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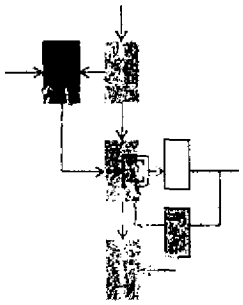

ON THE INTERACTION STRUCTURE OF LINEAR MULTI-INPUT FEEDBACK CONTROL SYSTEMS

Poh Kam Wong

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Electronic Systems Laboratory *Decisions and Control Sciences Group*

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Department of Electrical Engineering and Computer Science

September 1975 Report ESL-R-625

ON THE INTERACTION STRUCTURE OF LINEAR MULTI-INPUT FEEDBACK CONTROL SYSTEMS

by

Poh Kam Wong

This report is based on the unaltered thesis of Poh Kam Wong, submitted in partial fulfillment of the requirements for the degree of Master of Science at the Massachusetts Institute of Technology in September, 1975. The research was conducted at the Massachusetts Institute of Technology, Electronic Systems Laboratory, Decision and Control Sciences Group with support extended by NASA under grant NGL-22-009-(124).

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ON THE INTERACTION STRUCTURE OF LINEAR MULTI-INPUT FEEDBACK CONTROL SYSTEMS

by

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S.B., MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1974

SUBMITTED IN PARTIAL FULFILLMENT OF THE

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MASTER OF SCIENCE

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

SEPTEMBER, 1975

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ON THE INTERACTION STRUCTURE OF LINEAR MULTI-INPUT FEEDBACK CONTROL SYSTEMS

by

Poh Kam Wong

Submitted to the Department of Electrical Engineering and Computer Science on the 25th of August, 1975 in partial fulfillment of the requirements for the degree of Master of Science

ABSTRACT

The closely-related problems of designing reliable feedback stabilization strategy and coordinating decentralized feedbacks are considered. Two approaches are taken. A geometric characterization of the structure of control interaction (and its dual) is first attempted and a concept of structural homomorphism developed based on the idea of 'similarity' of interaction pattern. The idea of finding classes of individual feedback maps that do not 'interfere' with the stabilizing action of each other is next developed by identifying the structural properties of non-destabilizing and LQ-optimal feedback maps. Using this approach, we were able to firstly generalize and make transparent some known stability properties of LQ-feedback, and secondly to provide some partial solutions to the reliable stabilization and decentralized feedback coordination problems. A concept of coordination parametrization is introduced, and a scheme for classifying different modes of decentralization (information, control law computation, on-line control implementation) in control systems is developed.

THESIS SUPERVISOR: Michael Athans

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Dedicated to

All who have taught me

Never to let my studies

interfere with my education

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CHAPTER 1

INTRODUCTION

1.1 Problem Background and Brief Historical Review

Many significant control problems in the real world involve the use of multiple control inputs to regulate a dynamical system with a multiple number of desired outputs through a multiple number of measurement and monitoring devices. Typical examples range from nuclear reactor control and industrial chemical process control to multi-reservoir river basin regulation and electric power distribution.

Conceptually, the presence of a multiple number of physically distinct control input channels which can be actuated independently gives rise tp a consideration of the interaction between the input signals independently applied to the system from the different input channels. In particular, it gives rise to a consideration of those interaction characteristics that are in some sense inherently determined by the system structure itself, i.e. by the way the input channels are connected to the system dynamics. Dually, the use of multiple measurement channels providing information that can be independently processed raises the conceptual issue of observation interdependency, while the desire to similtaneously regulate a multiple number of distinct physical outputs leads to an immediate recognition of the problem of interference (or conflict) of output regulation objectives.

It is fair to say that what complicates most our understanding of many multivariable system control synthesis problems is not so much the size of the system involved, but rather their lack of modularity: one usually has a situation where the signal injected at one input channel necessarily interact and interfere with those from other inputs, making it difficult to determine how they eventually affect the outputs individually. It is such 'coupling' and 'cross-talk' that makes a simple decomposition of the overall system control sysnthesis task into smaller independent tasks impossible, i.e. one cannot specify what trajectory a particular output should be manipulated to follow without considering what undesirable interference this might necessarily impose on the achieving of other output objectives, and one cannot specify what signal should be injected at one input to achieve any given desired objective without knowing what the other input channel signals are to be.

On the other hand, the phenomenon of interaction leads naturally to the dual concept of redundancy: if two controllers both affect an output, there is in a sense a redundancy of controllability of that output, assuming that both controllers are accessible by the same control agent. Similarly, if one has access to two sensor to two sensor channels that overlap in their observability, there is a duplication of some (extractable) information about the system. The concept of redundancy in turn is intimately related to another fundamental engineering concept, that of reliability. Indeed, redundancy is the very key to construct a reliable system out of unreliable components: a redundantly-controlled output

remains controllable when one input fails, or is effectively compensated when extra disturbance is injected.

As we have emphasized, the structural concepts we have introduced so far arise naturally from considering the inputs, sensors and outputs as independent, distinct entities, i.e. from considering the internal structure of the input, sensor and output space. Indeed, it is fair to say that a major concern of 'classical' control theory has been the independent exploitation of every available degree of freedom (e.g. the number of feedback loop entry points available, leading to such concepts as 'single-degree-of-freedom configuration' feedback and 'loop shaping', see [4]). It is also precisely this detailed need to consider every degree of freedom individually and independently that made classical control methods unattractive for systems with a large number of degree of freedoms.

As we know, the 'modern' state-space optimal control theory that subsequently superceded the classical approach have succeeded in overcoming this problem of 'dimensionality' by going to the other extreme - that is, by 'ignoring' the internal structure altogether. The state-space approach allows one to mathematically 'lump' a high-dimensional problem into one conceptually equivalent to a unidimensional one (multiple scalar inputs become a single vector object, etc.),while the use of cost criterion optimization (e.g. the standard Linear Quadratic optimization approach) allows the implicit trade-off between different output objectives and simultaneous determination of all the control inputs. Thus, modern

optimal control allows one to 'solve' a multivariable synthesis problem without knowing anything about the structure of the system (as opposed to numerical data). In obtaining the simultaneous control strategies, one never has to worry about and have no control over, how the different control input signals so determined will interact. Such structural concepts as controllability and observability are used only to the extent of providing a test on solvability, and in any case their aggregate nature ('can all the control channels combined together control all the states?') is of no help in understanding the internal structure of control interaction.

It is precisely because conventional optimal control theory does not demand structural information for its problem formulation that it is incapable of exploiting such information. If one knows that the A, B matrices of a state-space equation are really a model of a chemical plant, and one really has a lot of intuitive and physical understanding of the behavior of the many subsystem chemical processes involved, all that insights are irrelevant in an optimal control design formulation, in so far as one cannot translate them into one's specification of the 'weightings' in the cost criterion and of the optimization constraints. (In fact, 'optimal' control problems are often formulated as if the choice of the cost criterion can be made independent of one's knowledge of the systme's properties and limitations.)

It is thus the case that one often has to ignore an awful

lot of information in solving a design problem using optimal control theory. In practice one often 'iterates' on different choices of cost weightings until the resultant behavior is 'intuitively' satisfactory; yet the theory itself provides no insight as to how the weightings should be 'iterated'.

In particular, the conceptual 'lumping' implicit in such a 'centralized' optimal control framework precludes all considerations that have to do with the fact that the control inputs (and sensor outputs) are physically distinct and independent entities. There are at least two contexts when such considerations are crucial. First is the case when physical failures of control inputs are important, and one wishes to design reliable system performance by exploiting control redundancy. Such system reliability problems cannot be addressed without an understanding of how the different failure-prone control channels structurally intertwine. (Dually, to understand how a sensor failure affects state reconstruction performance, one needs to know the internal interdependency structure of the measurement system as a whole.) The second case arises when the centralized control actuation assumption is invalid, as for example, when different decision agents have access to different control channels and compute their actuation signals independently. To analyze the large classes of such and related decentralized control problems, where the independence of the control channels (and the observation channels) is the essence of the matter, the internal pattern of observation and control overlapping becomes indeed the primary object of studies.

The recent trend of development towards a theory of decentralized control of large-scale systems with 'non-classical' information pattern, or team decision problems (see e.g. [M3], [C3]) has contributed much towards a better understanding of the theoretical issues that arise when more than one control agent act on the system. However, most contributions in the literature have focused attention on the effect of stochastic uncertainties on controller strategies, and on the technical conditions for guaranteeing existence and optimality of control strategies. The a priori, structurally determined aspects of control decentralization and information decentralization have not been given much considerations (exceptions being the recent papers of Aoki [A3,4] and Eckberg's thesis [E1]). The same lack of exploitation of structural information can be said of much of the linear differential games literature.

The true revolution in thinking about the role of the use of system structural information came with the more recent introduction of the 'geometric' approach to linear multivariable control, as initiated independently by Wonham [W5, 6], Morse [M6, 7] and Basile and Marro [B1, 2]. The geometric approach provided the first convenient mathematical tools for a coordinate-free, invariant description of system structural properties (see also [P2]). Furthermore, its philosophy of posing control problems as feasibility questions (as opposed to optimality) leads naturally to constructive proof in terms of algebraic system structural characterization. It is thus realized that the conventional charac-

terization of a system as controllable, observable, stable, etc. is too crude: oen needs to know which state is controllable by which control channel, which subspace a control can influence without disturbing other states outside of that subspace, etc. Indeed, the first major use of internal structural concepts in control problems occurred in the area of decoupling (or non-interacting) control (see [M6]). Although the decoupling problem has a long history (see, e.g. [H 4]), it is only after the introduction of geometric structural concepts that the essence of the problem is made transparent. Another class of problems for which structural concepts have provided transparent solution is the so called 'disturbance localization' problems (see [H4] for its classical origin and [W6] for its 'geometric' solution).

A fundamental conceptual contribution of the structural approach has been the lights it sheds on the role of (linear) feedback in linear system. It becomes obvious that, among other things, many control synthesis problem can be formulated as achieving certain structural configuration using feedback. The extent of structural changes achievable through feedback application relates intimately to the internal interaction structure of the input channels. It is well-known that an outstanding characteristic of many conventional multi-input, multi-output control problems is the high redundancy in the degree of freedom available for their solution. For example, in the 'standard' linear modal control (pole assignment) problem (see, e.g. [P4]), an infinite number of different control feedback gain matrices can achieve the same pole allocation; the problem specification is highly incomplete.

Conventional solutions ('dyadic' feedback, etc., all basically amount to reducing the multi-feedback problems into one with scalar feedback) is purely of mathematical convenience, and has absolutely no physical basis whatsoever. A (dual) example is the standard Linear Quadratic (LQ) problem, where whole subspaces of the parametric space of the output weighting matrix map into the same optimal feedback. The redundancy in solvability freedom of many such problems are intimately related to the degree of interaction or coupling of the input channels (and dually). The important recognition is that two different combinations of feedback from the individual input channels that yield the 'same' closed loop system behavior may well have widely different properties with regard to other considerations, e.g. stability under arbitrary feedback loop failures, or under plant structure perturbations, or under sensor failures, or under saturation effects previously ignored, etc.

In summary, we have attempted to indicate the role and importance of a suitable system structural characterization of control interaction (and its various dual versions) as a basis for tackling many classes of control problems. We have in particular identified two broad classes of problems, systems reliability and multi-agent control problems, for which we feel that a structural approach will provide new perspectives and means of attack. We have also pointed out the shortcomings of conventional optimal control theory, the LQ-theory in particular, and have suggested the importance of establishing a theoretical link whereby structural information can be incorporated in the optimzation formulation,

and whereby the solution can be structurally interpreted. Finally, we have reiterated, if implicitly, throughout the above discussion the possibility and usefulness of viewing many important classes of problems as dual to each other within the structural framework sketched, thus suggesting a conceptual unification that might deepen our insights.

1.2 Thesis Motivation.

The main motivation of the research reported in this thesis is to better understand the problem of system reliability synthesis in the context of multivariable control, and to relate that understanding to the system structure in a way naturally suggested by the 'duality' between the concept of interaction and redundancy. We are quickly led to recognize that the class of reliable synthesis problems can best be viewed as a subclass of the class of problems where it is natural to consider the input channels (or groups of them) as 'independent'.

This concept can be formalized in precise algebraic terms, and the idea of examining the intersection structure of various algebraic objects that can be generated by such independent 'invariants' in fact forms the conceptual basis of our approach to a structural characterization of control interaction in a linear multivariable system. More precisely, we will look at the lattice structure generated by the collection of individual controllable subspaces, and to examine how it gets modified under various 'feedback'. A new concept of structural homomorphism between linear multivariable systems also follows naturally from our framework, and it is hoped that it will provide a useful tool for studying the class of linear multi-input, multi-output systems.

While the above 'geometric' approach yields interesting structural insights, it cannot yet be used directly to solve our system reliability synthesis problem because a convenient characterization of possible modifications of the interaction structure under arbitrary feedback is not yet available. We are therefore motivated to take a different approach by focussing our attention on the feedback maps instead of the invariant

subspaces. In so doing we are firstly motivated by the fact that the class of LQ-feedback has 'nice' stability properties, and hence that it might be fruitful to identify other useful classes of feedback maps; secondly, the idea of viewing a reliable stabilization synthesis problem in terms of determining a set of 'coordinating parameters' that ensures non-interference of the individual control feedback actions on the stabilizing efforts of other controllers leads naturally to the idea of finding suitable parametrization on the feedback maps themselves.

This idea turns out to be very fruitful. By identifying first the class of feedback maps that has the fundamental property that any member of the class never destabilizes an originally open-loop stable system, and then viewing the LQ-feedback as simply a stabilizing feedback class, we were able to generalize and make more transparent some of the known structural stability properties of LQ-feedback. Moreover, it allows us to contribute significantly towards the solution of the reliable stabilization synthesis problem we started off with, as well as offering an intuitively interesting interpretation of 'coordination' and 'interference avoidance' in the reliability synthesis context.

The close connection between reliable synthesis and coordination of independent controllers leads us to extend the structural understandings we have achieved to the class of decentralized control problems. The fundamental motivation here is to define the very concept of decentralization in terms of the structure of the model. This new perspective allows us to clarify several concepts that have been used widely but confusingly in the literature, by pinning down at the very beginning <u>what exactly</u> is being decentralized (observation, control actuation, strategy computation, etc.), and <u>why</u>.

1.3 Thesis Organization

The thesis is organized as follows:

In Chapter 2, we first consider the problem of characterizing the structure of interaction of control inputs (and its dual) using the geometric approach and lattice theory. We then examine how this interaction structure is modified when arbitrary feedback maps are imposed. A new concept of structural homomorphism between multivariable systems is then introduced.

In Chapter 3, we take a different approach to the study of control interaction, and focus attention instead on the properties of feedback maps. We identify first a class of feedbacks that has the fundamental property that any member of the class never destabilize the originally open-loop stable system, and then show that the class of LQ-feedback can be naturally viewed as having the additional property that it always stabilizes the open-loop system. This allows us to generalize and make more transparent some known stability properties of LQ-feedback. Some geometric interpretations of the Riccati equation has also been obtained.

The properties of non-destabilizing and LQ-feedback maps are applied in Chapter 4 to study the problem of system reliability synthesis in the control context. After examining the various failure modes in control systems, we carefully formulate a class of reliable stabilization problem and show how it can be resolved by solving the class of reliable LQ-synthesis problems. Some results towards the solution of the latter are next presented as well as some illustrative numerical examples. The natural dual problem of reliable

state-reconstructor synthesis is next formulated. We then give a brief discussion on the role of control strategy redundancy in system design, and conclude with a brief recapitulation of the practical implications of our results in this and the last chapter.

The problem of structural characterization of decentralization in large-scale control systems is taken up in Chapter 5. We first present an informal scheme for classifying different modes of decentralization, and then show how several classes of decentralized control problems in the linear multivariable system context can be naturally formulated within this framework. The idea of non-destabilizing feedbacks are extended to tackle a class of decentralized feedback stabilization problems.

Finally, in Chapter 6, we summarize our conclusions and suggest future research topics.

1.4 NOTATION

Except for changes to improve clarity we follow the 'standard' geometric notation of [W6], to which we refer the readers for expository details of the underlying concepts; see also [G1]. Note that our notations in (viii) and (ix) assume the 'standard' results on LQ-optimization and the Riccati equation; see [W6] and [W3].

(i) small Roman letters x,y,z ... denote vectors, with the exception that the letters from i to n and also p,r denote integers.

$\underline{k} \equiv \{ 1,2,\ldots,k \}$

small Greek alphabets denote scalars ε R, except when argumented, in which case they denote polynomials (see (vi)).

(ii) Script letters $\mathcal{X},\mathcal{Y},\mathcal{Z}$ ..... denote vector spaces (finite dim. and over the real field) or subspaces. The following designation is fixed:

$\mathcal{X} = R^n$ denote the state-space

$\mathcal{U}_i = R^{m_i}$ " the ith control input space

$\mathcal{Y}_i = R^{p_i}$ " the ith observation output space

$\mathcal{Z}_i = R^{r_i}$ " the ith controlled output space

$\mathcal{U} = \mathcal{U}_1 x\ \mathcal{U}_2 x \ldots x\ \mathcal{U}_k$

$\mathcal{Y} = \mathcal{Y}_1 x\ \mathcal{Y}_2 x \ldots x\ \mathcal{Y}_k$

$\mathcal{Z} = \mathcal{Z}_1\ x \mathcal{Z}_2 x \ldots x\ \mathcal{Z}_k$

(iii) Capital Roman letters A,B,... denote maps between vector spaces. The same capital letters will be used to denote the matrix representation of the corresponding map . This is a 'standard' abusage; which is intended will always be clear from the context. A map $D : R^{n_1} \longrightarrow R^{n_2}$ will sometimes be equivalently denoted as $D \in R^{n_1 x n_2}$

F^T denotes the dual map (transpose matrix) of F. All maps whose range space dimension is less than its domain dimension will be defined in its dual form. This notation will be extremely convenient for mental clarity in dimension 'balancing' for 'messy' equations.

If $F : S \longrightarrow R$ and $\mathcal{D} \subset R$, then

$$F^{-1}\mathcal{D} \equiv \{ x \in S \mid F(x) \in \mathcal{D} \}$$

$$S^{\perp} \equiv \{ x^T \in X^{\perp} \mid x^T y = 0 \ \forall y \in S \} \quad S^{\perp} \equiv \text{annililator of } S$$

The following designations are fixed:

$A : X \longrightarrow X$ state dynamics map

$B_i : U_i \longrightarrow X$ i^{th} control map

$C_i^T : X \longrightarrow Y_i$ i^{th} observation map

$H_i^T : X \longrightarrow Z_i$ i^{th} controlled output map

$$[B_1, \ldots, B_k] : U_1 \times \ldots\ldots \times U_k \longrightarrow X$$

$$\begin{bmatrix} C_1^T \\ \cdot \\ \cdot \\ \cdot \\ C_k^T \end{bmatrix} : X \longrightarrow Y_1 \times \ldots \times Y_k$$

and correspond to the system (finite dim., time-inv., linear) :

$$\dot{x} = Ax + \sum_{i \in \underline{k}} B_i u_i$$

$$y_i = C_i^T x \qquad i \in \underline{k}$$

$$z_i = H_i^T x \qquad i \in \underline{k}$$

which we will simply denote as $\Sigma = (A, B_i, C_i^T, H_i^T, i \in \underline{k})$

(when the C_i^T's and H_i^T's are not relevant for a particular discussion we will shorten Σ to $\Sigma = (A, B_i, i \in \underline{k})$

the notation $y = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix}$ will also be used for $y \in Y_1 \times \ldots \times Y_k$

iv) Given $S \subseteq X$, $A S \subseteq S$,

$A \mid S$ = restriction of map A to S

$\overline{X} = X/S$ = factor space of A w.r.t. S

$\overline{A}, \overline{B}, \overline{C}$ = maps induced by maps A,B,C in $\overline{X}$ such that the following diagram commutes :

$$\begin{tikzcd}
 & X \arrow[r, "A"] \arrow[d, "P"] & X \arrow[dr, "C^T"] \arrow[d, "P"] & \\
U \arrow[ur, "B"] \arrow[dr, "\overline{B}"'] & & & Y \\
 & X/S \arrow[r, "\overline{A}"] & X/S \arrow[ur, "\overline{C}^T"'] &
\end{tikzcd}$$

i.e. $\overline{A}P = PA$, $\overline{C}^T P = C^T$

$\overline{B} = PB$

Note that $\overline{C}^T$ is well-defined iff $N(C^T) \subset S$

$\overline{x}$ = induced element in $\overline{X}$

v) Range space of map B = R(B)

Null space of map $C^T = N(C^T)$

$d(S)$ = dimension of space S

rk(F) = rank of map (matrix) F

vi) C^- (C^+) = open left (right)-half complex plane

$\pi(\lambda) = \det(\lambda I - A)$ characteristic polynomial of A (ch.p.)

