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~~A~~ GENERALIZED TENSOR FORMULATION OF ATMOSPHERE AND SEAS DYNAMICS

Lee M. Avis, Richard E. Turner, and Thomas H. Rees

Langley Research Center
Hampton, Va. 23665

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16. Abstract

A new generalized mathematical model has been developed for simulation of the dynamics and transport of both the atmosphere and seas. A nearly horizontal bottom coordinate surface conforms to the land-to-air interface and the sea-floor-to-water interface, which simplifies computations. General vertical motion of the other quasi-horizontal coordinate surfaces is allowed; thus, external gravity waves can be represented by the top coordinate surface, and meteorological fronts and inversion layers in the atmosphere and refractive layers in the seas can be represented with enhanced resolution by the internal quasi-horizontal coordinate surfaces. A tensor reformulation of the standard subgrid mixing theory departs significantly from the standard theory in allowing, as a solution under adiabatic conditions, rigid-body rotation of the atmosphere.

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A GENERALIZED TENSOR FORMULATION OF ATMOSPHERE AND SEAS DYNAMICS

Lee M. Avis, Richard E. Turner,
and Thomas H. Rees
Langley Research Center

SUMMARY

A new generalized mathematical model has been developed which is applicable to simulation of the dynamics and transport of both the atmosphere and seas. Considerable conceptual and computational simplification is achieved by requiring the nearly horizontal bottom coordinate surface to follow the land-to-air interface and the sea-floor-to-water interface. The model allows general vertical motion of the other quasi-horizontal coordinate surfaces; thus, external gravity waves can be simulated by letting the top coordinate surface ride with a top free surface. Also, the freedom of motion of the internal quasi-horizontal surfaces provides a potential for tracking meteorological fronts and inversion layers in the atmosphere and refractive layers in the seas with resolutions superior to those realizable by simulation models having fixed coordinate lines, for equal "horizontal" grid spacing and an equal number of "horizontal" surfaces.

The standard subgrid mixing theory used by meteorologists is reformulated in terms of a strain-rate tensor for both three-dimensional and hydrostatic quasi-two-dimensional dynamics. The tensor reformulation departs significantly from the standard theory in allowing, as a solution under adiabatic conditions, rigid-body rotation of the atmosphere.

INTRODUCTION

There are many advantages in developing computer models of the Earth's atmosphere or seas in coordinate systems allowing generally specified vertical motion of the coordinate gridpoints. In the atmospheric troposphere, coordinate gridpoints can be moved vertically to follow the motions of sharp temperature gradients occurring in frontal systems to obtain improved resolution of the frontal system. In regions where low-pressure cells penetrate into the stratosphere, the coordinate gridpoints can be moved vertically to follow the zone separating the turbulent troposphere from the thermally inverted stratosphere and thereby reduce false numerical diffusion between the two regions. The seas also have a time-dependent vertical structure that can be well represented by coordinate systems allowing vertical motion of the coordinate gridpoints.

In coastal regions where fresh-water rivers empty into the seas, strong vertical density stratifications occur and cause time-varying changes in the flow fields. Spurious numerical diffusion through such strong vertical density gradients can be reduced by moving gridpoints vertically to follow the density gradients. In the case of coastal-zone tidal waves, tidal currents can be simulated in conjunction with moving the upper level gridpoints vertically to follow the water's free surface. Vertical density stratifications occur along the common boundaries of adjacent oceans (the Arctic and Atlantic Oceans, for example), and the open seas are generally stratified because they are top heated.

The National Aeronautics and Space Administration is presently involved in developing computer models of the Earth's atmosphere and seas to aid in determining satellite sensor requirements and to aid in the interpretation of remotely sensed data in connection with environmental pollution problems. Since pollution occurs over a wide range of distance scales in the atmosphere and seas, a mathematical model of geophysical circulation allowing vertical motion of coordinate gridpoints and having validity over a wide range of scales is needed for the development of circulation models.

The literature shows that a number of successful attempts have been made to model the atmosphere and seas. References 1 to 4 are descriptions of global atmospheric models; reference 2 also describes a global seas model. A survey of small-scale models is given in reference 5.

Generally, atmospheric-circulation models have been in existence longer than ocean-circulation models and, consequently, have reached a higher level of sophistication. The atmospheric models given in references 2 and 4 use the "sigma coordinate" method where vertical motion of the gridpoints is used to keep constant mass ratios in the model vertical layers. Such a model cannot be converted easily to track surfaces of constant temperature, for example, and even though the "sigma coordinate" method allows vertical motion of the gridpoints, it cannot be considered a flexible technique.

The coordinate system and equations of motion of seas models (e.g., see ref. 2) generally differ considerably from the coordinate system and equations of motion of atmospheric-circulation models (e.g., see refs. 2 and 4). This fact probably increases the difficulty of communication between atmospheric and ocean researchers.

In this report an attempt is made to develop the equations of motion in a coordinate system that allows generally specified vertical motion of the gridpoints. The resulting equations of motion are cast in terms of dependent variables which represent either atmospheric or ocean circulation. Computer models developed from such a formulation need differ only in respect to the physics to be included in each model. Consequently, the terminology gap between atmospheric and oceanic researchers is bridged.

Present-day understanding of mixing processes is poor, and it is anticipated that turbulent mixing theories will become considerably more sophisticated, especially for pollution transport. Tensor calculus (having been developed from principles of invariance) is a good candidate for the description of better mixing theories and is employed in this report.

SYMBOLS

A	determinant of the quasi-horizontal metric tensor
a	determinant of the metric tensor
$\left(a_c\right)_i$	coriolis and centripetal acceleration along dx^i
C_V	heat capacity per mole at constant volume
D_T, D_S	elements in the subgrid diffusion model, defined by equations (D9)
E	internal energy per unit mass
g	acceleration due to gravity
g_i	gravity component along dx^i
g_{ij}, g^{ij}	covariant and contravariant elements of the metric tensor, respectively
h	physical height above reference level of parametric (grid) lines
K_H	horizontal kinematic viscosity
K_o	Von Karman constant
K_V	vertical kinematic eddy viscosity for momentum
$\hat{L}_{ij}$	physical components of the deviatoric strain-rate tensor

$\hat{\ell} = \sqrt{\hat{\ell}^p_n \hat{\ell}^n_p}$

$\hat{\ell}^i_j$	mixed tensor components of the deviatoric strain-rate tensor
ℓ^i_j	mixed tensor components of the strain-rate tensor
n_j	unit vector normal to $d\bar{S}$, positive outward
P	hydrostatic pressure
Q	general quantity which assumes the role of the dependent variables
$\bar{Q}$	heating rate in the thermodynamic equation (E5)
q	water-vapor mass fraction (specific humidity)
R	universal gas constant
r	Earth's radius
S	area
$\hat{S}$	quasi-source term for transport equation $\left(\text{for a scalar } Q, \hat{S}\left(\sqrt{a}\rho Q\right) = S'\left(\sqrt{a}\rho Q\right)\right)$
S'	source term for transport equation
S_j	physical vertical velocity of coordinate gridpoints relative to rotating Earth
s^j	contravariant components of gridpoint velocities relative to rotating Earth
T	temperature, $^{\circ}$K
T^i, T_i	general contravariant and covariant vectors, respectively
t	time, sec
U_i	physical components of velocity vector relative to rotating Earth
U_i'	physical components of turbulent velocity vector relative to rotating Earth

u_i	covariant tensor component of velocity vector relative to rotating Earth
V_i	physical components of fluid velocity relative to coordinate frame of reference
v^i	contravariant form of V_i
X^3	depth of atmosphere or seas in transformed coordinates
x^i	coordinate variables for reference coordinate system, where x^1 and x^2 lie along the real Earth surface and x^3 is perpendicular to the real Earth surface
$\tilde{z}_i$	physical coordinate in direction of x^i
α,β	indices restricted to the horizontal surface
γ	ratio of heat capacity at constant pressure to heat capacity at constant volume
Δ	coordinate increment
δ	physical height of Earth's terrain from a given reference level
$\tilde{\delta}$	characteristic δ on a grid interval
δ^i_j	mixed tensor components of Kronecker delta
ϵ_ϕ	error fraction in the square of a line element
ζ	generalized density defined by $\rho\sqrt{g_{33}}$
$d\nu$	coordinate volume element, $dx^1\,dx^2\,dx^3$
ξ	variable of integration
ρ	density of air or water
ω	Earth's spin rate
$\overline{\omega}$	mean molecular weight

Operators:

$\langle \ \rangle$ space-time average over a coordinate volume ν and time increment $\Delta\tau$,

$$\frac{1}{\nu\,\Delta\tau}\int_{\Delta\tau}\int_{\Delta\nu}(\ )\,dx^1\,dx^2\,dx^3\,dt$$

$(\ )'$ denotes turbulent part of $(\ )$

$(\ )^2$ square of $(\ )$

$(\ )_{,3}$ partial derivative of $(\ )$ with respect to x^3

$(\ )_{|j}$ covariant derivative of $(\ )$ with respect to x^j

$\frac{d}{dt}(\ )$ co-moving time derivative of $(\ )$

$\frac{\partial}{\partial t}(\ )$ partial derivative of $(\ )$ with respect to t

$\left\{ {i \atop j\ k} \right\}$ Christoffel symbols of the second kind

ANALYSIS

Introductory Remarks

A simple, generally valid mathematical model for simulating atmospheric and oceanic dynamics and pollution transport was developed with a general, time-varying vertical grid structure. A review of the literature (e.g., refs. 1 to 4, 6, and 7) indicated that no known existing published model with generally prescribed time-varying vertical grid structure had been applied to both atmospheric and oceanic circulation. Reference 1 describes an atmospheric-circulation model wherein the vertical coordinate is the fixed physical distance from sea level. References 2, 3, and 4 describe atmospheric-circulation models in the "sigma coordinate" wherein the ratio of pressure to surface pressure is used as the vertical coordinate. The "sigma coordinate" technique allows vertical motion of the gridpoints; unfortunately, the vertical motion cannot be prescribed generally. Reference 6 describes a coordinate system wherein general vertical motion of the gridpoints is allowed; however, the resulting transport equations contain elements from two coordinate systems and may present difficulties when modeling subgrid transport. The strongpoint of the approach given in reference 6 is that it is not limited by the small slope approximation for the horizontal coordinate surfaces. Reference 7 describes the "sigma coordinate" approach in detail. None of the references gave formulations intended to cover both atmospheric and oceanic circulation.

Some notable features of the new mathematical model are as follows:

(1) Combining density and the vertical scale factor to form a generalized density so that the atmosphere and seas have the same motion equations

(2) Distortion of the quasi-horizontal bottom coordinate surface to follow the real Earth terrain and provision for general vertical motion of the other quasi-horizontal coordinate surfaces

(3) Projection of gravity onto the quasi-horizontal coordinate surfaces (The gravity force is derived from the gravity potential specified along the quasi-horizontal parametric lines.)

(4) The representation of the intuitive subgrid diffusion (see ref. 3) as a true tensor, which corrects a deficiency in the subgrid diffusion formulation in allowing rigid-body rotation of the atmosphere under adiabatic conditions

(5) The extension of the atmospheric top to zero pressure as in the "sigma coordinate" approach (ref. 7), which has the ratio of pressure to surface pressure as the vertical coordinate (see also ref. 6)

(6) The use of generalized density rather than pressure as a dependent variable

The development of the mathematical model in tensor notation is accomplished in appendixes A to F. A brief outline of the development sequence in the appendixes which will serve to give them continuity is given as follows:

In appendix A, the transport equations for momentum and a general motion integral are derived from conservation integrals in which the volume elements are allowed to vary with time: that is, the coordinate points are allowed to move in some arbitrarily specified manner. As a direct result, the motion equations can simulate any conceivable mixture of Eulerian and Lagrangian viewpoints.

In appendix B, differential distances for real Earth topography are shown to be quadratically dependent on the slope of the real Earth surface. Thus, for small surface slopes the geometry is, to second order, unchanged by mountains and sea bottoms, and, to the order of the square of the slope of the real Earth geometry, the equations of motion for geophysical circulation depend on the real Earth topography only through the external forces, such as gravity.

In appendix C, one transport equation, valid for all motion integrals, including subgrid transport terms and appropriate source terms, is given.

In appendix D, the subgrid transport treatment of Washington and Kasahara (ref. 1) is converted into tensor notation for full three-dimensional turbulence transport and is then simplified to the usual two-dimensional stratified turbulence transport. The tensor elements of the strain-rate tensor differ from the strain-rate matrix of reference 1 with the result that the tensor treatment of subgrid momentum transport allows rigid-body rotation of the atmosphere under adiabatic conditions. An alternate tensor treatment of subgrid momentum transport, also allowing adiabatic rigid-body rotation, is reported in reference 8.

In appendix E, a tensor mathematical model for full three-dimensional dynamics with full three-dimensional Reynolds transport in physical components, as opposed to tensor components, is presented.

Finally, in appendix F, the usual hydrostatic assumption is applied to the momentum equations. (See ref. 1, for example.) As a direct result of the hydrostatic assumption, one can calculate pressure from the known density field. The hydrostatic assumption causes the atmosphere and oceans to be similar except in respect to computation of the vertical velocities.

In the nearly incompressible oceans, the vertical velocity v^3 may be computed from the mass-conservation equation, and because (see ref. 9) work of compression is safely ignored in the internal-energy balance, one need not compute the divergence of the velocity u^j.

Dynamic simulations of the atmosphere generally include work of compression in the internal-energy balance. If one chooses pressure rather than density as the dependent variable to be simulated, then the internal-energy constraint is automatically satisfied; however, then one must be careful to conserve mass. In the present report, density is chosen over pressure as the dependent variable and, consequently, mass is automatically conserved, whereas the internal-energy constraint is satisfied only if one computes the vertical velocity from a conservative form of the internal-energy equation.

Conservation Equations for the Atmosphere and Seas

Over a wide range of scales, the atmosphere and seas approximately satisfy the hydrostatic assumption, for which condition the atmosphere and seas have a common set of conservation equations for mass, momentum, and energy. The atmosphere has water vapor as an additional variable, whereas the seas have salt.

The coordinate system chosen to present the conservation equations is an approximately orthogonal coordinate set wherein the coordinate surfaces defined by

x^3 = Constant are nearly spherical, with the $x^3 = 0$ coordinate surface following the real Earth terrain. All other "horizontal" parametric surfaces are allowed to be positioned quite generally, or even to be in motion, but with the restriction that the slopes of all "horizontal" coordinate surfaces are not very much greater than the maximum slope of the real Earth terrain. The x^1 coordinate is the co-latitude angle, the x^2 coordinate is the longitude angle, and the x^3 coordinate direction is locally perpendicular to the quasi-horizontal coordinate surfaces.

In appendix B, it is shown that the differential geometry of the chosen coordinate system is sufficiently represented by the approximate metric tensor

$$g_{11} = (r)^2$$

$$g_{22} = \left(r \sin x^1\right)^2$$

$$g_{33} = g_{33}\left(x^1, x^2, x^3, t\right)$$

where r is the Earth's radius; and, thus, the determinate of the quasi-horizontal metric tensor becomes

$$A = g_{11}g_{22}$$

From appendix E, one has a common set of conservation equations for the atmosphere and seas wherein the real density ρ is combined with the vertical scale factor $\sqrt{g_{33}}$ to form a generalized density $\zeta = \rho\sqrt{g_{33}}$.

