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EARTH-MOON SYSTEM:  DYNAMICS AND PARAMETER
ESTIMATION

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By

W.J. Breedlove, Jr.

Semiannual Progress Report

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia

Under
Grant NSG 1152
February 17, 1975 - August 17, 1975

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A UNIFIED SPECIAL PERTURBATION MODEL FOR THE
MOTION OF THE EARTH-MOON SYSTEM

By
W.J. Breedlove, Jr.¹

SUMMARY

This report contains a theoretical development of the equations
of motion governing the Earth-Moon system. The Earth and Moon are
treated as finite rigid bodies and a mutual potential is utilized.
The Sun and remaining planets are treated as particles. Relativistic, non-rigid, and dissipative effects are not included.

The translational and rotational motion of the Earth and
Moon are derived in a fully coupled set of equations. Euler
parameters are used to model the rotational motions.

The mathematical model developed herein is intended for use
with data analysis software to estimate physical parameters of
the Earth-Moon system using primarily LURE type data.

The Appendix contains two program listings. Program ANEAMØ
computes the translational/rotational motion of the Earth and
Moon from analytical solutions. Program RIGEM numerically inte-
grates the fully coupled motions as described above.

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INTRODUCTION

The Lunar Laser Ranging Experiment (LURE), (ref. 1) has resulted in the placement of three ranging retroreflectors on the Moon. A series of measurements of the distance of these retroreflectors from several Earth-based observatories began in August 1969. These measurements, at present, allow the determination of the distance to the Moon with an accuracy of ± 8 cm. A resolution of ± 2 to 3 cm is expected within the next few years. Overall, a ± 10-cm accuracy over a 10-year period will soon be available.

The LURE data, in combination with other data types, can be used to determine parameters related to the internal composition of the Earth and Moon (ref. 24). Also, checks of current gravitational theories may be made (ref. 1). For example, data accuracies of ± 3 cm would make feasible the determination of the following parameters (refs. 1 and 2):

A. Geometrical and Orbital Parameters:
   - station coordinates
   - retroreflector coordinates
   - Earth and Moon orbital constants of integration

B. Geophysical Parameters:
   - station drift
   - polar wobble
   - rotation variations
   - Earth tide
   - universal time determination
   - orbital acceleration

C. Selenophysical Parameters:
   - physical librations
   - free librations
   - Moon tide
D. Systematic Error Sources:

fixed bias
zenith-distance bias
arbitrary periodic biases.

The original mathematical model (October 1973) of the LURE team (ref. 1) involved (1) the numerical integration of the Moon and major planets as point masses including relativistic effects, (2) the utilization of an analytical lunar physical libration theory based on Eckhardt's work, plus certain additive and planetary terms from the Improved Lunar Ephemeris and 3rd and 4th order terms in the lunar potential (refs. 19 and 23), and (3) determining the angular position and pole of the Earth from BIH data. Earth tides and dissipative effects were not modeled (ref. 2). Lunar orbital-rotational coupling was not fully modeled (ref. 22). Finally, the BIH data imposes limits on the accuracy achievable from this model.

The model described above reached its current state by appending additional effects to existing models. For example, Eckhardt's original libration theory did not include the 3rd and 4th order terms in the lunar potential. Also, the additive and planetary terms were appended to this original theory.

This model provided (October 1973) rms residuals in range of \pm 3 meters. An improved libration theory would considerably reduce this value. The LURE team suggested that a numerical integration of Euler's equations holds promise for future gains in accuracy for the rotational motion of the Moon (ref. 19). The above residuals imply the existence of unmodeled effects or modeling inaccuracies (ref. 2).

The determination of geometrical and orbital parameters, geophysical parameters, selenophysical parameters, and systematic error sources to an accuracy compatible with the observational accuracy thus awaits the development of a rigorous model of the Earth-Moon system. Previous models have been attempted in a piecewise fashion. Thus, there is a need for a consistent
theoretical, mathematical model incorporating Earth and Moon rotational, translational, and deformational motions in a coupled sense. This model should allow for an inhomogeneous Earth and Moon, dissipation effect, general relativity effects, and planetary perturbations. Secondly, there is a need for a "special" numerical model incorporating pertinent effects from the above theoretical model to be used in the parameter estimation process.

The "special" model envisioned at this point (although subsequent investigations may modify it) is of the following form.

1. Treat Sun and planets as perturbing point masses,
2. treat Moon as a tidally deformed body,
3. treat Earth as a tidally deformed body,
4. consider the coupled orbital-rotational motions of the Earth and Moon (ref. 22), and
5. consider the effects of relativity (refs. 25 and 26).

The governing equations of motion for this "special" model are to be numerically integrated (refs. 4 and 9).

The appropriate numerical integration routine to be used in this model should be investigated. One candidate is an extremely accurate Cowell type routine used by Oesterwinter and Cohen in a determination of planetary masses (ref. 7). This routine has been developed over a period of years by Cohen and Hubbard and has been used primarily for solutions to the planetary n-body problem. This scheme is based on the use of a 16th-order set of predictor-corrector formulae for integrating accelerations. Herrick (ref. 27) also espouses the use of numerical integration schemes that integrate accelerations directly.

GENERAL SYMBOLS AND NOMENCLATURE

vector
universal gravitational constant
(·), (¨)  first and second time derivatives
\( \dot{r}_i \)  radius vector from solar system barycenter to mass center of body \( i \) \((i = 1,11)\)
\( \ddot{r}_i \)  radius vector from Sun to mass center of body \( i \) \((i = 1,11)\)
\( \nabla_j \)  gradient operator with respect to coordinates of mass center of body \( j \) \((j = 1,11)\)
\( U \)  work function
\( m_j \)  mass of body \( j \) \((j = 1,11)\)
\( \beta_i \)  Euler parameters locating Earth body axes with respect to Earth reference axes \((i = 0,1,2,3)\)
\( \omega_i \)  absolute angular velocity components of Earth resolved along body axes \((i = 1,2,3)\)
\( \omega_i \)  absolute angular velocity components of Moon resolved along lunar body axes \((i = 1,2,3)\)
\( \beta_i \)  Euler parameters locating lunar body axes with respect to lunar reference axes \((i = 0,1,2,3)\)
\( r, \lambda, \phi \)  components of spherical polar coordinate system
\( \{ \} \)  vector-column
\( [ ] \)  matrix
\( \{ \}^T \)  transposed vector-row
\( \{( \) \( )^T\} \)  translating reference frame
\( [ ] \)  vector-row
\( (%) \)  dyadic

PHYSICAL MODEL

For the purposes of this study, the Sun, Mercury, Venus, Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto are modeled as particles. The Earth is modeled as a triaxial rigid body
and the Moon as an asymmetric rigid body. The Sun, Moon, and planets interact gravitationally with translational and rotational motions fully coupled. Non-rigid, dissipative, and relativistic effects are not considered here but anticipated for future inclusion in the model.

MATHEMATICAL MODEL

A system of 41 second-order ordinary differential equations has been derived to represent the physical model described in the previous section. These may be summarized as follows:

A. Motion of Sun with respect to center of mass of Solar System

$$\ddot{\mathbf{r}}_1 = G \sum_{j=2}^{11} m_j \frac{\mathbf{r}_j}{r_{1j}^3}$$  \hspace{1cm} (1)

B. Motion of Moon and planets with respect to the Sun

$$\ddot{\mathbf{r}}_i + G(m_1 + m_i) \frac{\dot{\mathbf{r}}_i}{r_{i1}^3} = \sum_{j=2}^{11} m_j \ddot{u}_{ij} U_{ij} \quad (i = 2, 3, \ldots, 11)$$  \hspace{1cm} (2)

C. Rotational motion of the Earth

$$\{\dot{\beta}'\} = \frac{1}{2} [\dot{\beta}'] \left( \begin{array}{c} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{array} \right) - \{f(t)\}$$  \hspace{1cm} (3)

$$+ \frac{1}{2} [\beta'] \left( \begin{array}{c} 0 \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{array} \right) - \{f(\dot{\varepsilon})\}$$
D. Rotational motion of the Moon

\[
\{\dot{\beta}^{'''}\} = \frac{1}{2} \left[ \dot{\beta}^{'''} \right] \begin{pmatrix}
0 \\
\ddot{\omega}_1 \\
\ddot{\omega}_2 \\
\ddot{\omega}_3
\end{pmatrix} - \left[ c(\beta^{''''}) \right]_A \begin{pmatrix}
0 \\
-\dot{\lambda}s\phi \\
\dot{\phi} \\
\dot{\lambda}c\phi
\end{pmatrix}
\]

\[
+ \frac{1}{2} \left[ \beta^{''''} \right] \begin{pmatrix}
0 \\
\ddot{\omega}_1 \\
\ddot{\omega}_2 \\
\ddot{\omega}_3
\end{pmatrix} - \left[ c(\beta^{''''}) \right]_A \]

\[
\times \frac{d}{dt} \begin{pmatrix}
0 \\
-\lambda s\phi \\
\phi \\
\lambda c\phi
\end{pmatrix}
\]

The "short-hand" notation \( s\phi = \sin(\phi) \) and \( c\phi = \cos(\phi) \) has been utilized in the above equation.

A full discussion of the above equations including rationale and derivations appears later in this report.

Equations (1) represent the motion of the Sun with respect to the mass center of the Solar System as forced by the Moon and planets.

Equations (2) represent the motion of the Moon and planets with respect to the Sun. The gradient is to be taken with respect to the coordinates locating each body with respect to the Sun. The force function \( U_{ij} \) may be written as

\[
U_{ij} = U_{ij}^P + U^I
\]

where \( U_{ij}^P \) represents the mutual gravitational interaction between all masses treated as particles and \( U^I \) is the mutual potential of the Earth and Moon regarded as finite rigid bodies.
Equations (3) represent the rotational motion of the Earth with respect to a defined "reference" coordinate frame. The Euler parameters \( \{\beta'\}^T = \{\beta'_0, \beta'_1, \beta'_2, \beta'_3\} \) represent the angular deviations of the Earth from this "reference" frame. The forcing torques enter through the angular acceleration terms,

\[
\begin{bmatrix}
0 \\
\dot{w}_1 \\
\dot{w}_2 \\
\dot{w}_3
\end{bmatrix}
\]

The function \( \{f(t)\} \) defines the reference frame. A solution of these equations represents the deviation of the Earth from a uniform rotation about a fixed axis in space, i.e., precession and nutation.

Equations (4) represent the rotational motion of the Moon with respect to a defined "reference" coordinate frame. The Euler parameters \( \{\beta''\}^T = \{\beta''_0, \beta''_1, \beta''_2, \beta''_3\} \) represent the angular deviations of the earth from this "reference" frame. The forcing torques enter through the angular acceleration terms,

\[
\begin{bmatrix}
0 \\
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3
\end{bmatrix}
\]

The rotation matrix \( [c(\beta'')]_A \) and the terms in \( -\dot{s}\phi, \dot{\phi}, \dot{s}\phi \) and their derivatives account for the motion of the "reference" frame. A solution of these equations represents the optical plus the physical librations.

**Rationale for Development of Equations**

The various existing analytical theories for the translational and rotational motion of the Earth and Moon are being looked at more and more critically due to ever increasing observational accuracy (ref. 1).
Many previously neglected effects must now be included in the mathematical models used to reduce the observational data. This, of course, leads to a better knowledge of these small effects and their causes. Currently unknown effects may also be discovered as the observational data is analyzed.

The possibility of using LURE data to determine various geophysical and selenophysical parameters was pointed out in the introduction and in reference 2.

Accordingly, this report describes work undertaken to develop a mathematical model of the motion of the Earth-Moon system that has as few restrictions on accuracy as possible. Thus, the coupled rotational-translational motions of both the Earth and Moon are included in this model.

A more immediate and specialized goal, however, is to be able to solve for the coupled rotational-translational motion of the Moon for use in the reduction of LURE data to estimate the low-order gravitational harmonic coefficients of the Moon. Thus, this problem will be emphasized in this report.

Several recent papers (refs. 1, 3, 4, 5) have pointed out the facts that (1) analytical theories for the lunar translational motion and (2) analytical theories for the lunar rotational motion are not accurate enough to be used in the reduction of LURE-type data. Attempts are therefore being made to numerically integrate (1) the equations of motion for the lunar orbit (ref. 3), and (2) the equations governing lunar rotation (refs. 5,6).

The above facts and the success of Cohen and Oesterwinter (ref. 7) in numerically integrating the motion of the solar system have prompted this attempt at a numerical integration of the equations of motion representing the coupled translational-rotational motions of the Earth and Moon.

The formulation of both the translational and rotational equations of motion as a system of second-order differential equations was dictated by the general observation that Class II (second-order) numerical integration methods are more efficient (ref. 8).
Since the force and torque evaluations at each integration step are very costly in computer time, it was decided to utilize Euler parameters rather than Euler angles in the rotational equations. The relation of the rates of change of those parameters to the angular velocity components is algebraic rather than trigonometric in the case of Euler angle rates. Although two additional second-order equations are thereby added to the system, no trigonometric functions need be evaluated at each step. A time saving is thereby accomplished in the integration of the rotational equations. This approach has been common practice in the simulation of aircraft and gyroscopic motions (refs. 9,10). Advantages arise also in problem formulation and parameter estimation when Euler parameters are utilized (ref. 11).

The reference axes used in the rotational equations were chosen so that the large angular rotation rates of the Earth and Moon with respect to inertial space did not have to be integrated. The reference axis for the Earth spins with respect to an inertial system about a fixed axis with a fixed rate equal to the mean sidereal rotation rate of the Earth. The reference axis for lunar rotation is centered at the Moon's mass center and its primary axis points to the Earth's center of mass. The axes of this system are parallel to the unit vectors of a spherical polar coordinate system that locates the Moon with respect to a mean equator and equinox of 1950.0 rectangular system centered at the Earth. This approach is similar in philosophy to the Enke method of celestial mechanics.

Coordinate Systems

The coordinate systems utilized in this study are standard and are summarized in table 1 and illustrated in figures 1 and 2. Transformation between coordinate systems is accomplished using orthogonal rotation matrices in the sense

\[ \{x'\} = [R_{xx^\prime}] \{x\} \]

where \([R]\) is a 3 x 3 rotation matrix for a rotation of \(\{x\}\) into \(\{x'\}\).
Table 1. Coordinate systems.

<table>
<thead>
<tr>
<th>No.</th>
<th>Origin of Frame</th>
<th>Axis Notation $(i = 1, 2, 3)$</th>
<th>Fundamental Plane</th>
<th>Fundamental Direction</th>
<th>Secondary Direction</th>
<th>Unit Vector Notation $(i = 1, 2, 3)$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Barycenter of Solar System</td>
<td>$X_1'$</td>
<td>Ecliptic of 1950.0</td>
<td>Intersection of ecliptic of 1950.0 and mean equator of Earth of 1950.0</td>
<td>$X_3'$ points toward North Pole of ecliptic of 1950.0</td>
<td>$\hat{t}_1'$</td>
<td>Primary inertial reference frame</td>
</tr>
<tr>
<td>2</td>
<td>Barycenter of Solar System</td>
<td>$X_1$</td>
<td>Mean equator of Earth of 1950.0</td>
<td>Same as 1</td>
<td>$X_3$ points toward North Pole of rotation of Earth of 1950.0</td>
<td>$\hat{t}_1$</td>
<td>Secondary inertial reference frame</td>
</tr>
<tr>
<td>3</td>
<td>Center of mass of Sun</td>
<td>$X_1'$</td>
<td>Same as 2</td>
<td>Same as 2</td>
<td>Same as 2</td>
<td>$\hat{t}_1'$</td>
<td>Translating frame with respect to $X_1$</td>
</tr>
<tr>
<td>4</td>
<td>Center of mass of Earth</td>
<td>$Y_1$</td>
<td>Same as 2</td>
<td>Same as 2</td>
<td>Same as 2</td>
<td>$\hat{y}_1$</td>
<td>Reference frame for Earth rotation. Rotates at uniform rate of $\dot{\omega}$ with respect to $X_1$.</td>
</tr>
</tbody>
</table>

(cont'd.)
<table>
<thead>
<tr>
<th>No.</th>
<th>Origin of Frame</th>
<th>Axis Notation ((i = 1,2,3))</th>
<th>Fundamental Plane</th>
<th>Fundamental Direction</th>
<th>Secondary Direction</th>
<th>Unit Vector Notation ((i = 1,2,3))</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Center of mass of Earth</td>
<td>(Y_i)</td>
<td>Plane of equatorial principal axes</td>
<td>Axis of minimum principal moment of inertia</td>
<td>Axis of minimum principal moment of inertia</td>
<td>(\hat{j}_i)</td>
<td>Earth &quot;body fixed&quot; frame</td>
</tr>
<tr>
<td>6</td>
<td>Center of mass of Moon</td>
<td>(Z_i)</td>
<td>Plane formed by radial and longitudinal unit vectors of spherical polar coordinate system locating the Moon with respect to (X_T) centered at the Earth</td>
<td>Axis points from Moon's mass center to Earth's mass center</td>
<td>Axis points opposite to longitudinal unit vector as described in column 4</td>
<td>(\hat{k}_i)</td>
<td>Reference frame for lunar rotation. An &quot;orbital reference frame&quot;.</td>
</tr>
<tr>
<td>7</td>
<td>Center of mass of Moon</td>
<td>(Z_i)</td>
<td>Plane of equatorial principal axes</td>
<td>Axis of minimum principal moment of inertia</td>
<td>Axis of minimum principal moment of inertia</td>
<td>(\hat{k}_i)</td>
<td>Moon &quot;body fixed&quot; frame</td>
</tr>
</tbody>
</table>

The superscript \(T\) on a set of axes indicates a frame translating with respect to the unsuper- scripted frame.
Figure 1. Coordinate reference frames.
Figure 2. Coordinate reference frames.
Equations of Motion

Translational Equations. Reference 13 provides the equations of motion for the particles and centers of mass with respect to an intertial reference frame, viz.

\[ m_i \ddot{r}_i = \ddot{\mathbf{v}}_i \mathbf{U} \quad (i = 1, 11) \]  \hspace{1cm} (6)

where

\[ \mathbf{U} = G \sum_{i>j=1}^{11} \frac{m_i m_j}{r_{ij}} \]  \hspace{1cm} (7)

and \( m_1 \) is the Sun's mass
\( m_2 \) is Mercury's mass
\( m_3 \) is Venus' mass
\( m_4 \) is Earth's mass
\( m_5 \) is the Moon's mass
\( m_6 \) is Mars' mass
\( m_7 \) is Jupiter's mass
\( m_8 \) is Saturn's mass
\( m_9 \) is Uranus' mass
\( m_{10} \) is Neptune's mass
\( m_{11} \) is Pluto's mass.

In equation (6),

\[ \ddot{r}_i = X_{1i} \frac{\partial}{\partial X_{1i}} + X_{2i} \frac{\partial}{\partial X_{2i}} + X_{3i} \frac{\partial}{\partial X_{3i}} \]  \hspace{1cm} (8)

\[ \ddot{\mathbf{v}}_i = \ddot{X}_{1i} \frac{\partial}{\partial X_{1i}} + \ddot{X}_{2i} \frac{\partial}{\partial X_{2i}} + \ddot{X}_{3i} \frac{\partial}{\partial X_{3i}} \]  \hspace{1cm} (9)

where \( X_{1i}, \ X_{2i}, \ X_{3i} \) are the coordinates of mass \( i \).
If the origin of coordinates is now translated to the Sun, the equations of motion for the Moon and planets are

\[ \ddot{r}_i + G(m_1 + m_i) \frac{\dot{r}_i}{r_{i1}^3} = G \sum_{j=2}^{11} m_j \frac{\dot{r}_j - \dot{r}_i}{r_{ij}^3} - \frac{\dot{r}_j}{r_{j1}^3} \tag{10} \]

where

\[ r_{i1} = \sqrt{(X'_{1i} - X_{11})^2 + (X'_{2i} - X_{21})^2 + (X'_{3i} - X_{31})^2} \]

\[ r_{ij} = \sqrt{(X'_{ij} - X'_{1i})^2 + (X'_{2j} - X'_{2i})^2 + (X'_{3j} - X'_{3i})^2} \], and

\[ \ddot{r}_i = x'_{1i} \ddot{r}_1 + x'_{2i} \ddot{r}_2 + x'_{3i} \ddot{r}_3 \quad (i = 2, \ldots, 11) \]

The terms on the right-hand side of equation (10) arise from the force function \( U^P_{ij} \) of equation (5). Reference 12 provides \( U^P_{ij} \) as

\[ U^P_{ij} = G \left( \frac{1}{r_{ij}^3} - \frac{\dot{r}_i \cdot \dot{r}_j}{r_{j1}^3} \right) \tag{11} \]

since

\[ m_j \ddot{r}_j U^P_{ij} = G m_j \left( \frac{\dot{r}_j - \dot{r}_i}{r_{ij}^3} - \frac{\dot{r}_j}{r_{j1}^3} \right) \]
The equations of motion for the Sun are

\[ \begin{align*}
\ddot{r}_1 &= G \sum_{j=2}^{11} \frac{m_j}{r_{1j}^3} \frac{\ddot{r}_j}{r_{1j}} \tag{12}
\end{align*} \]

where

\[ r_j = x_{1j}' \dot{x}_1 + x_{2j}' \dot{x}_2 + x_{3j}' \dot{x}_3 \]

These equations follow directly from equations (6) and (7).

As will be discussed later, the mutual gravitational potential of the Earth and Moon, treated as finite rigid bodies, may be important to LURE accuracy. Thus, \( U \) in equations (6) and (7) should be of the form

\[ U = G \sum_{i>j=1}^{11} \frac{m_i m_j}{r_{ij}} \]

\[ + \frac{G m_4 m_5}{r_{45}} \left\{ \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{1}{(r_{45})^n} p_{nm} (s\phi) \right\} \]

\[ \left[ x_{nm} \cos m\lambda + y_{nm} \sin m\lambda \right] \]

Thus, if the origin is shifted to the sun, the resulting potential may be written as

\[ U_{ij} = U_{ij}^P + U_{45}^I \]
where

\[ \mathbf{U}_{ij}^p = \mathbf{G} \left( \frac{1}{\mathbf{r}_{ij}} \cdot \mathbf{r}_{ij} \right) \quad \text{for } j = 2, \ldots, 11 \]

\[ \mathbf{U}_{45}^l = \frac{\mathbf{G}}{r_{45}} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left( \frac{1}{n_{45}} \right)^n \mathbf{P}_{nm}(s\phi) \]

\[ \cdot \left[ x_{nm} \cos m\lambda + y_{nm} \sin m\lambda \right] \]

Rotational Equations for the Earth. The rotational motion of a rigid earth must satisfy Euler's principal axis equations, viz.

\[ \begin{align*}
\dot{\omega}_1 &= \frac{M_1}{A} - k_1 \omega_2 \omega_3 \\
\dot{\omega}_2 &= \frac{M_2}{B} - k_2 \omega_1 \omega_3 \\
\dot{\omega}_3 &= \frac{M_3}{C} - k_3 \omega_1 \omega_2
\end{align*} \]

where

\[ k_1 = \frac{C - B}{A} \]
\[ k_2 = \frac{A - C}{B} \]
\[ k_3 = \frac{B - A}{C} \]

In the above, \( \omega_i \) are inertial angular velocity components in the \( y_i \) frames. The moment components \( M_i \) likewise are in this frame. \( A, B, C \) are the principal moments of inertia of the Earth. In order to orient the Earth with respect to the "reference" axes and the inertial axes the following sets of Euler parameters are introduced:
1) \{\beta\} represents a notation from \{X'_i\} to \{Y_i\},

2) \{\beta'\} represents a notation from \{Y_i\} to \{y_i\}, and

3) \{\beta''\} represents a notation from \{X'_i\} to \{y_i\},

where

\{\beta\}_T = [\beta_0, \beta_1, \beta_2, \beta_3].

