N76-10170

SOME RECENT INVESTIGATIONS IN NUMERICAL AVERAGING

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Our interest in averaging techniques stems from a requirement to conduct large-scale mission planning exercises on a system of computers that gets overloaded whenever such a study arises. These studies have been conducted at Aerospace using a Cowell propagator simply because of the necessity for accuracy in a high drag environment. However, due to limited computer resources, a project has been initiated to develop a better method, which will involve using an existing algorithm or developing a new one.

This paper presents two topics, the first of which concerns a feasibility study that was conducted using two programs, MAESTRO and PECOS, to see if averaging could be accurate. The second topic concerns some continuing investigations and some new ideas, generally untested. It is assumed that the idea of averaging with respect to trajectory propagation is familiar to everyone so it will not be reviewed here.

The MAESTRO program, developed by Chauncey Uphoff and Dave Lutzky of Analytical Mechanics Associates, was obtained by Aerospace for a feasibility test. It was initially designed as an interplanetary mission analysis tool for the Radio Astronomy Explorer-2 (RAE-2) lunar orbiter. Several choices of variables and propagating techniques are available in MAESTRO. For instance, a Cowell integrator, an Encke integrator, and several variation-of-parameter techniques, including the numerical averaging of the Gaussian variation-of-parameter (VOP) equations, are available.

The forces in the program are given in the radial, circumferential, and polar directions. MAESTRO has a fairly simple exponential atmosphere and has a precision averaging startup, which means that osculating initial conditions in several choices of reference frames can be input and the program will automatically provide the initial mean state.

The Aerospace version of MAESTRO has been merged with a program called PECOS. PECOS, itself, is a test bed for orbit planning simulations. The interplanetary calculations were stripped out of MAESTRO and the force evaluations from PECOS are now used by both MAESTRO and PECOS.

The output can be plotted by an auxiliary pr ·g · available at Aerospace. In particular, overlays of PECOS precision and MAESTRO averaging output can easily be accomplished for comparison purposes. Two versions of mean-to-osculating [illegible] are used.

On output, the Kozai form is used, and within the averaging quadrature, the Iszak form is used. There is no particular reason for this order except for the program's historical development. Fortunately, Iszak is very good for use under the quadrature because it is nonsingular at low eccentricity.

With present program limitations, only certain parameter sets and integration types from MAESTRO are available. For instance, the Cowell integration is not available. Only two out of the original eight techniques are being used. They include the integration of a parameter set which is nonsingular at low eccentricity. This set is similar to, but not exactly the same as, the equinoctial elements. The averaging technique uses the same elements.

The initial conditions for the test case that were used for the feasibility study were chosen from typical orbits of interest to Aerospace. They were initially compiled for Terry Harter to run at GSFC on a test using the Goddard Trajectory Determination System (GTDS). At Aerospace they were run by Stan Navickas in the Mission Analysis Department with the MAESTRO/PECOS software.

Table 1 shows that the four cases here have perigees ranging from 67 to 92 n.m. and apogees ranging from 156 to 223 n.m. The inclinations are all between 90° and 110.5°. The lifetimes range from about 4.5 days to 2 months: case A-1 has a 2-month lifetime and case B-1 has a 4.5-day lifetime, and those are the two cases that will be discussed here.

Figures 1, 2, and 3 are time histories of elements for case A-1, which had approximately a 2-month lifetime. The computer runs were limited to 7 days because it is so expensive to run the Cowell integration in PECOS. This is the orbit which does the least of the four test cases.

Table 1
Initial Conditions for Four Test Cases

Case	Perigee H_p(n.m.)	Apogee H_A (n.m.)	Inclination (deg)	C_D A/W(ft^2/lb)
A1	92	156	96.43	0.0007295 (0.002099)
A2	85	161	94.55	0.0007295 (0.002099)
B1	67	207	96.57	0.0062981 (0.001286)
B2	69	223	110.5	0.0062981 (0.001286)

The semimajor axis is plotted in figure 1. The oscillating line is the output from the Cowell technique in PECOS, and the line down the center is the mean element output from MAESTRO. There has been no correction to put the short-period variations back on.

Figure 2 is the eccentricity from the same trajectory. Again the oscillating curve is the output from the Cowell integration translated into Kepler elements. Figure 3 is the argument of perigee. Figures 4, 5, and 6 are for case B-1, which dies in 4.5 days. The dark line down the center is the mean output from MAESTRO. It can be seen that the program tracks the elements all the way to the end of the lifetime.

Figure 6 is the argument of perigee, and, as expected, the envelope of the oscillations grows rapidly as it approaches the end of the lifetime and the eccentricity drops. The orbit becomes circular, so the argument of perigee begins to oscillate more and more wildly. But the mean argument of perigee is still being tracked and follows the center of the oscillation envelope.

Figures 7, 8, and 9 give the comparison between the Cowell integration and the corrected average propagation at about 4 days into the case having a 4.5-day lifetime. The orbit is generally tracked. The phasing and the magnitude have tracked quite well at the end of 4 days. I am very hesitant to interpret the small errors there, because we have a problem with the interpolator in PECOS/MAESTRO right now. The step sizes in the integration for the averaging process are 90 minutes, but the output is roughly every 2 minutes.

If the output of mean elements were plotted on a finer scale, it would be seen that the result has a small wiggle. This wiggle is probably due to the interpolator. There is some suspicion that the errors in the oscillating elements are due to that wiggle, but that is still a qualitative judgment. The important thing is that the orbit is being tracked.

We interpreted these plots as proof that the averaging technique does work. At this point, there had been no attempt whatsoever to make this program efficient. MAESTRO was simply merged into a larger system using some of the logic from that larger system, so timings from these runs were totally meaningless. Using the results proving the feasibility of the technique, we decided to go ahead and attempt to modify MAESTRO to make it a stand-alone version that would be efficient, using essentially the same logic that it has always had, but using a more accurate force model than it had when it was developed for GSFC. The modifications and developments which will go into this further study will be discussed now.