The resultant mass-conservation equation in terms of physical velocities V_i is, from equation (E2),

$$\frac{\partial\zeta}{\partial t} + \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\,\zeta\,\frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) + \frac{\partial}{\partial x^3}\left(\zeta\,\frac{V_3}{\sqrt{g_{33}}}\right) = 0 \tag{1}$$

The momentum equation along the x^1 parametric line is, from equations (E19) and (E21),

$$\frac{\partial}{\partial t}\left(\zeta U_1\right) + \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\, \zeta U_1 \frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) + \frac{\partial}{\partial x^3}\left(\zeta U_1 \frac{V_3}{\sqrt{g_{33}}}\right)$$

$$= -\frac{\sqrt{g_{33}}}{\sqrt{g_{11}}} \frac{\partial P}{\partial x^1} + \frac{\zeta}{\sqrt{g_{11}}} U_3 \frac{\partial S_3}{\partial x^1} - \zeta \frac{\left(a_c\right)_1}{\sqrt{g_{11}}} - \zeta \frac{g}{\sqrt{g_{11}}} \frac{\partial h}{\partial x^1}$$

$$- \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\, \zeta \frac{\left\langle U_1{}' U_\alpha{}' \right\rangle}{\sqrt{g_{\alpha\alpha}}}\right) - \frac{\partial}{\partial x^3}\left(\zeta \frac{\left\langle U_1{}' U_3{}' \right\rangle}{\sqrt{g_{33}}}\right)$$

$$- \zeta\left(\frac{\left\langle U_2{}' U_2{}' \right\rangle}{\sqrt{g_{11}}} + V_2 \frac{U_2}{\sqrt{g_{11}}}\right) \cot x^1 \tag{2}$$

The momentum equation along the x^2 parametric line is, from equations (E22) and (E24),

$$\frac{\partial}{\partial t}\left(\zeta U_2\right) + \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\, \zeta U_2 \frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) + \frac{\partial}{\partial x^3}\left(\zeta U_2 \frac{V_3}{\sqrt{g_{33}}}\right)$$

$$= -\frac{\sqrt{g_{33}}}{\sqrt{g_{11}}} \frac{\partial P}{\partial x^2} + \frac{\zeta}{\sqrt{g_{22}}} U_3 \frac{\partial S_3}{\partial x^2} - \zeta \frac{\left(a_c\right)_2}{\sqrt{g_{22}}} - \zeta \frac{g}{\sqrt{g_{22}}} \frac{\partial h}{\partial x^2}$$

$$- \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\, \zeta \frac{\left\langle U_\alpha{}' U_2{}' \right\rangle}{\sqrt{g_{\alpha\alpha}}}\right) - \frac{\partial}{\partial x^3}\left(\zeta \frac{\left\langle U_2{}' U_3{}' \right\rangle}{\sqrt{g_{33}}}\right)$$

$$- \zeta\left(\frac{\left\langle U_1{}' U_2{}' \right\rangle}{\sqrt{g_{11}}} + V_1 \frac{U_2}{\sqrt{g_{11}}}\right) \cot x^1 \tag{3}$$

where the Greek indices take on the values 1 and 2 and, from reference 1,

$$\frac{\left(a_c\right)_1}{\sqrt{g_{11}}} = -2\omega U_2 \cos x^1$$

$$\frac{\left(a_c\right)_2}{\sqrt{g_{22}}} = 2\omega U_1 \cos x^1 \tag{4}$$

and, it should be noted, where the centrifugal term proportional to $(\omega)^2$ has been considered to be a part of the gravity term, which is the usual practice.

The internal-energy conservation equation is, from equation (E5),

$$\frac{\partial}{\partial t}(\zeta E) + \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^{\alpha}}\left(\sqrt{A}\, \zeta E \frac{V_{\alpha}}{\sqrt{g_{\alpha\alpha}}}\right) + \frac{\partial}{\partial x^3}\left(\zeta E \frac{V_3}{\sqrt{g_{33}}}\right)$$

$$= \zeta\bar{Q} - \frac{P}{\sqrt{A}} \frac{\partial}{\partial x^{\alpha}}\left(\sqrt{g_{33}}\sqrt{A}\, \frac{U_{\alpha}}{\sqrt{g_{\alpha\alpha}}}\right) - P \frac{\partial}{\partial x^3}\left(U_3\right)$$

$$- \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^{\alpha}}\left(\sqrt{A}\, \zeta \frac{\langle E' U_{\alpha}' \rangle}{\sqrt{g_{\alpha\alpha}}}\right) - \frac{\partial}{\partial x^3}\left(\zeta \frac{\langle E' U_3' \rangle}{\sqrt{g_{33}}}\right) \tag{5}$$

The final conservation equation for atmospheric water or ocean salt, or for an arbitrary pollutant, is, from equation (E6),

$$\frac{\partial}{\partial t}(\zeta q) + \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^{\alpha}}\left(\sqrt{A}\, \zeta q \frac{V_{\alpha}}{\sqrt{g_{\alpha\alpha}}}\right) + \frac{\partial}{\partial x^3}\left(\zeta q \frac{V_3}{\sqrt{g_{33}}}\right)$$

$$= \frac{1}{\sqrt{A}} \hat{S}\left(\sqrt{A}\, \zeta q\right) - \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^{\alpha}}\left(\sqrt{A}\, \zeta \frac{\langle q' U_{\alpha}' \rangle}{\sqrt{g_{\alpha\alpha}}}\right) - \frac{\partial}{\partial x^3}\left(\zeta \frac{\langle q' U_3' \rangle}{\sqrt{g_{33}}}\right) \tag{6}$$

where $\frac{1}{\sqrt{A}} \hat{S}\left(\sqrt{A}\, \zeta q\right)$ is the source term for water vapor resulting from evaporation condensation. For the seas, the source term would reflect salt dissociation precipitation; for pollution, the source term would represent scavenging and chemical reactions.

Subgrid Correlations

The subgrid mixing model is the tensor representation of an intuitive subgrid model developed by Smagorinsky and described in reference 3. Simply put, subgrid waves, or waves unresolvable on the grid spacing, have a sizable effect on the development of circulation patterns. It is implicit in Smagorinsky's model that these subgrid waves act like diffusion and tend to drive property gradients to zero and, therefore, represent the irreversible nature of circulating fluids. The horizontal subgrid action is similar to wind- or current-driven turbulence, whereas the vertical subgrid action is similar to thermally driven turbulence.

The horizontal subgrid mixing, then, is taken to be a function of the strain-rate tensor, which is a measure of the mechanical stirring of the fluid by the resolvable waves.

The vertical mixing in the atmosphere (being driven by small-scale convective overturning, as, for example, by cumulus convection) is taken to be a function of the thermal lapse rate. (See ref. 1.) The vertical mixing the oceans (which is top heated and, therefore, generally stable thermally) is modeled more like mechanical turbulence in reference 9 by specifying a very low eddy viscosity in the deep ocean and specifying relatively high eddy viscosity in the surface wind-driven currents where mechanical stirring is strong.

The subgrid mixing model begins with the deviatoric strain-rate tensor which is a measure of the stirring of the fluid. The deviatoric strain-rate tensor $\hat{\ell}^{i}_{.j}$ from appendix D is, for the quasi-two-dimensional case,

$$\left.\begin{aligned}
\hat{\ell}^{1}_{.1} &= \left[\left(U_1 \sin x^1\right)_{,1} - U_{2,2} - 2\cos x^1\, U_1\right]\Big/\sqrt{g_{22}} \\
\hat{\ell}^{1}_{.2} &= \sin x^1\left[U_{1,2} + \left(U_2 \sin x^1\right)_{,1} - 2\cos x^1\, U_2\right]\Big/\sqrt{g_{22}} \\
\hat{\ell}^{2}_{.1} &= \hat{\ell}^{1}_{.2}\Big/\left(\sin x^1\right)^2 \\
\hat{\ell}^{2}_{.2} &= -\hat{\ell}^{1}_{.1} \\
\hat{\ell}^{1}_{.3} &= U_{1,3}\Big/\sqrt{g_{11}} \\
\hat{\ell}^{3}_{.1} &= \hat{\ell}^{1}_{.3}\left(g_{11}/g_{33}\right) \\
\hat{\ell}^{2}_{.3} &= U_{2,3}\Big/\sqrt{g_{22}} \\
\hat{\ell}^{3}_{.2} &= \hat{\ell}^{2}_{.3}\left(g_{22}/g_{33}\right) \\
\hat{\ell}^{3}_{.3} &= 0
\end{aligned}\right\} \qquad (7)$$

Horizontal Subgrid Mixing

The strain-rate tensor is related to the subgrid horizontal-velocity correlation tensor as in reference 3 by

$$\left\langle U_\alpha ' U_\beta ' \right\rangle = -K_H \hat{L}_{\alpha\beta} \tag{8}$$

where $\hat{L}_{\alpha\beta}$ represents the physical components of the strain-rate tensor $\hat{\ell}^\alpha_\beta$ and is computed by

$$\hat{L}_{\alpha\beta} = \hat{\ell}^\alpha_{.\beta} \sqrt{g_{\alpha\alpha}/g_{\beta\beta}} \tag{9}$$

One simple interpretation of equation (8) is that velocity gradients in the fluid $\left(\text{measured by } \hat{L}_{\alpha\beta}\right)$ cause a potential for momentum to be transferred by means of the velocity correlation $\left\langle U_\alpha ' U_3 ' \right\rangle$, and K_H is simply a constant of proportionality. The horizontal subgrid mixing for a scalar Q, as in reference 1, is

$$\left\langle U_\alpha ' Q' \right\rangle = -\frac{K_H}{\sqrt{g_{\alpha\alpha}}} \frac{\partial Q}{\partial x^\alpha} \tag{10}$$

where gradients in Q cause a potential for Q to be transferred and, again, K_H is the constant of proportionality.

The kinematic eddy viscosity is related to the strain-rate tensor for mechanical stirring by

$$K_H = \sqrt{2}\left(K_o\ \Delta\right)^2 \left(\hat{\ell}^\alpha_{.\beta}\ \hat{\ell}^\beta_{.\alpha}\right)^{1/2} \tag{11}$$

where K_o is the Von Karman constant and Δ is the grid spacing. Equation (11) also has a simple interpretation. Mechanical stirring $\left(\text{measured by } \sqrt{\hat{\ell}^\alpha_{.\beta}\ \hat{\ell}^\beta_{.\alpha}}\right)$ causes the eddy viscosity, whereas $\sqrt{2}\left(K_o\ \Delta\right)^2$ is a constant of proportionality which allows one to tune his subgrid mixing model to best simulate the real world. Equation (11) implies that, for simulation cases where Δ approaches zero, all important mixing processes can be resolved on the grid.

Vertical Subgrid Mixing

In the case of vertical transport, the actual mixing process is generally thermally driven in the atmosphere. Reference 1 gives a parameterized description of the vertical kinematic eddy viscosity K_V, which is dependent on the temperature lapse rate. In the oceans, Cox (ref. 9) has two values of K_V. In the deep ocean where mechanical stirring is low, a small value of K_V is used; whereas for the upper ocean in the wind-driven currents where mechanical stirring is strong, a large value of K_V is used.

The deviatoric strain-rate tensor is related to the subgrid velocity correlations corresponding to vertical mixing of the horizontal momentum, as in reference 1, by

$$\left\langle U_\alpha' U_3' \right\rangle = -K_V \hat{L}_{\alpha 3} \tag{12}$$

where, again,

$$\hat{L}_{\alpha 3} = \hat{\ell}^{\alpha}_{.3} \sqrt{g_{\alpha\alpha}/g_{33}} \tag{13}$$

and equation (12) has the same interpretation as equation (10). The vertical subgrid mixing for a scalar Q, as in reference 1, is

$$\left\langle U_3' Q' \right\rangle = -\frac{K_V}{\sqrt{g_{33}}} \frac{\partial Q}{\partial x^3} \tag{14}$$

Vertical-Velocity Computations

The prime unknown remaining to be specified is the vertical velocity. In hydrostatic models such as those given herein, the vertical velocity cannot be computed from the vertical-momentum equation.

<u>Seas.</u>- In ocean-circulation models the work of compression $\frac{P}{\rho} u^j_{|j}$ is customarily neglected in the internal-energy conservation statement given in equation (5) which implies that

$$\frac{\partial U_3}{\partial x^3} = -\frac{1}{\sqrt{A}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\, \sqrt{g_{33}}\, \frac{U_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) \tag{15}$$

Clearly, at the bottom $\left(x^3 = 0\right)$, the vertical velocity U_3 vanishes, and so

$$U_3 = -\frac{1}{\sqrt{A}} \int_0^{x^3} \left[\sqrt{g_{33}} \frac{\partial}{\partial x^{\alpha}} \left(\sqrt{A} \frac{U_{\alpha}}{\sqrt{g_{\alpha\alpha}}}\right)\right] d\xi \tag{16}$$

where ξ is a variable of integration. The vertical velocity U_3 is not required to solve the ocean-circulation equations (as approximated herein); however, values of U_3 might be a desired output of the simulation. One has the freedom to assign V_3 or S_3 for input to the transport equations, and for computing ζ.

An additional approximation is employed, the Boussinesq approximation, which states that density variations are negligible except when multiplied by the acceleration of gravity.

Atmosphere.- In atmospheric simulations, the work of compression is usually considered in the internal-energy equation, and so computation of U_3 is required for solution of the equation for atmospheric models; however, when U_3 has been computed, one still has the freedom to assign values of either V_3 or S_3 and, consequently, move the vertical position of the horizontal coordinate surfaces to any desired position.

Several thoughts which should be considered before the vertical-velocity computational sequence is developed are given as follows:

(1) The pressure can be computed from the density profile by utilizing the hydrostatic assumption.

(2) The internal energy can be computed from pressure and density from the equation of state.

(3) Since internal energy can be computed independent of equation (5), and since equation (5) must be satisfied for consistency of internal energy, then equation (5) should be used to compute U_3.

(4) The physical velocity component V_3 can then be chosen arbitrarily to give any desired vertical coordinate spacing; or any desired vertical distribution of ζ can be chosen equally well provided $\int_0^{X^3} \zeta \, dx^3$ is properly conserved, because this integral is the total mass per unit cross section of area along the nearly vertical x^3 coordinate line. With these thoughts in mind, one can proceed generally to specify V_3 and then compute $\partial\zeta/\partial t$ and U_3; or one can specify $\partial\zeta/\partial t$ and then compute V_3 and U_3 equally well. In either case, the boundary conditions on V_3 at the bottom $(x^3 = 0)$ and the top $(x^3 = X^3)$ for a conservative-mass system are

$$V_3(0) = V_3\left(X^3\right) = 0 \tag{17}$$

In illustration of these considerations, V_3 is chosen, subject to the boundary conditions given in equation (17). Next, $\partial\zeta/\partial t$ is computed from the mass-conservation equation (1). The final task is to compute U_3 from equation (5). In order to solve equation (5) for U_3, one must know (ζE) and $\frac{\partial}{\partial t}(\zeta E)$.

The computation of (ζE) begins with the hydrostatic assumption where

$$P = \int_{x^3}^{X^3} \frac{g_3}{\sqrt{g_{33}}} \zeta \, d\xi \cong g \int_{x^3}^{X^3} \zeta \, d\xi \tag{18}$$

Then, from the equation of state,

$$P = \frac{\rho RT}{\overline{\omega}} = \frac{\zeta RT}{\sqrt{g_{33}}\,\overline{\omega}} \tag{19}$$

and since

$$E = \frac{C_v T}{\overline{\omega}} \tag{20}$$

equations (19) and (20) combine to give

$$\zeta E = \frac{C_v}{R} \sqrt{g_{33}}\, P \tag{21}$$

Consequently, the hydrostatic approximation allows one to convert equation (5) from (ζE) description to P dependence, and thereby simplify the solution, because P and $\partial P/\partial t$ can be computed from simple density integrals. The right-hand side of equation (5), being already in its proper form, can be denoted by $\frac{1}{\sqrt{A}} \hat{S}\left(\sqrt{A}\,\zeta E\right)$; then, substitute equation (21) into the left-hand side of equation (5) and get

$$\frac{C_v}{R}\left[\frac{\partial}{\partial t}\left(\sqrt{g_{33}}\,P\right) + \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\,\sqrt{g_{33}}\,P\,\frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) + \frac{\partial}{\partial x^3}\left(PV_3\right)\right] = \frac{1}{\sqrt{A}}\hat{S}\left(\sqrt{A}\,\zeta E\right) \tag{22}$$

The next step is to get an expression for $\partial P/\partial t$ from equations (1) and (18) which results in

$$\frac{\partial P}{\partial t} = -\frac{g}{\sqrt{A}} \int_{x^3}^{X^3} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\,\zeta\,\frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) d\xi + g\zeta \frac{V_3}{\sqrt{g_{33}}} \tag{23}$$

The first term in equation (22) expands to

$$\frac{\partial}{\partial t}\left(\sqrt{g_{33}}\,P\right) = \frac{\partial}{\partial t}\left(\sqrt{g_{33}}\right) P + \sqrt{g_{33}}\,\frac{\partial P}{\partial t} \tag{24}$$

Then, from equation (E11),

$$\frac{\partial}{\partial t}\left(\sqrt{g_{33}}\right) = \frac{\partial S_3}{\partial x^3} \tag{25}$$

Next, substitute equations (23) and (25) into equation (24) and get

$$\frac{\partial}{\partial t}\left(\sqrt{g_{33}}\,P\right) = -\frac{\sqrt{g_{33}}\,g}{\sqrt{A}} \int_{x^3}^{X^3} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\,\zeta\,\frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) d\xi + P\,\frac{\partial S_3}{\partial x^3} + g\zeta V_3 \tag{26}$$

Finally, substitute equation (26) into equation (22), expand $\frac{\partial}{\partial x^3}\left(PV_3\right)$ to

$$\frac{\partial P}{\partial x^3} V_3 + P\,\frac{\partial V_3}{\partial x^3}$$

set

$$\frac{\partial P}{\partial x^3} = -g\zeta$$

and

$$V_3 + S_3 = U_3$$

and then get

$$\frac{C_v}{R}\left[-\frac{\sqrt{g_{33}}\,g}{\sqrt{A}} \int_{x^3}^{X^3} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\,\zeta\,\frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) d\xi + P\,\frac{\partial U_3}{\partial x^3} + \frac{1}{\sqrt{A}}\,\frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\,\sqrt{g_{33}}\,P\,\frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right)\right]$$
$$= \frac{1}{\sqrt{A}}\,\hat{S}\left(\sqrt{A}\,\zeta E\right) \tag{27}$$

Equation (27) now involves U_3 as the only unknown; thus, replace the right-hand side of equation (5) for $\frac{1}{\sqrt{A}} \hat{S}\left(\sqrt{A}\,\zeta E\right)$, solve for $\partial U_3/\partial x^3$, and get

$$\left(\frac{C_v}{R} + 1\right) P \frac{\partial U_3}{\partial x^3} = \frac{C_v}{R} \frac{\sqrt{g_{33}}\, g}{\sqrt{A}} \int_{x^3}^{x^3} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\, \zeta \frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) d\xi$$

$$- \frac{C_v}{R} \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\, \sqrt{g_{33}}\, P \frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}}\right) - \frac{P}{\sqrt{A}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{g_{33}}\, \sqrt{A} \frac{U_\alpha}{\sqrt{g_{\alpha\alpha}}}\right)$$

$$+ \zeta\bar{Q} - \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^\alpha}\left(\sqrt{A}\, \zeta \frac{\langle E'U_\alpha' \rangle}{\sqrt{g_{\alpha\alpha}}}\right) - \frac{\partial}{\partial x^3}\left(\zeta \frac{\langle E'U_3' \rangle}{\sqrt{g_{33}}}\right) \tag{28}$$

Equation (28) is then integrated to get U_3 by use of the boundary condition, $U_3 = 0$ at $x^3 = 0$. Equation (28) was derived from the conservation equation for internal energy; it is a form of the Richardson equation in reference 1.

