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A COMPARISON OF MATRIX METHODS FOR CALCULATING EIGENVALUES IN ACOUSTICALLY LINED DUCTS

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SUMMARY

Three approximate methods – finite differences, weighted residuals, and finite elements – have been used to solve the eigenvalue problem which arises in finding the acoustic modes and propagation constants in an absorptively lined two-dimensional duct without airflow. The matrix equations derived for each of these methods were solved for the eigenvalues corresponding to various values of wall impedance. Two matrix orders, 20×20 and 40×40, were used. The cases considered included values of wall admittance for which exact eigenvalues were known and for which several nearly equal roots were present. Ten of the lower order eigenvalues obtained from the three approximate methods were compared with solutions calculated from the exact characteristic equation in order to make an assessment of the relative accuracy and reliability of the three methods. The best results were given by the finite-element method using a cubic polynomial. Excellent accuracy was consistently obtained, even for nearly equal eigenvalues, by using a 20×20 order matrix.

INTRODUCTION

There is considerable interest in developing reliable methods for calculating the propagation of sound along aircraft engine inlet and exhaust ducts. Research is being directed toward identifying and implementing techniques which can account for turbomachinery noise source distributions, variable and irregular distributions of cross-sectional area, realistic representations of flow fields including boundary layers and transonic flows, and variable wall impedance. The inclusion of these effects is necessary to improve current methods for predicting aircraft flyover noise, for designing acoustic liners carefully matched to the sound source for optimal attenuation, and for understanding the acoustic behavior of inlets containing high subsonic Mach number airflow. The purpose of this paper is to make an initial attempt to assess the relative merits of several currently used methods by solving the same problem by use of these methods. The methods to be considered are: finite difference, method of weighted residuals, and finite elements.

The finite-difference approach has been widely used for numerically analyzing sound propagation in ducts. Alfredson (ref. 1) used the method to analyze sound propagation within a closed circular cylinder driven at one end by a piston. He also analyzed the sound field produced at a right angle bend in a rectangular duct treated with sound-absorbing material. Baumeister and Bittner (ref. 2) use the method for calculating sound propagation in a one-dimensional hard-wall duct and a two-dimensional soft-wall rectangular duct for no mean flow. Baumeister and Rice (ref. 3) extended the method of reference 2 to include a uniform mean flow. Baumeister (ref. 4), in an attempt to reduce the size of the matrices used in reference 2, introduced a wave envelope method which reduced the matrix size of the conventional finite-difference method by an order of magnitude under certain limitations. Baumeister (ref. 5) removed some of the limitations of reference 4 and extended the method to include stepped noise source profiles and stepped axial impedance. Quinn (ref. 6) used the finite-difference approach to include variable cross-sectional area distributions.

The weighted residual approach, in contrast to the finite-difference approach, has been used in the literature mainly for calculating eigenvalues and mode shapes. The approach is especially well adapted for handling smoothly varying axial impedances and noise source profiles. Eversman et al. (ref. 7) use the weighted residual approach to calculate the modes in a nonuniform two-dimensional duct without flow. Results for several duct configurations were compared with a variational method, a stepped duct approximation, and an eigenfunction expansion method based on linearly tapered duct segments. Unruh and Eversman (refs. 8 and 9) applied the weighted residual approach to rectangular ducts of various widths having both hard and acoustically treated walls. A shear flow was included in the analysis. Convergence of the method was checked by investigating limiting cases of the rectangular duct for which exact results were known. They attempted to ascertain the applicability of the method to more complicated geometries.

The application of the finite-element method to acoustic-related problems has been very limited. Gladwell (ref. 10) used the method in conjunction with a variational approach to calculate the natural frequencies of the one- and two-dimensional Helmholtz equation. Both linear and cubic shape functions were used and rectangular elements were employed. The effects of nonrigid walls were not considered in the analysis. Arlett et al. (ref. 11) used the method to calculate the natural frequencies of the two- and three-dimensional Helmholtz equation. Tetrahedrons and rectangles were used as elements and only linear shape functions were considered. The results were compared with the conventional finite-difference technique which it was regarded to supersede. Here again, the effects of nonrigid walls were not considered. T. Shuku and K. Ishihara (ref. 12) use the finite-element method in conjunction with a variational approach to calculate the normal frequencies and modes of an irregularly shaped room. Triangular elements and cubic shape functions were used. Only rigid walls were considered in the analysis. Dean (ref. 13) used the finite-element method to study the generation and propagation of small-amplitude acoustic waves in a homogeneous, loss-free, compressible fluid. Isoparametric finite elements with curvilinear boundaries were used. Steady-state pressure distributions were found for a hemispherical region chosen to represent an infinite half space. Boundary reflections were suppressed through use of a radiation condition. The results were compared with exact solutions. Craggs (ref. 14) used the finite-element method to study the behavior of a coupled plate acoustic cavity system. Rectangular elements and cubic shape functions were used. Kapur and Munger (ref. 15) applied the method in conjunction with a Galerkin method to derive a solution of the basic equations of acoustics in a very general form. The resulting matrix equations included the effects of nonrigid boundaries and shear flow. No calculations were presented.

The numerical methods which are studied in this paper include finite differences, finite elements, and a weighted residual approach. These methods have been selected because they are useful not only for calculating eigenvalues and eigenvectors, but they can be generalized for solving propagation problems involving variable geometry, complicated mean flow fields, and variable wall impedance. Other analytical procedures with this dual capability, such as the wave envelope method, have not been treated in order to limit the scope of the present investigation. Moreover, since the emphasis here is on methods which can be used for propagation problems, various techniques which are useful only for calculating eigenvalues and eigenvectors, such as the integration of differential equations, shooting methods, or functional minimization, are not included.

The methods considered differ in the types of approximations used, the processes for deriving final equations, and the unknowns which must be calculated. Hence, it is useful to try to assess their relative strengths and weaknesses. Such information would be valuable in selecting a "best" method on which to focus effort for continued development and for pinpointing problems with one method which do not occur in the use of another method. The objective of this paper is to make an initial attempt at developing such information for arriving at an assessment of relative merits by applying all three methods, simultaneously, to the solution of the same problem. The problem that will be used is the one-dimensional eigenvalue problem which governs the eigenvalues and eigenfunctions in an acoustically treated rectangular duct.

Among the criteria which must be considered in selecting a best method are: computer storage requirements, computation time, accuracy, and reliability. The first two criteria will be touched on qualitatively by observing the structure of the final matrix equations which must be solved. Primary emphasis will be on the reliability and the accuracy. Accuracy will be ascertained by comparing the first 10 eigenvalues obtained by each method with eigenvalues obtained from the exact characteristic equation. Reliability will be measured by the ability of the methods to separate and calculate nearly equal eigenvalues and to give accurate results consistently for a wide selection of impedance values and frequencies. The simultaneous comparison of calculated results, although not an infallible basis for judging a best method, gives considerable insight into the behavior to be expected from each procedure under a wide variety of circumstances.

SYMBOLS

A,B	square matrices
$\overline{A}_m, \overline{A}_n$	vectors of unknown parameters
$\bar{a}_i, \bar{b}_i$	polynomial coefficients defined after equation (12)
b	duct width
c	ambient speed of sound in duct
F(y)	transverse acoustic pressure
$G(\lambda_0, \tau_0, \tau_1)$	$= \left(\tau_1\tau_0 + \lambda^2\right) \sin \lambda + i\left(\tau_1 + \tau_0\right)\lambda \cos \lambda$
$\bar{I}$	functional
Im( )	imaginary part of variable
i	$= \sqrt{-1}$
K	$= \omega/c$
K_x, K_y	propagation constants in x- and y-direction, respectively
m,n,r,s	integers
p	acoustic pressure
$\bar{p}$	steady-state acoustic pressure
t	time
x,y	distances along X- and Y-axis, respectively

α_j	arbitrary parameters
β_0, β_1	acoustic admittance of lower and upper wall, respectively
γ_n	$= n\pi \quad (n = 0, 1, 2, \ldots)$
Δ	length of subdivision of interval (0,1)
δ	first variation
η	$= Y/b$
λ	$= bK_y$
μ	eigenvalue parameter
ρ_0	ambient density of air
τ_0, τ_1	$= \rho_0 cKb\beta_0$ and $= \rho_0 cKb\beta_1$, respectively
ω	frequency

Primes denote derivatives.

STATEMENT OF THE PROBLEM

The duct to be analyzed and the Cartesian coordinate system to be used are shown in figure 1. The duct is a two-dimensional, infinite, parallel-plate wave guide of width b. The X-axis points along the length of the duct. The transverse coordinate y has its origin at the lower duct wall so that the interior of the duct corresponds to the interval $0 \leqq y \leqq b$. Airflow through the duct is not considered. The upper and lower walls of the duct are acoustically treated. The acoustic admittance of the lower wall (y = 0) is β_0; the acoustic admittance of the upper wall (y = b) is β_1.

The propagation of sound in such a treated duct can be expressed in terms of a set of functions called "normal modes" or "characteristic functions." These functions are building blocks for superimposing more general and more complicated sound fields. Associated with each characteristic function is a "characteristic number" or "eigenvalue" which must be found in order to compute the shape of the characteristic function and describe its propagation and decay along the duct. The eigenvalues for the duct in figure 1 are calcu-

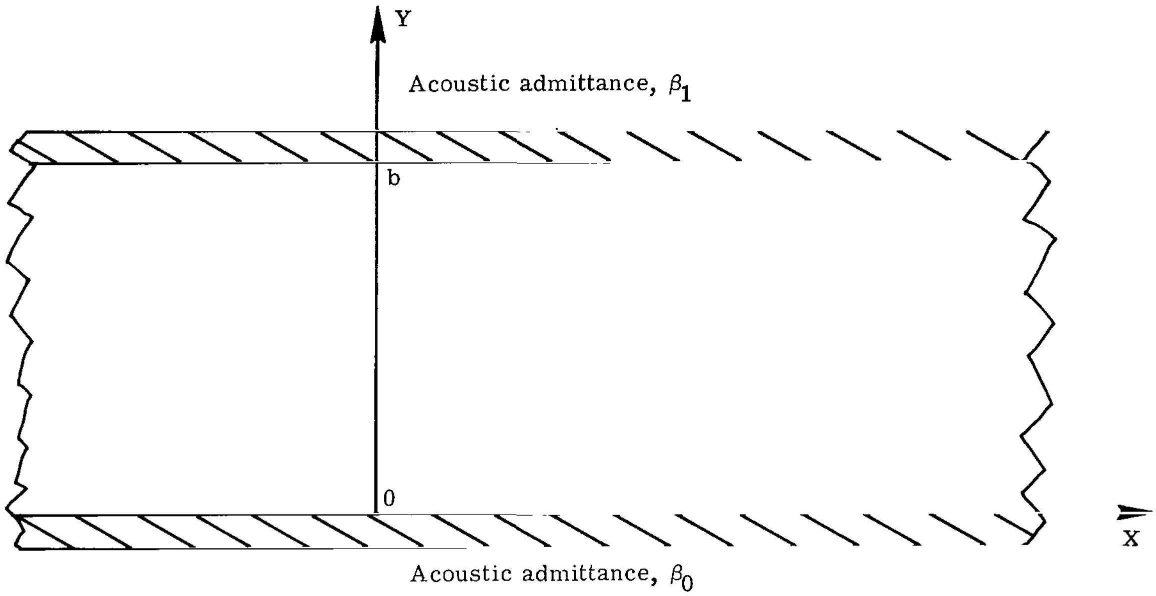

Figure 1.- Infinite parallel-plate wave guide and coordinate system.

lated by several numerical methods and the results are compared to assess the relative merits of these different methods.

