INFLUENCE OF SURFACE ROUGHNESS AND WAVINESS ON FILM THICKNESS AND PRESSURE DISTRIBUTION IN ELASTOHYDRODYNAMIC CONTACTS

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The Christensen theory of a stochastic model for hydrodynamic lubrication of rough surfaces was extended to elastohydrodynamic lubrication between two rollers. The Grubin-type equation including asperity effects in the inlet region was derived. Solutions for the reduced pressure at the entrance as a function of the ratio of the average nominal film thickness to the rms surface roughness (in terms of standard deviation $a$) were obtained numerically. Results were obtained for purely transverse as well as purely longitudinal surface roughness for cases with or without slip. The reduced pressure was shown to decrease slightly by considering longitudinal surface roughness. The transverse surface roughness, on the other hand, had a slight beneficial effect on the average film thickness at the inlet. The same approach was used to study the effect of surface roughness on lubrication between rigid rollers and lubrication of an infinitely wide slider bearing. Results of these two cases showed that the effects of surface roughness have the same trend as those found in elastohydrodynamic contacts. A comparison was made between the results using the stochastic approach and the results using the conventional deterministic method for the inlet pressure in a Hertzian contact assuming a sinusoidal roughness. It was found that the validity of the stochastic approach depends upon the number of wave cycles $n$ within the Hertzian contact. Using the flow balance concept, the perturbed Reynolds equation, which includes a single three-dimensional rigid asperity in one of the lubricating surfaces, was derived and solved for the perturbed pressure distribution. In addition, Cheng’s numerical scheme, for EHD contacts, was modified to incorporate a single two-dimensional elastic asperity, or a waviness profile, on the stationary surface. The perturbed pressures obtained by these three different models were compared.
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LIST OF SYMBOLS

a' = semi-major axis of the ellipsoidal asperity
b = half width of Hertzian contact
b' = semi-minor axis of the ellipsoidal asperity
\( \overline{b} \) = \( b'/b \)
c = integration constant
c_1 = maximum asperity height
\( \overline{c_1} \) = \( c_1/h_o \)
C_{UD} = 48(U_D/H_o^2)(\overline{b})
\( 1/E' \) = \( 1/2 \left( \frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2} \right) \)
E_1, E_2 = Young's modulus for rollers 1 and 2
f(\( \gamma \)) = probability density function of random variable \( \gamma \)
g(\( \delta^* \)) = probability density function of surface roughness distribution
\( = \begin{cases} 
35/96 \left( 1-\delta^{*2}/9 \right)^3 & -3 < \delta^* < 3 \\
0 & \text{elsewhere}
\end{cases} \) for polynomial distribution function
\( = \begin{cases} 
1/(\pi \sqrt{1-\delta^{*2}/9}) & -3 < \delta^* < 3 \\
0 & \text{elsewhere}
\end{cases} \) for sinusoidal distribution function
G = \( \alpha E' \)
G_2 = \( \int_0^\infty \frac{g(\delta^*)}{\left( 1 + \frac{3}{H} \delta^* \right)^3} d\delta^* \)
G_4 = \( \int_0^\infty \frac{\delta^* g(\delta^*)}{\left( 1 + \frac{3}{H} \delta^* \right)^3} d\delta^* \)
h = average film thickness
h_1 = smooth-film thickness of an EHD contact
h_c = average center film thickness of rigid rollers
\( h_g \) = smooth-film thickness between two rigid rollers

\( h_o \) = center film thickness at \( x = 0 \) for EHD contacts as well as for rigid roller; also used as the minimum film thickness for the slider bearing

\( h_T \) = local film thickness

\( h^* \) = average reference film thickness at \( p = dp/dx = 0 \)

\( h_{\min} \) = nominal film thickness at the exit of the slider bearing

\( H \) = \( h/h_o \)

\( H_1 \) = \( h_1/h_o \)

\( H_0 \) = \( h_o/R \)

\( H_T \) = \( h_T/h_o \)

\( K_D \) = \( (Q^*_W - Q^*_R)/Q^*_R \)

\( K_E \) = \( Q^*_R/Q^*_s \)

\( K_R \) = \( w^*_R/w^*_s \)

\( i \) = a dummy index

\( j \) = a dummy index

\( L \) = overall characteristic length of a slider bearing

\( m \) = slope of the inclined surface of a slider

\( m_1, m_2, m_3, m_4 \)

= mass of lubricants flowing into or out of the grid

\( M \) = the number of grids in the Y-direction such that \( y/a^' = \pm 5 \) at \( M \)

\( M(x) \) = see Eq. (2.10)

\( n \) = number of wave cycles within the Hertzian contact

\( N \) = the number of grids in the X-direction such that \( x/b^' = \pm 5 \) at \( N \)

\( p \) = local pressure

\( p_1 \) = smooth film pressure

\( P_{Hz} \) = maximum Hertzian pressure

\( \bar{p} \) = the expected or mean value of \( p \)

\( \text{P} \) = \( p/P_{Hz} \)
\[ P = \frac{p}{p_{Hz}} \text{ for EHD analyses} \]
\[ P = \frac{h^2 o^2}{6 \mu (u_1 + u_2) R} \text{ for the analysis of rigid rollers} \]
\[ P = \frac{h^2 o^2}{6 \mu u_1 l} \text{ for the analysis of a slider bearing} \]
\[ P_1 = \frac{p_1}{p_{Hz}} \]
\[ P_{Hz} = \frac{p_{Hz}}{E'} \]
\[ q = \frac{1 - e^{-\alpha p}}{\alpha} \]
\[ \bar{q} = \text{the expected or mean value of } q \]
\[ Q = \frac{R^2}{6 \mu_0 (u_1 + u_2)b} \cdot q = 48 \sqrt{\frac{W}{2\pi}} U \left( \frac{R}{E'} \right) \]
\[ Q_R = Q^* \text{ calculated by the stochastic theory for roughness model} \]
\[ Q_W = Q^* \text{ calculated by the waviness model} \]
\[ Q_S = Q^* \text{ for the smooth film analysis} \]
\[ R = \frac{R_1 R_2}{R_1 + R_2} \]
\[ R_1, R_2 = \text{radii of rollers 1 and 2} \]
\[ S = \frac{2(u_1 - u_2)}{(u_1 + u_2)} \]
\[ t = \text{time} \]
\[ u_1, u_2 = \text{velocities of surfaces 1 and 2} \]
\[ u = \frac{(u_1 + u_2)}{2} \]
\[ U = \mu_s (u_1 + u_2) / (2E'R) \]
\[ U_D = \mu_s (u_1 - u_2) / (2E'R) \]
\[ w = \text{load per unit width} \]
\[ W = w / E'R \]
\[ W_R^* = W^* \text{ calculated by stochastic theory for rigid rollers} \]
\[ W_S^* = W^* \text{ calculated by smooth film theory for rigid rollers} \]
\[ W^* = h^2 o^2 w / (6 \mu(u_1 + u_2) R^2) \text{ for the analysis of rigid rollers} \]
\[ W^*_S = h^2 o^2 w / (6 \mu u_1 l^2) \text{ for the analysis of a slider bearing} \]
$x$ = coordinate along the sliding direction

$x_3$ = distance between the asperity center and the contact center

$x^*$ = reference coordinate at $p = dp/dx = 0$

$X_3 = x_3/b$

$X = x/b$ for EHD analyses

$X = x/R$ for the analysis of rigid rollers

$X = x/l$ for the analysis of a slider bearing

$\Delta t$ = grid size in $x$-direction

$\Delta X = \Delta x/b$

$y$ = coordinate perpendicular to the sliding direction

$Y = y/R'$

$\Delta y$ = grid size in $y$-direction

$\Delta Y = \Delta y/a'$

$\mu$ = viscosity

$\mu_s = ambient \viscosity$

$\mu_e = \mu/\mu_s$

$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

$\sigma_1, \sigma_2 = standard \ deviation \ of \ roughness \ amplitudes \ of \ surfaces \ 1 \ and \ 2$

$\delta = \delta_1 + \delta_2$

$\delta_1, \delta_2 = roughness \ amplitude \ measured \ from \ the \ mean \ level \ of \ surfaces \ 1 \ and \ 2$

$\delta_{max} = \delta_1$

$\delta^{*} = \delta/\sigma$

$\delta^{*}_{max} = 3$

$\xi = dummy \ variable$

$\xi = \xi/b$

$v_1, v_2 = Poison's \ ratio \ of \ rollers \ 1 \ and \ 2$

$\alpha = pressure \ viscosity \ coefficient$
\bar{\alpha} = \alpha_{pHz}
\gamma = \text{random variable, also used as ellipticity ratio } a'/b'
\epsilon\{ \} = \text{the expected value of } \{ \}
\theta = \text{the phase angle of the waviness surface profile measured from } X = 0
\theta_1, \theta_2 = \theta \text{ of surfaces } 1 \text{ and } 2
\phi = \text{perturbed pressure}
\bar{\phi} = \phi_{pHz}
\bar{\delta}_{\text{max}} - \bar{\delta}_{\text{min}} = \text{the maximum and the minimum perturbed pressure measured from the average pressure profile}
\Delta s = \bar{\delta}_{\text{max}} - \bar{\delta}_{\text{min}}
\rho = \text{lubricant density}
\bar{\delta} = \bar{\delta}/h_o
\bar{\delta}_1 = \bar{\delta}_1/h_o
\bar{\delta}_2 = \bar{\delta}_2/h_o
\bar{\delta}_{\text{max}} = \bar{\delta}_{\text{max}}/h_o
\bar{\sigma} = \sigma/h_o
SUMMARY

The Christensen Theory of a stochastic model for hydrodynamic lubrication of rough surfaces is extended to elastohydrodynamic lubrication between two rollers. The Grubin-type equation including asperity effects in the inlet region is derived. Solutions for the reduced pressure at the entrance as a function of the ratio of the average nominal film thickness to the r.m.s. surface roughness (in terms of standard deviation $\sigma$), have been obtained numerically. Results were obtained for purely transverse as well as purely longitudinal surface roughness for cases with or without slip. The reduced pressure is shown to decrease slightly by considering longitudinal surface roughness. The transverse surface roughness, on the other hand, has a slight beneficial effect on the average film thickness at the inlet.

The same approach was used to study the effect of surface roughness on lubrication between rigid rollers and lubrication of an infinitely-wide slider bearing. Results of these two cases show that the effects of surface roughness have the same trend as those found in elastohydrodynamic contacts.

A comparison is made between the results using the stochastic approach and the results using the conventional deterministic method for the inlet pressure in a Hertzian contact assuming a sinusoidal roughness. It was found that the validity of the stochastic approach depends upon the number of wave cycles $n$ within the Hertzian contact. For $n$ larger than a critical number, which depends upon the ratio of asperity height to the nominal film thickness, $\delta_{\text{max}}/h_0$, the stochastic theory yields the same results as that obtained by the deterministic approach.
Using the flow balance concept, the perturbed Reynolds equation, which includes a single three-dimensional rigid asperity in one of the lubricating surfaces, is derived and solved for the perturbed pressure distribution. In addition, the Cheng's numerical scheme, for EHD contacts, is modified to incorporate a single two-dimensional elastic asperity, or a waviness profile, on the stationary surface. The perturbed pressures obtained by these three different models are compared. Qualitatively, the results for the single 2D elastic asperity and the waviness profile model by using Cheng's scheme, are mostly the same. However, some results obtained for the single 3D rigid asperity exhibit different trends when compared with the single 2D elastic asperity or the waviness profile. In the case of the waviness profile in which the local elastic deformation is allowed, the magnitude of the pressure fluctuation \( \Delta_s \), is found to increase when the pressure viscosity parameter \( G \), or the ratio of the asperity amplitude to the nominal film thickness, \( \delta_{\text{max}} / h_0 \), increases. On the other hand, \( \Delta_s \) is found to decrease as the magnitude of the Hertzian pressure \( P_{\text{Hz}} \), or the ratio of the nominal film thickness to the radius of the equivalent cylinder, \( h_0 / R \) increases.
CHAPTER I
INTRODUCTION

In conventional sliding bearings, there usually exists a high degree of conformity between bearing surfaces, and this enables a substantial load to be generated by the thin oil film. The performance of these bearings can be satisfactorily predicted by solving the Reynolds equation for the pressure distribution within the lubricant film. However, for highly loaded concentrated contacts, such as gears, cams and rolling contact elements, the lubrication phenomenon cannot be predicted by Reynolds equation alone. Local elastic deformation of the solid under high pressure becomes influential in determining the load capacity and film thickness of these contacts. The study of lubrication processes including the elastic effects is presently known as elastohydrodynamic lubrication (EHL).

To date, the theories for hydrodynamic and elastohydrodynamic lubrication have reached a very advanced stage. However, these theories are mostly based on the assumption that the lubricating surfaces can be described by smooth mathematical functions. In reality, surfaces are never perfectly smooth in a microscopic scale. In the hydrodynamic lubrication regime, the asperity heights of the rough surfaces are much smaller than the average lubricant film. Thus, the effect of surface roughness on hydrodynamic lubrication, in most cases, can be neglected. Hence, the smooth film hydrodynamic lubrication theories provides a very satisfactory prediction of lubrication performance. In elastohydrodynamic lubrication of concentrated contacts, there exist two distinctively different regimes, the full film and the partial film EHL. In the full film
regime, the average nominal film thickness is usually much greater than the asperity heights, and, in this regime, the behavior of the contact can be predicted quite satisfactorily by smooth-film EHL theories. In the partial film regime, the asperity heights are of the same order as the average nominal lubricant film. Thus, the effect of surface roughness in the regime must be considered.

EHD film thickness has been well accepted as an important bearing design parameter. The degree of asperity interactions, and the surface distress in the forms of wear, pitting and scuffing are associated with the ratio of the average nominal film thickness to the r.m.s. surface roughness (in terms of standard deviation σ), in EHD contacts. In many cases, bearing failures can be attributed to insufficient film thickness which leads to asperity contacts. Thus, there is a need to determine the surface on the film forming capability in EHD contacts. Therefore, the second chapter of this dissertation is focused on the effect of surface roughness on the average film thickness between lubricated rollers. The stochastic theory developed by Christensen [7] is extended to determine the surface roughness influence on the inlet film thickness of EHD contacts. In addition, the roughness effect on the load capacity in rigid rollers and infinitely-wide slider bearing is also studied.

The third chapter compares the difference between the roughness effect and waviness effect on the average film thickness in EHD contacts. It also provides some criteria to determine the applicability of the stochastic theory.

The pressure profile enables one to predict the stress distribution of the lubricated contact. In the fourth and fifth chapters, the effect of surface roughness on the pressure distribution are discussed. In
Chapter IV, comparisons are made between the effect on the perturbed pressure due to a single three-dimensional rigid asperity and due to a single two-dimensional elastic asperity within an EHD contact. In Chapter V, the surface profile before elastic deformation is assumed to be in the form of sinusoidal waviness. The effects of the following non-dimensional variables: the Hertzian pressure $P_{Hz}$, the nominal center film thickness $h_o/R$, the pressure viscosity parameter $G$, the number of wave cycles within the Hertzian contact, $n$, and the asperity height $h_o/R$, on the magnitude of the pressure fluctuation as well as the perturbed pressure profile are studied.
CHAPTER II

THE EFFECT OF SURFACE ROUGHNESS ON THE AVERAGE FILM THICKNESS BETWEEN LUBRICATED ROLLERS

2.1 INTRODUCTION

The inclusion of surface irregularities in lubrication analysis can be traced back to [1-3], in which the roughness is modelled as sinusoidal or saw-tooth waviness. Subsequently, Tseng and Saibel [6] introduced the stochastic concept based on random surface roughness analysis on lubrication. Their method deals with surfaces with one dimensional transverse roughness only. The stochastic model has been revived by Christensen and his colleagues [7,8,9,16,17] in studying the lubrication process between rigid surfaces containing surface roughness modelled as ridges oriented transversely or longitudinally. Recently, the effects striated roughness on both bearing surfaces have been obtained by Rhow and Elrod [18].

The effect of surface roughness on film thickness in EHD contact has not been fully explored. However, there have been some related work. For instance, Fowles [4] studied the EHD lubrication between identical sliding asperities. Lee and Cheng [5] have studied the effect of a single asperity on the film and pressure distribution during its entrance into an elastohydrodynamic contact. The load sharing between fluid film and asperity contacts as well as the traction in partial EHD contacts have been studied by Tallian [10], and Thompson and Bocci [12]. Johnson, Greenwood and Poon [11] have used an approximate analysis to ascertain the effect of roughness on the EHD film thickness. They concluded that, to a first approximation, the separation between two rough surfaces is very close to that calculated by the smooth film theory. Recently, pressure and traction rippling in EHD contact of rough surfaces have been calculated by Tallian [19], by using Christensen's stochastic model of hydrodynamic lubrication.
In the present analysis, Christensen's approach [7] is extended to determine the surface roughness influence on the inlet film thickness of EHD contacts. The Grubin-type hydrodynamic equation in the inlet region for the rough surfaces is derived and solved numerically. Results are compared with smooth-film theories. In addition, the load capacity in rigid rollers and in an infinitely-wide bearing is also presented for comparison.
2.2 GOVERNING EQUATION

Assuming that the lubricant is isothermal and incompressible and the side-leakage is negligible, the one-dimensional Reynolds equation governing the pressure in an EHD contact is

\[
\frac{\partial}{\partial x} \left( \frac{h_T^3}{12\mu} \frac{\partial p}{\partial x} \right) = \left( \frac{u_1 + u_2}{2} \right) \frac{\partial h_T}{\partial x} + \frac{\partial h_T}{\partial t}
\]  

(2.1)

where \( h_T \) is the local total film thickness consisting of the following three parts

\[
h_T = h + \delta_1 + \delta_2
\]  

(2.2)

In the above, \( h \) is the local average film thickness, and \( \delta_1, \delta_2 \), the roughness profile measured from the mean level of surface profiles 1 and 2 (Fig. 2.1).

2.2.1 Transverse Surface Roughness

In this case the asperities on both lubricating surfaces are straight ridges perpendicular to the direction of rolling. Equation (2.2) becomes

\[
h_T = h + \delta_1(x - u_1t) + \delta_2(x - u_2t)
\]  

(2.3)

With the relations,

\[
\frac{\partial}{\partial t} \delta_1(x - u_1t) = -u_1 \frac{\partial \delta_1}{\partial x}
\]  

(2.4)

\[
\frac{\partial}{\partial t} \delta_2(x - u_2t) = -u_2 \frac{\partial \delta_2}{\partial x}
\]  

(2.5)

Eq. (2.1) is simplified to

\[
\frac{\partial}{\partial x} \left[ \frac{h_T^3}{12\mu} \frac{\partial p}{\partial x} - \frac{u_1 + u_2}{2} \right] h + \frac{u_1 - u_2}{2} (\delta_1 - \delta_2) \]  

\[
= \frac{\partial h}{\partial t}
\]  

(2.6)
It is assumed here that there are enough numbers of asperities within the Hertzian zone such that \( h \) can be considered as a constant of time. Let the bracketed term in the left-hand side of Eq. (2.6) be denoted by

\[
M = \frac{h^3}{12\mu} \frac{\partial p}{\partial x} - \frac{u_1 + u_2}{2} h + \frac{u_1 - u_2}{2} (\delta_1 - \delta_2) \tag{2.7}
\]

For infinitely-wide slider bearing and rigid roller bearing, \( M \) is expressed as above. In elastohydrodynamic contact, the reduced pressure, \( q \), and viscosity, \( \mu \), which are respectively

\[
q = \frac{1 - e^{-dp}}{\alpha} \tag{2.8}
\]

\[
\mu = \mu_s e^{\alpha dp} \tag{2.9}
\]

are introduced. Then, \( M \) in an EHD contact is

\[
M = \frac{h_T^3}{12\mu_s} \frac{\partial q}{\partial x} - \frac{u_1 + u_2}{2} h + \frac{u_1 - u_2}{2} (\delta_1 - \delta_2) \tag{2.10}
\]

It is shown in appendix A that \( M \) in the case of EHD contact, rigid roller bearing, or infinitely-wide slider bearing is a stochastic quantity with a negligible variance comparing to the variance of the terms on the right hand side.