Reference 13 provides the following relation between the sets of Euler parameters representing the above successive rotations:

\{\beta''\} = [\tilde{\beta}'] \{\beta\} = [\beta] \{\beta'\} \tag{16}

where

\[\begin{bmatrix}
\beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\
-\beta_1 & \beta_0 & \beta_3 & -\beta_2 \\
-\beta_2 & -\beta_3 & \beta_0 & \beta_1 \\
-\beta_3 & \beta_2 & -\beta_1 & \beta_0
\end{bmatrix}
\tag{17}

and where

\[\begin{bmatrix}
\beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\
\beta_1 & \beta_0 & -\beta_3 & \beta_2 \\
\beta_2 & \beta_3 & \beta_0 & -\beta_1 \\
\beta_3 & -\beta_2 & \beta_1 & \beta_0
\end{bmatrix}
\tag{18}

The rotation matrices linking the above coordinate systems are
\{y_i\} = [R_{x'y}] \{x'_i\} = [c(\beta)] \{x'_i\}, \quad (19)

\{y_i\} = [R_{yy}] \{y_i\} = [c(\dot{\beta})] \{y_i\}, \quad (20)

\{y_i\} = [R_{x'y}] \{x'_i\} = [c(\beta')] \{x'_i\} \quad (21)

The rotation matrices introduced above have the following form where expressed in terms of Euler parameters (ref. 13):

\[ [c(\beta)] = \begin{bmatrix}
\beta_0 + \beta_1 - \beta_2 - \beta_3 \\
2(\beta_1 \beta_2 - \beta_0 \beta_3) \\
2(\beta_1 \beta_3 + \beta_0 \beta_2) \\
\end{bmatrix} \]

\[ \dot{\beta}_0 - \dot{\beta}_1 + \dot{\beta}_2 - \dot{\beta}_3 = (2\beta_2 \beta_3 + \beta_0 \beta_1) \]

\[ 2(\beta_1 \beta_3 - \beta_0 \beta_2) = 2(\beta_2 \beta_3 - \beta_0 \beta_1) \]

\[ \beta_0 - \beta_1 - \beta_2 + \beta_3 \]

\[ \quad (22) \]

The matrices \([\beta'], [\tilde{\beta'}], [c(\beta)], \) etc. are all orthogonal and hence their inverses are their transposes (ref. 13).

The angular velocity of the Earth can now be expressed in terms of the Euler parameters and rates. The inertial angular velocity of the Earth is

\[ \omega = \omega_1 \hat{j}_1 + \omega_2 \hat{j}_2 + \omega_3 \hat{j}_3 \quad (23) \]

or

\[ \ddot{\omega} = 2 \dot{\omega}_x/x' + 2 \ddot{\omega}_y/y' \quad (24) \]

The reference axes, \(y_i', \) rotate at a uniform rate \(\alpha\) about the \(\hat{j}_3\) axis so that

\[ \ddot{\omega}_{y'/x'} = \alpha \hat{j}_3 \quad (25) \]
Reference 13 provides the following relation between the Euler parameters, their rates, and the angular velocity components in the rotating system:

\[
\begin{pmatrix}
0 \\
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}
= 2[\beta']^{-1} \{\dot{\beta}'\}
\]

(26)

where

\[
\{\dot{\beta}'\}^T = [\dot{\beta}_0 \dot{\beta}_1 \dot{\beta}_2 \dot{\beta}_3]
\]

To provide a consistent four-parameter representation of the angular velocity of the Earth, the vector \(\dot{\omega}_{Y/X'}\) must be projected on the \(\{Y_1\}\) axes and certain "augmented" matrices must be introduced. Accordingly, the angular velocity of the Earth assumes the form

\[
\begin{pmatrix}
0 \\
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}
= [c(\beta')]_A \begin{pmatrix}
0 \\
0 \\
0 \\
\dot{\alpha}
\end{pmatrix}
+ 2[\beta']^{-1} \{\ddot{\beta}'\}
\]

(27)

where the "augmented" rotation matrix \([c(\beta')]_A\) is of the form

\[
[c(\beta')]_A = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & [c(\beta')] \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

(28)
The nature of the "reference" axes, $x_i$, provides a very simple form for $[c(\beta')]$, viz.

$$[c(\beta')] = \begin{bmatrix}
\cos (\alpha) & \sin (\alpha) & 0 \\
-sin (\alpha) & \cos (\alpha) & 0 \\
0 & 0 & 1
\end{bmatrix}$$  \hspace{1cm} (29)

where $\alpha = \alpha_0 + \dot{\alpha}t$. The combination

$$[c(\beta')]_A \begin{bmatrix}
0 \\
0 \\
0 \\
\dot{\alpha}
\end{bmatrix}$$

is now defined as $\{f(t)\}$ and has the form

$$\{f(t)\} = \begin{bmatrix}
0 \\
2\dot{\alpha}(\beta'_1\beta'_3 - \beta'_0\beta'_2) \\
2\dot{\alpha}(\beta'_2\beta'_3 + \beta'_0\beta'_1) \\
\dot{\alpha}(\beta'_0{}^2 - \beta'_1{}^2 - \beta'_2{}^2 + \beta'_3{}^2)
\end{bmatrix}$$  \hspace{1cm} (30)

A set of second-order equations can now be formed by differentiating equations (27) and solving for $\{\beta''\}$. These equations assume the form

$$\{\beta''\} = \frac{1}{2} [\beta'] \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} - \{f(t)\}$$  \hspace{1cm} (31)
In equation (31),

\[
\begin{pmatrix}
0 \\
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}
\}
\{ f(t) \}
\}
and
\[
\begin{pmatrix}
0 \\
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3
\end{pmatrix}
\}
\}

are provided in equations (27), (30), and (15) respectively. The term \{ f(t) \} is of the form

\[
\{ f(t) \} = \begin{pmatrix}
2\alpha (\beta_2 \dot{\beta}_3 + \beta_1 \dot{\beta}_2 - \beta_0 \dot{\beta}_2 - \dot{\beta}_0 \dot{\beta}_2) \\
2\alpha (\beta_2 \dot{\beta}_2 + \beta_2 \dot{\beta}_3 + \beta_0 \dot{\beta}_1 + \dot{\beta}_0 \dot{\beta}_1) \\
2\alpha (\beta_0 \dot{\beta}_0 - \beta_1 \dot{\beta}_1 - \beta_2 \dot{\beta}_2 + \beta_3 \dot{\beta}_3)
\end{pmatrix}
\]

(32)

and \([\dot{\beta}']\) is of the form

\[
[\dot{\beta}'] = \begin{pmatrix}
\ddot{\beta}_0 & -\ddot{\beta}_1 & -\ddot{\beta}_2 & -\ddot{\beta}_3 \\
-\ddot{\beta}_1 & \ddot{\beta}_0 & -\ddot{\beta}_3 & \ddot{\beta}_2 \\
-\ddot{\beta}_2 & \ddot{\beta}_3 & \ddot{\beta}_0 & -\ddot{\beta}_1 \\
-\ddot{\beta}_3 & -\ddot{\beta}_2 & \ddot{\beta}_1 & \ddot{\beta}_0
\end{pmatrix}
\]

(33)

**Rotational Equations for the Moon.** The derivation of the equations of rotational motion for the Moon proceeds along a path similar to that used for the Earth.

The rotational motion of a rigid Moon must satisfy Euler's equations, viz.
where all quantities here have analogous definitions to those of equation (15). Primes are used to distinguish variables related to the Moon from those related to the Earth (un-primed).

The inertia ratios \( k_i \) in equations (32) have a more familiar notation, viz.

\[
\begin{align*}
k_1 & = \alpha \\
k_2 & = -\beta \\
k_3 & = \gamma
\end{align*}
\]

These ratios are related by the constraint

\[
\alpha = \frac{\beta - \gamma}{1 - \beta \gamma}
\]

Euler parameters are now introduced to orient the Moon with respect to its "reference" axes and the inertial frame. For this purpose, define

\[
\{\beta''\} = \begin{pmatrix}
\beta'''
\beta'_0
\beta'_1
\beta'_2
\beta'_3
\end{pmatrix}
\]

which represents a rotation from \( \{z_i\} \) to \( \{z'_i\} \). The corresponding rotation matrix is
\[
\{z_i\} = [R_{zz}] \{z_i\} = [c(\beta''')] \{z_i\}
\] (37)

The inertial angular velocity of the Moon is

\[
\vec{\omega}' = \vec{\omega}_{Z/X'} + \vec{\omega}_{Z/Z}
\] (38)

In terms of the Euler parameters \(\{\beta''\}\) and their rates, the components of \(\vec{\omega}_{Z/Z}\) are

\[
\begin{pmatrix}
0 \\
\dot{\omega}_{1Z/Z} \\
\dot{\omega}_{2Z/Z} \\
\dot{\omega}_{3Z/Z}
\end{pmatrix} = 2[\beta''']^{-1} \{\dot{\beta'''}\}
\] (39)

The angular velocity \(\vec{\omega}_{Z/X'}\) of the "reference" frame is defined completely by the translational motion of the Moon with respect to the Earth.

To determine the angular velocity \(\vec{\omega}_{Z/X'}\), introduce the spherical polar coordinates \(r, \lambda, \phi\) as illustrated in figure 2. These are the coordinates of the Moon's center of mass with respect to \(\{X_i\}'\).

Now, define the "relative" rectangular coordinates, \(\Delta_i\), as

\[
\begin{align*}
\Delta_1 &= X_{15}' - X_{14}' = r \cos \phi \cos \lambda \\
\Delta_2 &= X_{25}' - X_{24}' = r \cos \phi \sin \lambda \\
\Delta_3 &= X_{35}' - X_{34}' = r \sin \phi
\end{align*}
\] (40)
Inverting these expressions provides

\[ r = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2} \]

\[ \lambda = \tan^{-1} \left( \frac{\Delta_2}{\Delta_1} \right) , \quad \text{and} \]

\[ \phi = \tan^{-1} \left( \frac{\Delta_3}{\sqrt{\Delta_1^2 + \Delta_2^2}} \right) . \]

Since the unit vectors \( \hat{k}_i \) are related to those of the spherical polar system by

\[ \hat{k}_1 = -\hat{x}_\lambda \\
\hat{k}_2 = -\hat{x}_\lambda \\
\hat{k}_3 = \hat{x}_\phi , \]

the inertial angular velocity of the axes \( \{Z_i\} \) can be written as

\[ \dot{\omega}_{Z/X} = \dot{\lambda} + \dot{\phi} = \dot{\lambda} \hat{I}_3 - \dot{\phi} \hat{x}_\lambda . \]

The above vector may be projected on the \( \{Z_i\} \) axes providing

\[ \dot{\omega}_{B/X} = \dot{\lambda} \left[ \cos \phi \hat{k}_3 - \sin \phi \hat{k}_1 \right] + \dot{\phi} \hat{k}_2 . \]

The components

\[ \omega_1' = -\dot{\lambda} \sin \phi \\
\omega_2' = \dot{\lambda} \cos \phi \\
\omega_3' = \dot{\phi} \]
can be related to the relative position coordinates \(\Delta_i\) and the relative velocity components \(\dot{\Delta}_i\). To do this, differentiate equations (40), solve for \(\dot{\Delta}_i\), and put the results in the matrix form

\[
\begin{pmatrix}
\Delta_1 \\
\Delta_2 \\
\Delta_3
\end{pmatrix} =
\begin{bmatrix}
c\phi c\lambda & -s\lambda & -c\lambda s\phi \\
c\phi s\lambda & c\lambda & -s\lambda s\phi \\
s\phi & 0 & c\phi
\end{bmatrix}
\begin{pmatrix}
\dot{r} \\
\dot{r_\phi} \\
\dot{r_\lambda c\phi}
\end{pmatrix}
\] (45)

Equation (45) can be inverted to provide

\[
\begin{pmatrix}
\dot{r} \\
\dot{r_\lambda c\phi} \\
\dot{r_\phi}
\end{pmatrix} =
\begin{bmatrix}
c\phi c\lambda & s\lambda c\phi & c\phi \\
-s\lambda & c\lambda & 0 \\
-c\lambda s\phi & -s\lambda s\phi & c\phi
\end{bmatrix}
\begin{pmatrix}
\Delta_1 \\
\Delta_2 \\
\Delta_3
\end{pmatrix}
\] (46)

In the above equations, the "short-hand" notation \(c\phi \equiv \cos \phi\) and \(s\phi \equiv \sin \phi\) has been utilized.

Now, the components of \(\dot{\omega}_{Z/X'}\) in the \(\{z_i\}\) frame are

\[
\begin{pmatrix}
\omega_{1Z/X'} \\
\omega_{2Z/X'} \\
\omega_{3Z/X'}
\end{pmatrix} = [c(\beta'')] \begin{pmatrix}
\ddot{\phi} \\
-\dot{\lambda} s\phi \\
\dot{\lambda} c\phi
\end{pmatrix}
\] (47)

The quantities \(-\dot{\lambda} s\phi, \ddot{\phi}, \dot{\lambda} c\phi\) follow from equations (46) and (40) as follows:
\[ \dot{\lambda s \phi} = \frac{\Delta_3}{r} \left[ \frac{\Delta_2 s \lambda + \Delta_2 c \lambda}{\sqrt{\Delta_1^2 + \Delta_2^2}} \right] \]

\[ \dot{\phi} = \frac{1}{r} \left[ -\Delta_1 c \lambda s \phi - \Delta_2 s \lambda s \phi + \Delta_3 c \phi \right] \quad (48) \]

\[ \dot{\lambda c \phi} = \frac{1}{r} \left[ -\Delta_1 s \lambda + \Delta_2 c \lambda \right] \]

where

\[ s \lambda c \phi = \Delta_2 / r \]
\[ s \phi = \Delta_3 / r \]
\[ c \lambda c \phi = \Delta_1 / r \]

\[ c \phi = \sqrt{\Delta_1^2 + \Delta_2^2} / r \]
\[ s \lambda = \Delta_2 / r \]

The absolute angular velocity of the Moon can now be obtained by adding (39) and (47) to obtain in augmented form,

\[ \begin{pmatrix} 0 \\ \omega_1^i \\ \omega_2^i \\ \omega_3^i \end{pmatrix} = 2[\hat{\beta}^{'''}] 1 \{\hat{\beta}^{''''}\} + [c(\hat{\beta}^{''''})]_A \begin{pmatrix} 0 \\ -\lambda s \phi \\ \phi \\ \lambda c \phi \end{pmatrix} \quad (49) \]

Equation (49) can be solved for \{\hat{\beta}^{''''}\} and differentiated to obtain the second order equation for \{\beta^{''''}\}. Thus,
\begin{equation}
\{\dot{\beta}''\} = \frac{1}{2} [\dot{\beta}'''] \begin{pmatrix}
0 \\
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix} - [c(\beta''')]_A \begin{pmatrix}
0 \\
-\lambda s \phi \\
\phi \\
\lambda c \phi
\end{pmatrix}
\end{equation}

\begin{equation}
+ \frac{1}{2} [\dot{\beta}'''] \begin{pmatrix}
0 \\
\dot{\omega}_1 \\
\dot{\omega}_2 \\
\dot{\omega}_3
\end{pmatrix} - [c(\beta''')]_A \frac{d}{dt} \begin{pmatrix}
0 \\
-\lambda s \phi \\
\phi \\
\lambda c \phi
\end{pmatrix}
\end{equation}

\begin{equation}
- \frac{d}{dt} [c(\beta''')]_A \begin{pmatrix}
0 \\
-\lambda s \phi \\
\phi \\
\lambda c \phi
\end{pmatrix}
\end{equation}

In equation (50), \( [\dot{\beta}'''] \) is of the form given in equation (18); \( [c(\beta''')]_A \) is of the form given in equations (22) and (28); \( [\dot{\beta}'''] \) is of the form given in equation (33); \( \{0 \ \dot{\omega}_1 \ \dot{\omega}_2 \ \dot{\omega}_3\} \) are given in equations (34); \( \{0 \ \omega_1 \ \omega_2 \ \omega_3\} \) are given in equations (49); and the elements of \( [c(\beta''')]_A \) are

\begin{align*}
c_{11A} &= c_{12A} = c_{13A} = c_{14A} = c_{21A} = c_{31A} = c_{41A} = 0 \\
c_{22A} &= 2(\beta_0 \dot{\beta}_0 + \beta_1 \dot{\beta}_1 - \beta_2 \dot{\beta}_2 - \beta_3 \dot{\beta}_3) \\
c_{33A} &= 2(\beta_0 \dot{\beta}_0 - \beta_1 \dot{\beta}_1 + \beta_2 \dot{\beta}_2 - \beta_3 \dot{\beta}_3) \\
c_{44A} &= 2(\beta_0 \dot{\beta}_0 - \beta_1 \dot{\beta}_1 - \beta_2 \dot{\beta}_2 + \beta_3 \dot{\beta}_3) \\
c_{23A} &= 2(\beta_1 \dot{\beta}_2 + \beta_1 \dot{\beta}_2 + \beta_0 \dot{\beta}_3 + \beta_0 \dot{\beta}_3)
\end{align*}

(cont'd.)
\[ c_{24A} = 2(\beta_1 \beta_3 + \beta_1 \beta_3 - \beta_0 \beta_2 - \beta_0 \beta_2) \]
\[ c_{32A} = 2(\beta_1 \beta_2 + \beta_1 \beta_2 - \beta_0 \beta_3 - \beta_0 \beta_3) \]
\[ c_{34A} = 2(\beta_2 \beta_3 + \beta_2 \beta_3 + \beta_0 \beta_1 + \beta_0 \beta_1) \]
\[ c_{42A} = 2(\beta_1 \beta_3 + \beta_1 \beta_3 + \beta_0 \beta_2 + \beta_0 \beta_2) \]
\[ c_{43A} = 2(\beta_2 \beta_3 + \beta_2 \beta_3 - \beta_0 \beta_1 - \beta_0 \beta_1) \]

All \( \beta \)'s in the above equations have a triple-prime superscript.

The derivatives of the polar coordinates and their rates in equation (50) are

\[ \frac{d}{dt} (\lambda c) = \dot{\Lambda}_1 \left[ - \frac{c\ddot{\lambda}}{r} + \frac{r\dot{s}\lambda}{r^2} \right] + \dot{\Lambda}_2 \left[ - \frac{s\ddot{\lambda}}{r} - \frac{r\dot{c}\lambda}{r^2} \right] \]

\[ - \dot{\Lambda}_1 \frac{s\ddot{\lambda}}{r} + \dot{\Lambda}_2 \frac{c\ddot{\lambda}}{r} \]

\[ \frac{d}{dt} (\phi) = \dot{\Lambda}_1 \left[ \frac{\dot{\lambda} s\lambda\phi}{r} + \frac{\dot{r} c s\lambda\phi}{r^2} - \frac{\phi \dot{c} c\lambda \phi}{r} \right] + \dot{\Lambda}_2 \left[ - \frac{\dot{c} s\lambda\phi}{r} + \frac{\phi \dot{c} c\lambda \phi}{r^2} + \frac{\dot{r} s\lambda s\phi}{r^2} \right] + \dot{\Lambda}_3 \left[ - \frac{\dot{\phi} s\phi}{r} - \frac{\dot{\phi} c \phi}{r^2} \right] \]

\[ - \dot{\Lambda}_1 \frac{c\ddot{\lambda} s\phi}{r} - \dot{\Lambda}_2 \frac{s\ddot{\lambda} s\phi}{r} + \dot{\Lambda}_3 \frac{c\ddot{\phi}}{r} \]

\[ \frac{d}{dt} (\lambda s\phi) = \dot{\lambda} \frac{\ddot{\phi}}{c\phi} + s \frac{\ddot{\phi}}{c\phi} \{ \frac{d}{dt} (\lambda c\phi) \} \]
where

\[ \dot{\mathbf{a}}_i = \dot{\mathbf{x}}_{15} - \dot{\mathbf{x}}_{14} \]

\[ \ddot{\mathbf{a}}_i = \ddot{\mathbf{x}}_{15} - \ddot{\mathbf{x}}_{14} \]  \hspace{1cm} (55)

are available from the integration of equations (10).

Forces and Torques

The gravitational forces between a set of particles was given in equations (10). If the Earth and Moon are considered as rigid bodies then mutual gravitational forces and torques arise and must be modeled. Also torques exerted on the Earth and Moon due to a point mass Sun must also be considered. These are developed in this section.

**Mutual Gravitational Potential.** There are two approaches in the literature for deriving the mutual forces and torques. Approach (A) involves deriving a mutual gravitational potential and then finding the gradient of this potential with respect to the translational and rotational variables to give the forces and torques (refs. 14, 15). Approach (B) involves a direct integration of a differential force and torque over both bodies (ref. 16).

Approach A appears to be more easily developed when higher order gravity harmonics than the second are included for each body. Approach B is more concise than A for the case when only second order terms are retained for either one or the other body. Also, the effect of Earth oblateness on lunar torques can readily be derived using this approach.

For generality and ease of extension to higher orders, Approach A will be followed here. The concise results of Approach B are presented in Appendix A.

For the purposes of this report the Earth and Moon will be modeled as follows:
Earth (ref. 17, {y_i})

\[ c_{20} = -1.082637 \times 10^{-3} \]
\[ c_{21} = s_{21} = 0 \]
\[ c_{22} = 1.5362 \times 10^{-5} \]
\[ s_{22} = -8.8149 \times 10^{-7} \]

These values provide the following moments of inertia:

\[ A = .33912 \text{ Ma}^2 \]
\[ B = .33906 \text{ Ma}^2 \]
\[ C = .34017 \text{ Ma}^2 \]

if the dynamical flattening \[ H = (C - A)/c = 3.27293 \times 10^{-3} \]
is adopted from reference 20.