ANALYSIS

Governing Equation and Boundary Conditions

The equations of momentum, continuity, and state for the propagation of sound in a perfect gas inside the duct can be combined in the case of no mean flow, to yield the linear two-dimensional wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{1}$$

in which p is the acoustic pressure, and c is the ambient speed of sound inside of the duct. The assumptions made in the derivation of equation (1) are given in most acoustic texts and need not be discussed here.

The characteristic functions of the duct are steady-state solutions to equation (1). Therefore, it is assumed that

$$p(x,y,t) = \bar{p}(x,y)\, e^{-i\omega t}$$

Equation (1) now becomes

$$\frac{\partial^2\bar{p}}{\partial x^2} + \frac{\partial^2\bar{p}}{\partial y^2} + K^2\bar{p} = 0$$

in which $K = \omega/c$. Since the duct under analysis is infinitely long (this leads to no reflected waves), it is convenient to separate out the x-dependence of the solution by assuming that

$$\bar{p}(x,y) = F(y)\, e^{iK_x x}$$

where K_x is a complex propagation constant in the x-direction. This procedure leads to the ordinary differential equation

$$\frac{d^2F}{dy^2} + K_y{}^2F = 0 \qquad (2)$$

where $K_y{}^2 = K^2 - K_x{}^2$.

The boundary condition relates the pressure and pressure gradients at each treated wall to the specific acoustic admittance there. At the lower wall ($y = 0$ plane), the following equation must hold:

$$\left.\frac{\partial p}{\partial y}\right]_{y=0} = -iK\rho_0 c\beta_0 p\Big]_{y=0}$$

or

$$F'(0) = -iK\rho_0 c\beta_0 F(0) \qquad (3a)$$

Likewise at the upper wall ($y = b$ plane), the following equation must hold:

$$\left.\frac{\partial p}{\partial y}\right]_{y=b} = iK\rho_0 c\beta_1 p\Big]_{y=b}$$

or

$$F'(b) = iK\rho_0 c\beta_1 F(b) \qquad (3b)$$

Equation (2), together with the boundary conditions (eqs. (3)), constitutes an eigenvalue problem, since there are only a discrete set of K_y values for which all the equations are satisfied. It is convenient to nondimensionalize the variables in equations (2) and (3) by introducing the nondimensional quantities: $\lambda = bK_y$, $\eta = Y/b$, $\tau_0 = \rho_0 cKb\beta_0$, and $\tau_1 = \rho_0 cKb\beta_1$.

In terms of these new quantities, the standard forms of the equations to be studied here for calculating the eigenvalues are

$$F'' + \lambda^2 F = 0 \tag{4a}$$

$$F'(0) = -i\tau_0 F(0) \tag{4b}$$

$$F'(1) = i\tau_1 F(1) \tag{4c}$$

A finite-difference, a finite-element, and a weighted-residual method are used to solve equations (4). All three methods lead to a common form of the matrix eigenvalue problem which can be solved by standard computer subroutines. The results are compared with eigenvalues obtained from the exact transcendental equation in order to assess the accuracy of the numerical methods.

After a discussion of the transcendental equation, whose solutions are used as the basis for comparison, a derivation is given of the final matrix equation for each method. The reader who is interested only in the comparison of the final calculations may omit the mathematical details with no loss of continuity and proceed directly to the "Presentation of Results."

The Exact Transcendental Equation

If F satisfies the eigenvalue problem defined by equations (4), then λ has to satisfy the following characteristic equation:

$$\left(\tau_1\tau_0 + \lambda^2\right) \sin \lambda + i\left(\tau_1 + \tau_0\right)\lambda \cos \lambda = 0 \tag{5}$$

This equation is easily derived by finding the general solution to equation (4a) and then applying the boundary conditions of equations (4b) and (4c).

Equation (5) must be solved numerically since exact solutions are generally not known. It is of interest to note several circumstances under which there are exact solutions to equation (5); $\lambda = 0$ is clearly always a solution to equation (5). It is, however, only acceptable as an eigenvalue to equations (4) when the additional condition $\tau_0\tau_1 + i(\tau_0 + \tau_1) = 0$ is also satisfied. (This restriction is easily derived by setting $\lambda = 0$ in equation (4a) and then applying the boundary conditions to the general solution of the reduced differential equation $F'' = 0$.) When $\tau_0 = -\tau_1$, equation (5) has exact solutions: $\lambda = \pi, 2\pi, 3\pi, \ldots$, and τ_1.

The solutions to equation (5) presented subsequently in the tables were obtained by an iterative process based upon a Newton-Raphson root-finding method. These results are used as a base against which to compare the other three methods which are discussed. These eigenvalues are presented in the tables of results under the heading "Exact," when $\tau_1 = -\tau_0$, since exact solutions to equation (5) are known. They are listed under the heading "Characteristic Equation" for all other values of τ_1 and τ_0.

Finite-Element Method

The finite-element method was originally designed as a tool for structural analysis. The theory and formulation have been progressively so refined and generalized that the method has been applied successfully to such fields as heat flow, seepage, hydrodynamics, and rock mechanics. For a general description of the method, see references 16 and 17.

As used here, the finite-element method for solving equations (4) proceeds in two stages:

(1) The reformulation of the problem as a variational problem, and

(2) The approximate solution of the variational problem using a polynomial representation of F.

Variational formulation.- In order to formulate a variational problem, first multiply both sides of equation (4a) by the variation of the function F, that is, δF, and integrate the results across the duct to obtain

$$\int_0^1 F''(\eta)\,\delta F(\eta)\,d\eta + \lambda^2 \int_0^1 F(\eta)\,\delta F(\eta)\,d\eta = 0 \tag{6}$$

The second integrand is the variation of $\frac{\lambda^2}{2}\left[F(\eta)\right]^2$. If the first integral is transformed by integration by parts, it takes the form

$$\left[F'(\eta)\,\delta F(\eta)\right]\Big|_0^1 - \int_0^1 F'(\eta)\,\delta F(\eta)\,d\eta$$

The integrand in this expression is the variation of $\frac{1}{2}\left[F'(\eta)\right]^2$. Thus, equation (6) becomes

$$\delta\left\{\frac{1}{2}\int_0^1 \left[F'(\eta)\right]^2 dy - \frac{\lambda^2}{2}\int_0^1 \left[F(\eta)\right]^2 d\eta\right\} - \left[F'(\eta)\ \delta F(\eta)\right]\Big|_0^1 = 0 \tag{7}$$

The last term of equation (7) represents the natural boundary conditions of the duct. If the duct has hard walls, $\tau_0 = \tau_1 = 0$ so that by equations (4b) and (4c), $F'(0) = F'(1) = 0$ and this additional term vanishes. If the boundary conditions (eqs. (4b) and (4c)) are incorporated into equation (7), it takes the form

$$\delta\bar{I} = 0 \tag{8a}$$

where

$$\bar{I} = \int_0^1 \left\{\left[F'(\eta)\right]^2 - \lambda^2\left[F(\eta)\right]^2\right\} d\eta - i\left\{\tau_0\left[F(0)\right]^2 + \tau_1\left[F(1)\right]^2\right\} \tag{8b}$$

The presence of the boundary values $F(0)$ and $F(1)$ in the functional is a somewhat unusual feature of this problem. In higher dimensions these terms will take the form of line or surface integrals over the boundary of the duct. These terms may be interpreted as accounting for the dissipation of energy at the treated walls.

Finite-element solution to variational problem. - To solve the variational problem by the finite-element method, the interval $0 \leq \eta \leq 1$ corresponding to the width across the duct is subdivided into N intervals or "elements" of equal length $\Delta = 1/N$ as shown in figure 2. Within the ith element the unknown function F is assumed to be approximated by a polynomial in η, $F_i(\eta)$ which has several arbitrary parameters α_j. The α_j parameters which define F in the different elements are allowed to be different. The α_j parameters are determined from the boundary conditions (eqs. (4b) and (4c)) at $\eta = 0$ and $\eta = 1$, the continuity of F and, perhaps, some of its higher derivatives at the adjoining ends of adjacent elements, and the stationarity of the functional $\bar{I}$ (eqs. (8a) and (8b)). This latter condition is imposed by requiring that $\frac{\partial\bar{I}}{\partial\alpha_j} = 0$ for each α_j accounting for the fact that all the α_j parameters are not independent as a result of the boundary and continuity conditions. The higher the order of the polynomial approximation used for F, the higher the order of the derivatives of F which may be kept continuous. In acoustics problems the velocity components, which must be continuous from physical considerations, are proportional to derivatives of the pressure field. Hence, it is desirable to require in the finite-element method that pressure gradients, F' in the present problem, be continuous across elements.

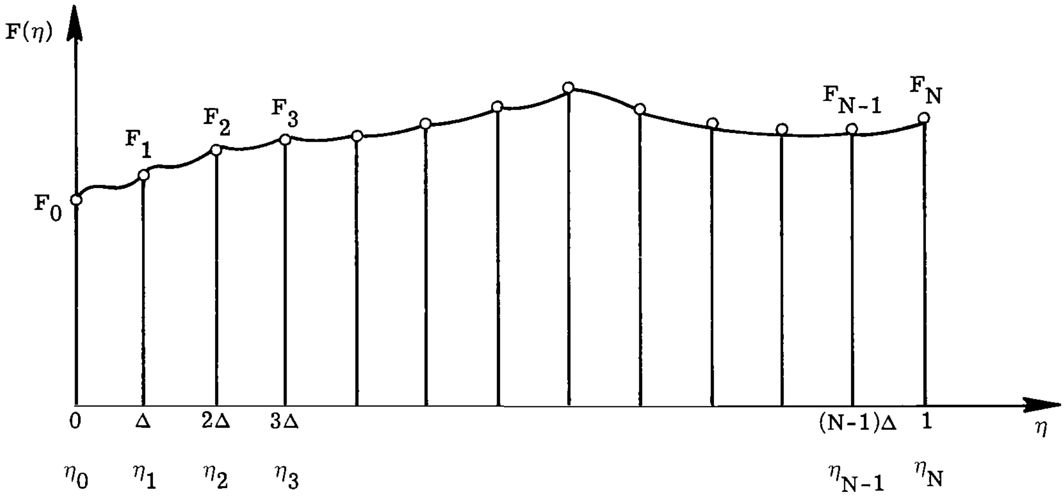

Figure 2. - Subdivision of interval for the finite-element and finite-difference approximations.