Re-arranging Eq. (2.7) and taking expected values on both sides, one obtains

\[
\epsilon \left[ \frac{M}{h_T^3} \right] = \frac{1}{12\mu} \epsilon \left[ \frac{dp}{dx} \right] - \frac{u_1 + u_2}{2} h \epsilon \left[ \frac{1}{h_T^3} \right] + \frac{u_1 - u_2}{2} \left[ \epsilon \left[ \frac{\delta_1}{h_T} \right] - \epsilon \left[ \frac{\delta_2}{h_T} \right] \right] \tag{2.11}
\]

where

\[
\epsilon \left\{ \right\} = \int_{-\infty}^{+\infty} \left\{ \right\} f(r) \delta \gamma \tag{2.12}
\]

and \( f(r) \) is the probability density distribution of the random variable \( \gamma \).
Since $M$ is a stochastic quantity with zero (or negligible) variance, $M$ and $\frac{1}{3h_T}$ can be considered to be (approximately) stochastically independent quantities.

Hence

$$e^{\frac{M}{h_T}} = M e^{\frac{1}{3h_T}} \quad (2.13)$$

Then Eq. (2.11) can be re-written as

$$M = \frac{1}{12\mu} \frac{dp}{dx} \frac{1}{e^{\frac{1}{3h_T}}} - \frac{u_1 + u_2}{2} \frac{h + \frac{u_1 - u_2}{2}}{e^{\frac{1}{3h_T}}} \quad (2.14)$$

where $\bar{p}$ is now the expected or mean value of $p$. Substituting Eq. (2.14) into Eq. (2.6) and re-arranging, one obtains the stochastic Reynolds equation for rigid rollers bearing and infinitely-wide slider bearing.

$$\frac{d}{dx} \left[ \frac{1}{12\mu} \frac{dp}{dx} \frac{1}{e^{\frac{1}{3h_T}}} \right] = \frac{u_1 + u_2}{2} \frac{dh}{dx} - \frac{u_1 - u_2}{2} \frac{d}{dx} \left[ \frac{e^{\frac{1}{3h_T}} - e^{\frac{1}{3h_T}}}{e^{\frac{1}{3h_T}}} \right] \quad (2.15)$$

Similarly, the stochastic Reynolds equation for EHD contact can be expressed as

$$\frac{d}{dx} \left[ \frac{1}{12\mu_0} \frac{dg}{dx} \frac{1}{e^{\frac{1}{3h_T}}} \right] = \frac{u_1 + u_2}{2} \frac{dh}{dx} - \frac{u_1 - u_2}{2} \frac{d}{dx} \left[ \frac{e^{\frac{1}{3h_T}} - e^{\frac{1}{3h_T}}}{e^{\frac{1}{3h_T}}} \right] \quad (2.16)$$

### 2.2.2 Longitudinal Surface Roughness

When the asperity ridges are parallel to the direction of rolling, $\delta_1$ and $\delta_2$ are independent upon $x$, $u_1$, $u_2$ and $t$. Again, assuming $\frac{dh}{dt} = 0$, then Eq. (2.1) can be simplified to

$$\frac{d}{dx} \left( \frac{h_T^3 \frac{dp}{dx}}{12\mu} \right) = \frac{u_1 + u_2}{2} \frac{dh}{dx} \quad (2.17)$$
For the cases of rigid rollers and infinitely-wide slider bearings, with longitudinal surface roughness, it is shown in [7] that the stochastic Reynolds equation is of the form

\[ \frac{d}{dx} \left[ \frac{1}{12 \mu} \frac{d}{dx} \epsilon \left\{ h^3 \right\} \right] = \frac{u_1 + u_2}{2} \frac{dh}{dx} \]  

(2.18)

Similarly, the stochastic Reynolds equation of an EHD contact can be shown to be

\[ \frac{d}{dx} \left[ \frac{1}{12 \mu_s} \frac{d}{dx} \epsilon \left\{ h^3 \right\} \right] = \frac{u_1 + u_2}{2} \frac{dh}{dx} \]  

(2.19)

If the roughness distribution is symmetric to zero mean

\[ \epsilon \left\{ h^3 \right\} = h^3 \left( 1 + 3 \frac{\sigma^2}{h^2} \right) \]  

(2.19a)

2.3 METHOD OF SOLUTION

2.3.1 Elastohydrodynamic Contacts

Using Grubin's approach, it is assumed that the average surface profile, \( h \), in the inlet region is governed by the deformation produced by a Hertzian elliptical pressure distribution in the contacting region. From [14], this profile is given by

\[ H - 1 = \frac{h - h_o}{h_o} = \frac{4W}{\pi H_o} \left[ |X| \sqrt{X^2 - 1} - \ln \left( |X| + \sqrt{X^2 - 1} \right) \right] \]  

(2.20)

Introducing dimensionless variables \( Q, X, H, H_T, \bar{\sigma}, U, \bar{\delta}_1, \bar{\delta}_2, \bar{\delta} \) and \( S \) as defined in the Nomenclature, Eq. (2.16), for the transverse roughness becomes,

\[ \frac{d}{dx} \left[ \frac{dQ}{dx} \frac{H^2}{h^T} \right] = \frac{dH}{dx} - \frac{S}{2} \frac{d}{dx} \left[ \frac{\epsilon \left\{ \frac{\delta}{3} \right\}}{H^T} - \frac{\epsilon \left\{ \frac{\delta}{3} \right\}}{H^T} \right] \]  

(2.21)

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The boundary conditions are:

\[
\frac{dq}{dx} = 0 \quad \text{at} \quad X = -1 \quad (\text{at} \quad H = 1) \quad (2.22)
\]

\[
Q = 0 \quad \text{at} \quad X = -\infty \quad (2.23)
\]

2.3.1.a Pure Rolling Case

In this case, \( S = 0 \), and Eq. (2.21) becomes

\[
\frac{d}{dx} \left[ \frac{dq}{dx} \frac{H^2}{H_o} \right] = \frac{dH}{dx} \quad (2.24)
\]

Integrating Eq. (2.24) twice with the two boundary conditions, one obtains

\[
Q^* = Q_{X=-1} = \int_{-\infty}^{-1} \frac{H - 1}{H_o} \cdot G_2 \, dx \quad (2.25)
\]

where

\[
G_2 = H^3 \cdot \left\{ \frac{1}{3} \right\}_{H_T}
\]

\[
= H^3 \int_{-\infty}^{\infty} \frac{g(\delta^*)}{h(1 + \delta^* (H_H^3)^3)} \, d\delta^*
\]

\[
= \int_{-\infty}^{\infty} \frac{g(\delta^*)}{1 + \delta^* (H_H^3)^3} \, d\delta^* \quad (2.26)
\]

and

\[
\delta^* = \bar{\delta}/\bar{\sigma}
\]

\[
\bar{\delta} = \delta_1 + \delta_2
\]

\( \bar{\sigma} \) is the composite standard deviation and is defined as

\[
\bar{\sigma}^2 = \sigma_1^2 + \sigma_2^2,
\]

\( g(\delta^*) \) is the roughness height distribution function.
Once $\bar{\sigma}$ and $g(\delta^*)$ are given, Eq. (2.25) can be evaluated numerically for $Q^*$. 

2.3.1.b Rolling and Sliding

If there is relative sliding between surfaces, $S$ will not be zero, and the last term in the stochastic Reynolds equation, Eq. (2.21), will not necessarily vanish. However, if the roughness distribution function of both surface profiles are the same, then

$$\epsilon \left\{ \frac{\delta}{H_T} \right\}_{1/3} = \epsilon \left\{ \frac{\delta}{H_T} \right\}_{2/3} \tag{2.27}$$

Using this relation, Eq. (2.21) becomes

$$\frac{d}{dx} \left[ \frac{dQ}{dx} \left( \frac{H_O^2}{\epsilon \left\{ \frac{1}{H_T} \right\}_{1/3}} \right) \right] = \frac{dH}{dx} \tag{2.28}$$

which is the same as Eq. (2.24) of the pure rolling case. $Q^*$ will accordingly be the same as that expressed in Eq. (2.25).

If, on the other hand, one of the contacting surfaces is considered rough while the other one smooth, then Eq. (2.21) becomes

$$\frac{d}{dx} \left[ \frac{dQ}{dx} \left( \frac{H_O^2}{\epsilon \left\{ \frac{1}{H_T} \right\}_{1/3}} \right) \right] = \frac{dH}{dx} \pm S \frac{d}{dx} \left[ \frac{\epsilon \left\{ \frac{\delta}{H_T} \right\}_{1/3}}{2} \right] \tag{2.29}$$

where the plus sign is for $\delta_1 = 0$, and minus sign for $\delta_2 = 0$.

(See Appendix B for the Fortran IV listing of the numerical analysis.)
Defining $G_4 = \frac{H^3}{\sigma} \left[ \frac{\delta}{H^3} \right]$

$$= \frac{H^3}{\sigma} \int_{-\infty}^{\infty} \frac{-\delta^* g(\delta^*)}{H^3} \frac{d\delta^*}{1 + \frac{\sigma}{H} \delta^*^3}$$

$$= \int_{-\infty}^{\infty} \frac{\delta^* g(\delta^*)}{1 + \frac{\sigma}{H} \delta^*^3}$$

Eq. (2.29), expressed in $G_2$ and $G_4$, becomes

$$\frac{d}{dx} \left[ \frac{H^2}{G_2} \frac{d\sigma}{dx} - H + \frac{G_4}{G_2} \right] = 0$$

Using boundary condition (2.22), one obtains

$$H^2 \frac{H^3}{G_2} \frac{d\sigma}{dx} - H + \frac{G_4}{G_2} = -1 + \frac{G_4}{G_2} \left( \frac{G_4}{G_2} \right)_{x=-1}$$

Integrating Eq. (2.32) between $-\infty$ and $-1$, one obtains the expression for $Q^*$

$$Q^* = \frac{1}{H^2} \left\{ \frac{1}{H^3} \left[ G_2 \frac{1}{x} + \frac{S\sigma}{2} \right] + \frac{1}{H^3} \left[ G_4 - G_2 \left( \frac{G_4}{G_2} \right)_{x=-1} \right] \right\}$$

Eq. (2.33) can be integrated numerically for $Q^*$ for various slide to roll ratios from the pure rolling case, $S = 0$, to the simple sliding case for which $u_1 = u$, $u_2 = 0$, and $S = 2$.

For EHD contacts with longitudinal surface roughness, the stochastic Reynolds equation following Eq. (2.19) becomes

$$\frac{d}{dx} \left\{ \frac{H^2}{G_2} \left[ 1 + \frac{1}{H^3} \left( \frac{\delta}{H} \right)^2 \right] \frac{d\sigma}{dx} \right\} = \frac{dH}{dx}$$
The above equation is valid for any rolling and sliding EHD contacts. Using boundary conditions, Eqs. (2.22) and (2.23), the reduced pressure $Q^*$ at $X = -1$ can be integrated as

$$Q^* = \int_{-\infty}^{1} \frac{H - 1}{H_0 H^2 \left[ 1 + 3 \left( \frac{\sigma}{H} \right)^2 \right]} \, dx$$  \hspace{1cm} (2.35)

### 2.3.2 Rigid Rollers

If the elastic deformation of the rollers is neglected, the smooth, average surface profile can be approximated by a parabolic profile

$$H = \frac{h}{h_0} = 1 + \frac{x^2}{2 \left( \frac{h_0}{R} \right)}$$  \hspace{1cm} (2.36)

where

$$X = \frac{x}{R}$$

$h_0$ = the average center film thickness

Using the same approach as developed in EHD contacts, but with different dimensionless variables for $P$, $X$, $X^*$, $H$, $H_T$, $H^*$, $\sigma$, $\bar{\sigma}_2$ and $\bar{\sigma}$ as defined in the Nomenclature, the stochastic Reynolds equation for rigid rollers with transverse surface roughness can be expressed as

$$\frac{d}{dx} \left[ \frac{dP}{dx} \frac{1}{H_T} \right] = \frac{\epsilon}{2} \frac{dH}{dx} \left[ \frac{H_T}{H_T} \right] - \frac{s}{2} \frac{d}{dx} \left[ \frac{1}{H_T} \right]$$  \hspace{1cm} (2.37)
This equation has the same form as Eq. (2.21) of the EHD contacts. However, the boundary conditions for the case of rigid rollers will be different from Eqs. (2.22) and (2.23). They are given by

\[
\frac{dP}{dx} = 0, \quad P = 0 \quad \text{at} \quad X = X^* \\
P = 0 \quad \text{at} \quad X = -\infty
\]  

(2.38)

Using these boundary conditions, Eq. (2.37) can be readily integrated to yield

\[
P(X) = \int_{-\infty}^{X} \left( \frac{H - H^*}{H^3} \right) G_2 \, d\xi \pm \frac{8\sigma}{2} \int_{-\infty}^{X} \frac{1}{3} \left[ \frac{G_4}{G_2} \left( \frac{G_4}{G_2} \right)^* G_2 \right] \, d\xi
\]

(2.39)

when \( \xi \) is a dummy variable for \( X \), and \( X^* \) and \( H^* \) are determined by imposing \( P(X^*) = 0 \). Eq. (2.39) can be integrated numerically to yield \( P \) for the pure rolling case (\( S = 0 \)), simple sliding case (\( S = 2 \)), as well as rolling and sliding case (any \( S \)). Once \( P(X) \) is found, the dimensionless load \( W^* \), can be determined by

\[
W^* = \left[ \frac{h^2}{6\mu (u_1 + u_2) R} \right] W = \int_{-\infty}^{X^*} P(X) \, dX
\]

(2.40)

For rigid rollers with longitudinal surface roughness, the stochastic Reynolds equation is

\[
\frac{d}{dx} \left[ H^3 \left[ 1 + 3 \left( \frac{\xi}{H} \right)^2 \right] \frac{dP}{dx} \right] = \frac{dH}{dx}
\]

(2.41)

Using boundary conditions, Eq. (2.38), one obtains

\[
P = \int_{-\infty}^{X} \frac{H - H^*}{H^3 \left[ 1 + 3 \left( \frac{\xi}{H} \right)^2 \right]} \, d\xi
\]

(2.42)
where $X$ and $H$ are determined by the condition $P(X^*) = 0$. The dimensionless pressure and load can be determined in the same manner as that for rigid rollers with transverse surface roughness.

2.3.3 Infinitely-Wide Slider

For an infinitely-wide slider, the smooth, average, surface profile can be represented by

$$H(X) = \frac{h(x)}{h_o} = 1 + \frac{mX}{h_o} (1 - X)$$

(2.43)

where

$h_o = h_{\text{min}}$ = minimum film thickness at the exit of the slider

$m$ = slope of the slider

$l$ = length of the slider

Introducing dimensionless variables $P$, $H_T$, $X$, $H$ and $\delta$ as defined in the Nomenclature, one obtains the stochastic Reynolds equation for an infinitely-wide slider with transverse surface roughness,

$$\frac{d}{dX} \left[ \frac{dP}{dX} \frac{1}{\epsilon \left( \frac{1}{H_T} \right)} \right] = \frac{dH}{dX} - \frac{d}{dX} \left[ \frac{\epsilon \left( \frac{\delta_1}{H_T} \right) - \epsilon \left( \frac{\delta_2}{H_T} \right)} {\epsilon \left( \frac{1}{H_T} \right)} \right]$$

(2.44)

The boundary conditions for Eq. (2.44) are

$$P(0) = P(1) = 0$$

(2.45)

For both surfaces having the same roughness characteristics,

$$\epsilon \left( \frac{\delta_1}{H_T} \right) = \epsilon \left( \frac{\delta_2}{H_T} \right)$$

(2.46)
Eq. (2.44) becomes

\[
\frac{d}{dx} \left[ \frac{dp}{dx} \frac{H^3}{G_2} \right] = \frac{dH}{dx}
\]  

(2.47)

Integrating twice, one obtains

\[
P(X) = \int_0^X \frac{G_2}{H^2} \, dg - C \int_0^X \frac{G_4}{H^3} \, dg
\]  

(2.48)

where \( C \) is evaluated by the boundary condition \( P(1) = 0 \). For one surface rough, and the opposing surface smooth, Eq. (2.44) takes the form

\[
\frac{d}{dx} \left[ \frac{dp}{dx} \frac{1}{\epsilon \left( \frac{1}{H_T} \right)^{3/2}} \right] = \frac{dH}{dx} + \frac{d}{dx} \left[ \frac{1}{\epsilon \left( \frac{1}{H_T} \right)^{3/2}} \right]
\]  

(2.49)

where the plus sign represents the case of a smooth surface sliding against a stationary rough surface, and the negative sign implies the rough sliding against smooth surface. Eq. (2.49), expressed in terms of \( G_2 \) and \( G_4 \), becomes

\[
\frac{d}{dx} \left[ \frac{dp}{dx} \frac{H^3}{G_2} \right] = \frac{dH}{dx} \pm \sigma \frac{d}{dx} \left( \frac{G_4}{G_2} \right)
\]  

(2.50)

Integration of the above equation yields

\[
P(X) = \int_0^X \frac{G_2}{H^2} \, dg \pm \sigma \int_0^X \frac{G_4}{H^3} \, dg - C \int_0^X \frac{G_2}{H^3} \, dg
\]  

(2.51)

where \( C \) is determined by the boundary condition, \( P(1) = 0 \).

For the infinitely-wide slider with longitudinal surface roughness, the stochastic Reynolds equation is

\[
\frac{d}{dx} \left[ H^3 \left( 1 + 3 \left( \frac{G_2}{H} \right)^2 \right) \frac{dp}{dx} \right] = \frac{dH}{dx}
\]  

(2.52)
which, after integrating twice, yields

\[ P(X) = \int_{-\infty}^{X} \frac{d\xi}{H^2 \left[ 1 + 3 \left( \frac{\xi}{H} \right)^2 \right]} - C \int_{0}^{X} \frac{d\xi}{H^3 \left[ 1 + 3 \left( \frac{\xi}{H} \right)^2 \right]} \]  

(2.53)

with \( C \) determined by \( P(1) = 0 \). In all the above cases, the load can be evaluated by

\[ W^* = \int_{0}^{1} P(X) dX \]  

(2.54)

2.3.4 Roughness Distribution Function

The roughness distribution function employed in this paper is the same as that used by Christensen [7], namely,

\[ g(\delta^*) = \begin{cases} 
\frac{35}{96} (1 - \frac{\delta^*}{9})^3 & -\delta_\text{max} < \delta^* < \delta_\text{max} \\
0 & \text{Elsewhere} 
\end{cases} \]  

(2.55)

This polynomial distribution function is an approximation to Gaussian distribution. The reason using this polynomial function is that the roughness height distribution function of many engineering surfaces is very close to Gaussian [20], and that a Gaussian distribution always implies a finite probability of having asperities of very large sizes which are very unlikely in practice.
2.4 DISCUSSION OF RESULTS

2.4.1 EHD Contacts

The effect of surface roughness on the pressure generation at the inlet of EHD contacts can be presented conveniently by using a quantity $K_E$, defined as the ratio of the inlet pressure calculated from the stochastic theory, $Q^*_R$, to that calculated from the Grubin's smooth-film theory, $Q^*_S$. Thus,

$$K_E = \frac{Q^*_R}{Q^*_S}$$

(2.57)

In Fig. (2.4), the ratio $K_E$ is plotted against a surface roughness parameter, $\bar{\delta}_{\text{max}}$, for the following four cases

1) pure rolling with transverse surface roughness
2) pure rolling with longitudinal surface roughness
3) rolling and sliding with $S = 0.2$ and $\delta_1 = 0$ (smooth surface is faster)
4) rolling and sliding with $S = 0.2$ and $\delta_2 = 0$ (rough surface is faster)

The dimensionless load and film thickness for the above cases are $W = 3 \times 10^{-5}$ and $H_o = \frac{h_0}{R} = 10^{-5}$.