A brief recapitulation of the chosen approach to computing the vertical velocity is given as follows: First, V_3, the vertical fluid velocity relative to the coordinate system, is specified (consistent with the boundary conditions of eq. (17)). Second, $\partial\zeta/\partial t$ is computed from equation (1). Third, U_3 is computed from equation (28). When V_3 is chosen arbitrarily, $\sqrt{g_{33}}$ will usually be time dependent, the variation of which can now be computed from equation (25) by

$$\frac{\partial}{\partial t}\left(\sqrt{g_{33}}\right) = \frac{\partial}{\partial x^3}\left(S_3\right) = \frac{\partial}{\partial x^3}\left(U_3 - V_3\right) \tag{29}$$

There are many other valid approaches for specifying the vertical structure of an atmospheric model, but the important point to be made is that one can choose the vertical strucdure in any desired manner. Equation (28) is the general link by which the vertical structure is made to evolve with internal energy properly conserved.

General Solution Considerations

The solution to a given circulation problem is typically straightforward in concept. The conservation equations describe the time evolution of the chosen dependent variables by means of dynamic equations driven by the dependent variables and subgrid correlations between the dependent variables. In the present model, approximations have been made on the terms containing real world terrain to gain simplicity in the governing equations while incurring only insignificant errors. The results indicate that the real world is, for practical grid spacings, equivalent to a spherical world where the gravity vector is oblique to the spherical surface. The resulting model allows one to specify general vertical grid structures for the model as a function of time. The model was developed specifically to simulate both atmosphere and seas motions.

CONCLUDING REMARKS

A generally valid, but simple, mathematical model suitable for simulating atmospheric and oceanic dynamic motions and pollution transport is presented. The mathematical model was formulated in general tensor notation, and in the conservation equations, the density ρ and vertical scale factor $\sqrt{g_{33}}$ have been grouped together to form a generalized density variable ζ suitable for oceanic or atmospheric simulation. For simplifying computations the quasi-horizontal bottom coordinate surface has been deformed to follow real Earth terrain, whereas the top coordinate surface moves vertically with the oceanic or atmospheric free surface. The intermediate coordinate surfaces are allowed to move as freely as one desires. Thus, external gravity waves can be represented by the top coordinate surface, and meteorological fronts and inversion layers in the atmosphere and refractive layers in the seas can be represented with enhanced resolution by the internal quasi-horizontal coordinate surfaces. The action of subgrid mixing has been represented as true tensor quantities, which corrects a deficiency in the intuitive subgrid mixing formulation in that rigid-body rotation of the atmosphere under adiabatic conditions is an allowed solution.

Langley Research Center
National Aeronautics and Space Administration
Hampton, Va. 23665
April 1, 1975

APPENDIX A

GENERAL TRANSPORT EQUATIONS IN COORDINATE SYSTEMS WITH TIME-DEPENDENT METRIC TENSORS

In this appendix, the conservation equations for continuous media are derived in tensor notation. The control volume is allowed to move relative to the rotating Earth. The resulting general transport equations are applicable to any conservable quantity of a circulation system.

Since the control volume may be in motion with respect to the Earth, the Earth relative velocities u_i are related to the control-volume relative velocities v_i by

$$u_i = v_i + s_i \tag{A1}$$

where s_i denotes the velocity of the control volume relative to the rotating Earth.

Consider a conservable intensive quantity Q inside a small coordinate volume ν. The total storage rate of Q inside ν is given by

$$\frac{\partial}{\partial t} \int_{\nu} (\rho Q) \sqrt{a}\, d\nu \tag{A2}$$

where ρ is the density of the fluid in ν and a is the determinant of the metric tensor. The outward flux of Q across the surface S of ν is given by the surface integral

$$\int_S \left(\rho Q v^j\right) n_j \, dS \tag{A3}$$

where n_j is the unit vector normal to dS positive outward. By the generalized Gauss' theorem of reference 10, the flux of (ρQ) across S is equal to the integrated divergence of $\left(\rho Q v^j\right)$ in ν; that is,

$$\int_S \left(\rho Q v^j\right) n_j \, dS = \int_{\nu} \left(\rho Q v^j\right)_{|j} \sqrt{a}\, d\nu \tag{A4}$$

Conservation of Q requires that the storage rate plus the transport rate equal the rate of creation of Q in ν, or

$$\frac{\partial}{\partial t} \int_{\nu} (\rho Q) \sqrt{a}\, d\nu + \int_{\nu} \left(\rho Q v^j\right)_{|j} \sqrt{a}\, d\nu = \int_{\nu} S'\left(\rho Q \sqrt{a}\right) d\nu \tag{A5}$$

where S' is the creation rate of $\rho Q\sqrt{a}$ in $d\nu$. If the requirement that the control volume be constant in the coordinate space (but not in the physical space) is imposed, then the time differentiation of the first term in equation (A5) can be carried inside the volume integral. Since the resultant equation must hold for an arbitrary ν, the integration can be dropped to give

$$\frac{\partial}{\partial t}\left(\rho Q\sqrt{a}\right) + \left(\rho Q v^j\right)_{|j}\sqrt{a} = S'\left(\rho Q\sqrt{a}\right) \tag{A6}$$

Equation (A6) may be rewritten as

$$\frac{\partial}{\partial t}\left(\rho Q\sqrt{a}\right) + \frac{\partial}{\partial x^j}\left(\rho Q v^j\sqrt{a}\right) = S'\left(\rho Q\sqrt{a}\right) \tag{A7}$$

Equation (A7) is the general transport equation for a conservable intensive scalar quantity Q.

The derivation of the conservation equation for tensors $\left(Z^{i\,\cdots}_{k\,\cdots}\right)$ yields additional terms. In the present report, such terms are grouped with the source term to form a pseudosource term $\hat{S}\left(\sqrt{a}\rho Z^{i\,\cdots}_{k\,\cdots}\right)$. Thus, a common form for all conservation equations is obtained, which is

$$\frac{\partial}{\partial t}\left(\sqrt{a}\rho Z^{i\,\cdots}_{k\,\cdots}\right) + \frac{\partial}{\partial x^j}\left(\sqrt{a}\rho Z^{i\,\cdots}_{k\,\cdots}v^j\right) = \hat{S}\left(\sqrt{a}\rho Z^{i\,\cdots}_{k\,\cdots}\right) \tag{A8}$$

The derivation of the conservation equation for momentum is presented in the following discussion. Newton's second law for the infinitesimal control volume $d\nu$, if considering only the force of the pressure gradient and gravity and neglecting viscous forces, is

$$\rho\sqrt{a}\,d\nu\left[\frac{du_i}{dt} + \left(a_c\right)_i\right] = \sqrt{a}\,d\nu\left(-P_{|i} + g_i\rho\right) \tag{A9}$$

where $\left(a_c\right)_i$ is the coriolis and centripetal acceleration arising from the Earth's rotation and du_i/dt is the comoving time derivation of u_i, which, from reference 11, may be expanded in terms of the velocity relative to the control volume $s^i = u^i - v^i$ as

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + v^j u_{i|j} - s^j_{|i}u_j \tag{A10}$$

Thus, equation (A9) can be rewritten as

$$d\nu\sqrt{a}\rho\left(\frac{\partial u_i}{\partial t} + v^j u_{i|j} - s^j_{|i}u_j\right) = \sqrt{a}\, d\nu\left[-P_{|i} + g_i\rho - \rho\left(a_c\right)_i\right] \tag{A11}$$

Equation (A11) can be rewritten in conservative form by the following procedure. First, in equation (A6), Q is set to unity. The resulting equation is the mass-conservation equation

$$d\nu\left[\frac{\partial}{\partial t}\left(\sqrt{a}\rho\right) + \left(\rho v^j\right)_{|j}\sqrt{a}\right] = 0 \tag{A12}$$

Next, equation (A12) is multiplied by u_i and added to equation (A11) to yield the conservative form:

$$d\nu\left[\frac{\partial}{\partial t}\left(\sqrt{a}\rho u_i\right) + \left(\rho u_i v^j\right)_{|j}\sqrt{a}\right] = S'\left(\sqrt{a}\rho u_i\right) d\nu$$

$$= \sqrt{a}\left[-P_{|i} + g_i\rho - \rho\left(a_c\right)_i + \rho s^j_{|i}u_j\right] d\nu \tag{A13}$$

Equation (A13) can then be rewritten in the form of equation (A8) by the following procedure. First, the term $\left(\rho u_i v^j\right)_{|j}$ is expanded by identities to yield

$$\sqrt{a}\left(\rho u_i v^j\right)_{|j} = \frac{\partial}{\partial x^j}\left(\sqrt{a}\rho u_i v^j\right) - \left\{{}_i{}^m{}_j\right\}\left(\sqrt{a}\rho v^j u_m\right) \tag{A14}$$

If the following terms are collected in the pseudosource term

$$\hat{S}\left(\sqrt{a}\rho u_i\right) = \sqrt{a}\left[-P_{|i} + g_i\rho - \rho\left(a_c\right)_i + \rho s^j_{|i}u_j\right] + \left\{{}_i{}^m{}_j\right\}\left(\sqrt{a}\rho v^j u_m\right) \tag{A15}$$

then the momentum-conservation equation (A13) takes the desired form

$$\frac{\partial}{\partial t}\left(\sqrt{a}\rho u_i\right) + \frac{\partial}{\partial x^j}\left(\sqrt{a}\rho u_i v^j\right) = \hat{S}\left(\sqrt{a}\rho u_i\right) \tag{A16}$$

which is an alternate form of momentum conservation expressed in equation (A13).

Since equation (A16) holds for the infinitesimal volume $d\nu$, it can be integrated over a finite volume ν. The result must still hold, although the resulting equation does not represent a true tensor. In numerical analysis, the solution to a given problem is often obtained by breaking the problem space into finite volumes. Equation (A5) for scalar transport, for example, might be solved by such a technique. The numerical approximation of equation (A5) causes some information (fine structure) about the state of the system to be lost or ignored. Unfortunately, the fine structure can have a significant role in the solution. These intuitive ideas can be symbolized by the relationship

$$Q = \langle Q \rangle + Q' \tag{A17}$$

where $\langle Q \rangle$ is the average value of Q taken over the gridcell volume and time step, and, thus, Q' is the difference between an instantaneous point value of Q and the average of Q over the four-dimensional space-time gridcell "volume." A model employing a discrete space-time grid represents the field of Q by estimating $\langle Q \rangle$. Although the space-time gridcell average of subgrid variations Q' vanishes (as can be seen by applying the operator $\langle\ \rangle$ to eq. (A17)), subgrid variations must still be accounted for in estimating $\langle Q \rangle$, as shown in the following example.

Applying the operator $\langle\ \rangle$ to equation (A6) and substituting the generic expression (A17) for Q, v^j, $Q_{|j}$, and $v^j_{|j}$ yields

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial t}\left(\rho Q \sqrt{a}\right)\right\rangle + \left\langle \left\langle \sqrt{a}\rho_{|j}\right\rangle \left(\langle Q\rangle + Q'\right)\left(\langle v^j\rangle + v^{j'}\right)\right\rangle \\
&+ \left\langle \left\langle \sqrt{a}\rho\right\rangle \left[\left\langle Q_{|j}\right\rangle + \left(Q_{|j}\right)'\right]\left(\langle v^j\rangle + v^{j'}\right)\right\rangle \\
&+ \left\langle \left\langle \sqrt{a}\rho\right\rangle \left(\langle Q\rangle + Q'\right)\left[\left\langle v^j_{|j}\right\rangle + \left(v^j_{|j}\right)'\right]\right\rangle = \left\langle S'\left(\sqrt{a}\rho Q\right)\right\rangle
\end{aligned} \tag{A18}$$

where subgrid variations of ρ and $\sqrt{a}$ are neglected. By definition of the operator $\langle\ \rangle$, $\langle Q \rangle$ is constant for general Q throughout the space-time gridcell; thus, equation (A18) becomes

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$$\left\langle \frac{\partial}{\partial t}\left(\rho Q\sqrt{a}\right)\right\rangle + \left\langle \sqrt{a}\rho_{|j}\right\rangle\left[\langle Q\rangle\left\langle v^j\right\rangle + \langle Q\rangle\left\langle v^{j'}\right\rangle + \left\langle v^j\right\rangle\langle Q'\rangle + \left\langle Q'v^{j'}\right\rangle\right]$$

$$+ \left\langle \sqrt{a}\rho\right\rangle\left[\left\langle Q_{|j}\right\rangle\left\langle v^j\right\rangle + \left\langle Q_{|j}\right\rangle\left\langle v^{j'}\right\rangle + \left\langle v^j\right\rangle\left\langle \left(Q_{|j}\right)'\right\rangle + \left\langle \left(Q_{|j}\right)' v^{j'}\right\rangle\right]$$

$$+ \left\langle \sqrt{a}\rho\right\rangle\left[\langle Q\rangle\left\langle v^j_{|j}\right\rangle + \langle Q\rangle\left\langle \left(v^j_{|j}\right)'\right\rangle + \left\langle v^j_{|j}\right\rangle\langle Q'\rangle + \left\langle Q'\left(v^j_{|j}\right)'\right\rangle\right]$$

$$= \left\langle S'\left(\sqrt{a}\rho Q\right)\right\rangle \tag{A19}$$

Applying the operator $\langle\ \rangle$ to equation (A17) yields

$$\langle Q\rangle = \langle\langle Q\rangle\rangle + \langle Q'\rangle = \langle Q\rangle + \langle Q'\rangle$$

Thus,

$$\langle Q'\rangle = 0 \tag{A20}$$

with the result that equation (A19) reduces to

$$\left\langle \frac{\partial}{\partial t}\left(\rho Q\sqrt{a}\right)\right\rangle + \left\langle \sqrt{a}\rho_{|j}\right\rangle\left[\langle Q\rangle\left\langle v^j\right\rangle + \left\langle Q'v^{j'}\right\rangle\right] + \left\langle \sqrt{a}\rho\right\rangle\left[\left\langle Q_{|j}\right\rangle\left\langle v^j\right\rangle\right.$$

$$\left. + \left\langle \left(Q_{|j}\right)' v^{j'}\right\rangle\right] + \left\langle \sqrt{a}\rho\right\rangle\left[\langle Q\rangle\left\langle v^j_{|j}\right\rangle + \left\langle Q'\left(v^j_{|j}\right)'\right\rangle\right] = \left\langle S'\left(\sqrt{a}\rho Q\right)\right\rangle \tag{A21}$$

The operator $\langle\ \rangle$ is an integral operator; therefore, for general Q,

$$\left\langle Q_{|j}\right\rangle = \langle Q\rangle_{|j} \tag{A22a}$$

(by neglecting variations in the Christoffel symbols of the second kind over a space-time gridcell). Similarly, by use of equation (A17),

$$\left(Q_{|j}\right)' = Q_{|j} - \left\langle Q_{|j}\right\rangle \tag{A22b}$$

and

$$(Q')_{|j} = \left(Q - \langle Q \rangle\right)_{|j} = Q_{|j} - \langle Q \rangle_{|j}$$

or, by use of equation (A22a),

$$(Q')_{|j} = Q_{|j} - \left\langle Q_{|j} \right\rangle$$

Thus, by use of equation (A22b),

$$\left(Q_{|j}\right)' = (Q')_{|j} \tag{A22c}$$

Applying equations (A22a) and (A22c) to equation (A21) yields

$$\left\langle \frac{\partial}{\partial t}\left(\rho Q \sqrt{a}\right) \right\rangle + \sqrt{a}\left(\langle \rho \rangle \; \langle Q \rangle \; \langle v^j \rangle\right)_{|j}$$

$$= -\sqrt{a}\left(\langle \rho \rangle \left\langle Q' v^{j'} \right\rangle\right)_{|j} + \left\langle S'\left(\sqrt{a}\rho Q\right) \right\rangle \tag{A23}$$

The first right-hand term of equation (A23) can be interpreted as the contribution to the flux of Q into the unit coordinate volume of the statistical correlation of the space-time fluctuations of Q and v^j. The modeling of this term is described in appendix D.