$\alpha(\lambda)$ = minimal polynomial of A (m.p.)

$\sigma(A)$ = spectrum of A

$X_g(A) = N(\alpha_g(A))$ where $\alpha_g(\lambda) \mid \alpha(\lambda)$

$X^-(A)$ ($X^+(A)$) = stable (unstable) subspace of A in

$\dot{\cup}$ = disjoint union

vii) $\langle A \mid R(B) \rangle \triangleq R(B) + A\,R(B) + \ldots + A^{n-1}R(B)$

$N(C^T,A) \triangleq \bigcap_{i \in \underline{K}} N(C^T A^{i-1})$

' S is A-inv $\Longleftrightarrow$ $S \subset X$, $AS \subset S$

S is (A,B)-inv $\Longleftrightarrow$ $A\,S \subset S + R(B)$

S is (C^T,A)-inv $\Longleftrightarrow$ $A(S \cap N(C^T)) \subset S$

$I(A,B;V) = \{ S \subset V \mid A\,S \subset S + R(B) \}$

$I(C^T,A;V) = \{ S \subset V \mid A(S \cap N(C^T)) \subset S \}$

If $S \in I(A,B;V)$

$\underline{F}^T(A,B;S) = \{ F^T : X \longrightarrow U \mid (A+BF^T)S \subset S \}$

R is (A,B)- c.s. $\Longleftrightarrow$ $R = \langle A+BF^T \mid R(B) \cap R \rangle$ for some $F^T : X \longrightarrow U$

viii) $S(A,B)$ = the stabilizable pair (A,B)

$D(C^T,A)$ = the detectable pair (C^T,A)

$C(A,B)$ = the controllable pair (A,B)

$O(C^T,A)$ = the observable pair (C^T,A)

ix) given $S(A,B)$ and $D(H^T,A)$, $Q > 0$, $R > 0$

$K = K(A,B,H^T,R,Q)$ denotes the unique positive semidefinite solution to the Riccati equation (algebraic) :

$$PA + A^TP - PBR^{-1}B^TP + HQH^T = 0$$

(Q and R will always be assumed of compatible dimension)

For $R = I$, $Q = I$, we will simplify to $K = K(A,B,H^T)$

Note that the notation $K = K(A,B,H^T,R,Q)$ <u>will only be used when $SD(A,B,H^T)$ and</u> $Q > 0$ and $R > 0$. Hence <u>whenver K occurs it will be unique, well-defined, and</u> ≥ 0.

CHAPTER 2

GEOMETRIC CHARACTERIZATION OF INTERACTION STRUCTURE

2.1 Introduction

In this chapter we consider the problem of developing convenient mathematical characterization of the structure of control input interaction (and its dual) in a finite dimensional, linear time-invariant multivariable system. As our basic tools we will take the 'geomatric' approach of Wonham and Morse [W6] and also Basile and Marro [B1,2] . However, the class of problem we want to formulate and examine is different from those previously considered by them.

The word 'interaction' has appeared frequently in the literature on multivariable control, although the underlying concept intended by different researchers is by no means homogeneous. On the whole, the word 'interaction' has been taken to mean one of two things:

1) 'state coupling' : the usual situation is this. We are given a state-space model with a distinguished partition in a distinguished coordinate frame:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_2$$

from 'physical' consideration one regards the pairs (A_{11},B_1), (A_{22},B_2) as two different subsystems and the matrices A_{12}, A_{21} as providing coupling between the state vectors x_1 and x_2. One then talks about the 'interaction' between the state vectors x_1 and x_2, and derive concepts such as 'weak interaction' and 'singular perturbations' based on conditions related to the 'magnitude' of the 'coupling' terms A_{12}, A_{21} or the 'time constants' of A_1 and A_2 (see e.g. [A2] and references cited in [A6]).

The recent 'hierarchical' approach of Rosenbrock [R5] which considers arbitrary interconnections of subsystems falls within this framework of analysis.

(2) Input-output transmission interaction

This interpretation is the standard one in the literature on decoupling (non-interaction). Given a square input-output transfer function $H(s)$, or equivalently a realization (C^T,A,B): $C^T(sI-A)^{-1}B = H(s)$, the system so represented is said to be interacting if more than one control input affects an output or more than one output is affected by the same input; i.e. $H(s)$ contain non-diagonal terms (up to a permutation). The classical decoupling problem consists of finding suitable compensator such that the resulted cascaded system transfer function becomes diagonal. The subsequent state-space algebraic approach of Wolovich and Gilbert [G2], and the more recent geometric formulation of Wonham and Morse [M6] have substantially expanded the problem's generality (and transparency), but the basic concept on interaction remains that of input-output transmission interference.

A different interpretation of interaction will be considered in this thesis. Our starting point is the recognition that two things are involved in input-output transmission: how the input channels get connected to the state dynamics, and how the output channels get connected to the state dynamics. This observation is or course irrelevant in those 'black box' situations where the inputs and outputs are the only physically significant (and knowable) quantities, so that any one particular state-space realization (C^T,A,B) has no physical meaning. However, many real-world physical systems cannot be so regarded: we often have a situation where the state space itself do represent physically significant quantities, and where the 'given' control inputs and observation outputs in the model often represent only the existing channel implementation, rather than the

full set of physically available/knowable quatities, while the set of 'given' controlled outputs may well simply represent a selection (or definitional combination) of the physically significant states, and are often non-identical to the observed variables. In those situation where the state-vector itself represents physical quantities, therefore, it becomes meaningful to study the input-to-state and state-to-output structures separately.

Adapting the latter viewpoint, we are therefore led to examine the internal structure of input-output transmissions, in terms of (1) the input-to-state interaction (how the controllable subspaces of different controllers intersect) and (2) the state-to-output interdependency (how the observable subspaces of different measurement channels intersect). It is thus seen that the input output transmission interaction can properly be viewed as the combined effect of input-to-state coupling structure and state-to-output structure. This accords with our intuition: suppose two systems have identical input-to-state structure as well as state dynamics, but different state-to-output structure (because they are required to regulate different quantities, for instance), and hence different input-output properties. One ought to be able to recognize their 'similarity' with respect to input-to-state behavior, but such a notion of 'structural' similarity cannot be gotten if we just look at the control input-controlled output properties - the different output structures simply mask and disguise that input similarity beyond recognition.

2.2 Lattice structure of control interaction

The basic idea in characterizing the structure of control interaction is to indentify the intersection pattern of the controllable subspaces of the individual controllers, and to determine how the internal cyclic structures of the controllable subspaces intertwine where they intersect. Characterization of observation interdependency structure follows by simply taking the dual. Next, allows arbitrary feedback maps and use the idea of (A,B)-inv subspaces to examine how the input-to-state interaction structure can get modified. The change in state-to-output observability intertwining under feedback is next to be characterized. By combining the last two characterizations we then obtain the input-to-output interaction properties. Finally, by restricting the class of allowed feedback maps, interaction modification under output feedback can be studied. Due to lack of time this program has been carried out only very partially, but even then we will see that the partial results we have so far obtained provide useful insights.

The underlying algebraic structure we will exploit is that of a lattice. (see [S1], [B5] for the basic definitions and properties of lattices)

Let $\mathcal{X}$ be a finite-dimensional vector space over the real field. It is known that the set of all subspaces of $\mathcal{X}$ forms a lattice (under the inf and sup operations of $\cap$ and $\cup$ respectively) ordered by set inclusion $\subset$. Moreover, it is known that this lattice is infinite, modular, complete and complemented. Now let A be a linear map: $\mathcal{X} \to \mathcal{X}$, and define $I(A) \triangleq \{ \mathcal{S} \mid A\mathcal{S} \subset \mathcal{S} \}$, i.e. I(A) is the set of all A-invariant subspaces of $\mathcal{X}$. It is easy

to show that I(A) forms a sublattice of the lattice of all subspaces of $\mathcal{X}$. I(A) is always modular and complete; however, it needs NOT be infinite or complemented. We have in fact the following results from Brickman, etc. [B6]:

Fact 2.2.1

(i) I(A) is finite iff A is cyclic

(ii) I(A) is complemented iff A is diagonalizable

(ii) I(A) is distributive iff A is cyclic

Note that (ii), (iii) imply that, if A is cyclic and diagonalizable, then I(A) in fact forms a Boolean algebra. In this special case, the atoms of the lattice I(A) are simply the n one-dimensional (distinct) eigensubspaces of A, where $n=d(\mathcal{X})$, and I(A) consists of the 2^n possible unions of these eigensubspaces.

Consider now the system $\Sigma=(A,B_i,C_i^T,\ i\in\underline{k})$

$$\dot{x} = Ax + \sum_{i\in\underline{k}} B_i u_i \qquad (2.1.1)$$

$$y_i = C_i^T x, \quad i\in\underline{k}$$

where $d(\mathcal{X})=n$, $d(\mathcal{U}_i)=m_i$, $\sum_{i\in\underline{k}} m_i=m$, $d(\mathcal{Y}_i)=p_i$, $\sum_{i\in\underline{k}} p_i=p$

(see Notation 1.4)

Note that we are regarding the k groups of inputs (the u_i's) as independent. For convenience we will refer to the ith group of control inputs as simply 'controller i'.

The controllable subspace of controller i is given by

$$\langle A|R(B_i)\rangle \triangleq R(B_i) + AR(B_i)+\ldots\ldots+A^{n-1}R(B_i)$$

$\langle A|R(B_i)\rangle$ has the following algebraic properties:

Fact 2.2.2

1) $\langle A|R(B_i)\rangle = \inf\{S \mid AS\subset S, R(B_i)\subset S\}$

2) $\langle A|\sum_i R(B_i)\rangle = \sum_i \langle A|R(B_i)\rangle$

3) $\langle A+B_iF_i|R(B_i)\rangle = \langle A|R(B_i)\rangle \quad \forall$ linear map $F_2: \quad X \longrightarrow U_i$

4) Let $AS\subset S$, and let P be the canonical projection: $X \longrightarrow X/S$

Then $P\langle A|R(B_i)\rangle = \langle \bar{A}|R(\bar{B}_i)\rangle$
where $PA=\bar{A}P$, and $R(\bar{B}_i)=(R(B_i)+S)/S$

5) Let $\hat{A} \triangleq A|_{\langle A|R(B_i)\rangle}$, and let $R\subset\langle A|R(B_i)\rangle$ decompose $\hat{A}$, i.e. $\hat{A}R\subset R$ and $\exists\ S\subset\langle A|R(B_i)\rangle$ s.t. $\hat{A}S\subset S$ and $R\oplus S = \langle A|R(B_i)\rangle$. If $Q: \langle A|R(B_i)\rangle \to \langle A|R(B_i)\rangle$ is the projection on R along S, then

$$R = \langle A|QR(B_i)\rangle$$

Note that (1) $\Rightarrow \langle A|R(B_i)\rangle$ is an A-invariant subspace of (A-inv for short).

The $\{\langle A|R(B_i)\rangle\}_{i\in\underline{k}}$ in general have arbitrary intersections. When $\langle A|R(B_i)\rangle\cap\langle A|R(B_j)\rangle \neq 0$, then any state $\in \langle A|R(B_i)\rangle\cap\langle A|R(B_j)\rangle$ will be controllable by both u_i and u_j. The first step towards a structural analysis therefore consists of the idea of isolating those 'states' that are interaction free'. More precisely, we want to isolate the maximal subspaces R_i such that $R_i\subset\langle A|R(B_i)\rangle$ and $R_i\cap\langle A|R(B_j)\rangle = 0$ for $i \neq j$. We can in fact continue this decomposition for subspaces controllable by only two controllers, etc. The result can be succinctly stated as:

Theorem 2.2.1

Let $\mathcal{X}_1 \triangleq \langle A|\sum_{i\in\underline{k}} R(B_i)\rangle$

$$\mathcal{X}_2 \triangleq \sum_{i\in\underline{k}} [\langle A|R(B_i)\rangle \cap \langle A|\sum_{j\in\underline{k}-\{i\}} R(B_j)\rangle] = \bigcap_{i\in\underline{k}} [\langle A|\sum_{j\in\underline{k}-\{i\}} R(B_j)\rangle]$$

$$\vdots$$

$$\mathcal{X}_j \triangleq \sum_{\substack{I\subset\underline{k}\\ |I|=j-1}} [\langle A|\sum_{i\in I} R(B_i)\rangle \cap \langle A|\sum_{i\in\underline{k}-I} R(B_i)\rangle] = \bigcap_{\substack{I\subset\underline{k}\\ |I|=j-1}} \langle A|\sum_{i\in\underline{k}-I} R(B_i)\rangle$$

$$\vdots$$

$$\mathcal{X}_k \triangleq \bigcap_{i\in\underline{k}} \langle A|R(B_i)\rangle \tag{2.2.2}$$

Then $A\mathcal{X}_i \subset \mathcal{X}_i$, $\quad \mathcal{X}_1 \supset \mathcal{X}_2 \supset \cdots \quad \mathcal{X}_k$ (2.2.3)

$$\mathcal{X}_i = \mathcal{S}_i \oplus \mathcal{X}_{i+1}$$

$$\mathcal{S}_i \simeq \sum_{I\subset\underline{k}}^{\oplus} (\bigcap_{j\in I} \langle A|R(B_j)\rangle + \mathcal{X}_{i+1}/\mathcal{X}_i)$$

$$\simeq \sum_{I\subset\underline{k}}^{\oplus} (\bigcap_{j\in I} \langle A|R(B_j)\rangle / [\bigcap_{j\in I} \langle A|R(B_j)\rangle \cap \mathcal{X}_{i+1}]) \quad , \ |I| = i \tag{2.2.4}$$

moreover, if $d(\langle A|R(B_i)\rangle) = n_i$, $\quad i\in\underline{k}$

$$d(\mathcal{X}_i) = \ell_i, \qquad i\in\underline{k}$$

then $\sum_{i\in\underline{k}} n_i = \sum_{i\in\underline{k}} \ell_i$ (2.2.5)

Proof See pp. 35

Remark $\mathcal{X}_j$ is the smallest A-inv subspace such that $x\in\mathcal{X}_j \Rightarrow x$ is controllable by at least j of the controllers. $\mathcal{S}_j$ is the subspace of all states controllable by exactly j controllers. Our result says that $\mathcal{S}_j$ is decomposable into a direct sum of subspaces, each controllable by a distinct combination of j out of the k controllers (there are $\binom{k}{j}$ such possible combinations, but of course many of these will yield the zero subspace)

Remark $\mathcal{X}_1$ is just the controllable subspace of the pair $(A,[B_1,\ldots,B_k])$ and is of dimension ℓ_1. Wonham has suggested that $(\sum_{i\in\underline{k}} n_i - \ell_1)$ can serve as a measure of the 'degree' of interaction between the subspaces $\langle A|R(B_i)\rangle$, $i\in\underline{k}$. Our results above suggests that the list

$(\ell_1, \ldots, \ell_k)$ provides a much sharper measure.

<u>Remark</u>

For i=1, in (2.24) we have the $S_{11}, \ldots, S_{1k}$ as the <u>interaction-free</u> subspaces, i.e. $S_{1i} \subset \langle A | R(B_i) \rangle$, $S_{1i} \cap \langle A | R(B_j) \rangle = 0$, $i \neq j$

We note the following special cases :

1) Complete non-interaction :

$$X_2 = 0 \iff \langle A | \sum_{i \in \underline{k}} R(B_i) \rangle = \langle A | R(B_1) \rangle \oplus \ldots \oplus \langle A | R(B_k) \rangle$$

2) Complete interaction :

$$X_1 = \ldots = X_k \iff \langle A | \sum_{i \in \underline{k}} R(B_i) \rangle = \langle A | R(B_i) \rangle \quad \forall\ i \in \underline{k}$$

3) Centralized interaction :

$$X_2 = \ldots = X_k \iff \langle A | \sum_{i \in \underline{k}} R(B_i) \rangle = S_{11} \oplus \ldots \oplus S_{1k} \oplus \bigcap_{i \in \underline{k}} \langle A | R(B_i) \rangle$$

where $\langle A | R(B_i) \rangle = S_{1i} \oplus \bigcap_{j \in \underline{k}} \langle A | R(B_j) \rangle$

4) 'Nested' interaction :

for some permutation : $(1, \ldots, k) \longmapsto (i_1, \ldots, i_k)$,

$$X_1 = \langle A | R(B_{i_1}) \rangle, \ldots, X_k = \langle A | R(B_{i_k}) \rangle \iff \langle A | R(B_{i_k}) \rangle \subset \ldots \subset \langle A | R(B_{i_1}) \rangle$$

5) 'Neighborhood' interaction :

$$X_2 = (\langle A | R(B_{i_1}) \rangle \cap \langle A | R(B_{i_2}) \rangle) \oplus \ldots \oplus (\langle A | R(B_{i_{k-1}}) \rangle \cap \langle A | R(B_{i_k}) \rangle$$
$$\iff \langle A | R(B_{i_m}) \rangle \cap \langle A | R(B_{i_j}) \rangle = 0, \quad |m-j| > 1$$

The following easy consequences of Theorem 2.2.1 are also noted :

<u>Corollary</u> 2.2.1

1) $R(B_i) \subset X_j \iff \langle A | R(B_i) \rangle \subset X_j \ \forall\ j \in \underline{k}$

2) $\sum_{i \in I} R(B_i) \subset \langle A | \sum_{i \in \underline{k}-I} R(B_j) \rangle \iff \langle A | \sum_{i \in I} R(B_i) \rangle \subset X_{|I|+1}$

The next thing we want to do is to introduce more structure into the original space X. The simplest thing to do is to allow a partition of X, so that certain subspace of X is made 'distinguished'. Briefly, the program is as follows (see W [6]). Partition the complex plane as $C = C_g \cup C_b$, where $C_g^* = C_g$, and where C_b roughly represents the undesirable domain of C that we might want the poles of A to <u>avoid</u>. Next partition the m.p. of A as $\alpha(\lambda) = \alpha_g(\lambda)\alpha_b(\lambda)$, where the zeros of $\alpha_g(\alpha_b) \in C_g(C_b)$. Then we have

$$X = X_g(A) \oplus X_b(A) \qquad \text{with} \quad X_g(A) \equiv N(\alpha_g(A))$$

and $A X_g \subset X_g$, $\quad A X_b \subset X_b \qquad X_b(A) \equiv N(\alpha_b(A))$

We can now consider the interaction structure w.r.t. distinguished $X_b \subset X$.

<u>Definition</u> 2.2.1

The system (2.2.1) is C_g - assignable w.r.t. u_i iff $X_b \subset \langle A | R(B_i) \rangle$

<u>Definition</u> 2.2.2

The system (2.2.1) is individually C_g - assignable w.r.t. $\{u_i\}_{i \in \underline{k}}$ iff $X_b \subset \bigcap_{u \in \underline{k}} \langle A | R(B_i) \rangle$

<u>Remark</u>

If $X_b = X^+$ (unstable subspace), def. 2.1.1 corresponds to the definition of stabilizability w.r.t. u_i.

By noticing the duality : $N(C^T,A) = \langle A^T | R(C) \rangle^{\perp}$, we see that the dual concept of observation redundancy (overlap of observability) can be analogously defined by simply taking the dual of Theorem 2.2.1.