These curves are obtained by changing the magnitude of $\bar{\delta}_{\text{max}}$. It is seen that, for pure rolling with longitudinal surface roughness, $K_E$ is reduced very slightly due to surface roughness effects. Even for $\bar{\delta}_{\text{max}}$ as high as 0.99, the reduction is only about 7.5%.

Contrast to the effect of longitudinal roughness, the transverse roughness has a much more pronounced effect on the integrated pressure. It tends to increase the dimensionless reduced pressure and hence also tends to increase the load capacity as $\bar{\delta}_{\text{max}}$ increases. For pure rolling, the transverse roughness causes an increase in $Q^*$ from 7% to 30% as $\bar{\delta}_{\text{max}}$ is increased from 0.6 to 0.99. The effect of relative sliding between a smooth surface and a rough surface in EHD contacts is shown in the two curves for $S = 0.2$. Even for such a small slip, there...
is already noticeable departure from the pure rolling case. For the case where
the smooth surface is faster, there is an additional pumping effect, compared to
the pure rolling case. The reverse is true if the roughness surface is faster.

The effect of $H_o$ is studied in Fig. (2.6). It is shown that the three curves
for $H_o = 10^{-5}, 5 \times 10^{-5}, 9 \times 10^{-5}$ almost coincide with one another. This indicates
that the roughness effect on EHL is almost entirely independent upon $H_o$.

Fig. (2.6) shows the integrated value of $Q^*$ against $H_o$ for the condition of pure
rolling with transverse roughness, for different ratio of $\delta_{\text{max}}$ ranging from
0.0 (smooth film theory) to 0.99. It is interesting to note that the curves are
parallel straight lines. Again it is readily seen that the magnitude of $Q^*$ depends
on the ratio of $\delta_{\text{max}}$. For $\delta_{\text{max}} = 0.99$, there is approximately a 20% gain in
the mean film thickness over that based on smooth film theory. For $\delta_{\text{max}} = 0.9,$
and 0.6, the gain in mean film thickness is about 15% and 5% respectively.

In the case of simple sliding of an EHD contact, i.e. $S = 2$, elastic deforma-
tion of asperities begins to be significant at $X = -1$. At this position, the
values of $\sigma$, the r.m.s. roughness amplitude and $g(\delta^*)$ the asperity distribution
function, will no longer be the same as those of the undeformed asperities. There-
fore Eq. (2.33) cannot be applied under this situation. One should notice that
Eq. (2.33) is valid for rolling and sliding case, only when the elastic deformation
of asperities near $X = -1$ can be assumed to be negligibly small. This occurs
only when $S$ is small.

2.4.2 Rigid Rollers

The effect of surface roughness on the dimensionless load of rigid rollers
is presented in terms of $K_R$ which is a quantity defined as the ratio of the
dimensionless load from the stochastic theory, $W^*_R$, to that of the smooth film
theory, $W^*_S$. Thus
\[ \frac{K_R}{W} = \frac{W_R}{W_S} \]

Results of \( K_R \) versus \( \delta_{\text{max}} \) for the following six cases are shown in Fig. (2.7):

1. simple sliding of a smooth roller against a stationary rough roller with transverse surface roughness, \( S = 2 \),
2. rolling and sliding with transverse roughness; \( S = 0.2, \delta_1 = 0 \) (smooth surface is faster),
3. pure rolling with transverse roughness,
4. rolling and sliding with transverse roughness; \( S = 0.2, \delta_2 = 0 \) (rough surface is faster),
5. simple sliding of a rough roller against a stationary smooth roller with transverse roughness; \( S = 2 \),
6. longitudinal surface roughness.

The center film thickness for the above cases is \( h_0/R = 5 \times 10^{-4} \).

Similar to elastic rollers, the effect of surface roughness on rigid rollers with longitudinal roughness is quite small. For \( \delta_{\text{max}} = 0.99 \), the reduction of \( K_R \) is only about 5%.

For pure rolling with transverse roughness, \( K_R \) is increased by about 16% when \( \delta_{\text{max}} = 0.99 \). For rolling and sliding with \( S = 0.2 \) and smooth surface moving faster, there is an increase in load carrying capacity, compared to the pure rolling case. The reverse is true when the rough surface is faster in the rolling and sliding case. When a smooth roller is sliding against a stationary rough roller, there is a substantial increase in \( K_R \). When \( \delta_{\text{max}} = 0.6 \) and 0.99, the gain in \( K_R \) are 8.5% and 39% respectively. However, when a rough roller is sliding against a stationary smooth roller, \( K_R \) is almost unaffected. When \( \delta_{\text{max}} \) is greater than 0.66, \( K_R \) begins to decrease slightly after a steady increase.
This small dip in $K_R$ is suspected to be caused by using the above mentioned polynomial function as the surface roughness distribution function. When a sinusoidal distribution function [17] which is defined as,

$$g(\delta^*) = \begin{cases} 1 & -3 < \delta^* < 3 \\ 0 & \text{Elsewhere} \end{cases}$$

is employed, the results which are not plotted here show that the dip disappears and that $K_R$ increases with the increase of $\delta_{max}$.

2.4.3 *Infinitely-Wide Slider Bearing*

In Fig. (2.8), $W^*$ is plotted against $\delta_{max}$ for the following four cases:

1. simple sliding of a smooth surface against a rough surface with transverse roughness,
2. both surfaces having the same roughness distribution functions with transverse roughness,
3. simple sliding of a rough surface against a smooth one, with transverse roughness,
4. longitudinal surface roughness.

Qualitatively, the roughness effect on a slider checks very well with that on rollers. When the roughness direction is longitudinally oriented, the load carrying capacity of the oil film is reduced.

In the case of transverse roughness, the effect on load capacity is always beneficial. For both surfaces having the same kind of roughness distribution function, $W^*$ is increased by 66% approximately, when $\delta_{max}$ is 0.99. For a smooth surface sliding against a rough one, the gain in $W^*$ is even larger. $W^*$ is increased by about 129% when $\delta_{max}$ is 0.99. For a rough surface sliding against a smooth one, $W^*$ is only increased slightly.
The authors' results agree very well to those obtained by Rhow and Elrod [18], who have studied the effects of two-sided straited roughness on the load-carrying capacity of an infinitely wide slider bearing. The only exception is that the authors' results show a small dip in $W^*$ when $\bar{\delta}_{\text{max}}$ is larger than 0.9. This small dip in $W^*$ is caused by employing the polynomial function as the surface roughness distribution. When the sinusoidal distribution is used, the dip in $W^*$ disappears.
2.5 CONCLUSIONS

1. Based on Christensen's stochastic model of hydrodynamic lubrication, a Grubin type elastohydrodynamic analysis at the inlet of a Hertzian contact indicates that surface roughness can have a noticeable effect on the level of mean film thickness between EHD contacts.

2. For longitudinal surface roughness, ridges parallel to the direction of rolling, the present analysis predicts an inlet pressure or inlet film thickness smaller than that predicted by the smooth-film EHD theory.

3. For transverse surface roughness, ridges perpendicular to the direction of rolling, the inlet mean film thickness level is increased noticeably due to the additional pumping by transverse ridges. The level of increase is mainly a function of $\delta_{\text{max}}/h_o$, the ratio of the maximum ridge height to the mean film thickness at the inlet and is not sensitive to other operating parameters. For $\delta_{\text{max}}/h_o$ approaching unity, which corresponds to $h_o/\sigma = 3$ for $\delta_{\text{max}} = 3\sigma$, one can expect an increase of 25% in mean film thickness compared to the smooth-film EHD film thickness for pure rolling. Results for small slide to roll ratios indicates that the case of smooth sliding over rough gives further enhancement in mean film thickness, whereas the case of rough sliding over smooth yields a slight reduction in mean film thickness comparing to pure rolling. For high slide to roll ratios, it was found that local elastohydrodynamic effect due to local pressure fluctuations will become significant, and the present analysis based on the Grubin approach will become invalid.

4. For rigid rollers and infinitely-wide slider bearings, load capacities calculated for rough surfaces show trends similar to those found in EHD contacts. However, the effects for infinitely-wide slider bearings are much stronger than that for rigid rollers.
Figure 2-1  EHD Contacts Between Two Rough Surfaces
Figure 2-2  Rigid Rollers With Rough Surfaces
Figure 2-3 An Infinitely-Wide Slider Bearing with Rough Surfaces
Fig. 2-4 The Effect of Transverse and Longitudinal Roughness on the Ratio of the Reduced Pressure for Rough Surfaces to that for Smooth Surfaces, $K_E = Q_R/Q_S$, of an EHD Contact.
**EHD CONTACTS**

**POLY. DIST.**

**PURE ROLLING**

\[ W = 3 \times 10^{-5} \]

---

**Fig. 2-5** The Effect of the Center Film Thickness, \( H_0 \), on the Ratio of the Reduced Pressure for Rough Surfaces to that for Smooth Surfaces, \( K_E = \frac{Q_R}{Q_S} \), of an EHD Contact
Fig. 2-6 The Effect of the Roughness Height to Center Film Thickness Ratio, $\delta_{\text{max}}$, on the Normalized Reduced Pressure, $Q^*_R$, of an EHD Contact.
RIGID ROLLERS

POLY. DIST.

$h_0/R = 5 \times 10^{-4}$

- - - - $S = 2$ $\delta_1 = 0$
- - - - $S = .2$ $\delta_1 = 0$
- - - - PURE ROLLING
- - - - $S = .2$ $\delta_2 = 0$
- - - - $S = 2$ $\delta_2 = 0$
- - - - LONG. ROUGHNESS

$K_R$ vs. $\bar{\delta}_{max}$

**Fig. 2-7** The Variation of the Normalized Load Ratio, $K_R = \frac{W^*}{W^*_{max}}$, with the Roughness Height, $\bar{\delta}_{max} = \frac{\delta_{max}}{h_0}$, for Rigid Rollers
Fig. 2-8 The Variation of the Normalized Load, $W^* = \frac{h_o^2}{6\mu u_1 \bar{F}^2}$, with the Roughness Height, $\delta_{\text{max}}$, for an Infinitely-wide Slider Bearing
CHAPTER III
WAVINESS AND ROUGHNESS IN ELASTOHYDRODYNAMIC LUBRICATION

3.1 INTRODUCTION

A stochastic theory for elastohydrodynamic lubrication of contact with two-sided roughness has been developed in Chapter II. The basic requirement of this theory is that the roughness pattern must be very dense within the contact zone. In other words, the largest wavelength in the roughness spectrum must be small compared to the contact width. If the largest wavelength is of the same order of the contact width, surface roughness becomes surface waviness. The stochastic theory may be invalid in this region.

In order to ascertain the conditions under which the stochastic theory becomes valid the reduced pressure based on the deterministic approach using a sinusoidal profile and that calculated from the stochastic theory using a surface roughness distribution equivalent to the sinusoidal profile is compared. The comparison is only made for the transverse roughness since the effect of longitudinal roughness is usually negligibly small comparing to the effect of transverse roughness.

3.2 GOVERNING EQUATION

In comparing the effect of waviness and roughness on the reduced pressure, the surface profile is assumed to be in the form of sinusoidal waviness. The reduced pressure is then solved by the deterministic approach for the waviness case. At the same time, a density distribution function equivalent to the chosen sinusoidal wave is evaluated. Using this equivalent roughness distribution function, the reduced pressure for the roughness case is then solved by the stochastic approach as illustrated in Chapter II. Then, the effect of surface waviness and roughness will be compared for the pure rolling case.
3.2.1 The Waviness Case

In the waviness case, the one-dimensional Reynolds equation governing the pressure in an EHD contact of an isothermal and incompressible lubricant is

$$\frac{\partial}{\partial x} \left( \frac{h_T^3}{12 \mu} \frac{\partial p}{\partial x} \right) = \frac{u_1 + u_2}{2} \frac{\partial h_T}{\partial x} + \frac{\partial h_T}{\partial t}$$

(3.1)

where $h_T$ is the local total film thickness consisting of the following three parts $h, \delta_1, \delta_2$

where $h$ = the nominal smooth part of the average film thickness

$\delta_1, \delta_2$ = roughness amplitude measured from the mean level of surfaces 1 and 2

$$\delta_1 = \delta_{\max_1} \sin \left[ (2n_1 \pi)(x - u_1 t) \right]$$

(3.2)

$$\delta_2 = \delta_{\max_2} \sin \left[ (2n_2 \pi)(x - u_2 t) \right]$$

(3.3)

$$\theta_1 = u_1 t$$

$$\theta_2 = u_2 t$$

It is assumed that the asperities on both surfaces are straight ridges perpendicular to the direction of rolling. With the relations

$$\frac{\partial \delta_1}{\partial t} = -u_1 \frac{\partial \delta_1}{\partial x}$$

(3.4)

$$\frac{\partial \delta_2}{\partial t} = -u_1 \frac{\partial \delta_2}{\partial x}$$

(3.5)

Eq. (3.1) is simplified as

$$\frac{d}{dx} \left( \frac{h_T^3}{12 \mu} \frac{dp}{dx} \right) = \left( \frac{u_1 + u_2}{2} \right) \frac{dh}{dx} - \left( \frac{u_1 - u_2}{2} \right) \frac{d}{dx} \left( \delta_1 - \delta_2 \right) + \frac{dh}{dt}$$

(3.6)

It is further assumed that there are enough number of asperities within the Hertzian contact zone such that $h$ can be considered as a constant of time.
i.e. \[ \frac{dh}{dt} = 0 \] (3.7)

Hence, in the case of pure rolling, one obtains
\[
d\left( \frac{h^3}{12\mu} \frac{dp}{dx} \right) = u \frac{dh}{dx}
\] (3.8)

with dimensionless variables \( Q_w, X, H, H_T, \sigma, U, \bar{\delta}_1, \bar{\delta}_2, \) and \( \bar{\theta} \) as defined in the Nomenclature, Eq. (3.8) is transformed to
\[
sicontdX \left( H_T \frac{dQ_w}{dx} \right) = \frac{1}{H_o^2} \frac{dH}{dx}
\] (3.9)

where \( H \), the average surface profile in the inlet region is governed by
\[
H - 1 = \frac{h - h_o}{h_o} = \frac{4W}{\pi H_o} \left[ \frac{\sqrt{X^2} - 1}{X} \right] - \frac{1}{2} \ln \left( \frac{\sqrt{X^2} + 1}{X} \right)
\] (3.10)

The boundary conditions are:
\[
\frac{dQ_w}{dx} = 0 \quad \text{at} \quad X = -1 
\] (3.11)
\[
Q_w = 0 \quad \text{at} \quad X = -\infty 
\] (3.12)

Integrating Eq. (3.10) twice with these two boundary conditions, one obtains
\[
Q_w^* = Q_w \bigg|_{X=-1} = \int_{-\infty}^{-1} \left( \frac{H - 1}{H_o^2 H_T^2} \right) dX 
\] (3.13)

In the pure rolling case
\[
\theta_1 = \theta_2 = \theta = ut 
\] (3.14)

In addition, it is assumed that
\[
n_1 = n_2 = n
\] (3.15)
\[
\delta_{\text{max}} = \delta_{1 \text{ max}} + \delta_{2 \text{ max}}
\] (3.16)
For given $h_o/R$, $W$, $\delta_{\text{max}}$, $n$ and $\theta$, Eq. (3.13) is solved numerically by Simpson's integration routine.

3.2.2 The Roughness Case

The probability density function corresponding to a sinusoidal wave is [17]

$$f(\delta^*) = \begin{cases} \frac{1}{\pi \sqrt{1 - \frac{1}{9} \delta^*^2}} & -3 < \delta^* < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(3.17)

From Chapter II, the corresponding reduced pressure at the inlet is

$$Q^*_R = -1 \int_{-\infty}^{-1} \left( \frac{H - 1}{H_o H^3} \right) G dX$$

(3.18)

where

$$G = \frac{1}{\pi^3} \int_{-3}^{3} \frac{1}{\sqrt{1 - \frac{1}{9} \delta^*^2}} \left[ 1 + \delta^*(\frac{\delta}{H}) \right]^3 d\delta^*$$

(3.19)

$$Q^*_R = Q^*_W$$

Hence $Q^*_R$ is evaluated for different values of $\delta_{\text{max}}$.

3.3 DISCUSSION OF RESULTS

Though the following examples are only connected with the pressure generation at the inlet of elastohydrodynamic contacts, yet, qualitatively, the general trends of the results will be relevant to other types of bearing and surface irregularity.

The comparison between waviness and roughness can be presented conveniently by using a quantity $K_D$ which is defined as

$$K_D = \frac{Q^*_W - Q^*_R}{Q^*_R}$$

(3.20)
where $Q^*_w$ and $Q^*_R$ stand for the inlet reduced pressure calculated from the waviness model and the roughness model respectively. Hence, by definition, $K_D$ is the fractional deviation of the inlet reduced pressure of the waviness model from that of the corresponding roughness model. The results of such a comparison are shown in Fig. 3.1 to Fig. 3.3. The dimensionless load and film thickness for these examples are $W = 3 \times 10^{-5}$ and $H_o = h_o/R = 10^{-5}$, while the ratio $\delta_{max}/h_o$ in these three figures are 0.3, 0.45 and 0.6 respectively. The phase angles chosen are 0, $\pi/2$, $\pi$ and $-\pi/2$ while the $n$-values are integers.

It is readily seen that $K_D$ heavily depends upon the phase angle $\theta$ for small $n$. Particularly, for $\theta = \pi/2$ and $-\pi/2$, the magnitude of $K_D$ even changes signs. However, the effect of phase angle on $K_D$ decreases rapidly as $n$ increases. Furthermore, for larger $\delta_{max}$ which means more pronounced asperity interaction, $K_D$ is larger for the same $n$. For smaller $\delta_{max}$, $K_D$ is smaller. The approximate values of $K_D$ at $n = 5$ and $n = 10$ for the extreme cases of $\theta = -\pi/2$ and $\pi/2$ are listed as follows:

<table>
<thead>
<tr>
<th>$\delta_{max}/h_o$</th>
<th>$n = 5$</th>
<th>$n = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>$K_D$</td>
<td>±3.5%</td>
<td>±6%</td>
</tr>
<tr>
<td></td>
<td>±10%</td>
<td>±10%</td>
</tr>
<tr>
<td></td>
<td>±1.5%</td>
<td>±1.5%</td>
</tr>
</tbody>
</table>

These results provide a better understanding to the statistical roughness theory. First, they show that the effect of phase angle vanishes with increasing $n$. Second, the effect of $n$ becomes less important when $n$ is greater than some critical value for a given $\delta_{max}$. The discrepancy between the waviness model and the roughness model becomes poorer as $\delta_{max}/h_o$ increases. It is quite evident that $n$ and $\delta_{max}/h_o$ are both important parameters that determine the validity of the statistical theory for roughness surfaces. For the particular numerical example used in these
calculations, it is found that for \( n > 10 \), waviness is equivalent to roughness and that the stochastic theory holds. Even for \( n = 5 \), one can still apply the stochastic theory with reasonable accuracy.
3.4 CONCLUSIONS

For a given $W$ and $H_o$

1. $\theta$, $n$ and $\delta_{\text{max}}/h_o$ are the parameters to determine the deviation of the stochastic, roughness made from the deterministic waviness model.