Linear approximation: The simplest approximate solution for the variational problem is obtained by assuming that in the ith element, F has the linear form

$$F(\eta) \approx F_i(\eta) = F_{i-1} + \frac{F_i - F_{i-1}}{\Delta}(\eta - \eta_{i-1}) \qquad (\eta_{i-1} \leqq \eta \leqq \eta_i) \tag{9}$$

where F_{i-1} and F_i are the values of F at the left and right end points of the element, respectively, $\eta_{i-1} = (i - 1)\Delta$ is the value of η at the beginning of the element, and Δ is the length of the element. The acoustic pressure F will then be continuous across the duct, but may have discontinuous derivatives at the end points of the element which implies discontinuities in the transverse velocity component. In order to render $\bar{I}$ stationary, it is required that

$$\frac{\partial \bar{I}}{\partial F_0} = \frac{\partial \bar{I}}{\partial F_1} = \frac{\partial \bar{I}}{\partial F_2} = \ldots \frac{\partial \bar{I}}{\partial F_N} = 0 \tag{10}$$

This requirement leads to $N + 1$ equations and $N + 1$ unknowns of the form

$$[A]\{F\} = \mu[B]\{F\} \tag{11}$$

where $\lambda = \frac{\sqrt{\mu}}{\Delta}$

$$\{F\} = \begin{Bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \cdot \\ \cdot \\ \cdot \\ F_{N-2} \\ F_{N-1} \\ F_N \end{Bmatrix}$$

and the matrices A and B are given by

$$A = \begin{bmatrix} a_{11} & -6 & & & & & & & \\ -6 & 12 & -6 & & & & & & \\ & -6 & 12 & -6 & & & & & \\ & & -6 & 12 & -6 & & & & \\ & & & -6 & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & & \cdot & \cdot & \cdot \\ & & & & & & -6 & 12 & -6 \\ & & & & & & & -6 & a_{N+1,N+1} \end{bmatrix}$$

where

$$a_{11} = 6(1 - i\,\Delta\tau_0)$$

$$a_{N+1,N+1} = 6(1 - i\,\Delta\tau_1)$$

$$B = \begin{bmatrix} 2 & 1 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & 1 & 4 & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & & \cdot & \cdot & \cdot \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 1 & 2 \end{bmatrix}$$

Cubic approximation: A more accurate solution to the variational problem (eqs. (8)) may be obtained by demanding that F and F' be continuous everywhere. This is done by taking F to be a cubic polynomial in each element. Assume that in the ith element

$$F(\eta) \approx F_i(\eta) = F_{i-1} + F'_{i-1}(\eta - \eta_{i-1}) + \bar{a}_i(\eta - \eta_{i-1})^2 + \bar{b}_i(\eta - \eta_{i-1})^3 \tag{12}$$

where

$$\bar{a}_i = \frac{3(F_i - F_{i-1})}{\Delta^2} - \frac{2F'_{i-1}}{\Delta} - \frac{F'_i}{\Delta}$$

$$\bar{b}_i = \frac{F'_i}{\Delta^2} + \frac{F'_{i-1}}{\Delta^2} - \frac{2(F_i - F_{i-1})}{\Delta^3}$$

In these expressions, F_{i-1}, F'_{i-1}, F_i, and F'_i are the values of F and F' at the two end points η_{i-1} and η_i, of the element, respectively. This form insures the continuity of both pressure and velocity in the calculated mode shapes. This cubic expression for $F_i(\eta)$ (eqs. (12)) is then substituted into equations (8). After the integration, it is found that $\bar{I} = \bar{I}(F_0,F'_0,F_1,F'_1, \ldots,F'_{N-1},F_N,F'_N)$. The boundary conditions are now built into the expression for $\bar{I}$. From equation (4a) one has $F'_0 = -i\tau_0 F_0$, and from equation (4b), $F'_N = i\tau_1 F_N$. Two of the four unknowns F_0, F'_0, F_N, and F'_N can therefore be eliminated from the expression for $\bar{I}$. The authors choose to eliminate F'_0 and F'_N. Then, one obtains $\bar{I} = \bar{I}(F_0,F_1,F'_1, \ldots,F_{N-1},F'_{N-1},F_N)$ in which all parameters are now independent. In order to render $\bar{I}$ stationary, it is required that

$$\frac{\partial \bar{I}}{\partial F_0} = \frac{\partial \bar{I}}{\partial F_1} = \frac{\partial \bar{I}}{\partial F_1'} = \ldots \frac{\partial \bar{I}}{\partial F_N} = 0$$

These relations lead to 2N linear equations in 2N unknowns of the form

$$[A]\{F\} = \mu [B]\{F\} \tag{13}$$

where $\lambda = \sqrt{\mu}$

$$\{F\} = \begin{Bmatrix} F_0 \\ F_1 \\ F_1' \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_{N-1} \\ F_{N-1}' \\ F_N \end{Bmatrix}$$

and the matrices A and B are given by

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 & & & & & & \\
a_{12} & 24/5\,\Delta & 0 & -12/5\,\Delta & 1/5 & & & & & \\
a_{13} & 0 & 8\,\Delta/15 & -1/5 & -\Delta/15 & 0 & & & & \\
0 & -12/5\,\Delta & -1/5 & 24/5\,\Delta & 0 & -12/5\,\Delta & 1/5 & & & \\
 & 1/5 & -\Delta/15 & 0 & 8\,\Delta/15 & -1/5 & \cdot & \cdot & & \\
 & & 0 & -12/5\,\Delta & -1/5 & \cdot & \cdot & \cdot & \cdot & \\
 & & & 1/5 & \cdot & \cdot & \cdot & \cdot & -\Delta/15 & 0 \\
 & & & & \cdot & \cdot & \cdot & \cdot & 0 & a_{2N-2,2N} \\
 & & & & & \cdot & -\Delta/15 & 0 & 8\,\Delta/15 & a_{2N-1,2N} \\
 & & & & & & 0 & a_{2N-2,2N} & a_{2N-1,2N} & a_{2N,2N}
\end{bmatrix}$$

where

$$a_{11} = \frac{12}{5\,\Delta} - \frac{12 i \tau_0}{5} - \frac{4\,\Delta \tau_0{}^2}{15}$$

$$a_{2N,2N} = \frac{12}{5\,\Delta} - \frac{12 i \tau_1}{5} - \frac{4\,\Delta \tau_1{}^2}{15}$$

$$a_{13} = \frac{1}{5} + \frac{i\,\Delta \tau_0}{15}$$

$$a_{12} = -\frac{12}{5\,\Delta} + \frac{i \tau_0}{5}$$

$$a_{2N-1,2N} = -\frac{1}{5} - \frac{i\,\Delta \tau_1}{15}$$

$$a_{2N-2,2N} = -\frac{12}{5\,\Delta} + \frac{i \tau_1}{5}$$

$$B = \left[\begin{array}{cccccccccc}
b_{11} & b_{12} & b_{13} & 0 & & & & & & \\
b_{12} & 52\,\Delta/35 & 0 & 9\,\Delta/35 & -13\,\Delta^2/210 & & & & & \\
b_{13} & 0 & 4\,\Delta^3/105 & 13\,\Delta^2/210 & -\Delta^3/70 & 0 & & & & \\
0 & 9\,\Delta/35 & 13\,\Delta^2/210 & 52\,\Delta/35 & 0 & 9\,\Delta/35 & -13\,\Delta^2/210 & & & \\
 & -13\,\Delta^2/210 & -\Delta^3/70 & 0 & 4\,\Delta^3/105 & 13\,\Delta^2/210 & -\Delta^3/70 & \cdot & & \\
 & & 0 & 9\,\Delta/35 & 13\,\Delta^2/210 & \cdot & \cdot & \cdot & \cdot & \\
 & & & -13\,\Delta^2/210 & \cdot & \cdot & \cdot & \cdot & -\Delta^3/70 & 0 \\
 & & & & \cdot & \cdot & \cdot & \cdot & 0 & b_{2N-2,2N} \\
 & & & & & \cdot & -\Delta^3/70 & 0 & 4\,\Delta^3/105 & b_{2N-1,2N} \\
 & & & & & & 0 & b_{2N-2,2N} & b_{2N-1,2N} & b_{2N,2N}
\end{array}\right]$$

where

$$b_{11} = \frac{26\,\Delta}{35} - \frac{22 i\,\Delta^2 \tau_0}{105} - \frac{2\,\Delta^3 \tau_0{}^2}{105}$$

$$b_{2N,2N} = \frac{26\ \Delta}{35} - \frac{22\ \Delta^2 i \tau_1}{105} - \frac{2\ \Delta^3 \tau_1^2}{105}$$

$$b_{13} = -\frac{13\ \Delta^2}{210} + \frac{i\ \Delta^3 \tau_0}{70}$$

$$b_{12} = \frac{9\ \Delta}{35} - \frac{13\ \Delta^2 i \tau_0}{210}$$

$$b_{2N-1,2N} = \frac{13\ \Delta^2}{210} - \frac{i\ \Delta^3 \tau_1}{70}$$

$$b_{2N-2,2N} = \frac{9\ \Delta}{35} - \frac{13\ \Delta^2 i \tau_1}{210}$$

A Weighted Residual Formulation

The method of weighted residuals is another general purpose method which has been widely used to obtain approximate solutions to the equations of mathematical physics. (See ref. 18.) In this paper a variant of the method is used to solve the eigenvalue problem. The unknown function $F(\eta)$ is expanded in a series of specified trial functions with adjustable coefficients which are chosen to give the best solution to the differential equation and boundary conditions in a global sense. The trial functions used do not satisfy the boundary conditions, a choice which is at variance with the usual practice. The boundary conditions are built into the equations in the manner described below. It is convenient to refer to this method of solution as MWR.

Define the function R, known as the "residual," by the equation

$$R = F'' + \lambda^2 F \tag{14}$$

If F and λ are solutions of the eigenvalue problem, the residual is identically zero. In the MWR one seeks to adjust the coefficients in the series representation of F in such a way that the residual is forced to be zero in an average sense. This adjustment is made by requiring that the weighted integral of the residual across the width of the duct vanish:

$$\int_0^1 R(\eta)\ W_n(\eta)\ d\eta = 0 \tag{15}$$

where the W_n functions are a prescribed set of weighting functions. Equation (15) may be interpreted as requiring that R be orthogonal to each W_n.

