General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

Produced by the NASA Center for Aerospace Information (CASI)
A FAMILY OF APPROXIMATE SOLUTIONS AND EXPLICIT ERROR ESTIMATES
FOR THE NONLINEAR STATIONARY NAVIER-STOKES PROBLEM

Ralph E. Gabrielsen
Ames Research Center
and
U. S. Army Air Mobility R&D Laboratory
Moffett Field, Calif. 94035

and

Steven Karel
U. S. Army Element, NASA
Washington, D. C. 20546
and
Ames Research Center
Moffett Field, Calif. 94035

January 1975
An algorithm for solving the nonlinear stationary Navier-Stokes problem is developed. Explicit error estimates are given.
Abstract. An algorithm for solving the nonlinear stationary Navier-Stokes problem is developed. Explicit error estimates are given.

1. Introduction. Since the separation problem of aerodynamics is at present intractable, it has been decided to undertake a closely related problem via a mathematical technique that is potentially adaptable to the separation problem. Specifically, the problem under consideration is the "nonlinear stationary Navier-Stokes problem" of fluid dynamics. The generalized Newton's method, as developed by Kantorovich [5, 7] is used. Its application to this problem is of definite value for those seeking practical solutions of related fluid flow problems. The following questions are considered:

(i) Under what conditions does the sequence of functions obtained by Newton's method converge to the solution?

(ii) How should the initial guess be made, as a function of \( v \), so as to guarantee convergence?

(iii) At what rate does the sequence of approximate solutions converge?

Given \( S \), a two-dimensional Green's domain, and \( f_1(x,y) \in C^1(S) \), \( f_2(x,y) \in C^1(S) \), the nonlinear stationary Navier-Stokes problem is:

*Ames Research Center and U.S. Army Air Mobility R&D Laboratory, Moffett Field, Calif. 94035.

\[
\begin{align*}
\left\{
\begin{array}{l}
uu_x + vu_y + q_x - v du + f_1(x, y) = 0, \\
uv_x + vv_y + q_y - v dv + f_2(x, y) = 0, \\
u_x + v_y = 0,
\end{array}
\right.
\end{align*}
\]

with boundary conditions
\[
u(\partial S) = -b_2(\partial S), \quad v(\partial S) = b_1(\partial S).
\]

An equivalent expression is
\[
(2)\quad P(\psi) = \nu \Delta \psi + \psi \Delta \psi - \psi \Delta \psi + f_1 - f_2 = 0
\]
\[
\psi_x(\partial S) = b_1(\partial S), \quad \psi_y(\partial S) = b_2(\partial S),
\]

where
\[
\psi_x = v, \quad \psi_y = -u,
\]

and \( P \) is a mapping from \( C^4(S) \) into \( C^0(S) \) with norm
\[
\|\psi\|_{C^4} \leq \sum_{n=0}^{N} \sum_{m=0}^{n} \max \left| \frac{\partial^n \psi}{\partial x^m \partial y^{n-m}} \right|.
\]

For clarity, the equivalence of (1) and (2) is now shown.

**Lemma (1) \( \iff \) (2).**

**Proof.**

(1) \( \iff \) (2) directly follows from [6, Theorem 6, p. 131].

(2) \( \iff \) (1): (2) can be readily rearranged into the form
\[
(-\nu \Delta u + uu_x + vu_y + f_1)_y = (-\nu \Delta v + uv_x + vv_y + f_2)_x,
\]

with
\[
u = \psi_y, \quad v = -\psi_x.
\]

Let
\[
\tilde{z} = (-\nu \Delta u + uu_x + vu_y + f_1) \hat{i} + (-\nu \Delta v + uv_x + vv_y + f_2) \hat{j}.
\]

Note the fact: if \( \hat{V} = a \hat{i} + b \hat{j} \in C^1(S) \), then
\[
\hat{F} \cdot \hat{V} = VF \iff a = b.
\]
Hence, $q + \hat{z} = v_q$.

Therefore, (2) + (1).

2. Main Results. We seek a solution to (2) by the generalized Newton's method. Consider the equation

$$P(\psi_0) + P'(\psi_0)(\psi - \psi_0) = 0.$$  

If $\psi_1$ is a solution of this equation, we can write a new equation

$$P(\psi_1) + P'(\psi_1)(\psi - \psi_1) = 0.$$  

Assume that for each $n \geq 0$, \[ \psi_{n+1} = \psi_n - \frac{P(\psi_n)}{P'(\psi_n)} \]

(3)

$$\psi_{n,x}(\alpha_3) = b_1, \quad \psi_{n,y}(\beta_3) = b_2.$$  

(See LEMMA 1 for explicit expression for $P'$.)