Setting $Q = u_i$ in equation (A23) and expressing the source term S' by equation (A13) yields the space-time averaged form of the momentum-conservation equation; thus,

$$\left\langle \frac{\partial}{\partial t}\left(\sqrt{a}\rho u_i\right) \right\rangle + \sqrt{a}\left(\langle \rho \rangle \; \langle u_i \rangle \; \langle v^j \rangle\right)_{|j}$$

$$= -\sqrt{a}\left(\langle \rho \rangle \left\langle u_i' v^{j'} \right\rangle\right)_{|j} + \left\langle S'\left(\sqrt{a}\rho u_i\right) \right\rangle$$

$$= -\sqrt{a}\left(\langle \rho \rangle \left\langle u_i' u^{j'} \right\rangle\right)_{|j} + \left\langle \sqrt{a}\left[-P_{|i} + g_i \rho - \rho\left(a_c\right)_i + \rho s^j_{|i} u_j\right] \right\rangle \tag{A24}$$

where $s^{j'} = u^{j'} - v^{j'}$ is neglected and $v^{j'}$ is replaced by $u^{j'}$.

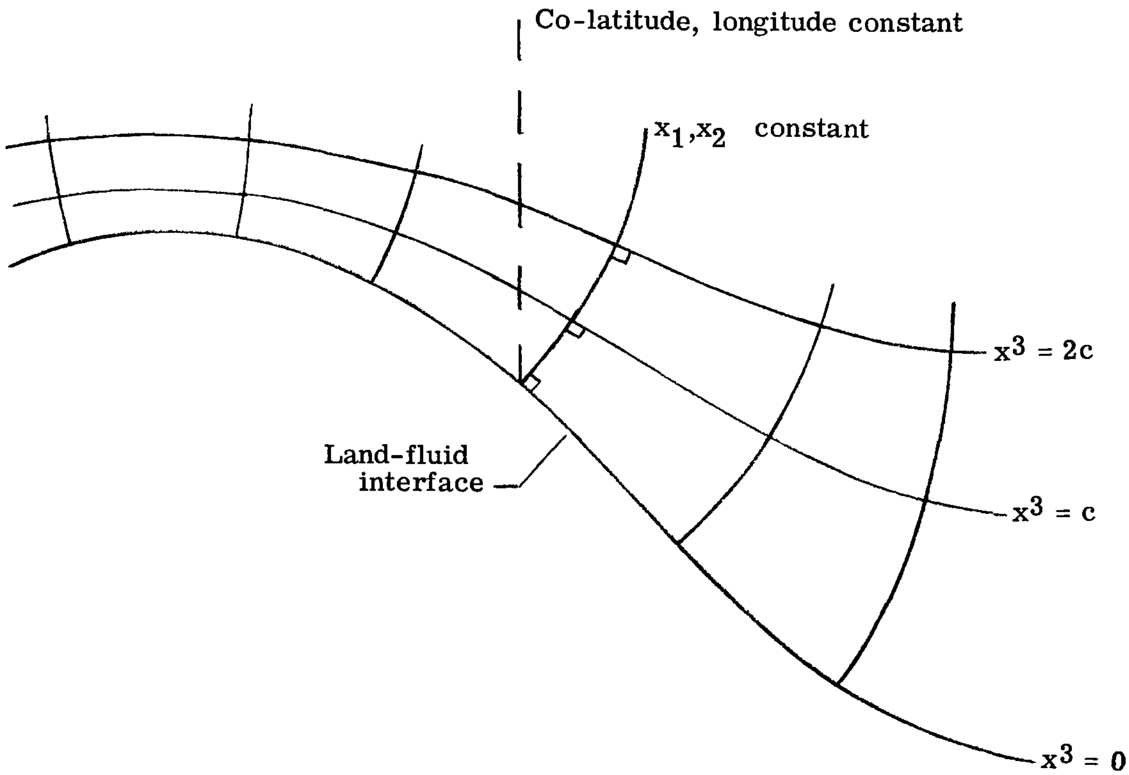
APPENDIX A

For convenience, in equations applied to a discretized space-time grid, the averaging brackets $\langle \; \rangle$ enclosing unprimed quantities are deleted, which should cause no misinterpretation because only averaged quantities are available to the finite difference treatment.

APPENDIX B

AN APPROXIMATE METRIC TENSOR FOR NEARLY SPHERICAL POLAR COORDINATE SYSTEMS

The coordinate system adopted is a variant of the standard co-latitude-longitude-radial system. At the land-to-air interface and the sea-floor-to-water interface, x^1 is the co-latitude angle, x^2 is the longitude angle, and x^3 is defined as zero. That is, the $x^3 = 0$ surface follows the land and sea-bottom terrain. The x^3 coordinate lines (which are the loci of constant x^1 and x^2) are locally normal to the $x^3 = 0$ surface, and, in general, the x^3 coordinate lines are locally normal to a surface of constant x^3. The loci of constant co-latitude and longitude, on the other hand, are radial, and so x^1 and x^2 generally deviate somewhat from the local co-latitude and longitude for non-zero x^3. The following sketch is a simplified (two-dimensional) illustration of the coordinate system:



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Let the local radial deviation of the $x^3 = 0$ surface from a true spherical surface be denoted by $\delta\left(x^1, x^2\right)$. Then, for a displacement dx^α, the first-order change in elevation (measured from a true sphere) of the $x^3 = 0$ surface is given by

$$d\delta = \frac{\partial\delta}{\partial x^1} dx^1 + \frac{\partial\delta}{\partial x^2} dx^2$$

The differential arc length squared for $dx^3 = 0$ is given by

$$(ds)^2 = \left(r\, dx^1\right)^2 + \left(r \sin x^1\, dx^2\right)^2 + (d\delta)^2$$

$$= \left[(r)^2 + \left(\frac{\partial\delta}{\partial x^1}\right)^2\right]\left(dx^1\right)^2 + \left[\left(r \sin x^1\right)^2 + \left(\frac{\partial\delta}{\partial x^2}\right)^2\right]\left(dx^2\right)^2$$

$$+ 2\left(\frac{\partial\delta}{\partial x^1}\right)\left(\frac{\partial\delta}{\partial x^2}\right) dx^1\, dx^2$$

The x^3 coordinate line is normal to the local Earth's surface; thus, for general dx^3,

$$(ds)^2 = \left[(r)^2 + \left(\frac{\partial\delta}{\partial x^1}\right)^2\right]\left(dx^1\right)^2 + \left[\left(r \sin x^1\right)^2 + \left(\frac{\partial\delta}{\partial x^2}\right)^2\right]\left(dx^2\right)^2$$

$$+ 2\left(\frac{\partial\delta}{\partial x^1}\right)\left(\frac{\partial\delta}{\partial x^2}\right) dx^1\, dx^2 + g_{33}\left(dx^3\right)^2 \tag{B1}$$

The symmetric covariant metric tensor, defined by

$$(ds)^2 = g_{ij}\, dx^i\, dx^j$$

becomes, at the Earth's surface,

$$g_{ij} = \begin{bmatrix} \left[(r)^2 + \left(\frac{\partial\delta}{\partial x^1}\right)^2\right] & \left(\frac{\partial\delta}{\partial x^1}\right)\left(\frac{\partial\delta}{\partial x^2}\right) & 0 \\ \left(\frac{\partial\delta}{\partial x^1}\right)\left(\frac{\partial\delta}{\partial x^2}\right) & \left[\left(r \sin x^1\right)^2 + \left(\frac{\partial\delta}{\partial x^2}\right)^2\right] & 0 \\ 0 & 0 & g_{33} \end{bmatrix} \tag{B2}$$

Similarly, the metric tensor can be written for positions not necessarily on the Earth's surface in terms of spatial derivatives of the altitude h above the ideal spherical Earth, where h reduces to δ at the Earth's surface. Thus,

$$g_{ij} = \begin{bmatrix} \left[(r)^2 + \left(\frac{\partial h}{\partial x^1}\right)^2\right] & \left(\frac{\partial h}{\partial x^1}\right)\left(\frac{\partial h}{\partial x^2}\right) & 0 \\ \left(\frac{\partial h}{\partial x^1}\right)\left(\frac{\partial h}{\partial x^2}\right) & \left[\left(r \sin x^1\right)^2 + \left(\frac{\partial h}{\partial x^2}\right)^2\right] & 0 \\ 0 & 0 & g_{33} \end{bmatrix} \tag{B3}$$

The formulation of the metric tensor by use of equation (B3) allows the "horizontal" coordinate lines to be positioned or moved vertically without restriction.

The square of the line element (which completely determines the differential geometry) becomes

$$\begin{aligned}(ds)^2 &= g_{ij}\, dx^i\, dx^j \\ &= \left[(r)^2 + \left(\frac{\partial h}{\partial x^1}\right)^2\right] dx^1\, dx^1 + \left[\left(r \sin x^1\right)^2 + \left(\frac{\partial h}{\partial x^2}\right)^2\right] dx^2\, dx^2 \\ &\quad + g_{33}\, dx^3\, dx^3 + 2\, \frac{\partial h}{\partial x^1} \frac{\partial h}{\partial x^2}\, dx^1\, dx^2\end{aligned}$$

Collecting terms in $\partial h / \partial x^\alpha$ gives

$$\begin{aligned}(ds)^2 &= (r)^2\, dx^1\, dx^1 + \left(r \sin x^1\right)^2 dx^2\, dx^2 + g_{33}\, dx^3\, dx^3 \\ &\quad + \left(\frac{\partial h}{\partial x^1}\, dx^1 + \frac{\partial h}{\partial x^2}\, dx^2\right)^2\end{aligned} \tag{B4}$$

where the first three right-hand terms are the square of the spherical polar line element, and the last right-hand term is the square of the height increment along the surface of constant x^3. Thus, if the square of the horizontal increment (the first two right-hand terms in eq. (B4)) is much larger than the square of the height increment along the surface of constant x^3, then the square of the line element is approximated by the square of the spherical polar line element, and the metric tensor reduces to the spherical polar metric tensor

$$g_{ij} \cong \begin{bmatrix} (r)^2 & 0 & 0 \\ 0 & \left(r \sin x^1\right)^2 & 0 \\ 0 & 0 & g_{33} \end{bmatrix} \tag{B5}$$

for

$$\left(\frac{\partial h}{\partial x^1} dx^1 + \frac{\partial h}{\partial x^2} dx^2\right)^2 << (r)^2 dx^1 dx^1 + \left(r \sin x^1\right)^2 dx^2 dx^2 \tag{B6}$$

The condition (B6) for the approximation of the metric tensor by equation (B5) is not very restrictive because, in a discrete grid model, the slope terms $\partial h/\partial x^\alpha$ are characteristic of a coordinate cell, rather than of a single point. For example, if $\tilde{\delta}$ is the maximum change in mean elevation at the $x^3 = 0$ surface from one coordinate cell to an adjacent cell, then the condition (B6) for equation (B5) to hold (at the $x^3 = 0$ surface) becomes

$$\left.\begin{aligned} (\tilde{\delta})^2 &<< (r)^2\left(\Delta x^1\right)^2 \\ (\tilde{\delta})^2 &<< \left(r \sin x^1\right)^2\left(\Delta x^2\right)^2 \end{aligned}\right\} \tag{B7}$$

where Δx^1 and Δx^2 are the cell horizontal coordinate dimensions in radians. Thus, the smaller the grid physical increments, $r\ \Delta x^1$ and $r \sin x^1\ \Delta x^2$, the more restrictive conditions (B7) become. More generally, if $\tilde{h}$ is the maximum change in mean elevation of a constant x^3 surface from one coordinate cell to an adjacent cell, then the condition (B6) for equation (B5) to hold becomes

$$\left.\begin{aligned} (\tilde{h})^2 &<< (r)^2\left(\Delta x^1\right)^2 \\ (\tilde{h})^2 &<< \left(r \sin x^1\right)^2\left(\Delta x^2\right)^2 \end{aligned}\right\} \tag{B8}$$

Generally, it is desirable and computationally efficient to maintain a uniform horizontal spatial resolution of all variables. The usual method of achieving uniform horizontal spatial resolution is to filter spatially all variables in the horizontal at half-wavelengths shorter than the maximum horizontal grid physical increments. An effect of the spatial (low-pass) filter is to make conditions (B8) much easier to satisfy than would be the case without filtering, especially near the poles, where $\sin x^1$ is small. Consider a complex Fourier expansion of h around the Earth along a parallel:

$$h = \sum_{n=0}^{\infty} h_n = \sum_{n=0}^{\infty} A_n(t) e^{-i \frac{2\pi}{\lambda_n} r \sin x^1 \, x^2} \tag{B9}$$

where λ_n is the physical wavelength associated with the nth Fourier term and $A_n(t)$ is a complex function of time. For simplicity, a perfect low-pass filter is assumed which truncates the series of equation (B9) such that $h_n = 0$ for $\lambda_n < 2r \, \Delta x^2$ (twice the grid physical increment at the equator). The filtered h, h', becomes

$$h' = \sum_{n=0}^{m} h_n = \sum_{n=0}^{m} A_n(t) e^{-i \frac{2\pi}{\lambda_n} r \sin x^1 \, x^2} \tag{B10}$$

where λ_m is the smallest λ_n greater than or equal to $2r \, \Delta x^2$. The maximum change in $|h'|$ over a grid increment $|\Delta h'|$ is realized if $|A_m(t)| = \frac{1}{2} \delta h$ and all other $A_n(t) = 0$, where δh = (Maximum h) - (Minimum h). Thus,

$$\left| \frac{\Delta h'}{\Delta x^2} \right| \lessapprox \left| \frac{\partial}{\partial x^2} \left(\frac{1}{2} \delta h \, e^{-i \frac{2\pi}{\lambda_m} r \sin x^1 \, x^2} \right) \right|$$

$$\lessapprox \left| \frac{1}{2} \delta h \left(-i \frac{2\pi}{\lambda_m} r \sin x^1 \right) \right| \leqq \left| \frac{1}{2} \delta h \frac{2\pi}{2r \, \Delta x^2} r \sin x^1 \right|$$

or

$$|\Delta h'| \leqq \frac{1}{2} \left| \delta h \, \pi \sin x^1 \right| \tag{B11}$$

Replacing $\tilde{h}$ in conditions (B8) by $|\Delta h'|$ yields the new condition

$$(\Delta h')^2 << \left(r \sin x^1 \right)^2 \left(\Delta x^2 \right)^2$$

or, by use of equation (B11),

$$\frac{\pi^2}{4}(\delta h)^2 << (r)^2\left(\Delta x^2\right)^2 \tag{B12}$$

The first inequality in conditions (B8) is retained:

$$\left(\tilde{h}\right)^2 << (r)^2\left(\Delta x^1\right)^2 \tag{B13}$$

Therefore, the inequalities (B12) and (B13) are sufficient conditions for equation (B5) for the metric tensor to hold when Fourier series terms for h of physical half-wavelenths shorter than the east-west equatorial physical grid increment are deleted. If $\Delta x^1 = \Delta x^2$, which is the usual case, then condition (B12) implies condition (B13), because $|\delta h| \geqq |\tilde{h}|$.

Consider the extreme case where $\delta h = 8$ kilometers, which is certainly sufficient for representing, by surfaces of constant x^3, terrain and sea-bottom topography, meteorological fronts, and refractive layers in the seas. If the left-hand side of inequality (B12) is required to be 1 percent of the right-hand side, then the minimum grid increment Δx^2 consistent with inequality (B12) is

$$\Delta x^2 = \frac{\pi}{2}\frac{\delta h}{r}\ (10)\ \text{radians} = \frac{\pi}{2}\frac{8}{6400}\ (10)\ \text{radians}$$

or

$$\Delta x^2 = 57.3\ \frac{\pi}{2}\frac{8}{6400}\ (10^{\circ}) = 1.12^{\circ}$$

which corresponds to a very fine grid.