The purpose of the current investigation is to increase the accuracy and, if possible, to decrease the computation time; this will of course involve a compromise between the two. Currently, we are developing MAESTRO as a stand-alone version with modification for accuracy in a high drag environment, to run on the CDC 7600. The next step will be to replace the independent variable with the true anomaly and to remove the fast variable. Another concurrent activity, being handled by Dave Lutzky of Vector Sciences, is treating the short period effects within the averaging quadrature and for output with a Fourier expansion from which the coefficients of the harmonics are computed automatically and numerically. This is work which was developed under funds from GSFC. The results will be inserted into MAESTRO to see if they will help to make the program more efficient and accurate.

Now I would like to discuss some of the work I am doing. It is all formal at this time, none of it has been implemented in the program yet.

One of the disadvantages of using time as the independent parameter is that the averaging period is always difficult to interpret. For instance, it is possible to use the mean motion, which is determined by the initial oscillating elements or by the initial mean elements, to determine the period. But, however it is done, it will probably be difficult to be consistent and to initially choose the right time.

Another problem is that the equations for the averaged elements have a term in their exact form which depends upon the time rate of the period. This is generally referred to as the Leibnitz term, and it depends upon the time rate of the period, the values of the true elements at both ends of the averaging period, and on the mean element at the midpoint of the averaging interval.

Still another problem arises if constant time steps are chosen: For instance, if the integration step is chosen as the initial period of the orbit, then, farther on down the line, the time step will no longer correspond to the period, and there will be drift with respect to perigee or any geometric event in the orbit. If the true anomaly is chosen as the independent parameter, none of the above disadvantages apply. Another effect is that we get automatic regularization of the orbit sampling. Points are automatically clustered near perigee where most of the action occurs for the high drag situation.

It is also possible to make repeated use of terms in the mean-to-osculating transformations, particularly if step size is chosen properly within the averaging quadrature. Also, the mean-to-osculating formulas appear in closed form, at least to first order in J_2. This is due simply to the fact that the disturbing potential can be expressed in a finite number of terms, depending upon the true anomaly.

In changing the propagation algorithm, we wanted to choose a state vector which is nonsingular for low eccentricity and inclination. For convenience, we chose to use the same state that MAESTRO now uses, except for the fast variable. If the independent variable is now chosen as the true anomaly, then the angle-time relationship must be tracked through an equation other than the one we had before. If the new fast variable is itself chosen as the time, we then have a new differential equation:

$$\frac{dt}{df} = h/r^2 + \text{perturbations.}$$

Because this leaves us with large oscillations, it is a poor choice for averaging. Suppose that, instead, the parameter Q (Stern, 1960) is chosen:

$$Q = M + \omega + \Omega - \bar{n}t$$

$$\bar{n} = (-2\epsilon)^{3/2} / \mu$$

where ϵ is the orbital energy. Then $\bar{n}$ determines the period of the true anomaly. The Q will always be a slow variable, and, in fact, for the Kepler problem, it is a constant.

The equations for the new system are given by

$$\frac{dE}{df} = \frac{dE}{dt}\frac{dt}{df} = F_E(E, t, f)\frac{dt}{df}$$

where E is any element

$$\dot{E} = F_E(E, t, f).$$

The exact averaged equations are

$$\bar{E} = \frac{1}{2\pi}\int_{f-\pi}^{f+\pi} E(f')\, df'$$

$$\frac{d\bar{E}}{df} = \frac{1}{2\pi}\int_{f-\pi}^{f+\pi} F_E(E, f', t)\, df' \frac{dt}{df'}$$

To make averaging efficient, the force evaluation is accomplished by approximating the elements in the integrand to the mean elements, plus a correction, as in the standard second-order averaging technique. Also, since the time no longer appears explicitly in the state, it must be the result of a similar correcting transformation. This means that the mean-to-osculating transformation is required at every step, as it is in all second-order averaging techniques.

Formally, the mean-to-osculating transformations are simple to derive. The equations needed are indicated below in abbreviated form:

$$E^* = \bar{E} + \delta E_p$$

If it is assumed that Δf is $2\pi/2K$, then values of the trigonometric functions of the harmonics of the true anomaly can be tabulated and never recalculated, because the same values will be repeated in true anomaly from revolution to revolution. This should save a significant amount of calculation and make for an efficient mean-to-osculating transformation.

In the equations,

$$\frac{dE}{df} = \text{CONST} + \sum_{1}^{N} (A_n \cos nf + B_n \sin nf)$$

and

$$\delta E_P = \sum_{1}^{N} (-A_n \sin nf + B_n \cos nf)/n,$$

A_n and B_n are derived from the Lagrange planetary equations with the J_2 perturbation only. The $[A_n, B_n]$ are slowly varying functions of the state vector and supposedly would not need to be recalculated more than once per revolution, and probably not that often. This depends upon the strength of the perturbations.

The evaluation of $d\bar{Q}/df$ is fairly obvious, and it is used here as an example: The forcing function, F_Q, is $(\dot{M} + \dot{\omega} + \dot{\Omega} - n) + (n - \bar{n}) - \dot{\bar{n}}t$. Time appears explicitly, which may create some difficulties, at least formally, although it is doubtful that numerically this will give any trouble. The factor dt/df, which appears in the integrand, is given by

$$\frac{dt}{df} = \frac{h}{r^2} + \text{perturbations.}$$

However, since all the parameters in the state are slow (there is no fast variable), probably the two-body rate, or just h/r^2, is good enough for the function dt/df. That will probably require some experimentation, but, since there is no fast variable, there is no purpose in carrying higher order terms in this factor. The term $\dot{\bar{n}}$ is related to the time derivative of the energy, and that relates (at least in its dominant term) to the time rate of the semimajor axis due to drag for the cases where drag is the most important perturbation aside from J_2. The quantity $(n - \bar{n})$ is proportional to the disturbing potential and $(\dot{M} + \dot{\omega} + \dot{\Omega} - n)$ is given by the Gaussian VOP equations. The MAESTRO propagator will be used as the basic software tool in the implementation of these techniques.