If $\lim_{n \to \infty} \psi_n$ exists, let $\psi^*$ be the limit. Then

$$P(\psi^*) + P'(\psi^*)(\psi^* - \psi^*) = 0$$

$$P(\psi^*) = 0.$$  

Thus $\psi^*$ is the desired solution. This is Newton's method.

Define

$$H_N = \max_{x',y'} \int_0 \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{\partial^2 G(x',y',x,y)}{\partial x_{y,n} \partial y_{x,m}} \, dx \, dy$$

where $G$ is the Green's function of the problem

$$\Delta \Delta \psi = 0 \text{ in } S, \quad \psi(\alpha S) = 0, \quad \psi_{n}(\beta S) = 0.$$  

[1], [2], [3], [4], [8].

Let $M_{\psi_0} = \max(\max|\Delta \psi_{0,y}|, \max|\Delta \psi_{0,x}|, \max|\psi_{0,x}|, \max|\psi_{0,y}|)$.

Therefore, based on the remarkable theory developed by Kantorovich, we obtain the following result:
THEOREM 1. If the initial $\psi_0$ is \* 

\[
\psi_0^x\bigg|_{\partial S} = b_1, \quad \psi_0^y\bigg|_{\partial S} = b_2, \quad M\psi_0H_3 < \nu,
\]

and

\[
\|P(\psi_0)\|_{C^0} \leq \frac{\nu^2(1 - \frac{1}{\nu} M\psi_0H_3)^2}{2H_4^2},
\]

then the Newton-Kantorovich sequence $\{\psi_m\}_{m=0}^{\infty}$ does in fact converge to the unique solution of (2).

Proof. By the theorem of Kantorovich [5, p. 708], it is sufficient to show that

\[
\|P(\psi_0)\| \leq \frac{1}{2\|P'(\psi_0)^{-1}\|^2\|P\|};
\]

this is shown by the following lemmas.

**Lemmas 1.** $P'(\psi)$, the Frechet derivative exists at all points $\psi_0$ in the domain, and

\[
P'(\psi_0)g = \nu\Delta g + \psi_y\Delta x_x + \Delta \psi_0 x_y - \Delta \psi_0 y_x - \psi_x \Delta y.
\]

**Proof:**

\[
P(\psi) = \nu\Delta \psi + \psi_0 \Delta x_x + \psi_x \Delta y_y + f_1 y - f_2 x.
\]

If

\[
\lim_{\|a\| \to 0} \frac{\|P(\psi+a) - P(\psi) - La\|}{\|a\|} = 0,
\]

for some linear operator $L$, then define $P'(\psi_0) = L$.

\[
P(\psi_0+a) - P(\psi_0) = \psi_0 \Delta a_x + \psi_y \Delta a_x + a_y \Delta \psi_0 x - a_x \Delta \psi_0 y - \psi_x \Delta y - a_x \Delta a_y.
\]

Let

\[
L(\psi_0)(a) = \nu\Delta a + \psi_0 \Delta a_x + a_y \Delta \psi_0 x - a_x \Delta \psi_0 y - \psi_x \Delta a_y.
\]

Then $L(\psi_0)$ is a linear operator. Therefore, $L(\psi) = P'(\psi)$ if

\[
\lim_{\|a\| \to 0} \frac{\|P(\psi_0+a) - P(\psi_0) - La\|}{\|a\|} = \lim_{\|a\| \to 0} \frac{\|a^y \Delta a_x - a^x \Delta a_y\|}{\|a\|} = 0.
\]

\[
a^y \Delta a_x - a^x \Delta a_y \in C^0.
\]
Therefore,
\[ \|a_y \Delta a - a_x \Delta a\| = \max_S \|a_y \Delta a - a_x \Delta a\| . \]
\[ a \in C^4 . \]

Therefore,
\[ \|a\| = \sum_{n=0}^{4} \sum_{m=0}^{n} \max_{x, y} \left| \frac{\partial^n a}{\partial x^m \partial y^{n-m}} \right| . \]

\[ |a_y \Delta a - a_x \Delta a| \leq |a_y|(|a_{xxx}| + |a_{xyy}|) + |a_x|(|a_{xyy}| + |a_{yyy}|) . \]

Therefore,
\[ \max_S |a_y \Delta a - a_x \Delta a| \leq \max_S |a_y| \left( \max_S |a_{xxx}| + \max_S |a_{xyy}| \right) + \max_S |a_x| \left( \max_S |a_{xyy}| + \max_S |a_{yyy}| \right) . \]