It is obvious that compatible sets of values of the constant λ and the function F which satisfy equation (15) are not necessarily approximate solutions to equations (4) since in its present form equation (15) takes no account of the boundary conditions. That is, equation (15) is satisfied for any λ and F which make $F'' + \lambda^2 F = 0$. Thus, equation (15) must be manipulated in some manner to include the influence of the boundary conditions before it can be used to find approximate values of λ and F satisfying equations (4).

For the present problem, this is done as follows. In explicit form, equation (15) is

$$\int_0^1 \left(F'' + \lambda^2 F\right) W_n \, d\eta = 0$$

By integrating the term $\int_0^1 F'' W_n \, d\eta$ twice by parts, the equation becomes

$$\int_0^1 \left(W_n'' + \lambda^2 W_n\right) F \, d\eta + \left(W_n F' - F W_n'\right)\Big|_0^1 = 0$$

Using the boundary conditions (eqs. (4b) and (4c)) finally gives

$$\int_0^1 \left(W_n'' + \lambda^2 W_n\right) F \, d\eta - \left(F W_n'\right)\Big|_0^1 + \left[i\tau_1 \, F(1) \, W_n(1) + i\tau_0 \, F(0) \, W_n(0)\right] = 0 \tag{16a}$$

This equation is the basis for finding approximate solutions for the eigenvalues and characteristic functions.

It is convenient to use as the trial functions for representing F, the set of functions $\cos \gamma_n \eta$, $\gamma_n = n\pi$ (n = 0, 1, 2, . . .). These functions are easy to handle mathematically; they are complete in the interval $0 < \eta < 1$ so that an infinite series of them which actually converges to the true eigenfunctions does exist, and they are the exact solutions of the problem for a hard walled duct, $\tau_1 = \tau_0 = 0$. Thus, it is assumed that the unknown F in equation (16a) can be represented in the form

$$F(\eta) = \sum_{m=0}^{\infty} \overline{A}_m \cos \gamma_m \eta \tag{16b}$$

where the $\overline{A}_m$ elements are constants. Assume also that $W_n(\eta) = \cos \gamma_n \eta$. Substituting these choices into equation (16a) and carrying out the indicated operations gives the following equations:

$$\sum_{m=0}^{\infty} \overline{A}_m \left\{ \left(\lambda^2 - \gamma_n^2\right) C_{mn} + i\left[\tau_0 + (-1)^{m+n}\tau_1\right] \right\} = 0 \tag{17}$$

in which

$$C_{mn} = \int_0^1 \cos n\pi\, \eta \cos m\pi\, \eta \, d\eta = \begin{cases} 1 & (n = m = 0) \\ \frac{1}{2} & (n = m \geqq 1) \\ 0 & (n \neq m) \end{cases}$$

In equation (17), n is a parameter which takes the values 0, 1, 2, 3, ... Thus, equation (17) represents an infinite number of linear equations for an infinite number of unknowns $\overline{A}_m$ ($m = 0, 1, 2, 3, \ldots$). Obviously, when $\tau_1 = \tau_0 = 0$, the exact solutions are $\lambda = \gamma_n$ for $n = 0, 1, 2, \ldots$, corresponding to $\overline{A}_n$ arbitrary; all other $\overline{A}_m$ values are 0, and $F = A_n \cos \gamma_n \eta$. These modes are, of course, the hardwall duct modes.

Approximate solutions to the system (eq. (17)) can be obtained by assuming that $\overline{A}_N = \overline{A}_{N+1} = \overline{A}_{N+2} \ldots = 0$, and writing out the equations for $n = 0, 1, 2, \ldots, N - 1$. The result is a set of linear homogeneous equations for $\overline{A}_0, \overline{A}_1, \overline{A}_2, \ldots, \overline{A}_{N-1}$. Such a system of equations has a nontrivial solution only if the determinant of the coefficient matrix vanishes. Therefore, the problem is to find those values of λ which make the determinant of the coefficient matrix vanish. The solution for the $\overline{A}_m$ values is then substituted into equation (16b) to obtain an approximate analytical representation of the corresponding eigenfunctions.

The analysis therefore leads to N equations and N unknowns of the form

$$[A]\{\overline{A}\} = \mu [B]\{\overline{A}\} \tag{18}$$

where $\lambda = \sqrt{\mu}$ and

$$\bar{A} = \begin{Bmatrix} \bar{A}_0 \\ \bar{A}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{A}_{N-3} \\ \bar{A}_{N-2} \\ \bar{A}_{N-1} \end{Bmatrix}$$

The elements of the A and B matrices are defined by the equations:

$$b_{11} = 1$$

$$b_{ss} = \frac{1}{2} \qquad (2 \leqq s \leqq N)$$

$$b_{rs} = 0 \qquad (r \neq s)$$

$$a_{11} = -i\,(\tau_0 + \tau_1)$$

$$a_{rs} = -i\left[\tau_0 + \tau_1(-1)^{r+s}\right] \qquad (r \neq s)$$

$$a_{ss} = a_{11} + \frac{\pi^2}{2}(s - 1)^2 \qquad (2 \leqq s \leqq N)$$

Finite-Difference Formulation

The finite-difference method for solving eigenvalue problems such as equations (4) has been well documented (ref. 19) and need not be discussed at length here. Briefly, the interval $0 \leqq \eta \leqq 1$ is subdivided into N intervals of equal length $\Delta = 1/N$ as in figure 2. Let F_i $(i = 0, 1, 2, . . ., N)$ be the values of $F(\eta)$ at $\eta = (i)\Delta$. The second derivative of F is replaced by the central difference approximation

$$F''(y_i) \approx \frac{F_{i+1} - 2F_i + F_{i-1}}{\Delta^2}$$

Then equation (4a) becomes

$$F_{i-1} + \left(\Delta^2\lambda^2 - 2\right)F_i + F_{i+1} = 0 \tag{19}$$

This equation is applied to points $i = 1, 2, 3, \ldots, N - 1$. The result is a set of matrix equations with $N + 1$ unknowns but only $N - 1$ equations. The two remaining equations are formed from the boundary conditions.

By using a forward difference approximation for F' at $\eta = 0$ and a backward difference for F' at $\eta = 1$, the following equations obtained from the boundary conditions (4b) and (4c) must also hold:

$$\left(-3 + 2\,\Delta i\tau_0\right)F_0 + 4F_1 - F_2 = 0 \tag{20}$$

$$\left(3 - 2\,\Delta i\tau_1\right)F_N - 4F_{N-1} + F_{N-2} = 0 \tag{21}$$

By using these two relations, F_0 and F_N are eliminated from the matrix equations obtained from equation (19). The analysis leads to $N - 1$ equations and $N - 1$ unknowns of the form

$$[A]\{F\} = \mu\{F\} \tag{22}$$

where $\lambda = \sqrt{\mu}$

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \cdot \\ \cdot \\ \cdot \\ F_{N-2} \\ F_{N-1} \end{Bmatrix}$$

and the A matrix is given by

$$
A = \begin{bmatrix}
a_{11} & a_{12} & & & & & & \\
a_{22} & -2a_{22} & a_{22} & & & & & \\
 & a_{22} & -2a_{22} & a_{22} & & & & \\
 & & a_{22} & -2a_{22} & \cdot & & & \\
 & & & \cdot & \cdot & \cdot & & \\
 & & & & \cdot & \cdot & \cdot & \\
 & & & & & \cdot & \cdot & \cdot \\
 & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & & \cdot & -2a_{22} & a_{22} \\
 & & & & & & & & a_{N-2,N-1} & a_{N-1,N-1}
\end{bmatrix}
$$

where

$$a_{22} = -\frac{1}{\Delta^2}$$

$$a_{11} = \frac{4\,\Delta i \tau_0 - 2}{-3\,\Delta^2 + 2i\,\Delta^3 \tau_0}$$

$$a_{12} = \frac{2 - 2i\,\Delta \tau_0}{-3\,\Delta^2 + 2i\,\Delta^3 \tau_0}$$

$$a_{N-2,N-1} = \frac{-2 + 2i\,\Delta \tau_1}{3\,\Delta^2 - 2i\,\Delta^3 \tau_1}$$

$$a_{N-1,N-1} = \frac{2 - 4\,\Delta i \tau_1}{3\,\Delta^2 - 2i\,\Delta^3 \tau_1}$$

The finite-difference formulation has the simplest matrices of all the methods tried. However, the shape of the solution between the discrete values F_i is not specified, whereas functional expressions for F are obtained from the finite-element method and the MWR.

PRESENTATION OF RESULTS

Equations (11), (13), (18), and (22) were solved by using a standard eigenvalue subroutine from the Langley library of subroutines for selected values of τ_0 and τ_1 for the purpose of obtaining some insight into the relative accuracy of the three approximate methods for calculating eigenvalues – finite element, method of weighted residuals, and finite differences – by comparing the first 10 eigenvalues obtained from these methods with results extracted from the exact transcendental equation. The results are given in tables I to XIV. Based on the error criterion used, the roots obtained from the characteristic equation are believed to be accurate to five significant figures. Each of the three approximate methods produces a system of matrix equations of the form $[A]\{x\} = \mu[B]\{x\}$. The same solution algorithm and same order matrices were used in solving this standard form to make the comparison. The order of the A and B matrices studied was 20×20 and 40×40. The results for these two different matrix orders give some indication of the improvement in accuracy to be expected as the size of the system of matrix equations is allowed to increase.

Since the eigenvalues are a function of the real and imaginary parts of τ_1 and τ_0 and the order of the A and B matrices, it is impossible to cover exhaustively every possible circumstance. Some selectivity must be exercised in the cases to be considered. The values used for $|\tau_{0,1}| = \rho_0 cKb|\beta_{0,1}|$ fall in the range acceptable for real liners in aircraft engine nacelles. Although negative real parts of τ_0 and τ_1 do not normally occur in noise reduction applications (this situation describes energy insertion at the walls), the equations and calculations remain valid. Several calculations using negative real parts of τ_0 and τ_1 are discussed. In general, the values of τ_0 and τ_1 were selected to be representative of conditions under which some knowledge of the nature of the roots can be deduced from the characteristic equation such as when the roots are known exactly or when several roots are nearly identical.

In comparing the results obtained from each of the methods used in this study, it must be kept in mind that each method is subject to a number of variations or refinements. For example, higher order difference approximations to F'' could be used in the method of finite differences or a different set of trial and weighting functions could be used in the MWR. The different variations within any one method could possibly produce a modified set of conclusions. However, it is not possible here to investigate all the many variations which are available.

The approximate methods investigated in this paper were selected because they are applicable to solving propagation problems in ducts including variable cross-sectional areas, variable wall impedances, and realistic mean flow fields for which exact methods (usually based upon separation of variable techniques) do not exist. Hence, establishing the relative merits of these approximate methods is of considerable interest. There are at least four criteria to be considered in the selection of a "best" method: storage requirements, computation time, accuracy, and reliability. The first two criteria, which depend upon the order and structure of the matrices used and the number of nonzero elements, are discussed qualitatively in the next section. The principal concerns of the remainder of the paper are the accuracy of the methods (determined by comparing approximate eigenvalues with solutions of the exact characteristic equation) and the reliability (determined by the consistency with which a method gives acceptable results as the distribution of eigenvalues in the complex plane varies). With regard to both of these criteria, values of τ_0 and τ_1 which give nearly equal eigenvalues are of particular concern. This situation occurs frequently in routine duct propagation studies and has frequently been the source of frustrating computational problems.