If condition (B12) is written in equation form, then

$$\epsilon_{\phi}\ \frac{\pi^2}{4}\ (\delta h)^2 = (r)^2\left(\Delta x^2\right)^2$$

where the error fraction in the square of the line element is $\epsilon_{\phi} << 1$; then, the maximum $|\delta h|$ is

$$|\delta h|\ (\text{kilometers}) = \sqrt{\epsilon_{\phi}}\ 80\ \frac{\Delta x^2\ (\text{degrees})}{1.12} = \sqrt{\epsilon_{\phi}}\ 71.4\ \Delta x^2\ (\text{degrees}) \tag{B14}$$

APPENDIX C

TRANSPORT EQUATIONS FOR GEOPHYSICAL CIRCULATION PROBLEMS

In appendix B and approximate metric tensor was presented for nearly spherical polar coordinate systems. In this appendix the simplications resulting from that metric tensor are applied to the transport equations developed in appendix A.

In principle, the metric tensor allows equations which are identical for all coordinate systems. The physical assumptions and solution techniques used by the analyst reduce the generality of the governing transport equations to special cases, and thereby introduce real differences into the transport equations for geophysical circulation problems. The specialization to atmospheric- and oceanic-circulation problems will be made so that one set of transport equations can describe both systems.

The prime difference, for modeling purposes, between the oceans and atmosphere lies in the different density characteristics. The atmosphere is compressible, whereas the oceans are essentially incompressible. However, if external gravity waves in the oceans are considered, then from the viewpoint of mass in a given fraction of a vertical column, the oceans and atmosphere are similar. Consider, then, a new variable ζ to replace ρ so that the similarity of the fluid bodies is emphasized. A good candidate for ζ is the grouping $\rho\sqrt{g_{33}}$ because ρ and $\sqrt{g_{33}}$ usually appear together in the transport equations, and ζ (generalized density) has similar behavior in the atmosphere and oceans.

From equation (B3), the determinate of the metric tensor is

$$|g_{ij}| = a = g_{33}\left\{\left[(r)^2 + \left(\frac{\partial h}{\partial x^1}\right)^2\right]\left[\left(r \sin x^1\right)^2 + \left(\frac{\partial h}{\partial x^2}\right)^2\right] - \left(\frac{\partial h}{\partial x^1}\right)^2\left(\frac{\partial h}{\partial x^2}\right)^2\right\} = g_{33}A \tag{C1}$$

where A is the determinate of the metric tensor for the quasi-horizontal coordinates. Applying the definition $\zeta = \rho\sqrt{g_{33}}$ and equation (C1) to equations (A12), (A23), and (A24) yields

$$\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta\right) + \left(\frac{1}{\sqrt{g_{33}}}\zeta v^j\right)_{|j}\sqrt{g_{33}}\sqrt{A} = 0 \tag{C2}$$

$$\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta Q\right) + \left(\frac{1}{\sqrt{g_{33}}}\,\zeta Q v^j\right)_{|j} \sqrt{g_{33}}\,\sqrt{A} = S'\left(\sqrt{A}\,\zeta Q\right) - \left(\frac{1}{\sqrt{g_{33}}}\,\zeta \left\langle Q' v^{j'}\right\rangle\right)_{|j} \tag{C3}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta u_i\right) + \left(\frac{1}{\sqrt{g_{33}}}\,\zeta u_i v^j\right)_{|j} \sqrt{g_{33}}\,\sqrt{A} \\
&= -\left(\frac{1}{\sqrt{g_{33}}}\,\zeta \left\langle u_i' u^{j'}\right\rangle\right)_{|j} \sqrt{g_{33}}\,\sqrt{A} + \sqrt{g_{33}}\,\sqrt{A}\left[-P_{|i} + g_i \frac{\zeta}{\sqrt{g_{33}}}\right. \\
&\left. - \frac{\zeta}{\sqrt{g_{33}}}\left(a_c\right)_i + \frac{\zeta}{\sqrt{g_{33}}}\, s^j_{|i} u_j\right]
\end{aligned} \tag{C4}$$

Equations (C2) and (C3) may be rewritten, respectively, as

$$\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta\right) + \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta v^j\right) = 0 \tag{C5}$$

and

$$\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta Q\right) + \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta Q v^j\right) = S'\left(\sqrt{A}\,\zeta Q\right) - \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta \left\langle Q' u^{j'}\right\rangle\right) \tag{C6}$$

where $v^{j'}$ was replaced by $u^{j'}$ as in equation (A24). Expanding

$$\left(\frac{1}{\sqrt{g_{33}}}\,\zeta u_i v^j\right)_{|j} \sqrt{g_{33}}\,\sqrt{A}$$

and

$$\left(\frac{1}{\sqrt{g_{33}}}\,\zeta \left\langle u_i' u^{j'}\right\rangle\right)_{|j} \sqrt{g_{33}}\,\sqrt{A}$$

in equation (C4) by identities similar to the procedure leading to equation (A14) yields

$$\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta u_i\right) + \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta u_i v^j\right) = \hat{S}\left(\sqrt{A}\,\zeta u_i\right) - \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle u_i' u^{j'}\right\rangle\right) \tag{C7}$$

where

$$\hat{S}\left(\sqrt{A}\,\zeta u_i\right) = \sqrt{A}\left[-\sqrt{g_{33}}\,P_{|i} + \zeta g_i - \zeta\left(a_c\right)_i + \zeta s^j_{|i} u_j + \left\{{}_i{}^m{}_j\right\}\left(\zeta v^j u_m\right) + \left\{{}_i{}^m{}_j\right\}\left(\zeta\left\langle u^{j'} u_m'\right\rangle\right)\right] \tag{C8}$$

By inspection of equations (C6), (C7), and (C8), it is evident that equations (C5) and (C7) can be obtained by replacing Q in equation (C6) with 1 and u_i, respectively, and by computing $\hat{S}\left(\sqrt{A}\,\zeta u_i\right)$ by equation (C8). Thus, it is allowable to speak of one transport equation (eq. (C6)) for all conservable quantities in the atmosphere or ocean.

From equation (B5), an approximation to the metric tensor is

$$g_{ij} \cong \begin{bmatrix} (r)^2 & 0 & 0 \\ 0 & \left(r \sin x^1\right)^2 & 0 \\ 0 & 0 & g_{33} \end{bmatrix} \tag{C9}$$

Thus, the determinant of the metric tensor is, approximately,

$$a \cong \left[(r)^2 \sin x^1\right]^2 g_{33} \tag{C10}$$

and the determinate of the metric tensor for the quasi-horizontal coordinates is, approximately,

$$A \cong \left[(r)^2 \sin x^1\right]^2 \tag{C11}$$

APPENDIX C

The full set of Christoffel symbols of the second kind are, by symmetry in the lower indices,

$$\left\{ {\alpha \atop \beta \ \gamma} \right\} \qquad \left\{ {3 \atop \beta \ \gamma} \right\} \qquad \left\{ {\alpha \atop \beta \ 3} \right\} \qquad \left\{ {3 \atop 3 \ \alpha} \right\} \qquad \left\{ {\alpha \atop 3 \ 3} \right\} \qquad \left\{ {3 \atop 3 \ 3} \right\}$$

In reference 10, page 67, it is shown that, for dx^3 locally normal to the surface of constant x^3,

$$\left.\begin{aligned}
\left\{ {3 \atop \beta \ \gamma} \right\} &= -\frac{1}{2g_{33}} \frac{\partial g_{\beta\gamma}}{\partial x^3} \\
\left\{ {\alpha \atop \beta \ 3} \right\} &= \frac{1}{2} g^{\alpha\gamma} \frac{\partial g_{\gamma\beta}}{\partial x^3} \\
\left\{ {3 \atop 3 \ \alpha} \right\} &= \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^\alpha} \\
\left\{ {\alpha \atop 3 \ 3} \right\} &= -\frac{1}{2} g^{\alpha\gamma} \frac{\partial g_{33}}{\partial x^\gamma} \\
\left\{ {3 \atop 3 \ 3} \right\} &= \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^3}
\end{aligned}\right\} \tag{C12}$$

For the metric tensor given by equation (C9), $\partial g_{\beta\gamma}/\partial x^3 = 0$ and $g^{\alpha\gamma} = 0$ for $\alpha \neq \gamma$, and $g^{\alpha\alpha} = 1/g_{\alpha\alpha}$; thus, equation (C12) becomes

$$\left.\begin{aligned}
\begin{Bmatrix} 3 \\ \beta\ \gamma \end{Bmatrix} &= 0 \\
\begin{Bmatrix} \alpha \\ \beta\ 3 \end{Bmatrix} &= 0 \\
\begin{Bmatrix} 3 \\ 3\ \alpha \end{Bmatrix} &= \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^{\alpha}} \\
\begin{Bmatrix} \alpha \\ 3\ 3 \end{Bmatrix} &= -\frac{1}{2g_{\alpha\alpha}} \frac{\partial g_{33}}{\partial x^{\alpha}} \\
\begin{Bmatrix} 3 \\ 3\ 3 \end{Bmatrix} &= \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^{3}}
\end{aligned}\right\} \tag{C13}$$

Thus, for the metric tensor given by equation (C9), the nonvanishing Christoffel symbols are contained in the set

$$\begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} \qquad \begin{Bmatrix} 3 \\ 3\ \alpha \end{Bmatrix} \qquad \begin{Bmatrix} \alpha \\ 3\ 3 \end{Bmatrix} \qquad \begin{Bmatrix} 3 \\ 3\ 3 \end{Bmatrix} \tag{C14}$$

The first subset in set (C14), $\begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix}$, expands to

$$\begin{aligned}
\begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} &\equiv \frac{1}{2} g^{\alpha j}\left(\frac{\partial g_{\beta j}}{\partial x^{\gamma}} + \frac{\partial g_{\gamma j}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{j}}\right) \\
&= \frac{1}{2} \frac{1}{g_{\alpha\alpha}}\left(\frac{\partial g_{\beta\alpha}}{\partial x^{\gamma}} + \frac{\partial g_{\gamma\alpha}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)
\end{aligned} \tag{C15}$$

by use of the relationships

$$\left.\begin{aligned}
g^{\alpha j} &= 0 \qquad (j \neq \alpha) \\
g^{\alpha\alpha} &= \frac{1}{g_{\alpha\alpha}}
\end{aligned}\right\} \tag{C16}$$

Of the derivatives of the metric elements in equation (C15), only $\partial g_{22}/\partial x^1$ does not vanish (in general) by use of equation (C9); thus, from equations (C9) and (C15), the nonvanishing $\left\{ {\alpha \atop \beta \ \gamma} \right\}$ are

$$\left.\begin{aligned} \left\{ {1 \atop 2\ 2} \right\} &= \frac{1}{2g_{11}}\left(-\frac{\partial g_{22}}{\partial x^1}\right) = -\sin x^1 \cos x^1 \\ \left\{ {2 \atop 1\ 2} \right\} &= \left\{ {2 \atop 2\ 1} \right\} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial x^1} = \cot x^1 \end{aligned}\right\} \tag{C17}$$

Then, the nonvanishing Christoffel symbols from equations (C13) and (C17) are

$$\left.\begin{aligned} \left\{ {1 \atop 2\ 2} \right\} &= -\sin x^1 \cos x^1 \\ \left\{ {2 \atop 1\ 2} \right\} &= \left\{ {2 \atop 2\ 1} \right\} = \cot x^1 \\ \left\{ {3 \atop 3\ \alpha} \right\} &= \left\{ {3 \atop \alpha\ 3} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^\alpha} \\ \left\{ {\alpha \atop 3\ 3} \right\} &= -\frac{1}{2g_{\alpha\alpha}} \frac{\partial g_{33}}{\partial x^\alpha} \\ \left\{ {3 \atop 3\ 3} \right\} &= \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^3} \end{aligned}\right\} \tag{C18}$$

From equation (B4) for the square of the line element $(ds)^2$,

$$\sqrt{g_{33}} = \frac{\partial s}{\partial x^3} \tag{C19}$$

The spherical polar system is obtained (from eq. (B4)) by setting $\partial h/\partial x^\alpha = 0$, for which case the surfaces of constant x^3 are concentric spheres. The vertical distance between any two such surfaces is, therefore, constant; hence, $\partial s/\partial x^3$ is constant over a surface of constant x^3, and, thus,

$$\frac{\partial\left(\sqrt{g_{33}}\right)}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial s}{\partial x^3}\right) = 0$$

or

$$\frac{\partial g_{33}}{\partial x^{\alpha}} = 0 \tag{C20}$$

for a spherical polar system. By use of equations (C18), the nonvanishing Christoffel symbols reduce to

$$\left.\begin{aligned}
\left\{ {}_{2}{}^{1}{}_{2} \right\} &= -\sin x^1 \cos x^1 \\
\left\{ {}_{1}{}^{2}{}_{2} \right\} &= \left\{ {}_{2}{}^{2}{}_{1} \right\} = \cot x^1 \\
\left\{ {}_{3}{}^{3}{}_{3} \right\} &= \frac{1}{g_{33}} \frac{\partial g_{33}}{\partial x^3}
\end{aligned}\right\} \tag{C21}$$

for a spherical polar system.

The Christoffel symbols are used herein only in expressing covariant derivatives of contravariant vectors and covariant vectors; the conditions under which the spherical polar Christoffel symbols of equations (C21) can replace the nonvanishing set of Christoffel symbols of equations (C18) are examined in the following discussion by comparing terms in the covariant derivatives of vectors. The nonvanishing set of Christoffel symbols of equations (C18) reduces to the spherical polar set if $\left\{ {}_{3}{}^{3}{}_{\alpha} \right\}$ and $\left\{ {}_{3}{}^{\alpha}{}_{3} \right\}$, or equivalently $\partial g_{33}/\partial x^{\alpha}$, can be neglected. From equations (C18),

$$\left\{ {}_{3}{}^{3}{}_{\alpha} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^{\alpha}} \tag{C22}$$

$$\left\{ {}_{3}{}^{\alpha}{}_{3} \right\} = -\frac{1}{2g_{\alpha\alpha}} \frac{\partial g_{33}}{\partial x^{\alpha}} \tag{C23}$$

From equation (C19),

$$\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{g_{33}}\right) = \frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial s}{\partial x^3}\right)$$

or

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{g_{33}}\right) = \frac{1}{2\sqrt{g_{33}}}\frac{\partial g_{33}}{\partial x^\alpha} = \frac{\partial}{\partial x^3}\left(\frac{\partial s}{\partial x^\alpha}\right)$$

Thus,

$$\frac{\partial g_{33}}{\partial x^\alpha} = 2\sqrt{g_{33}}\,\frac{\partial}{\partial x^3}\left(\frac{\partial s}{\partial x^\alpha}\right) \tag{C24}$$

By use of equations (C22) and (C23),

$$\left\{ {3 \atop 3 \quad \alpha} \right\} = \frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial x^3}\left(\frac{\partial s}{\partial x^\alpha}\right) \tag{C25}$$

$$\left\{ {\alpha \atop 3 \quad 3} \right\} = -\frac{\sqrt{g_{33}}}{g_{\alpha\alpha}}\frac{\partial}{\partial x^3}\left(\frac{\partial s}{\partial x^\alpha}\right) \tag{C26}$$

The line element ds, from equation (B4), is

$$ds = \left[(r)^2\,dx^1\,dx^1 + \left(r \sin x^1\right)^2 dx^2\,dx^2 + g_{33}\,dx^3\,dx^3 + \left(\frac{\partial h}{\partial x^1}\,dx^1 + \frac{\partial h}{\partial x^2}\,dx^2\right)^2\right]^{1/2} \tag{C27}$$

Thus,

$$\frac{\partial s}{\partial x^1} = \left[(r)^2 + \left(\frac{\partial h}{\partial x^1}\right)^2\right]^{1/2} \tag{C28}$$

$$\frac{\partial s}{\partial x^2} = \left[\left(r \sin x^1\right)^2 + \left(\frac{\partial h}{\partial x^2}\right)^2\right]^{1/2} \tag{C29}$$

and

$$\frac{\partial}{\partial x^3}\left(\frac{\partial s}{\partial x^1}\right) = \frac{\partial}{\partial x^3}\left[\left(\frac{\partial h}{\partial x^1}\right)^2\right] \Bigg/ \left(2\sqrt{(r)^2 + \left(\frac{\partial h}{\partial x^1}\right)^2}\right) \tag{C30}$$

$$\frac{\partial}{\partial x^3}\left(\frac{\partial s}{\partial x^2}\right) = \frac{\partial}{\partial x^3}\left[\left(\frac{\partial h}{\partial x^2}\right)^2\right] \Bigg/ \left(2\sqrt{\left(r \sin x^1\right)^2 + \left(\frac{\partial h}{\partial x^2}\right)^2}\right) \tag{C31}$$

or, by use of equation (B3) for the metric tensor,

$$\frac{\partial}{\partial x^3}\left(\frac{\partial s}{\partial x^\alpha}\right) = \frac{\partial}{\partial x^3}\left[\left(\frac{\partial h}{\partial x^\alpha}\right)^2\right] \Bigg/ \left(2\sqrt{g_{\alpha\alpha}}\right) \tag{C32}$$

