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Final Report

TASK NO. 4

RELATIVISTIC ASTROPHYSICS

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BY

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I. Introduction

This report follows two independent lines of research. The first is an investigation into the coordinate independence of gravitational radiation discussed in Section II and the second is an extension of the parameterized post Newtonian (PPN) approximation given in Section III. We conclude the report by speculating on the direction of future research indicated by the work completed here.

II. Gravitational Radiation in Asymptotic de Sitter Space

Although an investigation of gravitational radiation seems somewhat remote from the parameterized post Newtonian (PPN) analysis of gravitational theories, its importance as a definitive test between theories is just now being recognized.[1] Part of the reason for this budding importance comes from a partial failure at the level of the PPN approximation to distinguish between alternate and viable theories of gravitation.[2] Attempts to firmly root Einstein's general theory to experimental results has resulted in a plethora of theories that agree experimentally in the PPN approximation. Deviations between some of the theories are only expected in the "post" PPN approximation.[3] It is at this level of approximation that one begins to detect varying contributions to gravitational radiation.[1]

In general the PPN approximation is an expansion of the metric (Lagrangian) to fourth order in the velocity which is small in the neighborhood of the solar system. In fact near the sun one expects that the quantities

$$\rho(x) \sim v^2 \sim U(x) \sim \frac{p(x)}{\rho}$$

where ρ is the density, U is the potential and p is the pressure, are all of the same order of magnitude. Far from the solar system, we expect the metric to take its Minkowskiian form: $g_{\mu\nu} = (-1,1,1,1)$. [Greek symbols represent the space-time components 0,1,2,3 whereas latin symbols represent spatial components 1,2,3.] Thus if one carried out a consistent approximation scheme to successively higher orders, gravitational radiation effects would occur in the seventh order.[4] Therefore, a clear understanding of gravitational radiation seems necessary in order to interpret such an expansion.

A general unsolved problem exists with the understanding of gravitational radiation in the interpretation of coordinates in relativistic theories. Although the covariance of the equations seem to indicate an independence of the choice of coordinates in calculation, it does make a difference when one obtains the "physical components" necessary for comparison with experiment.[5] For instance, the formulation of gravitational radiation in one frame of reference (theory) may parrot the properties of gravitational radiation but in the end, fail the test of coordinate independence. This might occur if, for instance, a spacelike coordinate could become timelike and vice versa as inside the horizon of a blackhole.[6]

Here we have investigated the gravitational radiation conditions in an asymptotic de Sitter space, a theory conformally equivalent to general relativity. Physically, this is an interesting problem for at least two reasons:

(1) The results will help demonstrate the coordinate independence of gravitational radiation, i.e., a physical interpretation useful in the detection of gravitational radiation.

(2) It leads directly to an investigation of the group structure imposed on space-time by an asymptotic de Sitter universe.

The second case has its importance in the possible classification schemes for elementary particles.[7] The former is a necessary step towards correctly quantizing the gravitational field.[8]

Gravitational radiation in asymptotic spaces has been investigated by many, notably the works of Bondi, Van der Burg and Metzner[9] (BVM) in an empty, asymptotically flat space (i.e. asymptotically Minkowskiian) and of Hawking in a dust-filled, asymptotic conformally flat Friedmann universe with negative curvature.[10] Here we investigate a matter-less but asymptotic de Sitter universe. For completeness, we include the effect of the cosmological constant. We use the method of the BVM empty space approach which is well suited for this type of investigation.

To use the BVM method, it is necessary to transform their metric into a de Sitter-space metric. This can be done thru a conformal transformation of the metric[11]

$$g'_{\alpha\beta} = e^{2\sigma} g_{\alpha\beta} \tag{1}$$

where

$$\sigma = \ln \frac{4R^2}{4R^2-u^2-2ur} \tag{2}$$

where R is the "radius" of de Sitter space, $u \equiv t - r$ is the null coordinate and r is the ordinary radial coordinate. Since the BVM metric is axially symmetric, so will the de Sitter space metric be axially symmetric. After the correction of several misprints in the work of Bondi, et al,[12] one obtains the main field equations

$$\bar{R}_{11} = -\frac{4(\beta_1 - \frac{1}{2} r\gamma_1{}^2)}{r} + 2\sigma_{11} - 4\beta_1\sigma_1 - 2\sigma_1{}^2 \tag{3}$$

$$-2r^2\bar{R}_{12} = [r^4 e^{2(\gamma-\beta)} U_1]_1 - 2r^2(\beta_{12} - \gamma_{12} + 2\gamma_1\gamma_2 - \frac{2\beta_2}{r} - 2\gamma_1 \cot\theta)$$

$$+ 4r^2(\beta_2 + \frac{1}{2} r^2 e^{2(\gamma-\beta)} U_1)\sigma_1 \tag{4}$$

$$\bar{R}_{22} e^{2(\beta-\gamma)} + \bar{R}_{33} \frac{e^{2(\beta+\gamma)}}{\sin^2\theta} = 2V_1 + \frac{1}{2} r^4 e^{2(\gamma-\beta)} U_1{}^2 - r^2 U_{12} - 4rU_2$$

$$-r^2 U_1 \cot\theta - 4rU \cot\theta + 2e^{2(\beta-\gamma)}[-1$$

$$-(3\gamma_2 - \beta_2)\cot\theta - \gamma_{22} + \beta_{22} + \beta_2{}^2 + 2\gamma_2(\gamma_2 - \beta_2)]$$

$$-8r\sigma_0 + (2rV_1 + 6V - 4r^2 U_2 - 4r^2 U \cot\theta)\sigma_1$$

$$+6rV\sigma_{11} - 4r^2\sigma_{10} - 8r^2\sigma_0\sigma_1 \tag{5}$$

$$\frac{e^{2(\gamma+\beta)}}{\sin^2\theta}\ \overline{R}_{33} = 2r(r\gamma_o)_1 + (1-r\gamma_1)V_1$$

$$-(r\gamma_{11}+\gamma_1)V-r(1-r\gamma_1)U_2-r^2(\cot\theta-\gamma_2)U_1$$

$$+r(2r\gamma_{12}+2\gamma_2+r\gamma_1\cot\theta-3\cot\theta)U$$

$$e^{2(\beta-\gamma)}[-1-(3\gamma_2-2\beta_2)\cot\theta-\gamma_{22}+2\gamma_2(\gamma_2-\beta_2)]$$

$$-\sigma_o(4r-2r^2\gamma_1)+3Vr\sigma_{11}-2r^2\sigma_{o1}-4r^2\sigma_o\sigma_1$$

$$-\sigma_1(-2r^2\gamma_o+3r^2U\cot\theta-2r^2\gamma_2U-3V$$

$$+2r\gamma_1V-rV_1+r^2U_2) \qquad (6)$$

The supplementary field equations become

$$\overline{R}_{o2} = \beta_{o2}-\gamma_{o2}+2\gamma_o\gamma_2-2\gamma_o\cot\theta$$

$$-U(\beta_{22}+2\beta_2{}^2-2\beta_2\gamma_2+\beta_2\cot\theta)-\frac{V_{12}}{2r}+\frac{V_2}{2r^2}+(\gamma_1-\beta_1)\frac{V_2}{r}$$

$$+r^2e^{2(\gamma-\beta)}[\tfrac{3}{2}UU_{12}+\frac{3UU_2}{r}+2U(\gamma_{o1}+\frac{\gamma_o}{r})+\frac{1}{2}U_{o1}+2\gamma_{12}U^2$$

$$+(\gamma_o-\beta_o)U_1+\gamma_1UU_2+(2\gamma_2-\beta_2)UU_1+U_1U_2-\frac{U_{11}V}{2r}-\frac{UV_1+2U_1V}{r^2}$$

$$- \frac{\gamma_{11}UV+(\gamma_1-\beta_1)U_1V+\gamma_1UV_1}{r} - \frac{\gamma_1UV}{r^2} + \frac{2\gamma_2U^2}{r} + U(\frac{U_1}{2} + \frac{U}{r} + \gamma_1U)\cot\theta]$$

$$- \frac{1}{2} r^4e^{4(\gamma-\beta)}UU_1{}^2 - 2\sigma_o[\beta_2 - r^2e^{2(\gamma-\beta)}(\frac{U_1}{2} + \frac{U}{r} + \gamma_1U)]$$

$$-2\sigma_1\left\{ \frac{V_2}{2r} + r^2e^{2(\gamma-\beta)}[U(\frac{V}{r^2} + \frac{\gamma_1V}{r} - \gamma_o - U_2 - \gamma_2U) + \frac{U_1V}{2r}]\right\}$$

$$+Ue^{2(\gamma-\beta)}[-3Vr\sigma_1{}^2 + 2r^2\sigma_{1o} + 4r^2\sigma_o\sigma_1 + 2r\sigma_o$$

$$+\sigma_1(-rV_1 - V + r^2U_2 + r^2U\cot\theta)] \tag{7}$$

$$\bar{R}_{oo} = \frac{2\beta_{o1}V}{r} - \frac{VV_{11}}{2r^2} - \frac{\beta_{11}V^2}{r^2} - \frac{\beta_1V^2}{r^3} - \frac{\beta_1VV_1}{r^2} - \frac{V_o - 2\beta_oV}{r^2}$$

$$+ \frac{2\beta_{12}UV + \beta_2U_1V + \beta_1U_2V + 2\beta_1UV_2}{r} + \frac{2\beta_2UV}{r^2} - \frac{U_2V}{2r^2} + \frac{U_2V_1}{2r}$$

$$- \frac{2UV_2}{r^2} - \frac{U_1V_2}{2r} - \frac{2\gamma_1UV_2}{r} - 2\beta_{o2}U - 2\beta_oU_2 + 2\gamma_{o2}U + 2\gamma_oU_2$$

$$+ U_{o2} + UU_{22} + U_2{}^2 + 2(\gamma_2 - \beta_2)UU_2 + \frac{UV_{12}}{r}$$

$$+ (2\beta_2{}^2 - 2\beta_2\gamma_2 + \gamma_{22})U^2 + 2\gamma_o{}^2 - \cot\theta(2\beta_oU - 2\gamma_oU - U_o$$

$$- UU_2 - \gamma_2U^2 + \frac{UV}{2r^2} - \frac{UV_1}{2r} - \frac{\beta_1UV}{r}) + r^2e^{2(\gamma-\beta)}[-UU_{o1}$$