2. The effect of phase angle diminishes with increasing $n$.

3. The effect of $n$ vanishes as $n$ becomes large, and the stochastic theory for roughness surface is proved to be valid as $n$ approaches a critical value depending on $\delta_{\text{max}}/h_o$ for a given $W$ and $H_o$. 
Fig. 3-1 The Effect of the Number of Wave Cycles, $n$, and the Phase Angle, $\theta$, on the Percentage of Deviation of the Normalized Reduced Pressure, $K_D$, for Roughness to Thickness Ratio $\delta_{\text{max}}/h_o = 0.3$
$$W = 3 \times 10^{-5}, \quad h_o / R = 10^{-5}, \quad \delta_{\text{max}} / h_o = 0.45$$

$$K_D = \frac{(Q^*_W - Q^*_R)}{Q^*_R} \times 100\%$$

Fig. 3-2 The Effect of the Number of Wave Cycles, $n$, and the Phase Angle, $\theta$, on the Percentage of Deviation of the Normalized Reduced Pressure, $K_D$, for Roughness to Thickness Ratio $\delta_{\text{max}} / h_o = 0.45$
The Effect of the Number of Wave Cycles, $n$, and the Phase Angle, $\theta$, on the Percentage of Deviation of the Normalized Reduced Pressure, $K_D$, for Roughness to Thickness Ratio $\delta_{\text{max}}/h_0 = 0.6$

$$w = 3 \times 10^{-5}, \quad h_0/R = 10^{-5}, \quad \delta_{\text{max}}/h_0 = 0.6$$

$$K_D = \frac{(Q_W^{*} - Q_R^{*})}{Q_R^{*}} \times 100\%$$
CHAPTER IV
PRESSURE PERTURBATION IN EHD CONTACTS
DUE TO AN ELLIPSOIDAL ASPERITY

4.1 INTRODUCTION

Recently, there has been a growing interest in the effect of surface roughness on the bearing performance in thin film lubrication. In Chapter II, the stochastic theory is used to study the effect of surface roughness on the average EHD film thickness and the integrated pressure at the inlet of the lubricated Hertzian contacts. However, the stochastic theory is incapable of predicting any detailed local perturbations in pressure or deformation caused by the asperities. It was recently pointed out by Tallian [19] that the pressure ripples can rise to a very high level in rolling and sliding EHD line contacts. These ripples are very likely one of the chief attributing factors to contact fatigue.

In the last few years, there has been considerable interest in the basic event involving a single asperity entering an EHD contact [5] or the encounter between two identical asperities [4]. The work in this chapter is aimed towards gaining further understanding of the effect of a single asperity on pressure distribution in a line EHD contact. Special attention is given to the three-dimensional aspect of the asperity which is assumed to be ellipsoidal at the tip. The effects of ellipticity (aspect ratio) on the double amplitude of pressure fluctuations under various rolling and sliding condition is examined in detail.
4.2 MATHEMATICAL ANALYSIS

The present analysis consists of two parts. The first part studies the pressure fluctuations due to a single three-dimensional ellipsoidal asperity at the inlet region of an EHD contact assuming that the asperity shape is unaffected by the perturbed pressures. These pressure fluctuations are determined by solving a perturbed Reynolds equation, in which the unperturbed pressure profile is obtained by using a line contact EHD analysis [22]. Results are presented as the perturbed pressure profile, $\delta$, as well as the amplitude of the pressure ripple, $A_\delta$, as a function of ellipticity ratio, $\gamma$, maximum Hertzian pressure, $P_{HZ}$, nominal EHD film thickness $h_0/R$, asperity size, $b$, asperity height, $c_l/h_0$, pressure viscosity coefficient, $G$, slide to roll ratio, $S$, and the position of the asperity center $X_3$.

In the second part, the line contact EHD analysis [22] is modified to include a two dimensional asperity ridge on the stationary side of the lubricated contacts. In this approach, the elastic deformation of the asperity is included. Results which are presented as the double amplitude of the perturbed pressure as a function of $P_{HZ}$, $h_0/R$, $b$, $c_l/h_0$, $G$, and $X_3$, are compared with those obtained for the three dimensional ellipsoidal asperity with large ellipticity ratio for the case of simple sliding between a smooth surface and a stationary asperity ($S=2$).

4.2.1 Geometrical Configuration

The contact between two cylinders as shown in Fig. (4.1a) can be described by an equivalent cylinder near a flat surface as shown in Fig. (4.1b). As the contact width is very small compared to the dimension of the cylinder, the film thickness for a rigid cylinder, $h_g$, without the asperity is

$$h_g = h_0 + \frac{x^2}{2R} \quad (4.1)$$

where $x = \text{coordinate along the film}$
\[ R = \frac{R_1 R_2}{R_1 + R_2} \]

\( R_1 R_2 = \text{radius of rollers 1 and 2,} \)
\( h_0 = \text{undeformed center film thickness} \)

With elastic deformation, the film thickness profile, still without the asperity

\[ h_1 = h_0 + \frac{x^2}{2R} - \frac{4}{\pi E'} \int_{-\infty}^{x_f} \ln \left| \frac{\xi - x}{\xi} \right| p_1(\xi) d\xi \]  

(4.2)

where
\( h_1 = \text{smooth-film thickness} \)
\( h_0 = \text{center film thickness at } x=0 \)
\( E' = \left[ \frac{1}{2} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right) \right]^{-1} \)
\( E_1, E_2 = \text{Young's modulus for rollers 1 and 2} \)
\( \nu_1, \nu_2 = \text{Poisson's ratio of rollers 1 and 2} \)
\( \xi = \text{dummy variable for } x \)
\( p_1(\xi) = \text{smooth-film pressure profile} \)

Referring to Fig. (4.2) the height of a three-dimensional ellipsoidal asperity can be written as

\[ \delta = \delta \cos \eta \]  

(4.3)

As the contact width is very small compared to the radius of the cylinder, \( \eta \) is very small and

\[ \cos \eta \approx 1 \]  

(4.4)

Thus the asperity height function of a three-dimensional asperity can be approximately written as
\[ \delta(x,y) = \delta_1 = \begin{cases} 0 & \text{for } \left| \frac{x - x_3}{b'} \right| \leq 1 \text{ and } \left| \frac{y}{a'} \right| \leq 1 \\ \frac{c}{L} \left[ \left( \frac{x - x_3}{b'} \right)^2 + \left( \frac{y}{a'} \right)^2 \right] & \text{elsewhere} \end{cases} \]  

(4.5)

Similarly, the asperity height function of a two dimensional asperity ridge can be expressed as

\[ \delta(x) = \begin{cases} \frac{c}{L} \left[ \left( \frac{x - x_3}{b'} \right)^2 - 1 \right] & \text{for } \left| \frac{x - x_3}{b'} \right| \leq 1 \\ 0 & \text{elsewhere} \end{cases} \]  

(4.6)

The total local film thickness, \( h_T \), at any point under the surface asperity is

\[ h_T = h_1 + \delta \]  

(4.7)

4.2.2 Governing Equations

4.2.2.1 The Smooth-Film Case

Referring to [22], the two coupled equations governing the pressure and film distributions in an elastohydrodynamic line contact between two rollers with isothermal and incompressible lubricant are:

The Reynolds equation

\[ \frac{dp_1}{dx} = 6\mu (u_1 + u_2) \left( \frac{h_1 - h}{h_3} \right) \]  

(4.8)

where \( p_1, h_1, h \) and \( x \) are already defined

\[ \mu \equiv \text{viscosity} \]

\[ u_1, u_2 \equiv \text{velocity of rollers 1 and 2} \]

and the film thickness profile as described in Eq. (4.2). In non-dimensional form, Eqs. (4.8) and (4.2) become

\[ \frac{dp_1}{dx} = \left( \frac{48}{H_0^2} \right) U \left( \frac{H_1 - 1}{H_3^2} \right) \]  

(4.9)
Using the same numerical scheme as developed in [22], these two equations are solved by the Newton-Ralphson method. $P_1(X)$, $H_1(X)$ and $\bar{U}$ are solved for a given set of $h_o/R$, $P_{Hz}$, and $G$. These smooth-film results are used as the inputs to the perturbed Reynolds equation described later.

4.2.2.2 The Discretized Reynolds Equation

It is assumed here that the viscosity is an exponential function of the pressure with a pressure viscosity coefficient $\alpha$, i.e.

$$\mu = \mu_s e^{\alpha p}$$

(4.11)

In numerical form the steady Reynolds equation for an incompressible lubricant can be expressed by dividing the contact zone into small grids with irregular spacings. Referring to Fig. (4.3) and applying the principle of conservation of mass, one obtains for the $i$th grid

$$m_1 = m_2 = m_{\text{restored}}$$

(4.12)

where

$$m_1 = - \left( \frac{\rho h_1^3}{12 \mu_s} \frac{\partial p_1}{\partial x} \right)_{i-1/2} + \frac{u_1 + u_2}{2} \left( \rho h_1 \right)_{i-1/2}$$

$$= - \frac{\rho}{12 \mu_s} \left( h_1^3 e^{-\alpha p_1} \frac{\partial p_1}{\partial x} \right)_{i-1/2} + \frac{u_1 + u_2}{2} \left( h_1 \right)_{i-1/2}$$

(4.13)
Combining Eqs. (4.12) to (4.15) and re-arranging, one obtains

\begin{equation}
\frac{m_2}{12v_s} = -\frac{\rho}{12v_s} \left( h_1^3 \ e^{-\alpha p_1} \ \frac{\partial p_1}{\partial x} \right)_{i+1/2} + \rho \frac{u_1+u_2}{2} (h_1)_{i+1/2} \quad (4.14)
\end{equation}

\begin{equation}
m_{\text{restored}} = \frac{\partial}{\partial t} \left[ \rho (h_1) \ i \left( \frac{(\Delta x)_{i-1} + (\Delta x)_{i}}{2} \right) \right] = 0 \quad (4.15)
\end{equation}

Combining Eqs. (4.12) to (4.15) and re-arranging, one obtains

\begin{equation}
\frac{1}{12v_s} \left[ \left( h_1^3 \ e^{-\alpha p_1} \ \frac{\partial p_1}{\partial x} \right)_{i+1/2} - \left( h_1^3 \ e^{-\alpha p_1} \ \frac{\partial p_1}{\partial x} \right)_{i-1/2} \right] = \frac{u_1+u_2}{2} \left[ (h_1)_{i+1/2} - (h_1)_{i-1/2} \right] \quad (4.16)
\end{equation}

Thus \( p_1 \), the smooth-film pressure profile and \( h_1 \), the smooth-film thickness are functions of \( x \) only.

### 4.2.2.3 The Perturbed Reynolds Equation

The pressure distribution can be considered as the sum of the smooth-film pressure, \( p_1 \) and the perturbed pressure, \( \phi \). Define

\begin{equation}
p = p_1 + \phi \quad (4.17)
\end{equation}

Since \( p_1 \) is a function of \( x \) only, the derivatives of \( p \) are:

\begin{equation}
\frac{\partial p}{\partial x} = \frac{\partial p_1}{\partial x} + \frac{\partial \phi}{\partial x} \quad (4.18)
\end{equation}

\begin{equation}
\frac{\partial p}{\partial y} = \frac{\partial \phi}{\partial y} \quad (4.19)
\end{equation}

For a two-dimensional flow field, the principle of conservation of mass as applied to the \((i,j)\)th grid as shown in Fig. (4.4) yields
\[
\begin{align*}
\left[ \frac{(\Delta y)_{i-1} + (\Delta y)_i}{2} \right] (m_1 - m_2) & + \left[ \frac{(\Delta x)_{i-1} + (\Delta x)_i}{2} \right] (m_3 - m_4) \\
& = \frac{1}{\partial t} \left\{ (\rho h_T)_{i-1/2,j} \left[ \frac{(\Delta y)_{i-1} + (\Delta y)_i}{2} \right] \left[ \frac{(\Delta x)_{i-1}}{2} \right] \right. \\
& \left. + (\rho h_T)_{i+1/2,j} \left[ \frac{(\Delta y)_{i-1} + (\Delta y)_i}{2} \right] \left[ \frac{(\Delta x)_i}{2} \right] \right\} \\
\end{align*}
\]

where
\[
m_1 = \left\{ -\frac{\rho h_T^3}{12\mu_s} \frac{\partial p}{\partial x} + \rho \left( \frac{u_1 + u_2}{2} \right) h_T \right\}_{i-1/2,j}
\]

\[
= \left\{ -\frac{\rho h_T^2}{12\mu_s} e^{-\alpha \phi_1} \left( \frac{\partial p_1}{\partial x} + \alpha \frac{\partial p_1}{\partial x} \right) + \rho \left( \frac{u_1 + u_2}{2} \right) \left( h_1 + \delta \right) \right\}_{i-1/2,j}
\]

It is assumed that the value of $\alpha \phi$ is much smaller than unity such that $e^{-\alpha \phi}$ can be linearized as

\[
e^{-\alpha \phi} = (1 - \alpha \phi)
\]

by using Taylor's series expansion. Neglecting second order term, Eq. (4.21) can be re-written as

\[
m_1 = \left\{ -\frac{\rho h_T^3}{12\mu_s} \frac{\partial p_1}{\partial x} + \rho \left( \frac{u_1 + u_2}{2} \right) \left( h_1 + \delta \right) \right\}_{i-1/2,j}
\]

Likewise, $m_2, m_3,$ and $m_4$ can be shown to be

\[
m_2 = -\left\{ \frac{\rho h_T^3}{12\mu_s} e^{-\alpha \phi_1} \left( \frac{\partial p_1}{\partial x} + \alpha \frac{\partial p_1}{\partial x} \right) + \rho \left( \frac{u_1 + u_2}{2} \right) \left( h_1 + \delta \right) \right\}_{i+1/2,j}
\]

\[
m_3 = -\left\{ \frac{\rho h_T^3}{12\mu_s} e^{-\alpha \phi_1} \frac{\partial \phi_1}{\partial y} \right\}_{i,j-1/2}
\]

\[
m_4 = -\left\{ \frac{\rho h_T^3}{12\mu_s} e^{-\alpha \phi_1} \frac{\partial \phi_1}{\partial y} \right\}_{i,j+1/2}
\]

50
Substituting Eqs. (4.23) to (4.26) into the left-hand side of Eq. (4.20), one obtains

\[
\rho \left[ \frac{(\Delta y)_{i-1}^{+} + (\Delta y)_{i}^{-}}{2} \right] \left\{ \left( \frac{1}{12\mu_{s}} \right) \left[ \left( h_{1}^{3} e^{-\alpha \rho_{1} \frac{\partial}{\partial x}} \right)_{i+1/2, j} - \left( h_{1}^{3} e^{-\alpha \rho_{1} \frac{\partial}{\partial x}} \right)_{i-1/2, j} \right] \right. \\
- \left( \frac{1}{12\mu_{s}} \right) \left[ \left( h_{1}^{3} e^{-\alpha \rho_{1} \frac{\partial}{\partial x}} \right)_{i+1/2, j} - \left( h_{1}^{3} e^{-\alpha \rho_{1} \frac{\partial}{\partial x}} \right)_{i-1/2, j} \right] \right. \\
- \left( \frac{1}{12\mu_{s}} \right) \left[ \left( h_{1}^{3} - h_{1}^{3} e^{-\alpha \rho_{1} \frac{\partial}{\partial x}} \right)_{i+1/2, j} - \left( h_{1}^{3} - h_{1}^{3} e^{-\alpha \rho_{1} \frac{\partial}{\partial x}} \right)_{i-1/2, j} \right] \\
- \left( \frac{u_{1}^{+} + u_{2}^{-}}{2} \right) \left[ \frac{\delta i+1}{2} \frac{\delta i-1, j}{2} \right] \left\} \right. \\
+ \rho \left[ \frac{(\Delta x)_{i-1}^{+} + (\Delta x)_{i}^{-}}{2} \right] \left( \frac{1}{12\mu_{s}} \right) \left[ \left( h_{1}^{3} e^{-\alpha \rho_{1} \frac{\partial}{\partial y}} \right)_{i, j+1/2} \\
- \left( h_{1}^{3} e^{-\alpha \rho_{1} \frac{\partial}{\partial y}} \right)_{i, j-1/2} \right] (4.27)
\]

With the relations

\[
\frac{\partial h_{1}}{\partial t} = 0 \quad (4.28)
\]

and

\[
\frac{\partial}{\partial t} \delta(x - u_{2} t) = - u_{2} \frac{\partial \delta}{\partial x} \quad (4.29)
\]

the right-hand side of Eq. (4.20) can be re-written as

\[
\rho \left[ \frac{(\Delta y)_{i-1}^{+} + (\Delta y)_{i}^{-}}{2} \right] \left( - u_{2} \right) \left( \frac{\delta i+1, j}{2} - \frac{\delta i-1, j}{2} \right) \quad (4.30)
\]

Equating Eqs. (4.27) and (4.30) and simplifying, one obtains
Using the central difference approximation for the pressure gradients, Eq. (4.31) is discretized to

\[
\begin{align*}
&\left[\Delta y_{j-1} + \Delta y_j\right] \left\{ \left[ \left( h_T^3 e^{-\alpha p_1} \frac{dp_1}{dx} \right)_{i+1/2,j} - \left( h_T^3 e^{-\alpha p_1} \frac{dp_1}{dx} \right)_{i-1/2,j} \right] \\
&\quad + \left[ \left( h_T^3 e^{-\alpha p_1} \frac{dp_1}{dy} \right)_{i,j+1/2} - \left( h_T^3 e^{-\alpha p_1} \frac{dp_1}{dy} \right)_{i,j-1/2} \right] \right\} \\
&= \left[\Delta y_{j-1} + \Delta y_j\right] \left\{ \left[ \left( h_1^3 - h_T^3 \right) e^{-\alpha p_1} \frac{dp_1}{dx} \right]_{i+1/2,j} \\
&\quad - \left[ \left( h_1^3 - h_T^3 \right) e^{-\alpha p_1} \frac{dp_1}{dx} \right]_{i-1/2,j} + 6 \mu_s \left( u_1 - u_2 \right) \left[ \frac{\delta_{i+1,j} - \delta_{i-1,j}}{2} \right] \right\} \quad (4.31)
\end{align*}
\]
After re-arranging and the following non-dimensional variables are introduced:

\[ \Delta X = \Delta x/b', \quad \Delta Y = \Delta y/a', \quad \gamma = a'/b', \quad H_T = h_t/h_o, \quad H_1 = h_1/h_o, \quad \bar{c}_1 = c_1/h_o, \]
\[ X = x/b, \quad X_3 = x_3/b, \quad Y = y/a', \quad \bar{b} = b'/b, \quad p_1 = p_1/p_{Hz}, \quad \bar{\phi} = \phi/p_{Hz}, \]
\[ p_{Hz} = p_{Hz}/E', \quad G = \alpha E', \quad E' = \left[ \frac{1}{2} \left( \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \right) \right]^{-1}, \quad \bar{\alpha} = \alpha p_{Hz}, \]
\[ U_D = \frac{\mu_s (u_1 - u_2)}{2E'R}, \quad H_o = h_o/R, \quad C_{UD} = 48 U_D/H_o \cdot \bar{b}, \]
\[ \bar{\delta} = \delta/h = \frac{c_1}{h_o}\left[ \frac{(x-x_3)^2}{b^2} + \frac{(y/a)^2}{b^2} - 1 \right] = \bar{c}_1 \left[ \frac{(x-x_3)^2}{b^2} + (y)^2 - 1 \right] \]