So:
\[ \lim_{\|a\| \to 0} \frac{\|P(\psi_0 + a) - P(\psi_0) - L\|}{\|a\|} \leq \lim_{\|a\| \to 0} \frac{\|a\| (\|a\| + \|a\|) + \|a\| (\|a\| + \|a\|)}{\|a\|} \leq \lim_{\|a\| \to 0} \frac{\|a\| (\|a\| + \|a\|) + \|a\| (\|a\| + \|a\|)}{\|a\|} . \]

Therefore,
\[ \lim_{\|a\| \to 0} \frac{\|P(\psi_0 + a) - P(\psi_0) - L\|}{\|a\|} = 0 . \quad \text{Q.E.D.} \]

**LEMMA 2.** \( P''(\psi) \) exists at all points \( \psi_0 \) in \( C^4(S) \) and
\[ P''(\psi_0) \phi = \psi_y \Delta^2 x + \psi_y \Delta^2 x - \psi_x \Delta^2 y - \psi_x \Delta^2 y . \]

**Proof.** By definition, \( P''(\psi_0) \) exists if there is a bilinear operator \( B \to \)
\[ \lim_{\|\phi\| \to 0} \frac{\|P'(\psi_0 + \phi) - P'(\psi_0) - B\phi\|}{\|\phi\|} = 0 . \]
If so, $P''(\psi_0)$ is defined to be $B$.

$$P'(\psi_0 + \phi)g = v\Delta g + (\psi_0 + \phi)_y \Delta g_x + g_y \Delta (\psi_0 + \phi)_x - g_x \Delta (\psi_0 + \phi)_y - (\psi_0 + \phi)_x \Delta g_y .$$

$$[P'(\psi_0 + \phi) - P'(\psi_0)]g = \phi_y \Delta g_x + g_y \Delta \phi_x - g_x \Delta \phi_y - \phi_x \Delta g_y .$$

Now,

$$[P'(\psi_0 + \phi) - P'(\psi_0)](g_1 + g_2) = [P'(\psi_0 + \phi) - P'(\psi_0)]g_1 + [P'(\psi_0 + \phi) - P'(\psi_0)]g_2 ,$$

and

$$[P'(\psi_0 + \phi_1 + \phi_2) - P'(\psi_0)]g = [P'(\psi_0 + \phi_1) - P'(\psi_0)]g + [P'(\psi_0 + \phi_2) - P'(\psi_0)]g .$$

Therefore, $[P'(\psi_0 + \phi) - P'(\psi_0)]g$ is a bilinear operator of $\phi$ and $g$. Let

$$B\phi = P'(\psi_0 + \phi) - P'(\psi_0) .$$

Then it follows that

$$P''(\psi_0)\phi\theta = \phi_y \Delta \theta_x + \theta_y \Delta \phi_x - \theta_x \Delta \phi_y - \phi_x \Delta \theta_y .$$

**Lemma 3.** $P'(\psi_0)^{-1}$ exists.

**Proof.** Given $P(\psi_0) + P'(\psi_0)(\psi - \psi_0) = 0$, let $\tilde{\psi} = \psi - \psi_0$, then

$$P'(\psi_0)\tilde{\psi} = -P(\psi_0) .$$

Equivalently, $v\Delta \tilde{\psi} + \Delta \psi_0 \tilde{\psi} + \psi_0 \phi \Delta \tilde{\psi}_x - \psi_0 \Delta \psi \phi_x - \Delta \psi_0 \phi_x = -P(\psi_0)$. This equation can be abbreviated as $v\Delta \tilde{\psi} = \tilde{\psi}(\psi) + F$. Let $G$ be the Green's function for $\tilde{\psi}$ (see Theorem 1), then

$$\tilde{\psi} = \frac{1}{v} \int GF + \frac{1}{v} \int GF .$$

Define the linear operators

$$A\tilde{\psi} = \frac{1}{v} \int GF(\tilde{\psi}) .$$

$$B[-P(\psi_0)] = \frac{1}{v} \int GF .$$

Then

$$\left(I - \frac{1}{v} A\right)\tilde{\psi} = B[-P(\psi_0)] .$$

Under proper conditions, as shown later in the proof, $\left(I - \frac{1}{v} A\right)^{-1}$ exists. Then
\[ \tilde{\psi} = (I - \frac{1}{v} A)^{-1} B[-P'(\psi_0)] . \]