$$- 2(\gamma_{o1} + \frac{\gamma_o}{r})U^2 - 2(\gamma_o - \beta_o)UU_1 - 2U^2U_{12} - 2UU_1U_2 - 2\gamma_{12}U^3$$

$$- \frac{2}{r}\gamma_2U^3 - 3\gamma_2U^2U_1 + 2\beta_2U^2U_1 + \frac{UU_{11}V}{r} + \frac{4UU_1V}{r^2}$$

$$+ 2(\gamma_1-\beta_1)\frac{UU_1V}{r} + \frac{\gamma_{11}U^2V}{r} + \frac{\gamma_1U^2V_1}{r} + \frac{\gamma_1U^2V}{r^2} - \frac{3U^2U_2}{r} - \gamma_1U^2U_2$$

$$+ \frac{U^2V_1}{r^2} + \frac{U_1{}^2V}{2r} - U^2(U_1+\frac{U}{r}+\gamma_1U)\cot\theta] + \frac{1}{2}r^4e^{4(\gamma-\beta)}U^2U_1{}^2$$

$$- \frac{1}{2r^3}e^{2(\beta-\gamma)}[V_{22}+2\beta_{22}V+(2\beta_2-2\gamma_2+\cot\theta)(V_2+2\beta_2V)] + \frac{4\phi}{B}$$

$$- 2\sigma_0[2\beta_0+\frac{V}{2r^2}-\frac{V_1}{2r}-\frac{\beta_1V}{r}] \quad -2\sigma_0r^2e^{2(\gamma-\beta)}U(U_1+\frac{U}{r}+\gamma_1U)$$

$$- 2\sigma_1\left\{\frac{V_0}{2r}-\frac{\beta_0V}{r}-\frac{V^2}{2r^3}+\frac{VV_1}{2r^2}+\frac{\beta_1V^2}{r^2}-\frac{UV_2}{2r}-\frac{\beta_2UV}{r}\right.$$

$$\left. + r^2e^{2(\gamma-\beta)}[-\frac{UU_1V}{r}+U^2(U_2+\gamma_2U-\frac{V}{r^2}-\frac{\gamma_1V}{r}+\gamma_0)]\right\}$$

$$+ (\frac{Ve^{2\beta}}{r}-r^2U^2e^{2\gamma})\frac{e^{-2\beta}}{r^2}[-3Vr\sigma_1{}^2+2r^2\sigma_{10}+4r^2\sigma_0\sigma_1$$

$$+ 2r\sigma_0+\sigma_1(-rV_1+r^2U_2-V+r^2U\cot\theta)] \tag{8}$$

where

$$\phi = e^{\sigma} = B/(B-\mu^2-{}^2\mu r) \tag{9}$$

and

$$B = 4R^2 \tag{10}$$

For completeness we list the consistency field equation

$$\overline{R}_{01} = 2\beta_{10}-\frac{V_{11}}{2r}-\frac{\beta_{11}V}{r}-\frac{\beta_1V_1}{r}-\frac{\beta_1V}{r^2}+\beta_{12}U+\beta_2U_1$$

$$+ \frac{2\beta_2 U}{r} + \frac{U_{12}}{2} + \frac{U_2}{r} + \gamma_{12}U + \gamma_1 U_2 + 2\gamma_1\gamma_0$$

$$+ \left(\frac{U_1}{2} + \frac{U}{r} + \gamma_1 U\right)\cot\theta - \frac{e^{2(\beta-\gamma)}}{r^2}\left[\beta_{22} + 2\beta_2(\beta_2-\gamma_2) + \beta_2\cot\theta\right]$$

$$+ r^2 e^{2(\gamma-\beta)}\left[\frac{UU_{11}}{2} + \frac{U_1^{\,2}}{2} + \frac{2UU_1}{r} + \gamma_1 UU_1 - \beta_1 UU_1\right]$$

$$+ 4\sigma_{01} + 2\sigma_1\sigma_0 + \frac{2\sigma_0}{r} - \frac{3V}{r}\sigma_1^{\,2} + \sigma_1[U_2 + U\cot\theta]$$

$$- \sigma_1\left[\frac{2V_1}{r} + \frac{2\beta_1 V}{r} - 2\beta_2 U - r^2 e^{2(\gamma-\beta)} UU_1\right] \tag{11}$$

The origin of the names for the various field equations comes from the way in which a solution for the functions γ, β, U and V occur. An expansion is assumed for γ at some instant in retarded time u. Eqs. (3) - (5) then yield solutions for β, U and V at the same time. Eq. (6) then gives the time development of γ from which β, U and V can be found for all time. Eqs. (7) and (8) then represent relations between the expansion parameters and initial conditions (integration constants), and finally Eq. (11) must be trivially satisfied providing the solution for γ, β, U and V is correct.

Since we are working in a cosmological space, in particular de Sitter space, the field equations take the generalized form[13]

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} + \lambda\bar{g}_{\mu\nu} = \bar{T}_{\mu\nu} \tag{12}$$

where λ is the cosmological constant.

In empty space, they become

$$\bar{R}_{\mu\nu} = \lambda\, \bar{g}_{\mu\nu} \tag{13}$$

so that, in general, the empty space equations are not set equal to zero. In order to see how the field equations are solved, we note that the metric is now

$$\bar{g}_{\mu\nu} = \phi^2\, g_{\mu\nu} = \begin{bmatrix} \frac{\phi^2 V e^{2\beta}}{r} - r^2\phi^2 U^2 e^{2\gamma} & \phi^2 e^{2\beta} & r^2\phi^2 U e^{2\gamma} & 0 \\ \phi^2 e^{2\beta} & 0 & 0 & 0 \\ r^2\phi^2 U e^{2\gamma} & 0 & -r^2\phi^2 e^{2\gamma} & 0 \\ 0 & 0 & 0 & -r^2\phi^2 e^{-2\gamma}\sin^2\theta \end{bmatrix} \tag{14}$$

Since $\bar{g}_{11} = 0$, then

$$\bar{R}_{11} = \lambda\, \bar{g}_{11} = 0 \tag{15}$$

Thus (3) becomes

$$0 = -\frac{4(\beta_1 - \frac{1}{2} r\gamma_1{}^2)}{r} + 2\sigma_{11} - 4\beta_1\sigma_1 - 2\sigma_1{}^2 \tag{16}$$

From the definition of $\sigma = \ln \phi$, then

$$\sigma_1 = \frac{2\mu}{B}\, \phi \tag{17}$$

and then

$$\sigma_{11} = \sigma_1^{\,2} \tag{18}$$

Also

$$1 + r\sigma_1 = \frac{B-u^2}{B}\,\phi \tag{19}$$

Then (16) now becomes

$$0 = \beta_1(1+r\sigma_1) - \frac{1}{2}\,r\gamma_1^{\,2}$$

$$= \frac{B-u^2}{B}\,\phi\beta_1 - \frac{1}{2}\,r\gamma_1^{\,2} \tag{20}$$

In terms of the variables

$$\left.\begin{aligned} q &= r\phi \\ u' &= u \end{aligned}\right\} \tag{21}$$

(20) finally becomes (after dropping primes)

$$0 = \beta_1 q_1\left(\frac{B-u^2}{B}\right)\phi - \frac{1}{2}\,r\gamma_1^{\,2}q_1^{\,2} \tag{22}$$

where derivatives are with respect to q except for the function

$$q_1 \equiv \frac{\partial q}{\partial r} = \frac{B-u^2}{B}\,\phi^2 \tag{23}$$

which occurs in the charge of variables

$$\frac{\partial}{\partial r} = q_1\,\frac{\partial}{\partial q} \tag{24}$$

After using (23) in (22), we get finally

$$\beta_1 = \frac{1}{2} q \gamma_1^{\ 2} \tag{25}$$

Since $\overline{R}_{12} = \lambda \overline{g}_{12} = 0$, then in terms of the variables q, u, (4) now becomes

$$\left(\frac{B-u^2}{B}\right)[q^4 e^{2(\gamma-\beta)} U_1]_1 = 2q^2\gamma(2\gamma_1\gamma_2 - \gamma_{12} - 2\gamma_1 \cot\theta) + 2q^4 \left(\frac{\beta_2}{q^2}\right)_1 \tag{26}$$

Since

$$\overline{R}_{22} e^{2(\beta-\gamma)} + \overline{R}_{33} \frac{e^{2(\gamma+\beta)}}{\sin^2\theta} = -2q^2 e^{2\beta}\lambda \tag{27}$$

then (5) becomes

$$\begin{aligned} -2q^2 e^{2\beta}\lambda = {} & 2\left(\frac{B-u^2}{B}\right)^2 (\phi^3 V)_1 + \frac{1}{2} q^4 \left(\frac{B-u^2}{B}\right)^2 e^{2(\gamma-\beta)} U_1^{\ 2} \\ & - \frac{1}{q^2}\left(\frac{B-u^2}{B}\right)(q^4 U)_{12} - \frac{1}{q^2}\left(\frac{B-u^2}{B}\right)(q^4 U)_1 \cot\theta \\ & - 4\left(\frac{B-u^2}{B}\right)(q^2 \sigma_o)_1 \\ & + 2e^{2(\beta-\gamma)}[-1-(3\gamma_2-\beta_2)\cot\theta - \gamma_{22} + \beta_{22} + \beta_2^{\ 2} + 2\gamma_2(\gamma_2-\beta_2)] \end{aligned} \tag{28}$$