the governing equation becomes:
\[
\left[ \frac{(\Delta Y)_{j-1} + (\Delta Y)_j}{(\Delta X)_{i-1}} \right] \left[ \left( H_T^3 e^{-\bar{\alpha} p_1} \right)_{i,j} + \left( H_T^3 e^{-\bar{\alpha} p_1} \right)_{i-1,j} \right]
+ \bar{\alpha} \left( H_T^3 e^{-\bar{\alpha} p_1} \right)_{i-1,j} \left[ (p_1)_{i} - (p_1)_{i-1} \right] \right] \phi_{i-1,j} 
\]
\[ + \frac{1}{\sqrt{2}} \left[ \frac{(\Delta X)_{i-1} + (\Delta X)_i}{(\Delta Y)_{j-1}} \right] \left[ \left( H_T^3 e^{-\delta_{i-1,j}} \right)_{i,j} + \left( H_T^3 e^{-\delta_{i-1,j}} \right)_{i,j-1} \right] \hat{\phi}_{i,j} - \\
+ \left\{ \left[ (\Delta X)_{j-1} + (\Delta Y)_j \right] \left[ \frac{H_T^3 (e^{-\delta_{i,j}})}{i+1,j} + \frac{H_T^3 (e^{-\delta_{i,j}})}{i,j} \right] \frac{(p_1)_{i+1,j} - (p_1)_{i,j}}{(\Delta X)_i} - \frac{(p_1)_{i,j} - (p_1)_{i,j-1}}{(\Delta X)_{i-1}} \right\} \hat{\phi}_{i,j} - \\
+ \frac{1}{\gamma} \left[ (\Delta X)_{i-1} + (\Delta X)_i \right] \left[ \frac{H_T^3 (e^{-\delta_{i,j}})}{i+1,j} \frac{(p_1)_{i+1,j} - (p_1)_{i,j}}{(\Delta X)_i} + \frac{H_T^3 (e^{-\delta_{i,j}})}{i,j} \frac{(p_1)_{i,j} - (p_1)_{i,j-1}}{(\Delta X)_{i-1}} \right] \hat{\phi}_{i,j} + \\
+ \frac{1}{\gamma} \left[ (\Delta X)_{i-1} + (\Delta X)_i \right] \left[ \frac{(H_T^3 (e^{-\delta_{i,j}}))_{i+1,j} + (H_T^3 (e^{-\delta_{i,j}}))_{i,j}}{(\Delta Y)_{j-1}} \right] \hat{\phi}_{i,j+1} + \\
+ \left[ \frac{(\Delta Y)_{j-1} + (\Delta Y)_j}{(\Delta X)_i} \right] \left[ (H_T^3 (e^{-\delta_{i,j}}))_{i+1,j} + (H_T^3 (e^{-\delta_{i,j}}))_{i,j} \right] \hat{\phi}_{i+1,j} - \\
- \frac{1}{\gamma} \left[ (\Delta Y)_{j-1} + (\Delta Y)_j \right] \left[ \frac{H_T^3 (e^{-\delta_{i,j}})}{i+1,j} \frac{(p_1)_{i+1,j} - (p_1)_{i,j}}{(\Delta X)_i} \right] \hat{\phi}_{i+1,j} + \\
= \left[ (\Delta Y)_{j-1} + (\Delta Y)_j \right] \left\{ \left[ \left( H_T^3 - H_T^3 \right) e^{-\delta_{i+1,j}} \right]_{i+1,j} \right\} \hat{\phi}_{i+1,j} - \\
+ \left[ \left( H_T^3 - H_T^3 \right) e^{-\delta_{i,j}} \right]_{i,j} \left[ \frac{(p_1)_{i+1,j} - (p_1)_{i,j}}{(\Delta X)_i} \right] - \left\{ \left[ \left( H_T^3 - H_T^3 \right) e^{-\delta_{i,j}} \right]_{i,j} \right\} \hat{\phi}_{i,j} - \\
+ \left[ \left( H_T^3 - H_T^3 \right) e^{-\delta_{i,j}} \right]_{i-1,j} \left[ \frac{(p_1)_{i,j} - (p_1)_{i-1,j}}{(\Delta X)_{i-1}} \right] + c_{UD} \left( \delta_{i+1,j} - \delta_{i-1,j} \right) \right) } \text{(4.33)}
\]
4.2.3 Method of Solution

In Eq. (4.33), $\phi_{i-1,j}$, $\phi_i,j-1$, $\phi_i,j$, $\phi_i,j+1$, $\phi_{i+1,j}$ are the only unknowns. $H_1$ and $p_1$ considered as known are determined by the method outlined in [22] for the smooth lubricated contacts. Thus, Eq. (4.33) can be re-written in the following matrix form:

$$
\begin{bmatrix}
A_i
\end{bmatrix}\begin{bmatrix}
\phi_{i-1,j} \\
\phi_i,j \\
\phi_i,j+1
\end{bmatrix} +
\begin{bmatrix}
B_i
\end{bmatrix}\begin{bmatrix}
\phi_i,j
\end{bmatrix} +
\begin{bmatrix}
C_i
\end{bmatrix}\begin{bmatrix}
\phi_{i+1,j}
\end{bmatrix} = \begin{bmatrix}
R_i
\end{bmatrix}
$$

(4.34)

where $[A_i]$ and $[C_i]$ are $M \times M$ diagonal square matrix; $[B_i]$ is a $M \times M$ tri-diagonal square matrix; $[\phi_{i-1,j}]$, $[\phi_i,j]$, $[\phi_{i+1,j}]$ and $[R_i]$ are $M \times 1$ column matrix; $M$ and $N$ are the number of grids used in the x and y directions.

In prescribing the boundary conditions for Eq. (4.34), it is necessary to assume that the effect of the asperity on the pressure distribution at $|X-x_3/b'| \geq 5$ or $|y/a'| \geq 5$ is negligible. This assumption justifies the prescription of the contact. Thus the boundary conditions are $\phi = 0$ along $X-x_3/b' = \pm 5$ and along $y/a' = \pm 5$, or

$$
\begin{bmatrix}
\phi_{i-1,j} \\
\phi_{i,j} \\
\phi_{i,j+1}
\end{bmatrix} = \begin{bmatrix}
\phi_{i-1,j} \\
\phi_{i,j} \\
\phi_{i,j+1}
\end{bmatrix} = 0
$$

(4.35)

$\phi_{i,j} = 0$ for $j = M$

Along $y=0$ or $j=1$, the flow is considered to be symmetric, and this gives the following additional boundary condition,

$$
\phi_{i,j-1} = \phi_{i,j+1} \quad \text{for } j=1
$$

(4.36)

Eq. (4.34) is solved numerically by the Columnwise Matrix Inversion Method [25].

4.3 DEFORMED ASPERITY ON THE STATIONARY SURFACE

In the case of simple sliding of a smooth surface against a stationary asperity, the non-dimensional Reynolds equation and elasticity equation are respectively
\[ \frac{dP}{dx} = \left( \frac{48}{H_0^2} \right) U \mu \left( \frac{H_T - L_1}{H_T^3} \right) \]  

(4.37)

\[ H_T = H_1 + \frac{S}{h_o} \]  

(4.38)

In this case, \( S \) is a function of \( x \) only. Thus, \( \frac{\partial S}{\partial t} = 0 \). Therefore these coupled equations can be solved simultaneously using the same numerical scheme as described in [22]. The results are compared with those obtained by solving the perturbed Reynolds equation.

4.4 DISCUSSIONS OF RESULTS

The results of the perturbed pressure \( \tilde{p} \) are presented as \( \Delta_s \), which is defined as

\[ \Delta_s = (\bar{\tilde{p}}_{\text{max}} - \bar{\tilde{p}}_{\text{min}}) \quad \text{at } j=1 \]  

(4.39)

where the position \( j=1 \) is at the centerline of the asperity along the sliding direction. At this position, the effect of asperity on \( \tilde{p} \) is most severe. Since \( \tilde{p} \) is a function of \( P_{Hz} \), \( G \), \( h_o/R \), \( \bar{b} \), \( \gamma \), \( c_1/h_o \), \( x_3 \), and the slide to roll ratios, the effects of these variables on \( \Delta_s \) are studied separately.

1. Effect of \( h_o/R \)

With \( P_{Hz} = 0.003 \), \( G = 100 \), \( x_3 = -0.5 \), \( \bar{b} = 1/32 \), \( c_1/h_o = 0.3 \), the results of \( h_o/R \) versus \( \Delta_s \) are shown in Fig. (4.6). In the case of 2-D elastic asperity, the magnitude of \( \Delta_s \) increases as \( h_o/R \) increases from \( 1.0 \times 10^{-6} \) to about \( h_o/R = 5 \times 10^{-6} \) (approximate) after which the magnitude of \( \Delta_s \) decreases with further increase in \( h_o/R \). This phenomenon is attributable to the local elastic effect of the asperity. In general, one might imagine that the effect of roughness on \( \tilde{p} \) should be much more severe as the ratio \( h_o/R \) becomes smaller. However, in an elastohydrodynamic contact,
the elastic effect becomes more significant as \( h_o/R \) becomes smaller. Thus, for a fixed ratio of \( c_1/h_o \), the elastic effect tends to flatten out the asperity and produces a trend which shows a decrease in \( \Delta_s \) as \( h/R \) decreases. On the other hand, for the 3D rigid asperity analysis the elastic effect is not accounted; therefore, the magnitude of \( \Delta_s \) consistently increases with a decrease in \( h_o/R \).

Based on the results shown in Fig. 4.6, it appears that further comparisons between the 2D-elastic asperity and 3D-rigid asperity results are more meaningful for values of \( h_o/R \) greater than \( 1.0 \times 10^{-5} \), since below this value no good agreement is expected.

(2) Effect of \( P_{Hz} \)

In hydrodynamic lubrication, with a rigid asperity, it is expected that the effect of asperity on \( \Delta_s \) will become more severe as the load increases. However, for EHD contacts, this simple trend is only expected to be true if the load parameter \( P_{Hz} \) is sufficiently small. For very heavy loads, the elastic effect will iron out the asperity and this may decrease the magnitude of \( \Delta_s \). Such trend is demonstrated in Fig. (4.7). In the 2D-elastic asperity curve, the magnitude of \( \Delta_s \) increases with \( P_{Hz} \) until \( P_{Hz} = 0.0045 \). Beyond this point, the magnitude of \( \Delta_s \) flattens out and shows a tendency to decrease as load further increases. On the other hand, the 3D-rigid asperity curve shows a steady increase in \( \Delta_s \) with \( P_{Hz} \) even for very heavy loads. The lack of agreement at these heavy loads is definitely due to the local elastic effect.

Referring to Fig. (4.8), with the ratio \( h_o/R \) changing from \( 10^{-5} \) to \( 10^{-6} \), the discrepancy between the 3D-rigid asperity results and the 2D-elastic asperity is greatly enlarged. In this regime of extremely thin film, the local elastic effect is so overwhelming that one cannot expect any validity of the 3D-rigid asperity perturbation analysis.
The effect of pressure-viscosity dependence is examined by varying $G$ from 100 to 500 for $P_{Hz} = 0.003$, $h_o/R = 10^{-5}$, $X_3 = -0.5$, $\overline{b} = 1/32$, and $c_1/h_o = 0.3$. The reason for using a mild pressure-viscosity dependence is because recent studies in EHD traction has indicated that the effective pressure-viscosity coefficient within the Hertzian conjunction is only a small fraction of those measured under static equilibrium. For this reason, it is believed that values of $G$ around 500 should not be unreasonable to represent the effective pressure-viscosity dependence in the conjunction. It can be readily seen in Fig. 4.9 that the magnitude of $\Delta_s$ increases with $G$ for both 3D-rigid asperity and 2D-elastic asperity cases. However, the rigid-asperity model is slightly more influenced by $G$ than the elastic asperity model. It is also interesting to note that the pressure perturbation $\Delta_s$ is depending exponentially on the pressure-viscosity coefficient. This is somewhat expected since pressure is directly portionally to the viscosity.

With $P_{Hz} = 0.003$, $h_o/R = 10^{-5}$, $G = 100$, $\overline{b} = 1/32$, $c_1/h_o = 0.3$, Fig.(4.10) shows the relation between $\Delta_s$ and $X_3$, the location of the center of the asperity. Qualitatively, the effects on $\Delta_s$ due to rigid and elastic asperity have the same trend. The magnitude of $\Delta_s$ increases as $X_3$ moving toward the center of the contact. This trend is certainly expected, since a higher pressure level inevitably would lead to a stronger asperity interaction, hence higher pressure perturbation.

Fig. (4.11) shows the effect of $\overline{b}$, asperity size, on $\Delta_s$ for $P_{Hz} = 0.003$, $h_o/R = 10^{-5}$, $G = 100$, $X_3 = -0.5$, $c_1/h_o = 0.3$. From the two curves shown, a similar trend is seen to exist in the relation between $\Delta_s$ and the asperity size for both
the rigid model and the elastic model. In both cases, $\Delta$ increases with the asperity size; however, the increase in $\Delta$ is much larger in the case of rigid asperity model than that in the elastic-asperity model.

(6) Effect of $c_1/h_o$

With $P_{Hz} = 0.003$, $h_o/R = 10^{-5}$, $G = 100$, $X_3 = -0.5$, $\bar{b} = 1/32$, Fig. (4.12) shows the effect of asperity height, $c_1/h_o$, on $\Delta$ for both 3D-rigid-asperity and 2D-elastic asperity models. It is readily seen that the two curves almost coincide with each other and the magnitude of $\Delta$ for both cases increases with $c_1/h_o$. As $c_1/h_o$ becomes larger, the two curves begin to divert slightly.

(7) Effect of $\gamma$

As defined earlier, $\gamma$ is the ratio of half of the asperity length to half of the asperity width, known as the ellipticity ratio of the asperity. As the 2D elastic asperity model only describes straight asperity ridges, Fig. (4.13) only shows the results for the 3D-rigid-asperity model. It is seen that the magnitude of $\Delta$ increases with $\gamma$ as expected. When $\gamma$ is less than 1, $\Delta$ increases sharply with a small increase in $\gamma$. As $\gamma$ becomes larger the change of $\Delta$ becomes much more gentle. In fact, for $\gamma > 5$, the change of $\gamma$ is practically negligible.

(8) Effect of Slide to Roll Ratio $S$

Fig. (4.14) shows the effect of $S$ on $\Delta$ for $P_{Hz} = 0.003$, $G = 100$, $h_o/R = 10^{-5}$, $X_3 = -0.5$, $\bar{b} = 1/32$ and $c_1/h_o = 0.3$. For $\gamma = 1, 2$ and $10$, $\Delta$ decreases when $S$ increases from -2.0 to 0.3, and then increases when $S$ increases from 0.4 to 2.0. These trends show that between $S = 0.3$ and 0.4, the perturbed pressure $\Delta$ reaches a minimum.

This phenomenon can be explained by Fig. (4.15) in which the perturbed pressures $\xi$ around the asperity center are plotted for $S = 0.3$ and $S = 0.4$. In the case of $S = 0.3$, the value of $\xi$ is mostly negative for $x/b' \leq 0$, and positive for $x/b' \geq 0$. 59
In the case of $S = 0.4$, the pressure exhibits an opposite trend. Thus, one would expect that between $S = 0.3$, and 0.4, the trend for $\delta$ begins to reverse itself.

With $P_{Hz}$, $h_o/R$, $X_3$, $b$, $c_1/h_o$ and $\gamma$ having the same values as those used for Fig. (4.14), Fig. (4.16) shows the effect of $G$ on $S$. For $G = 100$ and 500, the qualitative trends of $\Delta S$ versus $S$ are the same. However, for $G = 100$, the minimum $\Delta S$ is located between $S = 0.3$ and 0.4, whereas, for $G = 500$, the minimum $\Delta S$ is shifted to a position between $S = 0.1$ and 0.2. Thus when the magnitude of $G$ is increased, the minimum $\Delta S$ is shifted toward the asperity center. This agrees very well with Lee and Cheng's [5] results in which the $G$ value is equal to 3180, the perturbed pressure is negligibly small when $S=0$.

The pressure profiles obtained from the smooth film theory and the perturbation theory are compared in Fig. (4.17). Again, it shows that in the case of pure rolling ($S=0$), the perturbed pressure can be neglected. For $|S| = 2$, the perturbed pressure is relatively more important.
4.5 CONCLUSIONS

The perturbed pressure distribution around an ellipsoidal asperity tip within an EHD line contact can be calculated by solving the perturbed Reynolds equation based on the assumption of a rigid asperity. The magnitude of the perturbed pressure $\Delta_p$ was found to be a function of the following dimensionless variables: The Hertzian pressure, $P_{Hz}$, the smooth film center film thickness $h_o/R$, the pressure viscosity parameter $G$, the position of the asperity center $X_3$, the size of the asperity $\bar{b}$, the height of the asperity $c_1/h_o$, the ellipticity ratio $\gamma$ and the slide to roll ratio $S$. $\Delta_p$ was shown to increase with an increase of $P_{Hz}$, $G$, $\bar{b}$, $c_1/h_o$, $\gamma$, or $X_3$. However, it decreases as $h_o/R$ increases. The manner in which $\Delta_p$ varies with $S$ is dependent upon the pressure viscosity parameter $G$. For a large $G$, $\Delta_p$ is at its minimum when $S$ approaches zero (pure rolling condition); it increases as the magnitude of $S$ increases. For a small $G$, the value of $S$ at which $\Delta_p$ reaches a minimum shifts from zero to some small positive values.

A comparison was made between the results obtained by the perturbation analysis based on an ellipsoidal asperity for $\gamma = 10$ and $S=2$, and those obtained by the 2D asperity analysis. The comparison indicated that the perturbation analysis which ignored the local elastic effect yielded a very good approximation on the magnitude of the pressure fluctuation for certain cases, provided that $h_o/R \geq 10^{-5}$, $P_{Hz} \leq 0.003$, $G \approx 100$, $c_1/h_o \leq 0.3$ and $\bar{b} \approx 1/32$. Beyond these limitations, the perturbation analysis would over-estimate the pressure fluctuation.
Figure 4-1  Two Lubricated Rollers and the Equivalent Roller-Plane System
Figure 4-2 A Single Ellipsoidal Rigid Asperity in an EHD Contact
Figure 4-3 One-Dimensional Grids for Smooth Contact
Figure 4-4  Two-Dimensional Grids for Rough Contact
Figure 4-5 Grid Spacing
Fig. 4-6 The Effect of the Nominal EHD Center Film Thickness, $h_o/R$, on the Double Amplitude of the Perturbed Pressure, $\Delta_s = \bar{E}_{ms} - \bar{E}_{mh}$. 

$h_o = 0.003$, $G = 100$, $X_3 = -0.5$, $\bar{b} = 1/32$, $c_1/h_o = 0.3$ 

3D-RIGID ASPERITY, $\gamma = 10$, $S = 2$ 

2D-ELASTIC ASPERITY 

$h_o/R \times 10^6$
Fig. 4-7 The Effect of the Normalized Hertzian Pressure, $P_{HZ} = \frac{p_{HZ}}{E'}$, on $\Delta s = \bar{s}_{max} - \bar{s}_{min}$, for $h_o/R = 10^{-5}$.
Fig. 4-8 The Effect of the Normalized Hertzian Pressure, $p_{Hz} = p_{Hz}/E'$, on $\Delta s = \bar{\Delta}s_{max} - \bar{\Delta}s_{min}$ for $h_o/R = 10^{-6}$
The Effect of the Pressure Viscosity Parameter, 
\[ G = \alpha' E' \], on \[ \Delta S = \bar{P}_{min} - \bar{P}_{min} \].