Proof of Theorem 2.2.1

$A\mathcal{X}_j \subset \mathcal{X}_j$ since A-inv subspaces form a lattice under the operation of intersection and sum.

That $\mathcal{X}_{j+1} \subset \mathcal{X}_j$ is easy to check

(2.2.4) follows from repeated application of Lemma 10.1(iii) of [W6], pp.249, and the fact that $(\mathcal{R}_1 + \mathcal{R}_2)/\mathcal{R}_2 \approx \mathcal{R}_1/(\mathcal{R}_1 \cap \mathcal{R}_2)$

To prove (2.2.5), note that $d(\mathcal{X}_i) = d(\mathcal{X}_{i+1}) + d(\mathcal{S}_i)$

so
$$\sum_{i\in\underline{k}} \ell_i = \sum_{i\in\underline{k}} d(\mathcal{X}_i) = \sum_{j=1}^{k} \sum_{i=j}^{k} d(\mathcal{S}_i)$$
$$= d(\mathcal{S}_1) + 2d(\mathcal{S}_2) + \ldots + kd(\mathcal{S}_k)$$

But the right hand side is just $\sum_{i\in\underline{k}} n_i$

2.3 Modification of interaction structure under State feedback

While $\langle A + B_i F_i^T \mid R(B_i)\rangle = \langle A \mid R(B_i)\rangle \quad \forall \; F_i : \mathcal{X} \longrightarrow \mathcal{U}_i$ it is NOT true in general that $\langle A + B_j F_j^T \mid R(B_i)\rangle = \langle A \mid R(B_i)\rangle$ for $F_j^T : \mathcal{X} \rightarrow \mathcal{U}_j$, $j \neq i$. That is, the controllable subspace of one controller can be altered by the feedback actions of other controllers. Similarly, $N(C_i^T, A + B_i K_i C_i^T) = N(C_i^T, A) \quad \forall \; K_i$, but $N(C_i^T, A + B_j K_j C_j^T) \neq N(C_i^T, A)$ in general if $i \neq j$, i.e. the unobservable subspace of one controller is alterable by the output feedback of other control agents.

In this section we therefore turn to examine how the interaction structure of $\Sigma = (A, B_i, i \in \underline{K})$ can be altered by arbitrary state feedback loops.

This problem has been examined by Aoki in [A3], [A4] for the case when k=2. We will therefore summarize some of their main results, generalizing where possible. We begin with

Lemma 2.3.1

$$\langle A + \sum_{j \neq i} B_j F_j^T \mid R(B_i)\rangle = \langle A \mid \sum_{i \in \underline{k}} R(B_i)\rangle \qquad \forall \; F_j^T : \mathcal{X} \longrightarrow \mathcal{U}_j$$

Remark

Even though the individual controllable subspaces are alterable by feedback, the sum of all the controllable subspaces is an invariance under feedback.

Lemma 2.3.2 [A4]

i) $\exists \; F_2^T$ such that $\langle A + B_2 F_2^T \mid R(B_1)\rangle = \langle A \mid R(B_1) + R(B_2)\rangle$

ii) If A is cyclic over $<A \mid R(B_1) + R(B_2)>$, and $<A \mid R(B_2)> \not\subset N(C_1^T)$

then $\exists$ K_1 such that $<A + B_1K_1C_1^T \mid R(B_2)> = <A \mid R(B_1) + R(B_2)>$

The above results show that the controllable subspace of anyone control can always be made 'maximal'. The inverse case, i.e. of minimizing the controllable subspace of one controller by the feedback actions of others can be solved (see [A4]) by Wonham and Morse's index algorithm [W7], by noticing that

$$\inf_{F_2} \left\{<A+B_2F_2^T|R(B_1)>\right\} = \inf \left\{ S \mid AS \subset S + R(B_2),\ S \supset R(B_1)\right\}$$

However the algorithm is fairly complicated. Our next result provides a partial characterization that is more explicit:

<u>Lemma</u> 2.3.3

If $A^{K+1}R(B_1) \subset (R(B_1) + \ldots + A^K R(B_1)) + R(B_2)$, $0 \leq k \leq n-2$ (2.3.1)

then $\exists$ F_2^T such that

$$<A+B_2F_2^T|R(B_1)> \subset R(B_1) + \ldots + A^K R(B_1)$$

<u>Proof</u>

$$(2.3.1) \Longleftrightarrow A(R(B_1)+\ldots+A^K R(B_1)) \subset (R(B_1)+\ldots+A^K R(B_1)) + R(B_2)$$

$$\Longleftrightarrow (R(B_1)+\ldots+A^K R(B_1)) \text{ is an } (A,B_2) \text{ - inv.}$$

$$\Longleftrightarrow \exists\ F_2^T \text{ s.t. } (A+B_2F_2^T)\ (R(B_1)+\ldots+A^K R(B_1)+\ldots+A^K R(B_1)) \subset (R(B_1)+\ldots + A^K R(B_1))$$

$$\Longrightarrow <A+B_2F_2^T|R(B_1)> \subset R(B_1)+\ldots+A^K R(B_1)$$

Example

Consider

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ a_n & \cdots & & a_2 & a_1 \end{bmatrix} \qquad b_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

In this case $AR(b_1) \subset R(b_1) + R(b_2)$, so by lemma 2.3.3. $\exists$ F_2^T s.t. $< A + B_2F_2^T | R(B_1)> = R(B_1)$. In fact, F_2^T is even unique, and $= [0, \ldots, 0, -1]$. This is in a sense the 'canonical' situation for a controllable subspace to be reduced to its minimum possible dimension. More generally, we have :

lemma 2.3.4

Let $<A | R(B_i) + \ldots + R(B_k)> = X$, $d(R(B_i))=1$

and let $[B_1, \ldots, A^{\sigma_1-1} B_1, B_2, \ldots A^{\sigma_2-1} B_2, \ldots, A^{\sigma_k-1} B_k]$ spans X

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 1$, $\sum_{i \in \underline{k}} \sigma_i = n$

are the controllability indices of $(A, [B_1, \ldots, B_k])$.

Then $\exists$ $F_1^T, \ldots, F_k^T$ such that

$< A + B_2F_2^T, \ldots, \quad + B_kF_k^T | R(B_1)> \subset R(B_1) + \ldots + A^{\sigma_1-1} R(B_1)$

$< A + B_1F_1^T + B_3F_3^T + \ldots + B_kF_k^T | R(B_2)> \subset (R(B_1) + \ldots + A^{\sigma_1-1} R(B_1)$

$+R(B_2) + \ldots + A^{\sigma_2-1} R(B_2))$

$< A + B_1F_1^T + \ldots + B_{k-1}F_{k-1}^T | R(B_k)> = X$

Proof

This result follows by a straight-forward application of Wolovich's 'canonical structure' theorem[W4] and hence is ommited.

2.4 Structural Homomorphisms

In this section we introduce a concept of 'structural' homomorphism (or similarity) for the class of systems $\Sigma = (A, B_i, H_i^T, i \in \underline{k})$ based on the idea that , if the lattice structure generated by the controllable (observable) subspaces of the individual controllers (sensors) of one system is similar to that being generated in another system, then the two systems will have similar interaction (interdependency) pattern. The basic motivation here is to provide a more precise means for studying classes of systems with similar interaction(interdependency) characteristics. What, for instance, is the class of objects we are really referring to when we talk about 'noninteracting' systems,'weakly interacting' systems, etc. ?

Without loss of generality we will assume in this section that the following assumptions hold:

1) $X = \langle A \mid R(B_1) + \ldots + R(B_k) \rangle$

2) $\bigcap_{i \in \underline{k}} N(C_i^T, A) = 0$

i.e. the system is controllable and observable if a central agent has access to all the control and observation channels.

We will first introduce the conceptual definitions, then discuss their interpretations.

<u>Definition</u> 2.4.1

Let I_1 , I_2 be two lattices. A lattice homomorphism f is a map $f : I_1 \longrightarrow I_2$ such that $\forall\ s_1, s_2 \in I_1$,

$$f(s_1 \cup s_2) = f(s_1) \cup f(s_2)$$

$$f(s_1 \cap s_2) = f(s_1) \cap f(s_2)$$

Definition 2.4.2

Let I_1, I_2 be bounded, complete lattices, and let $J_1 \subset I_1$, $J_2 \subset I_2$ be fixed finite sublattices such that

$$\bigcup_{s \in J_1} s = \sup I_1 \ , \qquad \bigcup_{s \in J_2} s = \sup I_2 \ , \qquad \text{and} \quad |J_1| = |J_2|$$

Further, let $R_1 \in I_1$, $R_2 \in I_2$ be fixed elements

Then (I_2, J_2, R_2) is said to be a structure homomorph of (I_1, J_1, R_1) if there exists a lattice homomorphism

$$f : I_1 \longrightarrow I_2$$

such that 1) $f|J_1$ is a bijection : $J_1 \longrightarrow J_2$, 2) $f(R_1) = R_2$

Definition 2.4.3

Given the two systems $\Sigma = (A, B_i, i \in \underline{k})$ and $\hat{\Sigma} = (\hat{A}, B_i, i \in \underline{k})$ with $X_b \in I(A)$, $\hat{X}_b \in I(\hat{A})$ two fixed subspaces, we say that the pair (Σ, X_b) is an input homomorph of $(\hat{\Sigma}, \hat{X}_b)$ iff

$(I(A), \{\langle A | R(B_i) \rangle\}_{i \in \underline{k}}, X_b)$ is a structure homomorph of

$(I(\hat{A}), \{\langle \hat{A} | R(\hat{B}_i) \rangle\}_{i \in \underline{k}}, \hat{X}_b)$

Remark

In words, definition 2.4.3 says that one system is input homomorphic to another if :

1) they have the same number of control input (groups)

2) whenever an invariant subspace is controllable by some subset of control inputs in one system, there is a corresponding invariant subspace controllable by a corresponding subset of control inputs in the other system.

3) a 'bad' invariant subspace in one system corresponds to a 'bad' invariant subspace in the other system.

Example

$$A = \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & -1 & 2 & & \\ & & -2 & -1 & & \\ & & & & 2 & \\ & & & & & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is input homomorphic to

$$\hat{A} = \begin{bmatrix} 3 & & & \\ & -2 & & \\ & & 3 & \\ & & & -1 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \hat{B}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \hat{x}_1 \qquad \langle A \mid R(B_1) \rangle \longmapsto \langle \hat{A} \mid R(\hat{B}_1) \rangle$$

etc.

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \longmapsto \hat{x}_2 \qquad \text{and} \qquad X^+ \approx \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} \longmapsto \begin{bmatrix} \hat{x}_1 \\ \hat{x}_3 \end{bmatrix} \approx \hat{X}^+$$

$$x_5 \longmapsto \hat{x}_3$$

$$x_6 \longmapsto \hat{x}_4$$

Lemma 2.4.1

If (Σ, X_b) is input-homomorphic to $(\hat{\Sigma}, \hat{X}_b)$, then

$$X_i \longmapsto \hat{X}_i \qquad \forall\ i \in \underline{k}$$

Remark

The dimension of X and $\hat{X}$ need not be the same; neither need $R(B_i), R(\hat{B}_i)$, nor $\langle A | R(B_i) \rangle$ and $\langle \hat{A} | R(\hat{B}_i) \rangle$, nor X_b and $\hat{X}_b$.

Remark

If $X_b = X^+$, $\hat{X}_b = \hat{X}^+$, two input-homomorphic systems not only have 'similar' interaction structure, but similar interaction structure w.r.t. stabilizability.

Definition 2.4.4

If an input homomorphism : $(\Sigma, X_b) \longrightarrow (\hat{\Sigma}, \hat{X}_b)$ is in fact an isomorphism, then the two pairs are said to be <u>input-equivalent</u>.

Remark

If (Σ, X_b) and $(\hat{\Sigma}, \hat{X}_b)$ are input-isomorphic, then $I(A)$ and $I(\hat{A})$ have the same cardinality; in particular, if A is cyclic($\Longrightarrow \hat{A}$ is cyclic), then $I(A), I(\hat{A})$ are finite, $d(\langle A | R(B_i) \rangle) = d(\langle \hat{A} | R(\hat{B}_i) \rangle)$ (up to a permutation), $d(X_b) = d(\hat{X}_b)$. However, $R(B_i)$ and $R(\hat{B}_i)$ still need not have the same dimension.

We can dually define the concept of state-to-output homomorphism in a straight-forward way.

Definition 2.4.5

Given the system $\Sigma = (A, C_i^T, i\varepsilon\underline{k})$ and $\Sigma = (\hat{A}, \hat{C}_i^T, i\varepsilon\underline{k})$, and $X_b \varepsilon I(A)$, $\hat{X}_b \varepsilon I(\hat{A})$, we say that the two systems are output-homomorphic iff there exists a structure homomorphism :

$$f : (I(A), \{ N(C_i^T, A)\}_{i\varepsilon\underline{k}}, X_b) \longrightarrow (I(\hat{A}), \{ N(\hat{C}_i^T, \hat{A})\}_{i\varepsilon\underline{k}}, \hat{X}_b)$$

Remarks dual to input-homomorph will apply here of course.

Remark

We have defined output-homomorphism in terms of observation outputs. It is entirely analogous to define controlled-output homomorph (i.e. in terms of $N(H_i^T, A)$)

We now combine the input and output sides to obtain a 'system' similarity characterization :

Definition 2.4.6

Two systems $\Sigma = (A, B_i, C_i^T, i\varepsilon\underline{k})$ and $\Sigma = (\hat{A}, \hat{B}_i, \hat{C}_i^T, i\varepsilon\underline{k})$ are <u>input-output</u> homomorphic iff there exists a structural homomorphism

$$f : (I(A), \{\langle A|R(B_i)\rangle, N(C_i^T, A)\}_{i\varepsilon\underline{k}}, X_b) \longrightarrow (I(\hat{A}), \{\langle \hat{A}|R(\hat{B}_i)\rangle, N(\hat{C}_i^T, \hat{A})\}_{i\varepsilon\underline{k}}, \hat{X}_b)$$

Remark

Given two systems Σ_1, Σ_2 that are both input-homomorphic and output-homomorphic it does not follow that the two systems are input-output homomorphic: the same homomorphic map has to work for both inputs and outputs in order for this to be true. Two input-output homomorphic systems not only have the same control interaction and observation interdependency pattern, but the same input-output transmission pattern.

Remark

It is obvious that the input(output) homomorphisms as defined are invariant under coordinate transformation(as well as input(output) transformation that preserves the individual control input's range space). However, they are in general not preserved under action of arbitrary feedback. This is so because the definition of homomorphisms we have used depend only on the fact that each controllable subspace is an invariant subspace; the internal cyclic structure of a controllable subspace has not been utilised. A stronger notion of similarity has therefore to be imposed to overcome this undesirable state of affairs; however, we feel that any such stronger notions must at least incorporate the features taken care of by our weaker notions.

One area where the above concepts might be useful is in the classification of large-scale linear systems in terms of their interaction pattern. If any system has one of the special input structure in pp32, for example, then any homomorphic image of it will also have the special structure. Other system structural concepts, e.g. 'prime' system [M 7], decentrally decouplable systems(systems decouplable by only feedback modification without input recombinations), 'non-minimal' phase systems [D2], etc. can be studied as equivalent classes of systems under structural homomorphisms (suitably strengthened).

Another area where the above concepts might turn out to be useful is that of model aggregation. The traditional definition of model aggregation (see [A2]) is not very useful since it is too restrictive --- it amounts to simply throwing away some eigenstates(i.e. it reduces a system modulo an invariant subspace). It might be useful to consider obtaining model aggregation by putting further constraints (yet to be identified) on a reduced homomorphic image of a system.

CHAPTER 3

STRUCTURAL PROPERTIES OF NON-DESTABILIZING AND LQ-OPTIMAL FEEDBACK

3.1 Introduction

In this chapter, an approach different from the previous chapter is taken to study the structural properties of linear multivariable systems under the application of multi-loop feedback. Rather than focusing our attention on the subspaces, we turn now instead to concentrate on the feedback maps. The motivation is, given a particular desired system property, what class of feedback maps can realize it? In particular, we want to know, what is a convenient characterization of a class of feedback that never destabilizes an originally (open-loop) stable system? that always stabilizes an originally (open-loop) unstable system?

The fundamental motivation behind this approach is the recognition that the class of feedback computable from solving some LQ-optimization problem has 'nice' stabilizing properties, and that these properties are intimately related to the system structure. We therefore seek to generalize the class of feedback maps to those that never destabilize an originally open-loop stable system, and to see what structural stability properties are guaranteed by these more general maps. Understanding along this line will turn out to be extremely useful for the reliable stabilization synthesis problems in Chapter 4. It also allows us to generalize and make more transparent some of the known stability properties of LQ-feedback.

3.2 Structural properties of non-destabilizing feedback

Our objective in this section is to identify and characterize a conveninet class of feedback maps that has the fundamental property that any member of the class never destabilizes the original open-loop stable system.

We begin by first recalling the standard 'Lyapunov' stability theorems:

lemma 3.2.1 (Lyapunov)

1) A is stable $\Longrightarrow$ ($Q \geq 0$, $PA + A^TP + Q = 0$ has a unique solution $P \geq 0$)

if in addition, $(Q^{1/2}, A)$ is observable, then $P>0$

2) $P \geq 0$, $Q \geq 0$, $(Q^{1/2}, A)$ detectable and $PA + A^TP + Q = 0 \Longrightarrow$ A is stable

The next lemma will provide a useful interpretation of non-destabilizing feedback maps later.

lemma 3.2.2

Let $\hat{A} \triangleq WK$, where $W \geq 0$, $K \geq 0$. Then $\hat{A}$ is diagonalizable and has a real, non-negative spectrum.

Proof since $W \geq 0$, it can be decomposed as $W=DD^T$ for some D. Hence $\det(sI-\hat{A}) = \det(sI-DD^TK) = \det(sI-D^TKD)$ but $D^TKD \geq 0$, so $\sigma(DD^TK) = \sigma(D^TKD) \cup \{0\} \subset R^+ \cup \{0\}$. That $\hat{A}$ is diagonalizable follows from the fact that the zero eigenvalues are simple.

The next lemma is a basic structural characterization of the Lyapunov equation.

<u>lemma</u> 3.2.3

Let A be stable, (H^T, A) detectable

and $PA + A^TP + HH^T = 0$

Then $N(P) = N(H^T, A)$

$R(P) = \langle A^T | R(H) \rangle$

<u>Proof</u> $$P = \int_0^\infty e^{A^Tt} HH^T e^{At}\, dt$$

so $x \in N(H^T,A) \Longrightarrow H^Te^{At}x = 0$

and thus $N(H^T,A) \subset N(P)$

Conversely, $x \in N(P) \Longrightarrow x^TPx = 0 \Longrightarrow \int_0^\infty |H^Te^{At}x|^2 dt = 0$

$\Longrightarrow x \in N(H^T,A)$

To complete the proof, note that $R(P) = R(P^T) = N(P)^\perp$

$= N(H^T,A)^\perp = \langle A^T | R(H) \rangle$

<u>Definition</u> 3.2.1

For any stable matrix A, define

$$LP(A) \triangleq \{K \geq 0 \mid KA + A^TK \leq 0\}$$

$$\text{and } LP^+(A) \triangleq \{K \geq 0 \mid KA + A^TK < 0\}$$

<u>Corollary</u> 3.2.1

if $P \in LP(A)$ and $P \not> 0$

Then $AN(P) \subset N(P)$

<u>Proof</u> $(PA + A^TP + HH^T)N(P) = 0 \Longrightarrow PA\,N(P) = 0 \quad A\,N(P) \subset N(P)$

Important Remark

LP (A) is a proper subset of the set of all positive semi-definite matrices (of dimension n). The following example illustrates the source of the difficulty:

$$A = \begin{bmatrix} -1 & 6 \\ 0 & -2 \end{bmatrix} \quad , \quad K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0$$

but $KA + A^T K = \begin{bmatrix} -2 & 6 \\ 6 & -4 \end{bmatrix} \not< 0$

Similarly, the sum of two stable matrices need NOT be stable.