Since

$$\frac{e^{2(\gamma+\beta)}}{\sin^2\theta}\,\overline{R}_{33} = -\,q^2e^{2\beta}\lambda \tag{29}$$

then (6) becomes

$$\begin{aligned}
-\,q^2e^{2\beta}\lambda = {} & 2\left(\frac{B-u^2}{B}\right)q(q\gamma_o)_1 + 2\left(\frac{B-u^2}{B}\right)\left[\frac{2u}{B-u^2} + \frac{2}{B}\left(\frac{B+u^2}{B-u^2}\right)q\right]\frac{(q^4\gamma_1)_1}{q} \\
& - \frac{4u}{B-u}\left(\frac{B-u^2}{B}\right)q^2\gamma_1 + \left(\frac{B-u^2}{B}\right)^2[(1-q\gamma_1)(\phi^3V)]_1 \\
& - q\left(\frac{B-u^2}{B}\right)(U_2+3U\cot\theta) + 2q\left(\frac{B-u^2}{B}\right)\gamma_2U \\
& q^2\left(\frac{B-u^2}{B}\right)U_1\cot\theta + q^2\left(\frac{B-u^2}{B}\right)\gamma_2U_1 + 2q^2\left(\frac{B-u^2}{B}\right)\gamma_{12}U \\
& + q^2\left(\frac{B-u^2}{B}\right)\gamma_1(U_2 + U\cot\theta) - 2\left(\frac{B-u^2}{B}\right)(q^2\sigma_o)_1 \\
& + e^{2(\beta-\gamma)}[-1-(3\gamma_2-2\beta_2)\cot\theta - \gamma_{22}+2\gamma_2(\gamma_2-\beta_2)]
\end{aligned} \tag{30}$$

Since $\overline{R}_{02} = \lambda q^2Ue^{2\gamma}$, then (7) becomes

$$\begin{aligned}
\lambda q^2Ue^{2\gamma} = {} & \beta_{2o}+q^3\sigma_o\left(\frac{\beta_2}{q^2}\right)_1 - q\sigma_o\gamma_{21} - \gamma_{20} + 2\gamma_2(q\sigma_o\gamma_1+\gamma_o) \\
& - 2(q\sigma_o\gamma_1+\gamma_o)\cot\theta - U(\beta_{22}+2\beta_2{}^2+2\beta_2\gamma_2+\beta_2\cot\theta) \\
& + \left(\frac{B-u^2}{B}\right)q^2e^{2(\gamma-\beta)}\left\{\frac{U_{o1}}{2} + \frac{(q^3\sigma_oU_1)_1}{2q^2} + \frac{2(q^2\sigma_o)_1U}{q^2} + (\gamma_oU)_1\right.
\end{aligned}$$

$$+ \frac{(q^2\gamma_o)_1 U}{q^2} + \frac{2(q^2\sigma_o\gamma_1)_1 U}{q} + \frac{(q^2\sigma_o)_1\gamma_1(q^2U)_1}{q^3} + \frac{3U(q^2U_2)_1}{2q^2}$$

$$+ U_1U_2 + (\gamma_1U)_2U + \frac{(q^4U^2)_1\cot\theta}{4q^4} + \frac{(\gamma_2q^2U^2)_1}{q^2} + \gamma_1U^2\cot\theta$$

$$- q\sigma_o\beta_1U_1 - \beta_oU_1 - \beta_2UU_1 + (\frac{B-u^2}{B})[- \frac{(U\phi^3V)_1}{q^2} - \frac{(q^2U_1)_1(\phi^3V)}{2q^3}$$

$$\left. - \frac{(q\gamma_1U\phi^3V)_1}{q^2} + \frac{\beta_1U_1(\phi^3V)}{q}]\right\} - \frac{1}{2}(\frac{B-u^2}{B})^2q^4e^{4(\gamma-\beta)}UU_1{}^2$$

$$+ (\frac{B-u^2}{B})[\frac{(\gamma_1-\beta_1)(\phi^3V_2)}{q} - \frac{1}{2}(\frac{\phi^3V_2}{q})_1] \tag{31}$$

Since $\overline{R}_{oo} = \lambda[\frac{(\phi^3V)e^{2\beta}}{q} - q^2U^2e^{2\gamma}]$, then (8) becomes

$$\lambda[\frac{(\phi^3V)e^{2\beta}}{q} - q^2U^2e^{2\gamma}] = \frac{4}{B}(\frac{B+2uq}{B-u^2}) - 4q\sigma_o{}^2\beta_1 - 4\sigma_o\beta_o$$

$$+ (\frac{B-u^2}{B})\left\{- \frac{(\phi^3V)_o}{q^2} + \frac{2(q^2\sigma_o)_1(\phi^3V)}{q^3} + \frac{2(\phi^3V)(q\beta_o)_1}{q^2} + \frac{2(\phi^3V)(q^3\sigma_o\beta_1)_1}{q^3}\right.$$

$$+ \frac{U}{2}(\frac{\phi^3V}{q})_1\cot\theta + \frac{U_2}{2}(\frac{\phi^3V}{q})_1 - \frac{(q^4U)_1(\phi^3V_2)}{2q^5} + \frac{U(\phi^3V_2)_1}{q}$$

$$+ \frac{(q^2\beta_2U)_1(\phi^3V)}{q^3} + \frac{(\beta_1U\phi^3V)_2}{q} + \frac{\beta_1U(\phi^3V)}{q}\cot\theta$$

$$- \frac{2\gamma_1U(\phi^3V_2)}{q} - (\frac{B-u^2}{B})[\frac{(\phi^3V)(\phi^3V)_{11}}{2q^2} + \frac{\beta_1(\phi^3V)(\phi^3V)_1}{q^2}$$

$$+ \frac{(q\beta_1)_1(\phi^3V)^2}{q^3}] + q^2e^{2(\gamma-\beta)}\left(-UU_{o1} - \frac{2(q^2\sigma_o)_1U^2}{q^2}\right.$$

$$- \frac{U(q^3\sigma_o U_1)_1}{q^2} - 2\gamma_{o1}U^2 - \frac{2U(Uq^3\sigma_o\gamma_1)_1}{q^2} - \frac{2U(qU)_{1\gamma o}}{q}$$

$$- 2U(UU_1)_2 - \frac{(U^3)_2}{q} - \frac{(q^3U^3)_1}{3q^3}\cot\theta - \frac{(q^2\gamma_2U^3)_1}{q^2} - (\gamma_1U)_2U^2$$

$$- \gamma_1U^3\cot \quad + (\frac{B-u^2}{B})[\frac{(q^4U_1)_1(\phi^3V)U}{q^5} + \frac{U^2(\phi^3V)_1}{q^2} + \frac{U_1{}^2(\phi^3V)}{q^2}$$

$$+ \frac{(\gamma_1U^2q\phi^3V)_1}{q^2} - \frac{2\beta_1UU_1(\phi^3V)}{q}] + 2(q\sigma_o\beta_1+\beta_o)UU_1+2\beta_2U^2U_1\Big)\Big\}$$

$$- 2q\sigma_o(\beta_1U)_2 - 2(\beta_oU)_2 + 2q\sigma_o(\gamma_1U)_2 + 2(\gamma_oU)_2 + q\sigma_oU_{12}$$

$$+ U_{o2} + \frac{1}{2}(U^2)_{22} + (\gamma_2U^2)_2 - 2\beta_2UU_2 + (2\beta_2{}^2-2\beta_2\gamma_2)U^2$$

$$+ 2(q\sigma_o\gamma_1+\gamma_o)^2 - \cot\theta[2(q\sigma_o\beta_1+\beta_o-q\sigma_o\gamma_1-\gamma_o)U$$

$$- q\sigma_oU_1-U_o-UU_2-\gamma_2U^2] + \frac{1}{2}(\frac{B-u^2}{B})^2q^4e^{4(\gamma-\beta)}U^2U_1{}^2$$

$$- \frac{e^{2(\beta-\gamma)}}{2q^3}\Big\{\phi^3V_{22}+2\beta_{22}(\phi^3V)+(2\beta_2-2\gamma_2+\cot\theta)[\phi^3V_2+2\beta_2\phi^3V\ ]\Big\} \tag{32}$$

Since $\overline{R}_{o1} = \lambda\phi^2e^{2\beta}$, then (11) becomes

$$\lambda e^{2\beta} = (\frac{B-u^2}{B})[\frac{2(q^2\sigma_o)_{11}}{q} + \frac{(q^2\beta_2U)_1}{q^2} + \frac{(q^2U_2)_1}{2q^2} + \frac{(q^2U)_1\cot\theta}{2q^2}$$

$$+ 2\beta_{1o}+2(q\sigma_o\beta_1)_1+(\gamma_1U)_2+\gamma_1U\cot\theta+2\gamma_1(\gamma_o+q\sigma_o\gamma_1)]$$

$$+ \left(\frac{B-u^2}{B}\right)^2 \left\{ - \frac{(\phi^3 V)_{11}}{2q} - \frac{(q\beta_1 \phi^3 V)_1}{q^2} + q^2 e^{2(\gamma-\beta)} \left[\frac{(q^4 U U_1)_1}{2q^4} + (\gamma_1 - \beta_1) U U_1 \right] \right\}$$

$$- \frac{e^{2(\beta-\gamma)}}{q^2} [\beta_{22} + 2\beta_2(\beta_2 - \gamma_2) + \beta_2 \cot\theta] \tag{33}$$

The complex of Eqs. (25), (26) and (28) for β, U, and V respectively, can be solved if we assume a power series expansion in $1/q$ for γ of the form

$$\gamma = \frac{f(u,\theta)}{q} + \frac{d(u,\theta)}{q^2} + \frac{g(u,\theta)}{q^3} + \frac{k(u,\theta)}{q^4} + \theta(q^{-5}) \tag{34}$$

The d term is dropped since it gives rise to a log term in the solution for U. The solution for (25) then becomes

$$\beta = H(u_1\theta) - \frac{\frac{1}{4} f^2}{q^2} - \frac{\frac{3}{4} fg}{q^4} - \frac{\frac{4}{5} fk}{q^5} + \theta(q^{-6}) \tag{35}$$

where H is an integration constant. The H term is also dropped since it gives rise to a log term in the solution for V. However, this term can be reduced to zero by a suitable coordinate transformation.[9] Using (34) and (35) in (26), we obtain

$$\left(\frac{B-u^2}{B}\right) U = L - (f_2 + 2f \cot\theta) \frac{1}{q^2} + (2N + 3ff_2 + 4f^2 \cot\theta) \frac{1}{q^3}$$

$$- \frac{1}{4} (12fN + 13f^2 f_2 + 14f^3 \cot\theta - 6g_2 - 12g \cot\theta) \frac{1}{q^4} + \theta(g^{-5}) \tag{36}$$

where L and N are integration constants.