Fig. 4-9
Fig. 4-10 The Effect of the Position of the Asperity Center, $X_3$, on 
$\Delta_s = \Phi_{\text{max}} - \Phi_{\text{min}}$
\[ p_{HZ} = 0.003 \]
\[ h_o/R = 10^{-5} \]
\[ G = 100 \]
\[ X_3 = -0.5 \]
\[ c_1/h_o = 0.3 \]

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**3D-RIGID ASPERITY, \( \gamma = 10, S = 2 \)**

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**2D-ELASTIC ASPERITY**

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**Fig. 4-11** The Effect of the Asperity Size, \( \bar{b} = b'/b \), on \( \Delta s: \bar{\Delta} \rightarrow \bar{\Delta}_{\text{ref}} \)
The Effect of the Asperity Height, $c_1/h_0$, on $\Delta_s$: $\tilde{E}_{\text{max}}$, $\tilde{E}_{\text{min}}$

Fig. 4-12
\( p_{HZ} = 0.003 \)
\( h_o/R = 10^{-5} \)
\( c = 100 \)
\( X_3 = -0.5 \)
\( \bar{b} = 1/32 \)
\( s = 2 \)

**Fig. 4-13** The Effect of the Ellipticity Ratio, \( \gamma = a'/b' \), on
\[ \Delta_S = \Phi_{max} - \Phi_{min} \]
Fig. 4-14 The Effect of the Slide to Roll Ratio, $S = 2(u_1-u_2)/(u_1+u_2)$, on $\Delta_s \cdot \psi_{max} - \psi_{min}$ for a Three-dimensional Rigid Asperity with Pressure Viscosity Parameter, $\alpha E' = 100$. 

- $PHZ = 0.003$
- $h_o/R = 10^{-5}$
- $G = 100$
- $X_3 = -0.5$
- $\bar{b} = 1/32$
- $c_1/h_o = 0.3$
Fig. 4-15 The Perturbed Pressure, $\overline{F}$, Around the Tip of a Three-dimensional Rigid Asperity

- $P_{HZ} = 0.003$
- $h_o/R = 10^{-5}$
- $C = 100$
- $X_3 = -0.5$
- $\bar{b} = 1/32$
- $c_1/h_o = 0.3$
- $\gamma = 10$

- $--- S = 0.3$
- $--- S = 0.4$
The Effect of Slide to Roll Ratio, $S = 2(u_1 - u_2)/(u_1 + u_2)$, on $\Delta s$, for a Three-dimensional Rigid Asperity with Pressure Viscosity Parameter, $\mathcal{E}' = 100$ and $500$. 

- $P_{HZ} = 0.003$
- $h_0/R = 10^{-5}$
- $X_3 = -0.5$
- $\bar{b} = 1/32$
- $c_1/h_0 = 0.3$
- $\gamma = 10$

Fig. 4-16

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\[ P_{HZ} = 0.003 \]
\[ \frac{h}{R} = 10^{-5} \]
\[ G = 100 \]

SMOOTH-FILM PRESSURE PROFILE

\[ X_3 = -0.5 \]
\[ \bar{b} = 1/32 \]
\[ c_1/h_o = 0.3 \]
\[ \gamma = 10 \]

Fig. 4-17 The Smooth-Film and the Perturbed Pressure Profiles for a Three-dimensional Rigid Asperity
CHAPTER V
THE EFFECT OF SINUSOIDAL WAVINESS ON THE PRESSURE FLUCTUATION WITHIN THE HERTZIAN CONTACT

5.1 Introduction

In the previous chapter, the emphasis has been placed on the pressure perturbation around a single asperity. Such analysis with a single asperity is useful only in determining qualitatively the effects of asperity geometry, lubricant property, speed, and load upon the perturbed pressure amplitude; but it is not suitable in making quantitative predictions of pressure fluctuations within these contacts. In order to simulate the effect of continuous transverse ridges on the pressure fluctuation in an elastohydrodynamic contact, one may assume these ridges can be represented by a series of sinusoidal waviness.

In the case the surface roughness is located on the stationary surface only, the pressure and deformation profiles become time-independent, and one can modify the method described in [22] to solve for the compatible pressure and film thickness profiles. The analysis in this chapter is confined to seek the EHD performance at the inlet of a Hertzian contact with a sinusoidal roughness on the stationary surface.

5.2 Mathematical Formulation

The roughness model is assumed to be continuous transverse ridges which can be represented by a sine wave on the stationary side of the contact. Thus the asperity height $\delta$, is a function of $x$ only. Therefore one would obtain the relations

$$\delta = c_1/h_0 \sin(2\pi X)$$

(5.1)
\[ \frac{\partial \delta}{\partial t} = 0 \]  

where \[ \delta = \delta/h_0 \]

\[ n = \frac{1}{4} x \frac{1}{b'/b} \]  
or  
\[ = \frac{1}{4} \frac{b'}{b} \]

\[ x = x/b \]

\[ h = \text{center film thickness of the smooth-film EHD contact} \]

\[ c_1 = \text{maximum asperity height} \]

\[ b' = \text{half of the asperity width} \]

\[ b = \text{half of the Hertzian contact width} \]

\[ x = \text{coordinate axis along the sliding direction} \]

The Reynolds equation and the elasticity equation are respectively

\[ \frac{dp}{dx} = \frac{48}{H_0^2} U \mu \left( \frac{H_T - \frac{1}{3}}{H_T} \right) \]  

(5.3)

\[ H_T = h_0 + \overline{\delta} \]  

(5.4)

\[ H_1 = 1 + \frac{16}{H_0^2} \frac{p_{Hz}^2}{E} \left( \frac{X}{2} - \frac{1}{\pi} \int_{-\infty}^{X} p(\overline{\xi}) \ln \left| \frac{\overline{\xi} - X}{\overline{\xi}} \right| d\overline{\xi} \right) \]  

(5.5)

where \[ P = p/p_{Hz}, \quad H_0 = h_0/R, \quad R = \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^{-1}, \]

\[ U = \frac{\mu s (u_1 + u_2)}{2E'R}, \quad H_T = h_T/h_0, \quad h_T = h_1 + \delta, \quad \overline{\mu} = \frac{\mu}{\mu_s} \]

\[ p_{Hz} = p_{Hz}/E', \quad E' = \left[ \frac{1}{2} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right) \right]^{-1} \]

\[ \overline{\xi} = \xi/b, \quad G = \alpha E' \]

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Using the same numerical scheme stated in [22], \( P(X) \), \( H_T(X) \) and \( U \) are solved for a given set of non-dimensional variables, namely nominal center film thickness \( h_o / R \), Hertzian pressure \( P_{Hz} \), pressure viscosity parameter \( G \), maximum asperity height \( c_1 / h_o \) and asperity width \( b'/b \).

5.3 Discussions of Results

The effect of the pressure viscosity parameter \( G \), the maximum asperity height \( c_1 / h_o \), the asperity width \( b'/b \), the nominal center film thickness \( h_o / R \), and the Hertzian pressure \( P_{Hz} \) on the magnitude of the perturbed pressure, \( \Delta_s \), (which is defined as the difference between the maximum and the minimum pressure ripples deviated from the nominal smooth-film pressure profile), and the actual pressure distribution will be discussed in the following sections.

5.3.1 The Effect of Pressure Viscosity Parameter \( G \)

For \( P_{Hz} = 0.003 \), \( h_o / R = 10^{-5} \), \( c_1 / h_o = 0.3 \) and \( n = 2( b'/b = 1/8) \). Fig. (5.1) shows the effect of \( G \) on \( \Delta_s \). Similar to the results obtained for the single asperity ridge in an EHD contact, as discussed in Chapter IV, the magnitude of \( \Delta_s \) increases with \( G \) for this case with a waviness surface profile. With the same set of values for \( P_{Hz} \), \( h_o / R \), \( c_1 / h_o \) and \( b'/b \), the effect of \( G \) on the pressure profile is shown in Fig. (5.2). As expected, the pressure fluctuation from the nominally smooth-film pressure profile is more pronounced when the value of \( G \) is larger and within the Hertzian contact zone. Since \( G \) represents a significant parameter, the effects of other parameters on \( \Delta_s \) are compared in the subsequent sections for \( G = 100 \) and \( G = 1,000 \). An attempt was made for cases with \( G \) greater than 1,000, some of the solutions failed to converge.

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5.3.2 The Effect of $c_1/h_o$

The asperity interaction will be more severe when the amplitude of the asperity is larger. This phenomenon has been shown earlier in the single asperity-ridge analysis and is exhibited again here in Fig. (5.3) for the case of wavy surface roughness with $P_{Hz} = 0.003$, $h_o/R = 10^{-5}$, and $n = 2(b'/b = 1/8)$. For both cases with $G = 100$ and $1,000$, the magnitude of the perturbed pressure increases with the ratio $c_1/h_o$. This phenomenon can also be observed in Fig. (5.4) with $G = 100$ and Fig. (5.5) with $G = 1,000$ in which the perturbed pressure profiles are shown.

5.3.3 The Effect of $n$

For $P_{Hz} = 0.003$, $h_o/R = 10^{-5}$ and $c_1/h_o = 0.3$, Fig. (5.6) shows the effect of $n$ on the magnitude of the perturbed pressure $\Delta_s$, for both $G = 100$ and $1,000$. It seems that the two curves shown here do not have the same qualitative trend. However, it is believed that the further increase of $n$ for the case with $G = 1,000$ will decrease the magnitude of $\Delta_s$ due to the elastic effect, so that the shape of the curve in this case will be similar to that with $G = 100$.

Let's recall that $n = 1/4 \times b/b'$. Thus increasing the value of $n$ implies the closer the distance between the center of each individual asperity ridge and the center of the EHD contact. Therefore the effect of $n$ actually consists of the effects of elastic deformation, the ratio $b'/b$ and the distance between the individual asperity center and the contact center.

The perturbed pressure profiles for $G = 100$ and $1,000$, due to the effect of $n$, are shown in Figs. (5.7) and (5.8) respectively. The results obtained for these two cases are the same qualitatively. For
G = 1,000 in which the elastic effect is more severe, the magnitude of the pressure ripple deviated from the nominally smooth-film pressure profile is much larger than those with G = 100.

5.3.4 $h/R$ Effect

The effect of $h/R$ on the magnitude of the perturbed pressure $\Delta_s$, for both $G = 100$ and 1,000 are shown in Fig. (5.9). In these examples, the values of $P_{Hz}$, $c_1/h_0$, $n$ and $b'/b$ are 0.003, 0.3, 2 and 1/8 respectively. It is observed that the magnitude of $\Delta_s$ increases with the ratio $h_o/R$. This increase is larger when the magnitude of $G$ is larger.

In Chapter IV, it has been shown that the magnitude of $\Delta_s$ decreases with the increase of $h_o/R$ when there is a single 3D rigid asperity within the contact zone. This opposing trend between the case of a single 3D rigid asperity and continuous elastic asperities can be explained in the following manner:

In the case of sinusoidal elastic asperities, the pressure amplitude is reduced by the local elastic deformation of the asperity. If $c_1/h_0$ is held constant, the reduction in pressure due to local elastic deformation becomes much greater as $h_o/R$ decreases, because at a smaller $h_o/R$, it requires only a very small pressure amplitude to flatten out the asperity. This phenomenon of elastic effect is best illustrated by Fig. (5.10) which shows the perturbed pressure and surface profiles for $h_o/R = 10^{-5}$, $5 \times 10^{-6}$ and $10^{-6}$ respectively, for the sinusoidal elastic asperities with $G = 100$. The shapes of these perturbed pressure and surface profiles are consistent with one another. However, the smaller the $h_o/R$, the more the asperity being flattened out. For the case of a 3D rigid asperity, the effect of local elastic deformation is ignored, and thus the opposing trend is found.
The perturbed pressure and surface profiles for the sinusoidal elastic asperities with $G = 1,000$ are shown in Fig. (5.11). The results for both $G = 100$ and 1,000 are shown to have the same characteristics and the magnitude of $\Delta_s$ is larger with $G = 1,000$ as expected.

5.3.5 $P_{Hz}$ Effect

For $h_o/R = 10^{-5}$, $c_i/h_o = 0.3$, $n = 2$ and $b'/b = 1/8$, the results of the magnitude of the perturbed pressure $\Delta_s$, versus the nominal value of the maximum Hertzian pressure $P_{Hz}$, are shown in Fig. (5.12). It is readily seen that, for both cases with $G = 100$ and 1,000, the increase in $P_{Hz}$ results in a decrease in $\Delta_s$. This phenomenon is due to the reduction of the pressure amplitude caused by the local elastic deformation of the asperity. The larger the $P_{Hz}$, the easier the asperity being flattened, and thus the smaller the $\Delta_s$.

The perturbed pressure profiles are shown in Figs. (5.13) and (5.14) respectively. The results for $G = 100$ and 1,000 are found to have the same trend qualitatively. As expected, the magnitude of $\Delta_s$ is larger in the case of $G = 1,000$. In addition the perturbed surface profiles for the case of $G = 100$ as shown in Fig. (5.13) illustrates the effect of the local elastic deformation of the asperity as explained previously. It tells that the larger the $P_{Hz}$, the more severe the asperity being deformed.
5.4 CONCLUSIONS

In the case of the simple sliding of a smooth surface against a stationary rough one, the magnitude of the pressure deviation from the nominally smooth-film profile and the perturbed pressure profile in the inlet of an elastohydrodynamic contact, can be determined quantitatively when the undeformed rough surface profile is given.

In the examples given in this chapter, the undeformed rough surface profile is simulated by continuous transverse ridges represented by a series of sinusoidal waviness. The effect of the following non-dimensional parameters on the magnitude of the pressure fluctuation $\Delta_s$ and the perturbed pressure profile are obtained: the pressure viscosity parameter $G$, the maximum height of the asperity $c_1/h_0$, the number of wave cycles within the contact zone, $n$, (which in turn determines the width of the asperity, $b'/b$), the nominal smooth-film center film thickness $h_0/R$ and the Hertzian pressure $P_{Hz}$.

The effects of $G$ and $c_1/h_0$ on the magnitude of $\Delta_s$ for this case with a waviness surface profile have the same characteristics compared to those obtained for the single asperity ridge in an EHD contact as presented in Chapter IV. Namely, the magnitude of $\Delta_s$ increases when the magnitude of $G$ or $c_1/h_0$ increases.

The results of $\Delta_s$ versus $P_{Hz}$ and $h/R$ are found to be consistent for both $G = 100$ and $1,000$. The magnitude of $\Delta_s$ decreases as the magnitude of $P_{Hz}$ or $h/R$ becomes larger. These results for the waviness surface profile have the opposing trend when compared to those obtained by the 3D rigid asperity analysis. The reason for the opposing trend is due to the pressure reduction caused by the local elastic deformation of the asperity for the waviness surface profile, whereas the effect of local elastic deformation is ignored for a 3D rigid asperity.
The effect of the number of wave cycles within the contact zone on the magnitude of the perturbed pressure $\Delta_s$ is found to be very complicated. The local elastic effect, the ratio $b'/b$, and the distance between each individual asperity center and the contact center are important factors affecting the magnitude of $\Delta_s$ caused by changing the magnitude of $n$. 
$P_{HZ} = 0.003$

$h_o / R = 10^{-5}$

$c_1 / h_o = 0.3$

$n = 2$

Fig. 5-1 The Effect of the Pressure Viscosity Parameter, $G = \phi E'$, on the Double Amplitude of the Perturbed Pressure
Fig. 5-2 The Effect of the Pressure Viscosity Parameter, $G = \alpha E'$, on the Perturbed Pressure Profiles

$P_{HZ} = 0.003, \quad h_o/R = 10^{-5}, \quad c_1/h_o = 0.3, \quad n = 2$

- Smooth-Film Pressure Profile
- $G = 100$
- $G = 500$
- $G = 1,000$
\[ P_{HZ} = 0.003 \]
\[ h_o/ R = 10^{-5} \]
\[ n = 2 \]

---

Fig. 5-3 The Effect of the Asperity Height to Film Thickness Ratio, \( c_1/h_o \), on \( \Delta_s = \delta_{max} - \delta_{min} \).
$P_{HZ} = 0.003, \quad h_o/R = 10^{-5}, \quad G = 100, \quad n = 2$

---

**Figure 5-4** The Effect of the Asperity Height to Film Thickness Ratio, $c_1/h_o$, on the Perturbed Pressure Profiles for $E' = 100$
$P_{Hz} = 0.003, \quad h_o/R = 10^{-5}, \quad G = 1,000, \quad n = 2$

Fig. 5-5 The Effect of the Asperity Height to Film Thickness Ratio, $c_1/h_o$, on the Perturbed Pressure Profiles for $E' = 1000$
\[ P_{HZ} = 0.003, \quad h_0 / R = 10^{-5}, \quad c_1 / h_0 = 0.3. \]

\[ - - \quad G = 1,000 \]

\[ - - - \quad G = 100 \]

![Graph showing the effect of the number of wave cycles, n, on \( \Delta \phi = \Phi_{max} - \Phi_{min} \).](image)

Fig. 5-6  The Effect of the Number of Wave Cycles, n, on \( \Delta \phi = \Phi_{max} - \Phi_{min} \).
Fig. 5-7 The Effect of the Number of Wave Cycles, $n$, on the Perturbed Pressure Profiles for Pressure Viscosity Parameter, $E' = 100$

$P_{HZ} = 0.003, \quad h_o/R = 10^{-5}, \quad G = 100, \quad c_1/h_o = 0.3$
$P_{HZ} = 0.003, \quad h_o/R = 10^{-5}, \quad G = 1,000, \quad c_1/h_o = 0.3$

---

Fig. 5-8 The Effect of $n_1$ on the Perturbed Pressure Profiles for Pressure Viscosity Parameter, $\alpha E' = 1000$
Fig. 5-9  The Effect of the Dimensionless Center Film Thickness, \( h_o/R \), on \( \Delta s = \bar{\delta}_{max} - \bar{\delta}_{min} \).
$P_{HZ} = 0.003$, $G = 100$,  
$c_1/h_0 = 0.3$, $n = 2$

Fig. 5-10 The Effect of the Dimensionless Center Film Thickness, $h_0/R$, on the Perturbed Pressure Profiles for $E' = 100$
SMOOTH-FILM PRESSURE PROFILE

$- - - - - - h_o/R = 10^{-6}$
$- - - - - - h_o/R = 5 \times 10^{-6}$
$- - - - - - h_o/R = 10^{-5}$

$P_{HZ} = 0.003$
$G = 1000$
$c_1/h_o = 0.3$
$n = 2$

Fig. 5-11 The Effect of the Dimensionless Center Film Thickness, $h_o/R$, on the Perturbed Pressure Profiles for $\alpha E' = 1000.$
$\frac{h_o}{R} = 10^{-5}$, $\frac{c_1}{h_o} = 0.3$, $n = 2$

$\longrightarrow \quad G = 1,000$

$\longrightarrow \longrightarrow \quad G = 100$

**Fig. 5-12** The Effect of the Normalized Hertzian Pressure, $P_{Hz} = P_{Hz}/E'$, on $\Delta_5 = \Phi_{max} - \Phi_{min}$. 

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\[ \frac{h_0}{R} = 10^{-5} \]
\[ \frac{c_1}{h_0} = 0.3 \]
\[ n = 2 \]

**Fig. 5-13** The Effect of the Normalized Hertzian Pressure, \( P_{Hz} = \frac{P_{Hz}}{E'} \), on the Perturbed Pressure and Film Profiles for \( E' = 100 \)
Fig. 5-14  The Effect of the Normalized Hertzian Pressure, $P_{Hz} = P_{Hz'}$, on the Perturbed Pressure Profiles for $\alpha E' = 1000$.

- $P_{Hz} = 0.003$
- $P_{Hz} = 0.0045$
- $P_{Hz} = 0.006$

$h_o / R = 10^{-5}$
$c_1 / h_o = 0.3$
$n = 2$
$G = 1,000$
REFERENCES


APPENDIX A

Justification of $M$ as a Random Quantity of Zero (or Negligible) Variance

When $\frac{\partial h}{\partial t}$ is assumed to be zero, Eq. (2.6) becomes

$$\frac{\partial M}{\partial x} = 0$$  \hspace{1cm} (A.1)

where $M$ is defined by Eqs. (2.7) and (2.10). Hence $M$ is a constant of $x$. However, this constant can be a stochastic quantity in the time domain.