Still to be considered are techniques for calculating an accurate argument of perigee at low eccentricity. There are some problems because there is a mix of the true mean anomaly and the mean argument of perigee in one of the transformations; some experimentation is still required.

The exponential instability in Q, which is due to the presence of $\dot{\bar{n}}t$, must also be investigated. This is a problem which will most likely be more apparent on computers with short word lengths. With the CDC computers and their essentially infinite word lengths, we have tried experimenting to see when we can make the equations blow up in a simple form, and we have been unable to find the instability numerically. That may give us trouble, though, and it will require more experimentation to see if there really is a problem.

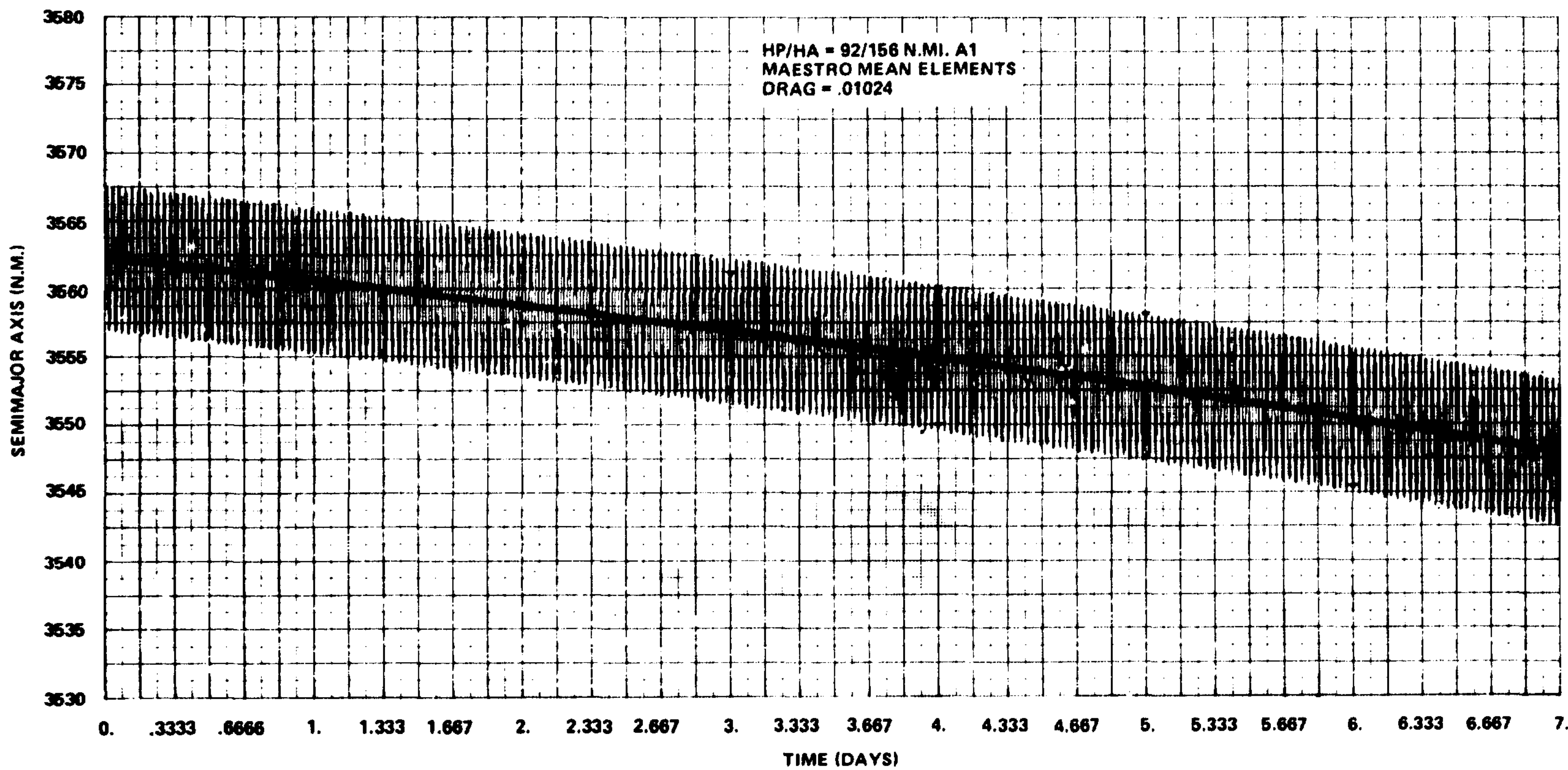

Figure 1. Comparison of PECOS and MAESTRO semimajor axis plots for case A-1.