Moon (refs. 18, 19, \{z_i\})

\[ c_{20} = -2.0272 \times 10^{-4} \] (ref. 18)
\[ c_{21} = s_{21} = 0 \] (ref. 19)
\[ c_{22} = 2.221 \times 10^{-5} \] (ref. 18)
\[ s_{22} = 0.0 \] (ref. 19)
\[ c_{30} = 3.9 \times 10^{-6} \] (ref. 19)
\[ c_{31} = 28.6 \times 10^{-6} \] (ref. 19)
\[ c_{32} = 6.0 \times 10^{-6} \] (ref. 19)
\[ c_{33} = 2.7 \times 10^{-6} \] (ref. 19)
\[ s_{31} = 8.8 \times 10^{-6} \] (ref. 19)
\[ s_{32} = 1.8 \times 10^{-6} \] (ref. 19)
\[ s_{33} = -1.4 \times 10^{-6} \] (ref. 19)
\[ c_{40} = 23.3 \times 10^{-6} \] (ref. 19)
\[ C_{41} = 11.1 \times 10^{-6} \quad \text{(ref. 19)} \]
\[ C_{42} = -2.48 \times 10^{-6} \quad \text{(ref. 19)} \]
\[ C_{43} = -0.17 \times 10^{-6} \quad \text{(ref. 19)} \]
\[ C_{44} = -0.25 \times 10^{-6} \quad \text{(ref. 19)} \]
\[ S_{41} = -2.61 \times 10^{-6} \quad \text{(ref. 19)} \]
\[ S_{42} = -3.28 \times 10^{-6} \quad \text{(ref. 19)} \]
\[ S_{43} = -0.45 \times 10^{-6} \quad \text{(ref. 19)} \]
\[ S_{44} = 0.27 \times 10^{-6} \quad \text{(ref. 19)} \]

These values provide the following moments of inertia:

\[ A' = 0.391753 M'a'^2 \]
\[ B' = 0.391842 M'a'^2 \]
\[ C' = 0.392 M'a'^2 \quad \text{(ref. 18)} \]

if the values of \( \beta \) and \( \gamma \) are taken to be:

\[ \beta = 631.1 \times 10^{-6} \]
\[ \gamma = 226.8 \times 10^{-6} \]

as in reference 18.

Reference 15 provides the general form of the mutual potential between two arbitrarily shaped rigid bodies in the form

\[
U_{45}^I = \sum_{\ell=2}^{\infty} U_{\ell}^{I} = G' r^{-1} \left\{ \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{1}{r^n} P_{nm}(\sin \phi) \right\} \left\{ X_{nm} \cos m\lambda + Y_{nm} \sin m\lambda \right\}
\]

(56)
where the term $Gr^{-1}$ has been included in $U_{ij}^p$, and where $U_{11}^T = 0$ due to the choice of coordinate system. Here, $M$ is the mass of the Earth and $M'$ is the mass of the Moon. The $X_{nm}$ and $Y_{nm}$ are functions of $a^p$, $a'^q$, $c_{pr}$, $c'_{qs}$, $s_{pr}$, $s'_{qs}$ where $a$ and $a'$ are the mean equatorial radii for masses $M$ and $M'$ and the $c$'s and $s$'s represent the harmonic coefficients for both bodies. Reference 15 provides the lower order values of $X_{nm}$ and $Y_{nm}$ as follows:

\[
X_{2j} = a^2 c_{2j} + a'^2 c_{2j} \quad (j = 0, 1, 2)
\]

\[
Y_{2j} = a^2 s_{2j} + a'^2 s_{2j} \quad (j = 1, 2)
\]

\[
X_{40} = 6a^2 a'^2 (c_{20} c_{21}^1 - 2c_{21} c_{21}^1 - 2s_{21} s_{21}^1 + 2c_{22} c_{22}^1 + 2s_{22} s_{22}^1)
\]

\[
Y_{41} = 3a^2 a'^2 (c_{20} c_{21}^1 + c_{21} c_{21}^1 - c_{21} c_{21}^1 - c_{22} c_{22}^1 + s_{21} s_{21}^1 - s_{22} s_{22}^1)
\]

\[
Y_{41} = 3a^2 a'^2 (c_{20} s_{21}^1 + s_{21} c_{21}^1 - c_{21} s_{21}^1)
\]

\[
X_{42} = a^2 a'^2 (c_{20} c_{22}^1 + c_{22} c_{20}^1 + c_{21} c_{21}^1 - s_{21} s_{21}^1)
\]

\[
Y_{42} = a^2 a'^2 (c_{20} s_{22}^1 + s_{22} c_{20}^1 + c_{21} s_{21}^1 + s_{21} c_{21}^1)
\]

\[
X_{43} = \frac{1}{2} a^2 a'^2 (c_{21} c_{21}^1 + c_{22} c_{22}^1 - s_{21} s_{21}^1 - s_{22} s_{22}^1)
\]

\[
Y_{43} = \frac{1}{2} a^2 a'^2 (c_{21} s_{21}^1 + c_{22} s_{22}^1 + s_{21} c_{21}^1 + s_{22} c_{22}^1)
\]

\[
X_{44} = \frac{1}{2} a^2 a'^2 (c_{22} c_{22}^1 - s_{22} s_{22}^1)
\]

\[
Y_{44} = \frac{1}{2} a^2 a'^2 (c_{22} s_{22}^1 + s_{22} c_{22}^1)
\]
The coordinate systems implicit in equation (56) is illustrated in figure 3.

If \( \{y_i\} \) is chosen as the primary reference, then \( c_{pr}, s_{pr} \) are the harmonic coefficients of the Earth referenced to \( \{y_i\} \) and \( c'_{qs}, s'_{qs} \) are those of the Moon referenced to \( \{y_i\} \). Likewise, \( r, \lambda, \phi \) and \( r', \lambda', \phi' \) are the spherical polar coordinates of the lunar mass center with respect to \( \{y_i\} \). If the \( \{z_i\} \) axes are chosen as the primary reference, then the above quantities are referred to \( \{z_i\} \) and \( \{z_i\}_T \).

The relative orientation of the above axis systems can be expressed as follows:

\[
\{y_i\} = \begin{bmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{bmatrix}, \quad \{z_i\} = [\lambda]\{z_i\}
\]

with a more precise definition of the \( \alpha's, \beta's, \) and \( \gamma's \) to be given later.

**Force on Moon Due to Earth.** In terms of spherical polar coordinates \( r, \lambda, \phi \) locating the Moon with respect to \( \{z_i\}_T \), the vector force may be written as

\[
\vec{F}_5 = m_5 \left[ \frac{\partial U_5}{\partial r} \frac{i}{r} + \frac{1}{r \cos \phi} \frac{\partial U_5}{\partial \lambda} i + \frac{1}{r} \frac{\partial U_5}{\partial \phi} i \phi \right].
\]

The general mutual potential may now be specialized to the problem at hand as follows:
Figure 3. Coordinate systems for mutual gravitational potential.
\[ U_2 = \frac{1}{\text{Gr}^{-1}} \left[ \left( \frac{a}{r} \right)^2 \left\{ c_20P_{20}(s\phi) + P_{21}(c_{21}c\lambda + s_{21}s\lambda) \\
+ P_{22}(c_{22}c_{22} + s_{22}s_{22}) \right\} + \left( \frac{a'}{r} \right)^2 \left\{ c_{20}'P_{20} \\
+ P_{22}(c_{22}'c_{22} + s_{22}'s_{22}) \right\} \right] \] (60)

\[ U_3 = \frac{1}{\text{Gr}^{-1}} \left[ \left( \frac{a'}{r} \right)^3 \left\{ c_{30}'P_{30} + P_{31}(c_{31}'c\lambda + s_{31}'s\lambda), \\
+ P_{32}(c_{32}'c_{22} + s_{32}'s_{22}) + P_{33}(c_{33}'c_{33} + s_{33}'s_{33}) \right\} \right] \] (61)

\[ U_4 = \frac{1}{\text{Gr}^{-1}} \left[ \left( \frac{a'}{r} \right)^4 \left\{ c_{40}'P_{40} + P_{41}(c_{41}'c\lambda + s_{41}'s\lambda) \\
+ P_{42}(c_{42}'c_{22} + s_{42}'s_{22}) + P_{43}(c_{43}'c_{33} + s_{43}'s_{33}) \\
+ P_{44}(c_{44}'c_{44} + s_{44}'s_{44}) \right\} + (a^2a'^2/r^4) \left\{ 6P_{40}(2c_{22}c_{22}' \\
+ 2s_{22}s_{22}' \right) + 3P_{41}(c_{21}c_{20}' - c_{21}s_{22}' + s_{21}s_{22}'c\lambda \\
+ 3P_{41}(s_{21}c_{20}' - c_{21}s_{22}'c\lambda + s_{21}c_{22}'), \right\} \right] \] (62)

Now, the partial derivatives of \( U_4 \) can be calculated and are

\[ \frac{\partial U_2}{\partial r} = -3G a^2r^{-4} \left\{ c_{20}P_{20} + P_{21}(c_{21}c\lambda + s_{21}s\lambda) \\
+ P_{22}(c_{22}c_{22} + s_{22}s_{22}) \right\} - 3G a'^2r^{-4} \left\{ c_{20}'P_{20} \right\} \] (63)

\[ \frac{\partial U_2}{\partial \lambda} = G a^2r^{-3} \left\{ c_{20}P_{20} + P_{21}(-c_{21}s\lambda + s_{21}c\lambda) \\
+ 2P_{22}(-c_{22}s\lambda + s_{22}c\lambda) \right\} + G a'^2r^{-3} \left\{ c_{20}'P_{20} \right\} \] (64)
\[ \partial U_2/\partial \phi = G a^2 r^{-3} \left\{ c_{20} P_{20} + P_{21} \left( c_{21} \lambda + s_{21} \lambda \right) + P_{22} \left( c_{22} c_{20} + s_{22} s_{20} \right) \right\} + G a^2 r^{-3} \left\{ c_{20} P_{20} + P_{22} \left( c_{12} c_{20} + s_{12} s_{20} \right) \right\} \]  

(65)

\[ \partial U_3/\partial r = -4 G a^3 r^{-5} \left\{ c_{30} P_{30} + P_{31} \left( c_{31} \lambda + s_{31} \lambda \right) + P_{32} \left( c_{32} c_{20} + s_{32} s_{20} \right) + P_{33} \left( c_{33} c_{30} + s_{33} s_{30} \right) \right\} \]  

(66)

\[ \partial U_3/\partial \lambda = G a^3 r^{-4} \left\{ c_{30} P_{30} + P_{31} \left( c_{31} \lambda + s_{31} \lambda \right) + 2P_{32} \left( c_{32} c_{20} + s_{32} s_{20} \right) + 3P_{33} \left( c_{33} c_{30} + s_{33} s_{30} \right) \right\} \]  

(67)

\[ \partial U_3/\partial \phi = G a^3 r^{-4} \left\{ c_{30} P_{30} + P_{31} \left( c_{31} \lambda + s_{31} \lambda \right) + P_{32} \left( c_{32} c_{20} + s_{32} s_{20} \right) + P_{33} \left( c_{33} c_{30} + s_{33} s_{30} \right) \right\} \]  

(68)

\[ \partial U_4/\partial r = -5G a^4 r^{-6} \left\{ c_{40} P_{40} + P_{41} \left( c_{41} \lambda + s_{41} \lambda \right) + P_{42} \left( c_{42} c_{20} + s_{42} s_{20} \right) + P_{43} \left( c_{43} c_{30} + s_{43} s_{30} \right) \right\} - 5GMM'a^2 a' r^{-6} \left\{ 6P_{40} \tilde{x}_{40} + 3P_{41} \left( \tilde{x}_{41} \lambda + \tilde{y}_{41} \lambda \right) + P_{42} \left( \tilde{x}_{42} c_{20} + \tilde{y}_{42} s_{20} \right) + \left( P_{43} / 2 \right) \left( \tilde{x}_{43} c_{30} + \tilde{y}_{43} s_{30} \right) + \left( P_{44} / 2 \right) \left( \tilde{x}_{44} c_{40} + \tilde{y}_{44} s_{40} \right) \right\} \]  

(69)

\[ \partial U_4/\partial \lambda = G a^4 r^{-5} \left\{ c_{40} P_{40} + P_{41} \left( c_{41} \lambda + s_{41} \lambda \right) + 2P_{42} \left( c_{42} c_{20} + s_{42} s_{20} \right) + 3P_{43} \left( c_{43} c_{30} + s_{43} c_{30} \right) + 4P_{44} \left( c_{44} c_{40} + s_{44} s_{40} \right) \right\} + G a^2 a' r^{-5} \left\{ 6P_{40} \tilde{x}_{40} + 3P_{41} \left( \tilde{x}_{41} \lambda + \tilde{y}_{41} \lambda \right) + 2P_{42} \left( \tilde{x}_{42} c_{20} + \tilde{y}_{42} c_{20} \right) + \left( 3P_{43} / 2 \right) \left( \tilde{x}_{43} s_{30} + \tilde{y}_{43} c_{30} \right) + 2P_{44} \left( \tilde{x}_{44} c_{40} + \tilde{y}_{44} c_{40} \right) \right\} \]  

(70)
\[ \frac{\partial U_4}{\partial \phi} = G \ r^{-5} \left\{ p_{40} \phi (a_0^4 c_0^4 + 6a_0^2 a^2 \tilde{x}_4^0) \\
+ p_{41} \phi \left[ (a_0^4 c_1^4 + 3a_0^2 a^2 \tilde{x}_4^1) c_1 \right. \\
+ \left. (a_0^4 s_1^4 + 3a_0^2 a^2 \tilde{y}_4^1) s_1 \right] \\
+ p_{42} \phi \left[ (a_0^4 c_2^4 + a_0^2 a^2 \tilde{x}_4^2) c_2 \right. \\
+ \left. (a_0^4 s_2^4 + a_0^2 a^2 \tilde{y}_4^2) s_2 \right] \\
+ p_{43} \phi \left[ (a_0^4 c_3^4 + \frac{1}{2} \tilde{x}_4^3) c_3 \right. \\
+ \left. (a_0^4 s_3^4 + \frac{1}{2} \tilde{y}_4^3) s_3 \right] \\
+ p_{44} \phi \left[ (a_0^4 c_4^4 + \frac{1}{2} \tilde{x}_4^4) c_4 \right. \\
+ \left. (a_0^4 s_4^4 + \frac{1}{2} \tilde{y}_4^4) s_4 \right] \right\} . \] (71)

In the above equations,

\[ \tilde{x}_4^0 = 6a_0^2 a^2 [2c_2^2 c_2^2 + 2s_2^2 s_2^2] \]

\[ \tilde{x}_4^1 = 3a_0^2 a^2 [c_2^1 c_0^1 - c_2^1 c_2^1 + s_2^1 s_2^1] \]

\[ \tilde{y}_4^1 = 3a_0^2 a^2 [s_2^1 c_2^1 - c_2^1 s_2^1 + s_2^1 c_2^1] \]

\[ \tilde{x}_4^2 = a_0^2 a^2 [c_2^0 c_2^2 + c_2^2 c_2^0] \]

\[ \tilde{y}_4^2 = a_0^2 a^2 [c_2^0 s_2^2 + s_2^2 c_2^0] \]

\[ \tilde{x}_4^3 = \frac{1}{2} a_0^2 a^2 [c_2^1 c_2^2 - s_2^1 s_2^2] \]

\[ \tilde{y}_4^3 = \frac{1}{2} a_0^2 a^2 [c_2^1 s_2^2 + s_2^1 c_2^2] \]

\[ \tilde{x}_4^4 = \frac{1}{2} a_0^2 a^2 [c_2^2 c_2^2 - s_2^2 s_2^2] \]

\[ \tilde{y}_4^4 = \frac{1}{2} a_0^2 a^2 [c_2^2 s_2^2 + s_2^2 c_2^2] \]
and

\[ P_{20\phi} = 3s\phi c\phi \]

\[ P_{21\phi} = 3c(2\phi) \]

\[ P_{22\phi} = -6s\phi c\phi \]

\[ P_{30\phi} = \frac{3}{2} c\phi (5s^2\phi - 1) \]

\[ P_{31\phi} = \frac{3}{2} s\phi + \frac{15}{2} s\phi (2c^2\phi - s^2\phi) \]

\[ P_{32\phi} = 15c\phi (c^2\phi - 2s^2\phi) \]

\[ P_{33\phi} = -45c^2\phi s\phi \]

\[ P_{40\phi} = \frac{1}{8} (140s^3\phi - 60s\phi)c\phi \]  \hspace{1cm} (73)

\[ P_{41\phi} = \frac{5}{2} \left[ (c^2\phi - s^2\phi)(7s^2\phi - 3) + 14s^2\phi c^2\phi \right] \]

\[ P_{42\phi} = \frac{15}{2} \left[ 14s\phi c^3\phi - 2s\phi c\phi (7s^2\phi - 1) \right] \]

\[ P_{43\phi} = 105[c^4\phi - 3s^2\phi c^2\phi] \]

\[ P_{44\phi} = -420c^3\phi s\phi \]

Since the harmonic coefficients \( c_{ij} \) and \( s_{ij} \) are referred to a coordinate system \( \{z_1^T\} \) that rotates with respect to the earth, they are functions of time. These functions may be evaluated by noting the definitions of the \( c_{ij} \) and \( s_{ij} \) (ref. 21) in terms of the inertia integrals, viz.
\[ c_{20} = \frac{1}{a^2 M} \left[ \frac{I_{11} + I_{22}}{2} - I_{33} \right] \]

\[ c_{21} = \frac{1}{a^2 M} I_{13} \]  

\[ s_{21} = \frac{1}{a^2 M} I_{32} \]  

\[ c_{22} = \frac{1}{4a^2 M} \left[ I_{22} - I_{11} \right] \]

\[ s_{22} = \frac{1}{2a^2 M} I_{12} \]  

and by noting the transformation laws of the inertia matrix, viz.

\[ \{z_i\} = [\ell]^T \{y_i\} \]

\[ \begin{bmatrix}
  I_{z_1 z_1} & I_{z_1 z_2} & I_{z_1 z_3} \\
  I_{z_2 z_1} & I_{z_2 z_2} & I_{z_2 z_3} \\
  I_{z_3 z_1} & I_{z_3 z_2} & I_{z_3 z_3}
\end{bmatrix}
\begin{bmatrix}
  A \\
  0 \\
  0 
\end{bmatrix}
\begin{bmatrix}
  0 \\
  B \\
  0 
\end{bmatrix}
[\ell]. \]  

\[ (l) \)  

The above equations provide the desired functions as follows:

\[ a^2 M c_{20} = \frac{1}{2} \left[ A + B + C - 3(\alpha''^2 A + \beta''^2 B + \gamma''^2 C) \right] \]

\[ a^2 M c_{21} = \alpha'' A + \beta'' B + \gamma'' C \]

\[ a^2 M s_{21} = \alpha'\alpha'' A + \beta'\beta'' B + \gamma'\gamma'' C \]  

\[ 4a^2 M c_{22} = A(\alpha'^2 - \alpha^2) + B(\beta'^2 - \beta^2) + C(\gamma'^2 - \gamma^2) \]

\[ 2a^2 M s_{22} = \alpha\alpha' A + \beta\beta' B + \gamma\gamma' C \]  

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Finally the components of the force on the Moon due to the Earth must be found along the \{x'_i\} axes. Accordingly,

\[
P_{sX'_i} = m_5 \left[ \frac{\partial U_{45}^I}{\partial r} \left( \hat{r}_x \cdot \hat{r}'_i \right) + \frac{1}{rc\phi} \frac{\partial U_{45}^I}{\partial \lambda} \left( \hat{r}_{\lambda} \cdot \hat{r}'_i \right) 
+ \frac{1}{r} \frac{\partial U_{55}^I}{\partial \phi} \left( \hat{r}_\phi \cdot \hat{r}'_i \right) \right].
\]

(78)

**Torque on Moon Due to Earth.** This torque can be derived by first expressing the mutual gravitational potential \(U_{45}^I\) in terms of the direction cosines relating the rotational orientation of the moon to the earth; and then by calculating the moment components as follows (ref. 22):

\[
\frac{1}{MM^r} M_{z1} = \alpha'''' U_{\alpha'}^I - \alpha'' U_{\alpha'}^I + \beta'''' U_{\beta'}^I - \beta'' U_{\beta'}^I 
+ \gamma'''' U_{\gamma'}^I - \gamma'' U_{\gamma'}^I.
\]

(79)

\[
\frac{1}{MM^r} M_{z2} = \alpha U_{\alpha}^I - \alpha'' U_{\alpha}^I + \beta U_{\beta}^I - \beta'' U_{\beta}^I 
+ \gamma U_{\gamma}^I - \gamma'' U_{\gamma}^I.
\]

\[
\frac{1}{MM^r} M_{z3} = \alpha' U_{\alpha}^I - \alpha U_{\alpha}^I + \beta' U_{\beta}^I - \beta U_{\beta}^I 
+ \gamma' U_{\gamma}^I - \gamma U_{\gamma}^I.
\]

A derivation of the above relations is presented in Appendix C.

In order to derive the torques, the term \(U_2^I\) and the second order coupling terms in \(U_4^I\) [see eqs. (81) and (82)] will be
treated together. Finally the term $U_3^1$ and the remaining terms in $U_4^\Theta C$ will be treated [see eqs. (86) and (87)].

The reference axes for the second order and coupling terms are $\{y_1\}$. Thus the $c_{ij}'$ and $s_{ij}'$ are functions of the orientation angles. These functions are

$$c_{20}' = \frac{1}{a''^2 M'} \left[ \frac{I_{y_1 y_1} + I_{y_2 y_2}'}{2} - I_{y_3 y_3}' \right]$$

$$= \frac{1}{a''^2 M'} \left[ \frac{A + B + C}{2} - \frac{3}{2} (A'\gamma^2 + B'\gamma'^2 + C'\gamma''^2) \right]$$

$$c_{21}' = \frac{1}{a''^2 M'} I_{y_1 y_3}'$$

$$= \frac{1}{a''^2 M'} \left[ \alpha yA' + \alpha'\gamma'B' + \alpha''\gamma''C' \right]$$

$$s_{21}' = \frac{1}{a''^2 M'} I_{y_3 y_2}'$$

$$= \frac{1}{a''^2 M'} \left[ \gamma\beta A' + \gamma'\beta'B' + \gamma''\beta''C' \right]$$

$$c_{22}' = \frac{1}{4a''^2 M'} \left[ A'(\beta^2 - \alpha^2) + B'(\beta'^2 - \alpha'^2) \right.$$

$$\left. + C'(\beta''^2 - \alpha''^2) \right]$$

$$s_{22}' = \frac{1}{4a''^2 M'} \left[ \alpha\beta A' + \alpha'\beta'B' + \alpha''\beta''C' \right]$$
The potential $U^I_2 + U^I_{ui}$, coupling thus assumes the form:

$$U^I_2 = G \frac{r^{-1}}{r^5} \left[ \left(\frac{a}{r}\right)^2 \left\{ c_{20} P_{20} + c_{22} P_{22} c_{22} \right\} + \left(\frac{a'}{r}\right)^2 \left\{ c_{20} P_{20} + P_{21} (c_{21} c_{21} + s_{21} s_{21}) + P_{22} (c_{22} c_{22} + s_{22} s_{22}) \right\} \right]$$

$$U^I_{ui}, \text{coupling} = G a^2 a'^2 r^{-5} \left[ P_{40} \left\{ 6 c_{20} c_{21} + 12 c_{22} c_{22} \right\} + P_{41} \left\{ -3 c_{22} c_{21} c_{21} + 3 c_{20} s_{21} + 3 c_{22} s_{21} \right\} \right] + P_{42} \left\{ c_{20} c_{22} + c_{22} c_{20} \right\} c_{22} + P_{43} \left\{ \frac{1}{2} c_{22} c_{21} \right\} c_{3} + P_{44} \left\{ \frac{1}{2} c_{22} c_{22} \right\} c_{4} + \frac{P_{22} s_{22}}{2} (a' \beta' + \beta' a'') \right]$$

Now, using equations (79) to (81) the torque components due to the terms in $U^I_2$ are:

$$M_{z1} = G M r^{-3} (B' - C') \left[ -3 P_{20} \gamma \gamma'' + P_{21} c \lambda (\alpha' \gamma' + \alpha' \gamma') + P_{21} s \lambda (\beta' \gamma + \gamma' \beta') + \frac{P_{22} c \lambda}{2} (\beta' \beta' - \alpha' \alpha'') \right]$$

$$M_{z2} = G M r^{-3} \left[ (C' - A') 3 P_{20} \gamma \gamma'' + P_{21} c \lambda (\alpha' \gamma' + \gamma' \alpha'') + P_{21} s \lambda (\beta \gamma' + \gamma \beta') + \frac{P_{22} c \lambda}{2} (\beta \beta' - \alpha \alpha'') \right] + \frac{P_{22} s \lambda}{2} (a \beta'' + \beta a'') \right]$$
\[
M_{23} = GMr^{-3} (B' - A') \left[ 3P_{20} \gamma \gamma + P_{21} c \lambda (\gamma a' + a \gamma')
- P_{21} s \lambda (\gamma b' + b \gamma') + \frac{P_{22} c^2 \lambda}{2} (a a' - b b') \right]
\]

The above components are for an arbitrary orientation of \( \{y_i\} \) with respect to \( \{z_i\} \). In the derivations presented in the literature, the Earth is treated as a particle so that the relative orientation of \( \{y_i\} \) and \( \{z_i\} \) is immaterial. To recover those results a special orientation of the \( \{y_i\} \) may be taken. If \( y_i \) is taken to be pointing at the Moon then \( \lambda = c \phi c \lambda = \frac{Y_1}{x} = 1 \), \( m = c \phi s \lambda = \frac{Y_2}{x} = 0 \), and \( n = s \phi = \frac{Y_3}{x} = 0 \). Also, \( \lambda' = \alpha, m' = \alpha', \) and \( n' = \alpha'' \), where \( \lambda', m', n' \) are the direction cosines of the Earth with respect to \( \{z_i\} \). This reduces equation (83) to a more recognizable form, viz.

\[
M_{21} = 3GMr^{-3} (C' - B') m' n'
\]

\[
M_{22} = 3GMr^{-3} (A' - C') \lambda' n' \tag{84}
\]

\[
M_{23} = 3GM (B' - A') \lambda' m'.
\]

Actually, equations (83) and (84) are identical as algebraic manipulation will show.