Example:

$$A_1 = \begin{bmatrix} -1 & 6 \\ 0 & -2 \end{bmatrix} \quad , \quad A_2 = \begin{bmatrix} -2 & 0 \\ 3 & -3 \end{bmatrix}$$ are both stable, but

$$A_1 + A_2 = \begin{bmatrix} -3 & 6 \\ 3 & -5 \end{bmatrix}$$ is NOT stable

Since the closed loop system matrix is a sum of the open-loop matrix and a feedback matrix, the above remarks underscore the fact that it is not quite trivial to find feedback that does not destabilize the open-loop system. That this can in fact be done is the content of Theorem 3.2.1:

Theorem 3.2.1

Let A be any stable matrix

Then $(A + (S-W)K)$ is a stable matrix

$\forall\ K \in LP(A)$ and $S = -S^T$, $W \geq 0$, $R(S) \subset R(W)$

If $K \in LP^+(A)$, the condition $R(S) \subset R(W)$ can be omitted.

To prove theorem 3.2.1 we need the following result by Wonham

([W6], pp. 82):

<u>lemma</u> 3.2.4

If $Q \geq 0$, $(Q^{1/2}, A)$ is observable (detectable) then $\forall$ $P = P^T \geq 0$, $R > 0$ and $\forall$ B, F^T, the pair $(\sqrt{Q+P+FRF^T}, A+BF^T)$ is observable (detectable)

<u>Proof of Theorem</u> 3.2.1

Let $Q \triangleq -(KA+A^TK)$

since $K \in LP(A)$, $Q \geq 0$

and A is stable $\Longrightarrow$ $(Q^{1/2}, A)$ is always detectable

We have

$$K(A + (S-W)K) + (A + (S-W)K)^T K + 2KWK + Q - \underbrace{(KSK+KS^TK)}_{\substack{\| \\ 0}} = 0$$

If $K \in LP^+(A)$, then $Q > 0$

so $(Q^{1/2}, A + (S-W)K)$ is observable

$\Longrightarrow$ $(A + (S-W)K)$ is stable

Otherwise, assume $R(S) \subset R(W)$

$\Longrightarrow$ $\exists$ V such that $S = VW$

$\Longrightarrow$ $(S-W)K = (V-I)WK$

By defining $B \triangleq (V-I)W^{1/2}$

$F^T \triangleq W^{1/2}K$

$P \triangleq 0$

We have from <u>lemma</u> 3.2.4 that

$(\sqrt{Q+KWK}, A + (S-W)K)$ is detectable

hence from lemma 3.2.1 (2), $A + (S-W)K$ is stable

Given $\Sigma = (A,B)$ with A stable,

If $u = (N-M)B^TKx$, with $N = -N^T$, $M>0$

and $K \in LP(A)$

then (3.2.1)

$$Bu = B(N-M)B^TKx$$

$$= (S-W)K\,x \qquad R(S) \subset R(W) \qquad (3.2.2)$$

Hence any feedback of the form (3.2.1) will never destabilize A. The following definition is convenient:

Definition 3.2.2

given $\Sigma = (A,B)$ with A stable,

The set $NDS(\Sigma) \triangleq \{G^T\ R^{nxm} \mid G^T = (N-M)B^TK,\ M>0,\ N=-N^T$

and $K \in LP(A)\ \}$

is called the class of non-destabilizing feedback of $\Sigma = (A,B)$, and any $u = G^Tx$, $G^T \in NDS(\Sigma)$ will be called an ndsf.

Remark

The following interpretation of the non-destabilizing action of the class $NDS(\Sigma)$ is useful. Note that from lemma 3.2.2, the term $(-WK)$ in (3.2.2) has non-positive eigenvalues, while the term (SK), being a product of a skew-symmetric matrix and a non-negative matrix, has all its eigenvalues with zero real part. Thus, roughly speaking, the first term provides 'negative' pole shifting while the second term provides 'complex conjugation'. Note that the restriction $K \in LP(A)$ conceptually define the 'directions' of permissible feedback in terms of the structure of A. It is a long-standing 'conjec-

ture' in the literature on 'interconnected' systems that the Lyapunov equation provides information on the structure of A ('k_{ij} can be regarded as an index of coupling between state x_i and x_j')

The next corollary is easy but has an interesting interpretation.

<u>Corollary</u> 3.2.2

Let A be stable

If $W \in LP(A^T)$

then (A-WK) is stable $\forall\ K \geq 0$

<u>Proof</u> $W \in LP(A^T) \Rightarrow (A^T-KW)$ stable $\forall\ K \geq 0$

$(A-WK)$ " "

<u>lemma</u> 3.2.5

Let A be stable

Then

(1) $P_1 \in LP(A),\ P_2 \in LP(A) \Rightarrow \alpha_1 P_1 + \alpha_2 P_2 \in LP(A)$

$\forall\ \alpha_1 \geq 0,\ \alpha_2 \geq 0$

(2) $P \in LP(A_1),\ P \in LP(A_2) \Rightarrow P \in LP(\alpha_1 A + \alpha_2 A_2)$

$\forall\ \alpha_1 \geq 0,\ \alpha_2 \geq 0$

(3) $P \in LP(A) \Rightarrow P \in LP(A+(S-W)P)$

$\forall\ W \geq 0,\ S=-S^T,\ R(S) \subset R(W)$

<u>Proof</u>

(1), (2) straight-forward

(3) We have $P(A+(S-W)P) + (A+(S-W)P)^T P = PA + A^T P - 2PWP$

≤ 0

We next give an example to illustrate the content of <u>Theorem 3.2.1</u>

<u>Example</u>

given $\Sigma = (A,b)$, suppose $\exists$ a basis such that A is of the form

$$A = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_\ell & & & \\ & & & \lambda_{\ell+1} & \omega & \\ & & & -\omega & \lambda_{\ell+1} & \\ & & & & & \ddots \end{bmatrix} \quad \lambda_i<0 \qquad \text{and} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Then any K of the form:

$$K = \begin{bmatrix} k_1 & & & & & \\ & \ddots & & & & \\ & & k_\ell & & & \\ & & & k_{\ell+1} & 0 & \\ & & & 0 & k_{\ell+1} & \\ & & & & & \ddots \end{bmatrix} \qquad \text{with } k_i \geq 0$$

is a member of LP(A),

and any feedback of the form $u = -[k_1 b_1, \ldots\ldots, k_n b_n]x$ is an ndsf.

This example can be generalized as follows:

Let $K = DD^T \geq 0$, A stable

If $D^TA = \hat{A}D^T$ for some $\hat{A}$ symmetric

Then $K \in LP(A)$

<u>Proof</u> $$DD^TA + A^TDD^T = D(\hat{A}+\hat{A}^T)D^T = 2D\hat{A}D^T \leq 0$$

since $\sigma(\hat{A}) \subset \sigma(A)$ and $\hat{A} = \hat{A}^T$

The class of ndsf identified in Theorem 3.2.1 can be further generalized.

<u>lemma</u> 3.2.6

Let A be stable, (H^T, A) - observable, and $M > 0$

Then

$$(A + P^{-1}H(\hat{M} + N)H^T) \text{ is stable} \tag{3.23}$$

$$\hat{M} < \tfrac{1}{2}M, \quad \hat{M} = \hat{M}^T, \quad N = -N^T$$

$$\text{where } PA + A^TP + HMH^T = 0 \tag{3.2.4}$$

<u>Proof</u>

(H^T, A) - observable $\Rightarrow P > 0$

from (3.2.4), we get

$$P(A + P^{-1}H(\hat{M}+N)H^T) + (A+P^{-1}H(\hat{M} + N)H^T)P + H(M-2\hat{M})H^T = 0$$

so $\hat{M} < \frac{1}{2}M \Rightarrow H(M-2\hat{M})H^T \geq 0$

and $(\sqrt{H(M-2\hat{M})H^T},\ A + P^{-1}H(\hat{M}-N)H^T)$ - observable

thus by <u>lemma</u> 3.2.1(2), (3.2.3) is true

The dual of <u>lemma</u> 3.2.6 is important enough to be stated as a theorem.

Theorem 3.2.2

Let A be stable, (A,B)-controllable, $M>0$

Then $(A + B(\hat{M}+N)B^TP^{-1})$ is stable (3.2.5)

$\forall \quad \hat{M} < \frac{1}{2}M, \quad N = -N^T$ and $PA^T + AP + BMB^T = 0$ (3.2.6)

Remark

Since the eigenvalues of $B(M + N)B^TP^{-1}$ all have nonnegative real parts, (3.2.5) can be loosely interpreted as identifying a class of 'positive' feedback that does not destabilize A , and hence provides a sort of 'stability margin' characterization of A. Note that (3.2.6) can be rewritten as

$$P^{-1}A + A^TP^{-1} + P\,BMB^TP^{-1} = 0$$

or $\quad P^{-1} \in LP(A)$

Thus (3.2.5) is of the form $(A + (S + W)K)$, (3.2.7)

with $K \in LP(A)$ and W suitable restricted as defined in Theorem 3.2.2. In this sense, (3.2.5) is a generalization of ndsf.

3.3 Structural properties of LQ-optimal feedback

It is well-known that, given $SD(A.B.H^T)$ and $Q>0$, $R>0$ (Q,R of appropriate dimentions), a unique, positive semidefinite solution matrix K exists to the algebraic Riccati equation:

$$KA + A^TK - KBR^{-1}B^TK + HQH^T = 0$$

moreover, the closed loop system

$$\dot{x} = (A - BR^{-1}BK)x \quad \text{is stable} \tag{3.3.1}$$

and $\quad N(H^T,A)=0 \qquad K \geq 0$

We will denote the unique K as $K(A,B,H^T,R,Q)$

The Linear Quadratic Optimization interpretation of K is that

$$u^* = -R^{-1}B^TK\,x$$

is the solution to:

$$\min_{u\in\mathcal{U}} \int (x^TQx + u^TRu)dt \tag{3.3.2}$$

subject to $\dot{x} = Ax + Bu$, $x(0) = \text{arbitrary}$

$$z = H^Tx$$

Structurally, the use of Q and R is equivalent to applyting the maps: $B \mapsto BR^{-\frac{1}{2}}$, $H^T \mapsto Q^{\frac{1}{2}}H^T$. For let $\hat{u} = R^{\frac{1}{2}}u$, $\hat{z} = Q^{\frac{1}{2}}z$. Then (3.3.2) is equivalent to the problem:

$$\min_{\hat{u}\in\mathcal{U}} \int (\hat{z}^T\hat{z} + \hat{u}^T\hat{u})dt \qquad \text{subject to} \qquad \begin{aligned} \dot{x} &= Ax + (BR^{-\frac{1}{2}})\hat{u} \\ \hat{z} &= (Q^{\frac{1}{2}}H^T)x \end{aligned}$$

Thus any LQ-problem can be reduced to one with $Q=I$, $R=I$ by appropriate input and output transformations. Note that if the input u is to be regarded as a direct sum of u_i, $i \in \underline{k}$ then the only transformation R that preserve the invariance of the u_i's are those that have the (appropriately dimensioned) block diagonal matrix representation. In what follows we will assume that such an initial transformation $B \mapsto BR^{-\frac{1}{2}}$, $H^T \mapsto Q^{\frac{1}{2}}H^T$ has already been made, so that $Q = I_p$, $R = I_m$, unless otherwise specified. We

will then abbreviate $K(A,B,H^T,I_p,I_m)$ to simply $K(A,B,H^T)$. Note that the notation $K(A,B,H^T)$ will always mean that $SD(A,B,H^T)$, and hence K will always be unique and well-defined.

We first note the following basic structural characterization of the Riccati equation:

Lemma 3.3.1

Let $K = K(A,B,H^T)$

Then $N(K) = N(H^T,A)$, $R(K) = \langle A^T | R(H) \rangle$

Proof A direct generalization of lemma 3.2.3.

This result state that the Riccati equation can be reduced by restricting to $(X/N(H^T, A))$.

Lemma 3.3.2

$$\sigma[A - BR^{-1}B^TK] \supset \sigma(A|N(H^T,A)) \cup \sigma(\bar{A})$$

where $PA = \bar{A}P$, $P: X \longrightarrow \bar{X}$ and $\bar{X} = X/(\langle A|R(B)\rangle + N(H^T,A))$

P = canonical projection

Proof see [K2]; the above is a mere translation into geometric terms.

Put simply, the uncontrollable and unobservable poles of A are not moved by the LQ-feedback. Conversely, if any eigensubspace is observed by H^T, then the corresponding pole is shifted.

Lemma 3.3.3

Let $K = K(A,B,H^T)$

Then $P^TKP = K(P^{-1}AP, P^{-1}B, H^TP)$

$\forall$ $(n \times n)$ - nonsingular P

Proof Straight-forward.

Lemma 3.3.4

Let $G^{T*} = -B^TK$, where $K = K(A,B,H^T)$

Then $N(H^T,A) = N(G^{T*},A)$

Proof

$N(G^{T*},A) \supset N(K) = N(H^T,A)$

Conversely, $(KA + A^TK - KBB^TK + HH^T)\ N\ (G^{T*},A) = 0$

$\Rightarrow H^TN(G^{T*},A) = 0$

$H^TA^KN(G^{T*},A) = 0\ ,\ k= 0,1,\ldots$

thus $N(G^{T*},A) \subset N(H^T,A)$

Lemma 3.3.4 result has a simple interpretation: given a B such that (A,B) is stabilizable , KB is such that $\left((KB)^T,A\right)$ is detectable (more precisely, $<A^T|KR(B)> = <A^T \mid R(H)\ )>\Big)$

Our next result shows that in the special case $N(H^T) = N(H^T,A) \neq 0$, the Riccati equation can be solved in a smaller dimension in a straight forward way:

Theorem 3.3.1

Let $N(H^T) = N(H^T,A)$ and H^T has full rank (i.e. H^T is epic)

and $K = K(A,B,H^T,\ R,Q)$

Then $K = H\hat{K}H^T$

where $\hat{K} = K(H^TAH(H^TH)^{-1},\ H^TB,\ I\ ,\ R,Q)$

Proof $N(K) = N(H^T,A) = N(H^T) \Rightarrow K = H\hat{K}H^T$ for some $\hat{K} > 0$

from $KA + A^TK - KBR^{-1}B^TK + HQH^T = 0$

we see that

$$H\hat{K}(H^TA) + (HA)^T\hat{K}H - H\hat{K}(H^TB)R^{-1}(H^TB)^T\hat{K}H + HQH^T = 0$$

now $N(H^T) = N(H^T,A) \Rightarrow \exists\ \hat{A}$ s.t. $H^TA = \hat{A}H^T$

since H^Thas full rank, $\hat{A}$ is even unique, and $= H^TAH(H^TH)^{-1}$

We therefore have

$H[\hat{K}\hat{A} + \hat{A}\hat{K} - \hat{K}\hat{B}R^{-1}\hat{B}^T\hat{K} + Q]H^T = 0$ where $\hat{B} = H^TB$

Hence if $\hat{K}$ is the positive definite solution to

$$\hat{K}\hat{A} + \hat{A}^T\hat{K} - \hat{K}\hat{B}R^{-1}\hat{B}^T\hat{K} + Q = 0$$

Then $H\hat{K}H^T$ will satisfy the higher-dimension equation, and since K is unique, $K = H\hat{K}H^T$.

<u>Remark</u> This result has a natural interpretation in terms of the concept of <u>aggregation</u>. Given the system

$$\dot{x} = Ax + Bu$$

define the aggregation vector $z = H^T x$. Then z satisfies

$$\dot{z} = \bar{A}z + \bar{B}u$$

iff $N(H^T) = N(H^T, A)$, or equivalently, $AN(H^T) \subset N(H^T)$

Note that $\hat{A}^T$ is just the matrix representation of $A^T \mid F(H)$: i.e. the $(\hat{A}, \hat{B})$ model is just the model (A,B) modulo $N(H^T)$. (If (λ, x) is an (eigenvalue, eigenvector) of A, then $(\lambda, H^T x)$ is the corresponding pair of $\hat{A}$ if $H^T x \neq 0$).

<u>Remark</u> Note that u^* can be written in the form

$$u^* = FH^T x = Fz$$

<u>Corollary</u> 3.3.1

u^* is output feedback iff $N(H^T) = N(H^T, A)$

<u>Remark</u> In the general case $N(H^T) \neq N(H^T, A) \neq 0$, transformation to a 'cannonical' form is needed to achieve the reduction. See [K2].

<u>Remark</u> By taking H^T to be $(1 \times n)$-(left) eigenvector of A, one has a case of single pole placement.

We next consider the problem of parametrization redundancy in LQ-optimization. Given an LQ-problem data $SD(A,B,Q)$, denote the corresponding (unique) optimal feedback gain matrix as $(G^T)^*$ ($u^* = G^{T*}x$). It is well-known that the map $(A,B,Q) \longmapsto G^{T*}$ is NOT injective with respect to the parameter space of $\left\{ Q = Q^T \in R^{nxn} \mid Q \geq 0, D(Q^{1/2}, A) \right\}$. The precise statement is as follows:

<u>Lemma</u> 3.3.7

Given $SD(A,B,Q^{1/2})$,

if $(A,B,Q^{1/2}) \longmapsto G^{T*}$

Then $(A,B,\hat{Q}) \longmapsto G^{T*} \quad \forall \ \hat{Q} \in \Xi$

where $\Xi \equiv \left\{ Q \in R^{nxn} \mid \hat{Q} = A^TY + YA + Q \geq 0,\ Y = Y^T \text{ and } YB = 0 \right\}$

moreover, if $RK(B) = m$, then Ξ is an $\frac{(n-m)(n-m+1)}{2}$ - dimension cone of R^{nxn}.

<u>Proof</u> This follows from a straight forward generalization of Theorem 7 in [M5] and hence is ommitted.

<u>Corollary</u> 3.3.2

$$\Xi = \left\{ Q \in R^{nxn} \mid Q + (A + BG^T)^TY + Y(A + BG^T) = 0,\ \ Y \geq 0 \right.$$
$$\left. N(Y) \supset R(B) \text{ and } \forall\ G^T \text{ s.t. } \quad (A + BG^T) \text{ is stable} \right\}$$

Next, we show that under certain conditions, addition of 'actuator dynamics' does not alter the original feedback solution, but merely appends extra actuator state feedback:

Lemma 3.3.8

Let $K = K(A,B,H^T)$

Let $\hat{A} \in R^{m\times m}$, $\hat{B} \in R^{m\times m}$, $\hat{H}^T \in R^{r\times m}$ be such that we have

$$SD\left([\hat{H}^T,\ H^T],\ \begin{bmatrix} \hat{A} & 0 \\ B & A \end{bmatrix},\ \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} \right)$$

If $P \in R^{n\times m}$ solves $AP - P\hat{A} = B$

$$P\hat{B} = B$$

$$H^T P = \hat{H}^T \tag{3.3.3}$$

Then $$\begin{bmatrix} P^T KP & P^T K \\ KP & K \end{bmatrix} = K\left(\begin{bmatrix} \hat{A} & 0 \\ B & A \end{bmatrix},\ \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix},\ [\hat{H}^T,\ H^T] \right) \tag{3.3.4}$$

Moreover, P exists $\Longrightarrow$ $H^T AB - \hat{H}^T \hat{A}\hat{B} = H^T B\hat{B}$ (3.3.6)

Proof

See pp. 63.

The next theorem shows the condition under which a Riccati solution can be decomposed as the sum of two other Riccati solutions.