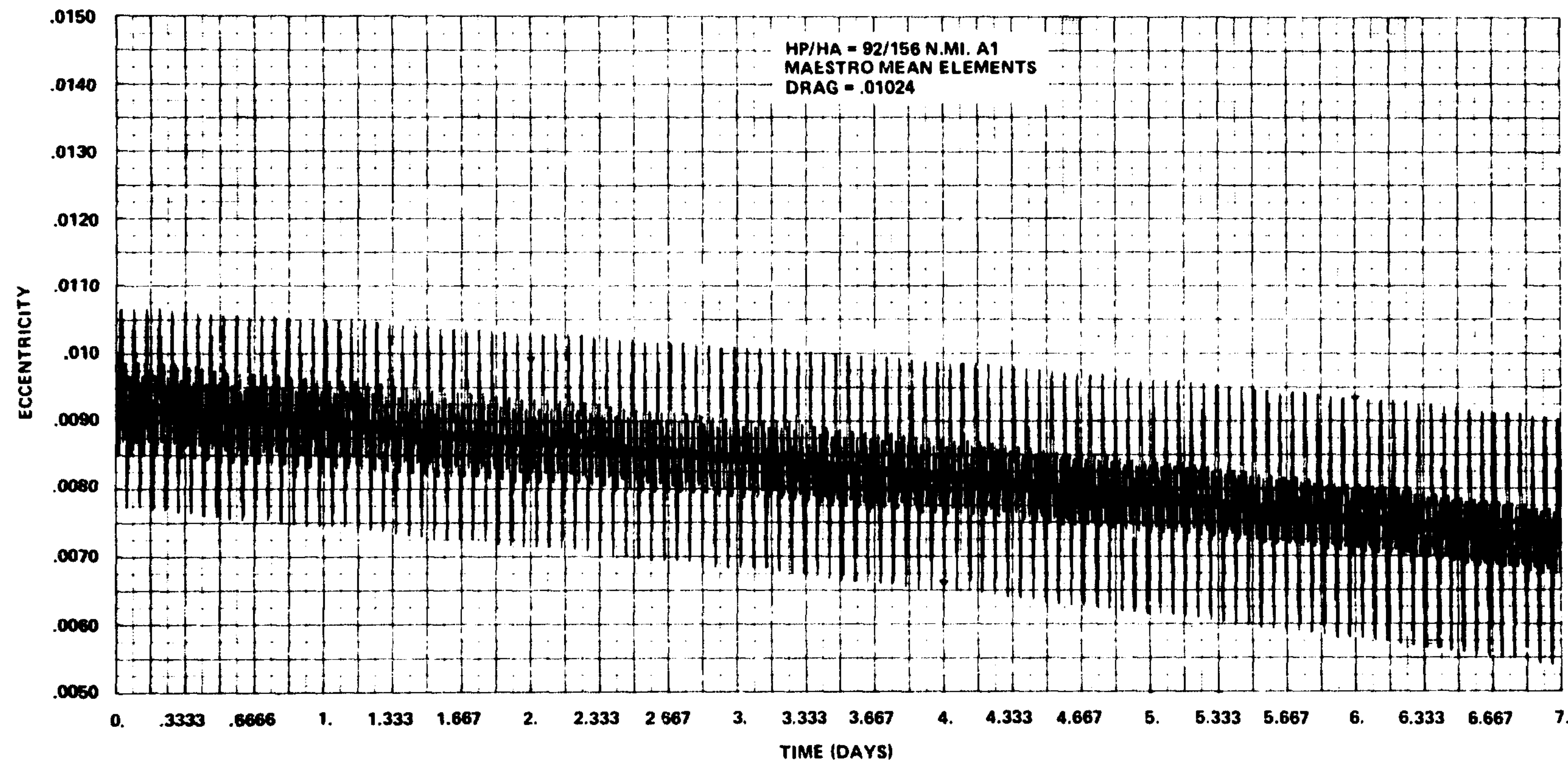

Figure 2. Comparison of PECOS and MAESTRO eccentricity plots for case A-1.

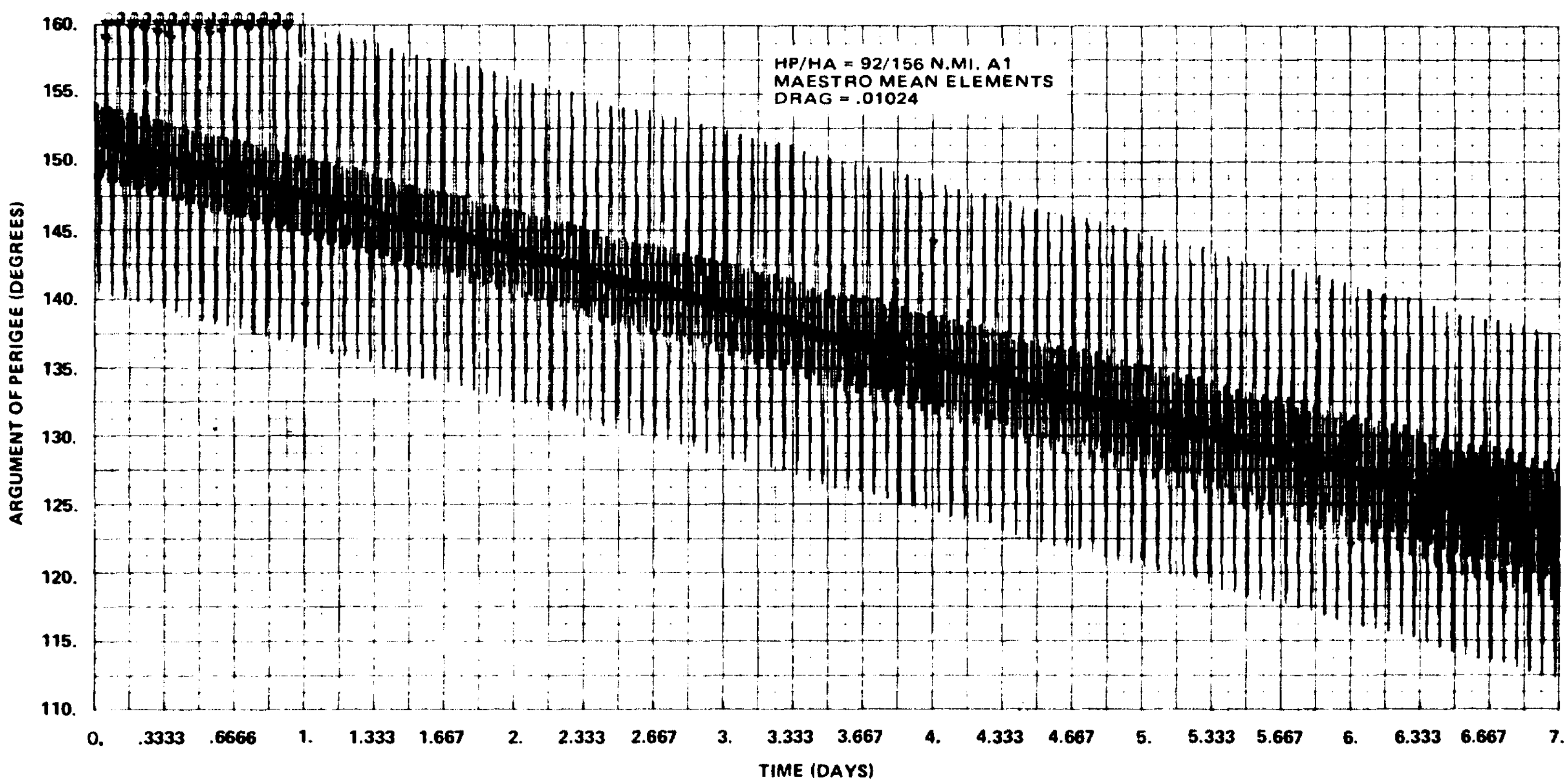

Figure 3. Comparison of PECOS and MAESTRO argument of perigee plots for case A-1.