Solution Technique, Storage Requirements, and Computer Time

The result of the analysis for each approach is the generalized eigenvalue problem, $[A]\{x\} = \mu[B]\{x\}$. These equations are readily solved by high-speed digital computers.

The eigenvalues for such systems of equations are extracted by numerical schemes that are either direct or iterative. The direct methods are more general and are commonly employed, although the iterative schemes are suitable for computations when only one or a few eigenvalues are needed. The four sets of eigenvalue equations generated in the text were solved by using a Langley program library subroutine which employed a direct method to extract the eigenvalues. The subroutine first reduced A to an upper Hessenberg matrix and at the same time reduced B to an upper triangular matrix by stabilized elementary transformations. The Lz algorithm is then applied to simultaneously reduce A and B to upper triangular form. The numerator and denominator of the eigenvalues are then the diagonal elements of the A and B matrices, respectively.

Storage requirements and computer time are factors to be considered in choosing a method for obtaining eigenvalues of matrix equations. The final matrix equations for the methods studied in this paper have their own special properties such as diagonal, symmetric, or banded form and number of nonzero elements. Storage requirements and solution time can be minimized by using solution techniques which take advantage of the special properties of each set of matrix equations. Unfortunately, the subroutine which was used to solve the eigenvalue equations for the results presented in this paper was very general and did not take advantage of any special characteristics of the equations. In particular, all zero elements had to be individually stored. Hence, no effort was made to compare them quantitatively.

The minimum time required to extract the eigenvalues of a set of matrix equations is not only related to the special character of the equations but also to the number of nonzero elements and the number of operations that must be carried out on each element of the matrix. Hence, the minimum computing time for a particular set of matrix equations is a very complex question and cannot be easily answered quantitatively.

However, since minimum computing time is a function of the number of operations that must be carried out on each element and the special character of each set of matrix equations, it is of interest to point out briefly the special structures of the A and B matrices for each method and to count the number of nonzero elements in each matrix. This is done in the accompanying table in which N is the order of the system of equations. If the matrix is symmetric, only the upper or lower triangular part of the matrix would have to be stored in an efficiently written program and if the matrix is banded, only the rectangular banded matrix needs to be stored.

Method		Finite difference	MWR	Linear finite element	Cubic finite element
A matrix	Form	Nonsymmetric Tridiagonal Complex	Symmetric Full Complex	Symmetric Tridiagonal Complex	Symmetric 7 diagonal Complex
	Nonzero elements	3N	$\frac{1}{2}(N^2 + N)$	2N	4N
B matrix	Form	Unit matrix	One diagonal Real	Symmetric Tridiagonal Real	Symmetric 7 diagonal Complex
	Nonzero elements	N	N	2N	4N
Total storage requirements		4N	$\frac{1}{2} N^2 + \frac{3N}{2}$	4N	8N

Calculations for $\tau_0 = -\tau_1$

Eigenvalues calculated for $\tau_1 = -\tau_0 = 0.0$, 1.0i, 3.0i, 5.0i, 5.0, and 10.0 are presented in tables I to VI. As can be seen from the characteristic equation, equation (5), the exact eigenvalues in this case are $n\pi$, $n = 1, 2, 3, \ldots$, and τ_1. Zero is also an eigenvalue if $\tau_0 = \tau_1 = 0.0$. If τ_1 is equal to an integral multiple of π, the characteristic included as subroutines in linear optimization programs based upon segmented duct concept (ref. 21). These programs require hundreds of eigenvalue calculations that must be made in a swift, efficient, reliable, mechanical manner. Thus, the ability of the three approximate methods to handle multiple or nearly equal eigenvalues is of special interest.

It may be shown by methods similar to those employed in reference 20, that when $\tau_0 = \tau_1 = 4.120 - 3.301i$, equation (5) has a double root at $\lambda = 4.212 - 2.251i$ and that when $\tau_0 = 0$ and $\tau_1 = 2.060 - 1.651i$, there is a double root at $\lambda = 2.106 - 1.125i$. Tables X and XI show the results obtained for slightly different admittance values which in each case would be expected to produce a pair of nearly equal eigenvalues. For both of these combinations of admittance values, the exact characteristic equation has complex roots with nonzero real and imaginary parts in contrast to the simpler structure of the roots encountered in previous examples.

Tables X and XI show, as before, that the finite-difference method gives the least accurate results for the higher order eigenvalues. The imaginary parts, in particular, are very inaccurate. The accuracy improves considerably as the order of the system increases. The nearly equal roots are picked out with no difficulty.

The MWR gives least accurate values for the lower order roots, which includes the nearly equal pair. As before, better results are obtained for the higher order eigenvalues. The accuracy of the lower order eigenvalues does not improve substantially by going to larger order matrices.

The cubic finite-element method gives excellent results for all 10 eigenvalues using 20×20 matrices. There is no substantial improvement in accuracy by using a larger order system.

Tables XII and XIII present some additional calculations for values of τ_0 and τ_1 which give nearly multiple roots. Table XII, which uses 40×40 matrices, is concerned with the nearly equal roots which occur when $\tau_0 = -\tau_1 =$ Integral multiple of π. The tendency of the MWR to give complex conjugate roots which only poorly approximate the exact answer is again evident. The ability of the cubic finite-element method to sort out the roots and to give excellent results is reconfirmed.

Table XIII pertains to values of τ_0 and τ_1 even closer to the multiple-root conditions investigated in tables X and XI and, in addition, some values which are close to higher order multiple-root conditions. Table XIII was obtained by using 20×20 matrices. The superiority of the cubic finite-element method is again clear.

Calculations for a True Zero Eigenvalue

As pointed out in the discussion of the exact characteristic equation, zero is a bona-fide eigenvalue if $\tau_0\tau_1 + i(\tau_0 + \tau_1) = 0$. This equation is obviously satisfied for the hard walled duct, $\tau_1 = \tau_0 = 0$, for which the corresponding eigenfunction is the plane wave. In general, the eigenfunction corresponding to the eigenvalue zero is a linear function of η across the duct, that is, $(1 - i\tau_0\eta)$; when normalized to unity at the wall, $\eta = 0$. Table XIV lists eigenvalues for the special case $\tau_0 = 1.0$, $\tau_1 = -0.5 - 0.5i$ for which zero is a true eigenvalue. All solution methods calculate the zero root to five decimal places except the MWR which gives a small nonzero complex number.

CONCLUDING REMARKS

Three approximate methods – finite differences, weighted residuals, and finite elements – have been used to solve the eigenvalue problem which arises in finding the acoustic modes and propagation constants in an absorptively lined two-dimensional duct without airflow. The final equations derived for all three methods assume a common matrix form. These equations were solved for the eigenvalues corresponding to various values of wall impedance. Two matrix orders, 20×20 and 40×40, were used in the calculations. The cases considered included values of wall admittance for which exact eigenvalues were known and for which several nearly equal roots were present. Ten of the lower order eigenvalues obtained from the three approximate methods were compared with solutions calculated from the exact characteristic equation. The cases studied permit an assessment of the relative accuracy and reliability of the three methods.

In general, it was found that the finite-difference and linear finite-element methods gave the least accurate results. High-order systems of equations are needed for good accuracy in the high-order eigenvalues. The method of weighted residuals gave better results than either of these two methods. Although the method of weighted residuals gave poor results for the two or three lowest order eigenvalues, it has the characteristic of yielding uniformly good accuracy for all the higher order eigenvalues. When nearly equal roots occurred, the method of weighted residuals was not found to be reliable or to give accurate results.

Best results in terms of both accuracy and reliability were judged to be given by the finite-element method by using a cubic polynomial. Excellent accuracy was obtained by using a 20×20 order matrix. The ability to use low-order matrices implies savings of computer time and storage. For the cases tested, nearly equal roots were consistently located and accurately calculated. This is a very favorable result as calculating multiple roots has been a stumbling block in analytical duct acoustics.

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TABLE I.- EIGENVALUES FOR $\tau_1 = -\tau_0 = 0.0$

Exact		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20×20 matrix									
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.14159	.00000	3.14032	.00000	3.14159	.00000	3.14517	.00000	3.14159	.00000
6.28319	.00000	6.27253	.00000	6.28319	.00000	6.31185	.00000	6.28319	.00000
9.42478	.00000	9.38647	.00000	9.42478	.00000	9.52168	.00000	9.42486	.00000
12.56637	.00000	12.46860	.00000	12.56637	.00000	12.79648	.00000	12.56690	.00000
15.70796	.00000	15.50214	.00000	15.70796	.00000	16.15824	.00000	15.71001	.00000
18.84956	.00000	18.46721	.00000	18.84956	.00000	19.62877	.00000	18.85560	.00000
21.99115	.00000	21.34251	.00000	21.99115	.00000	23.22905	.00000	22.00589	.00000
25.13274	.00000	24.10610	.00000	25.13274	.00000	26.97799	.00000	25.16422	.00000
28.27433	.00000	26.73628	.00000	28.27433	.00000	30.88963	.00000	28.33534	.00000
Eigenvalues for 40×40 matrix									
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
3.14159	.00000	3.14105	.00000	3.14159	.00000	3.14244	.00000	3.14159	.00000
6.28319	.00000	6.27881	.00000	6.28319	.00000	6.28998	.00000	6.28319	.00000
9.42478	.00000	9.40993	.00000	9.42478	.00000	9.44773	.00000	9.42478	.00000
12.56637	.00000	12.53088	.00000	12.56637	.00000	12.62080	.00000	12.56638	.00000
15.70796	.00000	15.63793	.00000	15.70796	.00000	15.81434	.00000	15.70801	.00000
18.84956	.00000	18.72713	.00000	18.84956	.00000	19.03352	.00000	18.84972	.00000
21.99115	.00000	21.79427	.00000	21.99115	.00000	22.28353	.00000	21.99160	.00000
25.13274	.00000	24.83493	.00000	25.13274	.00000	25.56958	.00000	25.13378	.00000
28.27433	.00000	27.84450	.00000	28.27433	.00000	28.89690	.00000	28.27649	.00000