EHD Contact

Re-arrange Eq. (2.10) to obtain

$$\frac{1}{12 \mu_s} \frac{\partial q}{\partial x} = \frac{M}{h_T^3} - \frac{u_1 + u_2}{2} \frac{h}{h_T} + \frac{u_1 - u_2}{2} \left( \frac{\delta_1 - \delta_2}{h_T^3} \right)$$  \hspace{1cm} (A.2)

At $x = -\infty$, the asperity effect is negligible so that $q = p = 0$. However, the pressure $p$, must reach a significant fraction of the Hertzian maximum at $x = -1$, such that $e^{-\alpha p} \ll 1$, i.e. $q \approx \frac{1}{\alpha}$. Therefore $q$ at $x = -1$, can be assumed to be a stochastic variable with extremely small variance. Integrate Eq. (A.2) from $x = -\infty$ to $-1$, and obtain

$$\frac{1}{12 \mu_s \alpha} = \int_{-\infty}^{-1} \frac{M}{h_T^3} \ dx - \frac{u_1 + u_2}{2} \int_{-\infty}^{-1} \frac{h}{h_T^3} \ dx + \frac{u_1 - u_2}{2} \int_{-\infty}^{-1} \frac{\delta_1 - \delta_2}{h_T^3} \ dx$$  \hspace{1cm} (A.3)

As $M$ is a constant of $x$, Eq. (A.3) can be re-written as

$$M = \frac{1}{12 \mu_s \alpha} \frac{1}{2} \int_{-\infty}^{-1} \frac{h - \delta_1 + \delta_2}{h_T^3} \ dx + \frac{u_2}{2} \int_{-\infty}^{-1} \frac{h + \delta_1 - \delta_2}{h_T^3} \ dx$$  \hspace{1cm} (A.4)

When there are enough asperities within the contact zone, the integrals in Eq. (A.4)
are independent of the precise arrangement of the roughness and also independent on time. Therefore M can be assumed to be a stochastic quantity with zero (or negligible) variance.

**Rigid Rollers**

Eq. (2.7) can be re-written as

\[
\frac{1}{12\mu} \frac{dp}{dx} = \frac{M}{h_T^3} - \frac{u_1 + u_2}{2} \frac{h}{h_T^3} + \frac{u_1 - u_2}{2} \left( \frac{\delta_1 - \delta_2}{h_T^3} \right) \quad (A.5)
\]

Integrate Eq. (A.5) with the boundary conditions:

\[
p = 0 \quad \text{at} \quad x = -\infty
\]

\[
p = \frac{dp}{dx} = 0 \quad \text{at} \quad x = x^*
\]

one obtains

\[
M = \frac{\frac{u_1}{2} \int_{-\infty}^{x^*} \frac{h - \delta_1 + \delta_2}{h_T^3} \, dx + \frac{u_2}{2} \int_{-\infty}^{x^*} \frac{h + \delta_1 - \delta_2}{h_T^3} \, dx}{\int_{-\infty}^{x^*} \frac{1}{h_T^3} \, dx} \quad (A.7)
\]

Even though \(x^*\) is a random quantity, its deviation from the mean is expected to be of the same order of the wavelength of the asperity, and is small compared with the size of the bearings. The integrals in Eq. (A.7) are only slightly affected by \(x^*\), and therefore can be considered as stochastic quantities with negligible variances. Thus, M is also a stochastic quantity with negligible variance.
Integrate Eq. (A.5) with the boundary conditions:

\[ p = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad L \]

one obtains

\[
M = \frac{u_1}{2} \left[ \int_0^L \frac{h - \delta_1 + \delta_2}{h_T^3} \, dx \right]
\]

Following the above argument, M is also a stochastic quantity with zero (or negligible) variance.
APPENDIX B

FORTRAN IV LISTINGS OF COMPUTER PROGRAMS

1. PROGRAM ROLSLIP: This program which is used in Chapters II and III, is to calculate the integrated pressure at the inlet of an EHD contact with random surface roughness by the stochastic approach.

2. PROGRAM RIGROL7: This program which is used in Chapter II, is to compute the load of a rigid roller bearing with random surface roughness by stochastic approach.

3. PROGRAM SLIDER5: This program which is used in Chapter II, is to compute the load of an infinitely-wide slider bearing with random surface roughness by stochastic approach.

4. PROGRAM DAPOL2: This program which is used in Chapters II and III, is to generate data for functions $G_2$, $G_4$ and $G_5$ for the corresponding $\sigma/h_0$. These data are input to programs ROLSLIP, RIGROL7 and SLIDER5. The data for functions $G_2$, $G_4$ and $G_5$ in the case when the asperity height distribution functions are in the form of polynomial function as well as sinusoidal function are tabulated respectively.

5. PROGRAM ROLWAV1: This program which is used in Chapter III, is to compute the integrated pressure at the inlet of an EHD contact with waviness surface roughness by the deterministic approach.

6. PROGRAM ASPERITY: This program which is used in Chapter IV, is to compute the perturbed pressure distribution due to a single 3D-Rigid Asperity.
PROGRAM ROLSL1P(INPUT,OUTPUT)
C THIS PROGRAM IS TO CALCULATE THE INTEGRATED PRESSURE AT THE INLET OF AN END
C CONTACT BY THE STOCHASTIC APPROACH
C THIS PROGRAM CONSISTS OF FIVE ROUTINES
C 1 - SUBROUTINE GRUBIN IS TO CALCULATE THE INLET INTEGRATED PRESSURE OF AN END CONTACT BY SMOOTH FILM GRUBIN THEORY
C 2 - SUBROUTINE CONC IS TO CALCULATE THE CONSTANT C FOR VARIOUS DSIG DATA
C 3 - SUBROUTINE FUNC IS TO EVALUATE FUNCTIONS F1 AND F2 FOR DIFFERENT S/H
C 4 - SUBROUTINE TABLE IS TO SET UP FUNCTIONS F1 AND F2 IN THE FORM OF ARRAYS
C 5 - SUBROUTINE INTF IS AN INTEGRATION ROUTINE TO CALCULATE THE INTEGRATED PRESSURES WLOAD1, WLOAD2, AND WLOAD3
C
C
C DATA CARD NO. 1
C KF = TOTAL NO. OF GRIDS
C KM = GRID NO. AT X=-2
C NSIG = NO. OF DSIG(1) DATA
C NHO = NUMBER OF HO DATA
C DATA CARD NO. 2
C W = LOAD PER UNIT WIDTH
C SLIP = SLIPAGE COEFFICIENT
C PI = 3.141593
C DATA CARD NO. 3
C USIG(1) = DATA FOR SIGMA/NO
C DATA CARD NO. 4
C UH0(1) = VARIOUS HO DATA
C DATA CARD NO. 5
C THESE DATA ARE THE OUTPUT FROM PROGRAM SUCH AS PROGRAM DATPOL2 WHICH EVALUATE THE INTEGRAL OF (DISTRIBUTION FUNCTION*HSS)/(1.+SH*HSS)**3 AND (DISTRIBUTION FUNCTION/(1.+SH*HSS))**3
C THIS DISTRIBUTION FUNCTION CAN BE POLYNOMIAL, GAUSSIAN, SINUSOIDAL OR ANY OTHER FUNCTION
C DSM(I) = THE ABSCISSA RANGING FROM 0 TO 0.333
C UV(I) = THE NONDIMENSIONAL DSM(I)
C DG4(I) = THE INTEGRAL OF 35*96*(1.-HSS**2/9)**3/(1.+SH*HSS)**3*HSS
C IN THE CASE OF POLYNOMIAL DISTRIBUTION
C DG2(I) = THE INTEGRAL OF 35*96*(1.-HSS**2/9)**3/(1.+SH*HSS)**3
C IN THE CASE OF POLYNOMIAL DISTRIBUTION
C DG5(I) = DG2(I)/DG4(I)
C
C THE OUTPUT OF THIS PROGRAM WGRUB, WLOAD1, WLOAD2, WLOAD3, W1, W2, W3 ARE ALL INTEGRATED PRESSURE AT THE INLET
C
C COMMON WLOAD1(14), WLOAD2(14), WLOAD3(14), DSIG(14), T1(14,301),
T2(14,301), XX(301), K*F, KM*F, HO*W, PI*F, S*X*F, F2*C, SLIP*,
2 C(301), F*WGRUB(14), DG4(201), DG2(201)
DIMENSION Q1(14), Q2(14,301), Q3(14), DSM(201), DV(201)
C DG5(201), DHG0(14), W1(14), W2(14), W3(14)
PRINT I
READ C*KF*KM*NSIG*NHO
READ J*W*SLIP*PI
READ J*(DSIG(I)*I=1*NSIG)
READ J*(DHO(I)*I=1*NHO)
DO 100 I=1,201
100 READ 5*(DSH(I)*OV(I)*DG4(I)*DG5(I)*DG2(I))
PRINT 51
PRINT 2, KF, KM, NSIG, NHO
PRINT 52
PRINT 4*W*SLIP*PI
PRINT 53
PRINT 4*(DSIG(I)*I=1*NSIG)
PRINT 54
PRINT 4*(DHO(I)*I=1*NHO)
XX(I) = -5.
DO 98 I=2,KM
98 XX(I) = XX(I-1) + 0.015
KMM = KM + 1
DO 99 I=KMM,KF
99 XX(I) = XX(I-1) + 0.01
PRINT 56
DO 97 I=1,NHO
IGruk = I
H0 = DHO(I)
CALL grubin(igrub)
PRINT 7,I,DHO(I),WGRUB(I)
DO 101 LL =1,NSIG
PRINT 8*DSIG(LL)
PRINT 6
CALL CUNS(LL)
DO 103 I=1,NHO
KK = I
H0 = DHO(KK)
S = DSIG(LL)*H0
CALL table
CALL int
W1(I) = wload1(I)/wgrub(I)
W2(I) = wload2(I)/wgrub(I)
W3(I) = wload3(I)/wgrub(I)
PRINT 7,I,DHO(I)*WLOAD1(I)*WLOAD2(I)*WLOAD3(I)*W1(I)*W2(I)*W3(I) + 7 WGRUB(I)
103 CONTINUE
101 CONTINUE
C
1 FORMAT(1H1)
2 FORMAT(2FL15.8)
3 FORMAT(1H1)
4 FORMAT(9(2X,E13.6))
5 FORMAT(1H1)
6 FORMAT(1H1)
}
FORMAT (15,8(2X,E13.6))
51 FORMAT (/3X,*KF*,3X,*KM*,X*,*NSIG*,2X,*NH0*)
52 FORMAT (/7X,*W*,11X,*SL1P*,13X,*PI*)
53 FORMAT (/6X,*DSIG(1)*)
54 FORMAT (/6X,*DHU(I)*)
55 FORMAT (/*/6X,*USIG=*,F10,E/)
56 FORMAT (/3X,*I*,6X,*DH0*,9X,*WGKUB*)
STOP
END

SUBROUTINE GRUBIN(IGRUB)
COMMON WLOAD1(14),WLOAD2(14),WLOAD3(14),DSIG(14),T1(14,301),
       1 T2(14+301),XX(301),KK,KF,KM,NF,H0,W,PI,S,X,F1,F2,C,SLIP,
       2 G(301),F,WGRU8(14),DG4(201),DG2(201)
       DIMENSION F3(301)
C
DO 400 J=1,KF
   X = XX(J)
   H=H0+4.*W/P1*(ABS(X)*SQR(DS(ABS((-X)**2-1.0))-ALOG(ABS(X))**SQR(DS(ABS(-X)**2-1.0))))
   F3(J) = (H-H0) / H**3
   WGKUB(IGRUB) = 0.
   KMK = KM - 2
   DO 401 J =1,KMK+2
   WGRUB(IGRUB) = WGRUB(IGRUB) + 0.015/3.*(F3(J)+4.*F3(J+1)+F3(J+2))
   KFF = NF - 2
   DO 402 J = KMK,KFF+2
   WGRUB(IGRUB) = WGRUB(IGRUB) + 0.01/3.*(F3(J)+4.*F3(J+1)+F3(J+2))
RETURN
400 CONTINUE

SUBROUTINE CONS(LL)
COMMON WLOAD1(14),WLOAD2(14),WLOAD3(14),DSIG(14),T1(14,301),
       1 T2(14+301),XX(301),KK,KF,KM,NF,H0,W,PI,S,X,F1,F2,C,SLIP,
       2 G(301),F,WGRUB(14),DG4(201),DG2(201)
C
   V = 1. + DSIG(LL) / 0.001665
   MV = V
   VM = MV
   G2 = (UG2(MV+1) - DG2(MV)) * (V-MV) + DG2(MV)
   G4 = (UG4(MV+1) - DG4(MV)) * (V-MV) + DG4(MV)
   C = G4 / G2
RETURN
END

SUBROUTINE TABLE
COMMON WLOAD1(14),WLOAD2(14),WLOAD3(14),DSIG(14),T1(14,301),
       1 T2(14+301),XX(301),KK,KF,KM,NF,H0,W,PI,S,X,F1,F2,C,SLIP,
       2 G(301),F,WGRUB(14),DG4(201),DG2(201)
DO 201 J=1,KF
   X = XX(J)
   CALL FUNC
   T1(KK+J) = F1
   T2(KK+J) = F2
201 CONTINUE
RETURN
END
SUBROUTINE FUNC

C

COMMON WLOAD1(14), WLOAD2(14), WLOAD3(14), DSIG(14), T1(14, 301),
1 T2(14, 301), XX(301), KK, KF, KM, NF, HO, WP, S, X, F1, F2, C, SLIP,
2 G(301), F, WGRUN(14), DG4(201), DG2(201)

C

H = MG + 4.0 * W / PI * (ABS(X) * SQRT(ABS((-(X)**2 - 1.0))) - ALOG(ABS(X) * SQRT(ABS
1 ((-X)**2 - 1.0))))
SH = S / H
V = 1. + S / H / 0.001665
MV = V
VM = MV
G2 = (UG2(MV + 1) - DG2(MV)) * (V - VM) + DG2(MV)
G4 = (UG4(MV + 1) - DG4(MV)) * (V - VM) + DG4(MV)
F1 = (H - HO) / HO * 3 * G2
F2 = (G4 - C * G2) / H * 3 * S
RETURN
END

SUBROUTINE INT

C

COMMON WLOAD1(14), WLOAD2(14), WLOAD3(14), DSIG(14), T1(14, 301),
1 T2(14, 301), XX(301), KK, KF, KM, NF, HO, WP, S, X, F1, F2, C, SLIP,
2 G(301), F, WGRUN(14), DG4(201), DG2(201)
DIMENSION P1(14), P2(14)

C

P1(KK) = 0.
P2(KK) = 0.
KK = KM - 2
D0 301 J = 1, KM, 2
P1(KK) = P1(KK) + 0.015 / 3.0 * (T1(KK, J) + 4.0 * T1(KK, J + 1) + T1(KK, J + 2))
P2(KK) = P2(KK) + 0.015 / 3.0 * (T2(KK, J) + 4.0 * T2(KK, J + 1) + T2(KK, J + 2))
301 CONTINUE
KFF = KF - 2
D0 302 J = KM, KFF, 2
P1(KK) = P1(KK) + 0.015 / 3.0 * (T1(KK, J) + 4.0 * T1(KK, J + 1) + T1(KK, J + 2))
P2(KK) = P2(KK) + 0.015 / 3.0 * (T2(KK, J) + 4.0 * T2(KK, J + 1) + T2(KK, J + 2))
302 CONTINUE
WLOAD1(KK) = P1(KK) - P2(KK) * SLIP
WLOAD2(KK) = P1(KK) + P2(KK) * SLIP
WLOAD3(KK) = P1(KK)
C
RETURN
END

000000000000000000000000 END OF RECORD
301 201 9 6
0.00003 0.1 3.14159265
0.05 0.1 0.15 0.2 0.25 0.3 0.31 0.32
0.33
0.00001 0.00002 0.00003 0.00005 0.00007 0.00009
C DATA FOR DS(R), DSY(1), UC(1), UC(1), 4G2(1), 4G5(1) ARE OMITTED
000000000000000000000000 END OF INFORMATION

110
PROGRAM RIGROL INPUT, OUTPUT
C
C THIS PROGRAM IS TO COMPUTE THE LOAD OF A RIGID ROLLER BEARING BY STOCHASTIC APPROACH.

C THIS PROGRAM CONSISTS OF 4 SUBROUTINES:
C 1 - SUBROUTINE TABLE IS TO SET UP A PRESSURE ARRAY CORRESPONDING TO THE GRID POSITIONS.
C 2 - SUBROUTINE CONS IS TO CALCULATE THE CONSTANT C.
C 3 - SUBROUTINE FUNC IS TO EVALUATE FUNCTIONS F1, F2, AND F.
C 4 - SUBROUTINE INT IS A SIMPSON INTEGRATION ROUTINE TO COMPUTE THE LOAD W(KK).
C THE OUTPUT OF THIS PROGRAM IS W(1) AND THE CORRESPONDING DSIG(I).