Using (36), the solution for V from (27) is

$$\left(\frac{B-u^2}{B}\right)^2(\phi^3 V) = \frac{1}{3}\left[\frac{12(B+u^2)}{B^2} - \lambda\right]q^3+\left[\frac{4u}{B} + L_2+L\cot\theta\right]q^2$$

$$+ \left(1+\frac{1}{2}f^2\lambda\right)q-2M-\left[N_2+N\cot\theta-f_2^{\,2}-4ff_2\cot\theta\right.$$

$$\left.-\frac{1}{2}f^2(1+8\cot^2\theta)+\lambda\left(\frac{3}{2}fg-\frac{1}{8}f^4\right)\right]\frac{1}{q}$$

$$-\frac{1}{2}\left[g_{22}+3g_2-2g+6N(f_2+2f\cot\theta)+9ff_2^{\,2}\right.$$

$$+\frac{1}{2}ff_{22}+\frac{51}{2}f_2f^2\cot\theta + 16f^3\cot^2\theta - \frac{1}{3}f^3$$

$$\left.+\frac{8}{5}fk\lambda\right]\frac{1}{q^2} + \theta(q^{-3}) \qquad (37)$$

If we replace the parameter $f \to C-c^3/6$, the solution represented by Eqs. (34), (35), (36) and (37) will then reduce to the assymptotic flat-space results[9] in the limit $B \to \infty$ and $\lambda \to 0$ providing we put $L = 0$. In asymptotic flat-space, the constant of integration L in the equation for U (Eq. (36)) must vanish in order to preserve the signature of the metric. In flat-space $V \sim r$ and $U \sim L$. Thus

$$g_{oo} = \frac{Ve^{2\beta}}{r} - U^2re^{2\gamma} \qquad (38)$$

would eventually change sign for large enough r. But in asymptotic de Sitter space, $\phi^3 V \sim \theta(q^3)$ and $U \sim L$; so that $\overline{g}_{oo}$ does not change sign for large q.

Thus we cannot a priori set L = 0 as in the flat-space case. The solution represented by γ, β, U and V (Eqs. (34) - (37)) are then substituted into the time development Eq. (30). This yields the following conditions on the expansion:

$$\frac{2}{3} f\lambda = L_2 - L \cot \theta \tag{39}$$

and

$$4\left(\frac{B-u^2}{B}\right)g_o = 2fM-N_2+N \cot \theta + \frac{4}{3} k\lambda$$

$$- 6gL_2-4g_2L-6gL \cot \theta \tag{40}$$

With these conditions, the form of γ is preserved and the development of the system is fully determined from initial conditions provided the functions f, k, N, M, L are known. The consistency field equation (33) is trivially satisfied to $\theta(q^{-4})$ by this solution for γ, β, U, and V.

This solution, along with the constraints (39) and (40), reduces the supplementary field equations (31) and (32) to inverse-square form. Setting these $\theta(q^{-2})$ terms equal to zero yields repsectively the following relations

$$\left(\frac{B-u^2}{B}\right)M_o = - \left(\frac{B-u^2}{B}\right)^2 f_o^2+\left(\frac{B-u^2}{B}\right)\left[\frac{1}{2} (f_{22o}+3f_{2o}\cot \theta -2f_o)\right.$$

$$\left.-ff_oL \cot \theta -2f_of_2L-2ff_oL_2\right]+ \frac{1}{2} \lambda(N_2+N \cot \theta)$$

$$+ \frac{1}{2} \lambda g(L_2-L \cot \theta)- \frac{1}{2} L(2M_2+3M \cot \theta -f_{222}$$

$$- 2f_{22}\cot\theta + 2f_2 + 4f_2\cot^2\theta + 2f\cot\theta + 2f\cot^3\theta)$$

$$- \frac{1}{2} L_2(3M - 4f_{22} - 10f_2\cot\theta - 2f\cot^2\theta + 4f)$$

$$- \frac{1}{2} L^2(-12ff_{22} + 2f_2^{\,2} + \frac{9}{2} ff_2\cot\theta + f^2 + \frac{25}{4} f^2\cot^2\theta)$$

$$- \frac{1}{2} LL_2(8N + \frac{19}{2} ff_2 + \frac{13}{2} f^2\cot\theta) - \frac{5}{8} L_2^{\,2} f^2 + \frac{3}{2} L_{22} f_2 \qquad (41)$$

$$- 3(\frac{B-u^2}{B})N_o = M_2 + (\frac{B-u^2}{B})(3ff_{20} + 4ff_o\cot\theta + f_o f_2)$$

$$- \lambda(g_2 + g\cot\theta) + L_2(2N + \frac{15}{4} ff_2 + \frac{5}{2} f^2\cot\theta)$$

$$+ L(3N_2 + 3N\cot\theta + 3ff_{22} + f_2^{\,2} + \frac{9}{4} ff_2\cot\theta$$

$$- 3f^2 - \frac{9}{2} f^2\cot^2\theta) \qquad (42)$$

where (42) was used in Eq. (31) in order to remove the N_o dependence from (41). Thus, the time development of M and N are known provided the functions g, L, M, N are given for one value of u and f is given as a function of u and θ. Thus, we are at the peculiar point that we need to know f, k and L as functions of u and θ. The reason we need to know L is that the constraint between f and L from Eq. (39) still leaves L unknown up to an arbitrary function of u. We can, however, obtain an equation for k_o provided we carry out the expansion of the time development field equation (30) to $\theta(q^{-3})$. But this would still leave L arbitrary.

An alternate approach to this problem is to require that the solution has a definite form in the limit of $q \to \infty$. This would be, of course, the "asymptotic" de Sitter space form of the metric

$$\bar{g}_{oo} \to \phi^2 = \frac{4u^2}{(B-u^2)^2} q^2 + \frac{4uB}{(B-u^2)^2} q + \frac{B^2}{(B-u^2)^2} \tag{43}$$

But from the solution (34) - (37) and the form of $\bar{g}_{oo}$ from the metric (14), we have

$$\begin{aligned} \bar{g}_{oo} = \frac{B^2}{(B-u^2)^2} \Big\{ & [\frac{4(B+u^2)}{B^2} - \frac{\lambda}{3} - L^2]\, q^2 \\ & + [\frac{4u}{B} + L_2 + L \cot\theta - 2fL^2]q + [1 + \frac{2}{3} f^2\lambda \\ & - \frac{2f^2(B+u^2)}{B^2} - 2f^2L^2 + 2L(f_2 + f \cot\theta)] + \theta(\frac{1}{q}) \Big\} \end{aligned} \tag{44}$$

Thus in the limit of large q, we must have

$$\frac{4u^2}{(B-u^2)^2} = \frac{B^2}{(B-u^2)^2} [\frac{4(B+u^2)}{B^2} - \frac{\lambda}{3} - L^2] \tag{45}$$

or

$$\frac{\lambda}{3} - \frac{4}{B} = L^2 \tag{46}$$

and similarly

$$2fL^2 = L_2 + L \cot\theta \tag{47}$$

$$\frac{2}{3} f^2\lambda - \frac{2f^2(B+u^2)}{B^2} - 2f^2L^2+2L(f_2+f\cot\theta) = 0 \qquad (48)$$

Eq. (46) immediately implies that L is independent of the coordinates and is therefore an invariant. But this implies with Eq. (47) that either L=0 or f α cot θ. The latter case does not have the correct regularity properties[9] for γ as $\theta \to 0$. Thus

$$\left.\begin{aligned} L &= 0 \\ \lambda &= \frac{12}{B} \end{aligned}\right\} \qquad (49)$$

Finally Eq. (48) implies that either f = 0 or B = $u^2/6$. Previously we ruled out f α cot θ on the basis of regularity for γ. But from Eq. (39) we must now conclude that either f = 0 or λ = 0. The consistent choice is

$$f = 0 \qquad (50)$$

Thus by comparison with the asymptotic flat-space case, the character of the solution is very different in asymptotic de Sitter space. Indeed, the mass aspect (Eq. (41)) now depends on a different "news" function that becomes, due to (40), (49) and (50)

$$M_o = \frac{6}{(B-u^2)} (N_2 + N \cot\theta) \qquad (51)$$

$$= -\frac{24}{B} g_o + \frac{12}{B-u^2} N \cot\theta + \frac{96}{B(B-u^2)} k$$

Similarly for N_o, we find

$$N_o = -\frac{BM_2}{3(B-u^2)} + \frac{4}{(B-u^2)}(g_2 + g\cot\theta) \tag{52}$$

On the other hand, the radiation condition[9,14] in asymptotic flat space expressed in (q,u) coordinates becomes

$$\frac{\partial}{\partial r}(r\gamma') = \frac{\partial}{\partial q}(q\gamma)\left(\frac{B+2uq}{B}\right) - \frac{2uq\gamma}{B} \tag{53}$$

where γ' is a function of $1/r$; so in the limit of $q \to \infty$, the right hand side will not vanish <u>unless</u> $f = 0$. Thus the condition for radiation in q-coordinates seems to be satisfied if $f = 0$, and that the description of mass or mass-loss now depends consistently on g_o in the supplementary conditions, i.e. Eq. (51). This different dependence of γ is not surprising since the asymptotic space cannot be described by an empty space axially symmetric static metric and therefore related to Weyl's form from which Bondi, et al.[9] could identify the mass function.

We conclude that radiation will occur in an asymptotic de Sitter space but that the nature of the solution requires that the $1/q$ dependence of the solution for γ begins with the $\theta(q^{-3})$ term. This implies that in the limit $B \to \infty$, we do not obtain the radiative flat-space results. That is, the requirement for radiation in asymptotic de Sitter space seems to be more restrictive than in asymptotic flat-space. We now are ready to return to the argument following Eq. (49) in which we concluded that $f = 0$. This choice was, however, only consistent with <u>a solution</u> in asymptotic de Sitter

space. Instead, if we had chosen $\lambda = 0$, then $B \to \infty$ which would then automatically restrict our solution to flat space. Thus, we are left with the unusual consequences of radiation in de Sitter space or no de Sitter space solution at all.

III. Gravitational Theories with Non-Zero Divergence of the Energy-Momentum Tensor

Until recently it was generally believed that any viable (acceptable) theory of gravitation must be a metric theory. In particular, any theory must satisfy the following two metric postulates:[15,16]

(1) There exists a metric of signature 2, which governs the proper-length and proper-time measures: $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$.

(2) The response of stressed matter is embodied in the divergenceless energy-momentum tensor.