The coupling terms in \( U_{i4} \) are handled similarly. Thus,
$$\begin{align*}
M_{z_1} &= \text{GMr}^{-5a^2}(B' - C') \left[ 6P_{40} \left\{ c_{20}(\alpha'y' + y'a') \right. \right. \\
&\quad + c_{22}(\beta'b'' - \alpha'a'') \right\} + 3P_{41} \left\{ -c_{22}(\alpha'y' + y'a')c\lambda \right. \\
&\quad + s\lambda(\gamma'b'' + y''\beta') \left( c_{20} + c_{22} \right) \right\} \\
&\quad + \frac{P_{42}}{2} c_{20} \left\{ c_{20}(\alpha'y' - \beta'b'') + s2\lambda(\alpha'y' + \alpha'y'') \right\} \\
&\quad - 3P_{42} c_{22} \left\{ c_{20}(\alpha'y' + \alpha'y'') \right\} \\
&\quad + s3\lambda(\beta'b' + \beta'\gamma') \left. \right\} - \frac{P_{44}}{2} c_{22} \left\{ c_{44}(\beta'b'' - \alpha'a'') \right. \\
&\quad + s4\lambda(\alpha'y' + \beta'\alpha') \left. \right\} \right]
\end{align*}$$

$$\begin{align*}
M_{z_2} &= \text{GMr}^{-5a^2}(C' - A') \left[ 6P_{40} \left\{ c_{20}(\gamma''y + \gamma'a'') \right. \right. \\
&\quad + c_{22}(\beta'b'' - \alpha'a'') \right\} + 3P_{41} \left\{ -c_{22}(\alpha'y' + \gamma'a') \right. \\
&\quad + s\lambda(c_{20} + c_{22})(\gamma'b'' + \gamma'b') \right\} + \frac{P_{43}}{2} c_{20} \left\{ (\beta'b'' - \alpha'a'') \right. \\
&\quad - \gamma''a' + (\alpha'y' + \beta'\alpha'') \left. \right\} - 3P_{42} c_{22} \left\{ c_{22}(2\lambda\gamma'y') \right. \\
&\quad + \frac{P_{43}}{2} c_{22} \left\{ (c_{22} + c_{22})(\gamma'y' + \gamma'\beta') \right. \\
&\quad + \frac{P_{44}}{4} c_{22} \left\{ (\beta'b'' - \alpha'a'') \right. \\
&\quad + c_{44}(\alpha'y' + \gamma'a') \left. \right\} \right]
\end{align*}$$

$$\begin{align*}
M_{z_3} &= \text{GMr}^{-5a^2}(B' - A') \left[ 6P_{40} \left\{ -c_{20}(\alpha'y' + y'a') \right. \right. \\
&\quad + c_{22}(\alpha'a' - \beta'b') \right\} + 3P_{41} \left\{ c_{22}c\lambda(\alpha'y' + y'a') \right. \\
&\quad - (c_{20} + c_{22})s\lambda(\gamma'y' + y'\beta') \right\} + \frac{P_{42}}{2} c_{22} \left\{ c_{20}(\alpha'y' \right. \\
&\quad - \beta'b') - s2\lambda(\alpha'y' + \beta'\alpha') \right\} - 3P_{42} c_{22} \left\{ c_{22}(2\lambda\gamma'y') \right. \\
&\quad + \frac{P_{43}}{2} c_{22} \left\{ -(\alpha'y' + y'\alpha)c_{33} - (\beta'y' + y'\beta)s3\lambda \right. \\
&\quad + \frac{P_{44}}{4} c_{22} \left\{ (\alpha'y' - \beta'b') \right. \\
&\quad c_{44} - s4\lambda(\alpha'b' + \beta'b') \left. \right\} \right]
\end{align*}$$
Finally, the torque components due to the $U_3^I$ potential terms and the $U_4^I$ potential terms not already treated, viz.

$$
U_3^I = \frac{G}{r} \left[ \left(\frac{a}{r}\right)^3 \left\{ P_{30}(\phi) C_{30} + P_{31}(\phi) (C_{31} \cos \lambda + s_{31} \sin \lambda)
+ P_{32}(\phi) (C_{32} \cos 2\lambda + s_{32} \sin 2\lambda)
+ P_{33}(\phi) (C_{33} \cos 3\lambda + s_{33} \sin 3\lambda) \right\} \right]
$$

$$
U_4^I = \frac{G}{r} \left[ \left(\frac{a}{r}\right)^4 \left\{ P_{40}(\phi) C_{40} + P_{41}(\phi) (C_{41} \cos \lambda + s_{41} \sin \lambda)
+ P_{42}(\phi) (C_{42} \cos 2\lambda + s_{42} \sin 2\lambda)
+ P_{43}(\phi) (C_{43} \cos 3\lambda + s_{43} \sin 3\lambda)
+ P_{44}(\phi) (C_{44} \cos 4\lambda + s_{44} \sin 4\lambda) \right\} \right]
$$

will be derived.

The general approach for calculating torques used earlier will now be modified. The Earth may be considered to be a particle in calculating the torques on the Moon arising from third and fourth degree terms in the lunar potential. Reference 23 provides a simple approach based on this fact that will be followed here.

Consider that the reference axes in equations (86) and (87) are the $\{z_i\}$ axes. Thus the $c_{ij}'s$ and $s_{ij}'s$ are constant and $r, \phi, \lambda$ are the spherical polar coordinates of the center of mass of the Earth with respect to the axes $\{z_i\}$.

For a point mass Earth, the torque on the finite Moon due to the Earth is equal and opposite to the torque on the Earth due to the Moon. Thus, reference 23 finds the torques to be
\[ M_{Z_{1}} = \frac{GM}{A} \begin{bmatrix} z_{3} \frac{\partial (U_{3} + U_{4})}{\partial z_{2}} - z_{2} \frac{\partial (U_{3} + U_{4})}{\partial z_{3}} \end{bmatrix} \]

\[ M_{Z_{2}} = \frac{GM}{B} \begin{bmatrix} z_{1} \frac{\partial (U_{3} + U_{4})}{\partial z_{3}} - z_{3} \frac{\partial (U_{3} + U_{4})}{\partial z_{1}} \end{bmatrix} \]

\[ M_{Z_{3}} = \frac{GM}{C} \begin{bmatrix} z_{2} \frac{\partial (U_{3} + U_{4})}{\partial z_{1}} - z_{1} \frac{\partial (U_{3} + U_{4})}{\partial z_{2}} \end{bmatrix} \]

where

\[
\begin{pmatrix}
  z_{1} \\
  z_{2} \\
  z_{3}
\end{pmatrix} = \begin{pmatrix}
  \cos \phi \cos \lambda \\
  \cos \phi \cos \lambda \\
  \sin \phi
\end{pmatrix} \equiv \begin{pmatrix}
  l_{1} \\
  l_{2} \\
  l_{3}
\end{pmatrix}
\]

Reference 23 then provides

\[
M_{Z_{1}} = \frac{3}{2} \frac{GMM'a'^{3}x^{-4}}{A} \left[ \begin{array}{c}
\lambda_{3}(1 - 5\lambda_{3}^{2})c_{30} - 10\lambda_{1}\lambda_{2}\lambda_{3}c_{31} \\
- 10\lambda_{2}(1 + \lambda_{3}^{2} - 2\lambda_{2}^{2})c_{32} - 60\lambda_{1}\lambda_{2}\lambda_{3}c_{33} \\
- \lambda_{3}(1 - 5\lambda_{3}^{2} + 10\lambda_{2}^{2})s_{31} + 20\lambda_{1}(\lambda_{3}^{2} - \lambda_{2}^{2})s_{32} \\
+ 30\lambda_{3}(\lambda_{1}^{2} - \lambda_{2}^{2})s_{33}
\end{array} \right]
\]
\[ M_{z_2} = \frac{3}{2} \frac{GM\alpha_3 r^{-4}}{B_1} \left[ -\ell_1 (1 - 5\ell_3^2) c_{30} + \ell_3 (1 - 5\ell_3^2 + 10\ell_1^2) c_{31} - 10\ell_1 (1 + \ell_3^2 - 2\ell_1^2) c_{32} - 30\ell_3 (\ell_1^2 - \ell_2^2) c_{33} + 10\ell_1 \ell_2 \ell_3 s_{31} + 20\ell_2 (\ell_1^2 - \ell_2^2) s_{32} - 60\ell_1 \ell_2 \ell_3 s_{33} + \frac{3GM\alpha_4 r^{-5}}{B_1} \right] \frac{5}{2} \ell_1^2 c_{41} + 3\ell_1^4 c_{43} \]

(90) (concl’d.)

\[ M_{z_3} = \frac{3}{2} \frac{GM\alpha_3 r^{-4}}{C_1} \left[ -\ell_2 (1 - 5\ell_3^2) c_{31}^1 + 40\ell_1 \ell_2 \ell_3 c_{32} + 30\ell_2 (3\ell_1^2 - \ell_2^2) c_{33} + \ell_1 (1 - 5\ell_3^2) s_{31} - 20\ell_3 (\ell_1^2 - \ell_2^2) s_{32} - 30\ell_1 (\ell_1^2 - 3\ell_2^2) s_{33} + \frac{3GM\alpha_4 r^{-5}}{C_1} \right] \left[ 5\ell_1^2 s_{42} - 140\ell_1^4 s_{44} \right] \]

In summary, the force components in the inertial frame \( \{X_i^1\} \) may be found from equations (78) and the preceding definitions found in equations (59) to (78). The torque components on the Moon due to the Earth resolved along the \( \{z_i\} \) axes are the sum of the torques in equations (84), (85), and (90).

**Torque on Moon Due to Sun.** Since the Sun is treated as a particle, the torque exerted on the Moon is of the same form as equations (84), viz.,
\[ M_{z1} = 3GM_1r_1^{-3}(C' - B') m_{\odot}^i n_{\odot}^i \]
\[ M_{z2} = 3GM_1r_1^{-3}(A' - C') \ell_{\odot}^i n_{\odot}^i \]
\[ M_{z3} = 3GM_1r_1^{-3}(B' - A') \ell_{\odot}^i m_{\odot}^i \]

where \( \ell_{\odot}^i, m_{\odot}^i, n_{\odot}^i \), are the direction cosines of the Sun with respect to \( \{z_1\} \).

**Force on Earth Due to Moon.** This force is derived in a manner analogous to the force on the Moon due to the Earth presented earlier. The reference axes are chosen to be the \( \{y_1\} \) set so that the \( c_{ij} \) and \( s_{ij} \) are constants and the \( c_{ij}' \) and \( s_{ij}' \) vary. An assumption made in the following is that only second degree terms in the lunar potential are important to the motion of the Earth.

The mutual potential then assumes the form

\[
U_{b5}^T = Gr^{-1} \left[ \frac{(a/r')^2}{2} \left\{ c_{20}P_{20}(s\phi) + P_{22}(s\phi)(c_{22}c2\lambda + s_{22}s2\lambda) \right\} \right.
+ \frac{(a'/r)^2}{2} \left\{ c_{20}'P_{20}(s\phi) + P_{21}(s\phi)(c_{21}c2\lambda + s_{21}s2\lambda) \right\} \]

where the \( c_{ij}' \) and \( s_{ij}' \) are functions of the mutual orientation angles as follows:

\[
c_{20}' = \frac{1}{a_{12}^{-2}M'} \left[ \frac{A + B + C}{2} - \frac{3}{2} (\gamma A + \gamma' B + \gamma'' C) \right]
\]
\[
c_{21}' = \frac{1}{a_{12}^{-2}M'} \left[ a\gamma A + a'\gamma' B + a''\gamma'' C \right]
\]
\[
s_{21}' = \frac{1}{a_{12}^{-2}M'} \left[ \gamma B + \gamma' B' + \gamma'' B'' \right]
\]

(93) (cont'd.)
\[ c_{22}^1 = \frac{1}{4a'^2M} \left[ (\beta^2 - \alpha^2)A + (\beta'^2 - \alpha'^2)B + (\beta''^2 - \alpha''^2)C \right] \]

\[ s_{22}^1 = \frac{1}{2a'^2M} \left[ \alpha\beta A + \alpha'\beta'B + \alpha''\beta''C \right] \]

The force acting on the Earth projected on the inertial axes \( \{X_i'\} \) is

\[ F_{4X_i}^i = m_4 \left[ \frac{\partial U_4^5}{\partial r} (\hat{r} \cdot \hat{r}_i') + \frac{1}{rc\phi} \frac{\partial U_4^5}{\partial \lambda} (\hat{\phi} \cdot \hat{r}_i') \right. \]

\[ + \left. \frac{1}{r} \frac{\partial U_4^5}{\partial \phi} (\hat{\phi} \cdot \hat{r}_i') \right] \]

where \( r, \phi, \lambda \) and the associated unit vectors are the spherical polar coordinates of the Earth with respect to \( \{x_i^T\} \).

The appropriate partial derivatives are

\[ \frac{\partial U_4^5}{\partial r} = -3Ga^2r^{-4} \left\{ c_{20}P_{20}(s\phi) + P_{22}(s\phi)(c_{22}c2\lambda + s_{22}s2\lambda) \right\} -3Ga'^2r^{-4} \left\{ c_{10}'P_{20}(s\phi) \right\} \]

\[ + P_{21}(s\phi)(c_{21}c\lambda + s_{21}s\lambda) \]

\[ + P_{22}(s\phi)(c_{22}c2\lambda + s_{22}s2\lambda) \] \( \quad \) \( (95) \)

\[ \frac{\partial U_4^5}{\partial \lambda} = Ga^2r^{-3} \left\{ c_{20}P_{20} + 2P_{22}(-c_{22}s2\lambda + s_{22}c2\lambda) \right\} + Ga'^3r^{-3} \left\{ c_{20}'P_{20} \right\} \]

\[ + P_{21}(-c_{21}s\lambda + s_{21}c\lambda) \]

\[ + 2P_{22}(-c_{22}'s2\lambda + s_{22}'c2\lambda) \] \( \quad \) \( (96) \)
\[ \frac{\partial U_{45}}{\partial \phi} = \text{Ga}^2 r^{-3} \left\{ c_22 \phi P_{20} + P_{22}(c_{22} c_2 \lambda + s_2 s_2 \lambda) \right\} \\
+ \text{Ga}^2 r^{-3} \left\{ c_11 \phi P_{20} + P_{21}(c_{21} c_2 \lambda + s_1 s_2 \lambda) \right\} \\
+ P_{22}(c_2 c_2 \lambda + s_2 s_2 \lambda) \]  

(97)

Torque on Earth Due to Moon and Sun. Again considering only second degree terms this torque is

\[ M_{y1} = 3GM_1 r_{15}^{-3} (C - B)m_\odot n_\odot \]
\[ + 3GM_3 r_{14}^{-3} (C - B)m_\odot n_\odot \]

\[ M_{y2} = 3GM_1 r_{15}^{-3} (A - C)\ell_\odot n_\odot \]
\[ + 3GM_4 r_{14}^{-3} (A - C)\ell_\odot n_\odot \]  

(98)

\[ M_{y3} = 3GM_1 r_{15}^{-3} (B - A)\ell_\odot m_\odot \]
\[ + 3GM_4 r_{14}^{-3} (B - A)\ell_\odot m_\odot \]

CONCLUDING REMARKS

This report presents a unified development of a physical model and a mathematical model of the Earth-Moon system. The Earth and Moon are considered to be rigid bodies. The equations of motion are formulated in a completely coupled fashion and the mutual potential of the Earth-Moon pair is incorporated in the development.

This model is intended as a basis for a more inclusive theoretical model including relativistic, non-rigid, and dissipative phenomena.
The models are being coded for use in data reduction packages to estimate physical parameters of the Earth-Moon system. The listing for two programs that have been developed to date are provided in Appendix C.

Program ANEAMØ evaluates (1) a truncated form of Brown's lunar theory, (2) Eckhardt's theory for lunar physical librations, and (3) Newcomb's expressions for the rotational motion of the Earth. Program RIGEM numerically integrates the rotational motion of the Earth and Moon and the translational motion of all the planets. More information on these may be found in the listings.
REFERENCES


APPENDIX A

FORCES AND TORQUES BY VECTOR-DYADIC METHOD

Reference 16 provides a derivation of the appropriate equations. These are summarized below as applied to the forces and torques on the Moon due to the Earth.

A. Force on Triaxial Moon Due to Spherical Earth

\[
\mathbf{F} = -\frac{GM'\mathbf{r}}{r^2} + \frac{3}{2} \frac{GM\mathbf{\theta}}{r^4} + \frac{3GMi\mathbf{r}}{r^6} \cdot (\mathbf{\hat{r}}_\theta - \mathbf{\hat{r}}_r) - \frac{15}{2} \frac{GMi\mathbf{r}}{r^6} \cdot (\mathbf{\hat{r}}_\theta - \mathbf{\hat{r}}_r) \cdot \mathbf{\hat{r}}_r
\]  

B. Force on Spherical Moon Due to Oblate Earth

\[
\mathbf{F} = -GM'M \left\{ \frac{\mathbf{r}}{r^2} + \frac{Ja^2}{r^4} \left[ \frac{\mathbf{r}}{r^2} - 5(\mathbf{\hat{r}}_3 \cdot \mathbf{\hat{r}}_r)^2 \mathbf{\hat{r}}_3 \\
+ 2(\mathbf{\hat{r}}_3 \cdot \mathbf{\hat{r}}_r) \mathbf{\hat{r}}_3 \right] \right\}
\]

C. Torque on Moon Due to Spherical Earth

\[
\mathbf{T} = -\frac{3GM}{r^3} \mathbf{r} \times \mathbf{\hat{r}}_r 
\]
D. Torque on Moon Due to Oblate Earth

\[ \mathbf{T} = \frac{-3GM}{r^3} \mathbf{i}_r \cdot \mathbf{I} \times \mathbf{i}_r \]

\[- \frac{5GMJ\alpha^2}{r^5} \left\{ 1 - 7 \left( \frac{J_3}{J_2} \cdot \mathbf{i}_x \right)^2 \right\} \mathbf{i}_r \cdot \mathbf{I} \times \mathbf{i}_r \]

(A4)

\[ + \frac{2}{5} J_3 \cdot \mathbf{i}_r \left( J_3 \cdot \mathbf{I} \times \mathbf{i}_r + \mathbf{i}_r \cdot \mathbf{I} \times J_3 \right) \]

\[- \frac{2}{5} J_3 \cdot \mathbf{I} \times J_3 \]

In the above equations,

\[ \theta = (A + B + C)/2, \]

\[ \mathbf{I} = A\hat{j}_1 \hat{j}_1 + B\hat{j}_2 \hat{j}_2 + C\hat{j}_3 \hat{j}_3 \]

\[ \mathbf{E} = \hat{j}_1 \hat{j}_1 + \hat{j}_2 \hat{j}_2 + \hat{j}_3 \hat{j}_3 \]
APPENDIX B

DERIVATION OF TORQUES FROM THE MUTUAL POTENTIAL

Consider the $\{y_i\}$ frame as the reference frame. The potential at any point of the moon $(y_1, y_2, y_3)$ is given by $\phi (y_1, y_2, y_3)$ in its most general form. The total mutual potential may be written as

$$U^I = \int \phi (y_1, y_2, y_3) \, dM'. \tag{B1}$$

Note that the subscript $45$ on $U^I$ has been omitted here.

Now, the force on a particle of mass $dM'$ at point $(y_1, y_2, y_3)$ resolved along the $\{y_i\}$ axes is

$$\mathbf{f} = f_{y_1} \mathbf{j}_1 + f_{y_2} \mathbf{j}_2 + f_{y_3} \mathbf{j}_3 \tag{B2}$$

where

$$f_{y_i} = \frac{\partial \phi}{\partial y_i}.$$

The components of this force along the axes $\{z_i\}$ are

$$\{f_{z_i}\} = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{bmatrix} \begin{bmatrix} f_{y_1} \\ f_{y_2} \\ f_{y_3} \end{bmatrix} \tag{B3}$$

The differential torques about the $\{z_i\}$ axes produced by these forces are

$$m_{z_1} = z_2 f_{z_2} - z_3 f_{z_2}$$

$$m_{z_2} = z_3 f_{z_1} - z_1 f_{z_3}$$

$$m_{z_3} = z_1 f_{z_2} - z_2 f_{z_1} \tag{B4}$$
These may be written as

\[ m_{z_1} = \alpha^{''}\phi_{y_1} z_2 - \alpha^{'}\phi_{y_1} z_3 \]
\[ + \beta^{''}\phi_{y_2} z_2 - \beta^{'}\phi_{y_2} z_3 \]
\[ + \gamma^{''}\phi_{y_3} z_2 - \gamma^{'}\phi_{y_3} z_3 \]

\[ m_{z_2} = \alpha\phi_{y_1} z_3 - \alpha^{''}\phi_{y_1} z_1 \]
\[ + \beta\phi_{y_2} z_3 - \beta^{''}\phi_{y_2} z_1 \]
\[ + \gamma\phi_{y_3} z_3 - \gamma^{''}\phi_{y_3} z_1 \]  

\[ m_{z_3} = \alpha^{''}\phi_{y_1} z_1 - \alpha\phi_{y_1} z_2 \]
\[ + \beta^{''}\phi_{y_2} z_1 - \beta\phi_{y_2} z_2 \]
\[ + \gamma^{''}\phi_{y_3} z_1 - \gamma\phi_{y_3} z_2 \]  

Now, if the differential torques are integrated over the body using

\[ M_{z_1} = \int_{M'} m_{z_1} dM' \]  

(B6)
then

\[ M_{z_1} = \alpha' \int \phi_y \frac{\partial y_1}{\partial \alpha} dM' - \alpha' \int \phi_y \frac{\partial y_1}{\partial \alpha'} dM' \]

\[ + \beta' \int \phi_y \frac{\partial y_2}{\partial \beta} dM' - \beta' \int \phi_y \frac{\partial y_2}{\partial \beta'} dM' \]

\[ + \gamma' \int \phi_y \frac{\partial y_3}{\partial \gamma} dM' - \gamma' \int \phi_y \frac{\partial y_3}{\partial \gamma'} dM' \]

(B7)

etc.