TABLE II.- EIGENVALUES FOR $\tau_1 = -\tau_0 = 1.0i$

Exact		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
0.00000	1.00000	0.00000	1.00082	0.00000	0.98663	0.00000	0.99989	0.00000	1.00000
3.14159	.00000	3.14017	.00000	3.14775	.00000	3.14517	.00000	3.14159	.00000
6.28319	.00000	6.27223	.00000	6.28634	.00000	6.31185	.00000	6.28319	.00000
9.42478	.00000	9.38607	.00000	9.42703	.00000	9.52168	.00000	9.42486	.00000
12.56637	.00000	12.46820	.00000	12.56799	.00000	12.79648	.00000	12.56690	.00000
15.70796	.00000	15.50166	.00000	15.70935	.00000	16.15824	.00000	15.71001	.00000
18.84596	.00000	18.84668	.00000	18.85066	.00000	19.62877	.00000	18.85560	.00000
21.99115	.00000	21.34209	.00000	21.99216	.00000	23.22905	.00000	22.00590	.00000
25.13274	.00000	24.10572	.00000	25.13359	.00000	26.97789	.00000	25.16422	.00000
28.27433	.00000	26.73594	.00000	28.27515	.00000	30.88963	.00000	28.33535	.00000
Eigenvalues for 40 × 40 matrix									
0.00000	1.00000	0.00000	1.00022	0.00000	0.99333	0.00000	0.99997	0.00000	1.00000
3.14159	.00000	3.14103	.00000	3.14459	.00000	3.14244	.00000	3.14159	.00000
6.28319	.00000	6.27877	.00000	6.28476	.00000	6.28999	.00000	6.28319	.00000
9.42478	.00000	9.40987	.00000	9.42587	.00000	9.44773	.00000	9.42478	.00000
12.56637	.00000	12.53180	.00000	12.56717	.00000	12.62080	.00000	12.56638	.00000
15.70796	.00000	15.63784	.00000	15.70863	.00000	15.81434	.00000	15.70801	.00000
18.84956	.00000	18.72702	.00000	18.85010	.00000	19.03352	.00000	18.84972	.00000
21.99115	.00000	21.79415	.00000	21.99163	.00000	22.28353	.00000	21.99160	.00000
25.13274	.00000	24.83481	.00000	25.13315	.00000	25.56958	.00000	25.13378	.00000
28.27433	.00000	27.84437	.00000	28.27471	.00000	28.89690	.00000	28.27649	.00000

TABLE III. - EIGENVALUES FOR $\tau_1 = -\tau_0 = 3.0i$

Exact		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
0.00000	3.00000	0.00000	3.02128	0.00000	2.90910	0.00000	2.99690	0.00000	3.00000
3.14159	.00000	3.13953	.00000	3.17354	.00000	3.14517	.00000	3.14159	.00000
6.28319	.00000	6.27027	.00000	6.30684	.00000	6.31185	.00000	6.28319	.00000
9.42478	.00000	9.38312	.00000	9.44340	.00000	9.52168	.00000	9.42486	.00000
12.56637	.00000	12.46469	.00000	12.58028	.00000	12.79648	.00000	12.56689	.00000
15.70796	.00000	15.49796	.00000	15.72003	.00000	16.15824	.00000	15.71002	.00000
18.84956	.00000	18.46311	.00000	18.85930	.00000	19.62877	.00000	18.85562	.00000
21.99115	.00000	21.33869	.00000	22.00014	.00000	23.22905	.00000	22.00593	.00000
25.13274	.00000	24.10267	.00000	25.14037	.00000	26.97790	.00000	25.16429	.00000
28.27433	.00000	26.73328	.00000	28.28163	.00000	30.88963	.00000	28.33550	.00000
Eigenvalues for 40 × 40 matrix									
0.00000	3.00000	0.00000	3.00577	0.00000	2.95435	0.00000	2.99926	0.00000	3.00000
3.14159	.00000	3.14094	.00000	3.15716	.00000	3.14244	.00000	3.14159	.00000
6.28319	.00000	6.27849	.00000	6.29500	.00000	6.28998	.00000	6.28319	.00000
9.42478	.00000	9.40941	.00000	9.43380	.00000	9.44773	.00000	9.42478	.00000
12.56637	.00000	12.53018	.00000	12.57326	.00000	12.62080	.00000	12.56638	.00000
15.70796	.00000	15.63709	.00000	15.71374	.00000	15.81434	.00000	15.70801	.00000
18.84956	.00000	18.72619	.00000	18.85431	.00000	19.03352	.00000	18.84972	.00000
21.99115	.00000	21.79325	.00000	21.99537	.00000	22.28353	.00000	21.99160	.00000
25.13274	.00000	24.83386	.00000	25.13637	.00000	25.56958	.00000	25.13378	.00000
28.27433	.00000	27.84340	.00000	28.27767	.00000	28.89690	.00000	28.27649	.00000

TABLE IV.- EIGENVALUES FOR $\tau_1 = -\tau_0 = 5.0i$

Exact		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
0.00000	5.00000	0.00000	5.09554	0.00000	4.75362	0.00000	4.98572	0.00000	5.00000
3.14159	.00000	3.13914	.00000	3.18996	.00000	3.14517	.00000	3.14159	.00000
6.28319	.00000	6.26775	.00000	6.33263	.00000	6.31185	.00000	6.28319	.00000
9.42478	.00000	9.37834	.00000	9.46922	.00000	9.52168	.00000	9.42486	.00000
12.56637	.00000	12.45844	.00000	12.60162	.00000	12.79648	.00000	12.56690	.00000
15.70796	.00000	15.59103	.00000	15.73952	.00000	16.15824	.00000	15.71003	.00000
18.84956	.00000	18.45613	.00000	18.87547	.00000	19.62877	.00000	18.85565	.00000
21.99115	.00000	21.33208	.00000	22.01536	.00000	23.22905	.00000	22.00599	.00000
25.13274	.00000	24.09667	.00000	25.15330	.00000	26.97789	.00000	25.16441	.00000
28.27433	.00000	26.72802	.00000	28.29423	.00000	30.88963	.00000	28.33569	.00000
Eigenvalues for 40 × 40 matrix									
0.00000	5.00000	0.00000	5.02609	0.00000	4.87503	0.00000	4.99658	0.00000	5.00000
3.14159	.00000	3.14089	.00000	3.16508	.00000	3.14244	.00000	3.14159	.00000
6.28319	.00000	6.27813	.00000	6.30787	.00000	6.28998	.00000	6.28319	.00000
9.42478	.00000	9.40867	.00000	9.44631	.00000	9.44773	.00000	9.42478	.00000
12.56637	.00000	12.56357	.00000	12.58382	.00000	12.62080	.00000	12.56638	.00000
15.70796	.00000	15.63574	.00000	15.72306	.00000	15.81434	.00000	15.70801	.00000
18.84956	.00000	18.72461	.00000	18.86220	.00000	19.03352	.00000	18.84972	.00000
21.99115	.00000	21.79151	.00000	22.00250	.00000	22.28353	.00000	21.99160	.00000
25.13274	.00000	24.83201	.00000	25.14257	.00000	25.56958	.00000	25.13378	.00000
28.27433	.00000	27.84150	.00000	28.28341	.00000	28.89690	.00000	28.27649	.00000

TABLE V.- EIGENVALUES FOR $\tau_1 = -\tau_0 = 5.0$

Exact		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20×20 matrix									
3.14159	0.00000	3.13756	0.00000	3.26246	0.00000	3.14517	0.00000	3.14159	0.00000
5.00000	.00000	4.89803	.00000	5.08994	.00000	5.01458	.00000	5.00000	.00000
6.28319	.00000	6.29129	.00000	6.02015	.00000	6.31185	.00000	6.28319	.00000
9.42478	.00000	9.39993	.00000	9.34471	.00000	9.52168	.00000	9.42486	.00000
12.56637	.00000	12.48188	.00000	12.51781	.00000	12.79648	.00000	12.56688	.00000
15.70796	.00000	15.51499	.00000	15.66934	.00000	16.15824	.00000	15.70999	.00000
18.84956	.00000	18.47932	.00000	18.81981	.00000	19.62877	.00000	18.85555	.00000
21.99115	.00000	21.35356	.00000	21.96440	.00000	23.22905	.00000	22.00578	.00000
25.13274	.00000	24.11589	.00000	25.11059	.00000	26.97789	.00000	25.16400	.00000
28.27433	.00000	26.74473	.00000	28.25326	.00000	30.88963	.00000	28.33496	.00000
Eigenvalues for 40×40 matrix									
3.14159	0.00000	3.14068	0.00000	3.19779	0.00000	3.14244	0.00000	3.14159	0.00000
5.00000	.00000	4.97206	.00000	5.04046	.00000	5.00343	.00000	5.00000	.00000
6.28319	.00000	6.28175	.00000	6.16375	.00000	6.28998	.00000	6.28319	.00000
9.42478	.00000	9.41213	.00000	9.38612	.00000	9.44773	.00000	9.42478	.00000
12.56637	.00000	12.53327	.00000	12.54231	.00000	12.62080	.00000	12.56638	.00000
15.70796	.00000	15.64057	.00000	15.68945	.00000	15.81434	.00000	15.70801	.00000
18.84956	.00000	18.72998	.00000	18.83499	.00000	19.03352	.00000	18.84972	.00000
21.99115	.00000	21.79728	.00000	21.97856	.00000	22.28353	.00000	21.99159	.00000
25.13274	.00000	24.83804	.00000	25.12211	.00000	25.56958	.00000	25.13378	.00000
28.27433	.00000	27.84764	.00000	28.26468	.00000	28.89690	.00000	28.27649	.00000

TABLE VI.- EIGENVALUES FOR $\tau_1 = -\tau_0 = 10.0$

Exact		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
3.14159	0.00000	3.13848	0.00000	3.21812	0.00000	3.14517	0.00000	3.14159	0.00000
6.28319	.00000	6.25076	.00000	6.51050	.00000	6.31185	.00000	6.28319	.00000
9.42478	.00000	8.97661	.00000	9.73502	-1.19494	9.52168	.00000	9.42489	.00000
10.00000	.00000	9.77614	.00000	9.73502	1.19494	10.12035	.00000	10.00000	.00000
12.56637	.00000	12.56031	.00000	12.08912	.00000	12.79648	.00000	12.56683	.00000
15.70796	.00000	15.56898	.00000	15.47475	.00000	16.15824	.00000	15.70989	.00000
18.84956	.00000	18.52336	.00000	18.69690	.00000	19.62877	.00000	18.85533	.00000
21.99115	.00000	21.39083	.00000	21.86445	.00000	23.22905	.00000	22.00536	.00000
25.13274	.00000	24.14750	.00000	25.03269	.00000	26.97789	.00000	25.16325	.00000
28.27433	.00000	26.24124	.00000	28.18200	.00000	30.88963	.00000	28.33368	.00000
Eigenvalues for 40 × 40 matrix									
3.14159	0.00000	3.14080	0.00000	3.17833	0.00000	3.14244	0.00000	3.14159	0.00000
6.28319	.00000	6.27585	.00000	6.39254	.00000	6.28998	.00000	6.28319	.00000
9.42478	.00000	9.34727	.00000	9.74090	-.71197	9.44773	.00000	9.42478	.00000
10.00000	.00000	9.85846	.00000	9.74090	.71197	10.02766	.00000	10.00000	.00000
12.56637	.00000	12.55091	.00000	12.33204	.00000	12.62080	.00000	12.56638	.00000
15.70796	.00000	15.65309	.00000	15.59519	.00000	15.81434	.00000	15.70801	.00000
18.84956	.00000	18.74133	.00000	18.77409	.00000	19.03352	.00000	18.84972	.00000
21.99115	.00000	21.80820	.00000	21.93096	.00000	22.28353	.00000	21.99159	.00000
25.13274	.00000	24.84871	.00000	25.08427	.00000	25.56958	.00000	25.13377	.00000
28.27433	.00000	27.85808	.00000	28.23164	.00000	28.89690	.00000	28.27648	.00000