Based upon this assumption, the Caltech program[15] has vigorously compiled and analyzed various theories which can be tested against experiment through the parametrized post-Newtonian (PPN) expansion of the metric.[17]

Unfortunately a number of difficulties have arisen in this program. Among other problems is that it is now known that there are theories which agree with the experimental values of the PPN parameters and hence the Einstein theory,[18] but may disagree theoretically with the Einstein theory in the as-yet-experimentally-inaccessible next-order expansions of the metric. Furthermore, there is at least one example of a nonmetric theory[19] which approximates a metric theory to first order and is in agreement with the Einstein theory to this same limit. Finally, we have shown that if we

replace the second condition for a metric theory (although perhaps not the more generalized integral form discussed by Thorne and Will[15] in the particular case of a Brans-Dicke gravitational theory,[20] we still obtain a consistent solution in the post-Newtonian expansion of the metric tensor,[21] $g_{\alpha\beta}$. As a result,we have published a paper[22] where we have modified the Brans-Dicke theory by assuming that the divergence of the energy-momentum tensor is proportional to the covariant derivative of the scalar curvature. Rastall argues that the form of the divergence of the energy-momentum tensor is not ruled out experimentally, at least for the Einstein theory.[17,23] However, no ad hoc additions to the usual Brans-Dicke field equations are required as in Rastall's case or as in the steady-state theories of which this is a natural possibility.[24] Very general approaches to scalar-tensor theories obtainable from Lagrangians have been considered by Bergmann[25] and Harrison.[26] In particular, Harrison shows that many theories thought distinct, such as the Einstein theory, Brans-Dicke theory, and some steady-state theories (among others), are in fact conformally equivalent. This implicit embodiment in the field equations of general relativity of various physically inequivalent but conformally equivalent versions of the scalar-tensor formalism has been recognized by others. Dicke[27] showed that even a unit transformation has this effect on the scalar-tensor theory. McCrea[28] recognized that formally the steady-state theory of Hoyle logically has the same structure as the Einstein equations if one adds a suitable term to $T^{\mu\nu}$. Indeed, Hoyle's original approach[24] to the steady-state theory is equivalent to the assumption that $T^{\mu\nu}{}_{;\mu} \neq 0$. However, his later identification[29] of this source with a scalar creation field does not apply here. Formally we differ in that no ad hoc additions to the field equation or $T^{\mu\nu}$

are required as in Hoyle's steady-state theory or as in Rastall's case. The extra degree of freedom of the scalar field determines uniquely the field equations for which we need only supply a reasonable assumption for the divergence of $T^{\mu\nu}$ such as done by Rastall. It is thus noteworthy that the Caltech group is now actively considering nonmetric as well as other possible theories of gravity.[30]

In a second paper,[31] we have discovered the formal equivalence of massive Brans-Dicke theories of gravitation with Brans-Dicke theories with non-zero divergence of the energy-momentum tensor. As an added bonus, we find that this equivalence constrains $\omega \lesssim 49$.

In order to demonstrate the general theory we carry out in detail the general calculation of the PPN parameters[4] for two different modifications, as yet unpublished, of the generalized scalar-tensor theories referred to as the Bergmann-Wagoner theory.[32] In the first case we let

$$T^{\mu\nu}{}_{;\nu} = \frac{\sigma}{8\pi} R^{;\mu} \tag{54}$$

where σ is an unknown coupling constant. As a matter of distinction, we note that this modification immediately implies a correction to the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0 \tag{55}$$

by adding a term proportional to

$$\propto \sigma \frac{\partial\rho}{\partial t} \tag{56}$$

to the right hand side of (55). This can, of course, be obviated by a priori requiring that $\sigma = \theta(v^2)$. However, Euler's equation

$$\rho(\vec{v} \cdot \nabla)\vec{v} + \rho\nabla U + \nabla p = 0 \tag{57}$$

will still be modified even if σ is a second order parameter. A modification of the form

$$T^{\mu\nu}{}_{;\nu} = \frac{\sigma}{8\pi}\, \phi^{;\nu}\phi_{;\nu}\phi^{;\mu} \tag{58}$$

where ϕ is the scalar field does not suffer the above problems since the right hand side of (58) is obviously sixth order since $\phi_{;\mu}$ is either second or third order depending on whether it is spatial or time derivatives, respectively. The modifications of the form (58) satisfy the usual Newtonian equations of perfect fluid hydrodynamics. Furthermore, in this case the coupling parameter σ is completely free.

In order to obtain the PPN parameters, we must work in the so called standard gauge so we can compare with the work of Will[4] and others. For us this is equivalent to choosing the Chandrasekhar-Nutku gauge:[33]

$$\begin{aligned} &\frac{1}{2}\frac{\partial g^{(2)}{}_{oo}}{\partial x^k} - \frac{1}{2}\frac{\partial g^{(2)}{}_{jj}}{\partial x^k} + \frac{\partial g^{(2)}{}_{ik}}{\partial x^i} = \frac{\partial \varepsilon^{(2)}}{\partial x^k} \\ &\frac{\partial g^{(3)}{}_{jo}}{\partial x^j} - \frac{1}{2}\frac{\partial g^{(2)}{}_{jj}}{\partial t} = \frac{\partial \varepsilon^{(2)}}{\partial t} \end{aligned} \tag{59}$$

From these it follows that[34]

$$\frac{1}{2}\frac{\partial^2 g^{(2)}{}_{ii}}{\partial t^2} - \frac{\partial^2 g^{(3)}{}_{io}}{\partial x^i \partial t} = -\frac{\partial^2 \varepsilon^{(2)}}{\partial t^2}$$

$$\frac{1}{2}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial t \partial x^j} - \frac{\partial^2 g^{(2)}{}_{ii}}{\partial t \partial x^j} + \frac{\partial^2 g^{(3)}{}_{io}}{\partial x^i \partial x^j} + \frac{\partial^2 g^{(2)}{}_{ij}}{\partial t \partial x^i} = 2\,\frac{\partial^2 \varepsilon^{(2)}}{\partial t \partial x^j}$$

$$-\frac{\partial^2 g^{(2)}{}_{oo}}{\partial x^j \partial x^i} + \frac{\partial^2 g^{(2)}{}_{kk}}{\partial x^j \partial x^i} - \frac{\partial^2 g^{(2)}{}_{ki}}{\partial x^k \partial x^j} - \frac{\partial^2 g^{(2)}{}_{jk}}{\partial x^i \partial x^k} = -\,2\,\frac{\partial^2 \varepsilon^{(2)}}{\partial x^i \partial x^j} \tag{60}$$

For the above we expanded

$$g_{oo} = -1 + g^{(2)}{}_{oo} + g^{(4)}{}_{oo} + \ldots$$

$$g_{ij} = \delta_{ij} + g^{2}{}_{ij} + \ldots$$

$$g_{io} = g^{(3)}{}_{io} + \ldots \tag{61}$$

The equations for the various orders of $R_{\mu\nu}$ become

$$R^{(2)}{}_{oo} = \frac{1}{2}\nabla^2 g^{(2)}{}_{oo} \tag{62}$$

$$R^{(4)}{}_{oo} = \frac{1}{2}\frac{\partial^2 g^{(2)}{}_{ii}}{\partial t^2} - \frac{\partial^2 g^{(3)}{}_{io}}{\partial x^i \partial t} + \frac{1}{2}\nabla^2 g^{(4)}{}_{oo} - \frac{1}{2} g^{(2)}{}_{ij}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial x^i \partial x^j}$$

$$-\frac{1}{2}\left(\frac{\partial g^{(2)}{}_{ij}}{\partial x^j}\right)\left(\frac{\partial g^{(2)}{}_{oo}}{\partial x^i}\right) + \frac{1}{4}\left(\frac{\partial g^{(2)}{}_{oo}}{\partial x^i}\right)\left(\frac{\partial g^{(2)}{}_{oo}}{\partial x^i}\right) + \frac{1}{4}\left(\frac{\partial g^{(2)}{}_{oo}}{\partial x^i}\right)\left(\frac{\partial g^{(2)}{}_{jj}}{\partial x^i}\right)$$

$$= -\frac{\partial^2 \varepsilon^{(2)}}{\partial t^2} + \frac{1}{2}\nabla^2 g^{(4)}{}_{oo} - \frac{1}{2} g^{(2)}{}_{ij} \frac{\partial^2 g^{(2)}{}_{oo}}{\partial x^i \partial x^j}$$

$$+ \frac{1}{2}\left(\nabla g^{(2)}{}_{oo}\right)^2 - \frac{1}{2}\frac{\partial g^{(2)}{}_{oo}}{\partial x^i}\frac{\partial \varepsilon^{(2)}}{\partial x^i} \tag{63}$$

$$R^{(3)}{}_{io} = \frac{1}{2}\frac{\partial^2 g^{(2)}{}_{jj}}{\partial x^i \partial t} - \frac{1}{2}\frac{\partial^2 g^{(3)}{}_{jo}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial^2 g^{(2)}{}_{ij}}{\partial x^j \partial t} + \frac{1}{2}\nabla^2 g^{(3)}{}_{oi}$$

$$= \frac{1}{4}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial t \partial x^i} - \frac{\partial^2 \varepsilon^{(2)}}{\partial t \partial x^i} + \frac{1}{2}\nabla^2 g^{(3)}{}_{io} \tag{64}$$

$$R^{(2)}{}_{ij} = -\frac{1}{2}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2 g^{(2)}{}_{kk}}{\partial x^i \partial x^j} - \frac{1}{2}\frac{\partial^2 g^{(2)}{}_{ik}}{\partial x^k \partial x^j} - \frac{\partial^2 g^{(2)}{}_{kj}}{\partial x^k \partial x^i} + \frac{1}{2}\nabla^2 g^{(2)}{}_{ij}$$

$$= -\frac{\partial^2 \varepsilon^{(2)}}{\partial x^i \partial x^j} + \frac{1}{2}\nabla^2 g^{(2)}{}_{ij} \tag{65}$$

We assume the energy momentum tensor has the form

$$T^{\mu\nu} = (\rho_o + \rho_o\,\pi + p)\, u^\mu u^\nu + p g^{\mu\nu} \tag{66}$$

ρ_o = energy density of rest mass

$\pi = \dfrac{\rho - \rho_o}{\rho_o}$ = specific interval energy density

ρ = total energy density

p = pressure; all quantities are measured in the frame comoving with the fluid element, and where the four-velocity is given by

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} \tag{67}$$

In order to obtain an expansion for $T^{\mu\nu}$, we require an expansion for u^o to 4th order and u^i to 3rd order:

$$\begin{aligned} u^o &= 1 + \frac{v^2}{2} - u + \theta(v^4) \\ u^i &= u^o v^i = v^i + \theta(v^3) \end{aligned} \tag{68}$$

Then to the required order

$$T^{(o)oo} = \rho_o \tag{69}$$

$$T^{(2)oo} = \rho_o(v^2 - 2u) + \rho_o\pi \tag{70}$$

$$T^{(2)ij} = \rho_o v^i v^j + p\delta_{ij} \tag{71}$$

$$T^{(1)oi} = \rho_o v^i \tag{72}$$

where the superscript in parentheses indicate order $(\rho/r^3)\theta(v^N)$.