Finally,

\[ M_{z_1} = \alpha' \ U_{\alpha'} - \alpha' \ U_{\alpha} \]

\[ + \beta' \ U_{\beta} - \beta' \ U_{\beta'} \]

\[ + \gamma' \ U_{\gamma} - \gamma' \ U_{\gamma'} \]

(B8)

(cont'd.)
and

\[ M_{z_3} = \alpha' \frac{U}{\alpha} - \alpha \frac{U}{\alpha}, \]

\[ + \beta' \frac{U}{\beta} - \beta \frac{U}{\beta}, \]

\[ + \gamma' \frac{U}{\gamma} - \gamma \frac{U}{\gamma}. \]  

(B8) (concl'd.)
APPENDIX C

PROGRAM LISTINGS
PROGRAM RIGEM (INPUT,OUTPUT,TAPE2=INPUT,TAPE3=OUTPUT)

THIS PROGRAM INTEGRATES THE TRANSLATIONAL MOTION OF THE
SUN, PLANETS, AND MOON. IT ALSO INTEGRATES THE FULLY COUPLED ROTATIONAL
MOTION OF THE EARTH AND MOON USING SUBROUTINE RA19S.

VARIABLES AND PARAMETERS

* X VECTOR OF COORDINATES
* V VECTOR OF VELOCITIES
* X(1-33) TRANSLATIONAL
* X(34-37) EARTH ROTATIONAL
* X(38-41) LUNAR ROTATIONAL

MULTI-CASE OPTION

ICODE=0 LAST CASE
ICODE=1 RETURN FOR NEW CASE
NOTE: CURRENTLY CODED TO READ NEW BETA TR. PRIME RATES ONLY

DIMENSION X(41),V(41),NORD(2),OMPL(1),D(4,4),F(41)
DIMENSION T(3,3),E(3,3),C(3,3),P(3,3),R(3,3),PP(3,3),DEL(3)
I,DAL(3),DELV(3),DALV(3),PD(3,3),EQ(3,3)
DIMENSION OMM(4),OMDM(4),RT(4)
INTEGER OMPL
COMMON/PARAM/NORD,AD,DEP,OMPL,INC,TMAX
COMMON/XOUT/OMM,OMDM,RT
ICODE=0
PI=3.14159265358979
DTR=PI/180.
RTD=180./PI

GET INITIAL CONDITIONS

77 CALL PROB(X,V,TT,TF,NV,NCLASS,NSS,NI,NOR,LL,ICODE)

INTEGRATION

4 CALL RA19S(X,V,TT,TF,TL,LL,NV,NI,NF,NS,NCLASS,NOR,NSS)
OUTPUT SECTION

000041  TWR=VJDEP+TF
000043  WRITE(3,100)
000047  WRITE (3,115) TWR
000055  WRITE(3,101)VJDEP,NOR,LL,NF,TF
000073  WRITE(3,105)
000077  WRITE(3,106)
00103   II=1
00104   DO 1 I=1,31,3
00106   WRITE(3,102)II
00113   WRITE(3,103)X(I),X(I+1),X(I+2)
00125   V(I)=V(I)*100.
00130   V(I+1)=V(I+1)*100.
00131   V(I+2)=V(I+2)*100.
00132   WRITE(3,104)V(I),V(I+1),V(I+2)
00144   II=II+1
00146   1 CONTINUE
00150   DO 75 I=1,33
00151    75 V(I)=V(I)/100.

EARTH_ORIENTATION

000155   IF (NORO(1),EQ.0) GO TO 76
000156   WRITE(3,107)
000161   WRITE (3,115) TWR
000167   WRITE(3,109)
000173   WRITE(3,111) (X(I+33),I=1,4)
000205   WRITE(3,111) (V(I+33),I=1,4)

EULER_PARAMETER_TESTS_FOR_EARTH

000217   WRITE (3,113)
000223   TEST1=X(34)**2+X(35)**2+X(36)**2+X(37)**2
000231   TEST2=X(34)*V(34)+X(35)*V(35)+X(36)*V(36)+X(37)*V(37)
000240   WRITE (3,112) TEST1,TEST2

MOON_ORIENTATION

000247   76 WRITE (3,109)
000253   WRITE (3,115) TWR
000261   WRITE(3,110)
Routine to calculate Earth's Selenographic Coordinates:

\[
\begin{align*}
D(2,2) &= X(38) ** 2 + X(39) ** 2 - X(40) ** 2 - X(41) ** 2 \\
D(3,3) &= X(38) ** 2 - X(39) ** 2 + X(40) ** 2 - X(41) ** 2 \\
D(4,4) &= X(38) ** 2 - X(39) ** 2 - X(40) ** 2 + X(41) ** 2 \\
D(2,3) &= 2 * (X(39) * X(40) - X(38) * X(41)) \\
D(2,4) &= 2 * (X(39) * X(41) - X(38) * X(40)) \\
D(3,2) &= 2 * (X(40) * X(41) - X(38) * X(39)) \\
D(3,4) &= 2 * (X(40) * X(41) + X(38) * X(39)) \\
D(4,3) &= 2 * (X(40) * X(41) + X(38) * X(39)) \\
DAL(I) &= X(I,3) - X(I,0) \\
RRR &= \sqrt{DAL(I) * DAL(I) + DAL(2) * DAL(2) + DAL(3) * DAL(3)} \\
CC &= DAL(1) / RRR \\
CS &= DAL(2) / RRR \\
SPH &= DAL(3) / RRR \\
CPH &= \sqrt{DAL(1) ** 2 + DAL(2) ** 2} / RRR \\
CL &= CC / CPH \\
SL &= CS / CPH \\
-- CALL FORCE (XVTFF) \\
SS &= \sqrt{D(2,2) ** 2 + D(3,2) ** 2} \\
SLONG &= \text{ATAN2}(D(3,2), D(2,2)) \\
SLAT &= \text{ATAN2}(D(4,2), SS) \\
-- ROUTINE TO CALCULATE PHYSICAL LIBRATIONS \\
T(1,1) &= CC \\
T(1,2) &= -CS \\
T(1,3) &= -SPH \\
T(2,1) &= SL \\
T(2,2) &= -CL \\
T(2,3) &= 0. \\
T(3,1) &= CL * SPH \\
T(3,2) &= -SL * SPH \\
T(3,3) &= CPH \\
TEP &= (V1DEP + TF - 2415020.0) / 36525. \\
EPS &= (23.452294 - .0130125 * TEP -.00000164 * TEP2 + .000000503 * TEP3) * DTR \\
E(1,1) &= 0. \\
E(1,2) &= 0. \\
E(1,3) &= 0. 
\end{align*}
\]
E(2,1) = 0.  $  E(2,2) = \cos(\text{EPS})  $  E(2,3) = -\sin(\text{EPS})$

E(3,1) = 0.  $  E(3,2) = \sin(\text{EPS})  $  E(3,3) = \cos(\text{EPS})$

C(1,1) = D(2,2)  $  C(1,2) = D(2,3)  $  C(1,3) = D(2,4)$

C(2,1) = D(3,2)  $  C(2,2) = D(3,3)  $  C(2,3) = D(3,4)$

C(3,1) = D(4,2)  $  C(3,2) = D(4,3)  $  C(3,3) = D(4,4)$

TT = VJDEP + TF - 2433282.5

XKAP = 0.063107 * TT * DTR / 3600.

OMEG = 0.063107 * TT * DTR / 3600.

XNU = 0.0548757 * TT * DTR / 3600.

P(1,1) = \sin(XKAP) * \sin(OMEG) + \cos(XKAP) * \cos(OMEG) * \cos(XNU)

P(1,2) = \cos(XKAP) * \sin(OMEG) - \sin(XKAP) * \cos(OMEG) * \cos(XNU)

P(1,3) = \cos(OMEG) * \sin(XNU)

P(2,1) = \sin(XKAP) * \cos(OMEG) + \cos(XKAP) * \sin(OMEG) * \cos(XNU)

P(2,2) = \cos(XKAP) * \cos(OMEG) - \sin(XKAP) * \sin(OMEG) * \cos(XNU)

P(2,3) = \sin(OMEG) * \sin(XNU)

P(3,1) = \cos(XKAP) * \sin(XNU)

P(3,2) = -\sin(XKAP) * \sin(XNU)

P(3,3) = \cos(XNU)

DO 50 K = 1, 3

DO 51 J = 1, 3

R(K, J) = 0.

DO 52 I = 1, 3

DO 53 L = 1, 3


DO 54 CONTINUE

DO 55 CONTINUE

C MATRIX PP BECOMES THE PRODUCT E(\text{TR}) * P  * T(\text{TR}) * C(\text{TR}) TR=TRANSPOSE

C

DO 70 I = 1, 3

DO 71 J = 1, 3

PP(I, J) = 0.

DO 72 L = 1, 3

PP(I, J) = PP(I, J) + R(I, L) * C(J, L)

DO 73 CONTINUE

DO 74 CONTINUE

APHI = ATAN2(-PP(3,1) - PP(3,2))

ST = SQRT(PP(3,1)**2 + PP(3,2)**2)

ATH = ATAN2(ST, PP(3,3))

APSI = ATAN2(-PP(1,3), PP(2,3))

APHI = APHI * RTD  $  ATH = ATH * RTD  $  APSI = APSI * RTD

APHI = AMOD(APHI, 360.)
IF (APHI .LT. 0.) APHI = APHI + 360.

APSI = AMOD(APSI, 360.)

IF (APSI .LT. 0.) APSI = APSI + 360.

WRITE (3, 120) APHI, ATHAPS.

APMEG = 259.183275 - 0.0529539222 * (365.25 * TEP) + 0.0001557 * (3.6525 ** 2) * TEP ** 2 + 0.00000005 * (3.6525 ** 3) * TEP ** 3

AMOON = 270.434358 + 3.1763965268 * (365.25 * TEP) - 0.000085 * (3.6525 ** 2) * TEP ** 2 + 0.000000039 * (3.6525 ** 3) * TEP ** 3

AI = 5549.3 / 3600.

AOMEG = AMOD(AOMEG, 360.)

IF (AOMEQ .LT. 0.) AOMEQ = AOMEQ + 360.

AMCON = AMOD(AMOON, 360.)

IF (AMOON .LT. 0.) AMOON = AMOON + 360.

WRITE (3, 121) AMOON, AI, AOMEQ

RHO = ATH - AI

SIG = APSI - AOMEQ

TAU = APHI - 180. - AMOON + APSI

WRITE (3, 118) TAU

WRITE (3, 119) RHO, SIG, TAU

C

EULER PARAMETER TEST FOR MOON

TEST3 = X(38) ** 2 + X(39) ** 2 + X(40) ** 2 + X(41) ** 2

TEST4 = X(38) * V(38) + X(39) * V(39) + X(40) * V(40) + X(41) * V(41)

CON1 = - V(38) * V(38) + V(39) * V(39) + V(40) * V(40) + V(41) * V(41)

CON2 = X(38) * F(38) + X(39) * F(39) + X(40) * F(40) + X(41) * F(41)

DCON = CON1 - CON2

WRITE (3, 113) DCON

WRITE (3, 112) TEST3, TEST4, DCON

WRITE (3, 111) TEST3, TEST4, DCON

WRITE (3, 111) TEST3, TEST4, DCON

WRITE (3, 111) TEST3, TEST4, DCON

WRITE (3, 111) TEST3, TEST4, DCON

WRITE (3, 111) TEST3, TEST4, DCON

WRITE (3, 111) TEST3, TEST4, DCON

WRITE (3, 111) TEST3, TEST4, DCON

WRITE (3, 111) TEST3, TEST4, DCON
SUBROUTINE RA19S(X,V,TI,TF,XL,L,NF,NS,NCLASS,NOR,NSS)
C PROGRAM BY E. EVERHART, PHYSICS AND ASTRONOMY DEPT. UNIVERSITY OF DENVER.
C DENVER, COLORAD 80210. PHONE (303)-753-2238 OR 753-2362.
C INTEGRATOR FOR ORDERS 7, 11, 15, 19. SINGLE PRECISION VERSION.
C THIS IS A VERSION OF INTEGRATOR RADAU.
C NV IS THE NUMBER OF DEPENDENT VARIABLES.
C NCLASS IS 1 FOR 1ST-ORDER DIFF EQ. AND 2 FOR 2ND ORDER DIFF EQ.
C IF FIRST DERIVATIVES ARE NOT PRESENT (CLASS II), THEN USE NCLASS=-2.
C X(NV) IS THE INITIAL POSITION VECTOR. IT RETURNS AS THE FINAL VALUE.
C V(NV) IS THE INITIAL VELOCITY VECTOR. IT RETURNS AS THE FINAL VALUE.
C IN THE CASE NCLASS IS UNITY, THEN V IS SIMPLY ZEROED.
C TI IS INITIAL TIME, TF IS FINAL TIME, NF IS NUMBER OF FUNCTION EVALUATIONS.
C NS IS THE NUMBER OF SEQUENCES.
C PROGRAM SET UP FOR A MAXIMUM OF 18 SIMULTANEOUS EQUATIONS.
C LL CONTROLS SEQUENCE SIZE. THUS SS=10.**(-LL) IS DESIRED SIZE OF A TERM.
C AS IN CONTROL SYSTEM I.
C IF LL=LT, THEN XL IS THE SPECIFIED CONSTANT SEQUENCE SIZE.
C WILL INTEGRATE IN A NEGATIVE DIRECTION IF TF.LT.TI.
000020 DIMENSION X(1),V(1),F(41),FJ(41),C(36),D(36),R(36)
  
000020 DIMENSION MC(BNW(10),NXI(36))
000020 LOGICAL J2,NPQ,NPER,NCLS
000020 DATA NW/0,0,1,3,6,10,15,21,28,36/
000020 DATA MC/,9,16,22,27,31,34,36/
000020 DATA ZEROLONE/0.,1./
000020 DATA NXI/2,3,4,5,6,7,8,9,3,6,10,15,21,28,36,4,10,20,35,56,84,5,15, 
  
000020 DATA HH/.212345823139152,.9093313559265,,101412040487296,  
000020 DATA X/35,70,126,6,21,36,126,7,28,84,8,36,9/
000020 DATA X/0.98555085798826426, .304535726646363905, .562025189752613855, 
000020 DATA X/8.010965821263391827, .960190142984531257, .056262560563922146, 
000020 DATA X/1.80240691736892364, .35262471711169637, .54715362633555383, 
000020 DATA X/7.342410177215410531, .85302946839095768, .977502613561287501, 
000020 DATA X/0.36257812883209460, .118079878789998700, .23717698414960385, 
000020 DATA X/3.81882765304705975, .53802959891899065, .690332420072362182, 
000020 DATA X/8.23883343837004716, .925612610290803955, .985587590351123451/
000020 KD=(NOR-3)/2
000022 KD2=KD/2
000024 KE=KD+1
000026 KF=KD+2
000030 PW=ONE/FLOAT(KD+3)
000033 NPER=.FALSE.
Program Description:

The program calculates the values of C(LC) and D(LC) using iterative methods. It initializes variables, sets tolerances, and performs calculations based on user-defined parameters.

Key Points:
- The program initializes variables such as R(LC), D(LC), C, and D.
- It uses a loop to calculate C(LC) and D(LC) based on the current values of C and D.
- User-defined parameters include N, LL, and T.
- The program includes conditional statements to handle different cases.

Code Snippet:

```plaintext
R(LC) = ONE/(H(K+1)-H(K))
 IF(K.EQ.3) GO TO 73
 DO 72 L=4,K
    LD=LA+L-3
    LE=LB+L-4
    JD=LD-LA
    C(LC)=W(JDM+I)*C(LE)/W(JDM)-H(K)*C(LE-I)
    D(LC)=(D(LE)+H(LE-I)*D(LE-I))*W(K-I)/W(K)
 72 CONTINUE
CONTINUE
SS=10.**(LL)
NL=NI+30
C SET IN A REASONABLE ESTIMATE TO T BASED ON EXPERIENCE. (SAME SIGN AS TF-TI)
IF(.NOT.NES) TP=((FLOAT(NOR)/11.)*0.5**(0.4*FLOAT(LL)))*DIR
IF(NES) TP=XL
IF(TP/TDIF.GT.0.5) TP=0.5*TDIF
NF=C
CALL FORCE(X,V,TM,F1)
C LOOP 58 FINDS THE BETA-VALUES FROM THE CORRECTED B-VALUES, USING D-C EFF
DO 58 K=1,NV
    BE(KE,K)=B(KE,K)/W(KE)
    DO 58 J=1,KD
    JD=J+1
    BE(J,K)=B(J,K)/W(J)
    DO 58 L=JD,KE
    N=NW(L)+J
    58 BE(J,K)=BE(J,K)+O(N)*B(L,K)
T=TP
TVAL=ABS(T)
T2=T**NCLASS
C LOOP 175 IS THE ITERATION LOOP WITH NL=NI PASS SESS AFTER THE FIRST SEQUENCE
```

Additional Notes:
- The program uses a combination of DO loops and conditional statements to perform calculations.
- The variable names and constants are user-defined and can affect the program's performance.
- The program includes error handling for conditional statements to ensure stability.

The program's main function is to calculate and update the values of C(LC) and D(LC) iteratively until convergence.
Q=S**(NCLASS-1)

IF(NPQ) GO TO 5100
DO 1300 K=1,NV
RES=B(KE,K)
TEMP=RES*U(KE)
DO 7340 L=1,KD
JR=KE-L
RES=B(JR,K)+S*RES
TEMP=B(JR,K)*U(JR)+STEP
Y(K)=X(K)+Q*(T4V(K)+T2*S*(FF(K)*W+RES))
Z(K)=V(K)+S*T*(F1(K)+S*TEMP)

GO TO 5200

DO 1400 K=1,NV
RES=B(KE,K)
DO 2340 L=1,KD
JR=KE-L
RES=B(JR,K)+S*RES
Y(K)=X(K)+Q*(T4V(K)+T2*S*(F1(K)*W+RES))
Z(K)=V(K)+S*T*(F1(K)+S*TEMP)
CONTINUE
CALL FORCE(Y,Z,TM+S*TFJ)
NF=NF+1
IF(J2) GO TO 702
DO 471 K=1,NV
TEMP=BE(JD,K)
RES=(FJ(K)-F1(K))/S
N=L
DO 134 L=1,JDM
N=N+1
RES=(RES-BE(L,K))*R(N)
B(E,JD,K)=RES
TEMP=RES-TEMP
B(JD,K)=B(JD,K)+TEMP*W(JD)
N=L
DO 471 L=1,JDM
N=N+1
B(L,K)=B(L,K)+C(N)*TEMP
GO TO 174
J2=.FALSE.
DO 271 K=1,NV
TEMP=BE(1,K)
RES=(FJ(K)-F1(K))/S
BE(1,K)=RES
GO TO 271
CONTINUE
IF(M.LT.NI) GO TO 175
HSUM=0.
VAL=TVAL**(-KE)
DO 635 K=1,NV
HSUM=HSUM+B(KE,K)**2
HSUM=VAL*SORT(HSUM)
IF( NSF ) GO TO 175
IF(ABS((HSUM-SM)/HSUM).LT.0.01) GO TO 176
SM=HSUM
175 CONTINUE
C THIS NEXT PART FINDS THE PROPER STARTING VALUE FOR T
176 IF( NSF ) GO TO 180
IF(.NOT.NES) TP=(SS/HSUM)**PW*DIR
IF(PES) TP=XL
IF(NES) GO TO 170
IF(TP/T.GT.ONE) GO TO 170
TP=0.8*TP
NCOUNT=NCOUNT+1
IF(NCOUNT.GT.10) RETURN
PRINT 89KDT,TP
GO TO 4000
170 PRINT 8,KD,T,TP
NSF=.TRUE.
C FIND POSITION (AND VELOCITY FOR CLASS II AND II$) AT THE END OF THE SEQUENCE.
180 DO 35 K=1,NV
RES=B(KE,K)
DO 34 L=1,KD
RES=RES+R(L,K)
X(K)=X(K)+V(K)*T+T2*(F1(K)*W1+RES)
34 RES=RES+B(L,K)*U(K)
35 CONTINUE
TM=TM+T
NS=NS+1
IF(NPER ) RETURN
CALL FORCE(X,V,TM,F1)
NF=NF+1
IF(NES) GO TO 341
TP=((SS/HSUM)**PW)*DIR
IF(TP/T.GT.SR) TP=SR*T
IF(NES) TP=XL
IF(DIR*(TM+TP).LT.DIR*TF-1.E-10) GO TO 77
TP=TF-TM
NPER=.TRUE.

C PREDICT B-VALUES FOR NEXT SEQUENCE.

Q =TP/T
DO 39 K=1,NV
RES=ONE
DO 39 J=1,KE
IF(NS.GT.1) BT(J,K)=B(J,K)-BT(J,K)
IF(J.EQ.KE) GO TO 740
M=MC(J)
JD=J+1
DO 40 L=J0,KE
B(J,K)=B(J,K)+XI(M)*B(L,K)
M=M+1
RES=RES*Q
TEMP=RES*B(J,K)
B(J,K)=TEMP+BT(J,K)
39 BT(J,K)=TEMP
NL=NI
GO TO 722
END
SUBROUTINE PROB (X,V,TI,TF,NV,NCLASS,NSS,NI,NOR,LL,IICODE)

C THIS SUBROUTINE PROVIDES CONSTANTS AND INITIAL CONDITIONS FOR RIGEM
C NORO(1)=0 NEGLECT EARTH ROTATION
C   =1 COMPUTE EARTH ROTATION
C NORO(2)=0 NEGLECT MOON ROTATION
C   =1 COMPUTE MOON ROTATION
C
C OMPL(1)=1 OMITS MERCURY, SATURN, URANUS, NEPTUNE, PLUTO
C OMPL(1)=0 OMITS NO PLANETS

DIMENSION XMASS(11),X(1),V(1),XMA(11),NORO(2),OMPL(1)
DIMENSION XS(41),VS(41)
INTEGER OMPL

COMMON/XMAS/XMASS
COMMON/INERT/BEM,GAM,ALPM,BEE,GAE,ALPE,E11,E12,E13,AMI1,AMI2,AMI3
COMMON/PARAM/NORO,AO,AD,VJDEP,OMPL,TINC,TMAX
DATA XMA /1.5983000, 408522, 332945.56192544, 2708807.1301, 13098700, 1047.3908, 3499.2, 22930, 19260, 1812000/
IF (IICODE.NE.0) GO TO 12
READ (2,120) NORO(1),NORO(2),OMPL(1)
READ (2,121) VJDEP,TINC,TMAX
READ (2,123) NI,NOR,LL
PI=3.14159265358979
DTR=PI/180. $ RTD=180./PI
BEM=.00063 $ GAM=.0002 $ ALPM=.0043
ALPE=.00322 $ BEE=.00327
GAE=.00054
AO=100.075542*DTR
AD=360.985647348*DTR
AMI1=5. $ AMI2=5. $AMI3=5.

