{\sqrt{x}} \right) dx + \int_{\varepsilon}^{a} \frac{g}{\sqrt{\varepsilon}} \delta (x - \varepsilon) dx \right)
\]

\[
= \lim_{\varepsilon \to 0} \left( \int_{\varepsilon}^{a} \frac{g}{\sqrt{\varepsilon}} \delta (x - \varepsilon) dx + \frac{g(\varepsilon)}{\sqrt{\varepsilon}} \right)
\]

\[
= \frac{g(a)}{\sqrt{a}} - \frac{g(\varepsilon)}{\sqrt{\varepsilon}} + \frac{g(\varepsilon)}{\sqrt{\varepsilon}} = \frac{g(a)}{\sqrt{a}}
\]

(B.19)

This shows that the singular contribution disappears and should not be taken into account.

Thus, consider the source integral given in Equation (B.9). According to Equations (B.15a) and (B.17), the integrant of Equation (B.9), \( \frac{1}{11 \cdot \|B\|} \), can be written as

\[
\frac{1}{\|B\|} H = - \frac{1}{\|B\|} \left\{ \left( \frac{a_{1} \cdot \bar{a}_{2} \cdot \bar{a}_{2} \cdot \bar{a}_{2}}{\|B\|} \right) H + \left( \frac{a_{1} \cdot \bar{a}_{2} \cdot \bar{a}_{2} \cdot \bar{a}_{2}}{\|B\|} \right) H \right\}
\]

therefore, Equation (B.9) can be rewritten as

\[
\sum_{h} \left( \frac{1}{\|B\|} \int_{\Omega} \left( \frac{\bar{a}_{1} \cdot \bar{a}_{2} \cdot \bar{a}_{2} \cdot \bar{a}_{2}}{\|B\|} \right) H \right) \hat{d} \eta
\]

(B.20)

Therefore, Equation (B.9) can be rewritten as

\[
b_{kk} = \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} \left( \frac{1}{\|B\|} \int_{\Omega} \left( \frac{\bar{a}_{1} \cdot \bar{a}_{2} \cdot \bar{a}_{2} \cdot \bar{a}_{2}}{\|B\|} \right) H \right) \hat{d} \eta = \frac{1}{\pi} \frac{1}{\|B\|} \left( S_{1} + S_{2} + S_{3} \right)
\]

(B.21)

\[
S_{1} = I_{51} (1, 1) - I_{52} (-1, 1) - I_{53} (-1, 1) + I_{54} (-1, 1)
\]

(B.22a)

\[
= - \int_{-1}^{1} \int_{-1}^{1} \left( \frac{\bar{a}_{1} \cdot \bar{a}_{2} \cdot \bar{a}_{2} \cdot \bar{a}_{2}}{\|B\|} \right) H \hat{d} \eta
\]

(B.22b)
First, according to Equation (B.12b), Equation (B.22b) can be rewritten as

\[ S_2 = I_{s2} (1, 1) - I_{s2} (1, -1) - I_{s2} (-1, 1) + I_{s2} (-1, -1) \]  \hspace{1cm} (B.23a)

\[ = \int_{-1}^{1} \int_{-1}^{1} H \left( \frac{\tilde{a}_1 \times \tilde{a}_2 \cdot \tilde{e}_n}{\| \tilde{e}_n \|} - \frac{\tilde{q} \cdot \tilde{p}_n \cdot \tilde{e}_n}{\| \tilde{q} \| \| \tilde{e}_n \|} - \frac{\tilde{q} \cdot \tilde{a}_2 \cdot \tilde{e}_n}{\| \tilde{q} \| \| \tilde{e}_n \|} \right) \frac{d \tilde{e}_n}{\| \tilde{e}_n \|} \frac{d \tilde{q}}{\| \tilde{q} \|} \]  \hspace{1cm} (B.23b)

\[ S_3 = I_{s3} (1, 1) - I_{s3} (1, -1) - I_{s3} (-1, 1) + I_{s3} (-1, -1) \]  \hspace{1cm} (B.24a)

\[ = -\int_{-1}^{1} \int_{-1}^{1} H \frac{\tilde{q} \cdot \tilde{a}_2 \cdot \tilde{e}_n}{\| \tilde{q} \| \| \tilde{e}_n \|} \frac{d \tilde{e}_n}{\| \tilde{e}_n \|} \frac{d \tilde{q}}{\| \tilde{q} \|} \]  \hspace{1cm} (B.24b)

Note that \( \sqrt{\tilde{q} \cdot \tilde{a}_1 \cdot \tilde{a}_2} \) can be expressed as \( \lambda (\eta) \sqrt{\eta^2 - \eta^2_0} \), where \( \lambda (\eta) \) is a regular function and \( \eta_0 \) is defined such that \( \tilde{q} \cdot \tilde{a}_1 \cdot \tilde{a}_2 = 0 \). Compared with Equation (B.18), it can be concluded that the portions along the intersection line of the element with the Mach forecone yield no contributions to the integral \( S_1 \). Therefore, in Equation (B.22a)

\[ I_{s1} (1, 1) - I_{s1} (-1, 1) = 0 \]  \hspace{1cm} (B.26)

or

\[ I_{s1} (1, -1) - I_{s1} (-1, -1) = 0 \]  \hspace{1cm} (B.27)

if the edge of \( \eta = 1 \) or \( \eta = -1 \), respectively, is completely outside the Mach forecone.

Otherwise, \( I_{s1} \) is given by Equation (5.10) if the corner point is inside the Mach forecone, or

\[ I_{s1} (\tilde{q}, \eta) = 0 \]  \hspace{1cm} if \( \tilde{q} \cdot \tilde{a}_1 \cdot \tilde{a}_2 > 0 \)

\[ = \left( \frac{\tilde{q} \cdot \tilde{a}_1 \cdot \tilde{a}_2}{\| \tilde{a}_1 \|} \right) \text{sign} \left( \frac{\tilde{q} \cdot \tilde{a}_1 \cdot \tilde{a}_2}{\| \tilde{a}_1 \|} \right) \]  \hspace{1cm} if \( \tilde{q} \cdot \tilde{a}_1 \cdot \tilde{a}_2 < 0 \)

if the corner point is outside the Mach forecone. Similarly, Equation (B.23b) can be rewritten as
therefore.

\[ I_{S2}(1,1) - I_{S2}(-1,1) = 0 \]  (B.30)

or

\[ I_{S2}(-1,1) - I_{S2}(-1,-1) = 0 \]  (B.31)

if the edge of \( \xi = 1 \), or \( \xi = -1 \), respectively, is completely outside the Mach forecone. Otherwise, \( I_{S2} \) is given by Equation (5.11) if the corner point is inside the Mach forecone, or

\[ I_{S2}(\xi^*, \eta) = \begin{cases} 0 & ; \overline{a_2} \cdot \overline{a_2} > 0 \\ - (q_x \overline{a_2} \cdot \overline{n}) \text{sign}(q_y \overline{a_2}) / \| \overline{a_2} \| & ; \overline{a_2} \cdot \overline{a_2} < 0 \end{cases} \]  (B.32)

if the corner point is outside the Mach forecone. Finally, considering Equation (D.10), Equations (B.24b) and (B.7) can be rewritten as

\[ S_3 = - \overline{a} \cdot \overline{n} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{2 \eta} \left( \frac{H_1}{\| \overline{a_1} \|} \right) \left( \frac{\overline{a_1} \cdot \overline{a_2} - \overline{a_1} \cdot \overline{a_3} - \overline{a_2} \cdot \overline{a_3} - \overline{a_2} \cdot \overline{a_1}}{\| \overline{a_1} \| \cdot \| \overline{a_2} \|} \right) d\eta \]  (B.33)

and

\[ C_{HK} = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{2 \eta} \left( \frac{H_1}{\| \overline{a_1} \|} \right) \left( \frac{\overline{a_1} \cdot \overline{a_2} - \overline{a_1} \cdot \overline{a_3} - \overline{a_2} \cdot \overline{a_3} - \overline{a_2} \cdot \overline{a_1}}{\| \overline{a_1} \| \cdot \| \overline{a_2} \|} \right) d\eta \]  (B.34)

with the same reason given above, one obtains

\[ I_{S3}(1,1) - I_{S3}(-1,1) = 0 \]  (B.35)

\[ I_{0}(1,1) - I_{P}(-1,1) = 0 \]  (B.36)

or

\[ I_{S3}(1,-1) - I_{S3}(-1,-1) = 0 \]  (B.37)

\[ I_{0}(1,-1) - I_{P}(-1,-1) = 0 \]  (B.38)
if the edge of $\eta=1$ or $\eta=-1$, respectively, is completely outside the Mach forecone. Otherwise, $I_{ss}$ and $I_D$ are given by Equations (5.12) and (5.8), if the corner point is inside the Mach forecone, or

$$I_{ss}(q^*, \eta^*) = \frac{\pi}{2} \text{sign} \left( \left( \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{q} \cdot \mathbf{n}} \right) \right)$$

(B.39)

$$I_D(q^*, \eta^*) = -\frac{1}{2} \text{sign} \left( \left( \frac{\mathbf{a} \cdot \mathbf{n}}{\mathbf{q} \cdot \mathbf{n}} \right) \right)$$

(B.40)

Note that the values of $q^*$ and $\eta^*$ in Equations (B.32), (B.28), (B.39) and (B.40) are evaluated such that $\frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{q} \cdot \mathbf{n}} = 0$.

### B.5 INTEGRAL EQUATIONS FOR DIAPHRAGM-ATTACHED CONFIGURATION

As mentioned in Subsection 5.3, if diaphragms are used, both values of the velocity potential and its normal derivative, $\phi_p$ and $\gamma_p$, of the diaphragm element are unknowns while two integral equations are obtained for each diaphragm element. Therefore, a system of equations, instead of Equation (5.3), may be obtained as follows.

$$\left( S_{hk} - a_{ih} \right) \begin{pmatrix} \phi_{hk}^u \\ \phi_{hk}^l \\ \phi_{hk}^p \\ \gamma_{hk}^p \end{pmatrix} = \begin{pmatrix} b_{hk}^i \\ b_{hk}^j \\ b_{hk}^k \\ b_{hk}^l \end{pmatrix}$$

(B.41)

where $\phi_{hk}^u$ and $\gamma_{hk}^u$ (or $\phi_{hk}^l$ and $\gamma_{hk}^l$) are the velocity potential and its normal at the centroid of the element on the upper (or lower) surface of the aircraft, while $\phi_{hk}^p$ and $\gamma_{hk}^p$ are those quantities for the diaphragm element, and $\int_{\mathcal{A}_K}$ is given by
where the superscript \( u \) (or \( \ell \) ) reminds that both the control point and dummy element are on the upper (or lower) surface and first subscript \( a \) (or \( d \) ) reminds that the control point is on the surface of aircraft (or diaphragm) second subscript \( m \) (or \( n \) ) reminds that the dummy element is on the surface of aircraft (or diaphragm). For example, \( C_{an}^{\ell} \) represents the submatrix formed by the doublet integrals, given by Equation (5.4), with control point on the upper surface of aircraft,
and dummy element on the upper surface of diaphragm. \( b_{dm} \) represents the submatrix formed by the source integrals, given by Equation (5.5) with control point on the lower surface of diaphragm and dummy element on the lower surface of aircraft.

Furthermore, if the problem is symmetric, the normal derivative of the velocity potential, \( \psi_p \), on the diaphragm is zero, while the velocity potential, \( \psi_p \), is unknown,

\[
\begin{align*}
\psi_u^v &= \psi_l^v, & \psi_u^v &= \psi_l^v \\
C_u^{am} &= C_l^{am}, & C_u^{an} &= C_l^{an} \\
C_u^{dm} &= C_l^{dm}, & C_u^{dn} &= C_l^{dn} \\
b_u^{am} &= b_l^{am}, & b_u^{an} &= b_l^{an} \\
b_u^{dm} &= b_l^{dm}, & b_u^{dn} &= b_l^{dn}
\end{align*}
\]

Therefore, Equation (B.41) reduced to

\[
\left[ \delta_{hk} - a_{hk} \right] \begin{bmatrix} \psi_u^v \\ \psi_p \end{bmatrix} = \begin{bmatrix} \left. b_{hk} \right| \psi_u^v \end{bmatrix} \begin{bmatrix} \chi_p^v \\ 0 \end{bmatrix}
\]

with

\[
\left[ a_{hk} \right] = \begin{bmatrix} C_u^{am} & C_u^{an} \\ C_u^{dm} & C_u^{dn} \end{bmatrix}
\]

and

\[
\left[ b_{hk} \right] = \begin{bmatrix} b_u^{am} & 0 \\ 0 & 0 \end{bmatrix}
\]
For the case that the problem is antisymmetric, \( \mathcal{J}_p \) is zero while \( \mathcal{Y}_p \) is unknown. Then Equation (B.41) reduced to Equation (B.46) with \([b_{hk}]\) given by (B.48) and \([a_{hk}]\) given by

\[
[a_{hk}] = \begin{bmatrix}
C_{am}^u & b_{an}^u \\
-C_{dm}^u & b_{dn}^u
\end{bmatrix}
\]  

(B.49)
C.1 BOUNDARY CONDITION

The boundary condition for oscillatory flow is given by

\[ \nabla_{XYZ} \cdot \nabla_{XYZ} \gamma = - \frac{2S}{\partial t} - \nabla_{XYZ} \gamma = 0 \]  

(C.1)

C.1.1 Subsonic Flow

By introducing variables \( X, Y, Z, T, \Omega \) and \( \phi \) given by Equations (2.2) and (3.3), and noting that \( \beta^2 = 1 - M^2 \), Equation (C.1) yields

\[ \nabla_{XYZ} \cdot \nabla_{XYZ} \phi + \frac{\theta}{M} \frac{2S}{\theta T} + \frac{1}{\beta} \frac{2S}{\partial X} + \frac{M^2}{\beta^2} \frac{2S}{\partial X} \frac{2\phi}{\partial X} = 0 \]  

(C.2)

The motion of the surface is assumed to consist of small harmonic oscillations around a rest configuration that is

\[ S = S_0 (x, y, z) + \tilde{S} (x, y, z) e^{i\alpha T} \]  

(C.3)

and

\[ \phi = \phi_0 (x, y, z) + \tilde{\phi} (x, y, z) e^{i\alpha T} \]  

(C.4)

Substituting Equations (C.3) and (C.4) into Equation (C.2) yields

\[ \nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi_0 + (\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \tilde{S} + \nabla_{XYZ} \tilde{S} \cdot \nabla_{XYZ} \phi_0) e^{i\alpha T} + \]

\[ + (\nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \tilde{S}) e^{i\alpha T} + \frac{\theta}{M} \hat{\alpha} S e^{i\alpha T} + \frac{1}{\beta} (\frac{2S_0}{\partial X} \frac{2\phi_0}{\partial X} e^{i\alpha T}) + \]

\[ + \frac{M^2}{\beta^2} \left( - \frac{2S_0}{\partial X} \frac{2\phi_0}{\partial X} + \frac{2S_0}{\partial Y} \frac{2\phi_0}{\partial Y} \right) e^{i\alpha T} + \frac{2S_0}{\partial X} \frac{2\phi_0}{\partial X} e^{2i\alpha T} \] = 0 \]  

(C.5)

Small terms can be dropped out by investigating the order of each quantity. Assuming*

*For instance, if \( z = ez_0(x, y) + es_2(x, y) e^{i\alpha T} \), then \( S_0 = z - ez_0(x, y), \tilde{S} = es_2(x, y) \).
\( S_0 = 0 (1) \) \hspace{2cm} (C.6)

\( \frac{\partial S_0}{\partial X} = 0 (\xi) \) \hspace{2cm} (C.7)

with

\[ | \nabla_{XY} S_0 | = 0 (1) \]

and

\[ \ddot{S} = 0 (\xi^2) \]

\[ \frac{\partial \ddot{S}}{\partial X} = 0 (\xi^2) \]

\[ | \nabla_{XYz} \ddot{S} | = 0 (\xi^2) \]

Also, [see, for instance, Equations (C.16) and (C.17)], it is noted that

\[ | \nabla_{XYz} \Phi_0 | = 0 (\xi) \] \hspace{2cm} (C.12)

\[ | \nabla_{XYz} \ddot{\Phi} | = 0 (\xi^2) \] \hspace{2cm} (C.13)

Investigating the order of each term of Equation (C.5) by Equations (C.6) to (C.13) and ignoring the terms which contain \( e^{i2\alpha T} \) (of order \( \xi^4 \)), one obtains the stationary and the oscillatory boundary conditions which are given as

\[ \nabla_{XYz} S_0 \cdot \nabla_{XYz} \Phi_0 + \frac{1}{\alpha} \frac{\partial S_0}{\partial X} + \frac{M^2}{\beta^2} \frac{\partial S_0}{\partial X} \frac{\partial \Phi_0}{\partial X} = 0 \] \hspace{2cm} (C.14)

and

\[ \nabla_{XYz} S_0 \cdot \nabla_{XYz} \ddot{\Phi} + \nabla_{XYz} \ddot{S} \cdot \nabla_{XYz} \Phi_0 + \frac{\rho}{M} i \omega \ddot{S} \]

\[ + \frac{1}{\beta} \frac{\partial \ddot{S}}{\partial X} + \frac{M^2}{\rho^2} \left( \frac{\partial \ddot{S}}{\partial X} \frac{\partial \Phi_0}{\partial X} + \frac{\partial \ddot{S}}{\partial X} \frac{\partial \ddot{\Phi}}{\partial X} \right) = 0 \] \hspace{2cm} (C.15)

Neglecting terms of order \( \xi^2 \) in Equation (C.14) and terms of order \( \xi^3 \) in Equation (C.15), one obtains

\[ C - 2 \]
\[ \nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi = -\frac{1}{\beta} \frac{\partial S_0}{\partial X} \]  \hspace{1cm} (C.16)

\[ \nabla_{XYZ} S_0 \cdot \nabla_{XYZ} \phi = -\left( i \frac{\beta}{M} - \alpha \right) S + \frac{1}{\beta} \frac{\partial S}{\partial X} \]  \hspace{1cm} (C.17)

In particular for:

\[ S = \pm \frac{1}{\kappa} \left( \bar{z} - \bar{z}_0 (x, y) - \bar{z}(x, y) e^{i \omega t} \right) \]  \hspace{1cm} (C.18)

(where the upper or lower sign holds on the upper or lower surface); one obtains

\[ S_0 = \pm \frac{1}{\kappa} \left( \bar{z} - \bar{z}_0 (x, y) \right) \]  \hspace{1cm} (C.19)

\[ \bar{S} = \pm \frac{1}{\kappa} \bar{z}(x, y) \]  \hspace{1cm} (C.20)

\[ \frac{1}{|\nabla_{XYZ} S_0|} = \frac{|\bar{S}_0|}{|\nabla_{XYZ} S_0|} = |\bar{z}| = \pm N_z \]  \hspace{1cm} (C.21)

Therefore,

\[ \frac{\partial \phi}{\partial N} = \frac{\nabla_{XYZ} S_0 - \nabla_{XYZ} \phi}{|\nabla_{XYZ} S_0|} = N_z \left( i K \frac{\bar{z}}{\kappa} + \frac{\partial \bar{z}}{\partial X} \right) \]  \hspace{1cm} (C.22)

where

\[ K = \beta \frac{\Omega}{M} = \omega \frac{L}{U_\infty} \]  \hspace{1cm} (C.23)

By using Equation (2.31), Equation (C.22) yields

\[ \frac{\partial \phi}{\partial N} = N_z \left( i K \frac{\bar{z}}{\kappa} + \frac{1}{\beta} \frac{\partial \bar{z}}{\partial X} \right) e^{-i \omega M X} \]  \hspace{1cm} (C.24)

Equation (C.20) gives the value of \( \frac{\partial \phi}{\partial N} \) to be used in Equation (4.3).