We now modify the Bergmann-Wagoner field equations[32]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = -\frac{8\pi}{\phi}[T_{M\mu\nu} + T_{\phi\mu\nu}] \tag{73}$$

and

$$\Box^2\phi = \frac{8\pi T_M}{3+2\omega(\phi)} + F(\phi)\phi^{;\rho}\phi_{;\rho} \tag{74}$$

which arises when we assume

$$T_M{}^{\mu}{}_{\nu;\mu} = \frac{\sigma}{8\pi} R_{;\nu} \tag{75}$$

where σ is a constant of 2nd order in v, and where $T_\phi{}^{\mu}{}_{\nu}$ has the general form

$$T_\phi{}^{\mu}{}_{\nu} = A(\phi)\phi^{;\mu}\phi_{;\nu} + B(\phi)\delta^{\mu}{}_{\nu}\phi_{;\rho}\phi^{;\rho} + C(\phi)\phi^{;\mu}{}_{;\nu} + \delta^{\mu}{}_{\nu}D(\phi)\Box^2\phi \tag{76}$$

From equation (73)

$$R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R = -\frac{8\pi}{\phi}[T_M{}^{\mu}{}_{\nu} + T_\phi{}^{\mu}{}_{\nu}] \tag{77}$$

or

$$R = \frac{8\pi}{\phi}[T_M + T_\phi] \tag{78}$$

where

$$T_M \equiv T_M{}^{\mu}{}_{\mu};\ T_\phi \equiv T_\phi{}^{\mu}{}_{\mu}$$

From equation (74) we can solve for T_M , and from equation (76) we can find T_ϕ.

$$R = +\frac{8\pi}{\phi}[\frac{3+2\omega(\phi)}{8\pi}\Box^2\phi - \frac{3+2\omega(\phi)}{8\pi}F(\phi)\phi^{;\rho}\phi_{;\rho}$$

$$+ (A+4B)\phi^{;\rho}\phi_{;\rho} + (C+4D)\Box^2\phi] \tag{79}$$

$$R_{;\nu} = + \frac{8\pi}{\phi}\left[\frac{2\omega'(\phi)}{8\pi}\phi_{;\nu}\Box^2\phi + \frac{3+2\omega(\phi)}{8\pi}(\Box^2\phi)_{;\nu}\right.$$

$$- \frac{3+2\omega(\phi)}{8\pi}F'(\phi)\phi_{;\nu}\phi^{;\rho}\phi_{;\rho} - \frac{2\omega'(\phi)}{8\pi}\phi_{;\nu}F(\phi)\phi^{;\rho}\phi_{;\rho}$$

$$- 2\,\frac{3+2\omega(\phi)}{8\pi}F(\phi)\phi^{;\rho}\phi_{;\rho;\nu} + (A'+4B')\phi_{;\nu}\phi^{'\rho}\phi_{'\rho}$$

$$+ 2(A+4B)\phi^{;\rho}\phi_{;\rho;\nu} + (C'+4D')\phi_{;\nu}\,\Box^2\phi$$

$$\left. + (C+4D)(\Box^2\phi)_{;\nu}\right] - \frac{8\pi}{\phi^2}\phi_{;\nu}\left[\frac{3+2\omega(\phi)}{8\pi}\Box^2\phi\right.$$

$$\left. + (C+4D)\Box^2\phi + (A+4B)\phi^{;\rho}\phi_{;\rho} - \frac{3+2\omega(\phi)}{8\pi}F(\phi)\phi^{;\rho}\phi_{;\rho}\right] \tag{80}$$

Equation (77) implies

$$(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R)\phi_{;\mu} = - 8\pi[T_M{}^{\mu}{}_{\nu;\mu} + T_{\phi}{}^{\mu}{}_{\nu;\mu}] \tag{81}$$

so that

$$\text{R.H.S.} = - 8\pi\left[+ \frac{\sigma}{8\pi}\frac{8\pi}{\phi}\right\}\phi_{;\nu}\Box^2\phi\left[\frac{2\omega'(\phi)}{8\pi} + C'+4D' - \frac{3+2\omega(\phi)}{8\pi\phi} - \frac{C+4D}{\phi}\right]$$

$$+ (\Box^2\phi)_{;\nu} \left[\frac{3+2\omega(\phi)}{8\pi} + C + 4D\right] + \phi_{;\nu}\phi^{;\rho}\phi_{;\rho}\left[- \frac{3+2\omega(\phi)}{8\pi} F'(\phi)\right.$$

$$\left. - \frac{2\omega'(\phi)}{8\pi} F(\phi) + A' + 4B' - \frac{A+4B}{\phi} + \frac{3+2\omega(\phi)}{8\pi\phi} F(\phi)\right]$$

$$\left. + \phi^{;\rho}\phi_{;\rho;\nu}\left[2(A+4B) - 2\,\frac{3+2\omega(\phi)}{8\pi} F(\phi)\right]\right\}$$

$$+ (A'+B')\phi^{;\rho}\phi_{;\rho}\phi_{;\nu} + (A+D')\phi_{;\nu}\Box^2\phi + (A+2B+C')\phi^{;\rho}\phi_{;\rho;\nu}$$

$$\left. + D(\Box^2\phi)_{;\nu} + C\Box^2(\phi_{;\nu})\right] \tag{82}$$

$$\text{L.H.S.} = (\Box^2\phi)_{;\nu} - \Box^2(\phi_{;\nu}) - \frac{1}{2}\,\phi_{;\nu}\left[+ \frac{8\pi}{\phi}\left\{\left[\frac{3+2\omega(\phi)}{8\pi} + C + 4D\right]\Box^2\phi\right.\right.$$

$$\left. + \left[(A+4B) - \frac{3+2\omega(\phi)}{8\pi} F(\phi)\right]\phi^{;\rho}\phi_{;\rho}\right] \tag{83}$$

Equating coefficients from both sides in (82) and (83)

$$(\Box^2\phi)_{;\nu} \Rightarrow 1 = -8\pi D - \frac{8\pi\sigma}{\phi}\left[\frac{3+2\omega(\phi)}{8\pi} + C + 4D\right] \tag{84}$$

$$\Box^2(\phi_{;\nu}) \Rightarrow -1 = -8\pi C \tag{85}$$

$$\phi_{;\nu}\Box^2\phi \Longrightarrow -\frac{4\pi}{\phi}\left[\frac{3+2\omega(\phi)}{8\pi} + C + 4D\right] = -8\pi(A+D')$$

$$-\frac{8\pi\sigma}{\phi}\left[\frac{2\omega'(\phi)}{8\pi} + C' + 4D' - \frac{3+2\omega(\phi)}{8\pi\phi} - \frac{C+4D}{\phi}\right] \tag{86}$$

$$\phi^{;\rho}\phi_{;\rho}\phi_{;\nu} \Longrightarrow \frac{4\pi}{\phi}\left[A + 4B - \frac{3+2\omega(\phi)}{8\pi}F\right]$$

$$= 8\pi(A'+B') - \frac{8\pi\sigma}{\phi^2}\left[A + 4B - \frac{3+2\omega(\phi)}{8\pi}F\right]$$

$$+ \frac{8\pi\sigma}{\phi}\left[A' + 4B' - \frac{2\omega'(\phi)}{8\pi}F - F'\frac{3+2\omega}{8\pi}\right] \tag{87}$$

$$\phi_{;\rho}\phi^{;\rho}{}_{;\nu} \Longrightarrow 0 = \frac{8\pi\sigma}{\phi}\left[2A + 8B - 2\,\frac{3+2\omega(\phi)}{8\pi}F\right] + 8\pi(A+2B+C') \tag{88}$$

Equations (84) - (88) can be solved for A, B, C, D, F giving

$$A = \frac{1}{8\pi}\,\frac{\omega(\phi)}{\phi(1+\frac{4\sigma}{\phi})}\ ,\quad C = \frac{1}{8\pi}\ ,\quad D = -\frac{1}{8\pi}\,\frac{1+\frac{2\sigma}{\phi}(\omega+2)}{(1+\frac{4\sigma}{\phi})}$$

$$B = -\frac{\omega(\phi)(1+\frac{2\sigma}{\phi})}{16\pi\phi(1+\frac{4\sigma}{\phi})^2} - \frac{\sigma\omega'(\phi)}{8\pi\phi(1+\frac{4\sigma}{\phi})}\ ,\quad F = -\frac{\omega'(\phi)}{3+2\omega(\phi)} \tag{89}$$

The field equations are then

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi}{\phi}\left[T_{M\mu\nu} + T_{\phi\mu\nu}\right] \tag{90}$$

$$\Box^2\phi = \frac{8\pi T}{3+2\omega(\phi)} - \frac{\omega'(\phi)}{3+2\omega(\phi)}\,\phi^{;\rho}\phi_{;\rho} \tag{91}$$

Rewriting (90), we get

$$R_{\mu\nu} = -\frac{8\pi}{\phi}\Big[T_{\mu\nu} + \frac{1}{8\pi}\frac{\omega\phi_{;\mu}\phi_{;\nu}}{\phi(1+\frac{4\sigma}{\phi})} + \frac{\phi_{;\mu;\nu}}{8\pi} - \frac{g_{\mu\nu}}{8\pi}\frac{\sigma}{\phi}\frac{\phi^{;\rho}\phi_{;\rho}}{\phi(1+\frac{4\sigma}{\phi})^2}$$

$$- g_{\mu\nu}\frac{(\omega+1) + \frac{2\sigma}{\phi}(\omega+2)}{(1+\frac{4\sigma}{\phi})(3+2\omega)}T - \frac{g_{\mu\nu}}{16\pi}\frac{(1-\frac{2\sigma}{\phi})\omega'(\phi)}{(1+\frac{4\sigma}{\phi})(3+2\omega)}\phi^{;\rho}\phi_{;\rho}\Big] \tag{91}$$