C THIS LOOP PROVIDES PLANETARY I.C. PER TABLE 10, P. 274, CEL. MECH.
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DO 1 I=1,11
1 XMASS(I)=XK2/XMA (I)

C THIS LOOP PROVIDES PLANETARY I.C. PER TABLE 10, P. 274

WRITE(3,102)
C THIS LOOP PROVIDES INITIAL EULER ANGLES AND RATES FOR EARTH-MOON

WRITE(3,108)
108 FORMAT (/30X,'EARTH INITIAL EULER PARAMETERS AND RATES',/)

DO 3 L=34,38,4.

READ(2,101) X(L),X(L+1),X(L+2),X(L+3)

XS(L)=X(L) $ XS(L+1)=X(L+1) $ XS(L+2)=X(L+2) $ XS(L+3)=X(L+3)

READ(2,101)V(L),V(L+1),V(L+2),V(L+3)

VS(L)=V(L) $ VS(L+1)=V(L+1) $ VS(L+2)=V(L+2) $ VS(L+3)=V(L+3)

WRITE(3,109) X(L),X(L+1),X(L+2),X(L+3)

WRITE(3,109) V(L),V(L+1),V(L+2),V(L+3)

IF(L.EQ.38) GO TO 3

3 CONTINUE

TI=0.
T T = TINC
NCLASS=+2
NV=41
NSS=0

WRITE (3,125)
WRITE (3,126)
WRITE (3,127) IXMASS(I)
WRITE (3,128) ALPEE,
WRITE (3,129) ALPBE,
WRITE (3,130) ALPBE,
WRITE (3,131) VJDEP,
WRITE (3,132) NI/NOR, LL
MULTI-CASE SEGMENT

IF (IICODE.EQ.0) GO TO 14

DO 13 I=1,41

X(I)=XS(I)

V(I)=VS(I)

READ (2,101) V(38),V(39),V(40),V(41)

WRITE (3,133)

WRITE (3,110)

WRITE (3,109) V(38),V(39),V(40),V(41)

TI=0.

TF=TINC

100 FORMAT (3X,16,3E25.14)

101 FORMAT (4E20.12)

102 FORMAT (*1*,40X,*INITIAL CONDITIONS AND PARAMETERS*,,/)

103 FORMAT (30X,*INITIAL POSITION AND VELOCITY OF PLANETS*,,/)

104 FORMAT (30X,*REFERRED TO MEAN EQUATOR AND EQUINOX OF 1950.0*,,/)

105 FORMAT (2X,*PLANETS*,16X,*X*,25X,*Y*,25X,*Z*)

106 FORMAT (25X,*XD*,25X,*YD*,25X,*ZD*,/)

107 FORMAT (3E25.14)

109 FORMAT (20X,4E20.12,/)

110 FORMAT (30X,*MOON INITIAL EULER PARAMETERS AND RATES*,,/)

120 FORMAT (3I15)

121 FORMAT (3E20.12)

122 FORMAT (3I15)

123 FORMAT (3I15)

124 FORMAT (*1*,50X,*PARAMETERS*,,/)

125 FORMAT (40X,*PLANETARY GRAVITY PARAMETERS*,,/)

126 FORMAT (40X,*MOON INITIAL EULER PARAMETERS AND RATES*,,/)

127 FORMAT (45X,12,2X,E19.12)

128 FORMAT (/*,40X,*EARTH INERTIA RATIOS*,,/15X,*ALPHA=*,E19.12,3X,1*BETA=*,E19.12,3X,

1*GAMMA=*,E19.12,/) /15X,*ALPHA=*,E19.12,3X,

129 FORMAT (*,40X,*MOON INERTIA RATIOS*,15X,*ALPHA=*,E19.12,3X,1*BETA=*,E19.12,3X,

123 FORMAT (20X,*INTEGRATION PARAMETERS*,,/20X,*NI=*,I3,2X,*NOR=*

14 IICODE=1

132 FORMAT (20X,*INTEGRATION PARAMETERS*,,/20X,*NI=*,I3,2X,*NOR=*1,13,2X,*LL=*,I3,/) /

133 FORMAT (*1*,50X,*NEW CASE*)

14 IICODE=1

RETURN

END
SUBROUTINE FORCE(X,V,T,M,F)
C THIS SUBROUTINE PROVIDES N BODY GRAVITATIONAL
C NOTATION
C XMASS(1)=SUN XMASS(6)=MARS
C XMASS(2)=MERCURY XMASS(7)=JUPITER
C XMASS(3)=VENUS XMASS(8)=SATURN
C XMASS(4)=EARTH XMASS(9)=UPANOUS
C XMASS(5)=MOON XMASS(IO)=NEPTUNE
C XMASS(II)=PLUTO
DIMENSION X(1),V(1),F(1),XMASS(II),Y(66)
DIMENSION OM(4),B(4,4),BD(4,4),OMD(4),FD(4),CM(4),BB(4,4),
BD(4,4),DALV(4),D(4,4),OMDM(4),DD(4,4),RT(4),RD(4),
INORO(2),PT(4),DEL(4),DAL(4),C(3,3),OMPL(1)
INTEGER OMPL
COMMON/XMAS/XMASS
COMMON/INERT/BEM,GAM,ALPM,BEE,BAE,ALPE,EL1,EL2,EL3,AMI1,AMI2,AMI3
COMMON/PARAM/NORO,AG,AD,VJDEP,OMPL,TIAC,TMAX
COMMON/XOIT/OMM,OMDM,RT
R(XI,XJ,YI,YJ,ZI,ZJ)= SQRT((XJ-XI)**2+(YJ-YI)**2+(ZJ-ZI)**2)
C I1 IS PLANET COUNTER
DO 1 1=1,41
Y(I)=X(I)
1 CONTINUE
II IS PLANET COUNTER
DO 60 I=1,41
60 F(I)=0.
IF (OMPL(I).EQ.1) 61,62
61 NN1=7 $ NN2=19 $ II=3
GO TO 63
62 NN1=4 $ NN2=31 $ II=2
DO 2 I=NN1,NN2,3
2 RR=R(Y(I),Y(I),Y(2),Y(I+1),Y(3),Y(I+2))
F(I)=-G*(XMASS(I)+XMASS(I))*YII/(RR*RR*RR)
JJ=2
IF (OMPL(I).EQ.1) JJ=3
G1=0.
G2=0.
DO 3 J=NN1,NN2,3
IF (I.EQ.JJ) GO TO 33
G1=G*XMASS(JJ)*(Y(J)-Y(I))/R(Y(I),Y(J),Y(I+1),Y(J+1),Y(I+2))
G2=G*XMASS(JJ)*Y(J)/R(Y(I),Y(J),Y(2),Y(J+1),Y(3),Y(J+2))**3+G1
000146 G2=G*XMASS(JJ)*Y(J)/R(Y(I),Y(J),Y(2),Y(J+1),Y(3),Y(J+2))**3+G2
000167  33 JJ=JJ+1
000171  3 CONTINUE
000173  F(I)=F(I)+G1-G2
000177  F(I+1)= -G*(XMASS(I)+XMASS(I+1))*Y(I+1)/(RR*RR*RR)
000207  JJ=2
000210  IF(CMPL(I).EQ.1) JJ=3
000213  G1=0.
000214  G2=0.
000215  DO 4 J=NN,NN+2,3
000217  IF(IIEQ.JJ)GO TO 44
000221  G1=G1+G*XMASS(JJ)*[Y(J+1)-Y(I+1))/R(Y(I),Y(J),Y(I+1),Y(J+1),Y(I+2)
000247  G2=G2+G*XMASS(JJ)*[Y(J+1)-Y(I+1)/R(Y(I),Y(J),Y(I+1),Y(J+1),Y(I+2)]**3
000271  44 JJ=JJ+1
000273  4 CONTINUE
000275  F(I+1)=F(I+1)+G1-G2
000301  F(I+2)= -G*(XMASS(I)+XMASS(I+1))*Y(I+2)/(RR*RR*RR)
000311  JJ=2
000315  [F(CMPL(I).EQ.1) JJ=3
000316  G1=0.
000317  G2=0.
000321  DO 5 J=NN,NN+2,3
000323  IF(IIEQ.JJ)GO TO 55
000351  G2=G2+G*XMASS(JJ)*[Y(J+2)-Y(I+2)/R(Y(I),Y(J),Y(I+1),Y(J+1),Y(I+2)
000373  55 JJ=JJ+1
000375  5 CONTINUE
000377  F(I+2)=F(I+2)+G1-G2
000403  2 JJ=JJ+1
000407  IF (CMPL(I).EQ.1) 64,65
000414  64 NN1=7 \$NN2=19 \$ JJ=3
000417  GO TO 66
000420  65 NN1=4 \$NN2=31 \$ JJ=2
000423  66 G1=0.
000424  DC 6 J=NN,NN+2,3
000426  G1=G1+G*XMASS(JJ)*[Y(I)-Y(J))/R(Y(I),Y(J),Y(I+1),Y(J+1),Y(I+2)
000451  F(I)=-G1
000455  C FORCES ON SUN (XMASS(I))
SUNS ACCELERATIONS SET EQUAL TO ZERO FOR TEST

000456  F(1)=0.
000457  IF (CMPL(1).EQ.1) 67,68
000464  67 NN1=7 $NN2=19 $ JJ=3
000467  GO TO 69
000470  68 NN1=4 $NN2=31 $ JJ=2
000473  69 G1=0.
000474  DO 7 J=NN1,NN2,3
000476  G1=G1+G*XMASS(JJ)*(Y(2)-Y(J+1))/R(Y(1),Y(J),Y(2),Y(J+1),Y(3),
          VY(J+2))^3
000521  7 JJ=JJ+1
000525  F(2)=-G1
000527  F(2)=0.
000530  IF (OMPL(1).EQ.1) 70,71
000534  70 NN1=7 $NN2=19 $ JJ=3
000537  GO TO 72
000540  71 NN1=4 $NN2=31 $ JJ=2
000543  72 G1=0.
000544  DO 8 J=NN1,NN2,3
000546  G1=G1+G*XMASS(JJ)*(Y(3)-Y(J+2))/R(Y(1),Y(J),Y(2),Y(J+1),Y(3),
          CY(J+2))**3
000571  8 JJ=JJ+1
000575  F(3)=-G1
000577  F(3)=0.

ROTATIONAL MOTION OF EARTH

X(34)=BETA PRIME 0
X(35)=BETA PRIME 1
X(36)=BETA PRIME 2
X(37)=BETA PRIME 3

COMPUTE BETA PRIME AND BETA PRIME DOT MATRICES B, D(INVERTED)

000600  T=VTDEP-2400000.5
000602  T=T-33282.+TM
000605  PI=3.14159265358979
000606  TWOPI=2.*PI
000610  AL=AO+AD*T
000613  F(37)=0.
000614  OM(4)=0.
000615  DO 10 J=1,3
000616  OM(J)=0.
000617  DEL(I)=0.
000620  F(I+33)=0.
CONTINUE

IF (NROD(I), EQ. 0) GO TO 1000

DO 100 I = 1, 4

BD(I, I) = V(I, 34)

100 B(I, I) = X(I, 34)

B(1, 2) = X(35)
B(1, 3) = X(36)
B(1, 4) = X(37)
B(2, 3) = X(37)
B(2, 4) = -X(36)
B(3, 4) = X(35)
B(1, 2) = V(I, 35)
B(1, 3) = V(I, 36)
B(1, 4) = V(I, 37)
B(2, 3) = V(I, 37)
B(2, 4) = -V(I, 36)
B(3, 4) = V(I, 35)

DO 101 I = 2, 4

J = I - 1

DO 102 J = 1, JJ

B(J, I) = -B(I, J)

BC(I, J) = -BD(J, I)

102 CONTINUE

DO 104 I = 1, 4

OM(I) = 2.*B(I, J)*V(J+33)+OM(I)

104 CONTINUE

CALCULATE OMEGA 1, 2, 3

FT(1) = 0.
FT(2) = X(35)*X(37) - X(34)*X(36)
FT(3) = X(36)*X(37) + X(34)*X(35)
FT(4) = X(34)**2 - X(35)**2 - X(36)**2 + X(37)**2

FT(2) = FT(2)*2.*AD
FT(3) = FT(3)*2.*AD
FT(4) = FT(4)*AD
OM(2) = OM(2) + FT(2)
OM(3) = OM(3) + FT(3)
OM(4) = OM(4) + FT(4)

CALCULATE MOMENTS ACTING ON EARTH PROJECTED ON BODY AXES Y

CALCULATE DIRECTION COSINES OF MOON WRT EARTH CENTERED AXES,

LITTLE Y, DEL
000750 DAL(1) = X(13) - X(10)
000753 DAL(2) = X(14) - X(11)
000755 DAL(3) = X(15) - X(12)

C CALCULATE BETA DOUBLE PRIME
000760 AL = AMOD(AL,TWOPI)
000763 CA = COS(AL/2.)
000767 SA = SIN(AL/2.)
000773 B1 = CA*X(34) - SA*X(37)
000774 B2 = CA*X(35) - SA*X(36)
000777 B3 = SA*X(35) + CA*X(36)
000778 B4 = SA*X(34) + CA*X(37)

C CALCULATE ELEMENTS OF ROTATION MATRIX C(BETA DOUBLE PRIME)
001013 C(1,1) = B1*B1 + B2*B2 - B3*B3 - B4*B4
001016 C(1,2) = 2.*B2*B3 + B1*B4
001019 C(1,3) = 2.*(B3*B4 + B1*B2)
001022 C(2,1) = 2.*B2*B3 + B1*B4
001025 C(2,2) = B1*B1 - B2*B2 + B3*B3 - B4*B4
001028 C(2,3) = 2.*B3*B4 + B1*B2
001031 C(3,1) = 2.*B2*B4 + B1*B3
001034 C(3,2) = 2.*B3*B4 - B1*B2
001040 DO 105 I = 1, 3
001042 DO 106 J = 1, 3
001045 DEL(I) = C(I,J)*DAL(J) OEL(I)
001050 CONTINUE
001060 RRR = SQRT(DEL(1)**2 + DEL(2)**2 + DEL(3)**2)
001063 DEL(I) = DEL(I)/RRR
001066 DEL(2) = DEL(2)/RRR
001069 DEL(3) = DEL(3)/RRR

C EM1G, EM2G, EM3G ARE GRAVITY GRADIENT TERMS
C EM1, EM2, EM3 ARE ALL OTHER TORQUES
001103 FMI = 0. $ EM2 = 0. $ EM3 = 0.
001106 EM1G = 3.*DEL(2)*DFL(3)*XMASS(5)*ALPF/RRR**3
001109 EM2G = -3.*DEL(1)*DEL(3)*XMASS(5)*BEE/(RRR**3)
001112 EM3G = 3.*DEL(1)*DEL(2)*XMASS(5)*GAF/(RRR**3)

C CALCULATE VALUES OF OMEGA1,2,3 DOT VECTOR
001135 OMD(1) = 0.
001138 OMD(2) = EM1G + EM1/EI1 - ALPE*OM(3)*OM(4)
001141 OMD(3) = EM2G + EM2/EI2 + BEE*OM(2)*OM(4)
001144 OMD(4) = EM3G + EM3/EI3 - GAEG*OM(2)*OM(3)
001147 FD(1) = 0.
001150 FD(2) = X(35)*V(37) + V(35)*X(37) - X(34)*V(36) - V(34)*X(36)
001153 FD(2) = 2.*AD*FD(2)
\[ FD(3) = X(36) \cdot V(37) + V(36) \cdot X(37) + X(34) \cdot V(35) + V(34) \cdot X(35) \]

\[ FD(4) = X(34) \cdot V(34) - X(35) \cdot V(35) - X(36) \cdot V(36) + X(37) \cdot V(37) \]

\[ FO(3) = 2.0 \cdot AD \cdot FD(3) \]

\[ FD(4) = X(34) \cdot V(34) - X(35) \cdot V(35) + X(36) \cdot V(36) - X(37) \cdot V(37) \]

\[ FD(4) = 2.0 \cdot AD \cdot FD(4) \]

\[ C1 \]

\[ O1 \]

\[ D1 \]

\[ O1 \]

\[ O1 \]

\[ 001276 \]

\[ CONTINUE \]

\[ ROTATIONAL MOTION OF MOON \]

\[ X(38) = BETA TRIPLE PRIME 0 \]

\[ X(39) = BETA TRIPLE PRIME 1 \]

\[ X(40) = BETA TRIPLE PRIME 2 \]

\[ X(41) = BETA TRIPLE PRIME 3 \]

\[ CALCULATE BETA TRIPLE PRIME AND BETA TRIPLE PRIME DOT MATRICES \]

\[ BB AND BBD (INVERTED) \]

\[ IF(NOR02.EQ.0) GO TO 2000 \]

\[ DAL(1) = X(13) - X(10) \]

\[ DAL(2) = X(14) - X(11) \]

\[ DAL(3) = X(15) - X(12) \]

\[ NORMALIZATION OF EULER PARAMETERS \]

\[ XNORM = X(38) \cdot X(38) + X(39) \cdot X(39) + X(40) \cdot X(40) + X(41) \cdot X(41) \]

\[ XNORM = \sqrt{XNORM} \]

\[ X(38) = X(38) / XNORM \]

\[ X(39) = X(39) / XNORM \]

\[ X(40) = X(40) / XNORM \]

\[ DO 200 I = 1, 4 \]

\[ BBD(1,1) = V(38) \]

\[ BBD(1,2) = V(39) \]

\[ BBD(1,3) = V(40) \]

\[ BBD(1,4) = V(41) \]

\[ BBD(2,1) = -V(40) \]

\[ BBD(2,2) = V(39) \]

\[ BBD(2,3) = V(40) \]

\[ BBD(2,4) = V(41) \]

\[ BBD(3,1) = V(39) \]

\[ BBD(3,2) = V(40) \]

\[ BBD(3,3) = V(41) \]

\[ BBD(3,4) = -V(40) \]

\[ BBD(4,1) = V(41) \]

\[ BBD(4,2) = -V(40) \]

\[ BBD(4,3) = V(39) \]

\[ BBD(4,4) = V(40) \]
BBD(3,4) = V(39)
DO 201 I = 2, 4
JJ = I - 1
DO 202 J = 1, JJ
BB(I, J) = BB(J, I)
BBD(I, J) = -BBD(J, I)
202 CONTINUE
DO 201 CONTINUE

C  CALCULATE O, OMEGA_{2,3} VECTOR FOR MOON
DO 2011 I = 1, 4
CMM(I) = 0.
DO 203 I = 1, 4
D 204 J = 1, 4
OMM(I) = 2. * BB(I, J) * V(J+37) + OMM(I)
204 CONTINUE
RRR = SORT(DAL(1)**2 + DAL(2)**2 + DAL(3)**2)

C  ANGLE PHI DEFINED FROM -PI/2 TO +PI/2
CC = DAL(1)/RRR
CS = DAL(2)/RRR
SPH = DAL(3)/RRR
CPH = SORT(DAL(1)**2 + DAL(2)**2)/RRR
CL = CC/CPH
SL = CS/CPH
DALV(1) = V(13) - V(10)
DALV(2) = V(14) - V(11)
DALV(3) = V(15) - V(12)
R1 = 0, R2 = RDDT, R3 = RLAMDOTCPH, R4 = PHIDOT
R1 = 0.
R2 = CC * DALV(1) + CS * DALV(2) + SPH * DALV(3)
R3 = -SL * DALV(1) + CL * DALV(2)
R4 = -CL * SPH * DALV(1) - SL * SPH * DALV(2) + CPH * DALV(3)
RT(1) = 0, RT(2) = -AMDOTSINPHI,
RT(3) = PHIDOT, RT(4) = AMDOTCOSPHI
RT(1) = R1
RT(2) = -R3 * SPH/(RRR * CPH)
RT(3) = R4/RRR
RT(4) = R3/RRR
C  CALCULATE AUGMENTED ROTATION MATRIX C(BETA TR. PRIME), D
DO 220 I = 1, 4
DO 205 J = 1, 4
D(I, J) = 0.
CONTINUE

D(2,2) = X(38)**2 + X(39)**2 - X(40)**2 - X(41)**2
D(3,3) = X(38)**2 - X(39)**2 + X(40)**2 - X(41)**2
D(4,4) = X(38)**2 - X(39)**2 - X(40)**2 + X(41)**2
D(2,3) = X(39)**2 - X(40)**2.* X(38)**2.* X(41)**2
D(2,4) = X(39)**2 - X(40)**2.* X(38)**2.* X(41)**2
D(3,4) = X(40)**2 - X(38)**2.* X(39)**2.* X(41)**2
D(4,3) = X(41)**2 - X(38)**2.* X(39)**2.* X(40)**2

CONTINUE

C CALCULATE MOMENTS ACTING ON MOON
RRR3 = RRR**3
RRR2 = RRR**2
AMM1 = X**3
AMM2 = X**2
AMM3 = X
AMM1G = D(4,2) * D(3,2) * XMASS(4) * ALPM / RRR3
AMM2G = D(4,2) * D(2,2) * XMASS(4) * BEM / RRR3
AMM3G = D(3,2) * D(2,2) * XWASS(4) * GAM / RRR3
FACT = (13.1763965268 * 3.14159265358978 / 180.)**2
FACT = FACT * 0.9905 * (0.0025637252**3) / XMASS(4)
AMM1G = AMM1G * FACT
AMM2G = AMM2G * FACT
AMM3G = AMM3G * FACT

C CALCULATE VALUES OF OMEGA1, OMEGA2, OMEGA3 DOT VECTOR.
OMDM(1) = 0.
OMDM(2) = AMM1/AMI1 + AMM1G - ALPM * OMM(3) * OMM(4)
OMDM(3) = AMM2/AMI2 + AMM2G + BEM * OMM(2) * OMM(4)
OMDM(4) = AMM3/AMI3 + AMM3G - GAM * OMM(2) * OMM(3)