---

*Note that \( z_0 \) is of order \( \varepsilon \) and \( z \) is of order \( \varepsilon^2 \).
C.1.2 Supersonic Flow

By introducing variable $x$, $y$, $z$, $T$, $\Omega$ and $\phi$ given by Equations (2.2) and (3.3), and noting that $\beta^2 = M^2 - 1$, Equation (C.1) yields

$$\nabla_{xyz} s \cdot \nabla_{xyz} \phi + \frac{\beta}{M} \frac{\partial s}{\partial T} + \frac{1}{\beta} \frac{\partial s}{\partial x} + \frac{M^2 - 1}{\beta^2} \frac{\partial s}{\partial x} \frac{\partial \phi}{\partial x} = 0$$  \hspace{1cm} (C.25)

with $\nabla_{xyz}$ defined by

$$\nabla_{xyz} = -\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z}$$  \hspace{1cm} (C.26)

By replacing $\nabla_{xyz}$ with $\nabla_{xyz} \phi_0$ and $\nabla_{xyz} \phi$ with $\nabla_{xyz} \phi$ and $\nabla_{xyz} \phi$, Equations (C.5) to (C.22) are valid for supersonic flow. Therefore, the boundary condition for supersonic flow can be rewritten as

$$\frac{\partial \phi}{\partial N} = N_z \left(\beta k \tilde{z} + \frac{\partial \tilde{z}}{\partial x}\right)$$  \hspace{1cm} (C.27)

By using Equation (2.39), Equation (C.27) yields

$$\frac{\partial \tilde{\phi}}{\partial N} = N_z \left(\beta k \tilde{z} + \frac{\partial \tilde{z}}{\partial x}\right) e^{i \lambda_{nx}}$$  \hspace{1cm} (C.28)

Equation (C.28) gives the value of $\frac{\partial \tilde{\phi}}{\partial N}$ to be used in Equation (6.3).

C.2 Pressure Coefficient

The pressure coefficient is given by the linear Bernoulli theorem, Equation (2.41), as

$$C_p = -\frac{2}{U_0^2} \left(\frac{\partial \phi}{\partial t} + U_{0x} \frac{\partial \phi}{\partial x}\right)$$

$$= -2 \left(-\frac{\beta}{M} \frac{\partial \phi}{\partial T} + \frac{1}{\beta} \frac{\partial \phi}{\partial x}\right)$$  \hspace{1cm} (C.29)

For oscillatory flow, setting
C.2.1 Subsonic Flow

By using Equation (2.31)

\[ \Phi = \Phi e^{i\alpha_1 t} \]  \hspace{1cm} (C.30)

\[ c_p = c_p e^{i\alpha} \]

one obtains

\[ \tilde{c}_p = -2 \left( \frac{\beta}{M} i \alpha \tilde{\Phi} + \frac{1}{\beta} \frac{\partial \tilde{\Phi}}{\partial x} \right) \]

\[ = -\frac{2}{\beta} \left( \frac{\partial \tilde{\Phi}}{\partial x} + i \kappa \beta \tilde{\Phi} \right) \]  \hspace{1cm} (C.31)

C.2.2 Supersonic Flow

By using Equation (2.39)

\[ \tilde{\Phi} = \tilde{\Phi} e^{i\alpha_2 x} \]  \hspace{1cm} (C.32)

\[ c_p \] for oscillatory subsonic flow can be rewritten from Equation (C.31) as

\[ \tilde{c}_p = -2 \left( \frac{\beta}{M} i \alpha \tilde{\Phi} + \frac{1}{\beta} \frac{\partial \tilde{\Phi}}{\partial x} \right) \]

\[ = -\frac{2}{\beta} \left( \frac{\partial \tilde{\Phi}}{\partial x} + i \kappa \beta \tilde{\Phi} \right) \]  \hspace{1cm} (C.33)

C.2.2 Supersonic Flow

By using Equation (2.39)

\[ \tilde{\Phi} = \tilde{\Phi} e^{-i\alpha_2 x} \]  \hspace{1cm} (C.34)

\[ c_p \] for oscillatory supersonic flow can be rewritten from Equation (C.31) as

\[ \tilde{c}_p = -2 \left( \frac{\beta}{M} i \alpha \tilde{\Phi} + \frac{1}{\beta} \frac{\partial \tilde{\Phi}}{\partial x} \right) \]

\[ = -\frac{2}{\beta} \left( \frac{\partial \tilde{\Phi}}{\partial x} + i \kappa \beta \tilde{\Phi} \right) \]  \hspace{1cm} (C.35)
APPENDIX D
USEFUL EQUATIONS

D.1 INTRODUCTION

In deriving the formulations presented in previous sections, several useful equations are applied. They are proven in this appendix.

For the subsonic flow theory, the following two equations are used.

\[
(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{a} \cdot \bar{d})(\bar{b} \cdot \bar{c}) \tag{D.1}
\]

\[
(\bar{q} \cdot \bar{g})(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2) + (\bar{q} \cdot \bar{a}_1)(\bar{q} \cdot \bar{a}_2)^2 \equiv \frac{1}{\bar{q}} x \bar{q}_1 |^2 \bar{q} x \bar{q}_2 |^2 \tag{D.2}
\]

For the supersonic flow theory, the following five equations are used.

\[
(\bar{a} \times \bar{b}) \otimes (\bar{c} \times \bar{d}) = (\bar{a} \otimes \bar{c})(\bar{b} \otimes \bar{d}) - (\bar{a} \otimes \bar{d})(\bar{b} \otimes \bar{c}) \tag{D.3}
\]

\[
(\bar{q} \otimes \bar{g})(\bar{q} \cdot \bar{a}_1 \times \bar{a}_2) + ((\bar{q} \cdot \bar{a}_1) \otimes (\bar{q} x \bar{a}_1)) + ^2 \equiv \frac{1}{\bar{q} x \bar{a}_1 |^2 \bar{q} x \bar{a}_2 |^2 \tag{D.4}
\]

\[
\frac{\partial \mathfrak{F}}{\partial \mathfrak{f}^1} = \frac{1}{\| \mathfrak{f} \|} \tag{D.5}
\]

\[
\frac{\partial \mathfrak{F}}{\partial \mathfrak{f}^2} = \frac{1}{\| \mathfrak{f} \|} \tag{D.6}
\]

\[
(\bar{a} \otimes \bar{a})(\bar{b} \times \bar{c}) \otimes (\bar{b} \times \bar{c}) - (\bar{a} \cdot \bar{b}) \bar{c}^2 \tag{D.7}
\]

\[
= (\bar{a} \otimes \bar{a})(\bar{b} \times \bar{c}) \otimes (\bar{b} \times \bar{c}) + \bar{a} \cdot \bar{b} (\bar{c} \times \bar{b}) \otimes (\bar{c} \times \bar{b}) \tag{D.7}
\]

D.2 PROOF OF EQUATION (D.1)

\[
(\bar{A} \cdot \bar{B})(\bar{C} \cdot \bar{D}) - (\bar{A} \cdot \bar{C})(\bar{B} \cdot \bar{D})
\]

\[
= (A_x B_x + A_y B_y + A_z B_z)(C_x D_x + C_y D_y + C_z D_z)
\]

\[
- (A_x C_x + A_y C_y + A_z C_z)(B_x D_x + B_y D_y + B_z D_z)
\]

\[
= A_x B_x C_x D_x + A_y B_y C_y D_y + A_z B_z C_z D_z + A_x B_x C_y D_y + A_x B_x C_z D_z
\]

\[
D - 1
\]
\[ + A_Y B_Y C_X D_X + A_Z B_Z C_X D_X + A_Y B_Y C_Z D_Z + A_Z B_Z C_Y D_Y - A_X C_X B_Y D_X \]
\[ - A_Y C_Y B_Y D_Y - A_X C_X B_Z D_Z - A_Y C_Y B_X D_X \]
\[ - A_Z C_Z B_X D_X - A_Y C_Y B_Z D_Z - A_Z C_Z B_X D_Y \]

while
\[
(A \times D) \cdot (B \times C)
\]
\[ = A_Y B_Y C_Z D_Z - A_Y B_Z C_Y D_Z - (A_Z B_Y C_Z D_Y - A_Z B_Z C_Y D_Y) \]
\[ + A_Z B_Z C_X D_X + A_X B_X C_Z D_Z - (A_Z C_Z B_X D_X + A_X C_X B_Z D_Z) \]
\[ + A_Z B_Y C_X D_Y + A_Y B_X C_Y D_X - (A_Y C_Y B_X D_X + A_X C_X B_Y D_Y) \quad (D.8) \]

\section*{D.3 PROOF OF EQUATION (D.2)}

Following is the proof of Equation (D.2):
\[
\left| \frac{q}{q} \times \hat{a}_1 \right|^2 \left| \frac{q}{q} \times \hat{a}_2 \right|^2 - \left\{ \left( \frac{q}{q} \times \hat{a}_1 \right) \left( \frac{q}{q} \times \hat{a}_2 \right)^2 + \left( \frac{q}{q} \times \hat{a}_1 \right) \left( \frac{q}{q} \times \hat{a}_2 \right)^2 \right\} 
\]
\[ = \left( \frac{q}{q} \cdot \hat{q} \right) (\hat{a}_1 \cdot \hat{a}_1) - (\hat{q} \cdot \hat{a}_0)^3 \left( \frac{q}{q} \cdot \hat{a}_2 \right)^2 - (\hat{q} \cdot \hat{a}_2)^2 - (\hat{q} \cdot \hat{a}_1)^2 \left( \frac{q}{q} \cdot \hat{a}_2 \right)^2 + (\hat{a}_2 \cdot \hat{a}_2)^2 \left( \frac{q}{q} \cdot \hat{a}_2 \right)^2 \left( \frac{q}{q} \times \hat{a}_2 \right)^2 \]
\[ = \left( \frac{q}{q} \cdot \hat{q} \right) (\hat{a}_1 \cdot \hat{a}_1) (\hat{a}_2 \cdot \hat{a}_2) + (\hat{q} \cdot \hat{a}_1)^2 \left( \frac{q}{q} \cdot \hat{a}_2 \right)^2 - (\hat{q} \cdot \hat{a}_2)^2 \left( \frac{q}{q} \cdot \hat{a}_1 \right)^2 \left( \frac{q}{q} \cdot \hat{a}_2 \right)^2 + (\hat{a}_2 \cdot \hat{a}_2)^2 \left( \frac{q}{q} \cdot \hat{a}_2 \right)^2 \]
\[ - \left( \frac{q}{q} \cdot \hat{q} \right) (\hat{a}_1 \cdot \hat{a}_2)^2 - (\hat{q} \cdot \hat{a}_1)^2 (\hat{q} \cdot \hat{a}_2)^2 - (\hat{q} \cdot \hat{a}_2)^2 (\hat{q} \cdot \hat{a}_1)^2 + 2(\hat{q} \cdot \hat{q}) (\hat{a}_1 \cdot \hat{a}_2) (\frac{q}{q} \cdot \hat{q}) (\hat{q} \cdot \hat{a}_1) (\frac{q}{q} \cdot \hat{a}_2) \]
\[ = \left( \frac{q}{q} \cdot \hat{q} \right) \left( (\hat{a}_1 \cdot \hat{a}_1) (\hat{a}_2 \cdot \hat{a}_2) - (\hat{a}_1 \cdot \hat{a}_2)^2 \right) - (\hat{q} \cdot \hat{a}_2)^2 (\hat{q} \cdot \hat{a}_1)^2 \left( \frac{q}{q} \cdot \hat{a}_2 \right)^2 + \left( \hat{a}_2 \cdot \hat{a}_2 \right)^2 \left( \frac{q}{q} \cdot \hat{a}_2 \right)^2 \]
\[ + (\hat{a}_2 \cdot \hat{a}_2) (\hat{q} \cdot \hat{a}_1)^2 - 2(\hat{a}_1 \cdot \hat{a}_2) (\hat{q} \cdot \hat{a}_1) (\frac{q}{q} \cdot \hat{a}_2) \left( \frac{q}{q} \cdot \hat{q} \right) \}

\textbf{D - 2}
\[
\begin{align*}
&= (\vec{q} \cdot \vec{q}) \left\{ (\vec{q} \cdot \vec{q}) (\vec{a}_1 \times \vec{a}_2) - (\vec{q} \cdot \vec{a}_1 \times \vec{a}_2)^2 - |\vec{a}_1(\vec{q} \cdot \vec{a}_2) - \vec{a}_2(\vec{q} \cdot \vec{a}_1)|^2 \right\} \\
&= (\vec{q} \cdot \vec{q}) \left\{ |\vec{q} \times (\vec{a}_1 \times \vec{a}_2)|^2 - |\vec{a}_1(\vec{q} \cdot \vec{a}_2) - \vec{a}_2(\vec{q} \cdot \vec{a}_1)|^2 \right\} \\
&= 0
\end{align*}
\]

since, by using Equation (D.1),
\[
|\vec{q} \times (\vec{a}_1 \times \vec{a}_2)|^2 = (\vec{q} \times (\vec{a}_1 \times \vec{a}_2)) \cdot (\vec{q} \times (\vec{a}_1 \times \vec{a}_2)) = (\vec{q} \cdot \vec{q}) (\vec{a}_1 \times \vec{a}_2) \cdot (\vec{a}_1 \times \vec{a}_2) - (\vec{q} \cdot \vec{a}_1 \times \vec{a}_2)^2
\]

and by using \(\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B})\), yields
\[
\vec{q} \times (\vec{a}_1 \times \vec{a}_2) = \vec{a}_1(\vec{q} \cdot \vec{a}_1) - \vec{a}_2(\vec{q} \cdot \vec{a}_1)
\]

D.4 PROOF OF EQUATION (D.3)

Note that
\[
(\vec{a} \times \vec{b}) \circ (\vec{c} \times \vec{d}) =
\]
\[
= (a_y b_z - a_z b_y)(c_y d_z - c_z d_y) - (c_z b_x - a_x b_z)(c_z d_x - c_x d_z)
- (a_x b_y - a_y b_x)(c_x d_y - c_y d_x)
= a_y b_z c_y d_z + a_z b_y c_z d_y - a_y b_z c_z d_y - a_z b_y c_y d_z
- a_z b_y c_x d_x + a_x b_z c_x d_x + a_x b_z c_x d_x + a_x b_z c_x d_x
- a_y b_x c_x d_y - a_y b_x c_y d_x + a_x b_y c_y d_x + a_y b_x c_y d_y
\]

while
\[
(\vec{a} \circ \vec{c}) \circ (\vec{b} \circ \vec{d}) = (\vec{a} \circ \vec{c} \circ \vec{b} \circ \vec{d})
\]
\[
= (a_x c_x - a_y c_y - a_z c_z)(b_x d_x - b_y d_y - b_z d_z)
- (a_x d_x - a_y d_y - a_z d_z)(b_x c_x - b_y c_y - b_z c_z)
= a_x c_x (b_x d_x - b_y d_y - b_z d_z) - a_y c_y (b_x d_x - b_y d_y - b_z d_z) - a_z c_z (b_x d_x - b_y d_y - b_z d_z)
- a_x d_x (b_y c_x - b_y c_y - b_z c_z) + a_y d_y (b_y c_x - b_y c_y - b_z c_z) + a_z d_z (b_y c_x - b_y c_y - b_z c_z)
\]

D.3
+\gamma dy (b_1 C_1 - b_2 C_2) + a_2 dz (b_x C_1 - b_y C_y - b_z C_z)

= a_y b_x C_1 dy + a_2 b_x C_2 dy - a_y b_2 C_2 dy - a_2 b_y C_y dz + a_2 b_x C_2 dz - a_x b_2 C_2 dz