If we expand all terms on right hand side with

$$\phi = \mathcal{G}(1+\varepsilon) \tag{92}$$

where $\varepsilon = \varepsilon^{(2)} + \varepsilon^{(4)} + \ldots$, $\mathcal{G}$ is a constant of order of the gravitational constant and once again the $\theta(v^N)$ is given in the parentheses; and further if we write

$$\omega(\phi) \approx \omega(\phi_0) + \frac{\varepsilon^{(2)}}{\mathcal{G}}\omega_0'(\phi_0)$$

$$\equiv \omega_0 + \frac{\varepsilon^{(2)}}{\mathcal{G}}\omega_0' \tag{93}$$

we find

$$R^{(2)}{}_{oo} = -8\pi\mathcal{G}\frac{\omega_o+2}{3+2\omega_o}T^{(o)}{}_{oo} \tag{94}$$

$$\begin{aligned} R^{(4)}{}_{oo} = {} & 8\pi\mathcal{G}T^{(o)oo}\varepsilon^{(2)}\left(\frac{\omega_o+2}{3+2\omega_o}\right) - 8\pi\mathcal{G}T^{(2)}{}_{oo} + \Gamma^{i}{}_{oo}\frac{\partial\varepsilon^{(2)}}{\delta x^{i}} \\ & - \frac{1}{2\mathcal{G}}\frac{\omega'_o(\overline{\nabla}\varepsilon^{(2)})^2}{(3+2\omega_o)} - 8\pi\mathcal{G}T^{(o)oo}g^{(2)}{}_{oo}\left(\frac{\omega_o+1}{3+2\omega_o}\right) \\ & - 8\pi\mathcal{G}T^{(o)oo}\frac{2\sigma G\omega_o}{3+2\omega_o} - 8\pi\mathcal{G}g^{(2)}{}_{oo}T^{(o)oo}\left(\frac{\omega_o+1}{3+2\omega_o}\right) \\ & + 8\pi\mathcal{G}T^{(2)oo}\left(\frac{\omega_o+1}{3+2\omega_o}\right) - 8\pi\mathcal{G}T^{(2)jj}\left(\frac{\omega_o+1}{3+2\omega_o}\right) \\ & - \frac{\partial^2\varepsilon^{(2)}}{\partial t^2} + 8\pi\mathcal{G}T^{(o)oo}\frac{\varepsilon^{(2)}\omega'_o}{\mathcal{G}(3+2\omega_o)^2} \end{aligned} \tag{95}$$

$$R^{(2)}{}_{ij} = -\frac{\partial^2\varepsilon^{(2)}}{\partial x^i\partial x^j} - 8\pi\mathcal{G}T^{(o)oo}\delta_{ij}\left(\frac{\omega_o+1}{3+2\omega_o}\right) \tag{96}$$

$$R^{(3)}{}_{oj} = -8\pi\mathcal{G}T^{(1)}{}_{oi} - \frac{\partial^2\varepsilon^{(2)}}{\partial t\partial x^{i}} \tag{97}$$

Using Equations (69) - (72) in Equations (94) - (97), we find

$$R^{(2)}{}_{oo} = \frac{1}{2}\nabla^2 g^{(2)}{}_{oo} = -8\pi\gamma\left(\frac{\omega_o+2}{2\omega_o+3}\right)T^{(o)oo} \tag{98}$$

$$R^{(2)}{}_{ij} = -\frac{\partial^2\varepsilon^{(2)}}{\partial x^i\partial x^j} + \frac{1}{2}\nabla^2 g^{(2)}{}_{ij}$$

$$= -8\pi\gamma T^{(o)oo}\delta_{ij}\frac{\omega_o+1}{2\omega_o+3} - \frac{\partial^2\varepsilon^{(2)}}{\partial x^i\partial x^j} \tag{99}$$

$$R^{(3)}{}_{oi} = \frac{1}{2}\nabla^2 g^{(3)}{}_{oi} - \frac{\partial^2\varepsilon^{(2)}}{\partial x^i\partial t} + \frac{1}{4}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial x^i\partial t}$$

$$= -8\pi\gamma T^{(1)}{}_{oi} - \frac{\partial^2\varepsilon^{(2)}}{\partial t\partial x^i} \tag{100}$$

Therefore

$$\nabla^2 g^{(2)}{}_{oo} = -16\pi\gamma T^{(o)oo}\left(\frac{\omega_o+2}{3+2\omega_o}\right) = -8\pi\gamma T^{(o)oo}\left(\frac{2\omega_o+4}{3+2\omega_o}\right) \tag{101}$$

$$\nabla^2 g^{(2)}{}_{ij} = -16\pi\gamma T^{(o)oo}\delta_{ij}\left(\frac{\omega_o+1}{3+2\omega_o}\right) \tag{102}$$

$$\nabla^2 g^{(3)}{}_{oi} = +16\pi\gamma T^{(1)oi} - \frac{1}{2}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial x^i\partial t} \tag{103}$$

But (101 and (102) imply that

$$\nabla^2 g^{(2)}{}_{ij} \equiv \gamma\delta_{ij}\, \nabla^2 g^{(2)}{}_{oo} \tag{104}$$

with

$$\gamma = \frac{\omega_o+1}{\omega_o+2} \tag{105}$$

By definition (agreement with Newton's Law of gravitation)

$$g^{(2)}{}_{oo} \equiv -\,2U = 4\mathcal{G}\left(\frac{\omega_o+2}{3+2\omega_o}\right)\int \frac{\rho(\bar{x}',t)d^3x'}{|\bar{x}-\bar{x}'|}$$

$$= +\,2G\int \frac{\rho(\bar{x}',t)d^3x'}{|\bar{x}-\bar{x}'|} \tag{106}$$

which implies that

$$\mathcal{G}\left(\frac{2\omega_o+4}{3+2\omega_o}\right) = G \tag{107}$$

where G is the measurable gravitational constant. To second order, Eq. (91) becomes

$$\nabla^2\varepsilon^{(2)} = -\,\frac{8\pi\mathcal{G}}{3+2\omega_o}\; T^{(o)oo} \tag{108}$$

By comparison with Eq. (101), we note that

$$(2\omega_o+4)\;\varepsilon^{(2)} = g^{(2)}{}_{oo} \tag{109}$$

Next we find that

$$\nabla^2 g^{(4)}{}_{oo} = g^{(2)}{}_{ij} \frac{\partial^2 g^{(2)}{}_{oo}}{\partial x^i \partial x^j} - \left(\nabla g^{(2)}{}_{oo}\right)^2 + \frac{\partial g^{(2)}{}_{oo}}{\partial x^j} \frac{\partial \varepsilon^{(2)}}{\partial x^j}$$

$$+ 16\pi \mathcal{G} T^{(o)00} \varepsilon^{(2)} \left(\frac{\omega_o+2}{3+2\omega_o} + \frac{\omega_o'}{\mathcal{G}(3+2\omega_o)^2} \right) - 16\pi \mathcal{G} T^{(2)oo} \left(\frac{\omega_o+2}{3+2\omega_o} \right)$$

$$+ 2\Gamma^i{}_{oo} \frac{\partial \varepsilon^{(2)}}{\partial x^i} - \frac{1}{\mathcal{G}} \frac{\omega_o'}{(3+2\omega_o)} (\overline{\nabla}\varepsilon^{(2)}) - 32\pi \mathcal{G} g^{(2)}{}_{oo} T^{(o)oo} \left(\frac{\omega_o+1}{3+2\omega_o} \right)$$

$$- 16\pi \mathcal{G} T^{(2)jj} \left(\frac{\omega_o+1}{3+2\omega_o} \right) - 16\pi \mathcal{G} T^{(o)oo} \frac{2\sigma\, \omega_o}{3+2\omega_o} \tag{110}$$

which gives

$$\nabla^2 g^{(4)}{}_{oo} = -2\nabla^2 U^2 (1+\Lambda) - 8\pi G\rho_o \left[\left(\frac{3+2\omega_o}{\omega_o+2} \right) V^2 + \left(\frac{\omega_o+1}{\omega_o+2} \right)^3 p/\rho_o \right.$$

$$\left. + \Pi + 2\left(-\frac{1}{2} \frac{2\omega_o+1}{\omega_o+2} + \Lambda \right) U \right] - 16\pi G\rho_o \frac{3+2\omega_o}{(2\omega_o+4)^2} 2\sigma G\omega_o \tag{111}$$

By comparison with Will's generalized PPN metric,[35] we find

$$\beta = 1 + \Lambda \qquad \beta_3 = 1$$

$$\beta_1 = \frac{3+2\omega_o}{2\omega_o+4} \qquad \beta_4 = \frac{\omega_o+1}{\omega_o+2}$$

$$\beta_2 = \frac{2\omega_o+1}{2\omega_o+4} - \Lambda \qquad \zeta = 0 \tag{112}$$

where

$$\Lambda \equiv \frac{\omega_o'}{G} \frac{1}{(2\omega_o+3)^2(2\omega_o+4)} \tag{113}$$

But the observable G now has a 2nd order correction

$$G^* = G\left(1+\sigma \frac{3+2\omega_o}{(\omega_o+2)^2} G\omega_o\right) \tag{114}$$

Finally

$$\nabla^2 g^{(3)}{}_{oi} = -\frac{1}{2}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial t \partial x^i} + 16\pi\rho_o v^i \tag{115}$$

In order to solve this equation, we use the so called superpotential[36]

$$g^{(2)}{}_{oo} = \nabla^2 \chi \tag{116}$$

where

$$\chi = G\int \rho_o(\bar{x}', t)|\bar{x}-\bar{x}'|\, d^3x' \tag{117}$$

and note that

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial t}\chi = -G\int \frac{\rho_o v'^i}{|\bar{x}-\bar{x}'|} d^3x' + G\int \frac{d^3x' \rho_o v'^j (x^j - x'^j)(x^i - x'^j)}{|\bar{x}-\bar{x}'|^3} \tag{118}$$