Theorem 3.3.2

Let $K_i = K(A,B_i,H_i^T)$, $i=1,2$

and $K = K(\ A,\ [B_1,B_2],\ \begin{bmatrix} H_1^T \\ H_2^T \end{bmatrix}\ )$

Then $K = K_1 + K_2$ if $\begin{cases} R(B_1) \subset N(H_2^T,A) \\ R(B_2) \subset N(H_1^T,A) \end{cases}$

or equivalently, $\begin{cases} \langle A \mid R(B_1) \rangle \subset N(H_2^T) \\ \langle A \mid R(B_2) \rangle \subset N(H_1^T) \end{cases}$

Proof

We have $K_i A + A^T K_i - K_i W_i K_i + H_i H_i^T = 0$, $W_i = B_i B_i^T$

Adding, we get ;

$$(K_1+K_2)A + A^T(K_1+K_2) - (K_1W_1K_1+K_2W_2K_2) + (H_1H_1^T+H_2H_2^T) = 0$$

If $K_iB_j=0$, $i \neq j$, then the above reduces to

$$(K_1+K_2)A + A^T(K_1+K_2) - (K_1+K_2)(W_1+W_2)(K_1+K_2) + (H_1H_1^T + H_2H_2^T) = 0$$

Hence $K = K_1+K_2$ by uniqueness .

$$\text{Now } R(B_i) \subset N(K_j) \implies R(B_i) \subset N(H_j^T, A)$$

$$\implies H_j^T R(B_i) = 0, \ldots, H_j^T A^{n-1} R(B_i) = 0$$

$$\implies \langle A|R(B_i)\rangle \subset N(H_j^T)$$

Proof of Lemma 3.3.8

$$K = K(A,B,H^T)$$

$$\Longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} = K\left(\begin{bmatrix} \hat{A} & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} \hat{B} \\ B \end{bmatrix}, [0 \ , \ H^T] \right)$$

Take $\hat{P} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \Longrightarrow \hat{P}^{-1} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$

From Lemma 3.3.3,

$$\hat{P}^T K \hat{P} = \begin{bmatrix} P^T K P & KP \\ P^T K & K \end{bmatrix} = K\left(\hat{P}^{-1} \begin{bmatrix} \hat{A} & 0 \\ 0 & A \end{bmatrix} \hat{P} \ , \ \hat{P}^{-1} \begin{bmatrix} B \\ B \end{bmatrix}, [0, \ H^T] \hat{P} \right) \qquad (3.3.5)$$

It is easy to check that (3.3.3) and (3.3.5) $\Longrightarrow$ (3.3.4) . The necessary condition (3.3.6) follows by trivial manipulation of (3.3.3).

3.4 Structural Stability Properties of LQ-feedback

In this section we combiné the results of 3.2 and 3.3 to obtain gereralizations of known stability properties of LQ-feedback as well as to put them in new perspectives and greater transparency.

We begin with

Definition 3.4.1

Given S(A,B), let

$$LQ(A,B) = \left\{ P \geq 0 \;\middle|\; P = K(A,B,H^T) \text{ for some } (H^T,A) \text{ detectable} \right\} \tag{3.4.1}$$

We will call the set LQ(A,B) the class of LQ-solutions, and the feedback

$$u = -R^{-1}B^TK \tag{3.4.2}$$

an LQ-feedback.

If $K \in LQ(A,B)$ then of course (A-WK) is stable for some $W = BR^{-1}B^T$. The next lemma is trivial to show but has important consequences.

lemma 3.4.2

$$K \in LQ(A,B) \implies K \in LP(A-BR^{-1}B^TK)$$

Proof

Obvious from the Riccati equation

Important Remark

For an unstable A, given any K such that $(A-BR^{-1}B^TK)$ is stable, in general it is NOT true that $K \in LP(A-BR^{-1}B^TK)$.

Example

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \qquad BR^{-1}B^T = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $\quad A - BR^{-1}B^TK = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$ is stable, BUT

$$K(A-BR^{-1}B^TK) + (A-BR^{-1}B^TK)^TK = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix} \not\leq 0$$

hence $K \notin LP(A-BR^{-1}B^TK)$

The importance of lemma 3.4.2 is that it shows that such an unfortunate state of affairs CANNOT occur if K is an LQ-solution. It is this distinguishing property of LQ(A,B), together with that of ndsf that is responsible for the following generalization :

<u>Theorem</u> 3.4.1 (Infinite Gain Margin Property)

Let $K \in LQ(A,B)$

Then $\quad (A - (B(N+\hat{R}^{-1})B^T + \hat{B}(\hat{N}+M)\hat{B}^T)K)$ is stable $\qquad$ (3.4.3)

$\forall \quad \hat{R} < R, \quad N = -N^T, \quad M>0, \quad \hat{N} = -\hat{N}^T, \ \hat{B}$ arbitrary $\qquad$ (3.4.4)

<u>Proof</u>

since $K \in LP(A-BR^{-1}B^TK)$

we have from Theorem 3.2.1 that

$(A - BR^{-1}B^TK + (S-\hat{W})K)$ is stable $\forall\ \hat{W}\geq 0,\ S=-S^T,\ R(S) \subset R(\hat{W})$

Take $\hat{W} = B(\hat{R}^{-1} - R^{-1})\ B^T + \hat{B}M\hat{B}^T$ with $\hat{R}<R,\ M>0$

and $S=BNB^T + \hat{B}\hat{N}\hat{B}^T$ with $N=-N^T$, and $\hat{N}=-\hat{N}^T$

The result follows.

Corollary 3.4.1

If $HH^T>0$, then the condition (3.4.4) can be relaxed to $\hat{R}\geq R$ (3.4.5)

Remark

For $\hat{B} \equiv 0$, theorem 3.4.1 is a generalization of the infinite gain margin property of LQ-feedback for single-input, single-output systems first noted by Anderson & Moore [A1], who showed that the feedback gain vector $g^T = -\frac{1}{r} b^T K$ can be multiplied by any scalar $\alpha \geq 1$ without destroying stability; the proof he used involve classical Nyquist techniques. Theorem 3.4.1 not only generalizes this property to multi-input, multi-output system, but allows more complicated alterations of the feedback gains; moreover, it makes the proof of this property much more transparent.

Remark

For $\hat{B} \neq 0$, theorem 3.4.1 allows for changes in the B matrix (e.g. if $N=\hat{N}$, $\hat{R}=M$, then the change in B is: $B \rightarrow B + \hat{B}$) without destroying stability. One useful interpretation is that the optimal gain computed for one nominal B_o remains stabilizing when B_o get perturbed so long as the change in B can be represented by those allowed in theorem 3.4.1. For example, let $B_o \longrightarrow B_1+B_o$. Then the feedback term becomes $B_1R^{-1}B_o^T + B_oR^{-1}B_o^T$. If $B_1 = B_o(\hat{N}+\hat{M})R$ for some $\hat{N} = -\hat{N}^T$, $\hat{M}>0$, then the system will remain stable. More complicated cases are allowed.

Alternatively, the case $\hat{B} \neq 0$ can be interpreted as allowing for the possibility of imposing extra controllers, and using these

extra feedbacks to 'tune' the closed-loop behavior of the original system. Then theorem 3.4.1 says that so long as such tuning has the feedback form allowed, the stability of the closed-loop system will not be destroyed by such additional 'fine-tuning'.

The inverse of the infinite gain margin property, i.e. how much the feedback 'gains' can be reduced without leading to instability is answered by the next proposition:

<u>Theorem</u> 3.4.2 (Gain reduction tolerance property)

Let $K \in LQ(A,B)$

Then $(A - B(\hat{R}^{-1}+N)B^TK)$ is stable (3.4.6)

$\forall\ \hat{R} > 0$ such that $\hat{R}^{-1} > \frac{1}{2}R^{-1}$ and any $N=-N^T$ (3.4.7)

If $HH^T>0$, then the condition (3.4.7) can be relaxed to

$$\hat{R}^{-1} \geq \frac{1}{2} R^{-1} \tag{3.4.8}$$

<u>Proof</u>

Let $\hat{A} \triangleq (A - B(\hat{R}^{-1}+N)B^TK)$

from the Riccati equation

$$KA + A^TK - KBR^{-1}B^TK + HH^T = 0$$

we get $K\hat{A} + \hat{A}^TK + KB(2\hat{R}^{-1}-R^{-1})B^TK + HH^T = 0$

hence from lemma 3.2.4, 3.4.6 and $D(H^T,A)$

$(\sqrt{KB(2\hat{R}^{-1} - R^{-1})B^TK + HH^T}, \hat{A})$ detectable

and lemma 3.2.1(2) $\Rightarrow$ $\hat{A}$ is stable

If $HH^T>0$, then $(H^T,\hat{A})$ is always detectable, and stability of $\hat{A}$ follows from (3.4.7)

Remark

This is a generalization of the known 'stability margin property', or gain reduction tolerance property of LQ-feedback. The interpretation is most transparent in the special case $R^{-1}=$ diag $(a_1,\ldots\ldots,a_k)$, and $\hat{R}^{-1} = \text{diag}(\hat{a}_1,\ldots\ldots,\hat{a}_k)$, $N\equiv 0$. Then the original individual LQ-feedbacks are of the form

$$u_i = -a_i b_i^T K \qquad\qquad i=1,\ldots\ldots k$$

The theorem says that in this special case, the system remains stable if the feedbacks are altered to:

$$u_i = -\hat{a}_i b_i^T K$$

so long as $\hat{a}_i \geq \frac{1}{2} a_i$

More complicated cases are allowed by the theorem.

Remark

As is clear from the proof, neither the condition (3.4.7) nor (3.4.8) are necessary.

By exploiting lemma 3.2.6, we can further generalize Theorem 3.4.2.

Theorem 3.4.3

Let $K \in LQ(A,B)$ and $K=K(A,B,R,H^T,Q)$ where (H^T,A) is observable.

Then $(A - B(\hat{R}^{-1}+ N)B^T K + K^{-1}H(\hat{Q}+\hat{N})H^T)$ is stable

$\forall$ $\hat{R}>0$ such that $\hat{R}^{-1} > \frac{1}{2}R^{-1}$

and $\forall$ $\hat{Q} = \hat{Q}^T$ such that $\hat{Q}< \frac{1}{2} Q$ and $\hat{N}=-\hat{N}^T$

Proof

Let $\hat{A} = (A-BR^{-1}B^TK)$, which is stable

Then

$$K\hat{A} + \hat{A}^TK + KBR^{-1}B^TK + HQH^T = 0$$

$$K\hat{A} + \hat{A}^TK + [KB,H]\begin{bmatrix} R^{-1} & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} B^TK \\ H^T \end{bmatrix} = 0$$

Thus by lemma 3.2.6, the result follows.

Remark

Although this result has no easily interpretable physical applications, it is suspected that it may be a 'time-domain' state-space version of the 'phase margin' property of LQ-feedback, a property first characterized by [A1] for single-input, single-output LQ-feedback system using Nyquist techniques. (In [A1] it is shown that a single-input LQ-regulator has a phase margin of at least 60 degrees.) Recalling the importance of phase margin for time - delay tolerance, this conjecture, if true, would further illustrate the structure of LQ-feedback.

Remark

The observability assumption (rather than just detectability) is only used to guarantee $K>0$, and hence the existence of K^{-1}. This assumption can be removed by the use of generalized inverse (in lieu of K^{-1}) and hence is not essential.

CHAPTER 4

SYSTEM RELIABILITY SYNTHESIS

4.1 Failure Modes in Control System.

In this Chapter we study the problem of designing reliable feedback control system and state reconstructors (or observers) and Kalman filters subject to control input and observation sensor failures.

The general class of system reliability synthesis problems is little researched in the context of system control design. The only previous work on this subject which the author is aware of are those of McFarlane, Belletruti [M1], [B3] and also Rosenbrock [R4]. However, the investigations of these authors were conducted in the frequency domain and involved heavy use of the Nyquist criteria and related techniques. (What we will call the reliable stabilization problem (RSP) was referred to by them as 'system with high integrity problems'.)

We will take an approach here that is very different in concept as well as method. Our investigations will be entirely in the time-domain state space, and we will exploit the concept of non-destabilizing feedback maps and other structural concepts in our problem formulations and solutions. In so doing, we will be able to provide an indirect partial solution to a long-standing problem in reliable stabilization synthesis, which in the next Chapter will be extended to solve a related coordinated decentralized stabilization problem.

Before we present the technical problem formulations, we first briefly review the context in which failure occurs in a control system.

We consider a general model of a tracking-regulator control system as sketched in Fig. 4.1

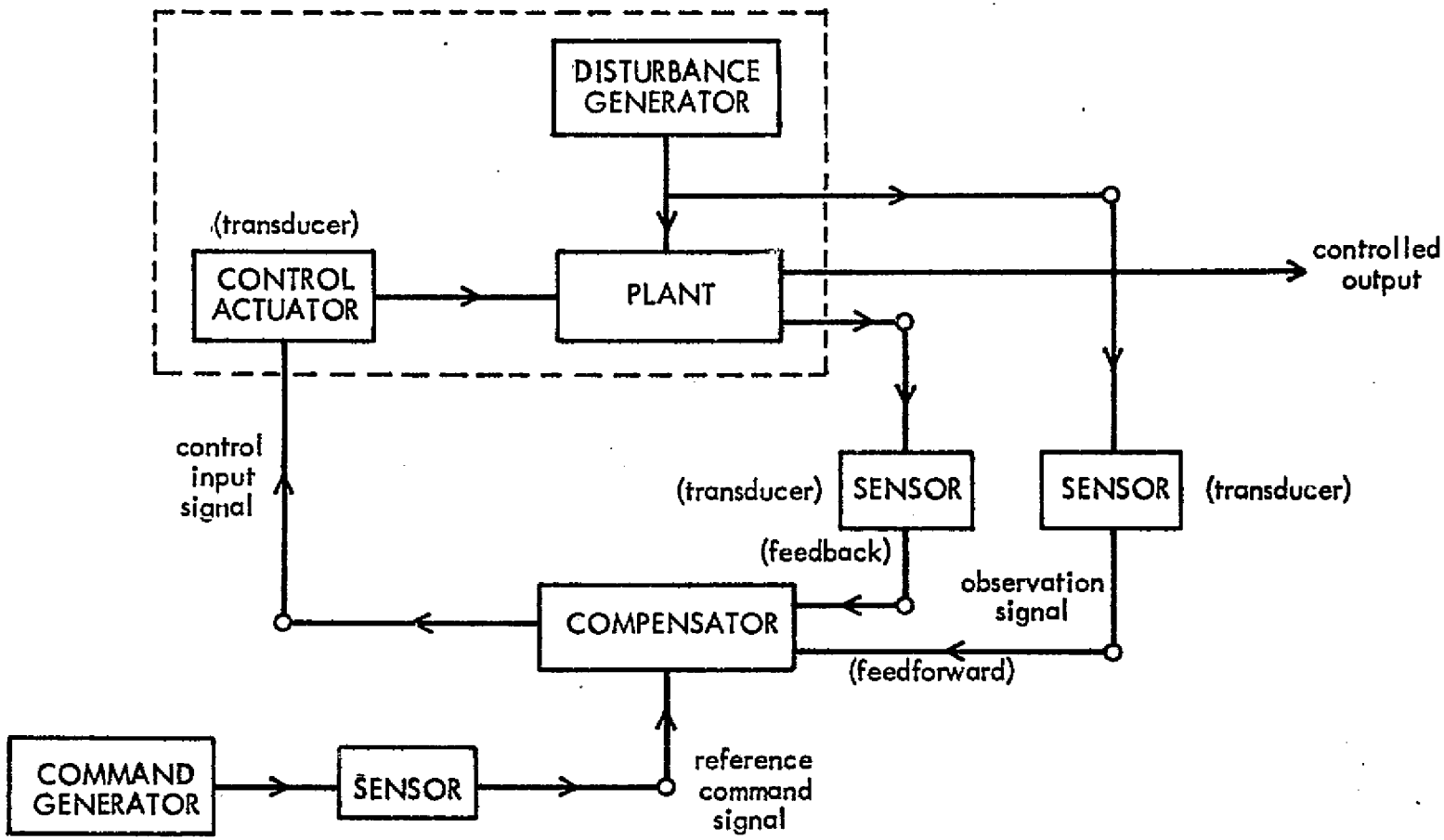
DISTURBANCE GENERATOR
(transducer)
CONTROL ACTUATOR
PLANT
controlled output
control input signal
(transducer)
SENSOR
SENSOR
(transducer)
(feedback)
observation signal
COMPENSATOR
(feedforward)
COMMAND GENERATOR
SENSOR
reference command signal

Fig. 4.1

What we have labelled 'compensator' could well have been called an 'information processor' since in real world situation it may be the model of something ranging from a simple electronic circuit to a minicomputer, and whose function is simply to compute a control input signal from the (feedforward and feedback) sensor signals. The map (or control strategy): (sensor inputs) $\longrightarrow$ (control signal) is usually fixed by the instrumentation (i.e. non-adaptive control). From a design point of view the compensator can be decomposed into several distinct functional devices; one version of such conceptual breakdown is shown in fig. 4.2. (See [F1], [D1])

For the purpose of failure analysis it is convenient to single out the following 'subsystem' modes of failure:

1) plant system failure
2) control actuator system failure
3) sensor system failure
4) compensator system failure
5) command generator failure

Note that, physically, the signal processing in the compensator operates on a different energy 'level' than the plant variables (and perhaps the command generator); the control actuators and observation sensors are the physical transducers that converts signals at one energy domain to the other. As such, the physical causes of failures are usually different for the different component systems.

It is usually the case that the plant and command generator systems are 'given', as far as the control engineer is concerned, and hence when failures in such system occurs (e.g. alterations of the plant dynamics, cutting of the command input channel) nothing can be done except repair, etc.

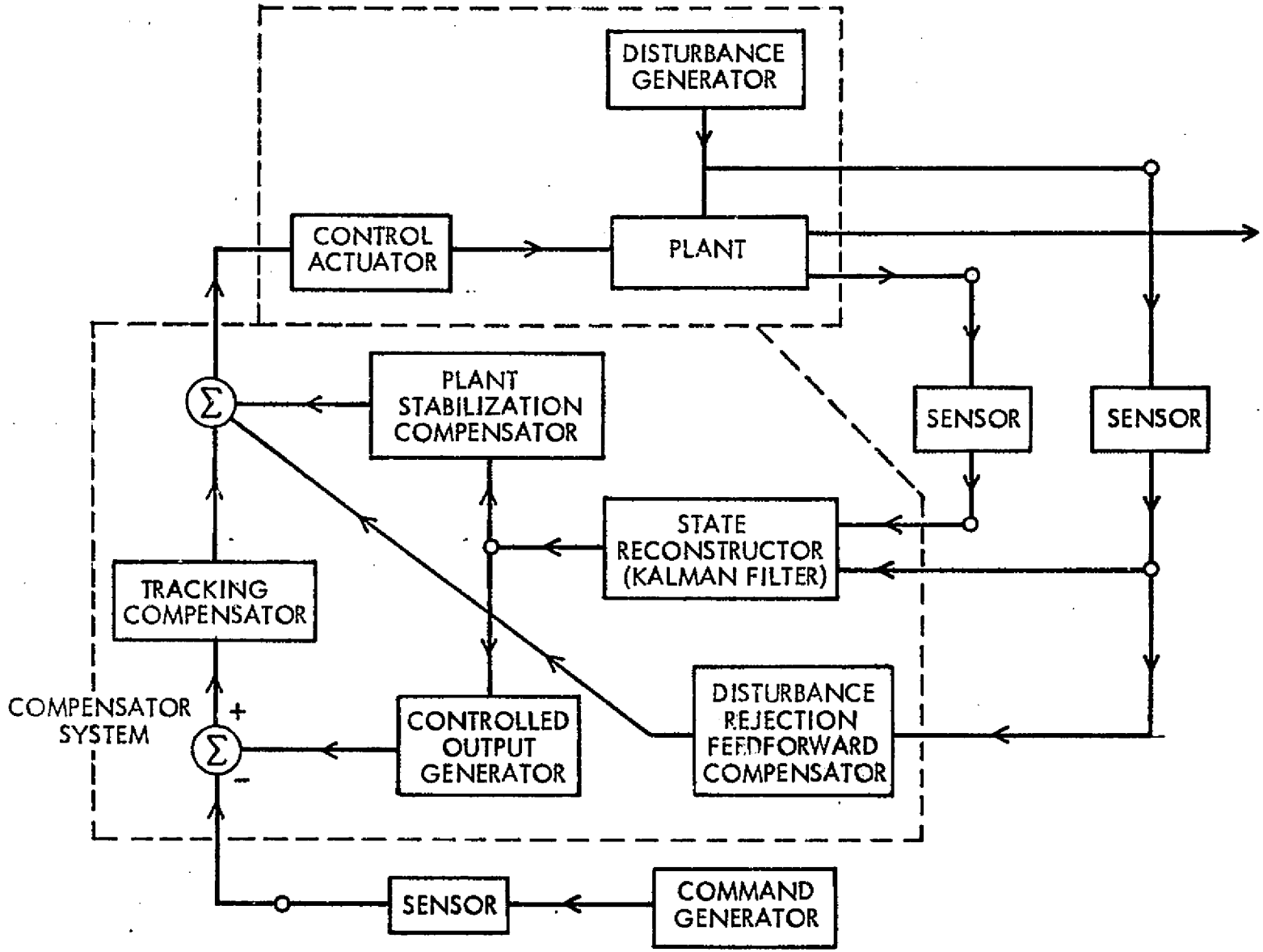
DISTURBANCE GENERATOR
CONTROL ACTUATOR
PLANT
PLANT STABILIZATION COMPENSATOR
SENSOR
SENSOR
STATE RECONSTRUCTOR (KALMAN FILTER)
TRACKING COMPENSATOR
COMPENSATOR SYSTEM
+
−
CONTROLLED OUTPUT GENERATOR
DISTURBANCE REJECTION FEEDFORWARD COMPENSATOR
SENSOR
COMMAND GENERATOR

Fig. 4.2

Similarly, as far as the control engineer is concerned, the component reliability characteristics of the other subsystems can also be regarded as given. The task of the control engineer, rather, is that of the following: given instrumentation components of fixed reliability characteristics, what is the best control implementation scheme that not only achieves the desirable system performance when all components are functioning normally, but that is also such that, when certain instruments (sensors, actuators,etc.) fail, the resultant performance deterioration will in some sense be minimized?