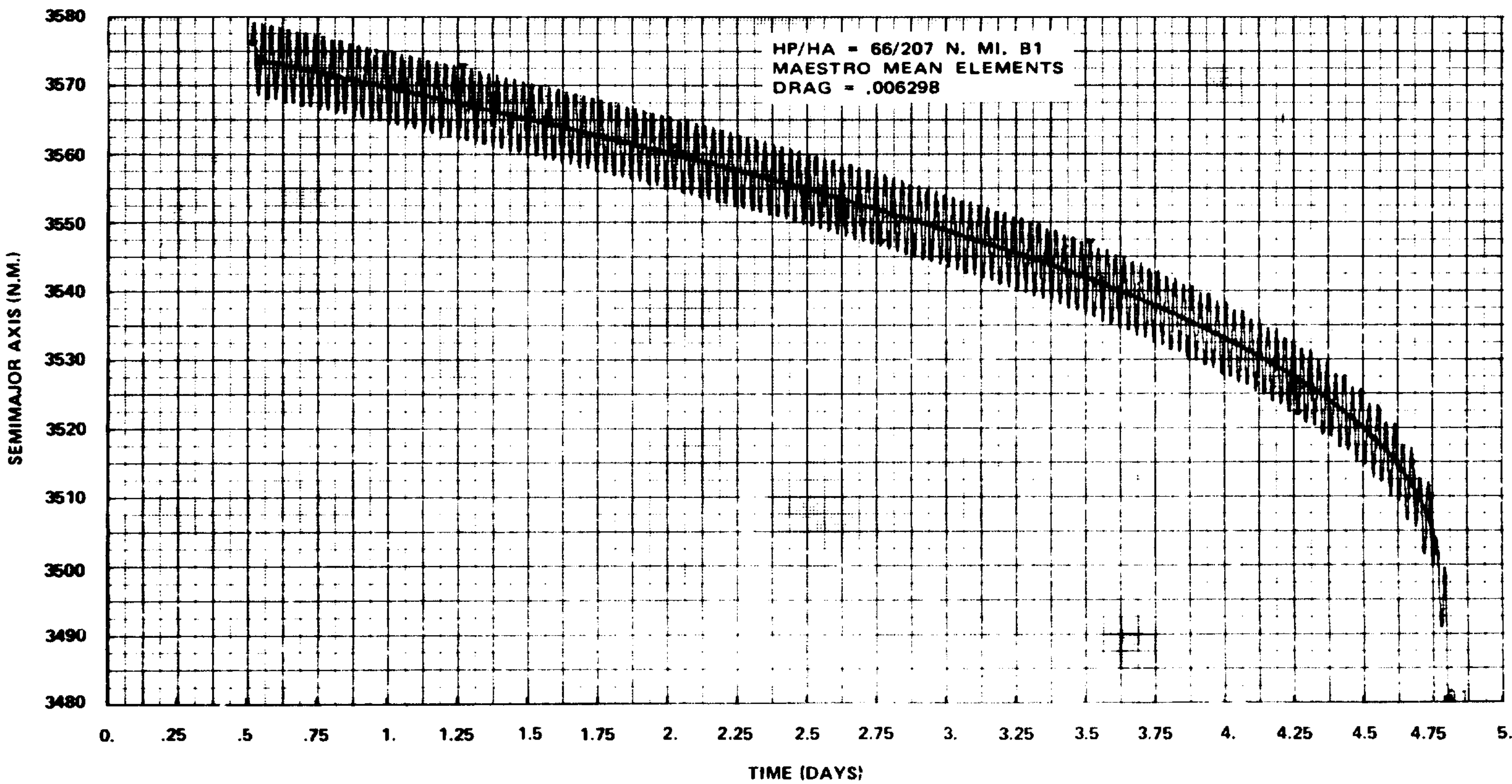

Figure 4. Comparison of PECOS and MAESTRO semimajor axis plots for case B-1.