Thus, equations (C25) and (C26) become, respectively,

$$\left\{ {3 \atop 3} {\;\atop \alpha} \right\}^{3} = \frac{1}{\sqrt{g_{33}}} \frac{\partial}{\partial x^3}\left[\left(\frac{\partial h}{\partial x^\alpha}\right)^2\right] \Bigg/ \left(2\sqrt{g_{\alpha\alpha}}\right) \tag{C33}$$

$$\left\{ {3 \atop 3} {\;\atop 3} \right\}^{\alpha} = -\frac{\sqrt{g_{33}}}{g_{\alpha\alpha}} \frac{\partial}{\partial x^3}\left[\left(\frac{\partial h}{\partial x^\alpha}\right)^2\right] \Bigg/ \left(2\sqrt{g_{\alpha\alpha}}\right) \tag{C34}$$

Consider the covariant derivatives of a general contravariant vector T^i and of the corresponding covariant vector T_i where

$$T^i_{|j} = \frac{\partial T^i}{\partial x^j} + T^k \left\{ {\;\atop k} {i \atop \;} {\;\atop j} \right\}$$

which is equivalent to

$$T^\alpha_{|j} = \frac{\partial T^\alpha}{\partial x^j} + T^k \left\{ {\;\atop k} {\alpha \atop \;} {\;\atop j} \right\} \tag{C35}$$

$$T^{3}_{|j} = \frac{\partial T^{3}}{\partial x^{j}} + T^{k}\begin{Bmatrix} 3 \\ k \quad j \end{Bmatrix} \tag{C36}$$

Also,

$$T_{i|j} = \frac{\partial T_{i}}{\partial x^{j}} - T_{k}\begin{Bmatrix} k \\ i \quad j \end{Bmatrix}$$

which is equivalent to

$$T_{\alpha|j} = \frac{\partial T_{\alpha}}{\partial x^{j}} - T_{k}\begin{Bmatrix} k \\ \alpha \quad j \end{Bmatrix} \tag{C37}$$

$$T_{3|j} = \frac{\partial T_{3}}{\partial x^{j}} - T_{k}\begin{Bmatrix} k \\ 3 \quad j \end{Bmatrix} \tag{C38}$$

From equations (C18), (C35), and (C37),

$$T^{\alpha}_{|\beta} = \frac{\partial T^{\alpha}}{\partial x^{\beta}} + T^{\gamma}\begin{Bmatrix} \alpha \\ \gamma \quad \beta \end{Bmatrix} \tag{C39}$$

$$T_{\alpha|\beta} = \frac{\partial T_{\alpha}}{\partial x^{\beta}} - T_{\gamma}\begin{Bmatrix} \gamma \\ \alpha \quad \beta \end{Bmatrix} \tag{C40}$$

Thus, $T^{\alpha}_{|\beta}$ and $T_{\alpha|\beta}$ do not depend on $\begin{Bmatrix} 3 \\ 3 \quad \gamma \end{Bmatrix}$ or $\begin{Bmatrix} \gamma \\ 3 \quad 3 \end{Bmatrix}$ and are the same as in spherical polar coordinates.

From equations (C35) and (C18),

$$T^{\alpha}_{|3} = \frac{\partial T^{\alpha}}{\partial x^{3}} + T^{k}\begin{Bmatrix} \alpha \\ k \quad 3 \end{Bmatrix}$$

$$= \frac{\partial T^{\alpha}}{\partial x^{3}} + T^{3}\begin{Bmatrix} \alpha \\ 3 \quad 3 \end{Bmatrix} \tag{C41}$$

or, from equation (C34),

$$T^{\alpha}_{|3} = \frac{\partial T^{\alpha}}{\partial x^3} - \frac{T^3 \dfrac{\sqrt{g_{33}}}{g_{\alpha\alpha}} \dfrac{\partial}{\partial x^3}\left[\left(\dfrac{\partial h}{\partial x^{\alpha}}\right)^2\right]}{2\sqrt{g_{\alpha\alpha}}} \tag{C42}$$

Multiplying equation (C42) by $\sqrt{g_{\alpha\alpha}}/\sqrt{g_{33}}$ yields

$$\frac{\sqrt{g_{\alpha\alpha}}}{\sqrt{g_{33}}} T^{\alpha}_{|3} = \frac{\sqrt{g_{\alpha\alpha}}}{\sqrt{g_{33}}} \frac{\partial T^{\alpha}}{\partial x^3} - \frac{1}{2g_{\alpha\alpha}} T^3 \frac{\sqrt{g_{33}}}{\sqrt{g_{33}}} \frac{\partial}{\partial x^3}\left[\left(\frac{\partial h}{\partial x^{\alpha}}\right)^2\right]$$

$$\cong \frac{\partial \tilde{T}_{\alpha}}{\partial \tilde{z}_3} - \frac{1}{2}\tilde{T}_3 \frac{\partial}{\partial \tilde{z}_3}\left[\left(\frac{\partial h}{\partial \tilde{z}_{\alpha}}\right)^2\right] \tag{C43}$$

where $\tilde{T}_i$ and $d\tilde{z}_i$ are the physical components of T^j $\left(\text{or } T_j\right)$ and dx^j, respectively, and the approximate metric tensor given by equation (C9) is assumed so that $\tilde{T}_i = \sqrt{g_{ii}}\, T^i$ and $\frac{\partial}{\partial x^3}\left(\sqrt{g_{\alpha\alpha}}\right) = 0$. The second right-hand term in equation (C43) is the contribution of $\left\{ {}_{3}{}^{\alpha}{}_{3} \right\}$; thus, in equation (C41) for $T^{\alpha}_{|3}$, $\left\{ {}_{3}{}^{\alpha}{}_{3} \right\}$ can be neglected if the square of the physical slope of the constant x^3 surface $\left(\partial h/\partial \tilde{z}_{\alpha}\right)^2$ is negligible; or, from equation (C41) $\left(\text{to the order } \left(\partial h/\partial \tilde{z}_{\alpha}\right)^2\right)$

$$T^{\alpha}_{|3} = \frac{\partial T^{\alpha}}{\partial x^3} + T^3 \left\{ {}_{3}{}^{\alpha}{}_{3} \right\}$$

$$\cong \frac{\partial T^{\alpha}}{\partial x^3} \tag{C44}$$

which is also the order of approximation of the metric tensor by use of equation (C9), as discussed in appendix B. In the remainder of this appendix, the symbol $\cong$ denotes approximations accurate to the order $\left(\partial h/\partial \tilde{z}_{\alpha}\right)^2$.

From equations (C36) and (C18),

$$T^3_{|\alpha} = \frac{\partial T^3}{\partial x^\alpha} + T^3 \begin{Bmatrix} 3 \\ 3 \quad \alpha \end{Bmatrix} \tag{C45}$$

or

$$T^3_{|\alpha} \cong \frac{\partial \tilde{T}_3}{\partial x^\alpha} \frac{1}{\sqrt{g_{33}}} + \tilde{T}_3 \frac{\partial}{\partial x^\alpha}\left(\frac{1}{\sqrt{g_{33}}}\right) + T^3 \begin{Bmatrix} 3 \\ 3 \quad \alpha \end{Bmatrix} \tag{C46}$$

where, again,

$$d\tilde{z}_i = \sqrt{g_{ii}}\, dx^i \tag{C47}$$

$$\tilde{T}_i = \sqrt{g_{ii}}\, T^i \cong \frac{1}{\sqrt{g_{ii}}}\, T_i \tag{C48}$$

with equation (C48) being exact for $i = 3$. Expanding the second right-hand term in equation (C46) gives

$$\tilde{T}_3 \frac{\partial}{\partial x^\alpha}\left(\frac{1}{\sqrt{g_{33}}}\right) = -\frac{1}{2} \tilde{T}_3 \frac{1}{g_{33}{}^{3/2}} \frac{\partial g_{33}}{\partial x^\alpha}$$

$$= -T^3 \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^\alpha} = -T^3 \begin{Bmatrix} 3 \\ 3 \quad \alpha \end{Bmatrix}$$

by use of equation (C22) for $\begin{Bmatrix} 3 \\ 3 \quad \alpha \end{Bmatrix}$. Therefore, equation (C46) becomes

$$T^3_{|\alpha} \cong \frac{1}{\sqrt{g_{33}}} \frac{\partial \tilde{T}_3}{\partial x^\alpha} \tag{C49}$$

which is equation (C45) for negligible $\begin{Bmatrix} 3 \\ 3 \quad \alpha \end{Bmatrix}$ and $\partial g_{33}/\partial x^\alpha$. From equations (C36) and (C33),

$$T^3_{|3} = \frac{\partial T^3}{\partial x^3} + T^\gamma \left\{ {}_{\gamma}{}^{3}{}_{3} \right\} + T^3 \left\{ {}_{3}{}^{3}{}_{3} \right\}$$

$$= \frac{\partial T^3}{\partial x^3} + T^\gamma \frac{1}{\sqrt{g_{33}}} \frac{\partial}{\partial x^3} \frac{\left[\left(\frac{\partial h}{\partial x^\alpha}\right)^2\right]}{2\sqrt{g_{\alpha\alpha}}} + T^3 \left\{ {}_{3}{}^{3}{}_{3} \right\} \tag{C50}$$

or, by using equations (C47), (C48), and (C18),

$$T^3_{|3} \cong \frac{\partial}{\partial x^3}\left(\frac{\tilde{T}_3}{\sqrt{g_{33}}}\right) + \frac{\tilde{T}_\gamma}{\sqrt{g_{\gamma\gamma}}} \frac{\partial}{\partial \tilde{z}_3} \frac{\left[\left(\frac{\partial h}{\partial x^\alpha}\right)^2\right]}{2\sqrt{g_{\alpha\alpha}}} + T^3 \left\{ {}_{3}{}^{3}{}_{3} \right\}$$

$$= \frac{\partial}{\partial \tilde{z}_3}\left(\tilde{T}_3\right) - \tilde{T}_3 \frac{1}{2\left(g_{33}\right)^{3/2}} \frac{\partial g_{33}}{\partial x^3}$$

$$+ \tilde{T}_\gamma \frac{1}{2} \frac{\partial}{\partial \tilde{z}_3}\left[\left(\frac{\partial h}{\partial \tilde{z}_\alpha}\right)^2\right] + \tilde{T}_3 \frac{1}{2\left(g_{33}\right)^{3/2}} \frac{\partial g_{33}}{\partial x^3}$$

$$= \frac{\partial}{\partial \tilde{z}_3}\left(\tilde{T}_3\right) + \tilde{T}_\gamma \frac{1}{2} \frac{\partial}{\partial \tilde{z}_3}\left[\left(\frac{\partial h}{\partial \tilde{z}_\alpha}\right)^2\right]$$

$$= \frac{\partial}{\partial \tilde{z}_3}\left(\tilde{T}_3\right) + T^\gamma \left\{ {}_{\gamma}{}^{3}{}_{3} \right\} \tag{C51}$$

Thus, by use of equation (C50),

$$T^3_{|3} \cong \frac{\partial T^3}{\partial x^3} + T^3 \left\{ {}_{3}{}^{3}{}_{3} \right\} \tag{C52}$$

to the order of $\left(\partial h/\partial \tilde{z}_\alpha\right)^2$; and, furthermore, by use of equations (C51) and (C47),

$$T^3_{|3} \cong \frac{\partial \tilde{T}_3}{\partial \tilde{z}_3} = \frac{1}{\sqrt{g_{33}}} \frac{\partial \tilde{T}_3}{\partial x^3} \tag{C53}$$

to the order of $\left(\partial h / \partial \tilde{z}_\alpha\right)^2$.

In summary, in the computation of the covariant derivatives of contravariant vectors $T^i_{|j}$, $\partial g_{33}/\partial x^\alpha$, $\left\{ {3 \atop 3} \, {\atop \alpha} \right\}$, and $\left\{ {\alpha \atop 3} \, {\atop 3} \right\}$ can be neglected with errors no greater than the order of the square of the physical slope of the constant x^3 surface $\left(\partial h / \partial \tilde{z}_\alpha\right)^2$. By use of equation (C40), $T_{\alpha|\beta}$ is not dependent on $\partial g_{33}/\partial x^\gamma$, $\left\{ {3 \atop 3} \, {\atop \gamma} \right\}$, or $\left\{ {\gamma \atop 3} \, {\atop 3} \right\}$. Now, in considering $T_{3|\alpha}$, $T_{\alpha|3}$, and $T_{3|3}$,

$$T_{i|j} = T^k_{|j} \, g_{ik} \cong T^i_{|j} \, g_{ii} \tag{C54}$$

by use of equation (C9) for the approximate metric tensor. Then, by use of equations (C49) and (C54),

$$T_{3|\alpha} \cong T^3_{|\alpha} \, g_{33} \cong \frac{\partial \tilde{T}_3}{\partial x^\alpha} \sqrt{g_{33}} \tag{C55}$$

According to equations (C38) and (C18),

$$T_{3|\alpha} = \frac{\partial T_3}{\partial x^\alpha} - T_3 \left\{ {3 \atop 3} \, {\atop \alpha} \right\} \tag{C56}$$

or, by equation (C48),

$$T_{3|\alpha} = \frac{\partial}{\partial x^\alpha} \left(\sqrt{g_{33}} \, \tilde{T}_3 \right) - T_3 \left\{ {3 \atop 3} \, {\atop \alpha} \right\} \tag{C57}$$

which is equation (C55) for negligible $\partial g_{33}/\partial x^\alpha$ and $\left\{ {3 \atop 3} \, {\atop \alpha} \right\}$. Therefore, equation (C56) can be written as

$$T_{3|\alpha} \cong \frac{\partial T_3}{\partial x^\alpha} \tag{C58}$$

From equations (C37) and (C18),

$$T_{\alpha|3} = \frac{\partial T_\alpha}{\partial x^3} - T_3 \begin{Bmatrix} 3 \\ \alpha \quad 3 \end{Bmatrix}$$

or, by use of equations (C56) and (C58),

$$T_{\alpha|3} = \frac{\partial T_\alpha}{\partial x^3} + T_{3|\alpha} - \frac{\partial T_3}{\partial x^\alpha} \cong \frac{\partial T_\alpha}{\partial x^3} \tag{C59}$$

From equation (C38),

$$T_{3|3} = \frac{\partial T_3}{\partial x^3} - T_\gamma \begin{Bmatrix} \gamma \\ 3 \quad 3 \end{Bmatrix} - T_3 \begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix} \tag{C60}$$

From equations (C54) and (C52),

$$T_{3|3} \cong T^3_{\ |3}\, g_{33} \cong \frac{\partial T^3}{\partial x^3} g_{33} + T^3 \begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix} g_{33}$$

or, by use of equation (C48),

$$T_{3|3} \cong \frac{\partial T^3}{\partial x^3} g_{33} + T_3 \begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix}$$

$$= \frac{\partial T_3}{\partial x^3} + T_3 g_{33} \frac{\partial}{\partial x^3}\left(\frac{1}{g_{33}}\right) + T_3 \begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix}$$

$$= \frac{\partial T_3}{\partial x^3} - T_3\left(\frac{1}{g_{33}}\right)\frac{\partial g_{33}}{\partial x^3} + T_3 \begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix}$$

or, by use of equations (C18),

$$T_{3|3} \cong \frac{\partial T_3}{\partial x^3} - T_3 \begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix} \tag{C61}$$

Thus, $\begin{Bmatrix} \gamma \\ 3 \quad 3 \end{Bmatrix}$ can be neglected in equation (C60).

In summary, in the computation of covariant derivatives of contravariant and covariant vectors, $\left\{ {3 \atop 3\ \alpha} \right\}$ and $\left\{ {\alpha \atop 3\ 3} \right\}$ can be neglected with errors no greater than the order of the square of the physical slope of the constant x^3 surface $\left(\partial h / \partial \tilde{z}_\alpha\right)^2$.

For the nearly spherical polar coordinate system with the exact metric tensor given by equation (B3), the nonnegligible Christoffel symbols are

$$\left.\begin{aligned} \left\{ {1 \atop 2\ 2} \right\} &= -\sin x^1 \cos x^1 \\ \left\{ {2 \atop 1\ 2} \right\} &= \left\{ {2 \atop 2\ 1} \right\} = \cot x^1 \\ \left\{ {3 \atop 3\ 3} \right\} &= \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial x^3} \end{aligned}\right\} \tag{C62}$$

and

$$\frac{\partial g_{33}}{\partial x^\alpha} \cong 0 \tag{C63}$$

for computation of covariant derivatives of contravariant or covariant vectors.

APPENDIX D

INTERPRETATION OF THE FINE-STRUCTURE INFORMATION AS A SUBGRID MIXING TENSOR

In appendix A the effects of uncertainties in the state of the fluid on mean fluid properties are described. In the present appendix, the effects of mean fluid properties on uncertainties in the state of the fluid are modeled.