In particular, from the control system point of view, the most critical criterion that any reasonable control design must satisfy is that of the controlled plant's stability: when possible at all, the control strategy ought to be such that any failure in the control instrumentation never destabilize the plant. This is especially in those common situations where tha stabilization of an originally unstable plant is the primary function of the control design in the first place.

Unfortunately,existing (time domain) multi-input control design techniques (LQ-optimization, pole assignment, etc.) do not have this desirable property. The reason for this is intimately related to the fact that, in such design techniques, all the feedback loops are 'simultaneously' determines rather than individually 'shaped', and the designer has no direct control over how the overall stabilization job is being 'distributed' among the various feedback loops. Indeed, it is clear that the problem of synthesizing reliable stabilization is related to the problem of being able to individually adjust the feedback loops without interfering with the stabilizing action of other loops.

Because of its fundamental importance, we will concentrate our attention in this Chapter on the reliable stabilization problem and its

natural dual, reliable state-reconstruction (or reliable Kalman filter design). This means that we will concern ourselves with the simplified 'regulator' model as ahown in fig. 4.3.

The two failure modes we now identify are

1) control actuator input channel failure, which is equivalent to the 'cutting' of a feedback loop.

2) sensor output channel failure, which is equivalent to the 'cutting' of a driving input to the state reconstructor (Kalman filter).

For convenience we will analyze the problem of reliable plant stabilization and reliable state reconstruction separately, since it will be obvious how the two can then be considered together. This allows us to replace what is properly a feedback of the state estimate (generated by the Kalman filter) by the state vector itself, 'as if' there is direct feedback. The analytic consequence of doing so is well known (so long as the state constructor does not fail) and will not be elaborated; we will simply summarize by saying that this just amounts to ignoring an extra driving term (state estimate error) which decays away and thus has no effect on the closed loop plant stability analysis.

Note that actuator failure of the type that does not lead to zero actuator output but rather to such phenomena as'getting stuck' at a non-zero bias, or saturation effects can be modelled as a feedback loop cut <u>plus</u> the injection of a non-zero bias driving term; so long as plant stability is ensured the steady state consequence is easy to analyse. Similar remarks apply to the analysis of actuator failure consequence on tracking and disturbance rejection performance.

A final introductory remark is in order. The reliability synthesis problems we are concerned with can best be described as 'operational'

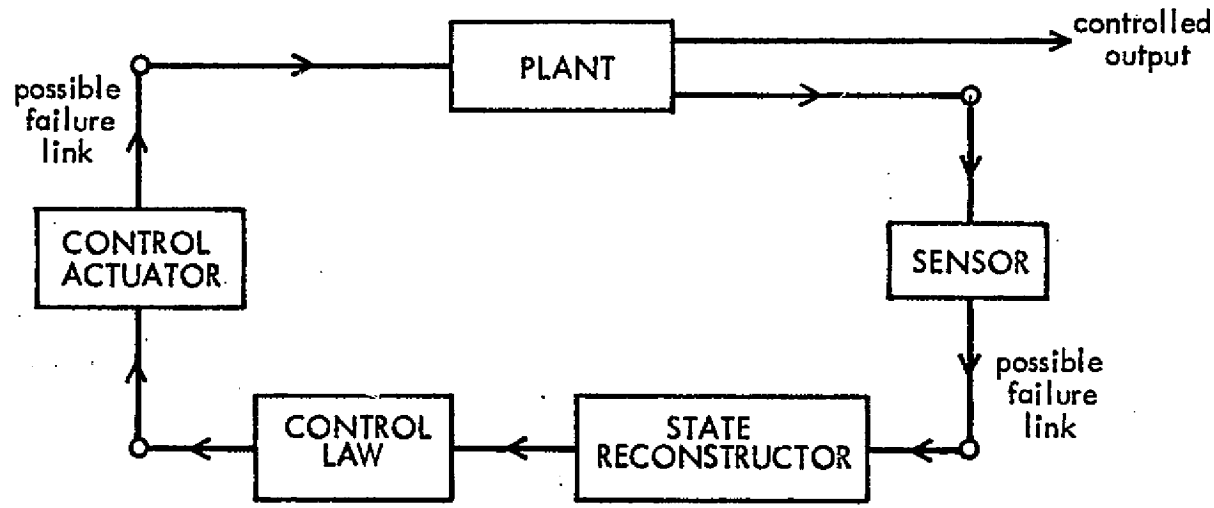
PLANT
controlled output
possible failure link
CONTROL ACTUATOR
SENSOR
possible failure link
CONTROL LAW
STATE RECONSTRUCTOR

Fig. 4.3

reliability that is automatically guaranteed throughout the system operation without any'outside' intervention, human or otherwise. Clearly, such additional 'adaptability' can be seperately considered, e.g. the design of failure detection system which can produce diagnosis signals to human operators who can then intervene by resetting the various loop gains and effecting repair, etc. The critical importance of our 'operational' reliability analysis lies in the fact that it automatically ensures that the plant does not get irreparably damaged be destabilization in the inevitable period of delay before such intervention can be effected. In this sense, a reliable synthesis is 'robust' against arbitrary failures.

4.2 Reliable Stabilization under feedback loop failures

Given the system $\Sigma = (A, B_i, i \in \underline{k})$, let the feedback law employed by the ith controller be :

$$u_i = G_i^T x \quad , \quad i \in \underline{k} \tag{4.2.1}$$

Subject to these multiple feedbacks, the resultant closed-loop system becomes :

$$\dot{x} = (A + \sum_{i \in \underline{k}} B_i G_i^T) x \tag{4.2.2}$$

Let the G_i's be chosen such that (4.2.2) is stable, i.e. all the feedback loops together stabilize A. If now a subset $I \subset \underline{k}$ of the control feedback loops are somehow broken, the resultant closed-loop system becomes

$$(A + \sum_{i \in \underline{k}-I} B_i G_i^T)$$

which <u>in general need no longer be stable</u> .

Conversely, given that $(A + \sum_{i \in I_1} B_i G_i T)$ and $(A + \sum_{i \in I_2} B_i G_i^T)$ are stable, where $I_1 \cap I_2 = \emptyset$, the combined system $(A + \sum_{I_1 \cup I_2} B_i G_i^T)$ need NOT be stable.

What we would like to have, from the viewpoint of reliability, is a multiple-feedback loop stabilization scheme that not only ensures stabilization when all the feedback loops are functioning, but that ensures stabilization even when an arbitrary number of the loops fail. The problem is not trivial because just ensuring that the <u>individual</u> feedback systems:

$$(A + B_i G_i^T) \quad , \quad i \in \underline{k} \tag{4.2.3}$$

are stable is not enough; superimposition of such feedback loops can destabilize the <u>collective</u> system (4.2.2) . (Examples demonstrating this fact will be given later.)

We are therefore led to formulate the following class of problems :

Reliable Stabilization Problem (RSP)

Given $\Sigma = (A, B_i, i \in \underline{k}) \in R^{n^2} \times \ldots \times R^{n \times m_k}$, characterize the set

$$RS(\Sigma) = \{ (G_1^T, \ldots, G_k^T) \in R^{m_1 \times n} \times \ldots \times R^{m_k \times n} \mid \forall \; I \subset \underline{k}, |I| \geq 1 \; (A + \Sigma B_i G_i^T) \text{ is stable} \} \tag{4.2.4}$$

Remark

In words, if $(G_1^T, \ldots, G_k^T) \in RS(\Sigma)$, then any combination of the feedback loops $u_i = G_i^T x$ will result in a stable system; conversely, with all the loops applied, breaking any number of the feedback loops will not destabilize the system, so long as at least one loop remains. It is this second interpretation that motivates us to call a member of $RS(\Sigma)$ a reliable stabilization.

Remark

By substituting $\mathcal{C}_g$-assignability for stabilizability, we can analogously define other assignability-reliable problems.

Lemma 4.2.1

$RS(\Sigma) \neq \emptyset$ only if $\mathcal{X}^+(A) \subset \cap \langle A \mid R(B_i) \rangle$, i.e. A is individually stabilizable.

Proof Obvious

Let S denote the set of all Σ such that Σ is individually stabilizable. Then lemma 4.2.1 means that we need only consider $\Sigma \in S$ as far as RSP is concerned. Is the converse also true for at least some $\Sigma \in S$?

The following special cases can be dispensed with right away :

Lemma 4.2.2

$$\Sigma = (A, B_i, i \in \underline{k}) \in S, \quad d(\mathcal{X}^+(A)) \leq 1 \implies RS(\Sigma) \neq \emptyset$$

Proof

If $d(\chi^+(A)) = 0$ then $G_1=\ldots=G_k=0$ is clearly a solution to RSP. If $d(\chi^+(A))=1$, take $N(G_i^T)=\chi^-(A)$ and such that $(A+B_iG_i^T)$ is stable $\forall\ i \in \underline{k}$. It is clear that this $\Longrightarrow$ $(A+\sum_{i\in I} B_iG_i^T)$ is stable $\forall\ I \subset \underline{k}$, $|I| \geq 1$. Hence all such $(G_i^T)_{i \in \underline{k}}$ are members of $RS(\Sigma)$.

The class of RSP solutions we are interested in (if there are any) are therefore those that have $N(G_i^T) \not\supset \chi^-(A)$ or dimension of $\chi^+(A)$ greater than one. More generally, we ask, is

$RS(\Sigma) \neq \emptyset$ for <u>all</u> $\Sigma \in S$?

A direct attack of this question using the 'geometric' tools appears to be difficult. We shall now show, however, that the question can be resolved by an indirect approach. The key idea is to exploit the fundamental properties of ndsf and LQ-feedback maps.

The following conceptual re-interpretation of RSP is useful. Since the RS property will not in general be satisfied if the G_i's are arbitrarily independently chosen, it is clear that solving RSP involves finding a suitable class of restrictions on the individual freedoms of the u_i's, i.e. of finding a suitable <u>parametrization</u> of the 'coordination' constraints on the individual controller's feedback map to achieve RS.

In this spirit, we have

<u>Lemma</u> 4.2.3

Let A be stable

If $G_i^T = (N_i-M_i)B_i^T(P+P_i)$, $i \in \underline{K}$ (4.2.5)

where $N_i = -N_i^T$, $M_i > 0$ are arbitrary, (4.2.6)

and $P \in LP(A)$

$P_i \in LP(A)$ and $P_iR(B_j) = 0$ $i \neq j$

Then $(G_1^T, \ldots, G_k^T) \in RS(\Sigma)$

Proof

From lemma 3.2.5 and Theorem 3.2.1 we have

$(A+(\sum B_i(N_i-M_i)B_i^T)(P+\sum P_i))$ is stable $\qquad I \subset \underline{K}$

and (4.2.6) $\Rightarrow$ $(A+\sum B_i(N_i-M_i)B_i^T(P+P_i))$ stable $\qquad I \subset \underline{K}$

Remark

The choices G_i^T may be interpreted to be 'coordinated' by the specification of P (fixed for all $i \in \underline{K}$), so that RS is guaranteed, while the condition $P \in LP(A)$ means that every $P \in LP(A)$ can coordinate the system to achieve RS.

Remark

The conditions (4.2.6) can be replaced by $M_i \geq 0$, $R(N_i) \subset R(M_i)$

Lemma 4.2.3 has identified a class of feedback maps that never destabilizes an open-loop-stable system in every possible combinations, i.e. the feedback loops can be independently shaped within the constraints imposed.

The condition $K_iR(B_j) = 0$, $i \neq j$ has the interesting interpretation of interference avoidance, i.e. if $K_1, K_2 \in LP(A)$ and $(A-W_1K_1)$ and $(A-W_2K_2)$ are both stable, then $(A-W_1K_1-W_2K_2)$ is also stable if either $K_1R(B_2)=0$ or $K_2R(B_1)=0$.

We next seek to extend these intuitively pleasing properties to the case when the open-loop system is unstable.

Theorem 4.2.1

Given $\Sigma = (A, B_i, i \in \underline{K}) \in S$

suppose $\exists\, K \geq 0$ such that for some $M_i > 0$, $N_i = -N_i^T$

$(A+L_iK)$ is stable $\qquad i \in \underline{K}$ (4.2.7)

where $L_i \triangleq B_i(N_i - M_i)B_i^T$

and $K \in LP(A+L_iK)$ (4.2.8)

If $G_i^T = (N_i - M_i)B_i^T K + (\hat{N}_i - \hat{M}_i)B_i^T K_i$

where $\hat{M}_i > 0$, $\hat{N}_i = -\hat{N}_i^T$ are arbitrary (4.2.9)

and $K_i \in LP(A+L_iK)$, $K_i R(B_j) = 0$, $i \neq j$ (4.2.10)

Then $(G_1^T, \ldots, G_k^T) \in RS(\Sigma)$

Proof

(4.2.8) $\Rightarrow (A + \sum_{i \in I} L_i K)$ is stable $\qquad I \subset \underline{K}$, $|I| \geq 1$

and (4.2.9) $\Rightarrow K_i \in LP(A + \sum_{j \in I} L_j K) \qquad I \supset \{i\}$

$\Rightarrow (A + \sum_{j \in I} L_j K + \sum_{j \in I} \hat{L}_j K_j)$ is stable

where $\hat{L}_i \equiv B_i(\hat{N}_i - \hat{M}_i)B_i^T$

Remark

The condition (4.2.9) can be replaced by $M_i \geq 0$, $R(N_i) \subset R(M_i)$

Remark

Recall that the condition $\Sigma \in S$ is necessary for $RS(\Sigma) \neq \emptyset$, and hence is not a restriction at all.

Remark

The 'coordinating parametric space' is now the set of all $K \geq 0$ that satisfies (4.2.7) and (4.2.10). Notice that once such a K is found, each control input can

generate ('shape') its own feedback loop independently so long as it respects the conditions specified by (4.2.9) and (4.2.10).

In order for Theorem 4.2.1 to be nonvacuous, we have to assure that there exists $K > 0$ that satisfies (4.2.7) and (4.2.8). For the case when A has only one unstable pole this is trivially true.

Corollary 4.2.1

If $d(\mathcal{X}^+(A)) = 1$ then any $K \gneq 0$ and $R(K) = \mathcal{X}^+(A)$ satisfies (4.2.7) and (4.2.8).

Proof

Projecting into the unstable subspace, we have a one-dimensional system, hence $K \mid \mathcal{X}^+(A) > 0$ clearly satisfies (4.2.7) and (4.2.8).

For $d(\mathcal{X}^+(A)) > 1$, the situation is much more complicated. However, the following is true:

Corollary 4.2.2

Let $P^T: \mathcal{X} \longrightarrow \mathcal{X}^+(A)$ be the canonical projection, let

$P^T A = \hat{A} P^T$, $P^T B_i = \hat{B}_i$

If $\exists \quad \hat{K} \geq 0$ such that

$$A_i = (\hat{A} + \hat{B}_i (N_i - M_i) \hat{B}_i^T \hat{K}) \text{ is stable} \qquad i \in \underline{k} \tag{4.2.10}$$

and $K \in LP(A_i)$

then $K = P\hat{K}P^T$ satisfies (4.2.7) and (4.2.8), i.e.

$A_i = (A + B_i (N_i - M_i) B_i^T K)$ is stable $\quad i \in \underline{k}$

and $K \in LP(A_i)$

Proof

This is a 'standard' geometric trick. $\hat{A}$ is just A reduced modulo the stable subspace; since the feedback map defined in (4.2.10) affects only the unstable subspace, leaving the stable subspace invariant, stability of $\hat{A}_i$ implies that of A_i.

Remark

The importance of this elementary result is that it shows that the 'coordination parametric space' can be reduced to just that part of the individual feedback maps that affect the unstable subspace - if the unstable subspace can be coordinated to be reliably stabilized, so is the whole state-space.

Recalling the stabilizing properties of LQ-feedback, we are motivated at this point to consider again an indirect approach to finding the class of $K > 0$ that provides RS coordination. More precisely, we formulate the following class of problems:

Reliable LQ-Stabilization Problem (RLQP)

Given $\Sigma = (A, B_i, i \in \underline{k}) \in S$

find conditions on H^T, $R_i > 0$ such that (if possible)

$G_i^{T*} = -R_i^{-1} B_i^T K$, where $K = K(A, [B_1, \ldots B_k], H^T, \begin{bmatrix} R_1 & & \\ & \ddots & \\ & & R_k \end{bmatrix})$

satisfy the reliable stabilization property, i.e. such that

$(G_i^{T*})_{i \in \underline{k}} \in RS(\Sigma)$

We will call a Riccati solution K that solves RLQP a RLQ solution for short.

Remark

The problem here is to start with some $(G_i^{T*})_{i \in \underline{k}}$ that always guarantee that $(A+\sum_{i\in\underline{k}} B_i G_i^{*T})$ is stable, but which may have any of the $(A+\sum_{i\in\underline{k}} B_i G_i^{T*})$, $I \subset K$, $|I| > 1$ unless suitable constraints are imposed.

Remark

The solution of this problem is of great significance by itself since, when possible, it allows a single LQ-feedback solution to automatically have the reliable stabilization property. We have already shown in Chapter 3 that LQ-feedback remains stabilizing under fairly general alterations in the actual gain implemented; if we can structurally ensure reliability synthesis automatically in LQ-feedback too, then we will have substantially increase the usefulness of the LQ-method as a stabilizing design technique.

Remark

A parametrization of the $HH^T \geq 0$, $R > 0$ that gives rise to RLQ will provide new significant interpretation of LQ-optimization in terms its role as providing 'structural' coordination.