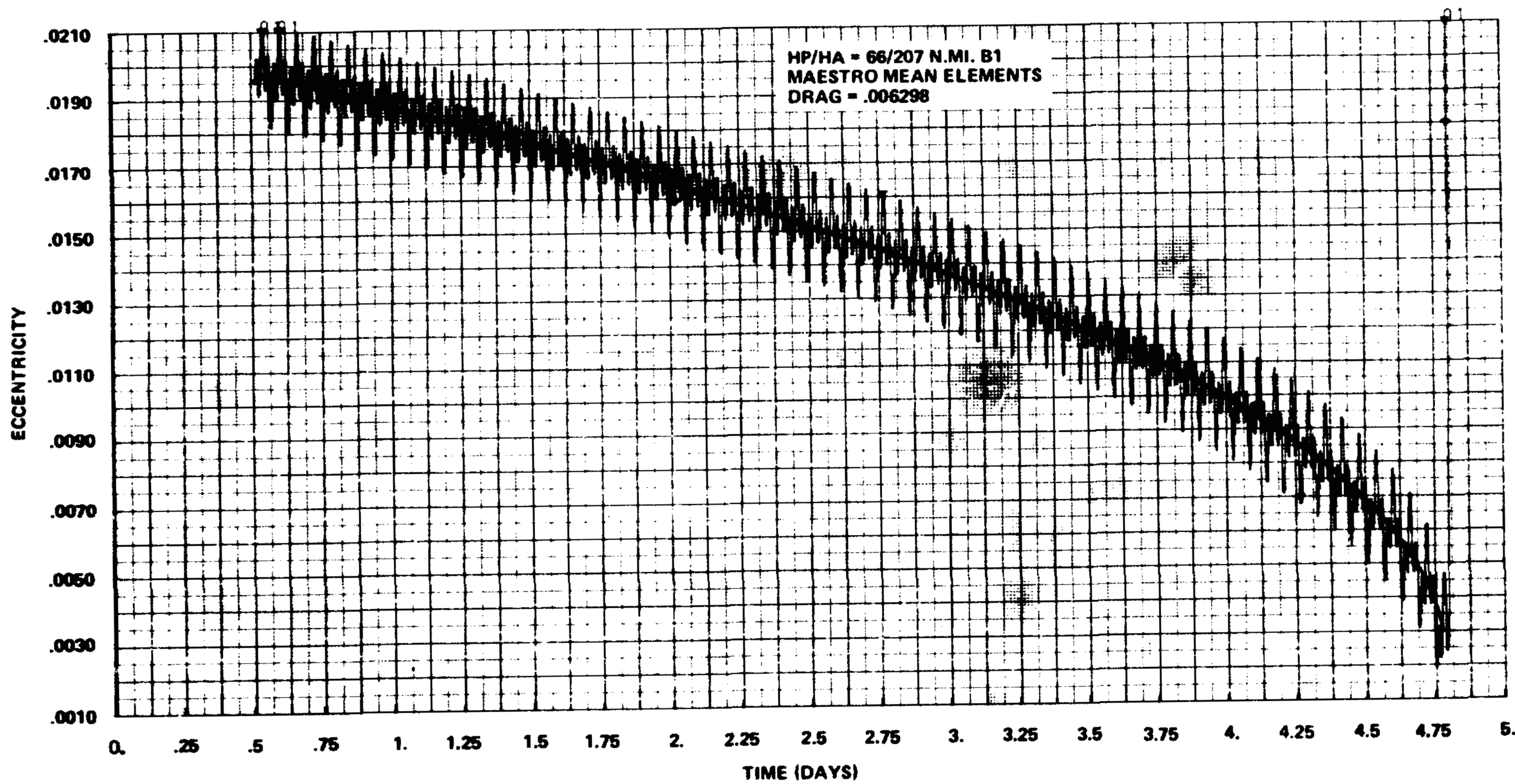

Figure 5. Comparison of PECOS and MAESTRO eccentricity plots for case B-1.

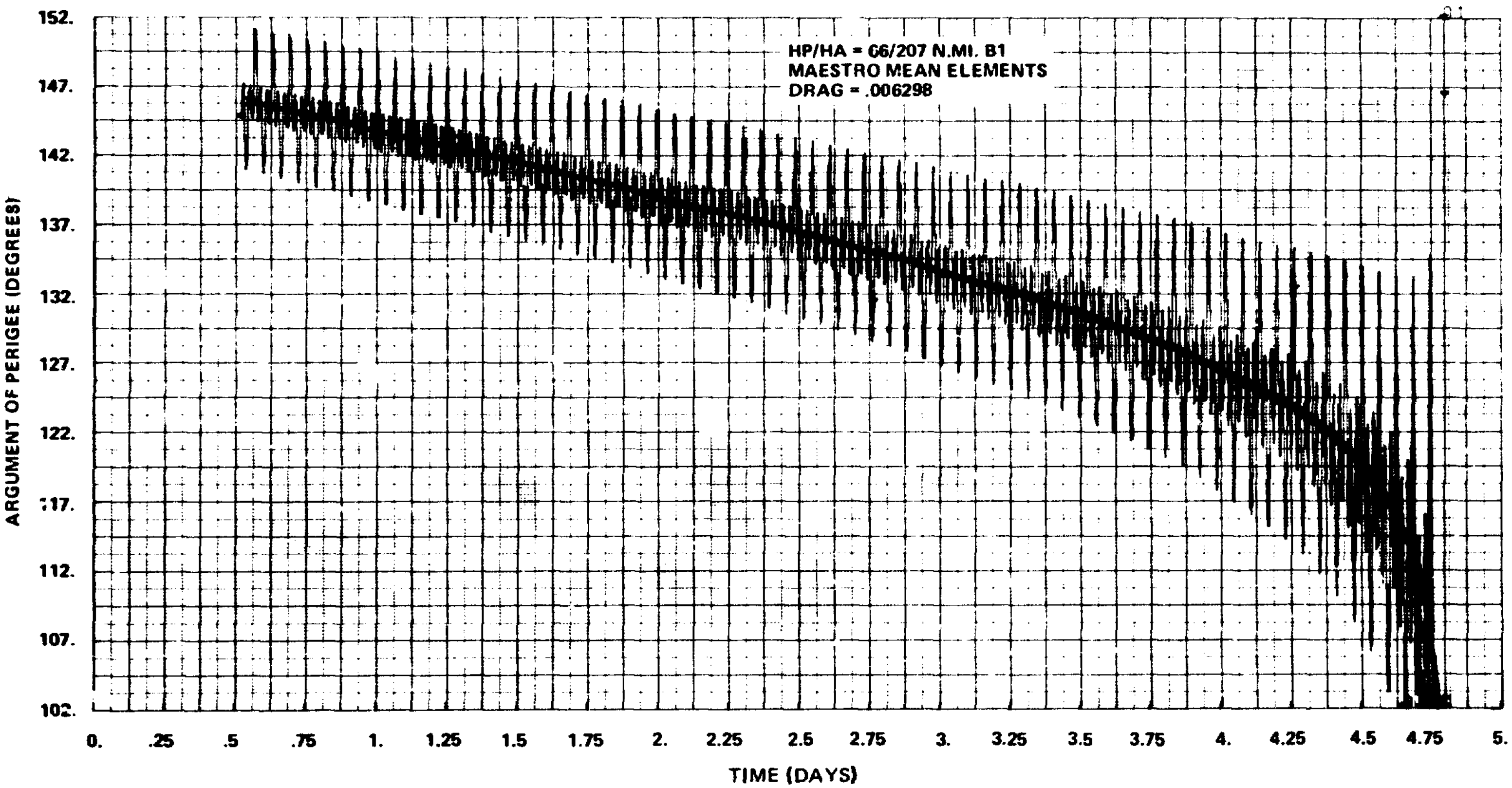

Figure 6. Comparison of PECOS and MAESTRO argument of perigee plots for case B-1.