\textbf{(D.13)}

\textbf{D.5 PROOF OF EQUATION (D.4)}

\[ \bar{q} \circ \bar{q} (\bar{q} \circ \bar{a}_1 \times \bar{a}_2)^2 + (\bar{q} \circ \bar{a}_1 \circ \bar{q} \times \bar{a}_2)^2 \]

\[ = (\bar{q} \circ \bar{q})(\bar{a}_1 \times \bar{a}_2) - (\bar{q} \circ \bar{a}_2)(\bar{a}_1 \times \bar{a}_2) + (\bar{q} \circ \bar{a}_1)(\bar{a}_2 \times \bar{a}_1) - (\bar{q} \circ \bar{a}_2)(\bar{a}_1 \times \bar{a}_2)
\]

\[ + \frac{1}{2} (\bar{a}_1 \circ \bar{a}_2)(\bar{a}_1 \circ \bar{a}_2) + (\bar{q} \circ \bar{a}_1)(\bar{q} \circ \bar{a}_2)^2 \]

\[ = (\bar{q} \circ \bar{q})^2 (\bar{a}_1 \circ \bar{a}_2)(\bar{a}_2 \circ \bar{a}_1) - (\bar{q} \circ \bar{a}_2)^2 - (\bar{q} \circ \bar{q})(\bar{a}_1 \circ \bar{a}_2)(\bar{a}_2 \circ \bar{a}_1) + (\bar{q} \circ \bar{a}_1)(\bar{a}_2 \circ \bar{a}_1)
\]

\[ = (\bar{q} \circ \bar{q})(\bar{a}_1 \circ \bar{a}_2)(\bar{a}_2 \circ \bar{a}_1) + (\bar{q} \circ \bar{a}_1)(\bar{a}_2 \circ \bar{a}_1)
\]

\[ = (\bar{q} \times \bar{a}_1 \circ \bar{q} \times \bar{a}_1) (\bar{q} \times \bar{a}_2 \circ \bar{q} \times \bar{a}_2)
\]

\[ = \| \bar{q} \times \bar{a}_1 \|^2 \| \bar{q} \times \bar{a}_2 \|^2
\]

\textbf{D.6 PROOF OF EQUATIONS (D.5) and (D.6)}

\text{Proof of Equation (D.6) is shown below. Proof of Equation (D.5) is similar; therefore, it is omitted. There are three different cases of } F_2 \text{. Considering the first case } \bar{a}_2 \circ \bar{a}_2 \geq 0 \text{, and noting that}

\[ \frac{\bar{q} \circ \bar{a}_2}{\gamma} \| \bar{a}_2 \| = 0 \]

\[ \text{and}
\]

\textbf{(D.14)}

\textbf{D - 4}
\[
\begin{align*}
\frac{\partial}{\partial \eta} (\bar{q} \times \bar{a}_2 \circ \bar{q} \times \bar{a}_2) & = 2 \left( \frac{\partial \bar{q}}{\partial \eta} \times \bar{a}_2 \right) \circ (\bar{q} \times \bar{a}_2) \\
& = 2 (\bar{a}_2 \times \bar{a}_2) \circ (\bar{q} \times \bar{a}_2) = 0 \quad \text{(D.15)}
\end{align*}
\]

\[
\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta} \left( \frac{1}{\|\bar{a}_2\|} \ln \left| \frac{\|\bar{q}\| \|\bar{a}_2\| + \bar{q} \circ \bar{a}_2}{\|\bar{q} \times \bar{a}_2\|} \right| \right)
\]

\[
= \frac{1}{\|\bar{a}_2\|} \ln \left| \frac{\|\bar{q}\| \|\bar{a}_2\| + \bar{q} \circ \bar{a}_2}{\|\bar{q} \times \bar{a}_2\|} \right| \\
= \frac{1}{\|\bar{a}_2\|} \frac{1}{\|\bar{q}\| \|\bar{a}_2\| + \bar{q} \circ \bar{a}_2} \left( \sqrt{\frac{\bar{q} \circ \bar{a}_2}{\|\bar{q}\|}} \sqrt{\bar{a}_2 \circ \bar{a}_2} + \bar{a}_2 \circ \bar{a}_2 \right) \frac{1}{\|\bar{q}\|} \\
= \frac{1}{\|\bar{q}\|} \quad \text{(D.16)}
\]

Consider the second case, \( \bar{a}_2 \circ \bar{a}_2 = 0 \)

\[
\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta} \left( \frac{\|\bar{q}\|}{\bar{q} \circ \bar{a}_2} \right)
\]

\[
= \frac{1}{\bar{q} \circ \bar{a}_2} \frac{\partial}{\partial \eta} \|\bar{q}\| \\
= \frac{1}{\bar{q} \circ \bar{a}_2} \frac{\bar{q} \circ \bar{a}_2}{\|\bar{q}\|} = \frac{1}{\|\bar{q}\|} \quad \text{(D.17)}
\]

since

\[
\frac{\partial}{\partial \eta} (\bar{q} \circ \bar{a}_2) = (\frac{\partial}{\partial \eta} \bar{q}) \circ \bar{a}_2 = \bar{a}_2 \circ \bar{a}_2 = 0
\]

(D.18)

Consider the third case, \( \bar{a}_2 \circ \bar{a}_2 < 0 \)
\[
\frac{\partial F_2}{\partial \eta} = \frac{\partial}{\partial \eta} \left( -\frac{1}{||a||} \sin^{-1} \left( \frac{\bar{b} \cdot \bar{a}_2}{||\bar{b} \times \bar{a}_2||} \right) \right) = -\frac{1}{||a||} \frac{\partial}{\partial \eta} \sin^{-1} \left( \frac{\bar{b} \cdot \bar{a}_2}{||\bar{b} \times \bar{a}_2||} \right)
\]

\[
= -\frac{1}{||\bar{a}_2||} \frac{1}{\sqrt{1 - \left( \frac{\bar{b} \cdot \bar{a}_2}{||\bar{b} \times \bar{a}_2||} \right)^2}} \frac{\bar{a}_2 \cdot \bar{a}_2}{||\bar{b} \times \bar{a}_2||}
\]

\[
= \frac{1}{||\bar{a}_2||} \frac{||\bar{b} \times \bar{a}_2||}{\sqrt{1 - \left( \frac{\bar{b} \cdot \bar{a}_2}{||\bar{b} \times \bar{a}_2||} \right)^2}} \frac{||\bar{a}_2||^2}{||\bar{b} \times \bar{a}_2||}
\]

\[
= \frac{1}{||\bar{b}||} \left( \frac{\bar{b} \cdot \bar{a}_2}{||\bar{b} \times \bar{a}_2||} \right) \frac{||\bar{a}_2||}{\sqrt{1 - \left( \frac{\bar{b} \cdot \bar{a}_2}{||\bar{b} \times \bar{a}_2||} \right)^2}}
\]

Combining (B.5), (B.6), and (B.7), yields

\[
\frac{\partial F_2}{\partial \eta} = \frac{1}{||\bar{b}||} \bar{a}_2 \cdot \bar{a}_2 \equiv \bar{0}
\]

(D.20)

Similarly,

\[
\frac{\partial F_1}{\partial \bar{b}} = \frac{1}{||\bar{b}||} \bar{a}_1 \cdot \bar{a}_1 \equiv \bar{0}
\]

(D.21)

D.7 PROOF OF EQUATION D.7

Consider the regular vector algebra rule

\[
\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b} (\bar{a} \cdot \bar{c}) - \bar{c} (\bar{a} \cdot \bar{b})
\]

(D.22)

and note

\[
\bar{a} \cdot \bar{b} = \bar{a}^\perp \cdot \bar{b} = \bar{a} \cdot \bar{b}^\perp
\]
One obtains
\[
\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b})
\]
\[
= \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b})
\]  \hspace{1cm} (D.23)

On the other hand, according to Equations (D.3) and (D.22)
\[
\vec{a} \times (\vec{b} \times \vec{c}) \circ \vec{a} \times (\vec{b} \times \vec{c})
\]
\[
= \vec{a} \circ \vec{a} (\vec{b} \times \vec{c}) \circ (\vec{b} \times \vec{c}) - (\vec{a} \circ (\vec{b} \times \vec{c}))^2
\]
\[
= \vec{a} \circ \vec{a} (\vec{b} \times \vec{c}) \circ (\vec{b} \times \vec{c}) - (\vec{a} \cdot \vec{b} \times \vec{c})^2
\]

while, according to Equations (D.3) and (D.22)
\[
(\vec{b} (\vec{a} \circ \vec{c}) - \vec{c} (\vec{a} \circ \vec{b})) \circ (\vec{b} (\vec{a} \circ \vec{c}) - \vec{c} (\vec{a} \circ \vec{b}))
\]
\[
= \vec{b} \circ \vec{b} (\vec{a} \circ \vec{c})^2 - 2 \vec{b} \circ \vec{c} \vec{a} \circ \vec{c} \vec{a} \circ \vec{b} + \vec{c} \circ \vec{c} (\vec{a} \circ \vec{b})^2
\]
\[
= \vec{a} \circ \vec{c} (\vec{c} \times \vec{a} \circ \vec{b} \times \vec{a}) + \vec{a} \circ \vec{b} (\vec{c} \times \vec{b} \circ \vec{c} \times \vec{a})
\]  \hspace{1cm} (D.24)

Combining Equations (D.22), (D.23) and (D.24) yields
\[
\vec{a} \circ \vec{a} (\vec{b} \times \vec{c}) \circ (\vec{b} \times \vec{c}) - (\vec{a} \cdot \vec{b} \times \vec{c})^2
\]
\[
= (\vec{a} \times (\vec{b} \times \vec{c})) \circ (\vec{a} \times (\vec{b} \times \vec{c}))
\]
\[
= (\vec{b} (\vec{a} \circ \vec{c}) - \vec{c} (\vec{a} \circ \vec{b})) \circ (\vec{b} (\vec{a} \circ \vec{c}) - \vec{c} (\vec{a} \circ \vec{b}))
\]
\[
= \vec{a} \circ \vec{c} (\vec{c} \times \vec{a} \circ \vec{b} \times \vec{a}) + \vec{a} \circ \vec{b} (\vec{c} \times \vec{b} \circ \vec{c} \times \vec{a})
\]  \hspace{1cm} (D.25)
APPENDIX E
LIST OF COMPUTER PROGRAM SOSSA ACTS

MAIN PROGRAM

COMPLEX AA, BB, SOURCA, SOURCB, VELPOT
COMMON/ZZZ1/NX(20), NY(20), NXY(20), NW, KSYM, KSYM*, NSYM, NSYM*
COMMON/ZZZ2/TAU, SPAN, TANGLE, TANGTE, CHORD, IZZZ, VMACH, REFLEN
COMMON/ZZZ99/KPRINT(10), NREAD, NWRITE

DIMENSION YK(3, 16, 16, 5), PC(3, 100), YPP(3, 100), YPM(3, 100), YMP(3, 100),
YMM(3, 100), KWAKE(100), SOURCA(100), SOURCB(100), AA(10000), VELPOT(1100), BB(10000)

EQUIVALENCE (YK(1, 1, 1, 1), AA(1))

NREAD = 5
NWRITE = 6
NXMAX = 15
NYMAX = 15
NTMAX = 100
NXMAXP = NXMAX + 1
NYMAXP = NYMAX + 1

READ (NREAD, 2) NCASE
FORMAT (1X, I2)
DO 999 ICASE = 1, NCASE
WRITE (NWRITE, 1)
FORMAT (*1////)
NLOOP = 1
CALL CODPT (NTOTAL, NXMAXP, NYMAXP, YK)
CALL CHECK(NTOTAL, NXMAX, NYMAX, NTMAX)
NT2S = NTOTAL**2
CALL GEOMET (NTOTAL, NXMAXP, NYMAXP, YK, YPP, YPM, YMP, YMM, KWAKE)
CALL VEC123(NTOTAL, PC, YPP, YPM, YMP, YMM)
CALL COEFF (NTOTAL, PC, YPP, YPM, YMP, YMM, KWAKE, NT2S, AA, SOURCA, SOURCB)

CONTINUE
DO 160 NNN = 1, NT2S
160 BB(NNN) = AA(1N)
CALL SOLUTN (NTOTAL, NT2S, BB, SOURCA, VELPOT)
CALL COEFPR (NTOTAL, PC, YPP, YPM, YMP, YMM, SOURCA, VELPOT)
IF (IZZZ.LT.500) GO TO 999
IF (INLOOP.EQ.2) GO TO 999
WRITE (NWRITE, 1)
DO 200 T = 1, NTOTAL
200 SOURCA(I) = SOURCB(I)
NLOOP = 2
GO TO 150
999 CONTINUE
STOP
END
C

SUBROUTINE COODPT(NTOTAL,NXMAXP,NYMAXP,YK)
DOUBLE PRECISION TODAY
COMMON/ZZZI/NXI(20),NYI(20),NXY(20),NW,KSYMMY,KSYMZ,NSYMXY,NSYMIZ
COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
COMMON/ZZZ3/NWAKES(20),NWAKE,KXINC R
COMMON/ZZZ4/ALFA,ALFABC,RED FRE,R,XLEZ,XTEZ,XNOSE,XTAIL,KCM
COMMON/ZZZ5/NBFY,NSBODY,NS,NT(20),KNORMAL(20),KDIAF(20),ISFAC E(20)
COMMON/ZZZ6/CHORD(10),CHAXIS(10)
COMMON/ZZZ7/KPRINT(10),NREAD,NWRITE
DIMENSION ACHORD(20),WNPP(2),WNP~(2),WNNP(2),WNMP(2)
DIMENSION YK(3,NXMAXP,NYMAXP,5),GG(20),KSFACE(20)
DIMENSION DFPP(2),DFPM(2),DFMP(2),DFMM(2)

C NSFX=12
C CALL DATE(TODAY)
C WRITE(NWRITE,111)TODAY
111 FORMATT(/11X,' DATE : ',A8//)
DO 19 I=1,3
READ(NREAD,10) GG
19 WRITE(NWRITE,29)GG
10 FORMAT(2A4)
20 FORMAT(1X,2A4)
READ(NREAD,18) GG(1),GG(2),NS,KSYMXY,KSYMZ
WRITE(NWRITE,28)GG(1),GG(2),NS,KSYMXY,KSYMZ
READ(NREAD,18) GG(1),GG(2),NWING,NSBODY1,NSBODY2,NSBODY3,NDIAF1
1,NDIAF2,NDIAF3
WRITE(NWRITE,28)GG(1),GG(2),NWING,NSBODY1,NSBODY2,NSBODY3,NDIAF
1,NDIAF2,NDIAF3
18 FORMAT(2A4,10I5)
28 FORMAT(1X,2A4,10I5)
READ(NREAD,12) GG(1),GG(2),UMACH,RED FRE
WRITE(NWRITE,22)GG(1),GG(2),UMACH,RED FRE
READ(NREAD,13) GG(1),GG(2),ALFA,ALFABC,IZZZ,KCM,MZLOOP
WRITE(NWRITE,23)GG(1),GG(2),ALFA,ALFABC,IZZZ,KCM,MZLOOP
12 FORMAT(2A4,3F8.3,3E12.5)
22 FORMAT(1X,2A4,3F8.3,3E12.5)
13 FORMAT(2A4,2F8.3,3I5)
23 FORMAT(1X,2A4,2F8.3,3I5)
READ(NREAD,12) GG(1),GG(2),SPAN,XLEZ,XTEZ,TANGLE,TANGTE,TAU
WRITE(NWRITE,22)GG(1),GG(2),SPAN,XLEZ,XTEZ,TANGLE,TANGTE,TAU
READ(NREAD,12) GG(1),GG(2),XNOSE,XTAIL,R
WRITE(NWRITE,22)GG(1),GG(2),XNOSE,XTAIL,R
15 FORMAT(2A4,8F8.3)
25 FORMAT(1X,2A4,8F8.3)
IF(UMACH.GT.1.0.OR.REDFRE.EQ.0.OR.NDIAF1.NE.0.OR.NDIAF3.NE.0)
GO TO 30
READ(NREAD,18) GG(1),GG(2),NWAKE,KXINCR
WRITE(NWRITE,28)GG(1),GG(2),NWAKE,KXINCR
30 CONTINUE
READ(NREAD,18) GG(1),GG(2),(KPRINT(K)),K=1,10
WRITE(NWRITE,28)GG(1),GG(2),(KPRINT(K)),K=1,10
C DO 21 ISFIX=1,NSFX
21 KSFACE(ISFIX)=0

E-2

ORIGINAL PAGE IS OF POOR QUALITY.
C

REFLEN=1.
C1ORD=XTEZ-XLEZ
BETA=SQRT(ABS(UMACH**2-1.))
KS=0
NSBODY=0
NSBTOT=0

C------------------------------------------WING------------------------------------------

C

IF(NWING.EQ.0)GO TO 199
DO 198 IWING=1,NWING
KS=KS+1
NSBODY=NSBODY+1
ISFIX=1+IWING
READ(NREAD,18) GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
1 KUNELE,KWNSHP,KWNTYP
WRITE(NWRITE,28)GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
1 KUNELE,KWNSHP,KWNTYP
NXY(KS)=NX(KS)*NY(KS)
KSFACE(ISFIX)=KS
ISFACE(KS)=ISFIX
NT(KS)=NSBTOT
NSBTOT=NSBTOT+NXY(KS)

C

IF(KWNTYP.EQ.1)GO TO 14
READ(NREAD,15) GG(1),GG(2),WNPP,WNPM,WNMP,WNNM
WRITE(NWRITE,25)GG(1),GG(2),WNPP,WNPM,WNMP,WNNM
14 CONTINUE
IF(ISFIX.EQ.1)SIGNZ=1.
IF(ISFIX.EQ.2)SIGNZ=-1.