We begin with the following partial characterization of RLPQ:

Lemma 4.2.4

Given $\Sigma = (A, B_i, i \in \underline{k}) \in S$

Let (H^T, A) be detectable, $R_0 = \begin{bmatrix} R_1 & & \\ & \ddots & \\ & & R_k \end{bmatrix} > 0$, $Q > 0$

and $K = K(A, [B_1, \ldots, B_k], H^T, R_0, Q)$

let $W_i = B_i R_i^{-1} B_i^T$

Then sufficient conditions for

$(A - \sum_{i \in I} W_i K)$ to be stable

$I \subset \underline{K}$

$|I| > 1$

are: i) $Q_I \geq 0$ and ii) $(Q_I^{1/2}, A)$ is detectable

where $Q_I = HQH^T - K(\sum_{i \in \underline{k}-I} W_i)K$

<u>Proof</u>

We have $KA + A^T K - K(\sum_{i \in \underline{K}} W_i)K + HQH^T = 0$

let $A_I = (A - \sum_{i \in I} W_i K)$

then $KA_I + A_I^T K + K(\sum_{i \in \underline{k}} W_i)K + Q_I = 0$

Hence $Q_I > 0$ and $(Q_I^{1/2}, A)$ - detectable is sufficient for A_I to be stable by lemma 3.2.4.

<u>Remark</u>

Since the conditions (i) and (ii) of <u>Lemma</u> 4.2.4 can only be checked <u>after</u> the Riccati solution K has been computed, it is of little practical use.

<u>Remark</u>

A necessary condition for $Q_I \geq 0$ is that $R(H) \supset K \sum_{i \underline{k}-I} R(B_i)$

Hence if we have $D(H^T, A)$ but $d(R(H)) < d(\sum_{i \in \underline{k}-I} R(B_i))$,

Then $Q_I \geq 0$ will never be satisfied.

Corollary 4.2.3

Let $P: X \rightarrow X^{+}(A)$ be the canonical projection

and denote $\hat{B}_i = PB_i$

If $R(\hat{B}_i) \subset R(\hat{B}_2) \quad \ldots \subset R(\hat{B}_k)$ (up to a permutation) (4.2.11)

Then $RLQ(\Sigma) \neq \emptyset$

Proof

Let $\hat{\Sigma} = (\hat{A}, \hat{B}_i, i \in \underline{k})$ be the aggregation model of Σ modulo $X^{-}(A)$. Since $X^{+}(\hat{A}) = \langle \hat{A} | R(\hat{B}_1) \rangle$ we can always find a LQ-solution $\hat{K}$ for some $\hat{M}_1 > 0$, and $D(\hat{H}^T, \hat{A})$ such that $(\hat{A} - \hat{B}_1 \hat{M}_1 \hat{B}_1^T \hat{K})$ is stable.

Since $R(\hat{B}_1) \subset R(\hat{B}_i) \qquad i = 2, \ldots, k$, it is clear that $M_i \geq 0$ can be chosen such that

$$\hat{B}_i \hat{M}_i \hat{B}_i^T = \hat{B}_1 \hat{M}_1 \hat{B}_1^T + W_i$$

for some $W_i \geq 0$

Hence from Theorem 3.2.1,

$$(\hat{A} - \sum_{i \in I} \hat{B}_i \hat{M}_i \hat{B}_i \hat{K}) \text{ is stable } \quad \forall\, I \subset \underline{k} \quad |I| > 1$$

Then from Corollary 4.2, $RLQ(\Sigma) = \emptyset$

That RLQP is not vacuous even for $\Sigma = (A, B_i, i \in \underline{k})$ with A totally unstable (i.e. $\chi^+(A) = \chi$) is shown by the existence of examples like the following:

<u>Example</u>

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Take $R_1 = R_2 = 1$

Then $K = \begin{bmatrix} 12.5 & -7 \\ -7 & 5 \end{bmatrix} \in RLQ(\Sigma)$

The question of interest is therefore: Is $RLQ(\Sigma) \neq \emptyset \quad \forall \; \Sigma \in S$?

Unfortunately, this remains at present an unresolved question.

We now look at some simple numerical examples. First consider the simplest possible case : one-dimensional state-space

Example 1

$A = [a]$, $a \geq 0$

$B_1 = [b_1]$, $B_2 = [b_2]$, $HH^T = [q]$, $R_1 = [1]$, $R_2 = [1]$

then

$$K = (1/(b_1^2 + b_2^2))\,(a + \sqrt{a^2+q(b_1^2+b_2^2)})$$

$$= \sqrt{q/(b_1^2+b_2^2)} \qquad a > 0$$

$$= (a/(b_1^2+b_2^2))(1+\sqrt{1+(q(b_1^2+b_2^2))/a^2})$$

thus $(a-b_1^2K) < 0 \iff q > (a/b_1)^2((b_2/b_1)^2-1)$

Note that $q > b_2^2K^2 \iff q > 4a^2b_2^2/b_1^4$

thus verifying the sufficient condition of Lemma 4.2.4. Note that the condition is not necessary.

We now present some two-dimensional examples.

Example 2

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad HH^T = \begin{bmatrix} 80 & 53 \\ 53 & 37 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We have

$$K = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \qquad \begin{matrix} G_1^T = \begin{bmatrix} -7 & -4 \end{bmatrix} \\ G_2^T = \begin{bmatrix} -5 & -5 \end{bmatrix} \end{matrix}$$

$$\left[A-W_1K-W_2K\right] = \begin{bmatrix} -20 & -13 \\ -17 & -13 \end{bmatrix} \qquad \text{with ch.p.} = s^2 + 33s + 39 = 0 \quad \text{i.e. STABLE}$$

$$A - W_1K = \begin{bmatrix} -15 & -8 \\ -7 & -3 \end{bmatrix} \qquad \text{with ch.p.} = s^2 + 18s - 11 = 0 \quad \text{i.e. UNSTABLE}$$

$$A - W_2K = \begin{bmatrix} -6 & -5 \\ -10 & -9 \end{bmatrix} \qquad \text{with ch.p.} = s^2 + 15s + 4 = 0 \quad \text{i.e. STABLE}$$

We check $Q_1, Q_2 \geq 0$?

$$Q_1 = HH^T - KW_2K = \begin{bmatrix} 55 & 28 \\ 28 & 12 \end{bmatrix} \quad \text{which is } \underline{\text{NOT}} \text{ positive semidefinite}$$

$$Q_2 = HH^T - KW_1K = \begin{bmatrix} 31 & 25 \\ 25 & 21 \end{bmatrix} \quad > 0$$

This example thus accords with our theory developed earlier. It also illustrates - examples are supposed to be simple and illuminating - the <u>structural reason</u> why stability fails when the second loop is cut. The first control input vector $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ has a <u>bigger</u> projection onto the stable subspace <u>than</u> the unstable subspace, and hence it ought to (negatively) feedback with more amplification of the unstable state - which it does not (see G_1^T). And the reason it does not is because matrix HH^T penalizes the stable state more than the unstable state! Note that the matrix K has the same 'penalty pattern' as Q. In fact, a short computation readily shows that for all matrix solution K of the form

$$K = \begin{bmatrix} a_1 & 1 \\ 1 & a_2 \end{bmatrix} \qquad a_1 > 0, \quad a_2 > 0$$

$A - W_1K$ will be <u>unstable</u> for all $a_2 < 4a_1 + 1$ (and stable $a_2 > 4a_1 + 1$)

(Note that $G_1^T = \begin{bmatrix} 2a_1 + 1, & 2 + a_2 \end{bmatrix}$)

Intuitively, as the unstable state is penalized more by HH^T, a_2 tends to increase, and as a_2 increases relative to a_1, it tends to put more feedback on the unstable state, thus compensating for the relative lack of controllability of the unstable pole by u_1.

This simple example suggests that the relative projection magnitude of b onto the various eigensubspaces as a measure of 'relative controllability' with respect to the poles is an important concept worth extending to the general n-dimension case.

The last example is adapted from Rosenbrock [R3] :

<u>Example 3</u>

$$A = \begin{bmatrix} 1/6 & 1/5 \\ -1/5 & -1/6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad HH^T = \begin{bmatrix} 2/3 & 0 \\ 0 & 4/3 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We get $\quad A - W_1K = \begin{bmatrix} -5/6 & 1/5 \\ -1/5 & -1/6 \end{bmatrix}$ stable

$$A - W_2K = \begin{bmatrix} 1/6 & 1/5 \\ -1/5 & -7/6 \end{bmatrix} \quad \text{unstable}$$

We check our sufficient conditions:

$$Q_1 = HH^T - KW_2K = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix} > 0$$

$$Q_2 = HH^T - KW_1K = \begin{bmatrix} -1/3 & 0 \\ 0 & 4/3 \end{bmatrix} > 0$$

<u>Remark</u> : This example was used by Rosenbrock in [R3] to illustrate the undesirability of traditional LQ-design.

4.3 The Dual Problem : Reliable State-Reconstructor and Kalman Filter Synthesis

We now consider the natural dual of the reliable stabilization problems treated in the previous section: that of designing reliable state reconstructors (or observer system) that are subject to arbitrary sensor input failures. In particular, special interests will be attached to the synthesis of reliable Kalman filters (a subclass of state reconstructors).

Since the duality theory for the pairs (A,B), (C^T,A) is well-known, we will omit all (dual) proofs in this section, and simply formulate the dual problems and state the main dual result . Before we can do that, however, we have t o briefly discuss a feature of state reconstructuon failures that has no analoque in the control context.

Consider as given the system $\Sigma = (A, B_i, C_i^T, i \in \underline{k})$ where the observation outputs $y_i = C_i^T x$ are to be considered as independent with respect to failures. A (full order,deterministic) state-reconstructor of Σ is just a dynamical system $\hat{\Sigma}$:

$$\dot{\hat{x}} = A\hat{x} + Bu + \sum_{i\in\underline{k}} F_i(y_i - \hat{y}_i) = (A - \sum_{i\in\underline{k}} F_i C_i^T)\hat{x} + Bu + \sum_{i\in\underline{k}} F_i y_i \qquad (4.3.1)$$

Define the error $e = x - \hat{x}$

Then $$\dot{e} = (A - \sum_{i\in\underline{k}} F_i C_i^T)e = Ae \qquad (4.3.2)$$

and hence if A is stable , $e(t) \longrightarrow 0$ as $t \longrightarrow \infty$, and $\hat{x}(t) \longrightarrow x(t)$.

In the stochastic case when noise disturbances on the plant and sensor inputs are taken into account, the state reconstructor takes the same form, but with the gain matrices F_i determined (through solving a Riccati equation) from the noise statistics (covariance of plant driving and sensor noises) . In this case one has of course a (full order) Kalman filter. For simplicity and consistency we will keep to our deterministic notation and refer to $\hat{\Sigma}$ as a state-reconstructor, although of course all our results will carry through without change to the Kalman filtering context, the formal manipulation for such deterministic-to-

stochastic 'translation' being standard. Also, the practical considerations of 'tuning' a Kalman filter should be borne in mind when interpreting the meaning of the gains F_i .

To consider the effect of sensor input failures, let one of the inputs, say y_j, be nulled. If nothing else is changed, the error equation will be altered to :

$$\dot{e} = (A - \sum_{i\varepsilon k} F_i C_i^T) e + F_j C_j^T \hat{x} \qquad (4.3.3)$$

i.e. a bias term $F_j \hat{y}_j$ appears which is due to the 'disappearance' of the sensor input y_j .

To proceed, therefore, we make the <u>assumption</u> that the input y_j can be somehow disconnected automatically when the input y_j fails (this requires the availability of, e.g. fault detection circuitry). When this assumption holds, the error system (4.3.3) will become the perfect dual of (4.2.2) subject to feedback failures. (Alternatively, we can interpret (4.3.3) without the bias term as the result of the jth <u>error comparator</u> channel failure).

We therefore have the following dual problem to RSP:

<u>Reliable State Reconstruction Problem (RSRP).</u>

Given $\Sigma = (A, C_i^T, i \varepsilon \underline{K})$, characterize the set

$$RSR(\Sigma) = \{ (F_1, \ldots, F_k) \mid (A - \sum_{i\varepsilon \underline{k}} F_i C_i^T) \text{ is stable } \forall \; I \subset \underline{k} \}$$

Denote by D the set of all $\Sigma = (A, C_i^T, i \varepsilon \underline{k})$ such that the $\Sigma_i = (A, C_i^T)$ are all individually detectable. Clearly a necessary condition for $RSQ(\Sigma) \neq 0$ is $\Sigma \varepsilon D$. The dual of Theorem 4.2.1 is the following:

Corollary 4.3.1

Given $\Sigma = (A, C_i^T, i \varepsilon \underline{k}) \varepsilon D$

Suppose $K \geq 0$ s.t. for some $M_i > 0$, $N_i = -N_i^T$

$(A+KL_i)$ is stable $\forall\ i \varepsilon \underline{k}$, where $L_i \equiv C_i(N_i - M_i)C_i^T$
and $K \varepsilon LP(A^T + L_i^T K)$

If $F_i = KC_i(N_i - M_i) + K_i C_i(N_i - M_i)$ where $M_i > 0$, $N_i = -N^T$ are arbitrary

and $K_i \ \varepsilon LP(A^T + L_i^T K)$, $K_i R(C_j) = 0$, $i \neq j$

then $(F_i \ldots\ldots, F_k) \ \varepsilon RSR(\Sigma)$

One can similarly define the dual of RLQP

4.4 The Role of Redundancy in Control Problems

The use of redundancy has played a long-established role in the field of system-maintainability and reliability engineering. There, one deals with the best way to connect (or put in standby) redundant components (each having identical functions) such that the resultant overall system performs a reliability characteristics that is better than the individual components'. In other words, the redundancy is at the instrumentation level.

We have used the concept of redundancy in a different and more abstract sense in this thesis. We have spoken of the redundancy of controllability (observability) rather than redundancy of control inputs (sensor outputs), and the reliability synthesis we are trying to achieve is at the level of strategy, not at the level of instrumentation of such a strategy, i.e. it is the control law itself that we are trying to make 'reliable'. This concept can (and should be) formalized and its utility extended to other strategy reliability problems than the one (stabilization) we have considered.

The close connection between such strategy reliability problems and the well-studies problems of strategy sensitivity must be noted. The latter addresses the question: given a particular solution to a control synthesis problem satisfying a set of criteria, how well satisfied is the set of criteria when the problem data and/or the implementation of that solution are slightly 'perturbed'? The most refined results to date of such insensitivity synthesis in the context of control systems have led to the enunciation of so-called 'Internal Model Principle' ([F1]), 'Steady-state Invariance Principle' ([N1]) and 'Perfect Control Principle' ([D2]), where the role of redundancy has further been clarified. It is hoped that similar advances will be made in the context of reliability synthesis, and

we believe that such advances will lead to a better understanding of the deep connection between system coordinability and redundancy.

c. 2

4.5 Ndsf and LQ-feedback : Some new perspectives

In this section we briefly recapitulate the new insights that our earlier results have shed on stabilizing feedback.

One new perspective is the following : the class of LQ-feedback can simply be regarded as a subclass of ndsf that always stabilizes the open-loop system. From this perspective, it becomes clear that some of the known desirable properties of LQ-feedback (infinite gain margin, compatible stabilization, tolerance to nonlinearities in the $(1/2, \infty]$, and perhaps phase margin and asymptotic 'Butterworth' pole pattern) really has very little to do with the fact that the feedback term comes from some LQ-optimization ---- they are simply structural properties of ndsf. LQ-optimization amounts to choosing a member of this class that stabilizes, but that may be other shorter means to that choice in certain situations.

The practical implications of this observation with respect to 'suboptimal' design is significant. First, in situations where physical intuitions are of no help in pinning down an a apriori preferred feedback choice, and where 'simulation trials' are resorted to, the above insight suggests that , rather than using an adhoc search, or else employing adhoc'dyadic' feedback technique, 'reduction to single control' technique, etc., which are of mathematical convenience but of little physical or structural justification, it could be more fruitful to limit the search to the class of ndsf.Recall the following fundamental of compatible ndsf : each feedback loop can be individually shaped without destroying the stability of the collective closed-loop system. This property, and its (dual) reliable stabilization interpretation, are of great practical importance.

Secondly, it suggests that, in situations where the 'Q' and 'R' terms of LQ-design are really of little physical intuitiveness, one may bypass

iterating on many trial Riccati equations, and instead iterate on the class of admissible feedback modifications allowed by Theorem 3.4.1, 3.4.2 for 'fine' tuning once a 'nominal' LQ-feedback has been obtained. Further, the initial choice of 'Q' and 'R' should be guided by considerations of their structural consequences on the resultant feedback maps. Similarly, sensitivity analysis with respect to structural perturbations of the feedback loop or to implementation imperfections may be carried out if such changes can be modelled by the class of allowed changes we have identified.

CHAPTER 5

STRUCTURE OF DECENTRALIZED CONTROL SYSTEMS

5.1

In this chapter we turn to study the structure of linear, multivariable decentralized model of large-scale dynamic control systems. Our primary objective is to obtain more precise structural characterizations of the very concept of 'decentralization' in the large-scale control system control problem context in terms of our previous interaction structure and feedback coordination concepts.

By a 'large-scale' dynamic system we simply mean a dynamic system with many control stations and many observation stations which are physically distinct and spatially separated. The control problem consists simply of determining a real-time implementation scheme that does not violate the natural physical constraints of the system being modelled. This qualification is important, since it is these extra considerations that justify distinguishing a problem as 'large-scale' as opposed to just any multivariable control problems.

Informally, a system 'plant' is large-scale because, firstly, the control input channels and sensor output channels are spatially separated from one another and among themselves, and to transmit information to where it can be actuated requires hardware (a communication network) that costs to build as well as to operate and maintain. There is also the inherent transmission delay, uncertainties in the information content and its further degradation in the transmission process. Secondly the system has to be operated in real-time, and continuous real-time human decision-making is costly, unavailable or unreliable --- hence the need to pre-program the control strategy as much as possible, thus reducing the task of human decision making to an initial problem formulation and off-line computation of control strategy and subsequent routine operations and periodic intervention and adaptation

where necessary; similarly, real-time computation limitation must be obeyed. In other words, the complexity of the control law computation must be suitably restrained. Thirdly, implementation of control strategy physically requires instrumentation that, once built can not be freely substituted later without incurring costs; the complexity of the control law implementation are constrained. Finally, human institutional constraints which although are external to the dynamic description of the system plant, nevertheless may constrain the admissible control strategy or its implementation. In a capitalist production firm the control ('management') strategy is 'hierarchical' not because that strategy is 'optimal' with respect to some abstract 'team decision' criterion --- as some people are trying to mystify --- but simply is an imposed institutional constraint that ensures that the owners of capitals have control over the management of production rather than the workers. Similarly, the 'bureaucratic' set-up in many real world control systems is an human institutional constraint that often has little to do with considerations of the efficiency of that particular mode of control strategy.

The above informal discussion serves merely to point out some of the more important <u>physical</u> origins of decentralization in the real-time <u>implementation</u> of many large-scale control systems.In any truly realistic optimal control problem formulation, the particular form of decentralization will of course follow from the solution itself. On the other hand, failure to realistically incorporate all possible implementation cost (communication network, computational software and hardware,etc.) into the optimization problem formulation will lead to highly unrealistic or inferior strategies; the 'optimality' with respect to an unrealistic cost criterion is deceptive.

Our objective in this chapter is therefore not to grapple with the issues of optimality, but rather that of feasibility : given a particular class of decentralization constraints, what are some of the meaningful control problems one can formulate that respect the imposed decentralization constraints? Given a particular decentralization constraint and a particular control objective , how to characterize (parametrize) the class of all feasible solutions ?

To be able to do so, we first need to have some meaningful scheme for classifying different possible kinds of decentralization constraints. This we will do in 5-2, albeit in an informal manner. We will then present in 5-3 some classes of linear multivariable decentralized control problems to illustrate the framework provided by 5.2

5.2 Classification of decentralised control problems

In this section we present informally a scheme for classifying several important types of decentralisation in control problems.

The basic components of a large-scale decentralised dynamic control problem are conceptualised as in Fig. 5.1.

The basic tasks we identify are:

1) information distribution problem

a) a priori structural information

the problem is either

i) given an initial common pool of available a priori information (system model, control objectives, stochastic description of noises, etc.), how should the information be shared by the different control agents

or ii) given a priori information that are decentralised to begin with, how should the information be pooled, exchanged, etc.

or both.

b) a posteriori (or real time) observation from sensor channels and communications of control actions. The problem is (1) who should be provided with which observation channels; (2) which control stations should communicate with which other control stations to exchange control information (what they have done, what they will do, etc.)