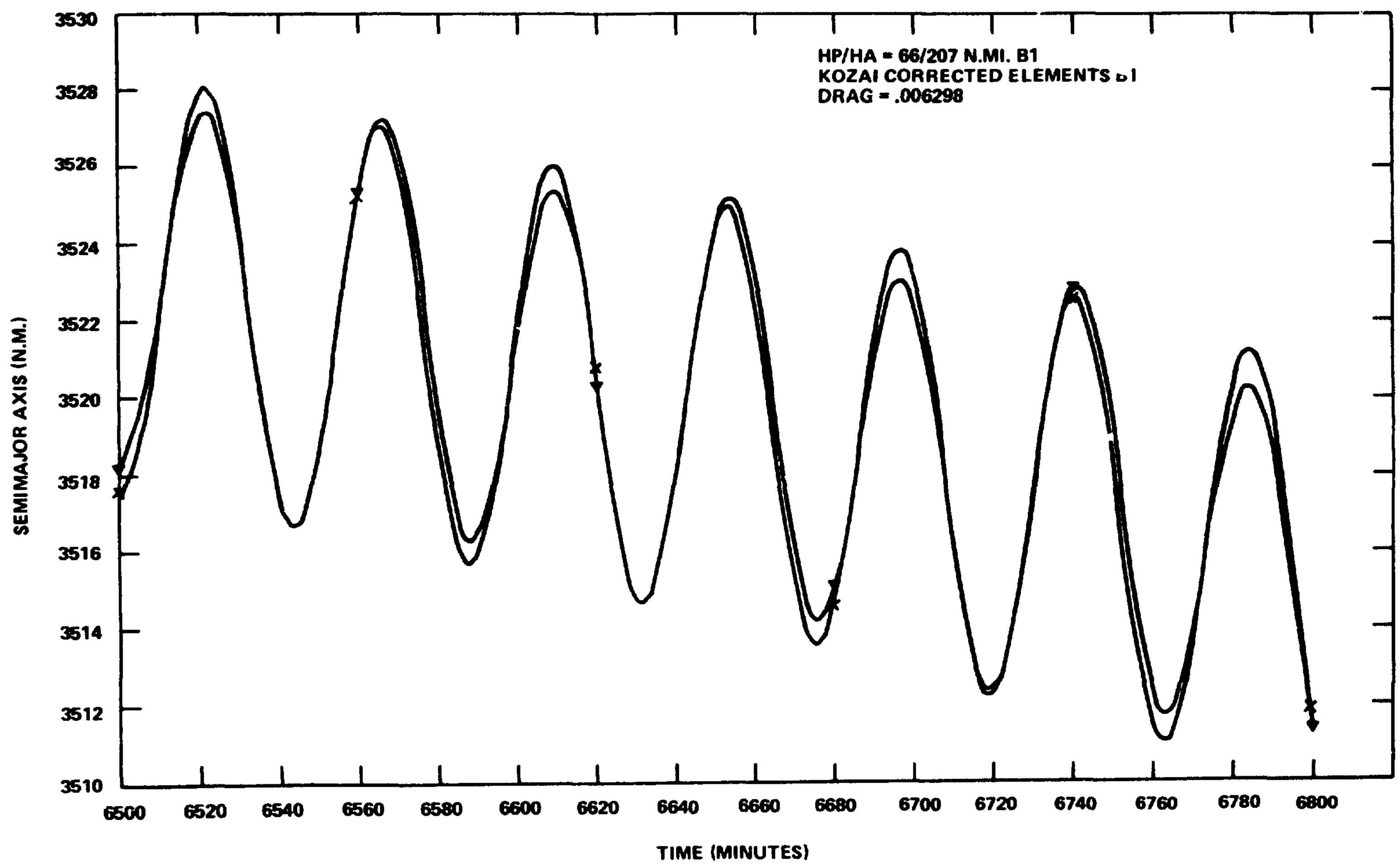

Figure 7. Comparison of Cowell integration and short period recovery for semimajor axis.