The approximation of the transport equations by finite difference equations neglects some fine-structure information about the fluid state. In an effort to recover the mutual coupling between the uncertain fine-structure fluid properties and the averaged (over a space-time gridcell) fluid properties, the pertinent fine-structure properties are expressed in terms of mean fluid properties by means of a tensor formulation of the standard subgrid mixing model (ref. 1). The tensor formulation of subgrid mixing corrects a deficiency in the standard treatment in allowing rigid-body rotation of the atmosphere under adiabatic conditions. The subgrid correlation tensor $\left\langle u^{j'} u_i' \right\rangle$ in equation (C8) will be approximated from the strain-rate tensor as in reference 3. For the global scale circulation, Smagorinsky has modeled the elements of $\left\langle u^{j'} u_i' \right\rangle$ in the horizontal space. The simplest tensor analogy for Smagorinsky's model is described by letting the strain-rate tensor be

$$\ell_{jk} = u_{j|k} + u_{k|j} \tag{D1}$$

the deviatoric strain-rate tensor be

$$\hat{\ell}_{jk} = \ell_{jk} - \frac{1}{3}\, \ell_P^P\, g_{jk} \tag{D2}$$

and the subgrid-velocity-correlation tensor be

$$\rho\left\langle u_j' u_k' \right\rangle = -\rho\left[\sqrt{2}\left(K_o\, \Delta\right)^2 \hat{\ell}\right]\hat{\ell}_{jk} \tag{D3}$$

where K_o is the Von Karman constant, Δ is the coordinate increment, and

$$\hat{\ell} = \left(\hat{\ell}_m^p \hat{\ell}_p^m\right)^{1/2} \tag{D4}$$

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Consider next the subgrid transport term $\langle u^{j\prime}Q'\rangle$, where the tensor analogy to Smagorinsky's model is

$$\rho\langle u_j'Q'\rangle = -\rho\left[\sqrt{2}\left(K_o\ \Delta\right)^2 \hat{\ell}\right]Q_{|j} \tag{D5}$$

The deviatoric strain-rate tensor can now be expanded explicitly.

From equation (D1) for the three-dimensional case, ℓ_{jk} is given by

$$\ell_{jk} = \begin{bmatrix} 2u_{1|1} & \left(u_{1|2} + u_{2|1}\right) & \left(u_{1|3} + u_{3|1}\right) \\ \left(u_{2|1} + u_{1|2}\right) & 2u_{2|2} & \left(u_{2|3} + u_{3|2}\right) \\ \left(u_{3|1} + u_{1|3}\right) & \left(u_{3|2} + u_{2|3}\right) & 2u_{3|3} \end{bmatrix} \tag{D6}$$

which, from equations (C62) applied to equation (D2), expands to

$$\hat{\ell}^{1}{}_{.1} = \frac{4}{3}\,\frac{U_{1,1}}{\sqrt{g_{22}}} - \frac{2}{3}\left(\frac{U_{2,2}}{\sqrt{g_{22}}} - \frac{U_{3,3}}{\sqrt{g_{33}}}\right) - \frac{2}{3}\cot x^1\left(\frac{U_1}{\sqrt{g_{11}}}\right) \tag{D7a}$$

$$\hat{\ell}^{1}{}_{.2} = \frac{U_{1,2}}{\sqrt{g_{11}}} + \frac{\left(U_2\sqrt{g_{22}}\right)_{,1}}{g_{22}}\left(\sin x^1\right)^2 - 2\sin x^1\cos x^1\left(\frac{U_2}{\sqrt{g_{22}}}\right) \tag{D7b}$$

$$\hat{\ell}^{2}{}_{.1} = \frac{1}{\left(\sin x^1\right)^2}\,\hat{\ell}^{1}{}_{.2} \tag{D7c}$$

$$\hat{\ell}^{1}{}_{.3} = \frac{U_{1,3}}{\sqrt{g_{11}}} + \frac{\left(U_3\sqrt{g_{33}}\right)_{,1}}{g_{33}}\left(\frac{g_{33}}{g_{11}}\right) \tag{D7d}$$

$$\hat{\ell}^{3}{}_{.1} = \left(\frac{g_{11}}{g_{33}}\right)\hat{\ell}^{1}{}_{.3} \tag{D7e}$$

$$\hat{\ell}^{2}_{.2} = \frac{4}{3}\frac{U_{2,2}}{\sqrt{g_{22}}} - \frac{2}{3}\left(\frac{U_{1,1}}{\sqrt{g_{11}}} + \frac{U_{3,3}}{\sqrt{g_{33}}}\right) + \frac{4}{3}\cot x^1\left(\frac{U_1}{\sqrt{g_{11}}}\right) \tag{D7f}$$

$$\hat{\ell}^{2}_{.3} = \frac{U_{2,3}}{\sqrt{g_{22}}} + \frac{\left(U_3\sqrt{g_{33}}\right)_{,2}}{g_{33}}\left(\frac{g_{33}}{g_{22}}\right) \tag{D7g}$$

$$\hat{\ell}^{3}_{.2} = \hat{\ell}^{2}_{.3}\left(\frac{g_{22}}{g_{33}}\right) \tag{D7h}$$

$$\hat{\ell}^{3}_{.3} = \frac{4}{3}\frac{U_{3,3}}{\sqrt{g_{33}}} - \frac{2}{3}\left(\frac{U_{1,1}}{\sqrt{g_{11}}} + \frac{U_{2,2}}{\sqrt{g_{22}}}\right) - \frac{2}{3}\cot x^1\left(\frac{U_1}{\sqrt{g_{11}}}\right) \tag{D7i}$$

The hydrostatic assumption neglects unbalanced forces in the quasi-vertical direction x^3. In order to specialize equations (D7) and (D3) to the quasi-two-dimensional case consistent with the hydrostatic approximation, the following conditions are imposed:

(1) The quasi-vertical transport of subgrid quasi-vertical momentum $\rho\left\langle u^{3'}u_3'\right\rangle$, or equivalently $\hat{\ell}^{3}_{.3}$, is neglected.

(2) The quasi-horizontal shear of quasi-vertical velocity is negligible compared with the quasi-vertical shear of quasi-horizontal velocity, or $\sqrt{\frac{1}{g_{\alpha\alpha}}}\,U_{3,\alpha} << \sqrt{\frac{1}{g_{33}}}\,U_{\alpha,3}$. Equations (D7) for $\hat{\ell}^{i}_{.j}$ become

$$\begin{aligned}\hat{\ell}^{1}_{.1} &= \frac{U_{1,1}}{\sqrt{g_{11}}} - \frac{U_{2,2}}{\sqrt{g_{22}}} - \cot x^1\left(\frac{U_1}{\sqrt{g_{11}}}\right)\\ &= \frac{1}{\sqrt{g_{22}}}\left[\left(U_1 \sin x^1\right)_{,1} - U_{2,2} - 2\cos x^1\, U_1\right]\end{aligned} \tag{D8a}$$

$$\hat{\ell}^1_{.2} = \frac{U_{1,2}}{\sqrt{g_{11}}} + \frac{1}{g_{22}}\left(U_2\sqrt{g_{22}}\right)_{,1}\left(\sin x^1\right)^2 - 2 \sin x^1 \cos x^1\left(\frac{U_2}{\sqrt{g_{22}}}\right)$$

$$= \frac{\sin x^1}{\sqrt{g_{22}}}\left[U_{1,2} + \left(U_2 \sin x^1\right)_{,1} - 2 \cos x^1 \, U_2\right] \qquad \text{(D8b)}$$

$$\hat{\ell}^2_{.1} = \frac{1}{\left(\sin x^1\right)^2}\hat{\ell}^1_{.2} \qquad \text{(D8c)}$$

$$\hat{\ell}^2_{.2} = -\hat{\ell}^1_{.1} \qquad \text{(D8d)}$$

$$\hat{\ell}^1_{.3} \cong \frac{U_{1,3}}{\sqrt{g_{11}}} \qquad \text{(D8e)}$$

$$\hat{\ell}^3_{.1} = \left(\frac{g_{11}}{g_{33}}\right)\hat{\ell}^1_{.3} \qquad \text{(D8f)}$$

$$\hat{\ell}^2_{.3} = \frac{U_{2,3}}{\sqrt{g_{22}}} \qquad \text{(D8g)}$$

$$\hat{\ell}^3_{.2} = \left(\frac{g_{22}}{g_{33}}\right)\hat{\ell}^2_{.3} \qquad \text{(D8h)}$$

The quantities D_T and D_S from reference 3 are related to $\hat{\ell}^1_{.1}$ and $\hat{\ell}^1_{.2}$ by

$$\left.\begin{aligned} &\hat{\ell}^1_{.1} + 2 \cos x^1 \frac{U_1}{\sqrt{g_{22}}} = D_T \\ \text{and} \quad &\frac{\hat{\ell}^1_{.2}}{\sin x^1} + 2 \cos x^1 \frac{U_2}{\sqrt{g_{22}}} = D_S \end{aligned}\right\} \qquad \text{(D9)}$$

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In Smagorinsky's subgrid mixing model, the second left-hand terms in equations (D9) cause rigid-body rotation stresses, contrary to physical principles. The second scalar of $\hat{\ell}^{i}_{.j}$ is taken in the tangent plane in equation (D3) for the quasi-two-dimensional case. Thus,

$$\hat{\ell} = \left(\hat{\ell}^{1}_{.1}\hat{\ell}^{1}_{.1} + \hat{\ell}^{1}_{.2}\hat{\ell}^{2}_{.1} + \hat{\ell}^{2}_{.1}\hat{\ell}^{1}_{.2} + \hat{\ell}^{2}_{.2}\hat{\ell}^{2}_{.2}\right)^{1/2} \tag{D10}$$

For rigid-body rotation west to east, $U_1 = 0$, $U_2 = c \sin x^1$, and $U_3 = 0$, where c is a constant. Substitution of these velocities into equations (D7) for the three-dimensional case and into equations (D8) for the quasi-two-dimensional case results in, respectively, $\hat{\ell}^{i}_{.j} = 0$ and $\hat{\ell}^{\alpha}_{.\beta} = 0$; thus, $\hat{\ell}$ and the subgrid velocity-correlation (or Reynolds stress) tensor vanish, by using equations (D3), (D4), and (D10) for both cases.

APPENDIX E

APPLICATION OF TENSOR TRANSPORT EQUATIONS TO THREE-DIMENSIONAL MOMENTUM AND SCALAR CONSERVATION EQUATIONS IN PHYSICAL COMPONENTS

In this appendix the tensor transport equations and associated source terms are converted into the more familiar physical components. The conversion begins with the mass transport equation (C5) which is given as follows:

$$\frac{\partial}{\partial t}\left(\sqrt{A}\ \zeta\right) + \frac{\partial}{\partial x^j}\left(\sqrt{A}\ \zeta v^j\right) = 0 \tag{E1}$$

Since A is not a function of time and v^j can be converted to V_j merely by multiplication by $\sqrt{g_{jj}}$, equation (E1) simplifies to

$$\frac{\partial \zeta}{\partial t} + \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\ \zeta\ \frac{V_j}{\sqrt{g_{jj}}}\right) = 0 \tag{E2}$$

The next transport equation from appendix C to be considered is the internal-energy equation. The first law of thermodynamics states that the increase in internal energy of a system results from heat being added directly to the system and/or from work being done on the system. If the heat added to a unit of mass m is $\bar{Q}$, and the work done on m is the pressure-volume work of compression $-\frac{P}{\rho}u^j_{|j}$, then the conservation equation for internal energy is, from equation (C6),

$$\frac{\partial}{\partial t}\left(\sqrt{A}\ \zeta E\right) + \frac{\partial}{\partial x^j}\left(\sqrt{A}\ \zeta E v^j\right) = \hat{S}\left(\sqrt{A}\ \zeta E\right) - \frac{\partial}{\partial x^j}\left(\sqrt{A}\ \zeta \langle E' u^{j'} \rangle\right) \tag{E3}$$

where

$$\hat{S}\left(\sqrt{A}\ \zeta E\right) = \sqrt{A}\ \zeta \bar{Q} - \sqrt{A}\sqrt{g_{33}}\ P u^j_{|j} \tag{E4}$$

and where $\bar{Q}$ includes radiation heating as well as latent heat release from condensation of water.

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The conversion of equations (E3) and (E4) consists of dividing by $\sqrt{A}$, expanding $u^j_{|j}$ to $\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^j}\left(\sqrt{a}\,u^j\right)$, and replacing v^j and u^j by $V_j/\sqrt{g_{jj}}$ and $U_j/\sqrt{g_{jj}}$, respectively, to get

$$\frac{\partial}{\partial t}(\zeta E) + \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta E\,\frac{V_j}{\sqrt{g_{jj}}}\right) = \frac{1}{\sqrt{A}}\,\hat{S}\left(\sqrt{A}\,\zeta E\right) - \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle E'\,\frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\right) \tag{E5}$$

where

$$\frac{1}{\sqrt{A}}\,\hat{S}\left(\sqrt{A}\,\zeta E\right) = \zeta\bar{Q} - \frac{P}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{g_{33}}\,\sqrt{A}\,\frac{U_j}{\sqrt{g_{jj}}}\right)$$

For the atmosphere, the specific humidity conservation equation has rainfall as the negative of its source term $\hat{S}\left(\sqrt{A}\,\zeta q\right)$; from equation (C6),

$$\frac{\partial}{\partial t}(\zeta q) + \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta q\,\frac{V_j}{\sqrt{g_{jj}}}\right) = \frac{1}{\sqrt{A}}\,\hat{S}\left(\sqrt{A}\,\zeta q\right) - \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle q'\,\frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\right) \tag{E6}$$

Consider next, the momentum equations (C7) and (C8)

$$\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta u_i\right) + \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta u_i v^j\right) = \hat{S}\left(\sqrt{A}\,\zeta u_i\right) - \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle u_i' u^{j\prime}\right\rangle\right) \tag{E7}$$

and

$$\hat{S}\left(\sqrt{A}\,\zeta u_i\right) = \sqrt{A}\left[-\sqrt{g_{33}}\,P_{|i} + \zeta g_i - \zeta\left(a_c\right)_i + \zeta s^j_{|i}u_j - \left\{{}_i{}^m{}_j\right\}\,\zeta\left(v^j u_m + \left\langle u^{j\prime} u_m'\right\rangle\right)\right] \tag{E8}$$

From equations (C62), the decomposition of the Christoffel symbols into the horizontal and vertical spaces yields

$$\left\{{}_i{}^m{}_j\right\} = \left\{{}_\beta{}^\alpha{}_\gamma\right\}\ \left\{{}_3{}^3{}_3\right\} \tag{E9}$$

where only g_{33} in $\left\{ {3 \atop 3}{}^{3}{}_{3} \right\}$ is a function of time. The term $\sqrt{A}\,\zeta s^{j}_{|i}u_j$ in equation (E8) expands to

$$\sqrt{A}\,\zeta s^{j}_{|i}u_j = \sqrt{A}\,\zeta\left(\frac{\partial s^j}{\partial x^i}u_j + s^k\left\{{}_{k}{}^{j}{}_{i}\right\}u_j\right)$$

$$= \sqrt{A}\,\zeta\left(\frac{\partial s^\alpha}{\partial x^i}u_\alpha + s^\beta\left\{{}_{\beta}{}^{\alpha}{}_{i}\right\}u_\alpha + s^3\left\{{}_{3}{}^{\alpha}{}_{i}\right\}u_\alpha\right.$$

$$\left. + \frac{\partial s^3}{\partial x^i}u_3 + s^\beta\left\{{}_{\beta}{}^{3}{}_{i}\right\}u_3 + s^3\left\{{}_{3}{}^{3}{}_{i}\right\}u_3\right)$$

$$= \sqrt{A}\,\zeta\left(\frac{\partial s^3}{\partial x^i}u_3 + s^3\left\{{}_{3}{}^{3}{}_{i}\right\}u_3\delta_i{}^3\right) \tag{E10}$$

because $s^1 = s^2 = 0$ everywhere, $\left\{{}_{3}{}^{\alpha}{}_{i}\right\} \cong 0$, and $\left\{{}_{3}{}^{3}{}_{i}\right\} \cong 0$ for $i \neq 3$ (by use of eqs. (C62)). For the u_3 momentum component, the first left-hand term in equation (E7) becomes

$$\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta u_3\right) = \frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta U_3\sqrt{g_{33}}\right)$$

$$= \sqrt{g_{33}}\,\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta U_3\right) + \sqrt{A}\,\zeta U_3\,\frac{\partial}{\partial t}\left(\sqrt{g_{33}}\right)$$

$$= \sqrt{g_{33}}\,\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta U_3\right) + \sqrt{A}\,\zeta U_3\sqrt{g_{33}}\,s^{3}_{|3}$$

which follows from $\left(\text{defining the physical increment } d\tilde{z}_3 = \sqrt{g_{33}}\,dx^3\right)$

$$\frac{\partial}{\partial t}\left(\sqrt{g_{33}}\right) = \frac{\partial}{\partial t}\left(\frac{\partial\tilde{z}_3}{\partial x^3}\right) = \frac{\partial}{\partial x^3}\left(\frac{\partial\tilde{z}_3}{\partial t}\right) = \sqrt{g_{33}}\,\frac{\partial}{\partial\tilde{z}_3}\left(S_3\right) = \sqrt{g_{33}}\,s^{3}_{|3} \tag{E11}$$

where the last step follows from

$$s^3_{|3} = \frac{\partial s^3}{\partial x^3} + s^j\begin{Bmatrix} 3 \\ j \quad 3 \end{Bmatrix} = \frac{\partial s^3}{\partial x^3} + s^3\begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix} = \frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial x^3}\left(\sqrt{g_{33}}\, s^3\right) = \frac{\partial S_3}{\partial \tilde{z}_3} \tag{E12}$$