TABLE VII. - EIGENVALUES FOR $\tau_1 = \tau_0 = 1.0i$

Characteristic equation		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
1.30654	0.00000	1.30566	0.00000	1.31250	0.00000	1.30680	0.00000	1.30654	0.00000
3.67319	.00000	3.66623	.00000	3.67775	.00000	3.67892	.00000	3.67319	.00000
6.58462	.00000	6.56267	.00000	6.58769	.00000	6.61764	.00000	6.58463	.00000
9.63168	.00000	9.57706	.00000	9.63376	.00000	9.73520	.00000	9.63178	.00000
12.72324	.00000	12.60494	.00000	12.72493	.00000	12.96230	.00000	12.72381	.00000
15.83411	.00000	15.60416	.00000	15.83541	.00000	16.29575	.00000	15.83627	.00000
18.95497	.00000	18.54564	.00000	18.95614	.00000	19.74810	.00000	18.96127	.00000
22.08166	.00000	21.40374	.00000	22.08262	.00000	23.33616	.00000	22.09669	.00000
25.21203	.00000	24.15425	.00000	25.21293	.00000	27.07664	.00000	25.24442	.00000
28.34486	.00000	26.77418	.00000	28.34564	.00000	30.98270	.00000	28.40789	.00000
Eigenvalues for 40 × 40 matrix									
1.30654	0.00000	1.30631	0.00000	1.30944	0.00000	1.30660	0.00000	1.30654	0.00000
3.67319	.00000	3.67107	.00000	3.67546	.00000	3.67455	.00000	3.67320	.00000
6.58462	.00000	6.57706	.00000	6.58610	.00000	6.59245	.00000	6.58462	.00000
9.63168	.00000	9.61208	.00000	9.63271	.00000	9.65618	.00000	9.63169	.00000
12.72324	.00000	12.68152	.00000	12.72405	.00000	12.77975	.00000	12.72325	.00000
15.83411	.00000	15.75647	.00000	15.83474	.00000	15.94309	.00000	15.83416	.00000
18.95497	.00000	18.82367	.00000	18.95552	.00000	19.14208	.00000	18.95514	.00000
22.08166	.00000	21.87475	.00000	22.08212	.00000	22.37773	.00000	22.08212	.00000
25.21203	.00000	24.90315	.00000	25.21245	.00000	25.65311	.00000	25.21309	.00000
28.34486	.00000	27.90303	.00000	28.34523	.00000	28.97224	.00000	28.34706	.00000

TABLE VIII.- EIGENVALUES FOR $\tau_1 = \tau_0 = -1.0i$

Characteristic equation		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
0.00000	1.54340	0.00000	1.54526	0.00000	1.53377	0.00000	1.54298	0.00000	1.54340
2.33112	.00000	2.33505	.00000	2.34163	.00000	2.33258	.00000	2.33112	.00000
5.95017	.00000	5.95028	.00000	5.95383	.00000	5.97450	.00000	5.95018	.00000
9.20843	.00000	9.18579	.00000	9.21063	.00000	9.29873	.00000	9.20850	.00000
12.40549	.00000	12.32763	.00000	12.40724	.00000	12.62670	.00000	12.40597	.00000
15.57977	.00000	15.39728	.00000	15.58110	.00000	16.01870	.00000	15.58170	.00000
18.74295	.00000	18.38680	.00000	18.74412	.00000	19.50828	.00000	18.74875	.00000
21.89990	.00000	21.27982	.00000	21.90085	.00000	23.12121	.00000	21.91416	.00000
25.05295	.00000	24.05681	.00000	25.05385	.00000	26.87866	.00000	25.08355	.00000
28.20345	.00000	26.69750	.00000	28.20422	.00000	30.79625	.00000	28.26253	.00000
Eigenvalues for 40 × 40 matrix									
0.00000	1.54340	0.00000	1.54391	0.00000	1.53868	0.00000	1.54330	0.00000	1.54340
2.33112	.00000	2.33205	.00000	2.33641	.00000	2.33147	.00000	2.33112	.00000
5.95017	.00000	5.94890	.00000	5.95196	.00000	5.95594	.00000	5.95017	.00000
9.20843	.00000	9.19822	.00000	9.20954	.00000	9.22984	.00000	9.20844	.00000
12.40549	.00000	12.37610	.00000	12.40634	.00000	12.45785	.00000	12.40551	.00000
15.57977	.00000	15.51717	.00000	15.58042	.00000	15.68354	.00000	15.57982	.00000
18.74295	.00000	18.62921	.00000	18.74351	.00000	18.92377	.00000	18.74311	.00000
21.89990	.00000	21.71283	.00000	21.90035	.00000	22.18858	.00000	21.90032	.00000
25.05295	.00000	24.76600	.00000	25.05338	.00000	25.48555	.00000	25.05397	.00000
28.20345	.00000	27.78539	.00000	28.20389	.00000	28.82121	.00000	28.20557	.00000

TABLE IX.- EIGENVALUES FOR $\tau_1 = \tau_0 = -2.1i$

Characteristic equation		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
0.00000	0.77848	0.00000	0.78230	0.00000	0.58277	0.00000	0.77843	0.00000	0.77848
.00000	2.48273	.00000	2.49160	.00000	2.44420	.00000	2.48096	.00000	2.48273
5.56104	.00000	5.57053	.00000	5.57751	.00000	5.58088	.00000	5.56104	.00000
8.96457	.00000	8.95728	.00000	8.97436	.00000	9.04780	.00000	8.96462	.00000
12.22616	.00000	12.16851	.00000	12.23378	.00000	12.43768	.00000	12.22659	.00000
15.43756	.00000	15.27930	.00000	15.44333	.00000	15.86414	.00000	15.43937	.00000
18.62500	.00000	18.29642	.00000	18.63010	.00000	19.37518	.00000	18.63054	.00000
21.79907	.00000	21.20934	.00000	21.80325	.00000	23.00229	.00000	21.81285	.00000
24.96490	.00000	24.00133	.00000	24.96881	.00000	26.76935	.00000	24.99457	.00000
28.12528	.00000	26.65374	.00000	28.12862	.00000	30.69346	.00000	28.18238	.00000
Eigenvalues for 40 × 40 matrix									
0.00000	0.77848	0.00000	0.77948	0.00000	0.68633	0.00000	0.77847	0.00000	0.77848
.00000	2.48273	.00000	2.48514	.00000	2.46383	.00000	2.48231	.00000	2.48273
5.56104	.00000	5.56255	.00000	5.56914	.00000	5.56575	.00000	5.56104	.00000
8.96457	.00000	8.95900	.00000	8.96949	.00000	8.98431	.00000	8.96457	.00000
12.22616	.00000	12.20305	.00000	12.22988	.00000	12.27627	.00000	12.22617	.00000
15.43756	.00000	15.38277	.00000	15.44043	.00000	15.53849	.00000	15.43760	.00000
18.62500	.00000	18.52045	.00000	18.62745	.00000	18.80239	.00000	18.62515	.00000
21.79907	.00000	21.62247	.00000	21.80112	.00000	22.08373	.00000	21.79949	.00000
24.96490	.00000	24.68953	.00000	24.96674	.00000	25.39288	.00000	24.96589	.00000
28.12528	.00000	27.71983	.00000	28.12688	.00000	28.73777	.00000	28.12736	.00000

TABLE X.- EIGENVALUES FOR $\tau_1 = \tau_0 = 4.1 - 3.3i$

Characteristic equation		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
4.05940	-2.38583	3.83750	-2.21795	3.58022	-2.83565	4.05883	-2.39839	4.05939	-2.38583
4.08004	-3.59351	4.13780	-3.51721	3.86405	-3.62235	4.06948	-3.60879	4.08004	-3.59351
4.35109	-2.11202	4.66144	-2.18904	4.61835	-1.65678	4.35390	-2.12420	4.35109	-2.11202
8.62554	-.96961	8.66603	-.92805	8.62339	-.89348	8.69659	-.99469	8.62557	-.96964
12.00460	-.68969	11.99051	-.62532	11.99948	-.63752	12.20232	-.72482	12.00492	-.68984
15.27002	-.54014	15.15506	-.45896	15.26517	-.50232	15.68049	-.58540	15.27717	-.54054
18.48940	-.44525	18.20467	-.35047	18.48483	-.41264	19.22072	-.50102	18.49456	-.44612
21.68482	-.37922	21.13956	-.27317	21.68084	-.35283	22.86611	-.44604	21.69788	-.38097
24.86602	-.33048	23.94751	-.21498	24.86227	-.30598	26.64518	-.40899	24.89434	-.33362
28.03804	-.29295	29.11508	-.13313	28.03473	-.27217	30.57725	-.38394	28.09212	-.29953
Eigenvalues for 40 × 40 matrix									
4.05940	-2.38583	3.98677	-2.30773	3.78101	-2.68000	4.05931	-2.38870	4.05940	-2.38583
4.08004	-3.59351	4.09572	-3.57183	3.97113	-3.61134	4.07756	-3.59716	4.08004	-3.59352
4.35109	-2.11202	4.44705	-2.16374	4.52468	-1.81815	4.35176	-2.11502	4.35109	-2.11202
8.62554	-.96961	8.63367	-.95628	8.62367	-.93074	8.64244	-.97553	8.62554	-.96961
12.00460	-.68969	11.99444	-.66899	12.00148	-.66398	12.05150	-.69789	12.00458	-.68970
15.27002	-.54014	15.22871	-.51343	15.26718	-.52118	15.36729	-.55058	15.27006	-.54015
18.48940	-.44525	18.39916	-.41323	18.48681	-.42947	18.66258	-.45793	18.48955	-.44527
21.68482	-.37922	21.52344	-.34242	21.68257	-.36622	21.96467	-.39416	21.68522	-.37927
24.86602	-.33048	24.60677	-.28937	24.86395	-.31870	25.28854	-.34771	24.86698	-.33057
28.03804	-.29295	27.64954	-.24797	28.03621	-.28296	28.64445	-.31251	28.04006	-.29310