C

WRITE(NWRITE,117)ISFIX,XLEZ,XTEZ
FORMAT(//2X,'PART ',I2,5X,'XINIT=',F6.1,5X,'XFIN=',F6.1)
WINGLN=SPAN-2.*R
WINGWD=WINGLN/2.*R
RLE=R
RTE=R
DXX=1./NX(KS)
DYY=1./NY(KS)
NXP=NX(KS)+1
NYP=NY(KS)+1
DO 118 IX=1,NXP
DO 118 IY=1,NYP
XX=(IX-1)*DXX
YY=(IY-1)*DYY
GO TO (1131,1132,1133,1134),KUNELE
1131 CONTINUE
CSI=XX
ETA=YY
GO TO 113
1132 CONTINUE
CSI=XX
ETA=1.-((1.-YY)**2
GO TO 113
1133 CONTINUE
CSI=XX
ETA=1.-((1.-YY)**2

ORIGINAL PAGE IS OF POOR QUALITY.
GO TO 113
CONTINUE
CSI=XX*XX
ETA=YY
113 CONTINUE
C IF(KWNTYP.NE.1)GO TO 1139
Y=WINGWD*ETA+R
XLE=XLEZ+TANGLE*(Y-XLE)
XTE=XTEZ+TANGETE*(Y-RTE)
DCHORD(IY)=XTE-XLE
CHAXIS(IY)=XLE+DCHORD(IY)*0.5
X=XLE+(XTE-XLE)*CSI
CONTINUE
C
1139 CONTINUE
IF(KWNTYP.NE.2)GO TO 1149
X1=(WNPP(1)-WNMP(1))*CSI+WNMP(1)
Y1=(WNPP(2)-WNMP(2))*CSI+WNMP(2)
X2=(WNPM(1)-WNMM(1))*CSI+WNMM(1)
Y2=(WNPM(2)-WNMM(2))*CSI+WNMM(2)
X3=(WNPP(1)-WNMP(1))*ETA+WNMP(1)
Y3=(WNPP(2)-WNMP(2))*ETA+WNMP(2)
X4=(WNPM(1)-WNMM(1))*ETA+WNMM(1)
Y4=(WNMP(2)-WNMM(2))*ETA+WNMM(2)
DET=(X1-X2)*(Y3-Y4)-(X3-X4)*(Y1-Y2)
IF(DET.EQ.0.)GO TO 1147
X=\frac{(X1-X2)*(Y3-Y4)-(X3-X4)*(Y1-Y2)}{DET}
Y=\frac{(Y1-Y2)*(X4*Y3-X3*Y4)-(Y3-Y4)*(X2*Y1-X1*Y2)}{DET}
GO TO 1149
1147 CONTINUE
X=X4
Y=Y4
1149 CONTINUE
GO TO (1151,1152,1153),KWNSHP
1151 CONTINUE
C FOR CIRCULAR BICONVEX
C
XC=0.5*(XLE+XTE)
PL=0.5*(XTE-XLE)
H=TAU*PL
-AT)=2.*H
ETACR=1.-ATAU/SPAN
XXC=X-XC
Z=0.
IF(H.EQ.0. OR IY.EQ.0. OR IX.EQ.1. OR IX.EQ.NX)GO TO 115
Z=SIGNZ*(SQR((PL*PL+H*H)/2.*H)**2-XXC**2)-(PL*PL-H*H)/(2.*H)
IF(ETA.GE.ETACR)Z=SIGNZ*ETA**2
GO TO 115
1152 CONTINUE
TAUBAR=TAU*.75*SQR(3.)*(XTE-ZLEZ)
Z=SIGNZ*TAUBAR*SQR(1.-CSI)*SQR(1.-ETA**2)
GO TO 115
1153 CONTINUE
Z=SIGNZ*TAU*(XTE-ZLEZ)*CSI*(1.-CSI)*SQR(1.-ETA**2)
115 CONTINUE
YK(I,IX,IY,KS)=X
YK(2, IX, IY, KS) = Y
YK(3, IX, IY, KS) = Z
118 CONTINUE
KWAKES(KS) = 1
198 CONTINUE
199 CONTINUE

C ----------NOSE--------------------------
C
540 IF (NBODY2.EQ.O) GO TO 599
DO 598 IBODY2 = 1, NBODY2
KS = KS + 1
N3BODY = N3BODY + 1
ISFIX = 4 + IBODY2
READ(NREAD, 18) GG(1), GG(2), NX(KS), NY(KS), KNORML(KS), KDIAF(KS),
1 KNSEL, KNSSHP, KNSTYP
WRITE(NWRITE, 28) GG(1), GG(2), NX(KS), NY(KS), KNORML(KS), KDIAF(KS),
1 KNSEL, KNSSHP, KNSTYP
NXY(KS) = NX(KS)*NY(KS)
KSFACE(ISFIX) = KS
ISFACE(KS) = ISFIX
NT(KS) = NSBTOT
NSBTOT = NSBTOT + NXY(KS)
IF (ISFIX.EQ.5) SIGNZ = +1
IF (ISFIX.EQ.6) SIGNZ = -1
NXP = NX(KS) + 1
NYP = NY(KS) + 1
XINIT = XLEZ - XNOSE
XFIN = XLEZ
WRITE(NWRITE, 117) ISFIX, XINIT, XFIN
DXX = 1./NX(KS)
XZERO = XLEZ
RR = R
SLOPE = XNOSE/R
CONST = 1.5
TINCR = 0.5*3.14159/NY(KS)
DO 550 JX = 1, NXP
XX = (IX - 1)*DXX
IF (KNSHEL.EQ.1) CSI = XX
IF (KNSHEL.EQ.2) CSI = XX**2
X = XINIT + (XFIN - XINIT)*CSI
IX = JX
IF (KNSHEL.EQ.3) GO TO 540
IF (KNSHEL.EQ.4) GO TO 540
IF (KNSHEL.EQ.5) GO TO 540
DO 550 JY = 1, NYP
THET = (IY - 1)*TINCR
THETA = (+0.5*3.14159 - THET) * SIGNZ
ETA = COS(THETA)
CONTINUE
540 CONTINUE
550 CONTINUE
\[ YY = RR \times ETA \]

Go to (541, 542, 543), KNSTYP

541 CONTINUE

\[ ZZ = RR \times \sin(THETA) \]

Go to 544

542 CONTINUE

\[ ZZ = 0.01 \times RR \times \sqrt{(1. - ETA)} \times 4.0 \]

Go to 544

543 CONTINUE

\[ ZZ = 0. \]

544 CONTINUE

\[ YK(1, IX, IY, KS) = X \]

\[ YK(2, IX, IY, KS) = YY \]

\[ YK(3, IX, IY, KS) = ZZ \]

550 CONTINUE

\[ KWALES(KS) = 0 \]

598 CONTINUE

599 CONTINUE

--- WING-DIAPHRAGM ---

IF (NDIAF1.EQ.0) GO TO 999

DO 998 IDIAF1 = 1, NDIAF1

\[ KS = KS + 1 \]

ISFIX = 8 + IDIAF1

READ (NREAD, 18) GG(1), GG(2), NX(KS), NY(KS), KNORML(KS), KDIAF(KS),

\[ KDFELE, KDFSHP \]

WRITE (NWRITE, 28) GG(1), GG(2), NX(KS), NY(KS), KNORML(KS), KDIAF(KS),

\[ KDFELE, KDFSHP \]

\[ NXY(KS) = NX(KS) \times NY(KS) \]

\[ KSFACES(ISFIX) = KS \]

\[ ISFACE(KS) = ISFIX \]

\[ NT(KS) = NSBTOT \]

\[ NSBTOT = NSBTOT + NXY(KS) \]

\[ DYY = 1. / NY(KS) \]

\[ NYP = NY(KS) + 1 \]

\[ NXP = NX(KS) + 1 \]

\[ XTIPC = 0.5 \times (XTEZ + WINGWD \times TANGL + XLEZ + WINGWD \times TANGL) \]

\[ DXTIP = 1.0 \times (XTEZ + WINGWD \times TANGL - XLEZ - WINGWD \times TANGL) \]

\[ XDIAC = XTIPC \]

\[ DXDIA = DXTIP \]

IF (NBDY2.EQ.0) GO TO 992

\[ DXDIA = XLEZ - XNOSE \times \text{BETA} \times (R + WINGWD) \]

\[ XTIPLE = XLEZ + WINGWD \times TANGL \]

IF (XTIPLE.DXDIA).LE.0.) GO TO 992

\[ XDIAC = 0.5 \times (XTEZ + WINGWD \times TANGL + DXDIA) \]

\[ DXDIA = 1.0 \times (XTEZ + WINGWD \times TANGL - DXDIA) \]

992 CONTINUE

IF (NBDY2.EQ.0.AND. NDIAF2.NE.0) GO TO 9909

Go to 9910

9909

\[ DXDIA = XTEZ + WINGWD \times TANGL - XLEZ - WINGWD \times \text{BETA} \]

\[ XDIAC = XTEZ + WINGWD \times TANGL - DXDIA \times 0.5 \]

9910 CONTINUE

\[ KDFLEL = DXDIA / (2. \times \text{BETA}) \]

DO 990 IY = 1, NYP

\[ YY = (IY - 1) \times DYY \]

IF (KDFELE.EQ.1) ETA = YY**1

ORIGINAL PAGE IS OF POOR QUALITY.
IF(KDFELE.EQ.2) ETA=YY**2
IF(KDFELE.EQ.3) ETA=YY**3
YYY=DIAFLN*ETA
ACHORD(IY)=(DIAFLN-YYY)*2.*BETA*1.005
IF(IY.EQ.1.AND.DXDIA.EQ.DXTIP) ACHORD(IY)=DXTIP
DXX=ACHORD(IY)/NX(KS)
Y=YYY+WINGWD+R
IF(IY.EQ.NYP)Y=Y+1.002
DO 990 IX=1,NXP
X=(IX-1)*DXX+DIAAC-ACHORD(IY)/2.
IF(IY.EQ.1.AND.IX.EQ.NXP)X=XTEZ+WINGWD*TANGTE
YK(1,IX,IY,KS)=X
YK(2,IX,IY,KS)=Y
YK(3,IX,IY,KS)=0.
CONTINUE
KWAKES(KS)=0
CONTINUE
CONTINUE

--------- TRIANGULAR DIAPHRAGM ---------

IF(NDIAF3.EQ.0) GO TO 1399
DO 1398 IDIAF3=1,NDIAF3
KS=KS+1
ISFIX=12+IDIAF3
READ(NREAD,18) GG(1),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
1 KDFELE,KDFSHP
WRITE(NWRITE,28) GG(ll),GG(2),NX(KS),NY(KS),KNORML(KS),KDIAF(KS),
1 KDFELE,KDFSHP
NXY(KS)=NX(KS)*NY(KS)
KSFACE(ISFIX)=KS
KSFACE(KS)=ISFIX
NT(KS)=NSBTOT
NSBTOT=NSBTOT+NXY(KS)
DYY=1./NY(KS)
NYP=NY(KS)+1
NXP=NX(KS)+1
READ(NREAD,15) GG(1),GG(2),DFPP,DFPM,DFMP,DFMM
WRITE(NWRITE,15) GG(1),GG(2),DFPP,DFPM,DFMP,DFMM
DXX=1./NX(KS)
DO 1396 IX=1,NXP
DO 1396 IY=1,NYP
XX=(IX-1)*DXX
YY=(IY-1)*DYY
GO TO (1302,1303),KDFELE
ETA=YY
GO TO 1394
1303 ETA=YY**2
1304 CSI=XX
X1=(DFPP(1)-DFMP(1))*CSI+DFMP(1)
Y1=(DFPP(2)-DFPM(2))*CSI+DFMP(2)
X2=(DFPM(1)-DFMM(1))*CSI+DFMM(1)
Y2=(DFPM(2)-DFMM(2))*CSI+DFMM(2)
X3=(DFPP(1)-DFPM(1))*ETA+DFPM(1)
Y3=(DFPP(2)-DFPM(2))*ETA+DFPM(2)
X4=(DFMP(1)-DFMM(1))*ETA+DFMM(1)
Y4=(DFMP(2)-DFMM(2))*ETA+DFMM(2)
DET = (X1 - X2) * (Y3 - Y4) - (X3 - X4) * (Y1 - Y2)

IF (DET .EQ. 0.) GO TO 1347
X = ((X1 - X2) * (X4 * Y3 - X3 * Y4) - (X3 - X4) * (X2 * Y1 - X1 * Y2)) / DET
Y = ((Y1 - Y2) * (X4 * Y3 - X3 * Y4) - (Y3 - Y4) * (X2 * Y1 - X1 * Y2)) / DET
GO TO 1349

1347 CONTINUE
X = X4
Y = Y4

1349 CONTINUE
YK(1, IX, IY, KS) = X
YK(2, IX, IY, KS) = Y
YK(3, IX, IY, KS) = 0.

1396 CONTINUE
KWAKES(KS) = 0

1398 CONTINUE
1399 CONTINUE

WRITE(NWRITE, 31) KWNELE, KNSELE, KBDELE, KTELE, KWNSHP, KNSSH, KBDSHP


C

C

C

IF (KS .NE. NS) CALL DEBUG(1020)
IF (NSBODY .LT. NS) KKDIAF = 1
IF (KSYM .EQ. 0) NSYMMZ = 1
IF (KSYM .NE. 0) NSYMMZ = 2
IF (KSYM .EQ. 0) NSYMMZ = 1
IF (KSYM .NE. 0) NSYMMZ = 2
IF (KSYM .NE. 0 .AND. KKDIAF .EQ. 1) NSYMMZ = 1
IF (MZLOOP .EQ. 1) NSYMMZ = 1

C

C

C

NTOTAL = NSBTOT

RETURN

END

C

SUBROUTINE CHECK(NTOTAL, NXMAX, NYMAX, NTMAX)
COMMON/ZZZ1/NX(20), NY(20), NXY(20), NW, KSYM, KSYMZ, NSYMM, NSYMMZ
COMMON/ZZZ2/NSFX, NSBODY, NS, NT(20), KNORM(20), KDIAF(20), ISFACE(20)

C

DO 100 IS = 1, NS
IF (NX(IS).GT.NXMAX) CALL DEBUG(1011)
IF (NY(IS).GT.NYMAX) CALL DEBUG(1002)
IF (NTOTAL.GT.NTMAX) CALL DEBUG(1003)

100 CONTINUE
IF (ALFA .NE. 0 .AND. (IZZZ.EQ. 12 .OR. IZZZ.EQ. 13 .OR. IZZZ.EQ. 14))
ICALL DEBUG(1004)
IF (KSYM .NE. -1 .AND. (IZZZ.EQ. 13 .OR. IZZZ.EQ. 14))
ICALL DEBUG(1005)
RETURN

END

ORIGINAL PAGE IS OF POOR QUALITY.
SUBROUTINE GEOMET(NTOTAL,NXMAXP,NYMAXP,YK,YPP,YPM,YMP,YMM,KWAKE)

THIS SUBROUTINE IS FOR QUADRILATERAL ELEMENTS

COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYM,Y,KSYMZ,NSYMMY,NSYMMZ
COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZ,UMACH,REFLEN
COMMON/ZZZ10/KWAKES(20)
COMMON/ZZZ12/NSFX,NSBOD,NST(20),KNORM(20),KDIAP(20),ISFACE(20)
COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
DIMENSION YK(3,NXMAXP,NYMAXP,NST),YPP(3,NTOTAL),YPM(3,NTOTAL),
YMP(3,NTOTAL),YMM(3,NTOTAL),KWAKE(NTOTAL)

DO 1999 IS=1,NS
ISFIX=ISFACE(IS)
NXX=NX(IS)
NYY=NY(IS)
DO 999 IX=1,NXX
DO 999 IY=1,NNX

IND=IX+NX(IS)*(IY-1)+NT(IS)
IF(KNORM(IS).EQ.-1)GO TO 906

IXM=IX
IXM=IX+1
IXMP=IX+1
IXM=IX
IYMM=IY
IYMP=IY
IYMP=IY+1
CONTINUE

GO TO 907

CONTINUE

IXM=IX
IXM=IX
IXMP=IX+1
IXMP=IX+1
IYMM=IY+1
IYMP=IY
IYMP=IY+1
CONTINUE

DO 908 K=1,3
YPP(K,IND)=YK(K,IXPP,IYPP,IS)
YPM(K,IND)=YK(K,IXPM,IYPM,IS)
YMP(K,IND)=YK(K,IXMP,IYMP,IS)

E-9
YMM(IND) = YK(K, IYMM, IYMM, IS)

C

KWAKE(IND) = 0

IF (KWAKES(IS).EQ.1.AND.IX.EQ.NXX) KWAKE(IND) = 1

CONTINUE

IF (KSYMMY.NE.0) GO TO 701

NTOTAL = 2 * NTOTAL
NS = 2 * NS
NTHALF = NTOTAL / 2
NSHALF = NS / 2

DO 2990 IS = 1, NSHALF
JS = IS + NSHALF
KNORML(IS) = KNORML(IS)
KDIAF(IS) = KDIAF(IS)
NT(IS) = NT(IS) + NTHALF
NXY(IS) = NXY(IS)
NX(IS) = NX(IS)
NY(IS) = NY(IS)
ISFACE(IS) = ISFACE(IS) + 100

CONTINUE

DO 300 IR = 1, NTHALF
IL = IR + NTHALF
DO 299 K = 1, 3
GO TO (2991, 2992, 2991), K

CONTINUE

YP(K, IL) = + YPM(K, IR)
YMP(K, IL) = + YMM(K, IR)
YPM(K, IL) = + YPP(K, IR)
YMM(K, IL) = + YMP(K, IR)

GO TO 299

CONTINUE

YP(K, IL) = - YPM(K, IR)
YMP(K, IL) = - YMM(K, IR)
YPM(K, IL) = - YPP(K, IR)
YMM(K, IL) = - YMP(K, IR)

CONTINUE

KWAKE(IL) = KWAKE(IR)

CONTINUE

IF (KPRINT(1).EQ.1)
1 CALL PRINTA(NTOTAL, NXMAXP, NYMAXP, YK, PC, YPP, YPM, YMP, YMM, 1)
IF (KPRINT(2).EQ.1)
1 CALL PRINTA(NTOTAL, NXMAXP, NYMAXP, YK, PC, YPP, YPM, YMP, YMM, 2)
IF (KPRINT(3).EQ.1)
1 CALL PRINTA(NTOTAL, NXMAXP, NYMAXP, YK, PC, YPP, YPM, YMP, YMM, 3)
RETURN

END
SUBROUTINE VEC123(NTOTAL,PC,YPP,YPM,YMP,YMM)
COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYM,SYMZ,NSYMXY,NSYMZ
COMMON/ZZZ2/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZ,UMACH,REFLEN
COMMON/ZZZ11/ALFA,ALFABC,REDFRE,BOOYR,XLEZ,XTEZ,XNOSE,XTAIL
COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
DIMENSION YPP(3,NTOTAL),YPM(3,NTOTAL),YMP(3,NTOTAL),YMM(3,NTOTAL)
DIMENSION PC(3,NTOTAL)