C CALCULATE VALUES FOR DDT OF RT(I) IE RD(I)
C CONVERT RT TO O - DOT SINPHI, + PHIDOTLOOTCOSPHI
C CALCULATE TIME DERIVATIVE OF ROTATION MATRIX
RD(1) = 0.
RD(2) = DALV(1) * (R2*SL / RRR2 - CL*RRT(4) / (CPH*RRR))
RD(2) = RD(2) - DALV(2) * (F(13) - F(10)) * SL / RRR
RD(2) = RD(2) + (F(14) - F(11)) * CL / RRR
RD(12) = PD(2)*SPP/CPH + RT(3)*RT(4)/(CPH*CPH)
RD(2) = RD(2) - RD(2)
RD(3) = DALV(1) * (-RT(2)*SL / RRR + R2*CL*SPP / RRR2 - RT(3)*CC / RRR)
RD(3) = DALV(2) * (RT(2)*CL/RRR - RT(3)*CS/RRR + R2*SL*SPP / RRR2) +
1RD(3)
RD(3) = DALV(3) * (-RT(3)*SPP/RRR - R2*CPH/RRR2) + RD(3)
RD(3) = -(F(13) - F(10) * CL * SPH / RRR + RD(3))
RD(3) = -(F(14) - F(11) * SL * SPH / RRR + RD(3))
RD(3) = (F(15) - F(12) * CP * RRR + RD(3))
RD(4) = DALV(11) * [RT(23) + CL / (RRR * SPH)]^2
RD(4) = DALV(2) * (RT(2) * SL / (RRR * SPH))^2
RD(4) = (F(14) - F(11)) * CL / RRR + RD(4)
RD(4) = OALV(i) * (RT(23) * CL / (RRR * SPH))

DO 209 J = 1, 4
DWD(J) = 0.
CONTINUE

208 CONTINUE

210 DO 210 I = 1, 4
FT(I) = 0.
CONTINUE

C CALCULATE ACCELERATIONS

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINUE

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINUE

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINUE

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINUE

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINUE

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
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FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
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DO 210 I = 1, 4
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
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FT(I) = FT(I) + OMDM(I)
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DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINUE

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINUE

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINUE

DO 210 I = 1, 4
FT(I) = FT(I) + OMDM(I)
CONTINU...
002353  217 CONTINUE
002355  216 CONTINUE
002357  2000 RETURN
002360  END
PROGRAM ANEAMO (INPUT, OUTPUT, PUNCH, TAPE2=INPUT, TAPE3=OUTPUT)

ANALYTIC EARTH AND MOON ROTATION AND TRANSLATION

THIS PROGRAM CALLS SUBROUTINES LUTH AND LURD PROVIDING RESULTS
FROM BROWN'S LUNAR THEORY AND ECKHART'S ROTATIONAL THEORY OF THE
MOON'S MOTION AT JULIAN DATES VJIN TO VJF IN INCREMENTS OF VJINC

THIS PROGRAM CALLS SUBROUTINE EARRO WHICH USES STANDARD PRECESSION-
NUTATION FORMULAE AND SIDERAL TIME FORMULAE TO PROVIDE EARTH
ORIENTATION

THIS PROGRAM ALSO PROVIDES I/O FUNCTIONS AND CALLS THE
TRANSFORMATION SUBROUTINES ICOND AND AXANG
AXANG- CONVERTS A ROTATION MATRIX INTO AXIS AND
ANGLE OF ROTATION AND THEN INTO EULER
PARAMETERS
ICOND- CONVERTS EULER PARAMETERS BETA DOUBLE PRIME
INTO PARAMETERS BETA PRIME

ALL ANGLES ARE IN DEGREES, PARALLAX IS IN DEGREES, ALL
COORDINATES ARE IN KILOMETERS
REFERENCES TO MEAN EQUINOX AND ECLIPTIC OF DATE

REFERENCES
1) IMPROVED LUNAR EPHEMERIS (ILE)
2) SAO STANDARD EARTH VOL. 1 L966
3) ECKHART A.J. VOL. 70 NO. 7 P. 466
4) WILLIAMS ET AL. LUNAR PHYSICAL LIBRATIONS AND LASER RANGING

MULTI-CASE OPTION  ICODE=1 READ NEW TITLE AND DATES
STOP

CARD OUTPUT OPTION FOR PARAMETER ESTIMATION

IPT=0 NO CARD OUTPUT
IPT=1 OUTPUT PHYSICAL LIBRATIONS ON CARDS AND OUTPUT INITIAL
VALUES ON PRINTER

CARD OUTPUT CONSISTS OF--
VJD  RHO  SIGMA  TAU
IN FORMAT (1X,4E19.12)
DIMENSION XEC(3), XEQ(3), VV(6), R(3,3), C(3), \$\beta$$\alpha$(4), T(36,2),
1 CC(31,2),
1 RR(31,2), NL(29), NLP(29), NF(29), ND(29), RS(3,3), BETDP(4), SBETA(4)
1, S(3,3), P(3,3), XMO(3,3), BETM(14), BETP(14), BTP(4), N(3,3)
1, LUKK(14), LKOA(14), LKOB(14), PKUK(11), PKOA(11), PKOB(11)
2, PLARG(23), PRLATE(23)
DIMENSION XIND(21), YDEP(21,4), TDER(1), FDER(1,4), DER(1,4),
1, IDER(1,4), WK(420)
2, LATK(26), LATA(26), LATB(26), ADA(12), ADK(12), ADB(12)
REAL LKUK, LKOA, LKOB
COMMON /LUCUN/ NL, NLP, NF, ND, PLARG, PRLATE
COMMON /PERT/ LUKK, LKOA, LKOB, PKUK, PKOA, PKOB
1, LATK, LATA, LATB, ADK, ADA, ADB

PI = 3.14159265358979
DTR = PI/180.
RTD = 180./PI
NSKIP = 0
MNPTS = 21 $ NDER = 21 $ NCVS = 4 $ MMAX = 1 $ MDER = 1 $ \$ I\nu = -1

INPUT

INPUT CONSTANTS FOR ECKHARDT'S THEORY
J IS THE $\beta$-Gamma INDEX- SEE SUBROUTINE LURO

J = 2
READ (2,119) (TT(I,J), I=1,21, J=1,36)
READ (2,119) (CC(L,J), L=1,31)
READ (2,119) (RR(H,J), H=1,31)
READ (2,119) (PLARG(I), I=1,23)
READ (2,120) (PLRATE(I), I=1,23)
DO 6 LL=1,29
6 READ(2,121) NL(LL), NLP(LL), NF(LL), ND(LL)

INPUT OF SMALL PERIODIC TERMS IN LONGITUDE, LATITUDE AND
PARALLAX FROM THE ILE
LONGITUDE LIST (IALPHA)
PARALLAX LIST (IGAMMA)
LATITUDE LIST (IBETA)

INPUT JULIAN DATES, MULTI-CASE CODE AND OUTPUT OPTION
DO 30 I=1,14
READ (2,161) LKOK(I),LKO(A(I),LKO(B(I)
30 CONTINUE
DO 31 I=1,11
READ (2,161) LKOK(I),LKO(A(I),LKO(B(I)
31 CONTINUE
DO 32 I=1,26
READ (2,166) LATK(I),LATA(I),LATB(I)
C INPUT OF ADDITIVE TERMS IN LONGITUDE(I=1,7),NODE(I=8,9)
AND GAMMA(I=10,12)
C
DO 33 I=1,12
READ (2,167) ADK(I),ADA(I),ADB(I)
READ (2,900)
C LUNAR THEORY
CALL LUTH (VJD,VV,DA,DB,DC,XEC,XEQ,TU)
IF(NSKIP.EQ.1) GO TO 20
IF (IFLAG.EQ.1.AND.IPT.EQ.1) GO TO 20
WRITE (3,901)
WRITE (3,900)
WRITE (3,902)
WRITE (3,101)VJD
WRITE (3,102)VV(1),VV(2),VV(3),VV(4),VV(5),VV(6)
C CALCULATE DELAY ARGUMENTS
XL=MOON+S MEAN ANOMALY
XCAPL=SUN+S MEAN LONGITUDE
XLPR=SUN+S MEAN ANOMALY
F=MEAN ANOMALY OF MOON+LUNAR_ARGUMENT_OF_PERIGEE
20 XL=VV(I)-VV(3)
XCAPL=VV(1)-VV(5)
XLPR=XCAPL-VV(2)
F=VV(1)-VV(4)
XL=AMOD(XL,360.)
XLPR=AMOD(XLPR,360.)
XCAPL=AMOD(XCAPL,360.)
C ECKHARDT'S THEORY FOR LUNAR ROTATION

004520  21 CALL EARRD(VJD,R,THETA,S,N,P)
004523  CALL LURD(XL,XLPR,F,VV,P,J,TAU,CI,RHO,CC,RR,TT,RS,XMO,XEO,SLONG
004525                                                   I,SLAT,TU)
004528  IF (NSKIPEQ.1) GO TO 22
004529  IF (IPT=1) 48,49,48
004532  49 PUNCH 50,VJD,RHO,CI,TAU
004534  IF (IFLAG=EQ.00) GO TO 48
004535  GO TO 47
004538  48 IF (NSKIPEQ.1) GO TO 22
004540  WRITE(3,113)
004543  WRITE(3,101) VJD
004546  WRITE(3,111)
004549  WRITE(3,124)
004552  CALL AXANG(RS,DEL,C,BETA)
004555  WRITE(3,123) (BETA(I),I=1,4)
004558  22 CALL AXANG(XMO,DEL,C,BETA)
004561  IF (NSKIPEQ.1) GO TO 23
004564  WRITE(3,160)
004567  WRITE(3,123) (BETA(I),I=1,4)
004569  7 CONTINUE
DO 10 I=1,4
10 SBETA(I)=BETA(I)
SSLON=SLONG $ SSLAT=SLAT

LOOP TO CALCULATE EULER PARAMETER RATES BETA TRIPLE PRIME BY NUMERICAL DIFFERENTIATION.

NSKIP=1
JKJ=1
Iw=-1
DELT=.0005
TDER(1)=VJD $ TSTO=TDER(1)
VJD=VJD-10*DELT
GU TO 2
23 XIND(JKJ)=VJD
DO 24 L=1,4
24 YDEP(JKJL)=BETA(L)
JKJ=JKJ+i
VJD=VJD+DELT
IF (JKJ.GT.21) GO TO 25
GO TO 2
25 CALL SPLDER (MNPTS,NDER,NCVS,MMAX,MDER,XIND,YDEP,TDER,FDER,
1DER1,DER2,IN,WK,IERR)
DO 27 L=1,4
27 BTPD(L)=DER1(1,L)
NSKIP=0
VJD=TSTO
WRITE(3,141)
WRITE(3,142) (BTPD(I),I=1,4)

EULER PARAMETER TESTS
TEST1=SBETA(1)**2+SBETA(2)**2+SBETA(3)**2+SBETA(4)**2
TEST2=SBETA(1)*BTPD(1)+SBETA(2)*BTPD(2)+SBETA(3)*BTPD(3)+
1 SBETA(4)*BTPD(4)
WRITE (3,164)
WRITE (3,165) TEST1,TEST2
WRITE (3,162)
WRITE (3,163) SSLON,SSLAT

PRECESSION-NUTATION CALCULATIONS FOR EARTH ORIENTATION
40 CALL EARRO(VJD,R,THETA,S,N,P)
IF (NSKIP.EQ.1) GO TO 41

WRITE (3,115)
WRITE (3,101) VJD

THETA= THETA*RTD.
WRITE (3,140) THETA

THETA=THETA*DTR
WRITE(3,128)
DO 12 L=1,3
K=1
WRITE (3,129) P(L,K),P(L,K+1),P(L,K+2)

CONTINUE
WRITE(3,150)
DO 13 L=1,3
K=1
WRITE (3,129) N(L,K),N(L,K+1),N(L,K+2)

CONTINUE
WRITE (3,124)
DO 14 L=1,3
K=1
WRITE (3,116) R(L,K),R(L,K+1),R(L,K+2)

CONTINUE
41 CALL AXANG(R,DEL,C,BETA)

IF (NSKIP.EQ.1) GO TO 42
WRITE(3,117)
WRITE (3,118) (BETA(I),I=1,4)

CALL ICOND(VJD, BETA, THETA, S, N, P)

DO 11 I=1,4
SBETA(I)=BETA(I)
IF (NSKIP.EQ.1) GO TO 43
WRITE(3,130)
WRITE (3,118) (BETA(I),I=1,4)

C LOOP TO CALCULATE EULER PARAMETER RATES BETA PRIME DOT BY NUMERICAL DIFFERENTIATION.

NSKIP=1
JKJ=1
DELT=.00005
001302   IW=-1
001303   TDER(1)=VJD $ TSTO=TDER(1)
001305   VJD=VJD-10*DELT
001310   GO TO 40
001311   43 XIND(JKJ)=VJD
001313   DO 45 L=1,4
001315   45 YDEP(JKJ,L)=BETA(L)
001325   JKJ=JKJ+1
001326   VJD=VJD+DELT
001330   IF (JKJ.GT.21) GO TO 44
001333   44 CALL SPLDER(MNPTS,NDER,NCVS,MMAX,MDER,XIND,YDEP,TDER,FDER,
001351     1 DER1,DER2,IVVJWK,IERR)
001351   DO 46 L=1,4
001353   46 BETDP(L)=OERI(1,L)
001361   NSKIP=0
001362   VJD=TSTO
001363   WRITE(3,127)
001366   WRITE(3,126) (BETDP(I),I=1,4)
001367   C EULER PARAMETER TESTS
001401   TEST1=SBETA(1)**2+SBETA(2)**2+SBETA(3)**2+SBETA(4)**2
001407   TEST2=SBETA(1)*BETDP(1)+SBETA(2)*BETDP(2)+SBETA(3)*BETDP(3)
001416   WRITE (3,164)
001421   WRITE (3,165) TEST1,TEST2
001431   47 IF (VJD.GE.VJF) GO TO 3
001434   VJD=VJD+VJINC
001436   IFLAG=1
001437   GO TO 2
001437   3 IF (ICODE.EQ.1) GO TO 60
001441   50 FORMAT (1X,4EI9.12)
001441   100 FORMAT (3E20.10,215)
001441   101 FORMAT (* *,44X,*JULIAN DATE= *,E19.12,/) 
001441   102 FORMAT (* *,*MGRID= *,F10.5,5X,*GAMMA= *,F10.5,5X,*GAMMA PRIME=*, 
001441     1F10.5,/* *,*OMEGA= *,F10.5,5X,*D= *,F10.5,5X,*OBLIQUITY= *,F10.5, 
001441     2///) 
001441   107 FORMAT (50X,*DELAUNAY ARGUMENTS*,/) 
001441   103 FORMAT (* *,*L= *,F10.5,5X,*CAP L= *,F10.5,5X,*L PRIME= *,F10.5,5X 
001441     1,*F= *,F10.5,///) 
001441   108 FORMAT (50X,*ECLIPTIC LONG. AND LAT.*,/) 
001441   104 FORMAT (* *,*LONGITUDE= *,F10.5,5X,*LATITUDE= *,F10.5,5X,*PARALLAX
SUBROUTINE LURO(ARGL, ARGLP, ARGF, VV, P, J, TAU, SIG, RHO, C, R, T, RM, XM0, 
1 XEQ, SLONG, SLAT, TAU)

C
C
C THIS SUBROUTINE PROVIDES THE PHYSICAL LIBRATION IN LONGITUDE (TAU), 
C IN NODE (ISIGMA), AND IN INCLINATION (RHO) FROM ECKHARDT'S LUNAR 
C LIBRATION TABLES—REF: THE MOON VI P264.
C
C ADDITIVE AND PLANETARY TERMS ARE INCLUDED PER REF—
C LUNAR PHYSICAL LIBRATIONS AND LASER RANGING, WILLIAMS, ET AL.

C INPUT
C
T(I, J) = TAU COEFFICIENTS
I = TERM NO. J = COEFFICIENTS (BETA)
C
J = 1 BETA = 0.0006268 GAMMA = 0.0002300 (GT 1 ARC-SEC)
C
J = 2 BETA = 0.00063 GAMMA = 0.0002 + ADDITIVE/PLANETARY
C TERMS (GT 3 ARC-SEC)

C
C ARGL = MEAN ANOMALY OF THE MOON =
C ARGLP = MEAN ANOMALY OF THE SUN =
C ARGF = MOON - OMEGA
C ARGD = MEAN ELONGATION OF MOON FROM SUN (D)
C
C C(I, J) = SIGI COEFFICIENTS
C R(I, J) = RHO COEFFICIENTS

000005 DIMENSION T(36, 2), C(31, 2), R(31, 2), NL(29), NLP(29), NF(29), ND(29)
000005 DIMENSION RM(3, 3), EC(3, 3), RR(3, 3), VV(6), PLARG(23), PLRATE(23)
000005 DIMENSION XMO(3, 3), XEO(3), XMI(3, 3), P(3, 3), XEQ(3)
000005 COMMON/LUCON/NL, NLP, NF, ND, PLARG, PLRATE
000005 ARGD = VV(5)
000006 IF (J.EQ.1) XI = 5521.5
000006 IF (J.EQ.2) XI = 5549.3
000003 PI = 3.14159265358979
000003 TTOP = 2.*PI
000003 RTD = 180./PI
000004 DTR = PI/180.
000005 TTT = TU*36525.
000006 ARGL = ARGL*DTR
000006 ARGLP = ARGLP*DTR
000006 ARGF = ARGF*DTR
000006 ARGD = ARGD*DTR
000006 TAU = 0. S SIGI = 0. S RHO = 0.
000006 DO 1 I = 1, 13
000006 ARG = NL(I)*ARGL + NLP(I)*ARGLP + NF(I)*ARGF + ND(I)*ARGD

C
C C
C
C
TAU = TAU + T(I,J) * SIN(ARG)

APG = NL(I+IDIV) * ARG + NLP(I+IDIV) * ARG + NF(I+IDIV) * ARG + ND(I+IDIV) * ARGD

SIGI = SIGI + C(I,J) * SIN(ARG)

DO 2 I = 1, 8

APG = NL(I+IDIV) * ARG + NLP(I+IDIV) * ARG + NF(I+IDIV) * ARG + ND(I+IDIV) * ARGD

SIGI = SIGI + C(I+8,J) * SIN(AA)

RHO = RHO + R(I,J) * COS(AA)

DO 50 I = 1, 23

AA = PLAGR(I) + PLRATE(I)*TTT

AA = AA * TW01

AA = AMOD(AA, TW01)

IF (AA LT 0.) AA = AA + TW01

TAU = TAU + T(I+13,J) * SIN(AA)

SIGI = SIGI + C(I+8,J) * SIN(AA)

RHO = RHO + R(I+8,J) * COS(AA)

TAU = TAU/3600. $ RHO = RHO/3600. $ SIGI = (SIGI/XI) * RTD

TH = RHO + XI/3600.

PSI = VV(4) + SIG

PSI = AMOD(PSI, 360.) $ PHI = AMOD(PHI, 360.) $ TH = AMOD(TH, 360.)

CPH = COS(PHI) $ SPH = SIN(PHI)

CPS = COS(PSI) $ SPS = SIN(PSI)

CTH = COS(TH) $ STH = SIN(TH)

RM - ECLIPTIC OF DATE TO BODY ROTATION MATRIX

RM(1,1) = CPH*CPS-SPH*CTH*SPS

RM(1,2) = -CPH*SPS-SPH*CTH*CPS $ RM(1,2) = -RM(1,2)

RM(1,3) = -SPH*STH

RM(2,1) = CPS*SPH*CTH*SPS $ RM(2,1) = -RM(2,1)

RM(2,2) = -SPS*SPH*CTH*CPS

RM(2,3) = CPS*STH $ RM(2,3) = -RM(2,3)

RM(3,1) = -SPS*STH

RM(3,2) = CPS*STH

RM(3,3) = CTH
VV(6) = VV(6) * DTR
EC(1,1) = 1. $ EC(1,2) = 0. $ EC(1,3) = 0.
EC(2,1) = 0. $ EC(3,1) = 0. $ EC(2,2) = COS(VV(6))
EC(2,3) = SIN(VV(6)) $ EC(3,2) = - SIN(VV(6)) $ EC(3,3) = COS(VV(6))
DO 12 I = 1, 3
DO 13 L = 1, 3
13 RR(I, L) = 0.
12 CONTINUE
DO 10 I = 1, 3
DO 19 L = 1, 3
19 RR(I, L) = 0.
10 CONTINUE
C
MULTIPLY MATRIX RR BY P TO REFER ANGLES TO MEAN EQUATOR AND
EQUINOX OF 1950.0
RM- MEQEO50 TO BODY ROTATION MATRIX.
C
DO 30 I = 1, 3
DO 31 L = 1, 3
31 RM(I, L) = 0.
30 CONTINUE
DO 32 K = 1, 3
32 RM(I, L) = RR(I, K) * EC(K, L) + RM(I, L)
31 CONTINUE
30 CONTINUE
DO 33 I = 1, 3
33 XE05(I) = 0.
32 CONTINUE
DO 34 K = 1, 3
34 XE05(I) = P(K, I) * XEO(K) + XEO5(I)
33 CONTINUE
DO 35 I = 1, 3
35 XEO(I) = XEO5(I)
34 CONTINUE
C
C CALCULATION OF EULER PARAMETERS FOR ROTATION FROM UP CASE Z
C TO LOW CASE Z FOR USE AS INITIAL CONDITION IN RIGEM
C
RRR = SORT(XE0(1)**2 + XE0(2)**2 + XE0(3)**2)
CC = XEO(1)/RRR
CS = XEO(2)/RRR
SPH = XEO(3)/RRR
CPH = SORT(XE01**2 + XE0(2)**2) / RRR
SL = XEO2/(RRR*CPH)
CL = XEO(1)/(RRR*CPH)

000631 XM1(1,1) = -CL*CPH $ XM1(1,2) = SL $ XM1(1,3) = -CL*SPH
000636 XM1(2,1) = -CPH*SL $ XM1(2,2) = -CL $ XM1(2,3) = -SL*SPH
000641 XM1(3,1) = -SPH $ XM1(3,2) = 0. $ XM1(3,3) = CPH

000644 DO 20 I = 1, 3
000651 DO 21 K = 1, 3
000652 21 XM0(I, K) = 0.
000660 20 CONTINUE
000662 DO 22 I = 1, 3
000663 DO 23 K = 1, 3
000664 DO 24 L = 1, 3
000665 24 XM0(I, K) = XM0(I, K) + RM(I, L) * XM1(L, K)
000704 23 CONTINUE
000706 22 CONTINUE

C CALCULATION OF EARTHS SELENOGRAPHIC COORDINATES

000710 SS = SQRT(XMO(1,1)**2 + XMO(2,1)**2)
000716 SLONG = ATAN2(XMO(2,1), XMO(1,1))
000724 SLAT = ATAN2(XMO(3,1), SS)
000732 SLONG = SLONG * RTD $ SLAT = SLAT * RTD
000735 RETURN
000736 END
SUBROUTINE LUTH (VJD, VV1, OA, OB, OC, XEC(1:3), XEO(1:3), TU)

THIS SUBROUTINE PROVIDES AN APPROXIMATE VERSION OF BROWN'S LUNAR THEORY (REF. 1.5)

INPUT

THE CALLING PROGRAM SHOULD PROVIDE THE JULIAN DATE (VJD)

VARIABLES

VV1=MEAN LONGITUDE OF MOON, MEASURED IN ECLIPTIC FROM MEAN EQUINOX
OF DATE TO MEAN ASC. NODE OF LUNAR ORBIT THEN ALONG ORBIT (DEGREES) (MOON)

VV2=SUN'S MEAN LONGITUDE OF PERIGEE (DEG) (GAMMA)

VV3=MEAN LONGITUDE OF LUNAR PERIGEE, MEASURED IN ECLIPTIC FROM MEAN EQUINOX
OF DATE TO MEAN ASCENDING NODE OF LUNAR ORBIT THEN ALONG ORBIT (DEG) (GAMMA PRIME)

VV4=LONG. OF MEAN ASC. NODE OF LUNAR ORBIT ON ECLIPTIC MEAS. FROM
MEAN EQUINOX OF DATE (DEG) (OMEGA)

VV5=MEAN ELONG. OF MOON FROM SUN XEC(1,2,3)=ECLIPTIC RECTANGULAR
COORDINATES (DEG) (D)

OA, OB, OC=MOUN'S ECLIPTIC LONGITUDE LATITUDE PARALLAX (DEG., DEG., DEG.)