Expanding further, by use of equation (E10),

$$\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta u_3\right) = \sqrt{g_{33}}\,\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta U_3\right) + \sqrt{A}\,\zeta\left(\frac{\partial s^3}{\partial x^3}\,u_3 + s^k\begin{Bmatrix} 3 \\ k \quad 3 \end{Bmatrix}\,u_3\right)$$

$$= \sqrt{g_{33}}\,\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta U_3\right) + \sqrt{A}\,\zeta\left(\frac{\partial s^3}{\partial x^3}\,u_3 + s^3\begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix}\,u_3\right)$$

$$= \sqrt{g_{33}}\,\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta U_3\right) + \sqrt{A}\,\zeta s^j_{|3}\,u_j \tag{E13}$$

Then, the third momentum equation in physical components is, from equations (E7) to (E10) and (E13),

$$\sqrt{g_{33}}\,\frac{\partial}{\partial t}\left(\sqrt{A}\,\zeta U_3\right) + \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta U_3\sqrt{g_{33}}\,\frac{V_j}{\sqrt{g_{jj}}}\right)$$

$$= \sqrt{A}\left[-\sqrt{g_{33}}\,P_{|3} + \zeta g_3 - \zeta\left(a_c\right)_3\right.$$

$$\left. + \begin{Bmatrix} 3 \\ 3 \quad 3 \end{Bmatrix}\,\zeta\left(V_3U_3 + \left\langle U_3'U_3'\right\rangle\right)\right]$$

$$- \frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle U_3'\,\frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\sqrt{g_{33}}\right) \tag{E14}$$

The second term on the left-hand side of equation (E14) expands to give

$$\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\sqrt{g_{33}}\,U_3\,\frac{V_j}{\sqrt{g_{jj}}}\right) = \sqrt{g_{33}}\,\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta U_3\,\frac{V_j}{\sqrt{g_{jj}}}\right)$$

$$+ \sqrt{A}\,\zeta U_3\,\frac{V_j}{\sqrt{g_{jj}}}\,\frac{1}{2\sqrt{g_{33}}}\,\frac{\partial}{\partial x^j}\left(g_{33}\right) \tag{E15}$$

From equation (C63), the first two terms of the second summation on the right-hand side of equation (E15) are negligible; thus, from equations (C62),

$$\sqrt{A}\,\zeta U_3\,\frac{V_3}{\sqrt{g_{33}}}\,\frac{1}{2\sqrt{g_{33}}}\,\frac{\partial}{\partial x^3}\left(g_{33}\right) = \sqrt{A}\,\zeta U_3 V_3\left\{{}_{3}{}^{3}{}_{3}\right\} \tag{E16}$$

Thus, the right-hand term in equation (E14)

$$\sqrt{A}\left\{{}_{3}{}^{3}{}_{3}\right\}\,\zeta\left(V_3 U_3\right)$$

can be canceled with the second term of the expansion (E15) of the second left-hand term of equation (E14). Similarly, cancellations can be made for the following term in equation (E14):

$$\sqrt{A}\left\{{}_{3}{}^{3}{}_{3}\right\}\,\zeta\left(\left\langle U_3' U_3'\right\rangle\right)$$

and a portion of an expansion (following the format of eq. (E15)) of the right-hand term

$$-\frac{\partial}{\partial x^j}\,\sqrt{A}\,\zeta\left\langle U_3'\,\frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\sqrt{g_{33}}$$

to leave, after division of equation (E14) by $\sqrt{g_{33}}\sqrt{A}$,

$$\frac{\partial}{\partial t}\left(\zeta U_3\right) + \frac{1}{\sqrt{A}}\,\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta U_3\,\frac{V_j}{\sqrt{g_{jj}}}\right) = -P|_3 + \zeta\,\frac{g_3}{\sqrt{g_{33}}} - \zeta\,\frac{\left(a_c\right)_3}{\sqrt{g_{33}}} - \frac{1}{\sqrt{A}}\,\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle U_3'\,\frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\right) \tag{E17}$$

Thus,

$$\frac{1}{\sqrt{A}}\hat{S}\left(\sqrt{A}\,\zeta U_3\right) = -P_{|3} + \zeta\frac{g_3}{\sqrt{g_{33}}} - \zeta\frac{\left(a_c\right)_3}{\sqrt{g_{33}}} - \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle U_3'\,\frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\right) \tag{E18}$$

For the momentum component along dx^1, $\sqrt{g_{11}} = R$ is constant. Substituting $U_1 = u_1/\sqrt{g_{11}}$ into equations (E7) and (E8) yields, by use of equations (E10) and (C63),

$$\frac{\partial}{\partial t}\left(\zeta U_1\right) + \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta U_1\frac{V_j}{\sqrt{g_{jj}}}\right) = \frac{1}{\sqrt{A}}\hat{S}\left(\sqrt{A}\,\zeta U_1\right) \tag{E19}$$

and

$$\begin{aligned}\frac{1}{\sqrt{A}}\hat{S}\left(\sqrt{A}\,\zeta U_1\right) = &-\frac{\sqrt{g_{33}}}{\sqrt{g_{11}}}\frac{\partial P}{\partial x^1} + \frac{\zeta g_1}{\sqrt{g_{11}}} - \zeta\frac{\left(a_c\right)_1}{\sqrt{g_{11}}} + \frac{\zeta}{\sqrt{g_{11}}}\frac{\partial S_3}{\partial x^1}U_3\\ &+ \frac{\sqrt{g_{\alpha\alpha}}}{\sqrt{g_{11}}\sqrt{g_{\beta\beta}}}\left\{{}_1{}^{\alpha}{}_{\beta}\right\}\zeta\left(V_\beta V_\alpha + \left\langle U_\beta' U_\alpha'\right\rangle\right)\\ &- \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle U_1'\,\frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\right)\end{aligned} \tag{E20}$$

Equation (E20) can be written more explicitly by using the result of appendix C; that is, of $\left\{{}_1{}^{\alpha}{}_{\beta}\right\}$, only $\left\{{}_1{}^{2}{}_{2}\right\} = \cot x^1$ is nonzero. Thus, equation (E20) reduces to

$$\begin{aligned}\frac{1}{\sqrt{A}}\hat{S}\left(\sqrt{A}\,\zeta U_1\right) = &-\frac{\sqrt{g_{33}}}{\sqrt{g_{11}}}\frac{\partial P}{\partial x^1} + \frac{\zeta g_1}{\sqrt{g_{11}}} - \frac{\zeta\left(a_c\right)_1}{\sqrt{g_{11}}} + \frac{\zeta}{\sqrt{g_{11}}}\frac{\partial S_3}{\partial x^1}U_3\\ &- \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle U_1'\,\frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\right) + \frac{\zeta}{\sqrt{g_{11}}}\cot x^1\left(V_2U_2 + \left\langle U_2'U_2'\right\rangle\right)\end{aligned} \tag{E21}$$

Conversion of the momentum component along dx^2 is accomplished in a similar manner, except that now extra terms are obtained from the partial derivatives in the horizontal direction because $U_2 = u_2/\sqrt{g_{22}}$ and $\sqrt{g_{22}} = r \sin x^1$. When the substitution for u_2 has been made in equations (E7) and (E8) and the results converted again to the form of equation (E7), there follows, from equations (E10) and (C63),

$$\frac{\partial}{\partial t}\left(\zeta U_2\right) + \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\left(\sqrt{A}\,\zeta U_3 \frac{V_j}{\sqrt{g_{jj}}}\right) = \frac{1}{\sqrt{A}}\hat{S}\left(\sqrt{A}\,\zeta U_2\right) \tag{E22}$$

and

$$\begin{aligned}\frac{1}{\sqrt{A}}\hat{S}\left(\sqrt{A}\,\zeta U_2\right) = &-\frac{\sqrt{g_{33}}}{\sqrt{g_{22}}}\frac{\partial P}{\partial x^2} + \frac{\zeta g_2}{\sqrt{g_{22}}} - \frac{\zeta\left(a_c\right)_2}{\sqrt{g_{22}}}\\ &+ \frac{\sqrt{g_{\alpha\alpha}}}{\sqrt{g_{\beta\beta}}\sqrt{g_{22}}}\left\{{2}\,{}^{\alpha}\,{\beta}\right\}\zeta\left(V_\beta U_\alpha + \left\langle U_\beta' U_\alpha'\right\rangle\right)\\ &+ \frac{\zeta}{\sqrt{g_{22}}}\frac{\partial S_3}{\partial x^2}U_3 - \frac{1}{\sqrt{A}}\frac{\partial}{\partial x^j}\frac{1}{\partial x^j}\left(\sqrt{A}\,\zeta\left\langle U_2' \frac{U_j'}{\sqrt{g_{jj}}}\right\rangle\right)\\ &- \frac{\frac{\partial}{\partial x^1}\left(\sqrt{g_{22}}\right)}{\sqrt{g_{22}}\sqrt{g_{11}}}\zeta\left(U_2V_1 + \left\langle U_2'U_1'\right\rangle\right)\end{aligned} \tag{E23}$$

Equation (E23) can be simplified further by expanding $\left\{{2}\,{}^{\alpha}\,{\beta}\right\}$ and noticing from appendix C that

$$\left\{{2}\,{}^{1}\,{2}\right\} = -\sin x^1 \cos x^1$$

and

$$\left\{{2}\,{}^{2}\,{1}\right\} = \cot x^1$$

are the nonvanishing terms. When these Christoffel symbols are substituted into equation (E23), the term containing $\left\{ {2 \atop 2\;1} \right\}$ cancels the last term and the simplified result is

$$\frac{1}{\sqrt{A}} \hat{S}\left(\sqrt{A}\, \zeta U_2\right) = -\frac{\sqrt{g_{33}}}{\sqrt{g_{22}}} \frac{\partial P}{\partial x^2} + \frac{\zeta g_2}{\sqrt{g_{22}}} - \frac{\zeta (a_c)_2}{\sqrt{g_{22}}} + \frac{\zeta}{\sqrt{g_{22}}} \frac{\partial S_3}{\partial x^2} U_3$$

$$- \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^j}\left(\sqrt{A}\, \zeta \left\langle U_2' \frac{U_j'}{\sqrt{g_{jj}}} \right\rangle\right) - \frac{\zeta}{\sqrt{g_{11}}} \cot x^1 \left(V_1 U_2 + \left\langle U_1' U_2' \right\rangle\right) \tag{E24}$$

This completes the set of conservation equations governing the motion of a fluid with three momentum components. The system allows for the time dependence of $\sqrt{g_{33}}$, which can be computed from $\frac{\partial}{\partial t}\left(\sqrt{g_{33}}\right) = \sqrt{g_{33}}\, s^3_{\mid 3}$ by equation (E11) if s^3 is specified.

APPENDIX F

RELATIONSHIP BETWEEN THREE-DIMENSIONAL MOMENTUM EQUATIONS AND HYDROSTATIC MOMENTUM EQUATIONS

When the full set of transport equations derived in appendix E are integrated on a digital computer to simulate the seas or atmosphere, the horizontal scale is generally much greater than the vertical scale. Consequently, weather prediction or ocean simulation would be prohibitively expensive if approximations were not made to alleviate the short time steps associated with integrating the vertical momentum equation.

The classical method to obtain a long time step is to neglect accelerations along the gravity vector, computing pressure by the hydrostatic approximation and then computing vertical velocity from the energy equation while conserving internal energy. (See ref. 1.) The loss of similarity between the resulting numerical model and the real world is associated with buoyant or convective overturning. (See ref. 3.) Consequently (for example), a hydrostatic circulation model is expected to be a poor vehicle for simulating thunderstorms.

The momentum equation derived in appendix A will be specialized to the hydrostatic approximation in this appendix. The hydrostatic momentum equation can be derived from equation (A13) which is revised as follows:

$$\frac{\partial}{\partial t}\left(\sqrt{a}\,\rho u_i\right) + \sqrt{a}\left(\rho u_i v^j\right)_{|j} - \sqrt{a}\left[\rho s^j_{|i} u_j - \rho\left(a_c\right)_i\right] = \sqrt{a}\left(-P_{|i} + \rho g_i\right)$$

Equation (A13) has three components which span the physical three-space. The hydrostatic approximation employed herein is that the right-hand (force) term of equation (A13) for the nearly vertical x^3-direction is zero. Then, equation (A13) becomes

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\left(\sqrt{a}\,\rho u_\alpha\right) + \sqrt{a}\left(\rho u_\alpha u^j\right)_{|j} - \sqrt{a}\left[\rho s^j_{|\alpha} u_j - \rho\left(a_c\right)_\alpha\right] = \sqrt{a}\left(-P_{|\alpha} + \rho g_\alpha\right) \\
&\frac{\partial}{\partial t}\left(\sqrt{a}\,\rho u_3\right) + \sqrt{a}\left(\rho u_3 v^j\right)_{|j} - \sqrt{a}\left[\rho s^j_{|3} u_j - \rho\left(a_c\right)_3\right] = 0
\end{aligned}\right\} \qquad \text{(F1)}$$

The hydrostatic assumption

$$P_{|3} = \rho g_3 \qquad \text{(F2)}$$

and the condition that the pressure P vanish at the top $\left(x^3 = X^3\right)$ of the atmosphere or seas yield

$$P = +\int_{\xi=x^3}^{X^3} \rho g_3 \, d\xi \tag{F3}$$

where the gravity-acceleration components g_i in equations (F1) and (F3) are

$$g_i = -g \frac{\partial h}{\partial x^i} \tag{F4}$$

The third component of equation (F1) is no longer needed because the x^3 velocities can now be computed from the internal-energy conservation equation (E3).

REFERENCES

1. Washington, Warren M.; and Kasahara, Akira: January Simulation Experiment With the Two-Layer Version of the NCAR Global Circulation Model. Mon. Weather Rev., vol. 98, no. 8, Aug. 1970, pp. 559-580.
2. Manabe, Syukuro: Climate and the Ocean Circulation. Part 1 – The Atmospheric Circulation and the Hydrology of the Earth's Surface. Mon. Weather Rev., vol. 97, no. 11, Nov. 1969, pp. 739-774.
3. Smagorinsky, Joseph; Manabe, Syukuro; and Holloway, J. Leith, Jr.: Numerical Results From a Nine-Level General Circulation Model of the Atmosphere. Mon. Weather Rev., vol. 93, no. 12, Dec. 1965, pp. 727-768.
4. Arakawa, Akio: Design of the UCLA General Circulation Model. Grant NGR-05-007-328, Dep. Meteorol., Univ. of Calif., July 1, 1972. (Available as NASA CR-131531.)
5. Mathis, Joe J., Jr.; and Grose, William L.: A Review of Methods for Predicting Air Pollution Dispersion. NASA SP-322, 1973.
6. Kasahara, Akira: Various Vertical Coordinate Systems Used for Numerical Weather Prediction. Mon. Weather Rev., vol. 102, no. 7, July 1974, pp. 509-522.
7. Phillips, N. A.: A Coordinate System Having Some Special Advantages for Numerical Forecasting. J. Meteorol., vol. 14, no. 2, Apr. 1957, pp. 184-185.
8. Williams, Gareth P.: Friction Term Formulation and Convective Instability in a Shallow Atmosphere. J. Atmos. Sci., vol. 29, no. 5, July 1972, pp. 870-876.
9. Cox, Michael, D.: A Mathematical Model of the Indian Ocean. Deep-Sea Res. & Oceanogr. Abstr., vol. 17, no. 1, Feb. 1970, pp. 47-75.
10. Synge, J. L.; and Schild, A.: Tensor Calculus. Univ. of Toronto Press, 1949.
11. Huckins, Earle K., III; and Turner, Richard E.: A General Form of the Co-Moving Tensorial Derivative. NASA TN D-7464, 1974.

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