ALFAR=ALFA*3.14159/180.
SINALF=SIN(ALFAR)
COSALF=COS(ALFAR)
BETA=SQRT(ABS(1.-UMACH*UMACH))
DO 100 IND=1,NTOTAL
   YPP1=YPP(1,IND)
   YPP3=YPP(3,IND)
   YPP(1,IND)=(YPP1*COSALF+YPP3*SINALF)/BETA
   YPP(3,IND)=-YPP1*SINALF+YPP3*COSALF
   YPM1=YPM(1,IND)
   YPM3=YPM(3,IND)
   YPM(1,IND)=(YPM1*COSALF+YPM3*SINALF)/BETA
   YPM(3,IND)=-YPM1*SINALF+YPM3*COSALF
   YMP1=YMP(1,IND)
   YMP3=YMP(3,IND)
   YMP(1,IND)=(YMP1*COSALF+YMP3*SINALF)/BETA
   YMP(3,IND)=-YMP1*SINALF+YMP3*COSALF
100 CONTINUE
DO 200 IND=1,NTOTAL
   DO 199 K=1,3
      PC(K,IND)=(YPP(K,IND)+YPM(K,IND)+YMP(K,IND)+YMM(K,IND))/4.
   C
      PI(K,IND)=(YPP(K,IND)+YPM(K,IND)-YMP(K,IND)-YMM(K,IND))/4.
   C
      P2(K,IND)=(YPP(K,IND)-YPM(K,IND)+YMP(K,IND)-YMM(K,IND))/4.
   C
      P3(K,IND)=(YPP(K,IND)-YPM(K,IND)-YMP(K,IND)+YMM(K,IND))/4.
199 CONTINUE
200 CONTINUE
IF(KPRINT(4).EQ.1)
   CALL PRINTA(NTOTAL,NXMAXP,NYMAXP,YK,PC,YPP,YPM,YMP,YMM,4)
RETURN
END

SUBROUTINE DEBUG(K)
COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
WRITE(NWRITE,1)K
1 FORMAT(/2X,'ERROR CODE =',I6) CALL EXIT END

ORIGINAL PAGE 15 OF POOR QUALITY
SUBROUTINE PRINTA(NTOTAL,NXP,NYP,YK,PC,YPP,YPM,YMP,YMM,NPRINT)
COMMON/ZZZ1/NX(20),NY(20),NXY(20),NW,KSYM~Y~KSY~MZ,NSYM~Y,NSYMMl
COMMON/ZZZZ/TAU,SPAN,TANGLE,TANGTE,CHORD,IZZZ,UMACH,REFLEN
COMMON/ZZZ12/NSFX,NSODY,NS,NT(20),KNORML(20),KDIAF(20),ISFACE(20)
COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE
DIMENSION YK(3,NXP,NYP,5),PC(3,NTOTAL),YPP(3,NTOTAL),YPM(3,NTOTAL),YMM(3,NTOTAL)

NY4=4*(NY(1)-1)
GO TO(I,2,3,4),NPRINT
CONTINUE
WRITE(NWRITE,110)
FORMAT(//2X,'SPECIFICATIONS OF THE PROBLEM'/)
DO 112 II=1,NS
IF(NXY(II).EQ.O)GO TO 112
WRITE(NWRITE,113)ISFACE(II)
WRITE(NWRITE,114)NX(II),NY(II)
CONTINUE
FORMAT(/2X,'PART ',IZ)
WRITE(NWRITE,116)TANGLE,TANGTE,CHORD,REDFRE,R
RETURN
CONTINUE
WRITE(NWRITE,222)
FORMAT(/4X,'X',10X,'Y',10X,'Z'/)
WRITE(NWRITE,221)((YK(J,J),J=1,3),IX=1,NXX),IY=1,NYY)
FORMAT(1X,F10.5,1X,F10.5,1X,F10.5)
CONTINUE
RETURN
WRITE(NWRITE,332)
FORMAT(//2X,'IND',16X,'PP',28X,'PM',28X,'MP',28X,'MM')
CONTINUE
RETURN
WRITE(NWRITE,440)
FORMAT(/2X,'IND',4X,'XPC',7X,'YPC',7X,'ZPC')
DO 444 I=1,NTOTAL
WRITE(NWRITE,445)\(PC(K,I),K=1,3)
RETURN
END
SUBROUTINE COEFPRT(NTOTAL,PC,YPP,YPM,YMP,YMM,SOURCE,VELPOT)
COMPLEX SOURCE,VELPOT,UNIMAG,AA,CL,CM,ACL,ACM,VELPTE
COMPLEX VELPTE,PHI1,PHI2,PHI3,COEF1,COEF2,COEF3
COMMON/ZZZ1/NX(20),NY(20),NXY(20),N,KSYYY,KSYYY,NSYYY,NSYYY
COMMON/ZZZ2/TAU,SPAN,TANG,STACK,CHORD,IZZ,UMACH,REFLEN
COMMON/ZZZ11/ALFA,ALFABC,REDRE,BOUYR,XLEZ,XTEZ,XNUSE,XTAIL,KCM
COMMON/ZZZ12/NFX,NSODY,NS,NT(20),KNORM(20),KDIAF(20),ISFACE(2
COMMON/ZZZ13/CHORD(10),CMAXIS(10)
COMMON/ZZZ99/KPRINT(10),NREAD,NWRITE

DIMENSION YPP(3,NTOTAL),YPM(3,NTOTAL),YMP(3,NTOTAL),YMM(3,NTOTAL)
DIMENSION SOURCE(NTOTAL),VELPOT(NTOTAL),PC1(3,NTOTAL)

DIMENSION ACL(10),ACM(10),VELPTE(10),AA(1)

BETA2S=ABS(1.-UMACH**2)
UNIMAG=(1.,1.)
BETA=SQRT(BETA2S)

IF(UMACH.GT.1.)SGNEXP=-1.
IF(UMACH.LT.1.)SGNEXP=+1.

IF(KCM.EQ.11)AXISMX=XLEZ*(XTEZ-XLEZ)*0.15
IF(KCM.EQ.12)AXISMX=XLEZ*(XTEZ-XLEZ)*0.20
IF(KCM.EQ.13)AXISMX=XLEZ*(XTEZ-XLEZ)*0.25
IF(KCM.EQ.14)AXISMX=XLEZ*(XTEZ-XLEZ)*0.30
IF(KCM.EQ.15)AXISMX=XLEZ*(XTEZ-XLEZ)*0.35
IF(KCM.EQ.16)AXISMX=XLEZ*(XTEZ-XLEZ)*0.35
IF(KCM.EQ.17)AXISMX=XLEZ*(XTEZ-XLEZ)*0.35
IF(KCM.EQ.18)AXISMX=XLEZ*(XTEZ-XLEZ)*0.35

DO 600 IND=1,NTOTAL
VELPOT(IND)=SOURCE(IND)*CEXP(SGNEXP*UNIMAG*REDRE*UMACH**2
1.*PC1(1,IND)/BETA)

CONTINUE

IF(KCM.EQ.0)GO TO 300

IF(KSYM2.EQ.0)NZLOOP=2
IF(KSYM2.NE.0)NZLOOP=1

NYLOOP=NY(1)

NX1=NX(1)
NY1=NY(1)
NXY1/NX1*N

DO 200 METHOD=1,2
DO 170 IY=1,NYLOOP
DO 170 IZ=1,NZLOOP
ITEMO=NX1*IY
ITEM1=ITEMO-1
ITEM2=ITEMO-2
KY=IY*NY1*(IZ-1)
X1=PC1(1,ITEMO)
\begin{verbatim}
X2=PC(1,ITEM1)
X3=PC(1,ITEM2)
XTE=0.5*(YPP(1,ITEMO)+YPM(1,ITEMO))
DET=(X1-X2)*(X2**2-X3**2)-(X2-X3)*(X1**2-X2**2)
GO TO (153,152),METHOD
152 PHI1=VELPOT(ITEM0)*CEXP(UNIMAG*RED FRE*BETA*PC(1,ITEM0))
PHI2=VELPOT(ITEM1)*CEXP(UNIMAG*RED FRE*BETA*PC(1,ITEM1))
PHI3=VELPOT(ITEM2)*CEXP(UNIMAG*RED FRE*BETA*PC(1,ITEM2))
GO TO 154
153 PHI1=VELPOT(ITEM0)
PHI2=VELPOT(ITEM1)
PHI3=VELPOT(ITEM2)
154 CONTINUE
COEF1=+(PHI1-PHI2)*(X2**2-X3**2)-(PHI1-PHI3)*(X1**2-X2**2)/DET
COEF2=-(PHI1-PHI2)*(X2-X3)+(PHI2-PHI3)*(X1-X2)/DET
COEF3=PHI1-COEF1*X1-COEF2*X1**2
VELPTE(KY)=COEF3*COEF1*XTE+COEF2*XTE**2
C WRITE(NWRITE,181)PHI3,PHI2,PHI1,VELPTE(KY)
IF (METHOD.EQ.2)
VELPTE(KY)=VELPTE(KY)*CEXP(-UNIMAG*RED FRE*BETA*XTE)
C WRITE(NWRITE,181)VELPTE(KY)
170 CONTINUE
C
CL=0.
CM=0.
AREAXY=0.
ACHORD=0.
DO 190 IY=1,NYLOOP
ACL(IY)=0.
ACM(IY)=0.
DO 199 IX=1,NX1
DO 199 IY=1,NYLOOP
DO 199 IZ=1,NZLOOP
I=IX+NX1*(IY-1)+NXY1*(IZ-1)
C C
TWO EDGES OF EACH ELEMENT ARE ASSUMED TO BE PARALLEL TO X-AXIS
C
DX=0.5*(YPP(1,I)+YMP(1,I)-YMP(1,I)-YMM(1,I))*BETA
DY=0.5*(YPP(2,I)+YMP(2,I)-YMP(2,I)-YMM(2,I)).
DXY=DX*DY
AREAXY=AREAXY+DXY
IF (KSYM.M.EQ.-1)COEFFT=4.
IF (KSYM.M.EQ.+0)COEFFT=2.*((3.-IZ+2)
IF (KSYM.M.EQ.+1)COEFFT=0.
C
IF (KCM.EQ.1)AXISMX=XL EZ+(XTE-ZL EZ)*0.00
IF (KCM.EQ.2)AXISMX=XL EZ+(XTE-ZL EZ)*0.50
IF (KCM.EQ.3)AXISMX=0.5*(CHAXIS(IY)+CHAXIS(IY+1))
IF (KCM.EQ.4)AXISMX=XL EZ+(XTE-ZL EZ)*0.25
ACL(IY)=ACL(IY)+COEFFT*VELPOT(I)*UNIMAG*RED FRE*DXY
ACM(IY)=ACM(IY)+COEFFT*VELPOT(I)*1.-UNIMAG*RED FRE*(BETA*PC(1,1)
-AXISMX))*DXY
IF (IX.NE.NX1)GO TO 199
ACHORD=ACHORD+(0.5*(ACHORD(IY)+DCHORD(IY+1))).*DXY
ACL(IY)=ACL(IY)+COEFFT*VELPTE(IY)*DXY
ACM(IY)=ACM(IY)-COEFFT*VELPTE(IY)*10.5*(YPP(1,1)+YMP(1,1))
* BETA-AXISMX)*DXY
199 CONTINUE
\end{verbatim}
ACHORD=ACHORD/AREAXY
WRITE(NWRITE,182)AREAXY,ACHORD,AXSIXM,IZZZ,KCM
182 FORMAT(/5X,3F9.5,5X,5F16)
181 FORMAT(5X,2F9.5,3X,2F9.5,3X,2F9.5,3X,2F9.5)
WRITE(NWRITE,101)
101 FORMAT(/)
DO 195 IY=1,NYLCOP
WRITE(NWRITE,102)ACL(IY),ACM(IY)
ABSCL=CABS(ACL(IY))
ABSCM=CABS(ACM(IY))
REALCL=REAL(ACL(IY))
REALCM=REAL(ACM(IY))
AIMGCL=AIMAG(ACL(IY))
AIMGCM=AIMAG(ACM(IY))
FSAGCL=ATAN2(AIMGCL,REALCL)*180./3.14159
FSAGCM=ATAN2(AIMGCM,REALCM)*180./3.14159
WRITE(NWRITE,102)ABSCL,FSAGCL,ABSCM,FSAGCM
195 CONTINUE
C
WRITE(NWRITE,107)CL,CM,REDRE
107 FORMAT(/5X,'TOTAL LIFT COEFFICIENT = ',2E15.5/5X,
'TOTAL MOMENT COEFFICIENT = ',2E15.5,5X,'FREQUENCY = ',F7.3/)
ABSCL=CABS(CL)
ABSCM=CABS(CM)
REALCL=REAL(CL)
REALCM=REAL(CM)
AIMGCL=AIMAG(CL)
AIMGCM=AIMAG(CM)
FSAGCL=ATAN2(AIMGCL,REALCL)*180./3.14159
FSAGCM=ATAN2(AIMGCM,REALCM)*180./3.14159
WRITE(NWRITE,107)ABSCL,FSAGCL,ABSCM,FSAGCM,REDRE
C
DO 400 IS=1,NS
NXX=NX(IS)
NYY=NY(IS)
DO 10 IY=1,NYY
DO 10 IX=1,NXX
IF(IX.EQ.NXX)GO TO 10
I=IX+NX(IS)*(IY-1)+NT(IS)
J=I+1
DDX=PC(1,J)-PC(1,I)
XPCM=0.5*(PC(1,J)+PC(1,I))
SOURCE(I)= -2.*CEXP(-UNIMAG*REDRE*BETA*XPCM)*
1 (SOURCE(J)*CEXP( SGNEXP*UNIMAG*REDRE*PC(1,J)/BETA)
1 -SOURCE(I)*CEXP( SGNEXP*UNIMAG*REDRE*PC(1,I)/BETA))/
1 (ODX*BETA)
SOURCE(I)= -2. *((VELPOT(J)-VELPOT(I))/(BETA*DDX)+0.5*UNIMAG*REDRE
C
1) CONTINUE
400 CONTINUE
SUBROUTINE SOLUTN(NTOTAL, NT2S, AA, SOURCE, VELPOT)

COMPLEX SOURCE, AA

COMMON/ZZZ1/NX(20), NY(20), NXY(20), NW, KSYMXY, KSYMXY, NSYMXY, NSYMXY
COMMON/ZZZ2/TAU, SPAN, TANGLE, TANGT, CHORD, IZZZ, U1ACH, REFLEN
COMMON/ZZZ99/KPRINT(10), NREAD, NWRITE
DIMENSION AA(NT2S), SOURCE(NTOTAL), VELPOT(NTOTAL)

IF(KPRINT(5).EQ.1) CALL PRINTB(NTOTAL, NT2S, AA, SOURCE, VELPOT, 1)
IF(KPRINT(6).EQ.1) CALL PRINTB(NTOTAL, NT2S, AA, SOURCE, VELPOT, 2)
TOL=0.001
CALL CGGELG(SOURCE, AA, NTOTAL, 1, TOL, IER)

IF(IER.NE.0) WRITE(NWRITE, 101) IER
IF(IER.NE.0) CALL DEBUG(1001)
FORMAT(46X,'---------- IER = '14,1X,14,'----------')
IF(KPRINT(7).EQ.1) CALL PRINTB(NTOTAL, NT2S, AA, SOURCE, VELPOT, 3)
RETURN
END
SUBROUTINE PRINTB(NTOTAL, NT2S, AA, SOURCE, VELPOT, NPRINT)

COMPLEX SOURCE, VELPOT, AA, UNIMAG

COMMON/ZZ1I/NX(20), NY(20), NXY(23), NW, KSYM1, KSYM2, NSYM1, NSYM2

COMMON/ZZ2/T4U, SPAN, TANGLE, TANGTE, CHORD, IZZZ, UACH, REFLEN

COMMON/ZZ11/ALFA, ALFABC, REDFRE, BODYR, XLEZ, XTEL, XNOSE, XTAIL, KCM

COMMON/ZZ12/NSFX, NSBODY, NS, NT(20), KNORML(20), KDIAF(20), ISFACE(20)

COMMON/ZZ99/KPRINT(10), NREAD, NWRITE

DIMENSION SOURCE(NTOTAL), VELPOT(NTOTAL), AA(NT2S)

DIMENSION ABSVAL(100), FASEAN(100)

NY4=4*(NY(1)-1)

GO TO (1, 2, 3, 4, 5, 6), PRINT

1 CONTINUE

WRITE(NWRITE,110)

110 FORMAT(/2X,'DISTRIBUTION OF AA(I,J)'/)

DO 11 I=1, NTOTAL

WRITE(NWRITE,111)I

N1=I

N2=NTOTAL*NTOTAL

111 FORMAT(2X,'INDEX=', I2)

WRITE(NWRITE,112)(AA(K), K=N1, N2, NTOTAL)

112 FORMAT(8E15.6)

CONTINUE

RETURN

2 CONTINUE

WRITE(NWRITE,221)

221 FORMAT(/2X,'THE DISTRIBUTION OF SOURCE'/)

CONTINUE

NSBTOT=0

DO 229 IS=1, NS

IF(NXY(IS).EQ.0)GO TO 229

WRITE(NWRITE,223)ISFACE(IS)

223 FORMAT(5X,'FOR PART ', I3)

IND=NSBTOT

NSBTOT=NSBTOT+NXY(IS)

IFIN=NSBTOT

NX=NXY(IS)

DO 226 IX=1, NXX

WRITE(NWRITE,228)

228 IF(NPRINT.GE.4 .AND. (IX.EQ.NXX))GO TO 226

IND=IND+1

WRITE(NWRITE,227)(SOURCE(KK), KK=IND, IFIN, NXX)

226 CONTINUE

227 FORMTAT(1X, 8E15.5)

228 FORMAT(/)

229 CONTINUE

RETURN

3 CONTINUE

WRITE(NWRITE,330)

330 FORMAT(/2X,'THE DISTRIBUTION OF THE VELOCITY POTENTIAL'/)

GO TO 225

4 CONTINUE

WRITE(NWRITE,440)

440 FORMAT(/2X,'THE DISTRIBUTION OF CP'/)

GO TO 225

5 CONTINUE

WRITE(NWRITE,550)

E-17
FORMAT(/1112X,'DISTRIBUTION OF ABSOLUTE VALUES AND PHASE-ANGLES OF CPI')

DO 555 KK=1,NTOTAL

ABSVAL(KK)=CABS(SOURCE(KK))
REALCP=REAL(SOURCE(KK))
AIMGCP=AIMAG(SOURCE(KK))
FASEAN(KK)=90.