Any particular solution of the information distribution problem fixes a particular <u>information pattern</u>. For each given fixed information pattern, we can consider the next task:

2) (Off-line) Control strategy computation problem

The problem is roughly, given a particular information set, computes (off-line) a control law, that is a map: (observations) $\longmapsto$ (control values) from among the class of all such maps admissible by the information constraints, causality, and other physical constraints, etc.

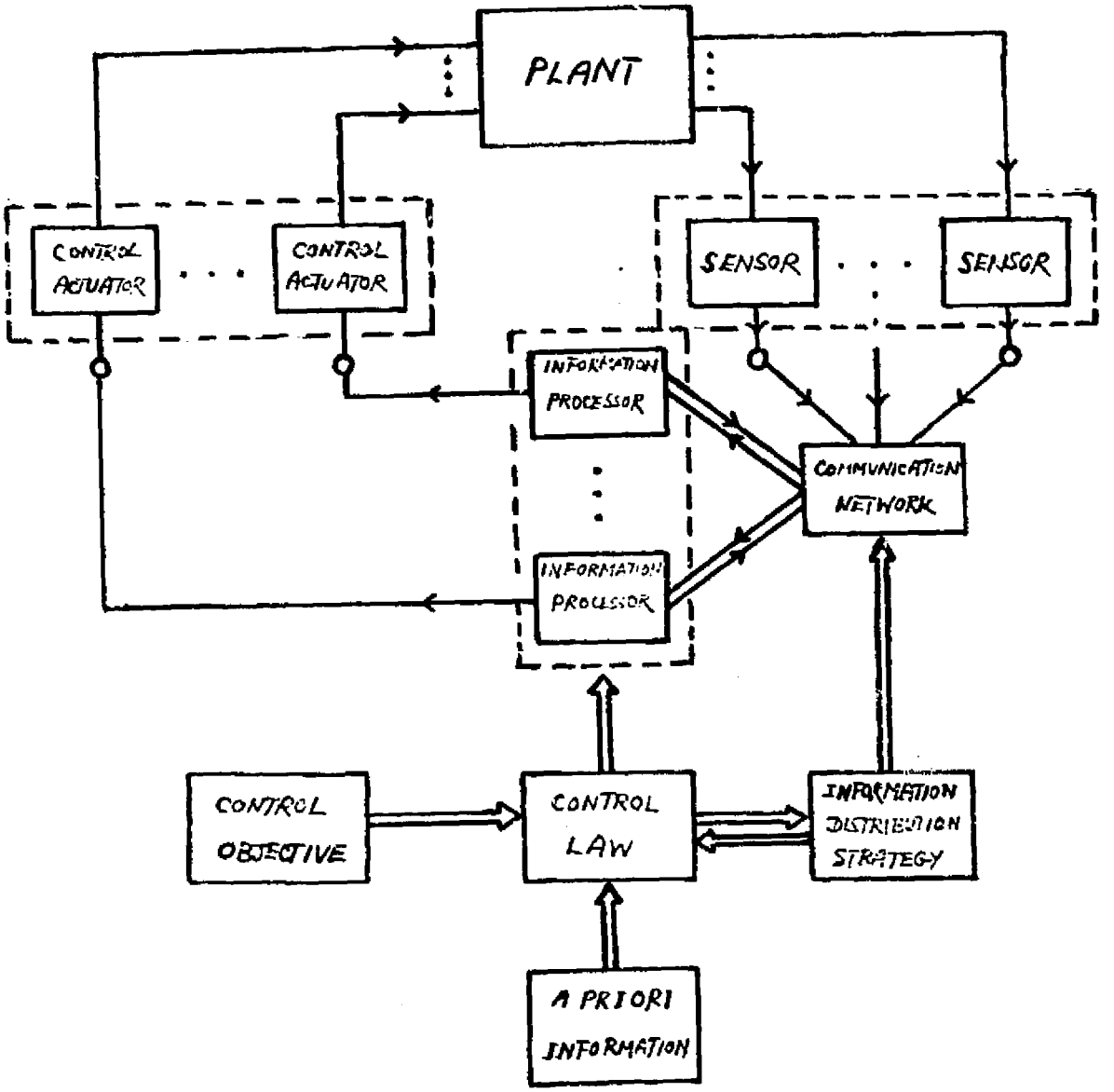

FIG 5.1

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Given a fixed information pattern and an admissible pre-computed control law, one can go on to task 3:

3) On-Line Control Implementation Problems

The problem here is to simply execute the on-line information processing and control actuation

Although other tasks (eg. adaptive control, learning, i.e. aggregation of a posteriori information into a priori structural knowledge, etc) can be identified , we will confine ourselves to the above three. We further remark that the above task identification represents merely an attempt at conceptual clarification, since in practice, such rigid demarcation cannot be made, and often the solution of the earlier task are constrained and defined by the limitations of the later tasks, e.g.control law computation may suggest or dictate what information distribution pattern should be made; similarly, limitations on on-line information processing capacity may constrain the level of complexity permitted in the selection of control laws.

We make the observation here that, conceptually, each of these three tasks can be carried out by the same or different agents. Thus it is useful to identify 3 classes of agents: information (a priori or a posteriori) distribution agents, control decision-making (strategy-determining) agents, and on-line control implenentation agents. The complexity of the control problem depends on the arrangement of task distribution among the agents. The problem is simplest when there is only a single agent performing all the tasks (centralized control case), but the nature of many 'large-scale' system problems often dictates that the agents be different - in such a case we have a 'decentralized' control problem. The 'decentralization' can be due to different agents performing the different classes of task (e.g. the control actuation operator is different from the control decision-maker), or to having different agents performing the same class of tasks (e.g. different control operators 'manning' different control stations).

Our informal discussion above has thus suggested the following (non-exhaustive) conceptual classification of 'decentralization' in control problems.

	Information			Off-line	On-line
	a priori	a posteriori		control law	control
		observation	control communication	computation	implementation
Case 1	Cen.	Cen.	Not needed	Cen.	Cen.
Case 2	"	"	"	Dec.	"
Case 3	"	Dec.	None	Cen.	Dec.
Case 4	"	"	Allowed	"	"
Case 5	Dec.	Dec.	None	Dec.	Dec.
Case 6	Dec	"	Allowed	"	"

Case 1

This is the traditional centralized control problem situation.

Case 2

This is the situation when the control law computation problem is too big and complex to be solved centrally, and decentralization ('decomposition') is introduced at various computational levels.

Case 3

This is the situation when measurement information are localized either because it is too costly, infeasible or else irrelevant to transmit observation information from every sensor channel to every control implementation agent. However, the control strategy is still centrally computed by a single decision-making agent with access to all a priori information. The control law determination task here is much more complicated than previous cases since the constraint of decentralized a posteriori information pattern has to be respected. Also, the control law determination problem tends to become 'ill-posed' unless

explicit penalty or constraint on the cost of computational and implementational complexity is imposed, since otherwise the decision-maker with all the a priori information, will try to assign complicated control strategies to the control implementation agents to allow them to generate as much information from their constrained observation channels as possible; the performance of the overall control scheme is also highly sensitive to any local control agent failure since the individual strategies of all the control agents are strongly 'interlocked'.

case 4

This is the same as case 3 but with certain channels for control information exchange available. The control strategy of each controller will be made a function of both the aposteriori state observation as well as observation of the control values of other controllers.

case 5

This is the situation when there are more than one decision making agent, each having a different a priori information set, and each responsible for computing the control strategy for one subgroup of the control implementation channels, subject to decentralized observation information constraints. The complexity of the problem depends on how much overlap there is between the aprior information sets of the decision making agents. Too much apriori information overlap will lead to each decision agent trying to'second guess' the strategy of other agents, and thus leads to complicated control laws.

case 6

This is case 5 with additional control information exchange. As in case 4,

the allowance of real-time control value communication will lead to extremely complicated individual control strategies unless computational and memory storage capacities of the individual agents are constrained to prevent them from trying to 'guess' the control strategies and 'informational states' of the other agents.

In summary, then, we can identify completely the decentralization pattern of a control problem in terms of :

1) a priori information distribution pattern
2) aposteriori observation channel distribution pattern
3) control implementation(actuation) channel distribution pattern
4) online control information communication pattern

(In the case when there are multiple number of independent decision-making agent , we have assumed that they all share the same set of control objectives ; when this is not true (e.g. game of conflicts) we have to specify in addition the controlled output-channel distribution pattern .)

The concept of coordination can be naturally interpreted within the above framework. It follows simply from the fact that certain set of control objectives are impossible to achieve with certain given fixed decentralization pattern (either the problem admits no physically implementable solution or else the computational task is too complex). One is thus faced with the problem of finding rules for relaxing the originally given decentralization pattern until a feasible solution exists. Conceptually, this process of relaxation of problem constraints involves the introduction of a coordinating agent .

The process of coordination can be carried out at the off-line control-strategy computational level or at the online strategy-implementation level. Most of the existing literatures on 'hierarchical, multilevel' approaches to decomposition of offline computation (see, e.g., [C3],[M4]) fall into the first category. We will develop a slightly different notion of coordination in 5.4.

5.3 Some Classes of Linear Multivariable Decentralized Control Problems

In this section, we review and re-interpret or generalize where appropriate several classes of decentralized control problems in the linear multivariable system context to illustrate the framework of 5.2.

We begin with a case of extreme decentralization pattern:

Decentralized Pole Assignability Problem

Given $\Sigma = (A, B_i, C_i T, i \in \underline{k})$ and $\Lambda \subset \mathcal{C}$, $|\Lambda| = n$, let there be k independent decision-making agents such that:

1) each agent knows the a priori structure Σ and the specification Λ .
2) agent i has access to only the channels u_i and y_i
3) each agent were to independently compute an output feedback law :

$$u_i = F_i y_i$$

i.e. agent i does not know what F_j is, $i \neq j$

Under the assumptions (1),(2),(3), for what structure of Σ is it possible for

$$\sigma(A + \sum_{i \in \underline{k}} B_i F_i C_i^T) = \Lambda \qquad \forall \ \Lambda \subset \mathcal{C} \qquad ?$$

Remark

This is a case of decentralized control law computation with decentralized observation and control actuation channels and no communication of control values among the k agents allowed.

This formulation is a re-interpretation of a problem first posed by Eckberg[E1,Chap.2]. We can generalize this problem by relaxing the exact pole assignment requirement.

Decentralized $\mathcal{C}_g$-Assignability Problem

Given Σ and $\mathcal{C}_g \subset \mathcal{C}$, and assumptions (1),(2),(3) for what structure of Σ is it possible for

$$\sigma(A+\sum_{i\in\underline{k}} B_i F_i C_i^T) \subset \mathcal{C}_g \quad ?$$

Remark

The restriction to output feedback can be relaxed by allowing dynamic compensation in the standard way (see, e.g. [W6]) without destroying the nature of the decentralization.

A further generalization is to allow state reconstructors to be used by the independent agents. The consequence of this for the case when no control communications are allowed is well-known (see[B1]): each agent regards the control input signals of other agents as unknown disturbances, and the largest unreconstructible subspace $\mathcal{V}_i$ of agent i is just

$$\mathcal{V}_i = I(A,[B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_k] \; ; \; N(C_i^T))$$

Letting $N\begin{bmatrix} C_i^T \\ \hat{C}_i^T \end{bmatrix} = \mathcal{V}_i$

we have that each agent has effectively enlarged its observation channels, and can effectively apply the feedback $u_i = \hat{F}_i \begin{bmatrix} C_i^T \\ \hat{C}_i^T \end{bmatrix} x$

We can modify the above extreme decentralization pattern by allowing exchange of the implemented control values between different agents in real-time, but at the same time impose the assumption that the agents are of such limited computational and memory-storage capacities that they are not capable of trying to 'guess' the control strategies and information states of other agents. Under such a controller complexity constraint, control information exchange amounts to allowing the independent agents to remove unknown biases in their state-reconstruction from measurements that are affected by the control signals of other agents. Note that transmission of an agent's entire control vector to other agents is not necessary; only that part of his control signal that affects the observable subspace of another agent need be transmitted. (One version of this problem has been considered by Aoki in [A5].) With control communication and under the assumption that each agent uses a state-reconstructor, therefore, we are effectively allowing each agent to feedback his entire observable subspace (and by the 'deterministic separation' principle (see [W6]) it can be argued that no more dynamic information processing are necessary).

The constraint of independent gain computation turns out to be very severe; essentially only systems whose control interaction structure is either interaction -free (i.e. $X_2=0$) or else is <u>nested</u> (see pp.32) and whose observation channels are independent can achieve such decentralized pole-assignment(see [E1] for details) .

We therefore turn next to consider a less extreme decentralized control problem.

Centralized Pole-assignability with decentralized channels Problem

Given $\Sigma=(A,B_i,C_i^T,\ i\in\underline{k})$ and $\Lambda\subset\mathbb{C}$, $|\Lambda|=n$, does there exists $(F_1,\ldots,F_k)$ such that

$$(A+\sum_{i\in\underline{k}} B_iF_iC_i^T) = \Lambda \quad \forall \quad \Lambda\subset\mathbb{C} \quad ?$$

Remark

This is the case of centralized control law computation by a single decision-making agent who has access to all a priori structure information, but with observation and control actuation channels decentralized (so that control implementation has to be decentralized), and no control information communication allowed.

Remark

This problem can be generalized to allow dynamic compensation in the standard way.

When complete pole-assignability is not required, we have the less stringent problem :

Centralized Stabilizability with decentralized channels Problem

Given Σ , does there exists $(F_1,\ldots,F_k)$ such that

$$\sigma(A+\sum_{i\in\underline{k}} B_iF_iC_i^T) \subset \mathbb{C}^- \quad ?$$

The dynamic feedback version of this latter problem has been posed and solved by Wang and Davison [W1].

5.4 Decentralized Feedback Coordination

In the last section we have considered problems which either have completely decentralized control strategy computation or else completely centralized control strategy computation. In this section we formulate a class of problems that have the control law computational task partially centralized and partially decentralized. The basic idea is that the centralized part of the control strategies represent the coordinating constraints imposed by a central decision-making agent who has access to all a priori structural information (and is thus called the coordinator), while the decentralized part of the control strategies represents the degree of freedom or independence that each local control actuation agent can exercise. The task of the coordinator, in other words, is to parametrize the degree of freedom of each of the local control agent and assign that a priori constraint information to the individual agents.

The motivation is as follows. In any realistic synthesis problem there are always a large number of feasible solutions that satisfy a particular set of control objectives(keeping a system stabilized, localizing a particular disturbance,etc.). While the use of cost-criterion optimization in theory allows a central agent to pick out exactly one choice(the optimal solution), in practice the difficulties of incorporating all the relevant cost considerations necesitates further trial and error 'hedging' about the nominal solution. In a large-scale system it is desirable that such further 'fine-tuning' be done decentrally by the local agents rather than by the central agent. It is therefore desirable that the central agent does not completely fix the individual control strategies of the local agents. In this spirit, we have:

Lemma 5.4.1

Given $\Sigma = (A, B_i, C_i^T, i\varepsilon\underline{k})$

If there exists $(K_1^*, \ldots, K_k^*)$ such that

$(A + \sum_{i\varepsilon\underline{k}} B_i K_i^* C_i^T)$ is stable, then

$$(A + \sum_{i\varepsilon\underline{k}} B_i ((N_i - M_i) B_i^T C_i L_i + K_i^*) C_i^T) \text{ is also stable} \tag{5.4.1}$$

$$\forall L_i \geq 0 \text{ such that } L_i C_i^T R(B_j) = 0 \ , \ i \neq j$$

$$\text{and } N_i = -N_i^T \ , \ M_i \geq 0 \text{ arbitrary} \tag{5.4.2}$$

Proof

This is a straight-forward generalization of Theorem 3.2.1

Remark

The tasks of the central coordinator are:

1) Select a feasible $(K_1^*, \ldots, K_k^*)$. This can be done offline.

2) Transmit to each agent i the specification K_i^* and the constraint specification (5.4.2) . This can be done offline too.

The task of the local agent i is to simply implement K_i^* and to perform 'fine-tuning' within the constraints imposed by (5.4.2)

Although the content of lemma 5.4.1 is too crude (the constraint (5.4.2) is sufficient but highly unnecessary) and no doubt has to be strengthened for it to be of practical significance, it is the conceptual interpretation it gives rise that we feel is of importance.

CHAPTER 6

CONCLUSION

In this thesis, a study of the two closely related problems of designing reliable multi-loop feedback stabilization strategies and of coordinating decentralized feedbacks has been initiated. Although only partial solutions have been obtained so far, to the problem posed, we feel that the approaches we have taken shed much new insights that not only have generalized our understanding of the properties of multi-loop feedback interaction, but have also provided a very useful conceptual framework for the eventual solution of the originally posed problems and other related problems where feedback interaction is important.

The main contribution of this thesis has been the exploitation of the idea of parametrizing directly the class of feedback maps that realizes a particular control objective rather than indirectly through looking at the subspaces they leave invariant, etc. Such an approach provides an explicit and therefore powerful means for generating feasible solutions that moreover have easy structural interpretation. Despite the elementary nature of our techniques (Lyapunov stability theory, LQ-feedback), our results in Chapter 3 and 4 have in fact provided a parametrization of an important class of feedback maps that have the <u>independent loop-adjustment property</u>. The significance of this property with respect to practical design (interactive search, 'fine tuning', protection against failure) have already been pointed out. In addition, our approach leads very naturally to the idea of coordination parametrization. The 'fixed' part of the individual controller's feedback map can be interpreted as the coordination constraint <u>selected</u> by the coordinating

agent, while the remaining degrees of freedom can be independently implemented by each individual control agent. The significance and attractiveness of this concept in the decentralized control context is immediate, and no doubt can be more extensively and vigorously developed in a formal context.

As a side benefit, our identification of the structure of the class of non-destabilizing feedback maps has allowed us to make transparent as well as to generalize the structural stability properties of LQ-feedback.

Our second hoped-for contribution lies in the introduction of a concept of 'structural' homomorphism based on the idea of classifying systems according to their control interaction (observation interdependency) pattern. The relevance of this concept for system structural classification and for refining our understanding of model aggregation has already been pointed out.

Finally, the informal decentralization pattern classification scheme we presented in 5.2 represents an initial effort at clarifying the conceptual basis for modeling the structural aspects of large-scale decentralized control systems, and we hope that the framework provided can be contributive towards a more precise and formal treatment.

Suggestions For Further Research

Lack of time did not allow us to pursue in depth the many interesting venues for further research opened up by this thesis work. As suggestion for further research topics, we list the following:

1) To obtain a complete solution to RLQ in its full generality. If RLQ(Σ) is non-empty for only a subset of all $\Sigma \in S$, obtain a parametrization of this subset in terms of the interaction structure ideas. If one system does not admit an RLQ synthesis, does this imply that every system structurally homomorphic to it also does not admit RLQ synthesis? If not, what other (stronger) forms of 'structural equivalence' can we impose?

2) Generalize the class of non-destabilizing state feedback maps to allow for dynamic feedback compensation, i.e. what is a convenient parametrization of a class of dynamic compensation and output feedback maps that also have the non-destabilizing property? The stabilizing property? It is known that by putting an extra cost term involving a quadratic form of the control <u>derivatives</u>, one can get a generalized LQ-problem that yields an optimal <u>dynamic</u> feedback solution. Can this idea be used to generate a class of reliably stabilizing <u>dynamic</u> feedback maps?

3) Consider other types of system reliability problems: e.g. <u>reliable</u> disturbance-localization synthesis, <u>reliable</u> regulation and tracking with internal stability, etc.

4) Formalize and extend the notion of viewing the selection of cost weighting matrices as a coordination strategy in a more general framework.

5) Investigate the relation (if any) between Theorem 3.4.3 and the frequency domain characterization of <u>phase margin</u> properties.

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