Then (115) becomes

$$\nabla^2 g^{(3)}{}_{oi} = \frac{1}{2}\nabla^2 \int \frac{G\rho_o v'^i}{|\bar{x}-\bar{x}'|} d^3x' + \frac{1}{2}\nabla^2 \int \frac{G\rho_o \bar{v}' \cdot |\bar{x}-\bar{x}'|(x^i - x'^i)}{|\bar{x}-\bar{x}'|^3}$$

$$+ 16\pi\left(\frac{3+2\omega_o}{2\omega_o+4}\right) G\rho_o v^i \tag{119}$$

with the solution

$$g^{(3)}{}_{oi} = -\frac{7}{2}\left(\frac{7\omega_o+10}{7\omega_o+14}\right)\int\frac{G\rho_o v'^i}{|\bar{x}+\bar{x}'|}\,d^3x'$$

$$-\frac{1}{2}\int\frac{G\rho_o v'^j(x^j-x'^j)(x^i-x'^i)}{|\bar{x}-\bar{x}'|}\,d^3x' \qquad (120)$$

Again by comparison with Will,[35] we find

$$\Delta_1 = \frac{7\omega_o+10}{7\omega_o+14}\ , \quad \Delta_2 = 1 \qquad (121)$$

By comparison with the Brans-Dicke scalar tensor theories, the only difference is in the parameters

$$\beta = \beta_{BD} + \Lambda$$

and

$$\beta_2 = \beta_{2BD} - \Lambda$$

Experimentally[32] we find that

$$-0.46 < \Lambda < 0.64$$

As an aside we note that the gauge equations are both satisfied providing σ is second order.

As a second example, we assume the same Bergmann-Wagoner field equation (73) and (74) but with second form of the divergence of the energy-momentum tensor given by Eq. (58).

This assumption leads to field equations

$$R_{\mu\nu} = -\frac{8\pi}{\phi}\left[T_{\mu\nu} + \frac{\omega(\phi)}{8\pi\phi}\,\phi^{;\rho}\phi_{;\mu} + \frac{\phi_{;\mu;\nu}}{8\pi}\right.$$

$$\left. - g_{\mu\nu}\frac{\omega+1}{3+2\omega}T_M - \frac{g_{\mu\nu}}{16\pi}\,\frac{\omega'+2\sigma\phi}{3+2\omega}\,\phi^{;\rho}\phi_{;\rho}\right] \tag{122}$$

and

$$\Box^2\phi = \frac{8\pi}{3+2\omega(\phi)}T_M - \frac{\omega'(\phi)+16\pi\sigma\phi}{3+2\omega(\phi)}\,\phi^{;\rho}\phi_{;\rho} + \frac{2\phi^2\lambda'(\phi)-2\phi\lambda(\phi)}{3+2\omega(\phi)} \tag{123}$$

where prime denotes differentiation with respect to the scalar field ϕ. We then expand each side of the equations as before

$$\nabla^2 g^{(2)}{}_{oo} = -16\pi\mathcal{G}\left(\frac{\omega_o+2}{3+2\omega_o}\right)T^{(o)oo} \tag{124}$$

$$\nabla^2 g^{(2)}{}_{ij} = -16\pi\mathcal{G}\left(\frac{\omega_o+1}{3+2\omega_o}\right)T^{(o)oo}\delta_{ij} \tag{125}$$

$$\nabla^2 g^{(3)}{}_{oi} = +16\pi\mathcal{G}T^{(1)oi} - \frac{1}{2}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial t\partial x^i} \tag{126}$$

$$\nabla^2 g^{(4)}{}_{oo} = g^{(2)}{}_{ij}\frac{\partial^2 g^{(2)}{}_{oo}}{\partial x^i\partial x^j} - (\overline{\nabla} g^{(2)}{}_{oo})^2$$

$$+ 16\pi\mathcal{G}T^{(o)oo}\varepsilon^{(2)}\left[\frac{\omega_o+2}{3+2\omega_o} - \frac{\omega_o'}{\mathcal{G}(3+2\omega_o)^2}\right]$$

$$- 16\pi\mathcal{G}\, T^{(2)oo}\left(\frac{\omega_o+2}{3+2\omega_o}\right) + 32\pi\mathcal{G}\, g^{(2)}{}_{oo}T^{(2)oo}\left(\frac{\omega_o+2}{3+2\omega_o}\right)$$

$$- \frac{\omega_o'+2\sigma\mathcal{G}^{-1}}{3+2\omega_o}\,(\nabla\varepsilon^{(2)})^2 - 16\pi\mathcal{G}\left(\frac{\omega_o+1}{3+2\omega_o}\right) T^{(2)kk} \tag{127}$$

From which we obtain

$$g^{(2)}{}_{oo} = 2\mathcal{G}\left(\frac{2\omega_o+4}{3+2\omega_o}\right)\int \frac{\rho_o(\bar{x}',t)}{|\bar{x}-\bar{x}'|}\, d^3x' \tag{128}$$

$$\equiv - 2U$$

which implies that

$$\mathcal{G} = G\left(\frac{3+2\omega_o}{2\omega_o+4}\right) \tag{129}$$

By (124) and (125)

$$\nabla^2 g^{(2)}{}_{ij} = \gamma\delta_{ij}\,\nabla^2 g^{(2)}{}_{oo} \tag{130}$$

which implies

$$\gamma = \frac{\omega_o+1}{\omega_o+2} \tag{131}$$

Eq. (127) then yields

$$\nabla^2 g^{(4)}{}_{oo} = - 2\nabla^2 U^2(1+\Lambda+\Omega) - 16\pi G\rho_o\left\{\frac{\Pi}{2} + \frac{v^2}{2}\left(\frac{2\omega_o+3}{\omega_o+2}\right)\right.$$

$$+ \frac{3}{2}\left(\frac{\omega_o+1}{\omega_o+2}\right)\frac{p}{\rho_o} - \left(\frac{2\omega_o+1}{2\omega_o+4} - \Lambda+\Omega\right)U\Bigg\} \tag{132}$$

where

$$\Lambda \equiv \frac{\omega_o'}{G}\,\frac{1}{(3+2\omega_o)^2(2\omega_o+4)} \tag{133}$$

$$\Omega \equiv \frac{2\sigma}{G(3+2\omega_o)^3} \tag{134}$$

By comparison with generalized PPN metric, we get

$$\left.\begin{aligned} \beta &= 1 + \Lambda + \Omega & \beta_1 &= \frac{2\omega_o+3}{2\omega_o+4} \\ \beta_2 &= \frac{2\omega_o+1}{2\omega_o+4} - \Lambda + \Omega & \beta_3 &= 1 \\ \beta_4 &= \frac{\omega_o+1}{\omega_o+2} & \zeta &= 0 \end{aligned}\right\} \tag{135}$$

Similarly from Eq. (126) we find

$$\Delta_1 = \frac{7\omega_o+10}{7\omega_o+14} \qquad \Delta_2 = 1 \tag{136}$$

We note that Ω allows an added degree of freedom in comparison with experiment. At present there is no experiment which can detect its value. The most probable test will come from lunar laser ranging which depends on the combination of parameters $\beta + \beta_2$. In this case Λ drops out leaving only

an Ω dependent term. At present the experiment is still consistent with zero oscillation in the radial variable and therefore no strict limitation on Ω is possible.[37]

IV. Conclusion

Obviously these types of modifications can be extended to more divergences of $T^{\mu\nu}$ then those described by Eqs. (54) and (58). Such an analysis is the subject of another paper which is forthcoming. In this paper, we consider the coupling $\sigma = \sigma(\phi)$ and consider the general consistency of the field equations with the Bianchi identities (which were used to obtain Eq. (81)), gauge conditions and the Newtonian limit of the perfect fluid equations of hydrodynamics. We note that the examples described above represent viable gravitational theories since their PPN parameters fall with experimental limits of those parameters as listed by Ni.[32,37] We should also note that the technique of modification indicated here and applied to scalar-tensor theories is in general applicable to other theories such as vector-metric theories[38] or double metric theories.[39] The degree of freedom from a second field allows unique field equations.

A more interesting problem is presented by the "new" PPN parameters of Will and Nordtvedt[40] in which for the usual Bergman-Wagoner theory

$$\gamma = 1 \quad , \beta = 1 + \Lambda$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0.$$

The parameters α_i now represent preferred frame effects whereas non-zero

ζ_i indicate non-conservative theories. However, in the modification given by Eq. (58),

$$\left.\begin{aligned} \gamma &= 1 \quad , \beta = 1 + \Lambda + \Omega \\ \alpha_i &= 0 \quad , i = 1, 2, 3 \\ \zeta_j &= 0 \quad , j = 1, 3, 4 \\ \zeta_2 &= 2\beta + 2\beta_2 - 3\gamma - 1 \\ &= 4\,\Omega \end{aligned}\right\}$$

But this says that these types of theories are non-conservative gravitational theories. As far as we know, these theories which <u>appear</u> to globally violate conservation of total momentum are the first such theories that are also viable alternatives to general relativity. The exact nature of this result is the subject of a forthcoming investigation.

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12. The following misprints or omissions occur in Ref. 9: In the list of three index symbols: $\Gamma^0{}_{00}$ let $e^{2(\beta-\gamma)} \to e^{2(\gamma-\beta)}$, $\Gamma^2{}_{33}$ let $e^{-2(\gamma-\beta)} \to e^{-2(\gamma+\beta)}$, and the correct order of expansions for $\Gamma^3{}_{03} = -\frac{c_0}{r} - \frac{C_0 - \frac{1}{2}c^2 c_0}{r^3}$, $\Gamma^1{}_{22} = r(c_0-1)+(2M-c_{22}-2c_2 \cot\theta + 2c\cot^2\theta + 2cc_0)$.

In the supplementary condition for R_{00} replace the term $\frac{\beta_2 U V_1}{r} \to \frac{\beta_2 U_1 V}{r}$. The constant of integration in Eq. (32) is $N + \frac{5}{6}cc_2 + \frac{2}{3}c^2 \cot\theta$ instead of the N discussed in the text. The term $8c(3cc_2)$ should be replaced by $8c(3cc_2\cot\theta)$ in Eq. (33).

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