IF(REALCP.NE.0.)FASEAN(KK)=ATAN2(AIMGCP,REALCP)*180./3.14159

CONTINUE

NSBTOT=0

DO 559 IS=1,NS

IF(NXY(IS).EQ.0.)GO TO 559
WRITE(NWRITE,553)IS,FACE(IS)

FORMAT(/115X,'FOR PART ',12)

IND=NSBTOT
NSBTOT=NSBTOT+NXY(IS)

IF(NX(IS).EQ.0.)GO TO 556

WRITE(NWRITE,558)

FORMAT(/1X,8E15.5)

CONTINUE

RETURN

WRITE(NWRITE,601)

FORMAT(/1112X,'THE DISTRIBUTION OF CORRECTED VELOCITY POTENTIAL')

DO 600 I=1,NTOTAL

SOURCE(I)=VELPOT(I)

GO TO 225

END
SUBROUTINE CCGELG(R,A,M,N,EPS,IER)

COMPLEX WAKE,SO,R,PWVI,TB,A
DIMENSION A(1),R(1)
IF(M)23,23,1

SEARCH FOR GREATEST ELEMENT IN MATRIX A
1 IER=0
PIV=0.
MM=M*M
NM=N*M
DO 3 L=1,MM
TB=CABS(A(L))
IF(TBB-PIV)3,3,2
2 PIV=TBB
I=L
3 CONTINUE
TOL=EPS*PIV
A(I) IS PIVOT ELEMENT. PIV CONTAINS THE ABSOLUTE VALUE OF A(I).

START ELIMINATION LOOP
LST=1
DO 17 K=1,M

TEST ON SINGULARITY
IF(PIV)23,23,4
4 IF(IER)7,5,7
5 IF(PIV-TOL)6,6,7
6 IER=K-1
7 PIV=1./A(I)
J=(I-1)/M
I=I-J*M-K
J=J+1-K
I+K IS ROW-INDEX, J+K COLUMN-INDEX OF PIVOT ELEMENT

PIVOT ROW REDUCTION AND ROW INTERCHANGE IN RIGHT HAND SIDE R
DO 8 L=K,M+1,M
LL=L+1
TB=PIV*R(LL)
R(LL)=R(L)
8 R(L)=TB

IS ELIMINATION TERMINATED.
IF(K-M)9,18,18

COLUMN INTERCHANGE IN MATRIX A
9 LEND=LST+M-K
IF(J)12,12,10
10 II=J*M
DO 11 L=LST,LEND
TB=A(L)
LL=L+II
A(L)=A(LL)
11 A(LL)=TB
C ROW INTERCHANGE AND PIVOT ROW REDUCTION IN MATRIX A

12 DO 13 L=LST,MM,M
   LL=L+I
   TB=PIVI*A(II)
   A(II)=A(LL)
13 A(LL)=TB

C SAVE COLUMN INTERCHANGE INFORMATION
   A(LST)=J

C ELEMENT REDUCTION AND NEXT PIVOT SEARCH
   PIV=0.
   LST=LST+1
   J=0
   DO 16 II=LST,LEND
      PIVI=-A(II)
      IST=II+M
      J=J+1
   15 CONTINUE
   DO 16 L=I,J,MM,M
      LL=L-J
      A(LL)=A(L)+PIVI*A(II)
      TBB=CAOS(A(LL))
      IF(TBB-PIV)15,15,14
14 PIV=TBB
15 CONTINUE
16 R(LL)=R(LL)+PIVI*R(L)
17 LST=LST+M
END OF ELIMINATION LOOP

C BACK SUBSTITUTION AND BACK INTERCHANGE

18 IF(M-1)23,22,19
19 IST=MM+M
   LST=M+1
   DO 21 I=2,M
      II=LST-I
      IST=IST-LST
      L=IST-M
      L=A(L)+.5
   DO 21 J=I,M
      TB=R(J)
      LL=J
   DO 20 K=IST,MM,M
      LL=LL+1
      TB=TB-A(K)*R(LL)
      K=J+L
      R(J)=R(K)
20 R(K)=TB
21 CONTINUE
22 RETURN

C ERROR RETURN
23 IER=-1
RETURN
END
SUBROUTINE COEFF(NTOTAL, PC, YPP, YPM, YMP, YMM, KWAKE, NT2S, AA, SOURCE, ISCTRAN)

THIS SUBPROGRAM IS TO CALCULATE DOUBLET AND SOURCE FOR SUBSONIC FLOW

COMPLEX SOURCE, SCTRA~, AA, UNIMAG, XNUNST, CMPXDB, CMPXSC, UIOMER UNST
COMPLEX XNTRAN

COMMON/ZZZ1/NX(20), NY(20), NXY(20), NW, KSYMMY, KSYMZZ, NSYMYY, NSYMZZ
COMMON/ZZZ2/TAU, SPAN, TANGLE, TANGTE, CHORD, IZZZ, UMACH, REFLEN
COMMON/ZZZ10/KWAKES(20), NWAKE, KXINC
COMMON/ZZZ11/ALFA, ALFABC, REDFRE, BODYR, XLEZ, XTEZ, XNOSE, XTAIL, KCM
COMMON/ZZZ12/NSFX, NSBODY, NS, NT(20), KNORML(20), KDIAF(20), ISFACE(20)
COMMON/ZZZ13/ACHORD(10), XCHORD(10)
COMMON/ZZZ88/QV(3), QQ, I, J, IMSYMY, IMSYMZZ
COMMON/ZZZ99/KPRINT(10), NREAD, NWRITE

DIMENSION YPP(3, NTOTAL), YPM(3, NTOTAL), YMP(3, NTOTAL), YMM(3, NTOTAL),
1 PC(3, NTOTAL), SOURCE(NTOTAL), SCTRA~(NTOTAL), KWAKE(NTOTAL), AA(NT2S)

DIMENSION A1(3, 2), A2(3, 2), A1A2(2), A2A2(2), QCRA1(3), QCRA2(3)
DIMENSION A1V(3), A2V(3), QNORM(3), UN(3), SN(3), SNUN(3), A1A2(2, 2)
DIMENSION PZ(3), Q(3, 4), QCRA1(3, 2), QCRA2(3, 2), A1A2(3, 2)
DIMENSION VMM(3), VDD(3), VMD(3), YTEPZ(3), YTENZ(3), SIGNPT(3)
DIMENSION AVA1(3), AEA2(3), AICRA2(4), QM(3, 4)
DIMENSION WPP(3), WPM(3), WMP(3), WPC(3)

DOTPROlX1, Y1, Z1, X2, Y2, Z2) = X1*X2+Y1*Y2+Z1*Z2

PROMIX(XX1, YY1, ZZ1, XX2, YY2, ZZ2, XX3, YY3, ZZ3) = (YY2*ZZ3-YY3*ZZ2)*XX1
- (XX2*ZZ3-XX3*ZZ2)*YY1+(XX2*YY3-XX3*YY2)*ZZ1

IF(UAMACH.GT.1.)CALL DEBUG(400)
BETA=SQR(l-UMACH**2)
WRITE(NWRITE,2)NSBODY, NS, NT(I), I=1, NS, KNORML(J), J=1, NS, KDIAF
1(K), K=1, NS, ISFACE(I), II=1, NS, NXYK, KK=1, NS, NSYMYY, NSYMZZ

FORMAX3, 2515)

IF(IZZ.ZE.100.AND.IZZ.ZE.200) WRITE(NWRITE, 9911)
IF(IZZ.ZE.200.AND.IZZ.ZE.300) WRITE(NWRITE, 9922)
IF(IZZ.ZE.300.AND.IZZ.ZE.400) WRITE(NWRITE, 9933)

9911 FORMAT(/2X,'-------- OSCILLATION IN FIRST BENDING MODE --------/'UNST
9922 FORMAT(/2X,'-------- OSCILLATION IN TRANSLATION -----------/'UNST
9933 FORMAT(/2X,'-------- OSCILLATION IN PITCH ---------------/'UNST

IF(IZZ.EQ.301) XPITCH=XLEZ+(XTEZ-XLEZ)*0.00
IF(IZZ.EQ.302) XPITCH=XLEZ+(XTEZ-XLEZ)*0.50
IF(KCM.EQ.11) XPITCH=XLEZ+(XTEZ-XLEZ)*0.15
IF(KCM.EQ.12) XPITCH=XLEZ+(XTEZ-XLEZ)*0.20
IF(KCM.EQ.13) XPITCH=XLEZ+(XTEZ-XLEZ)*0.25
IF(KCM.EQ.14) XPITCH=XLEZ+(XTEZ-XLEZ)*0.30
IF(KCM.EQ.15) XPITCH=XLEZ+(XTEZ-XLEZ)*0.325
IF(KCM.EQ.16) XPITCH=XLEZ+(XTEZ-XLEZ)*0.350
IF(KCM.EQ.17) XPITCH=XLEZ+(XTEZ-XLEZ)*0.1875
IF(KCM.EQ.18) XPITCH=XLEZ+(XTEZ-XLEZ)*0.4500
UNIMAG=(0.,1.)

OMEGA=REDPRE*UMACH/BETA

ALFRBC=ALFABC*3.14159/180.

SINABC=SIN(ALFRBC)

COSABC=COS(ALFRBC)

DO 6 NN=1,NT2S

AA(NN)=0.

INSERT THE KRONECKER DELTA (SWITCH FOR THE LOWER SIDES OF DIAPHRAGMS)

DO 9 IS=1,NS

NXYS=NXY(IS)

DO 9 II=1,NXYS

I=II+NT(IS)

J=I

NNN=I+(J-1)*NTOTAL

AA(NNN)=1.

CONTINUE

DO 10 I=1,NTOTAL

SOURCE(I)=0.

SCTRAN(I)=0.

CONTINUE

SIGNPT(1)=1.0

CONST=+0.5/3.14159

DO 180 JFACE=1,NS

JSBTOT=NXY(JFACE)

DO 180 JJ=1,JSBTOT

I=NT(JFACE)+JJ

DO 1002 K=1,3

AA1(K,1)=0.5*(YPP(K,J)-YMP(K,J))

AA1(K,2)=0.5*(YPM(K,J)-YMM(K,J))

AA2(K,1)=0.5*(YPP(K,J)-YPM(K,J))

AA2(K,2)=0.5*(YMP(K,J)-YMM(K,J))

A1(K,1)=AA1(K,1)

A1(K,2)=AA1(K,2)

A2(K,1)=AA2(K,1)

A2(K,2)=AA2(K,2)

CONTINUE

DO 1003 L=1,2

A1A1(L)=DOTPRO(A1(1,L),A1(2,L),A1(3,L),A1(1,L),A1(2,L),A1(3,L))

A2A2(L)=DOTPRO(A2(1,L),A2(2,L),A2(3,L),A2(1,L),A2(2,L),A2(3,L))

DO 1004 K=1,3

IF(A1A1(1).EQ.0.)A1(K,1)=A1(K,2)

IF(A1A1(2).EQ.0.)A1(K,2)=A1(K,1)

IF(A2A2(1).EQ.0.)A2(K,1)=A2(K,2)

IF(A2A2(2).EQ.0.)A2(K,2)=A2(K,1)

CONTINUE

DO 1006 L=1,2

A1A1(L)=DOTPRO(A1(1,L),A1(2,L),A1(3,L),A1(1,L),A1(2,L),A1(3,L))
A2A2(L) = DOTPRO(A2(1,L), A2(2,L), A2(3,L), A2(1,L), A2(2,L), A2(3,L))

DO 1007 L = 1, 2
DO 1007 M = 1, 2

A1A2(L, M) = DOTPRO(A1(1, L), A1(2, L), A1(3, L), A2(1, M), A2(2, M), A2(3, M))

A1CRA2(1) = SQRT(A1A1(2)*A2A2(1) - A1A2(2, 1)**2)
A1CRA2(2) = SQRT(A1A1(1)*A2A2(1) - A1A2(1, 1)**2)
A1CRA2(3) = SQRT(A1A1(1)*A2A2(2) - A1A2(1, 2)**2)
A1CRA2(4) = SQRT(A1A1(2)*A2A2(2) - A1A2(2, 2)**2)

DO 1008 K = 1, 3
AVAI(K) = 0.5*(A1(K, 1) + A1(K, 2))
AVAZ(K) = 0.5*(A2(K, 1) + A2(K, 2))

YNORM(1) = AVAI(2)*AVAZ(3) - AVAI(3)*AVAZ(2)
YNORM(2) = AVAI(1)*AVAZ(3) - AVAI(3)*AVAZ(1)
YNORM(3) = AVAI(1)*AVAZ(2) - AVAI(2)*AVAZ(1)

SN(1) = YNORM(1)/BETA
SN(2) = YNORM(2)
SN(3) = YNORM(3)
ASN = SQRT(SN(1)**2 + SN(2)**2 + SN(3)**2)
AYN = SQRT(YNORM(1)**2 + YNORM(2)**2 + YNORM(3)**2)

DO 1010 K = 1, 3
UN(K) = YNORM(K)/AYN
SNUN(K) = SN(K)/ASN

DO 170 IFACE = 1, NS
ISBTOT = NXY(IFACE)
DO 160 II = 1, ISBTOT
I = NT(IFACE) + II

DO 160 ISYMMY = 1, NSYMMY
DO 160 ISYMZ = 1, NSYMZ
SIGNPT(2) = 3.*-2*ISYMMY
SIGNPT(3) = 3.*-2*ISYMZ

DO 1102 K = 1, 3
PZ(K) = PC(K, J) - PC(K, I)*SIGNPT(K)
QDOTUN = DOTPRO(QUN(1), UN(2), UN(3), PZ(1), PZ(2), PZ(3))

DO 1110 K = 1, 3
Q(K, 1) = YPM(K, J) - PC(K, I)*SIGNPT(K)
Q(K, 2) = YPP(K, J) - PC(K, I)*SIGNPT(K)
Q(K, 3) = YMP(K, J) - PC(K, I)*SIGNPT(K)
Q(K, 4) = YMP(K, J) - PC(K, I)*SIGNPT(K)

CONTINUE

DO 1110 K = 1, 3
QM(K, 1) = 0.5*(Q(K, 1) + Q(K, 2))
QM(K, 2) = 0.5*(Q(K, 2) + Q(K, 3))
QM(K, 3) = 0.5*(Q(K, 3) + Q(K, 4))
QM(K, 4) = 0.5*(Q(K, 4) + Q(K, 1))

CONTINUE

E-23
DO 1112 K=1,3
KP1=K*1
KP2=K*2
IF(KP1.GT.3)KP1=KP1-3
IF(KP2.GT.3)KP2=KP2-3
QMCRA1(K,1)=QM(KP1,2)*A1(KP2,1)-QM(KP2,2)*A1(KP1,1)
QMCRA1(K,2)=QM(KP1,4)*A1(KP2,2)-QM(KP2,4)*A1(KP1,2)
QMCRA2(K,1)=QM(KP1,1)*A2(KP2,1)-QM(KP2,1)*A2(KP1,1)
QMCRA2(K,2)=QM(KP1,3)*A2(KP2,2)-QM(KP2,3)*A2(KP1,2)
1112 CONTINUE

RC=SQR DOTPRO(KP1(1),KP2(2),KP3(3),KP1(1),KP2(2),KP3(3))
UIOMER=UNIMAG*OMEGA*RC
CMOXDB=CEXP(-UIOMER)*CMEXP(UNIMAG*OMEGA*UMACH*PC(1,1))
CMOXSC=CMXCEXP(-UIOMER)*CEXP(-UNIMAG*OMEGA*UMACH*PC(1,1))

DO 155 ICORNR=1,4
GO TO (5502,5504,5506,5508),ICORNR
5502 CONTINUE
SIGN12=-1.
ICSI=1
IETA=2
GO TO 5510
5504 CONTINUE
SIGN12=+1.
ICSI=1
IETA=1
GO TO 5510
5506 CONTINUE
SIGN12=-1.
ICSI=2
IETA=1
GO TO 5510
5508 CONTINUE
SIGN12=+1.
ICSI=2
IETA=2
5510 CONTINUE

DO 5520 K=1,3
QV(K)=Q(K,ICORNR)
AIV(K)=A1(K,IETA)
AZV(K)=A2(K,ICSI)
QCRA1(K)=QMCRA1(K,IETA)
QCRA2(K)=QMCRA2(K,ICSI)
5520 CONTINUE

DO 5520 K=1,3
QO=SQR DOTPRO(QV(1),QV(2),QV(3),QV(1),QV(2),QV(3))
CALL LOG(ICORNR,AIV,QCRA1,ALOG1,1)
CALL LOG(ICORNR,A2V,QCRA2,ALOG2,2)
TANP=0.
IF(QDOTUN.EQ.0.)GO TO 555
HNUMER=DOTPRO(QCRA1(1),QCRA1(2),QCRA1(3),QCRA2(1),QCRA2(2),QCRA2(3))
DENOM=QO*QDOTUN*A1GRA2(ICORNR)