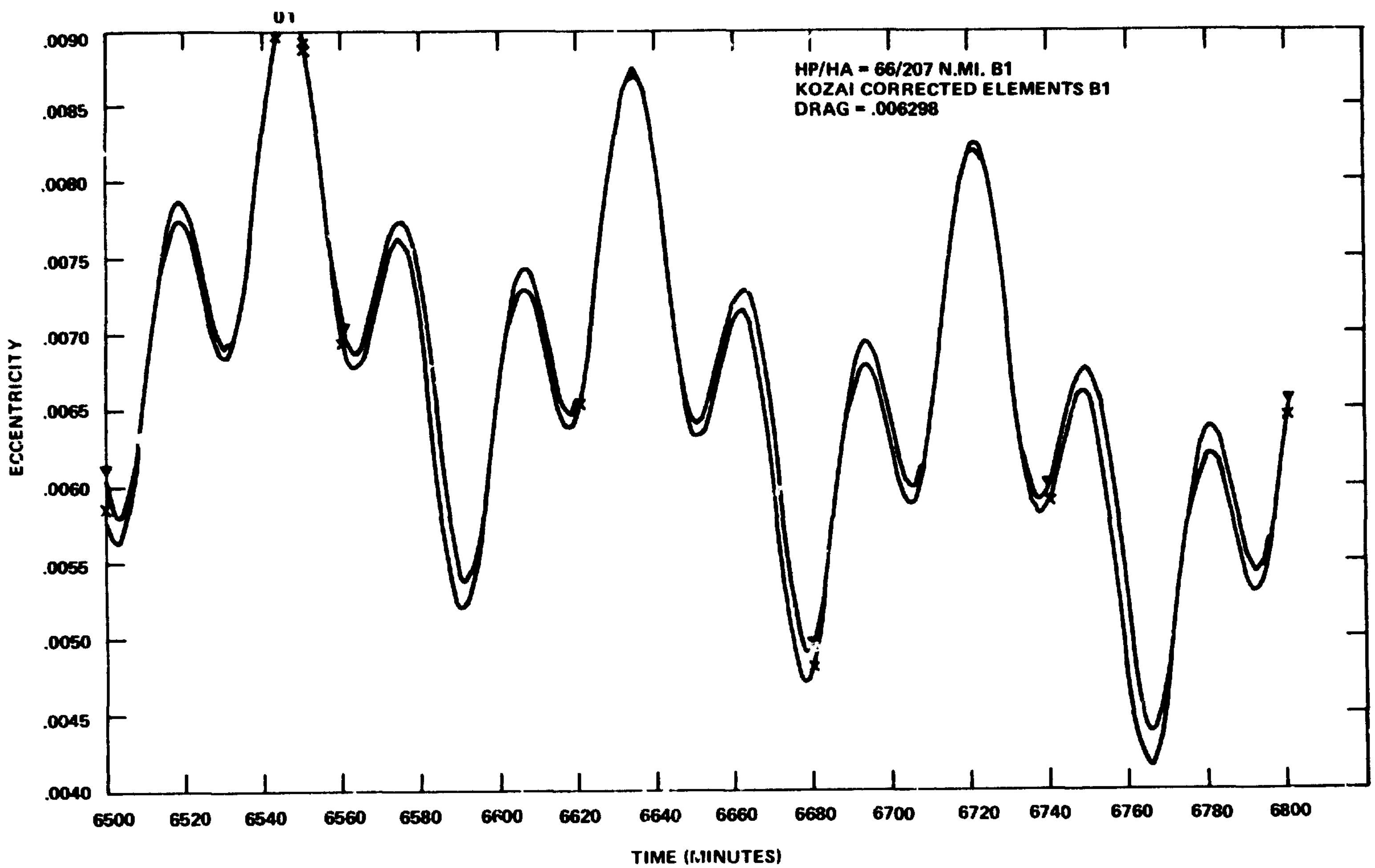

Figure 8. Comparison of Cowell integration and short period recovery for eccentricity.

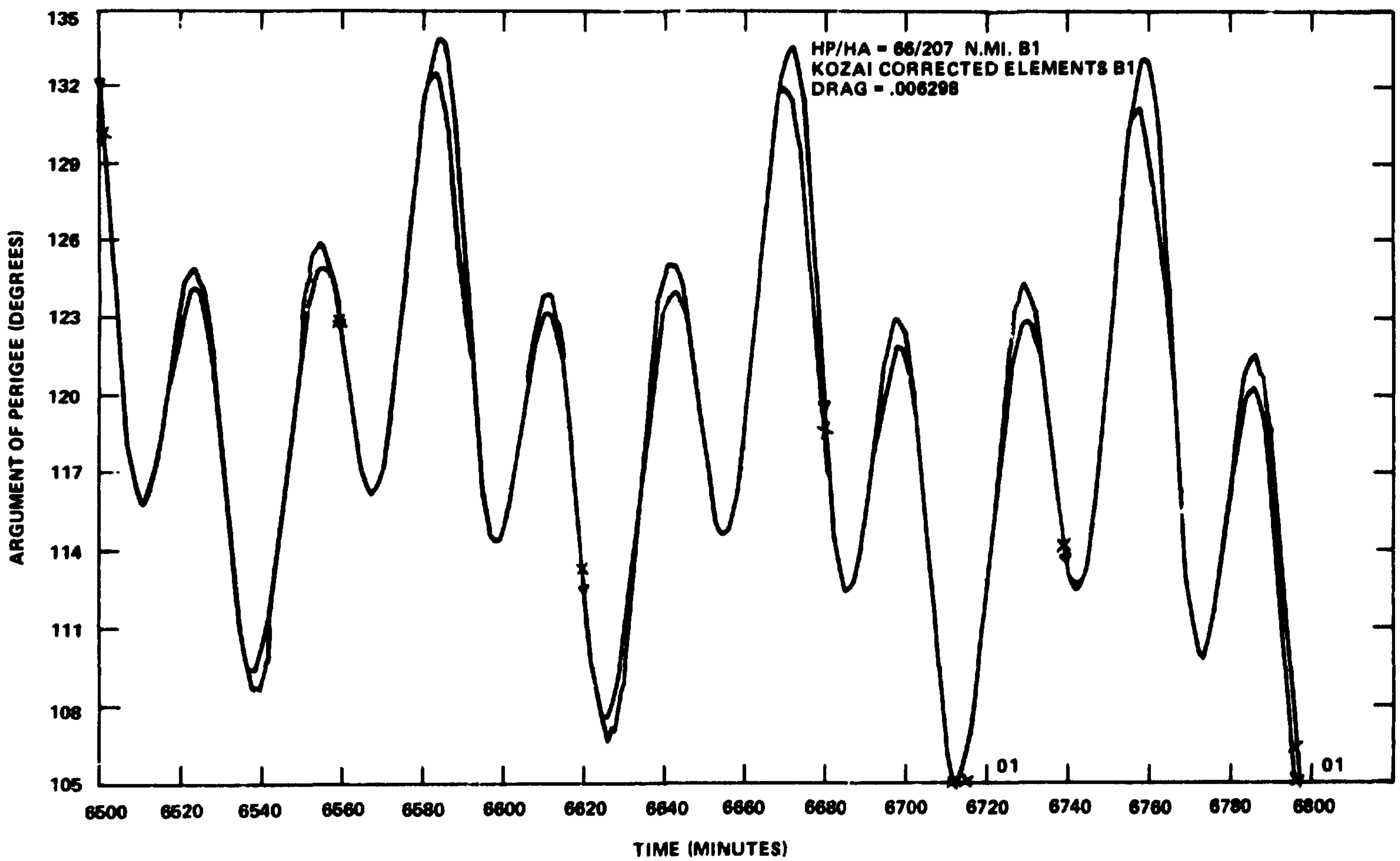

Figure 9. Comparison of Cowell integration and short period recovery for argument perigee.

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