TABLE XI.- EIGENVALUES FOR $\tau_1 = 2.0 - 1.6i$ and $\tau_0 = 0.0$

Characteristic equation		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
1.83408	-1.21857	1.82413	-1.20386	1.77037	-1.31520	1.83385	-1.21979	1.83408	-1.21857
2.34205	-.98715	2.36291	-.99025	2.35621	-.89039	2.34274	-.98891	2.34205	-.98715
6.01206	-.33637	6.01261	-.32716	6.01042	-.32426	6.03691	-.34060	6.01206	-.33637
9.25048	-.21718	9.22650	-.20285	9.24918	-.20973	9.34188	-.22371	9.25055	-.21719
12.43718	-.16120	12.35680	-.14264	12.43616	-.15573	12.65998	-.17007	12.43766	-.16125
15.60516	-.12836	15.41924	-.10624	15.60532	-.12400	16.04621	-.13966	15.60711	-.12847
18.76413	-.10670	18.40382	-.08160	18.76432	-.10305	19.53210	-.12054	18.76999	-.10694
21.91805	-.09130	21.29319	-.06380	21.91743	-.08817	23.14256	-.10785	21.93240	-.09178
25.06885	-.07980	24.06740	-.05227	25.06830	-.07703	26.89830	-.09922	25.09959	-.08068
28.21760	-.07090	26.70580	-.03960	28.21708	-.06839	30.81475	-.09334	28.27697	-.07277
Eigenvalues for 40 × 40 matrix									
1.83408	-1.21857	1.83141	1.21460	1.80435	-1.27113	1.83403	-1.21886	1.83408	-1.21857
2.34205	-.98715	2.34749	-.98786	2.34716	-.93484	2.34222	-.98757	2.34205	-.98715
6.01206	-.33637	6.01085	-.33350	6.01118	-.33035	6.01795	-.33737	6.01206	-.33637
9.20516	-.21718	9.23981	-.21282	9.24980	-.21349	9.27214	-.21871	9.25048	-.21718
12.43718	-.16120	12.40694	-.15539	12.43665	-.15851	12.48991	-.16327	12.43719	-.16120
15.60516	-.12836	15.54141	-.12120	15.60472	-.12623	15.70942	-.13096	15.60521	-.12836
18.76413	-.10670	18.64896	-.09830	18.76376	-.10493	18.94555	-.10983	18.76429	-.10671
21.91805	-.09130	21.72932	-.08179	21.91773	-.08981	22.20746	-.09500	21.91849	-.09133
25.06885	-.07980	24.78000	-.06927	25.06857	-.07850	25.50228	-.08406	25.06987	-.07985
28.21760	-.07090	27.79743	-.05943	28.21734	-.06973	28.83629	-.07571	28.21971	-.07095

TABLE XII. - EQUAL AND NEARLY EQUAL EIGENVALUES FOR 40 × 40 MATRICES

τ_1	τ_0	Exact	MWR	Cubic finite element
3.00000 + 0.00000i	-3.00000 + 0.00000i	3.00000 + 0.00000i	3.06559 + 0.20703i	3.00000 + 0.00000i
		3.14159 + 0.00000i	3.06559 - 0.20703i	3.14159 + 0.00000i
3.10000 + 0.00000i	-3.10000 + 0.00000i	3.10000 + 0.00000i	3.11612 + 0.22473i	3.10000 + 0.00000i
		3.14159 + 0.00000i	3.11612 - 0.22473i	3.14159 + 0.00000i
3.14159 + 0.00000i	-3.14159 + 0.00000i	3.14159 + 0.00000i	3.13716 + 0.22846i	3.14159 + 0.00000i
		3.14159 + 0.00000i	3.13716 - 0.22846i	3.14159 + 0.00000i
3.20000 + 0.00000i	-3.20000 + 0.00000i	3.20000 + 0.00000i	3.16674 + 0.23038i	3.20000 + 0.00000i
		3.14159 + 0.00000i	3.16674 - 0.23038i	3.14159 + 0.00000i
3.50000 + 0.00000i	-3.50000 + 0.00000i	3.50000 + 0.00000i	3.31933 + 0.17317i	3.50000 + 0.00000i
		3.14159 + 0.00000i	3.31933 - 0.17317i	3.14159 + 0.00000i
3.60000 + 0.00000i	-3.60000 + 0.00000i	3.60000 + 0.00000i	3.37047 + 0.10903i	3.60000 + 0.00000i
		3.14159 + 0.00000i	3.37047 - 0.10903i	3.14159 + 0.00000i
3.70000 + 0.00000i	-3.70000 + 0.00000i	3.70000 + 0.00000i	3.31454 + 0.00000i	3.70000 + 0.00000i
		3.14159 + 0.00000i	3.52898 + 0.00000i	3.14159 + 0.00000i

TABLE XII.- Concluded

τ_1	τ_0	Exact	MWR	Cubic finite element
9.42480 + 0.00000i	-9.42480 + 0.00000i	9.42480 + 0.00000i	9.40155 + 0.73519i	9.42479 - 0.00002i
		9.42478 + 0.00000i	9.40155 - 0.73519i	9.42479 + 0.00002i
9.71240 + 0.00000i	-9.71240 + 0.00000i	9.71240 + 0.00000i	9.56896 + 0.74274i	9.71240 + 0.00000i
		9.42478 + 0.00000i	9.56896 - 0.74274i	9.42478 + 0.00000i
9.40000 + 0.20000i	-9.40000 - 0.20000i	9.40000 + 0.20000i	9.40655 - 0.62671i	9.40000 + 0.20000i
		9.42478 + 0.00000i	9.36659 + 0.85667i	9.42478 + 0.00000i
9.40000 - 0.20000i	-9.40000 + 0.20000i	9.40000 - 0.20000i	9.40655 + 0.62671i	9.40000 - 0.20000i
		9.42478 + 0.00000i	9.36659 - 0.85667i	9.42478 + 0.00000i
9.40000 + 0.50000i	-9.40000 - 0.50000i	9.40000 + 0.50000i	9.43120 - 0.49952i	9.40000 + 0.50000i
		9.42478 + 0.00000i	9.33486 + 1.07274i	9.42478 + 0.00000i
10.00000 + 0.00000i	-10.00000 + 0.00000i	10.00000 + 0.00000i	9.74090 + 0.71197i	10.00000 + 0.00000i
		9.42478 + 0.00000i	9.74090 - 0.71197i	9.42478 + 0.00000i

TABLE XIII.- NEARLY EQUAL EIGENVALUES FOR 20 × 20 MATRICES

τ_1	τ_0	Characteristic equation	MWR	Cubic finite element
4.12000 - 3.30000i	4.12000 - 3.30000i	4.16304 - 2.24954i	3.60762 - 2.82417i	4.16303 - 2.24954i
		4.26194 - 2.25104i	4.60435 - 1.62190i	4.26194 - 2.25104i
7.44000 - 3.79000i	7.44000 - 3.79000i	7.45536 - 2.74214i	6.66647 - 3.89034i	7.45538 - 2.74235i
		7.54082 - 2.79492i	7.88682 - 1.84131i	7.54075 - 2.79473i
10.67000 - 4.12000i	10.67000 - 4.12000i	10.70633 - 3.01156i	9.53056 - 4.90826i	10.70795 - 3.01133i
		10.71889 - 3.19771i	11.10472 - 1.83781i	10.71691 - 3.19828i
2.06000 - 1.65000i	0.00000 + 0.00000i	2.08152 - 1.12477i	1.90829 - 1.33034i	2.08152 - 1.12477i
		2.13097 - 1.12552i	2.25154 - 0.92042i	2.13097 - 1.12552i
5.33000 - 2.06000i	0.00000 + 0.00000i	5.31568 - 1.61272i	5.01863 - 2.16940i	5.31568 - 1.61272i
		5.39352 - 1.49162i	5.50883 - 1.03003i	5.39352 - 1.49162i
11.69000 - 2.43000i	0.00000 + 0.00000i	11.66229 - 1.91260i	10.93670 - 3.67069i	11.66594 - 1.91233i
		11.73692 - 1.94549i	11.84190 - 1.06593i	11.73343 - 1.94656i

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TABLE XIV.- EIGENVALUES FOR $\tau_1 = -1/2 - i/2$ and $\tau_0 = 1.0$

Characteristic equation		Finite difference		MWR		Linear finite element		Cubic finite element	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
Eigenvalues for 20 × 20 matrix									
0.00000	0.00000	0.00000	0.00000	0.06105	-0.11444	0.00000	0.00000	0.00000	0.00000
2.98610	-.19436	2.98634	-.19244	2.98236	-.19686	2.98914	-.19496	2.98610	-.19436
6.20341	-.08345	6.19575	-.08039	6.20174	-.08428	6.23098	-.08457	6.20341	-.08345
9.37164	-.05417	9.33760	-.04993	9.37049	-.05476	9.46688	-.05584	9.37172	-.05417
12.52654	-.04026	12.43416	-.03498	12.52572	-.04065	12.75442	-.04251	12.52705	-.04027
15.67611	-.03207	15.47648	-.02594	15.67542	-.03242	16.12355	-.03492	15.67813	-.03210
18.82302	-.02667	18.44757	-.01984	18.82246	-.02693	19.59875	-.03015	18.82800	-.02673
21.96841	-.02282	21.32723	-.01544	21.96790	-.02308	23.20215	-.02698	21.98302	-.02294
25.11284	-.01995	24.09415	-.01211	25.11242	-.02015	26.95312	-.02483	25.14409	-.02018
28.25665	-.01773	26.72685	-.00952	28.25624	-.01793	30.86631	-.02336	28.31715	-.01822
Eigenvalues for 40 × 40 matrix									
0.00000	0.00000	0.00000	0.00000	0.04325	-0.08094	0.00000	0.00000	0.00000	0.00000
2.98610	-.19436	2.98597	-.19384	2.98428	-.19558	2.98682	-.19450	2.98610	-.19436
6.20341	-.08345	6.19987	-.08259	6.20258	-.08388	6.20995	-.08372	6.20341	-.08345
9.37164	-.05417	9.35802	-.05294	9.37108	-.05446	9.39420	-.05456	9.37164	-.05417
12.52654	-.04026	12.49264	-.03867	12.52614	-.04046	12.58045	-.04078	12.52654	-.04026
15.67611	-.03207	15.60802	-.03014	15.67578	-.03224	15.78184	-.03272	15.67616	-.03207
18.82302	-.02667	18.70285	-.02442	18.82275	-.02679	19.00620	-.02745	18.82318	-.02667
21.96841	-.02282	21.77407	-.02030	21.96817	-.02294	22.25986	-.02375	21.96885	-.02283
25.11284	-.01995	24.81783	-.17169	25.11264	-.02005	25.54862	-.02102	25.11388	-.01996
28.25665	-.01773	27.82985	-.01471	28.25646	-.01782	28.87801	-.01893	28.25879	-.01773

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