E-24
IF(DENOM.NE.0.)TANP=ATANP(HNUMER,DENOM)
CONTINUE
C
COEFF1=DOTPROD(UN(1),UN(2),UN(3),QCRA1(1),QCRA1(2),QCRA1(3))
COEFF2=DOTPROD(UN(1),UN(2),UN(3),QCRA2(1),QCRA2(2),QCRA2(3))
SUB
C
SRCINT=+CONST*SIGN12*
1*(-COEFF1*ALOG1+COEFF2*ALOG2-QDOTUN*TANP)
SUB
C
DBTINT=-CONST*SIGN12*TANP
C
SGNINT=1.
IF(ISYMMY.EQ.2)SGNINT=SGNINT*KSYMYY
IF(ISYMMZ.EQ.2)SGNINT=SGNINT*KSYMZZ
C
NNN=I+(J-1)*NTOTAL
C
AA(NNN)=AA(NNN)-SGNINT*DBTINT*CMRXDB
C
ARG=ABS((PC(2,J)-BODYR)/(O.5*SPAN-BODYR))
IF(IIZED.EQ.101)XNUNST=-UNIMAG*SNUN(3)*
1((J-1.89104*ARG+1.70255*ARG**2-1.13698*ARG**3+0.25387*ARG**4)
UNST
UNST
IF(IIZED.GE.200.AND.IIZED.LT.300)XNUNST=+UNIMAG*REDFRE*SNUN(3)
UNST
UNST
XNUNST=+(UNIMAG*(PC(1,J)*BETA-XPITCH)*REDFRE+1.0)*SNUN(3)
UNST
C
FOLLOWING IS FOR FLUTTER
IF(IIZED.LT.SJQ)GO TO 499
XNUNST=+(UNIMAG*(PC(1,J)*BETA-XPITCH)*REDFRE+1.0)*SNUN(3)
XNTRAN=+UNIMAG*REDFRE*SNUN(3)
SCTRAN(I)=SCTRAN(I)+SGNINT*SRCINT*XNTRAN*CMRXSC
499 CONTINUE
C
SOURCE(I)=SOURCE(I)+SGNINT*SRCINT*XNUNST*CMRXSC
C
150 CONTINUE
IF(I.GT.0)GO TO 155
IF(J.GT.5)GO TO 155
WRITE(NWRITE,66011,J,ISYMMY,ISYMMZ,ICORNR,DBTINT,SRCINT,COEFF1,
1ALOG1,COEFF2,ALOG2,SOURCE(I)
66011 FORMAT(1X,5I4,BE13.4)
155 CONTINUE
160 CONTINUE
170 CONTINUE
180 CONTINUE
C
CALL TIME(10)
C
FOLLOWING IS THE EVALUATION OF WAKE COEFFICIENTS
C
FACTOR=1.0
IXYWAK=0
C
IF(KXINCR.EQ.20)XINCRZ=2.0*CHORD/NWAKE
IF(KXINCR.EQ.21)XINCRZ=2.5*CHORD/NWAKE
IF(KXINCR.EQ.22)XINCRZ=3.0*CHORD/NWAKE
IF(KXINCR.EQ.23)XINCRZ=4.0*CHORD/NWAKE
IF(KXINCR.EQ.101)XINCRZ=0.04*CHORD/FACTOR
E-25
IF(KXINCR.EQ.102)XINCR=0.06*CHORO/FACTOR
IF(KXINCR.EQ.103)XINCR=0.08*CHORO/FACTOR
IF(KXINCR.EQ.104)XINCR=0.10*CHORO/FACTOR

WRITE(6,188)KXINCR,NWAKE,XINCR
188 FORMAT(115X,111,F14.4)
XINCR=XINCRZ
DO 999 I=WAKE=1,NWAKE
XINCR=XINCR*FACTOR
WAKE1=WAKE2
WAKE2=WAKE1+XINCR

DO 888 J=1,NTOTAL
IF(WAKE(J).EQ.0)GO TO 888

WPP(1)=YPP(1,J)+WAKE2
WPP(2)=YPP(2,J)
WPP(3)=YPP(3,J)
WPM(1)=YPM(1,J)+WAKE2
WPM(2)=YPM(2,J)
WPM(3)=YPM(3,J)
WMP(1)=YPP(1,J)+WAKE1
WMP(2)=YPP(2,J)
WMP(3)=YPP(3,J)
WMM(1)=YPM(1,J)+WAKE1
WMM(2)=YPM(2,J)
WMM(3)=YPM(3,J)

DO 770 K=1,3
WPC(K)=(WPP(K)+WPM(K)+WMP(K)+WMM(K))/4.
A1V(K)=0.5*(YPP(K)-WPM(K))
A2V(K)=0.5*(WMP(K)-WMM(K))
CONTINUE

A2VA2V=DOTPRO(A2V(1),A2V(2),A2V(3),A2V(1),A2V(2),A2V(3))

AICSA2=SQRT(A1VA1V*A2VA2V-A1VA2V**2)

YNORM(1)=A1V(2)*A2V(3)-A1V(3)*A2V(2)
YNORM(2)=A1V(3)*A2V(1)-A1V(1)*A2V(3)
YNORM(3)=A1V(1)*A2V(2)-A1V(2)*A2V(1)
ABNORM=SQRT(YNORM(1)**2+YNORM(2)**2+YNORM(3)**2)
DO 6601 K=1,3
UN(K)=YNORM(K)/ABNORM
CONTINUE

DO 777 I=1,NTOTAL
NNN=I+(J-1)*NTOTAL
DO 777 ISYMMY=1,NSYMMY
DO 777 ISYMMZ=1,NSYMMZ
SIGNPT(2)=3.-2*ISYMMY
SIGNPT(3)=3.-2*ISYMMZ
CONTINUE

DO 6602 K=1,3
6602  PZ(K)=WPC(K)-PC(K,I)*SIGNPT(K)
QDOTUN=DOTPROD(UN(1),UN(2),UN(3),PZ(1),PZ(2),PZ(3))
C
DO 6610 K=1,3
Q(K,1)=WPM(K)-PC(K,I)*SIGNPT(K)
Q(K,2)=WP(K)-PC(K,I)*SIGNPT(K)
Q(K,3)=WP(K)-PC(K,I)*SIGNPT(K)
Q(K,4)=WM(K)-PC(K,I)*SIGNPT(K)
6610 CONTINUE
D
DO 6611 K=1,3
QM(K,1)=0.5*(Q(K,1)+Q(K,2))
QM(K,2)=0.5*(Q(K,2)+Q(K,3))
QM(K,3)=0.5*(Q(K,3)+Q(K,4))
QM(K,4)=0.5*(Q(K,4)+Q(K,1))
6611 CONTINUE
C
DO 6612 K=1,3
KP1=K+1
KP2=K+2
IF(KP1.GT.3)KP1=KP1-3
IF(KP2.GT.3)KP2=KP2-3
QMCRA1(K,1)=QM(KP1,2)*AIV(KP2)-QM(KP2,2)*AIV(KP1)
QMCRA1(K,2)=QM(KP1,4)*AIV(KP2)-QM(KP2,4)*AIV(KP1)
QMCRA2(K,1)=QM(KP1,1)*A2V(KP2)-QM(KP2,1)*A2V(KP1)
QMCRA2(K,2)=QM(KP1,3)*A2V(KP2)-QM(KP2,3)*A2V(KP1)
6612 CONTINUE
C
RC=SQR(DOTPROD(PZ(1),PZ(2),PZ(3),PZ(1),PZ(2),PZ(3)))
WAKE=0.
C
DO 666 ICORNR=1,4
GO TO (7702,7704,7706,7708),ICORNR
7702 CONTINUE
SIGN12=-1.
ICSI=1
IETA=2
GO TO 7710
7704 CONTINUE
SIGN12=+1.
ICSI=1
IETA=1
GO TO 7710
7706 CONTINUE
SIGN12=-1.
ICSI=2
IETA=1
GO TO 7710
7708 CONTINUE
SIGN12=+1.
ICSI=2
IETA=2
7710 CONTINUE
C
DO 7720 K=1,3
QV(K)=Q(K,ICORNR)
QCRA1(K)=QMCRA1(K,IETA)
QCRA2(K)=QMCRA2(K,ICSI)
7720 CONTINUE
C

C QQ=SQRT(DOTPRO(QV(1),QV(2),QV(3),QV(1),QV(2),QV(3)))

C TANP=0.
IF(QDOTUN.EQ.0.)GO TO 7750
HNUMER=-DOTPRO(QCRA1(1),QCRA1(2),QCRA1(3),QCRA2(1),QCRA2(2),QCRA2(3))
1
DENOM=QQ*QDOTUN*AICSA2
IF(DENOM.NE.0.)TANP=ATANP(HNUMER,DENOM)
CONTINUE

7750

WAKE=WAKE-SIGN12*TANP*CONST
CONTINUE

666 CONTINUE

IXYWAK=IXYWAK+1
IF(IXYWAK.EQ.1)WRITE(6,2244)I,J,ISYMMY,ISYMMZ,WAKE
2244 FORMAT(415,E16.5)

C

C SGNINT=1.
IF(ISYMMY.EQ.2)SGNINT=SGNINT*KSYM*Y
IF(ISYMMZ.EQ.2)SGNINT=SGNINT*KSYM
Z

C

XXX=WPC(1)-PC(1,J)

C

AA(NNN)=AA(NNN)-WAKE*SGNINT*(1.+UNIMAG*OMEGA*RC)*
1
CEXP(-UNIMAG*REDFR*(XXX+UMACH*RC)/BETA)

777 CONTINUE
888 CONTINUE
999 CONTINUE
RETURN
END
SUBROUTINE LOG(ICORNR, AV, QCRAV, ALOG, ICOOR)
COMMON/ZZZ83/QV(3), QQ, I, J, ISYMMY, ISYMZ
DIMENSION AV(3), QCRAV(3)

DOTPRO(X1, Y1, Z1, X2, Y2, Z2) = X1*X2 + Y1*Y2 + Z1*Z2

AVAV = DOTPRO(AV(1), AV(2), AV(3), AV(1), AV(2), AV(3))
QQAV = DOTPRO(QV(1), QV(2), QV(3), AV(1), AV(2), AV(3))
QXA = SQRT(DOTPRO(QCRAV(1), QCRAV(2), QCRAV(3),
                 QCRAV(1), QCRAV(2), QCRAV(3)))
ALOG = ASINH(QQAV/QXA) / SQRT(AVAV)
RETURN
END

FUNCTION ASINH(X)
ABX = ABS(X)
XSQ = X*X
IF(ABX.LE.0.001) ASINH = X*(1. - 0.1666666*XSQ + 0.015*XSQ*XSQ)
IF(ABX.GT.1.01) ASINH = (X/ABX) * ALOG(ABX + SQRT(1. + XSQ))
RETURN
END

FUNCTION ATANP(HNUMER, DENOM)
ADENOM = ABS(DENOM)
ATANP = ATAN2(HNUMER, ADENOM)
IF(DENOM.LT.0.) ATANP = -ATANP
RETURN
END
SUBROUTINE COEFF(NTOTAL, PG, YPP, YMM, YNP, SUMD, NT2S, AA, SOURCE, 
SCTRAN)

THIS SUBPROGRAM IS TO CALCULATE DUFFIE'S UNST END SUPERSONIC FLOW

COMPLEX SOURCE, SCTRAN, AA, UNIVAG, XUNST, XTRAN, CMPXDB, CMPYSC
COMPLEX COMGA, SINHU, COSHU

COMMON/ZZL/NX(20), NY(20), NXY(20), NWKSYMM, KSYAM, NSYMM
COMMON/F377/TAU, SPAN, XX, YY, CHORD, 1777, UMACH, REFLN
COMMON/ZZZ0/TAU, SPAN, XX, YY, CHORD, 1777, UMACH, REFLN
COMMON/ZZZ1/ALPH, ALPHA, ALPHA, RPEN, RPONP, XEZ, XTEZ, XNOSE, XTAIL, CTR
COMMON/ZZZ2/NSXY, SNSPXY, NS, NT(1), KNDPLM(2), KDIAM(2), ISFACE(2)
COMMON/ZZZ5/AKCOF, KJINT(4), MDFURG(4,2), OM(3,4)
COMMON/ZZZ7/V(3), CNT, KNOFS(5), SIGNV1(4,2), SIGNV2(4,2)
COMMON/ZZZ9/V(3), ISY, ISYMM
COMMON/ZZZ99/KPI INT(4)

DIMENSION YPP(3,NTOTAL), YMM(3,NTOTAL), YNP(3,NTOTAL), SUMD(3,NTOTAL), NT2S(3,NTOTAL), 
SOURCE(NTOTAL), SCTRAN(NTOTAL), SUMD(3,NTOTAL), AA(NT2S), 

DORTP0D(X1, Y1, Z1, X2, Y2, Z2) = X1*X2+Y1*Y2+Z1*Z2

SUPP0D(XX, YY, ZZ) = XX**2-YY**2-ZZ**2

IF(UMACH.EQ.1.) UMACH=SOR(T(?,)
IF(UMACH.LT.1.) CALL DEROUG(500)
RTE=SORT(U MACH**2-1.)
WITYF(6,?) NSPXY, NS, NT(1), I=1, NS), (KNDPLM(J), J=1, NS), (KDIAM(K), 
K=1, NS), (ISFACE(II), II=1, NS), (NXY(KK), KK=1, NS), NSYMM, NSYMM
FORMAT(3X, 5F5)

IF(1777.GE.100.AND.1772.LT.200) WRITE(6,9011)
IF(1777.GE.200.AND.1772.LT.300) WRITE(6,9022)
IF(1777.GE.300.AND.1772.LT.403) WRITE(6,9033)

901 FORMAT(//2X,---- OSCILLATION IN FIRST FENDING MODE ----///)
902 FORMAT(//2X,---- OSCILLATION IN TRANSLATION ----///)
903 FORMAT(//2X,---- OSCILLATION IN PITCH: --------///)

XPITCH=XLFF2+(XTEZ-XLEF) * CTK
IF(1777.EQ.201) XPITCH=XLFF2+(XTEZ-XLEF)*7.97
IF(1777.EQ.301) XPITCH=XLFF2+(XTEZ-XLEF)*7.97
CD IF(KCM.EQ.11) XPITCH=XLFF2+(XTEZ-XLEF)*0.15
CD IF(KCM.EQ.12) XPITCH=XLFF2+(XTEZ-XLEF)*0.20
CD IF(KCM.EQ.13)XPITCH=XLEZ+(XTFZ-XLEZ)*0.25
CD IF(KCM.EQ.14)XPITCH=XLEZ+(XTFZ-XLEZ)*0.30
CD IF(KCM.EQ.15)XPITCH=XLEZ+(XTFZ-XLEZ)*0.375
CD IF(KCM.EQ.16)XPITCH=XLEZ+(XTFZ-XLEZ)*0.350
CD IF(KCM.EQ.17)XPITCH=XLEZ+(XTFZ-XLEZ)*0.1875
CD IF(KCM.EQ.18)XPITCH=XLEZ+(XTFZ-XLEZ)*0.4500

UNIMAG=(0.,1.)
C CMEGA=(RREN*UNIMAG*CPED)*UMACH/RETA

ALFRBC=ALFMAC+3.14159/180.
SINAHC=SIN(ALFRBC)
COSAHC=COS(ALFRBC)

DO 6 NN=1,NT2S
AA(0)=0.

INSERT THE KRONNECKER DELTA (SWITCH FOR THE LOWER SIDES OF DIAPHRAGMS)

DO 9 IS=1,NS
NXYS=NXY(1S)
DO 9 II=1,NXYS
I=I+NT(1S)
J=1

LOWER SIDES OF DIAPHRAGMS

IF((KNORML(1S),EQ.-1).AND.(1S.GT.NSBODY))J=1-NXY(1S)
NNN=I+(J-1)*NTOTAL
AA(0)=1.
CONTINUE

CONST=-1./3.14159

DO 10 I=1,NTOTAL
SUMPRI(I)=0.
SOURCE(I)=0.
STRAIN(I)=0.

SIGNPT(1)=1.0

DO 180 JFACk=1,NS
JSATOT=NXY(JFACk)
DO 180 JJ=1,JSATOT
J=NT(JFACk)+JJ

DO 1002 K=1,3
AA1(K,1)=-0.5*(YMP(K,J)-YMP(K,J))
AA1(K,2)=0.5*(YPP(K,J)-YMM(K,J))
AA2(K,1)=-0.5*(YPP(K,J)-YMP(K,J))
AA2(K,2)=0.5*(YPP(K,J)-YMM(K,J))
AA1(K,3)=AA1(K,1)
AA1(K,2)=AA1(K,2)
AA2(K,1)=AA2(K,2)

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A2(K, 2) = A2(K, 2)

CONTINUE

DO 1003 L = 1, 2
A1A1(L) = OCTPC(A1(1, L), A1(2, L), A1(3, L), A1(1, L), A1(2, L), A1(3, L))
A2A2(L) = OCTPC(A2(1, L), A2(2, L), A2(3, L), A2(1, L), A2(2, L), A2(3, L))

DO 1004 K = 1, 3
IF(A1A1(1).EQ.0.) A1(K, 1) = A1(K, 2)
IF(A1A1(2).EQ.0.) A1(K, 2) = A1(K, 1)
IF(A1A1(3).EQ.0.) A1(K, 3) = A1(K, 2)
IF(A2A2(1).EQ.0.) A2(K, 1) = A2(K, 2)
IF(A2A2(2).EQ.0.) A2(K, 2) = A2(K, 1)

CONTINUE

DO 1006 L = 1, 2
A1A1(L) = OCTPC(A1(1, L), A1(2, L), A1(3, L), A1(1, L), A1(2, L), A1(3, L))
A2A2(L) = OCTPC(A2(1, L), A2(2, L), A2(3, L), A2(1, L), A2(2, L), A2(3, L))

DO 1007 M = 1, 2

A1A2(L, M) = OCTPC(A1(1, L), A1(2, L), A1(3, L), A1(1, M), A2(2, M), A2(3, M))

C

DO 1008 K = 1, 3
ADV(K) = 0.5 * (A(K, 1) + A(K, 2))

C

YNORM(1) = ADV(1) * ADV(2) * ADV(3) * ADV(2) * ADV(3)
YNORM(2) = ADV(1) * ADV(2) * ADV(3) * ADV(2) * ADV(3)
YNORM(3) = ADV(1) * ADV(2) * ADV(3) * ADV(2) * ADV(3)

SN(1) = YNORM(1) / BETA
SN(2) = YNORM(2)
SN(3) = YNORM(3)
ASN = SORT(SN(1))**2 + SN(2) + SN(3)**2
AYN = SORT(YNORM(1))**2 + YNORM(2) + YNORM(3)**2

DO 1010 K = 1, 3
UN(K) = YNORM(K) / AYN
SNUN(K) = SN(K) / ASN

C

CONTINUE

C

SUNUN = SUM(SN(1), UN(2), UN(3), UN(1), UN(2), UN(3))

C

DO 170 IFACE = 1, 4

C

IF(KCIAF(IFACE).EQ.KDIAF(IFACE).AN.D.NSRODY.LT.NSIG TO 170
ISBTOT = NXY(IFACE)

C

DO 160 I1 = 1, ISBTOT
I = NT(IFACE) + I

C

ORIGINAL PAGE IS OF POOR QUALITY
DO 160 ISYMMY=1, NSYMMY
DO 160 ISYMMZ=1, NSYMMZ
SIGNPT(2)=-2*ISYMMY
SIGNPT(3)=-2*ISYMMZ

DO 1101 ICORNR=1, 4
DO 1101 ICOR2=1, 2
SIGNV1(ICORNR, ICOR2) = 0.
1101
SIGNV2(ICORNR, ICOR2) = 3.