Therefore,
\[ P'(\psi_0)(I - \frac{1}{v} A)^{-1} B[-P'(\psi_0)] = -P(\psi_0) ; \]
\[ P'(\psi_0)(I - \frac{1}{v} A)^{-1} B = I , \]
and \((I - \frac{1}{v} A)^{-1} B\) is a right inverse. Also, \((I - \frac{1}{v} A)^{-1} B P'(\psi_0)\tilde{\psi} = \tilde{\psi} ;\)
\((I - \frac{1}{v} A)^{-1} B P'(\psi_0) = I , \) and \((I - \frac{1}{v} A)^{-1} B\) is a left inverse. Therefore,
\[(I - \frac{1}{v} A)^{-1} B = [P'(\psi_0)]^{-1} . \]

We now show the conditions under which \((I - \frac{1}{v} A)^{-1}\) exists.

\[ A \psi = \int_S G(\psi_x \Delta \psi_{0y} - \psi_y \Delta \psi_{0x} + \psi_{0x} \Delta \psi_y - \psi_{0y} \Delta \psi_x) dS \]
\[ \|A\| = \sup_{\|\psi\| \leq 1} \|A\psi\| \]

so
\[ |A| = \sup_{\|\psi\| \leq 1} \left[ \max_{x', y' \in S} \left| \int_S G(\psi_x \Delta \psi_{0y} - \psi_y \Delta \psi_{0x} + \psi_{0x} \Delta \psi_y - \psi_{0y} \Delta \psi_x) dS \right| + \max_{x', y' \in S} \left| \int_S G(\psi_x \Delta \psi_{0y} - \psi_y \Delta \psi_{0x} + \psi_{0x} \Delta \psi_y - \psi_{0y} \Delta \psi_x) dS \right| \right] \]
\[ |A| \leq \sup_{\|\psi\| \leq 1} \left[ \max_{x', y' \in S} \int_S \left| G_x \right| + \left| G_y \right| + \ldots \right] \cdot \left[ \max_{x', y' \in S} \left| \psi_x \right| + \left| \psi_y \right| + \ldots \right] \]
\[ |A| \leq \max_{\|\psi\| \leq 1} \left( \max_{x', y' \in S} \int_S \left| G_x \right| + \left| G_y \right| + \ldots \right] \cdot \left[ \max_{x', y' \in S} \left| \psi_x \right| + \left| \psi_y \right| + \ldots \right] \).

Therefore,
\[ \|A\| \leq M_{\psi_0} \left( \max_{x', y' \in S} \int_S \left| G_x \right| + \left| G_y \right| + \ldots \right) dx dy \).
\[ \|A\| \leq M_{\psi_0} H_3 . \]

Therefore, \((I - \frac{1}{v} A)^{-1}\) exists if \( M_{\psi_0} H_3 < v \). \((I - \frac{1}{v} A)^{-1} = \sum_{n=0}^{\infty} \frac{1}{v^n} A^n\) exists.
LEMMA 4. \( \|P^n(\psi_0)\| \leq 1 \).

Proof.

\[
\|P^n(\psi_0)\| = \max_{\|\phi\| \leq 1} |\phi_y \Delta^2 x + \phi_y \Delta^2 x - \phi_x \Delta y - \phi_x \Delta y| \\
\leq \max(|\phi_y| + |\phi_{xxx}| + |\phi_{xyy}| + |\phi_x| + |\phi_{xxy}| + |\phi_{yyy}|) \leq 1 .
\]

This completes the proof of Theorem 1.

THEOREM 2. Under the hypothesis of Theorem 1, the error estimate for the \( m \)th approximate solution is expressed by

\[
\|\psi - \psi_m\| \leq \left(2^{m-m-1}\right) \left[\frac{H}{\nu(1 - \frac{1}{\nu} \psi_0 H_3)}\right] 2^{m+1-1} \|P(\psi_0)\|^{2m} .
\]

Proof. By the Kantorovich theory, this result follows from the hypothesis of Theorem 1.

COROLLARY 1. As a function of \( \nu \), for fixed \( m \),

\[
\|\psi - \psi_m\| = O\left(\frac{2^{m+1+1}}{\nu}\right) .
\]

COROLLARY 2. If the hypothesis of Theorem 1 is satisfied, then for a given \( \epsilon > 0 \), there exists a denumerably infinite number of linear equations and solutions \( \psi_m \) as specified by (3) such that the entire family of \( \psi_m \)'s are within the \( \epsilon \)-neighborhood of the exact solution \( \psi \) of (2), i.e.,

\[
\|\psi - \psi_m\|_{C^0} < \epsilon .
\]

Proof. Follows directly from Theorem 2.
REFERENCES