XEO(1,2,3)=EQUATORIAL RECTANGULAR COORDINATES

V6=OBLIQUITY OF THE ECLIPTIC (DEG) (OBLIQUITY)

DIMENSION XEC(3), XEO(3), VV(6), LKOK(14), LKOa(14), LKOB(14),
PKOK(11), PKOA(11), PKOB(11), LATK(26), LATA(26), LATB(26), ADA(12), ADB(12), ADK(12)

REAL LKOK, LKOa, LKOB

COMMON/PERT/LKOK, LKOa, LKOB, PKOK, PKOA, PKOB,
LATK, LATA, LATB, ADA, ADB

DMSTR(D, VM, S) = 3.141592653589793 / 180. \* (D + (VM + S/60.) / 60. )

PI = 3.141592653589793

TU = (VJD - 2415020. ) / 36525.

TU2 = TU * TU

TU3 = TU2 * TU

COR1 = 1336. \* 2. \* PI

COR2 = 11. \* 2. \* PI

COR3 = 5. \* 2. \* PI

COR4 = 1236. \* 2. \* PI

V1 = DMSTR(270., 26., 3.69) + (COR1 + DMSTR(307., 52., 59.31
101)) \* TU - DMSTR(0., 0., 4.08) \* TU2 + DMSTR(0., 0., 0068) \* TU3

V2 = DMSTR(281., 13., 15., 15.) + DMSTR(0., 0., 0., 6189.03) \* TU
1 + DMSTR(0., 0., 1.63) * TU2 + DMSTR(0., 0., 0.012) * TU3

V3 = DMSTR(334., 19., 46.75) + (CORR + DMSTR(109., 2., 2.52))

1 + TU - DMSTR(0., 0., 37.17) * TU2 - DMSTR(0., 0., 0.045) * TU3

V4 = DMSTR(259., 10., 59.79) - (CORR + DMSTR(134., 8., 31.23))

V5 = DMSTR(350., 44., 15.65) + (CORR + DMSTR(307., 6., 51.18))

1 + TU - DMSTR(0., 0., 5.17) * TU2 + DMSTR(0., 0., 0.008) * TU3

V6 = DMSTR(23., 27., 8.26) - DMSTR(0., 0., 46.845) * TU - DMSTR(0., 0., 0.0059) * TU2 - DMSTR(0., 0., 0.0181) * TU3

V1 = V1 * 180. / PI
V2 = V2 * 180. / PI
V3 = V3 * 180. / PI
V4 = V4 * 180. / PI
V5 = V5 * 180. / PI
V6 = V6 * 180. / PI

R360 = 360.

V1 = AMOD(V1, 360.)
V2 = AMOD(V2, 360.)
V3 = AMOD(V3, 360.)
V4 = AMOD(V4, 360.)
V5 = AMOD(V5, 360.)
V6 = AMOD(V6, 360.)

IF(V1.LE.0.) V1 = V1 + 360.
IF(V2.LE.0.) V2 = V2 + 360.
IF(V3.LE.0.) V3 = V3 + 360.
IF(V4.LE.0.) V4 = V4 + 360.
IF(V5.LE.0.) V5 = V5 + 360.

VV(1) = V1
$ VV(2) = V2
$ VV(3) = V3
$ VV(4) = V4
$ VV(5) = V5
$ VV(6) = V6

V1 = V1 * PI / 180.
V2 = V2 * PI / 180.
V3 = V3 * PI / 180.
V4 = V4 * PI / 180.
V5 = V5 * PI / 180.
V6 = V6 * PI / 180.

C CALCULATION OF ADDITIVE TERMS

TE1 = .53733431 - (10104982. E-12) * TU * 36525.

1 + 191. E-16 * TU * TU * 36525. **2

TE1 = TE1 * 2. * PI

AT = 14.27 * SIN(TE1) * PI / (3600. * 180.)

TE2 = .71995354 - (147094228. E-12) * TU * 36525.

1 + 43. E-16 * TU * TU * 36525. **2

TE2 = TE2 * 2. * PI

A0 = 95.96 * SIN(TE2) * PI / (3600. * 180. * 0)

TE3 = .48398132 - (147269147. E-12) * TU * 36525.

1 + 43. E-16 * TU * TU * 36525. **2
TE3 = TE3 * 2 * PI
000522
ATUI = 15.58 * SIN(TE3) * PI / (3600 * 180)
000531
TE4 = 71995354 - (147094228.E-12) * TU * 36525 * 1 + (43.E-16) * TU * TU * 36525 ** 2
000541
TE4 = TE4 * 2 * PI
000543
ATLI = 7.261 * SIN(TE4) * PI / (3600 * 180)
000552
TE5 = 52453688 - (147162675.E-12) * TU * 36525 * 1 + (43.E-16) * TU * TU * 36525 ** 2
000562
TE5 = TE5 * 2 * PI
000564
ATL2 = 1.86 * SIN(TE5) * PI / (3600 * 180)
000573
ATL2 = 0. $ TO3 = 0. $ DGC = 0.
000577
DO 13 I = 1, 17
000590
ADARG = (ADA(I) + ADB(I) * TU * 36525) * 2 * PI
000612
ATL2 = ATL2 + ADK(I) * SIN(ADARG)
000626
DO 14 I = 8, 9
000640
ADARG = (ADA(I) + ADB(I) * TU * 36525) * 2 * PI
000662
AT03 = AT03 + SIN(ADARG) * ADK(I)
000685
DO 15 I = 10, 12
000708
DGC = DGC + ADK(I) * COS(ADARG)
000731
VI = VI + ATL1 + ATL2
000754
V4 = V4 + AT0 + AT01 + AT02 + AT03

C  CALCULATION OF PERIODIC TERMS
C

000704
VA = VI - V3
000706
V8 = VI - V5
000710
VC = V8 - V2
000712
VD = V1 - V4
000714
V5S = SIN(V5)
000716
V5C = COS(V5)
000723
V52 = 2 * V5
000725
V52S = SIN(V52)
000727
V52C = COS(V52)
000732
V54S = SIN(V54)
000734
V54C = COS(V54)
000736
VAS = SIN(VA)
000740
VAC = COS(VA)
000742
VA2 = 2 * VA
000745
VA2S = SIN(VA2)
000747
VA2C = COS(VA2)
000751
VA3 = 3 * VA
000754  VA3S = SIN(VA3)
000756  VA3C = COS(VA3)
000760  VBS = SIN(VB)
000762  VBC = COS(VB)
000764  VCS = SIN(VC)
000766  VCC = COS(VC)
000770  VDS = SIN(VD)
000772  VDC = COS(VD)
000774  VD2 = 2*VD
000777  VD2S = SIN(VD2)
010001  VD2C = COS(VD2)
010003  V111 = VA - V52
010005  V11S = SIN(V111)
010007  V11C = COS(V111)
010011  V121 = VA2 - V52
010013  V12S = SIN(V121)
010015  V131 = VA + VC - V52
010020  V13S = SIN(V131)
010022  V13C = COS(V131)
010024  V141 = VA + V52
010026  V14S = SIN(V141)
010030  V14C = COS(2*VA + V52)
010036  V14CC = COS(V141)
010040  V151 = VC - V52
010042  V15S = SIN(V151)
010044  V15C = COS(V151)
010046  V161 = VA - VC
010050  V16S = SIN(V161)
010052  V16C = COS(V161)
010054  V171 = VA + VC
010056  V17S = SIN(V171)
010060  V17C = COS(V171)
010062  V181 = VD2 - V52
010064  V18S = SIN(V181)
010066  V191 = VA + VD2
010070  V19S = SIN(V191)
010072  V211 = VA - VD2
010074  V21C = COS(V211)
010076  V21S = SIN(V211)
011000  V311 = VA - V54
011002  V31S = SIN(V311)
011004  V31C = COS(V311)
011006  V411 = VA2 - V5*4
V41S = \sin(V41)
V51S = \sin(V51)
V61S = \sin(V61)
V71S = \sin(V71)
V81S = \sin(V81)
V91S = \sin(V91)
V10S = \sin(V10)
V11S = \sin(V11)
V12S = \sin(V12)
V13S = \sin(V13)
V14S = \sin(V14)
V15S = \sin(V15)
V16S = \sin(V16)
V17S = \sin(V17)
V18S = \sin(V18)
V19S = \sin(V19)
V20S = \sin(V20)
V21S = \sin(V21)
V22S = \sin(V22)
V23S = \sin(V23)
V24S = \sin(V24)
V25S = \sin(V25)
V26S = \sin(V26)
V27S = \sin(V27)
V28S = \sin(V28)
V29S = \sin(V29)
V30S = \sin(V30)
V31S = \sin(V31)
V32S = \sin(V32)
V33S = \sin(V33)
V34S = \sin(V34)
V35S = \sin(V35)
V36S = \sin(V36)
V37S = \sin(V37)
V38S = \sin(V38)
V39S = \sin(V39)
V40S = \sin(V40)
V41S = \sin(V41)
V42S = \sin(V42)
V43S = \sin(V43)
V44S = \sin(V44)
V45S = \sin(V45)
V46S = \sin(V46)
V47S = \sin(V47)
V48S = \sin(V48)
V49S = \sin(V49)
V50S = \sin(V50)
V51S = \sin(V51)
V52S = \sin(V52)
V53S = \sin(V53)
V54S = \sin(V54)
A11S = \sin(A11)
A12S = \sin(A12)
A13S = \sin(A13)
A14S = \sin(A14)
001251 A15 = VA3 - V52
001253 A15S = SIN(A15)
001255 V61C = COS(V611)
001257 V12C = COS(V121)
001261 V41C = COS(V141)
001263 A12 = COS(A12)
001265 V51C = COS(V511)
001267 AA1 = VA2 - VC $ AA2 = VA2 + VC $ AA3 = VA3 - V52
001275 AA1C = COS(AA1) $ AA2C = COS(AA2) $ AA3C = COS(AA3)
001303 AA4 = VC + V5 $ AA5 = VA + V5 $ AA6 = VD2 - V52
001311 AA4C = COS(AA4) $ AA5C = COS(AA5) $ AA6C = COS(AA6)
001341 OA = OA/3600. $ OA = OA + 180.
001343 OA = OA + V1
001345 OC = 3422.70 + 186.5398 * VAC + 34.3117 * V11C + 28.2333 * V52C
001351 PKARG = (PKOA(I) + PKOB(I) * T1 * 36525.) * 2. * PI
001377 OC = OC + PK0K*I) * COS(PKARG)
001398 OC = OC * OC * OC / (6. * 206265. ** 2)
001408 UC = OC + OC3
001412 ALKAR = (LKOA(I) + LKUB(I) * T1 * 36525.) * 2. * PI
001426 DO 10 I = 1, L
001431 OB = (OA + V1)
001440 OA = (OA + V1)
001452 OC = OC + V52S + 192.72 $ V12S - 52.53 $ V12S - 34.72
001460 OC = OC + V52S - 112.79 $ V52S + 237.36 $ V14S + 126.98 $ V15S - 165.06 $ V15S
001463 OC = OC + V52S + 192.72 $ V12S - 52.53 $ V12S - 34.72
001466 OC = OC + V52S - 112.79 $ V52S + 237.36 $ V14S + 126.98 $ V15S - 165.06 $ V15S
001478 OC = OC + V52S + 192.72 $ V12S - 52.53 $ V12S - 34.72
001482 OC = OC + V52S - 112.79 $ V52S + 237.36 $ V14S + 126.98 $ V15S - 165.06 $ V15S
**C**  **CALCULATION OF ECLIPTIC AND MEAN EQUINOX OF DATE COORDINATES**

002043  HPX=HP*OBC*OAC
002045  HPY=HP*OBC*DAS
002050  HPZ=HP*UBS
002052  XEC(1)=HPX  $  XEC(2)=HPY  $  XEC(3)=HPZ
002056  V6S=  SIN(V6)
002060  V6C=  COS(V6)

**C**  **CALCULATION OF MEAN EQUATOR AND EQUINOX OF DATE COORDINATES**

002062  RADGX=HPX
002064  RADGY=HPY*V6C-HPZ*V6S
002070  RADGZ=( HPY)*V6S+HPZ*V6C
002072  XEQ(1)=RADGX $ XEQ(2)=RADGY $ XEQ(3)=RADGZ
002100  OA=UA*180. / PI
002103  OB=OB*180. / PI
002106  OC=OC*180. / PI
002109  OA<AMOD(0A,R360)
002112  OB<AMOD(0B,R360)
002115  OC<AMOD(0C,R360)
002118  RETURN
002121  END

OA=AMOD(OA,R360)
OB=AMOD(0B,R360)
OC=AMOD(0C,R360)
SUBROUTINE EARRO(VJD,R,THETA,S,N,P)

C THIS SUBROUTINE UTILIZES NEWCOMS ANALYTICAL EXPRESSIONS TO CALCULATE
C ROTATION MATRIX S. INPUT IS THE JULIAN DATE VJD. OUTPUT CONSISTS OF
C THE DIRECTION COSINES RELATING THE X AND W AXIS SYSTEMS OF THE
C REFERENCE IN THE FORM X = (SNP)W

REF: SAO STANDARD EARTH 1966

REAL N(3,3)
DIMENSION S(3,3), P(3,3), R(3,3)
PI=3.14159265358979
TWOP=2.*PI
DTR=PI/180.
RTD=180./PI

CALCULATE MODIFIED JULIAN DATE AND S

XMJD=VJD-2400000.5
T=XMJD-3282.0
T2=T*T
ARG1=(12.1128 - .052954 *T)*DTR
ARG2=ARG1*2.
ARG3=(280.0812 + .985647 *T)*DTR*2.
ARG4=(64.3824 + 13.1763 *T)*DTR*2.
ARG1=AMUD(ARG1,TWOPI)
ARG2=AMOD(ARG2,TWOPI)
ARG3=AMUD(ARG3,TWOPI)
ARG4=AMOD(ARG4,TWOPI)
THETA= (100.075542 +360.985647348 *T+.29E-12*T2-4.392E-3*
ASIN(ARG1)+.053E-3* SIN(ARG2)-.325E-3* SIN(ARG3)-.05E-3*
BSIN(ARG4)*DTR
THETA=AMOD(THETA,TWOPI)

CTH=COS(THETA)
STH=SIN(THETA)
S(1,1)=CTH
S(2,2)=CTH
S(1,2)=STH
S(2,1)=-STH
S(3,1)=0.
S(3,2)=0.
S(3,3)=1.
S(2,3)=0.
R1=ARG1

CALCULATE NUTATION MATRIX N
\[ \begin{align*}
R_2 &= \text{ARG}3 \\
R_3 &= \text{ARG}4 \\
\text{DELNU} &= -76.7 \times 10^{-6} \sin(R_1) + 9 \times 10^{-6} \sin(2R_1) - 5.7 \times 10^{-6} \sin(R_2) - 9 \times 10^{-6} \sin(R_3) \\
\text{DELMU} &= -33.3 \times 10^{-6} \sin(R_1) + 4 \times 10^{-6} \sin(2R_1) - 2.5 \times 10^{-6} \sin(R_2) - 4 \times 10^{-6} \sin(R_3) \\
\text{DELEP} &= 44.7 \times 10^{-6} \cos(R_1) - 4 \times 10^{-6} \cos(2R_1) + 2.7 \times 10^{-6} \cos(R_2) + 4 \times 10^{-6} \cos(R_3) \\
\text{CNU} &= \cos(\text{DELNU}) \\
\text{CMU} &= \cos(\text{DELMU}) \\
\text{CEP} &= \cos(\text{DELEP}) \\
\text{N}(1,1) &= \text{CNU} \times \text{CMU} \\
\text{N}(1,2) &= \text{CNU} \times \text{SMU} \\
\text{N}(1,3) &= -\text{SNU} \\
\text{N}(2,1) &= \text{CMU} \times \text{SEP} \times \text{SNU} - \text{SMU} \times \text{CEP} \\
\text{N}(2,2) &= \text{SMU} \times \text{SEP} \times \text{SNU} + \text{CEP} \times \text{CMU} \\
\text{N}(2,3) &= \text{SEP} \times \text{CNU} \\
\text{N}(3,1) &= \text{CMU} \times \text{SNU} \times \text{CEP} + \text{SEP} \times \text{SMU} \\
\text{N}(3,2) &= \text{SMU} \times \text{SNU} \times \text{CEP} - \text{SEP} \times \text{CMU} \\
\text{N}(3,3) &= \text{CEP} \times \text{CNU}
\end{align*} \]

**C**

**CALCULATE PRECESSION MATRIX, P**

\[ \begin{align*}
\text{XKAP} &= 0.063107 \times T \times \text{DTR}/3600. \\
\text{UMEG} &= 0.063107 \times T \times \text{DTR}/3600. \\
\text{XNU} &= 0.0548757 \times T \times \text{DTR}/3600. \\
\text{P}(1,1) &= \sin(\text{XKAP}) \times \sin(\text{UMEG}) + \cos(\text{XKAP}) \times \cos(\text{UMEG}) \times \cos(\text{XNU}) \\
\text{P}(1,2) &= -\cos(\text{XKAP}) \times \sin(\text{UMEG}) - \sin(\text{XKAP}) \times \cos(\text{UMEG}) \times \cos(\text{XNU}) \\
\text{P}(1,3) &= -\cos(\text{UMEG}) \times \sin(\text{XNU}) \\
\text{P}(2,1) &= \sin(\text{XKAP}) \times \cos(\text{UMEG}) + \cos(\text{XKAP}) \times \sin(\text{UMEG}) \times \cos(\text{XNU}) \\
\text{P}(2,2) &= \cos(\text{XKAP}) \times \cos(\text{UMEG}) - \sin(\text{XKAP}) \times \sin(\text{UMEG}) \times \cos(\text{XNU}) \\
\text{P}(2,3) &= -\sin(\text{UMEG}) \times \sin(\text{XNU}) \\
\text{P}(3,1) &= \cos(\text{XKAP}) \times \sin(\text{XNU}) \\
\text{P}(3,2) &= -\sin(\text{XKAP}) \times \sin(\text{XNU}) \\
\text{P}(3,3) &= \cos(\text{XNU})
\end{align*} \]

**C**

**CALCULATE FINAL ROTATION MATRIX, R**

\[ \begin{align*}
\text{DO } 1 & \text{ K=1,3} \\
\text{DO } 2 & \text{ J=1,3} \\
\text{R}(K,J) &= 0. \\
\text{DO } 3 & \text{ I=1,3} \\
\text{DO } 4 & \text{ L=1,3} \\
\text{R}(K,J) &= \text{R}(K,J) + S(K,I) \times N(I,L) \times \text{P}(L,J) \\
\text{CONTINUE}
\end{align*} \]
000675  3 CONTINUE
000677  2 CONTINUE
000701  1 CONTINUE
000703  RETURN
000704  END
SUBROUTINE ICOND(VJD, BETA, THETA, S, N, P)

C THIS SUBROUTINE PROVIDES THE TRANSFORMATION FROM BETA
DOUBLE PRIME EULER PARAMETERS TO BETA PRIME PARAMETERS

BETA ENTERS AS BETA DOUBLE PRIME
BETA RETURNS AS BETA PRIME

DIMENSION BETA(4), B(4,4), BT(4,4), BETAS(4), OM(4), BETAD(4), BETDP(4)

REAL N(3,3)

XMJD=VJD-2400000.5
T=XMJD-33282.
PI=3.14159265358979
TWOPI=2.*PI
DTR=PI/180.
RTD=180.*PI

BETA MATRIX, B

ALPHA=(100.075542 +360.985647348 *T)*DTR
ALPHA=AMOD(ALPHA, TWOPI)
ALPHA=ALPHA/2.
CA2= COS(ALPHA)
SA2= SIN(ALPHA)
B(1,1)=CA2 $ B(1,2)=0. $ B(1,3)=0. $ B(1,4)=-SA2
B(2,1)=0. $ B(2,2)=CA2 $ B(2,3)=-SA2 $ B(2,4)=0. 
B(3,1)=0. $ B(3,2)=SA2 $ B(3,3)=CA2 $ B(3,4)=0. 
B(4,1)=SA2 $ B(4,2)=0. $ B(4,3)=0. $ B(4,4)=CA2

TRANSPOSE B TO GET BETA INVERSE, BT

DO 5 I=1,4
5 BETAS(I)=BETA(I)

DO 1 I=1,4
DO 2 J=1,4
2 BT(I,J)=B(J,I)
1 CONTINUE

CALCULATE BETA PRIME
BETA(I) = 0.

DO 4 J = 1, 4

4 BETA(I) = BETA(I) + BT(I, J) * BETAS(J)

3 CONTINUE

RETURN

END
SUBROUTINE AXANG(R, DEL, C, BETA)

C  THIS SUBROUTINE CALCULATES THE AXIS AND ANGLE OF REVOLUTION FOR
C AND ROTATION MATRIX R. IT ALSO PROVIDES THE EULER PARAMETERS
C BETA
C
C REF: KORN AND KORN

000007 DIMENSION R(3,3), C(3), BETA(4)
000007 PI=3.14159265358979
000010 TP=R(1,1)+R(2,2)+R(3,3)
000013 CDEL=(TP-1. )/2.
000016 SDEL= SQRT(1. -CDEL*CDEL)
000022 DEL= ATAN2(SDEL, CDEL)

C DEL IS SUPPLIED IN RANGE ZERO TO PI

000031 C(1)=(R(3,2)-R(2,3))/2. *SDEL
000035 C(2)=(R(1,3)-R(3,1))/2. *SDEL
000041 C(3)=(R(2,1)-R(1,2))/2. *SDEL
000045 C(1)=-C(1) $ C(2)=-C(2) $ C(3)=-C(3)

C SEQUENCE TO KEEP CALCULATED ROTATION AXIS GENERALLY ALIGNED WITH
C BODY ROTATION AXIS

000051 IF (C(3).GE.0. ) GO TO 1
000052 C(1)=-C(1) $ C(2)=-C(2) $ C(3)=-C(3)
000056 DEL=2. *PI-DEL
000060 DEL=DEL/2.

C CALCULATE EULER PARAMETERS

000062 BETA(1)= COS(DEL)
000067 BETA(2)=C(1)* SIN(DEL)
000075 BETA(3)=C(2)* SIN(DEL)
000103 BETA(4)=C(3)* SIN(DEL)

000111 RETURN
000112 END