DO 1102 K=1, 3
PZ(K) = PC(K, J) - PC(K, I) * SIGNPT(K)
PZDOPZ = DOTPRO((PZ(1), PZ(2), PZ(3), PZ(1), PZ(2), PZ(3)))
QDOTUN=DOTPRO((UN(1), UN(2), UN(3), PZ(1), PZ(2), PZ(3)))
IF(I.EQ.J) GOTO 1105
IF((ABS(QDOTUN)/SORT(PZDOPZ)) .LT. 1.0E-4) QDOTUN=0.

CONTINUE

DO 1110 K=1, 3
Q(K, 1) = YM(K, J) - PC(K, I) * SIGNPT(K)
Q(K, 2) = YM(K, J) - PC(K, I) * SIGNPT(K)
Q(K, 3) = YM(K, J) - PC(K, I) * SIGNPT(K)
Q(K, 4) = YM(K, J) - PC(K, I) * SIGNPT(K)
CONTINUE

DO 1111 K=1, 3
QM(K, 1) = 0.5*(Q(K, 1) + O(K, 2))
QM(K, 2) = 0.5*(Q(K, 2) + O(K, 3))
QM(K, 3) = 0.5*(Q(K, 3) + O(K, 4))
QM(K, 4) = 0.5*(Q(K, 4) + O(K, 1))
CONTINUE

DO 1112 K=1, 3
KP1=K+1
KP2=K+2
IF(KP1 .GE. 3) KP1=KP1-3
IF(KP2 .GE. 3) KP2=KP2-3
QMCA1(K, 1) = QM(KP1, 2) * A1(KP2, 1) - QM(KP2, 2) * A1(KP1, 1)
QMCA1(K, 2) = QM(KP1, 4) * A1(KP2, 2) - QM(KP2, 4) * A1(KP1, 2)
QMCA2(K, 1) = QM(KP1, 1) * A2(KP2, 1) - QM(KP2, 1) * A2(KP1, 1)
QMCA2(K, 2) = QM(KP1, 3) * A2(KP2, 2) - QM(KP2, 3) * A2(KP1, 2)
CONTINUE

DO 1120 ICORNR=1, 4
DO 1120 ICOR2=1, 2
MDDEBUG(ICORNR, ICOR2) = 0

MKODE = 0
DO 1130 ICORNR=1, 4
KODE(ICORNR) = 0
KCODE2(ICORNR) = 0
KCODE2(ICORNR) = 0
KCODE2(ICORNR) = 0
OSUPQ(ICORNR) = SUPPCG(O(1, ICORNR), O(2, ICORNR), O(3, ICORNR)).
Q1, ICCRNK), Q(2, ICCRNK), Q(3, ICCRNK)
IF(QSURQ(IICCNP), LE.0, CP.G(1, ICCRNK), GE.0) KODE(IICCNP) = 1
MKODE = MKODE + KODE(IICCNP)
1130 CONTINUE
C
IF(MKODE .EQ. 0) GO TO 2299
C
FOLLOWING IS TO CHECK THE DOUBLE INTERSECTIONS ALONG ETA DIRECTION AND
TO EVALUATE THE CRITICAL VECTORS OF Q, A1 AND SIGNV1(I,2) AT ANY CORN
POIN T OF WHICH THE ELEMENT IS PARTIALLY OUTSIDE THE MACH FORECONE
NKO D = 0
DO 2220 IA2 = 1, 2
ISIDE = (IA2 - 1) * 2 + 1
IF(KODE(ISIDE) .EQ. 0 .AND. KODE(ISIDE + 1) .EQ. 0) GO TO 2220
DO 2210 K = 1, 2
2210 AAA2(K) = AAA2(K, IA2)
C
CALL CPVECT(ISIDE, AAA2, AAI, 2)
C
2220 CONTINUE
C
FOLLOWING IS TO CHECK THE DOUBLE INTERSECTIONS ALONG CSI DIRECTION AND
TO EVALUATE THE CRITICAL VECTORS OF Q, A2 AND SIGNV1(I, 1)
C
KODE(5) = KODE(1)
DO 2240 IAA = 1, 2
ISIDE = (IAA - 1) * 2 + 2
IF(KODE(ISIDE) .EQ. 0 .AND. KODE(ISIDE + 1) .EQ. 0) GO TO 2240
DO 2230 K = 1, 2
2230 AAA1(K) = AAA1(K, IAA)
C
CALL CRVECT(ISIDE, AAA1, AAA2, 1)
C
2240 CONTINUE
2299 CONTINUE
C
IF(MKODE .EQ. 4 .AND. NKO D .EQ. 0) GO TO 160
C
IF(KODE(*) .EQ. 0 .OR. KODE(2) .EQ. 0 .OR. KDBINT(1) .EQ. 1) GO TO 3310
KODES2(1) = 1
KODES2(2) = 1
3310 IF(KODE(3) .EQ. 0 .OR. KODE(4) .EQ. 0 .OR. KDBINT(3) .EQ. 1) GO TO 3330
KODES2(3) = 1
KODES2(4) = 1
3320 IF(KODE(1) .EQ. 0 .OR. KODE(2) .EQ. 0 .OR. KDBINT(4) .EQ. 1) GO TO 3330
KODES1(1) = 1
KODES1(4) = 1
3330 IF(KODE(2) .EQ. 0 .OR. KODE(3) .EQ. 0 .OR. KDBINT(2) .EQ. 1) GO TO 3340
KODES1(2) = 1
KODES1(3) = 1
3340 IF(KODE(1) .EQ. 1 .AND. KCODE(2) .EQ. 0) KODES3(1) = 1
IF(KODE(2) .EQ. 1 .AND. KCODE(1) .EQ. 0) KODES3(2) = 1
IF(KODE(3).EQ.1.AND.KODE(4).EQ.0)KODE53(3)=1
IF(KODE(4).EQ.1.AND.KODE(3).EQ.0)KODE53(4)=1
IF(KOBINT(1).NE.1)GO TO 3350
KODE53(1)=1
KODE53(2)=1
3350 IF(KOBINT(3).NE.1)GO TO 3360
KODE53(3)=1
KODE53(4)=1
3360 CONTINUE
C
PC=0.
PZSPPZ=SUPPZ(PZ(1),PZ(2),PZ(3),PZ(4),PZ(5),PZ(6))
IF(PZSPPZ.GT.0.1)PC=SORT(PZSPPZ)
COSHU=(CEXP(C*MEGA*PC)+CEXP(-C*MEGA*PC))*0.5
SINHU=(CEXP(C*MEGA*PC)-CEXP(-C*MEGA*PC))*0.5
CMPXSC=COSHU*CEXP(C*MEGA*UMACH*PC(1,2))
C
DO 155 ICO=Q,NQ=1,4
GO TO (5502,5504,5506,5508),ICORN
5502 CONTINUE
SIGN12=-1.
ICSI=1
IETA=2
GO TO 5510
5504 CONTINUE
SIGN12=+1.
ICSI=1
IETA=1
GO TO 5510
5506 CONTINUE
SIGN12=-1.
ICSI=2
IETA=1
GO TO 5510
5508 CONTINUE
SIGN12=+1.
ICSI=2
IETA=2
5510 CONTINUE
C
DO 5520 K=1,3
QV(K)=Q(K,ICORN)
AV(K)=AV(K,IETA)
A2V(K)=A2(K,ICSI)
OCRA1(K)=OMCRA1(K,IETA)
OCRA2(K)=OMCRA2(K,ICSI)
5520 CONTINUE
C
QQ=SORT(ABS(OSUPQ(ICORN)))
C
ALOG1=0.
IF(KODE51(ICORN).EQ.1)GO TO 553
CALL LOG(ICORN,AV,OCRA1,ALOG1,1)
CONTINUE
C
ALOG2=0.
IF(KODES2(1CORNR),EQ,1)GO TO 5540
CALL LCG(1CORNR, A2V, QCRA2, ALOG2, 2)
CONTINUE

TANP=0.
IF(QDOTUN,EQ,0.)GO TO 5550
IF(KOC(1CORNR),EQ,1)GO TO 5544
HNUMER= SUPP(1CPA1(1), QCPA2(2), QCRA1(3), QCRA2(1), QCRA2(2)).
DENOM=QQ2*QDOTUN*ALOG2(1CORNR)
IF(DENOM,LE,0.)TANP=TANP(HNUMER, DENOM)
GO TO 5550
CONTINUE

IF(KODES2(1CORNR),EQ,0.)GO TO 5550
IF(MEBUG(1CORNR, 2). EQ, 1)CALL DEBUG(505)
SGNP=SIGNV1(1CORNR, 2)*SIGNV2(1CORNR, 2)*QDOTUN
IF(SGNP,LT,0.)SGNTAN=-1.
IF(SGNP,GT,0.)SGNTAN=+1.
IF(SGNP,GT,0.)SGNTAN=+1.
TANP=SGNTAN*`57.375
CONTINUE

C
C0EFF1= SUPP((UN1), UN(2), UN(3), QCPA1(1), 2, QCPA1(2), QCRA1(3), QCRA1(3))
C0EFF2= SUPP((UN1), UN(2), UN(3), QCPA2(1), QCRA2(2), QCRA2(2))

SRCINT=-CONST*SIGN12/SUNDNO*

1 (-COEFF1*ALOG1+COEFF2*ALOG2-QDOTUN*TANP)

SGINT=1.
IF(ISYMXY,EQ,2)SGINT=SGINT*YSYMXY
IF(ISYMXY,EQ,2)SGINT=SGINT*YSYMXY

IF(JFACE.GT.NSBCDY)GO TO 748
AIRCRAFT

NNN=I+(J-1)*NTOTAL

AA(NNN)=AA(NNN)-SGINT*DBTINT*CMPXDB

ARG=ABS((R(P, J)-BODYP)/(Q, 5*SP*N-20DYF))

1 F(I2Z, EQ, 1)XNUNST=-(RRED+UNIMAG*CRE3)*SNU1(3)

F(I2Z, GT, 200, AND, I2Z, LT, 300)XNUNST+=(RRED+UNIMAG*CRE3)*SNU1(3)

F(I2Z, GE, 300, AND, I2Z, LT, 400)XNUNST=+(RRED+UNIMAG*CRE3)*(P(*, J)*BETX*PITC)*XNUNST

FOLLOWING IS FOR FLUTTER
IF(I2Z, LT, 532)GO TO 499
XNUNST = (RRFD + UNIMAG * CREO) * (PC(1, J) * ROTA - XPITCH) + 1.0) * SNUN(3)
XNTRAN = (RRFD + UNIMAG * CREO) * SNUN(3)
SCTRAN(I) = SCTRAN(I) + SGNINT * SRCINT * XNTRAN * CMPXSC

CONTINUE

SOURCE(I) = SOURCE(I) + SGNINT * SRCINT * XNUNST * CMPXSC

GO TO 149

CONTINUE

DIAPHRAGM

IF (KSYMZ .NE. C) GO TO 149
NP = J
JCHI = J
IF (KDIAF(IFACE) .EQ. -1) JPHI = J - NXY(JFACE)
IF (KDIAF(IFACE) .EQ. +1) JCHI = J + NXY(JFACE)
NP = I + (JPHI - 1) * NTOTAL
NCHI = I + (JCHI - 1) * NTOTAL

AA(NPHI) = AA(NPHI) - SGNINT * DRTINT * CMPXDB
AA(NCHI) = AA(NCHI) - SGNINT * SRCINT * KDIAF(IFACE)

GO TO 150

CONTINUE

IF (KSYMZ .EQ. -1) GO TO 1491

SYMMETRIC : CHI = 0

NP = I + (J - 1) * NTOTAL
AA(NPHI) = AA(NPHI) - SGNINT * DRTINT * CMPXDB

GO TO 150

CONTINUE

ANTISYMMETRIC : PHI = 0

NCHI = I + (J - 1) * NTOTAL
AA(NCHI) = AA(NCHI) - SGNINT * SRCINT * CMPXSC

CONTINUE

IF (I .GT. 0) GO TO 155

IF (KODE .EQ. 0) GO TO 155
WRITE(6, 66) I, J, ISYMZ, ISYMM, ICORNP, KODE(ICORNP), SRCINT, DBTINT,
LALOG1, COEFF1, LALOG2, COEFF2

66 FORMAT(1X, 111, 7E15.6)

CONTINUE

CONTINUE

CONTINUE

CONTINUE

RETURN

END
SUBROUTINE CRVECT(ISIDE, V1, V2, ICOOR)
COMMON/7Z255/NKDE, KDATA(4), MDERUG(4, 2), OM(3, 4)
COMMON/7Z277/QV1(3), CO, KCODE(5), SIGMV(4, 2), SIGMV2(4, 2)
COMMON/7Z283/IJ, ISYMYY, ISYMYZ
DIMENSION GDOR(2), COV2(3), QQC(3), QCX(2), VI(3), V2(3, 2)
SPV1=V1(1)**2-V1(2)**2-V1(3)**2
IF(SPV1.EQ.0.) GO TO 1999
SPQ2OZ=OM(1, ISIDE)**2-OM(2, ISIDE)**2-OM(3, ISIDE)**2
SPV1=V1(1)**2-OM(1, ISIDE)-V1(2)**2-OM(2, ISIDE)-V1(3)**2-OM(3, ISIDE)
DISCPM=SPV1QZ**2-SPQ1QZ*SPV1
IF(FSPCPM.LE.0.) GO TO 1999
AVCOOR=-SPV1Z/SPV1
XAVEG=OM(1, ISIDE)+AVVOR*V1(1)
SQDIS=SQRT(DSCPM)/AP(SPV1)
GDOR(1)=AVCOOR+SQDIS
GDOR(2)=AVCOOR-SQDIS
QCX(1)=OM(1, ISIDE)+GDOR(1)*V1(1)
QCX(2)=OM(1, ISIDE)+GDOR(2)*V1(1)
IF((QCX(1).GT.0.) .AND. (QCX(2).GE.0.) ) GO TO 100
GDOR(1)=AVCOOR-SQDIS
GDOR(2)=AVCOOR+SQDIS
QCX(1)=OM(1, ISIDE)+GDOR(1)*V1(1)
QCX(2)=OM(1, ISIDE)+GDOR(2)*V1(1)
CONTINUE
DO 300 IEND=1, 2
IF((GDOR(IEND).GT.1.) .OR. (GDOR(IEND).LT.-1.) .OR. (QCX(IEND).GT.0.)) GO TO 300
IF(ISIDE.EQ.1) ICPNR=ISIDE+(2-IEND)
IF(ISIDE.EQ.2) ICPNR=ISIDE+(IEND-1)
IF(ISIDE.EQ.3) ICPNR=ISIDE+(IEND-2)+3
IF(MDERUG(ICPNR, ICOOR).EQ.1) CALL DEBUG(510)
IF(KDEQ(ICPNR).EQ.0.) GO TO 300
DO 200 K=1, 3
CRV2(K)=0.5*(V2(K, 1)+V2(K, 2))+0.5*GDOR(IEND)*((V2(K, 1)-V2(K, 2))
QC(K)=QV(K, ISIDE)+GDOR(IEND)*V1(K)
QSUPV1=QQC(1)*V1(1)-QOC(2)*V1(2)-QOC(3)*V1(3)
QSUPV2=QQC(1)*CRV2(1)-QCOC(2)*CRV2(2)-QCOC(3)*CRV2(3)
SIGNV1=ICPD0, ICPNR, ICOOR)=QSUPV1/ABS(QSUPV1)
SIGNV2=ICPD0, ICPNR, ICOOR)=QSUPV2/ABS(QSUPV2)
MDERUG(ICPD0, ICPNR, ICOOR)=1
CONTINUE
IF(KDEQ(ISIDE).EQ.0.) .OR. (KDEQ(ISIDE+1).EQ.1.) GO TO 1999.
IF(AVCOOR.GT.1.) .OR. (AVCOOR.LT.-1.) .OR. (XAVEG.GE.0.) .OR. (SPQ1QZ.LE.0.)
GO TO 1999
KDATA=1
NKDE=1
CONTINUE
RETURN
END
SUBROUTINE LOG (ICCRNR, AV, QCRAV, ALOG, ICCGR)
COMMON /ZZZT/ OQV(5), OQ, KCODE(5), SIGNV1(4, 2), SIGNV2(4, 2)
DIMENSION AV(3), QCRAV(3)

SUPPO(X1, Y1, Z1, X2, Y2, Z2) = X1*Y2 - Y1*Z2 - Z1*Z2

AVSPAV = SUPPO(AV(1), AV(2), AV(3), AV(1), AV(2), AV(3))
AVNORM = 1.
IF (AVSPAV .NE. 0.) AVNORM = SQRT(ABS(AVSPAV))
IF (KODE(ICCGR), EQ, 1) GO TO 10
QSUPAV = SUPPO(QV(1), QV(2), QV(3), AV(1), AV(2), AV(3))
QCRAVSP = SUPPO(QCRAV(1), QCRAV(2), QCRAV(3), QCRAV(1), QCRAV(2), QCRAV(3))

IF (QCRAVSP .LE. 0.) CALL DEBUG(520)
OXA = SQRT(-QCRAVSP)
IF (AVSPAV .GE. 0.) ALOG = ASINH(QCSPAV / QXA) / AVNORM
IF (QSUPAV .LT. 0.) ALOG = -ALOG
IF (AVSPAV .EQ. 0.) ALOG = -AVSPAV
IF (AVSPAV .LT. 0.) ALOG = -AVSPAV
1 ALOG = -ASIN(QSUPAV / SQRT(QSUPAV*QSUPAV - QCSPAV*QCSPAV)) / AVNORM
RETURN
10 CONTINUE
IF (AVSPAV .GE. 0.) ALOG = 0.
IF (AVSPAV .LT. 0.) ALOG = -SIGNV1(ICCGR) * 1.570796 / AVNORM
RETURN
END