C DATA CARD NO. 1
C N1 = NO. OF UNIFORM GRIDS FROM X=-1 TO X=0
C NF = NO. OF UNIFORM GRIDS FROM X=0 TO X=XA
C KO = N1 + 1
C KF = NF + 1
C ND = NO. OF DSIG(I) DATA
C DATA CARD NO. 2
C XA(1), XA(2) = INITIAL VALUES ASSIGNED TO XA(I), ASSUMING THAT THE PRESSURE AT THIS LOCATION IS 0.
C HO = CENTER FILM THICKNESS.
C SLIP = SLIPPING COEFFICIENT.
C PI = 3.141593
C DATA CARD NO. 3
C DSIG(I) = SIGMA / HO, DATA.
C DATA CARD NO. 4
C THESE DATA ARE THE OUTPUT FROM PROGRAM SUCH AS PROGRAM DATPOL2 WHICH EVALUATE THE INTEGRAL OF ( DISTRIBUTION FUNCTION*HSS)/(1+SH*HSS)**3 AND ( DISTRIBUTION FUNCTION**3)/(1+SH*HSS)**3.
C THIS DISTRIBUTION FUNCTION CAN BE POLYNOMIAL, GAUSSIAN, SINUSOIDAL OR ANY OTHER FUNCTION.
C DSH(I) = THE ABSCISSA RANGING FROM 0 TO 0.333
C UV(I) = THE NONDIMENSIONAL DSH(I)
C DG1(I) = THE INTEGRAL OF 35/96*(1-HSS**2/9)**3/(1+SH*HSS)**3*HSS
C IN THE CASE OF POLYNOMIAL DISTRIBUTION.
C DG2(I) = THE INTEGRAL OF 35/96*(1-HSS**2/9)**3/(1+SH*HSS)**3
C IN THE CASE OF POLYNOMIAL DISTRIBUTION.
C DG5(I) = DG2(I) / DG4(I)
C
C COMMON XX(121), P(20, 121), CSIG(10), W(10), HO, HA, KO, KF, H, SO, SF, X
C F = F1 + F2 + DELA + PI + C + DG4(201) + DG6(201) * SLIP
C DIMENSION XX(21), DSIG(10), UV(LU), DG5(201), XH(I), IK(10)
C
C PRINT 1
READ 2, NI, NF, KO, KF, ND
READ 4, XA(1), XA(2), HO, SLIP, PI
READ *(DSIG(I), I=1, ND)
DO 95 I = 1, 201
95 READ *(DSH(I), UV(I), DG4(I), DG6(I), DG5(I), DG2(I)
PRINT 51
PRINT 2, NI, NF, KO, KF, ND
PRINT 2
PRINT 9*XA(1),XA(2),HO,SLIP,PI
PRINT 3
PRINT *(DSIG(I)+I=1,ND)
SO = 1./NI
DO 99 J=1,K0
99 XX(J) = -1. + (J-1)*SO
DO 100 II=1,ND
KK = II
PRINT *II,DSIG(II)
XA(1) = 0.01
XA(2) = 0.02
DO 101 I=1,2
K = I
HA = 1. + 0.5*XA(I)*2/HC
SF = XA(I) / NF
PRINT 54
PRINT 3*I,XA(I),HA,SO,SF
CALL CUNS
CALL TABLE
CONTINUE
XA(3) = (XA(1)*P(2*KF)-XA(2)*P(1*KF)) / (P(2*KF)-P(1*KF))
DO 102 I=3,20
K = I
HA = 1. + 0.5*XA(I)*2/HC
SF = XA(I) / NF
PRINT 54
PRINT 3*I,XA(I),HA,SO,SF
CALL CUNS
CALL TABLE
DIFF = P(I*KF) / P(I,K0)
IF (AE>(DIFF) .LE. 0.001 ) GO TO 103
PRUD = P(I*KF) .AF P(I-1,KF)
IF (PRUD .GE. 0.) GO TO 104
XA(I+1) = (XA(I-1)*P(I,KF)-XA(I)*P(I-1,KF)) / (P(I,KF)-P(I-1,KF))
GO TO 102
104 CONTINUE
P(I-1,KF) = P(I-2,KF)
XA(I-1) = XA(I-2)
XA(I+1) = (XA(I-1)*P(I,KF)-XA(I)*P(I-1,KF)) / (P(I,KF)-P(I-1,KF))
CONTINUE
102 CONTINUE
CONTINUE
CALL INT
XB(II) = XA(K)
IK(II) = K
100 CONTINUE
PRINT 7
DO 105 II=1,ND
105 PRINT 8*II,DSIG(II),W(II),XB(II),IK(II)
FORMAT(1H1)
FORMAT(205)
FORMAT(15,8(2X,E13.6))
FORMAT(8F10.8)

112
SUBROUTINE TABLE

C COMMON XX(121),P(20*121),CSIG(10),W(10),H0,HA,K0,KF,K,KK,S0,SF,X
C F,F1,F2,DELA,PI,C,DG4(201),DG5(201),SLIP
C DIMENSION E(121)

C PRINT 21
J = 1
X = XX(J)
CALL FUNC
E(J) = F
P(K,J) = 0.
DO 201 J = K0+1, K
X = XX(J)
CALL FUNC
E(J) = F
X = (XX(J-1)+XX(J))/2.
CALL FUNC
P(K,J) = P(K,J-1) + S0/2.*3.*(E(J-1)+4.*F+E(J))
201 CONTINUE
K01 = K0 + 1
DO 202 J = K01+1, KF
XX(J) = XX(J-1) + SF
X = XX(J)
CALL FUNC
E(J) = F
X = (XX(J-1)+XX(J))/2.
CALL FUNC
P(K,J) = P(K,J-1) + SF/2.*3.*(E(J-1)+4.*F+E(J))
202 CONTINUE
DO 203 J = K01, K0+4
203 PRINT *2,J,XX(J),P(K,J)
DO 204 J = K01, KF+2
204 PRINT *2,J,XX(J),P(K,J)
C FORMAT(/3X,*J,F10.2,3F20.10)
FORMAT(15*4(3X,E17.10))
RETURN
END
SUBROUTINE CONS
COMMON XX(121), P(20,121), CSIG(10), w(10), H0, HA, KO, KF, K, KK, S0, SF, X,
C F, FI, F2, DFLA, P1, C, DG4(201), DG2(201) + SLIP

V = 1. + DSIG(KK) / HA / 0.001665
MV = V
VM = MV
G2 = (DG2(MV+1)-DG2(MV)) * (V-VM) + DG2(MV)
G4 = (DG4(MV+1)-DG4(MV)) * (V-VM) + DG4(MV)
C = G4 / G2
PRINT 41, KK, K, C

41 FORMAT (3X, 3K8, 5X, 6K8, 10X, 2I5, 3X, E17.10)
RETURN
END

SUBROUTINE FUNC
COMMON XX(121), P(20,121), CSIG(10), w(10), H0, HA, KO, KF, K, KK, S0, SF, X,
C F, FI, F2, DFLA, P1, C, DG4(201), DG2(201) + SLIP

C = 0.01 0.002 0.0005 -0.1 3.14159265
RETURN
END

SUBROUTINE INT
COMMON XX(121), P(20,121), CSIG(10), w(10), H0, HA, KO, KF, K, KK, S0, SF, X,
C F, FI, F2, DFLA, P1, C, DG4(201), DG2(201) + SLIP

C = 0.5 * X ** 2 / H0
V = 1. + DSIG(KK) / H / 0.001665
MV = V
VM = MV
G2 = (DG2(MV+1)-DG2(MV)) * (V-VM) + DG2(MV)
G4 = (DG4(MV+1)-DG4(MV)) * (V-VM) + DG4(MV)
F1 = (M-HA) / H ** 3 * G2
F2 = (F4-C ** G2) / H ** 3
F = F1 + DSIG(KK) * F2 * SLIP
RETURN
END

SUBROUTINE INT
COMMON XX(121), P(20,121), CSIG(10), w(10), H0, HA, KO, KF, K, KK, S0, SF, X,
C F, FI, F2, DFLA, P1, C, DG4(201), DG2(201) + SLIP

C = 0.5 * X ** 2 / H0
K02 = K0 - 2
DO 301 J = 1, K02, 2
301 CONTINUE
KFF = KF - 2
DO 302 J = K0, KFF, 2
302 CONTINUE
PRINT 31, KK, W(KK)
C = 0.01 0.02 0.0005 -0.1 3.14159265
RETURN
END

0000V00000000-0000000000000000 END OF RECORD
40 10 41 51 10
0.01 0.01 0.0005 -0.1 3.14159265
0.06 0.1 0.14 0.18 0.22 0.26 0.3 0.31
0.32 0.33
C DATA FOR DSH(I), DV(I), DG4(1), 4G2(I), DG5(I) ARE OMITTED
00000000000000000000000000000000 END OF INFORMATION

114
PROGRAM SLIDER(I=INPUTOOUTPUT)

C THIS PROGRAM IS TO COMPUTE THE LOAD OF AN INFINITELY-WIDE SLIDER BEARING BY
C STOCHASTIC APPROACH

C THIS PROGRAM CONSISTS OF 3 SUBROUTINES
C 1 - SUBROUTINE TABLE IS TO FORM THE T1, T2, T3 ARRAYS WHICH WILL BE USED TO
C COMPUTE THE PRESSURE PROFILE
C 2 - SUBROUTINE FUNC IS TO EVALUATE THE FUNCTIONS F1, F2 AND F3
C 3 - SUBROUTINE INT IS THE INTEGRATION ROUTINE TO CALCULATE THE PRESSURE
C PROFILE AND THEN THE TOTAL LOAD
C THE OUTPUT OF THIS PROGRAM ARE: WLOAD1, WLOAD2 and WLOAD3 AND THE CORRESPONDING
C DSIG(I)
C WLOAD1 = LOAD IN SLIDING ROUGH SURFACE*FIXED SMOOTH SURFACE CASE
C WLOAD2 = LOAD IN FIXED ROUGH SURFACE*SLIDING SMOOTH SURFACE CASE
C WLOAD3 = LOAD IN BOTH SURFACES WITH SAME ROUGHNESS DISTRIBUTION

C DATA CARD NO. 1
C KF = TOTAL NO. OF GRID POINTS
C NF = NO. OF DSIG(I) DATA
C PI = 3.141593

C DATA CARD NO. 2
C DSIG(I) = DATA FOR SIGMA / H(MIN) RATIO

C DATA CARD NO. 3
C THESE DATA ARE THE OUTPUT FROM PROGRAM SUCH AS PROGRAM DatPol2 WHICH EVALUATE
C THE INTEGRAL OF ( DISTRIBUTION FUNCTION*HSS)/(1+SH*HSS)**3 AND
C ( DISTRIBUTION FUNCTION/(1+SH*HSS)**3
C THIS DISTRIBUTION FUNCTION CAN BE POLYNOMIAL, GAUSSIAN, SINUSOIDAL OR ANY
C OTHER FUNCTION
C DHSH(I) = THE ABSICSA RANGING FROM 0 TO 0.333
C DV(I) = THE NONDIMENSIONAL DSH(I)
C DUG4(I) = THE INTEGRAL OF ( (1-HSS)*2/9)/(1+SH*HSS)**3*HSS
C IN THE CASE OF POLYNOMIAL DISTRIBUTION
C DUG2(I) = THE INTEGRAL OF ( (1-HSS)*2/9)/(1+SH*HSS)**3
C IN THE CASE OF POLYNOMIAL DISTRIBUTION
C DUG5(I) = DUGZ(I)/DUG4(I)

C COMMON DS(201),DV(201),DG5(201),DG2(201),WLOAD1(14),
1 WLOAD2(14),WLOAD3(14),SIG(14),T1(14*101),T2(14*101),T3(14*101),
2 KS*KF*NS*X+F1+F2+F3+C1+C2+C3,P1+XX(101)
PRINT 1
READ 2,KF,NF,F1
PRINT 2,KF,NF,F1
READ 3,DSIG(I),I=1,NF
PRINT 4,DSIG(I),I=1,NF
D0 100 I=1,201
100 XX(I) = (J-1)*PI/100
100 XX(I) = (J-1)*PI/100
WREF = LGCG(2.5) - 2.*3.
DO 101 I=1*NF
  KK = I
  S = DSIG(KK)
  CALL TABLE
  C1 = (-T1(KK,KF) + S*T3(KK,KF)) / T2(KK,KF)
  C2 = (-T1(KK,KF) - S*T3(KK,KF)) / T2(KK,KF)
  C3 = -T1(KK,KF) / T2(KK,KF)
  CALL INT
  CONTINUE

  PRINT *,'G',WREF
  UC 102 I=1*NF
  PRINT '/I*DSIG(I)*WLOAD1(I)*WLOAD2(I)*WLOAD3(I)
  CONTINUE

C 1 FORMAT(1H1/2X,*KF*3X,*NF*10X,*PI/)
  2 FORMAT(215,F20.14)
  3 FORMAT(20F5.3)
  4 FORMAT(X,*DSIG(I),I=1*NF)*5X*20(X*F5.3/)
  5 FORMAT(F10.6,F10.2,F30.10)
  6 FORMAT(/X,*WREF=*,E17.10/3X,*I*6X,*USIG*12X,*WLOAD*,14X*
       *WLOAD2*14X,*WLOAD3*/)
  7 FORMAT(IS*5X*,F20.14)
  STOP
  END

SUBROUTINE TABLE

C COMMON DSIG(201),WV(201),CC4(201),DG5(201),DG2(201),WLOAD1(14),
  WLOAD2(14),WLOAD3(14),USIG(14),T1(14,101),T2(14,101),T3(14,101),
  KK,KF,NF,S,X*F1,F2,F3,C1,C2,C3,PI,XX(101)

C DIMENSION E1(I01),E2(I01),E3(I01)
  J = 1
  X = XX(J)
  CALL FUNC
  E1(J) = F1
  E2(J) = F2
  E3(J) = F3
  T1(KK,J) = 0.
  T2(KK,J) = 0.
  T3(KK,J) = 0.
  DO 201 J =2*KF
       X = XX(J)
  CALL FUNC
  E1(J) = F1
  E2(J) = F2
  E3(J) = F3
  X = (XX(J-1)+XX(J))/2.
  CALL FUNC
  T1(KK,J) = T1(KK,J-1) + 0.01/2.*E1(J-1) + 4.*F1 + E1(J))
  T2(KK,J) = T2(KK,J-1) + 0.01/2.*E2(J-1) + 4.*F2 + E2(J))
  T3(KK,J) = T3(KK,J-1) + 0.01/2.*E3(J-1) + 4.*F3 + E3(J))
  CONTINUE
  RETURN
  END
SUBROUTINE FUNC
COMMON DSH(201),DV(201),CC(201),DG5(201),DG2(201),WLOAD1(14),
1 WLOADC(14),WLOAD3(14),DS1G(14),T1(14*101),T2(14*101),T3(14*101),
2 KK,KF,NF,SX,F1,F2,F3,C1,C2,C3,P1,XX(101)
H = 2. * X
SH = S/H
V = 1. * SH/0.001665
VM = V
VM = VM
D2 = (ALOG(DG2(MV+1))-ALC(G(DG2(MV)))) * (V-VM) + ALOG(DG2(MV))
G2 = EAP(D2)
D4 = (ALOG(-DG4(MV+1))-ALC(S(-DG4(MV)))) * (V-VM) + ALOG(-DG4(MV))
G4 = -EXP(D4)
F1 = GC/H*#2
F2 = GC/H*#3
F3 = GC/H*#3
RETURN
END

SUBROUTINE INT
COMMON DSH(201),DV(201),CC(201),DG5(201),DG2(201),WLOAD1(14),
1 WLOADC(14),WLOAD3(14),DS1G(14),T1(14*101),T2(14*101),T3(14*101),
2 KK,KF,NF,SX,F1,F2,F3,C1,C2,C3,P1,XX(101)
DIMENSION P1(14*101),P2(14*101),P3(14*101)
PRINT J1,KK,DS1G(KK)
DO 300 J=1,KF
P1(KK,J) = T1(KK,J) + C1*F2(KK,J) - S0*P3(KK,J)
P2(KK,J) = T1(KK,J) + C2*F2(KK,J) + S0*P3(KK,J)
P3(KK,J) = T1(KK,J) + C3*F2(KK,J)
PRINT J2,J,P1(KK,J),P2(KK,J),P3(KK,J)
CONTINUE
WLOAD1(KK) = 0.*
WLOAD2(KK) = 0.*
WLOAD3(KK) = 0.*
KFF = KF*2
DO 302 J=1,KFF,2
WLOAD1(KK) = WLOAD1(KK) + 0.*P1(KK,J) + 4.*P1(KK,J+1) + 1.*P1(KK,J+2)
WLOAD2(KK) = WLOAD2(KK) + 0.*P2(KK,J) + 4.*P2(KK,J+1) + 2.*P2(KK,J+2)
WLOAD3(KK) = WLOAD3(KK) + 0.*P3(KK,J) + 4.*P3(KK,J+1) + 3.*P3(KK,J+2)
CONTINUE
PRINT 7,KK,DS1G(KK),WLOAD1(KK),WLOAD2(KK),WLOAD3(KK)
C 31 FORMAT(/X*KK=\$14.5X,US1G=\$7.4/3X*P1*,10X*P2*,18X*P3*,18X*P4*)
C 32 FORMAT(15*13X,17.10))
6 FORMAT(/3X*I*6X*DS1G*12X*WLOAD1*14X*WLOAD2*14X*WLOAD3*/)
7 FORMAT(13*5X*F3*3.5(3X,E17.10))
C RETURN
END
118
PROGRAM DATPULC (INPUT, OUTPUT, PUNCH)

C
C THIS PROGRAM IS TO CALCULATE DATA FOR FUNCTIONS G2 AND G4 FOR THE CASE WHEN
C THE ASPERITY HEIGHT FUNCTION IS IN THE FORM OF POLYNOMIAL
C FUNCTION 35/96* (1-HSS**2/9)**3 FOR HSS = -3 TO 3
C G2 = THE INTEGRAL OF 35/96* (1-HSS**2/9)**3/(1+SH**HSS)**3 FOR HSS = -3 TO 3
C G4 = THE INTEGRAL OF 35/96* (1-HSS**2/9)**3/(1+SH**HSS)**3*SH FOR HSS = -3 TO 3
C SINCE HSS RANGES FROM -3 TO 3, SH MUST BE LESS THAN 1/3. THUS SH VALUES
C USED IN THIS PROGRAM RANGES FROM 0 TO 0.333
C SH(I) = I=1*201, WITH 200 UNIFORM GRIDS OF SIZE 0.001665
C V(I) = NONDIMENSIONAL SH(I)

C
C THIS PROGRAM CONSISTS OF THREE SUBROUTINES
C 1 - SUBROUTINE MAIN IS TO CALCULATE R11 AND R12 FROM SUBROUTINES INT1
C AND INT2
C 2 - SUBROUTINE INT1 IS TO INTEGRATE THE FUNCTION
C (1+HSS**2/9)**3/(1+SH**HSS)**3 FOR HSS = -3 TO 3
C 3 - SUBROUTINE INT2 IS TO INTEGRATE THE FUNCTION
C (1+HSS**2/9)**3/(1+SH**HSS)**3 FOR HSS = -3 TO 3
C
C THE RESULTS OF G2(I)+G4(I)+G5(I) TOGETHER WITH THE CORRESPONDING SH(I)*V(I)
C VALUES WHICH ARE THE OUTPUT OF THIS PROGRAM ARE IN THE FORM OF DATA CARD
C
C
COMMON A*B*R11*R12*SIMPSCA
DIMENSION G2(201),G4(201),G5(201),V(201),SH(202)
PRINT 1
SH(1)=0.0
DO 10 I=1,201
CALL MAIN (SH(I))
G2(I) = 35/96*SH(I)
G4(I) = 35/96*SH(I)
G5(I) = G2(I) / G4(I)
V(I)=1.0*SH(I)/0.001665
PRINT 3
PRINT 4*1,SH(I),V(I),G4(I),G5(I),G2(I)
SH(I+1)=SH(I)+0.001665
10 CONTINUE
PRINT <
DO*20 I=1,201
PRINT >,SH(I),V(I),G4(I),G5(I),G2(I)
PUNCH >,SH(I),V(I),G4(I),G5(I),G2(I)
20 CONTINUE
1 FORMAT(1H1)
2 FORMAT(5X*SH**8X*V**16X*G4*16X*G5*16X*G2/)
3 FORMAT(/2X*1A5,5X*SH**5X*G4**7X*G5**15X*G2/)
C*G2(I)**
4 FORMAT(X*13+2X*FY,+1.3(2X,E17.10/)
5 FORMAT(F10.6,F10.2,F32U.10)
STOP
END
SUBROUTINE MAIN(SH)
COMMON A,B,R11,R12,SIMPSIN
DIMENSION D(3),E(3),SM(3),SIM(3)
DATA (D(1),1=1,2)/-3.0,-2.8/,(E(1),1=1,2)/-2.8,3.

C DO 100 J=1,2
A=D(J)
B=E(J)
CALL INT1(SH)
SM(J)=SIMPSIN
CALL INT2(SH)
SIM(J)=SIMPSIN
PRINT 11,J,A,B,SM(J),SIM(J)
100 CONTINUE
R11=SM(1)+SM(2)
R12=SIM(1)+SIM(2)
PRINT 12,R11,R12
C
11 FORMAT(X,**J=*,11,4x,**A=*,FB,3,4x,**B=*,F6.3,4x,**SM=*,E17.10,4x)
C,**SIM=*,E17.10/)
12 FORMAT(X,**R11=*,E17.10,5x,**R12=*,E17.10/)
RETURN
END

SUBROUTINE INT1(SH)
COMMON A,B,R11,R12,SIMPSIN
PRINT 41
HSS=A
FA=(1.0-HSS**2/9.0)**3/(1.0+SH**HSS)**3*HSS
HSS=B
FB=(1.0-HSS**2/9.0)**3/(1.0+SH**HSS)**3*HSS
F1=FA+FB
F2=0.0
HSS=(A+B)/2.0
F4=(1.0-HSS**2/9.0)**3/(1.0+SH**HSS)**3*4.0*HSS
N=2
T=(B-A)/N
SIMPSIN=T/3.0*(F1+F2+F4)
DO 10 K=1,500
T=(B-A)/N
TN=T/2.0
F1=F1+T/4.0
F2=F2+F1/2.0
F4=F1+F2
DO 20 J=1,N
HSS=A-J**T
F4=(1.0-HSS**2/9.0)**3/(1.0+SH**HSS)**3*4.0*HSS + F4
20 CONTINUE
SIMP=SIMPSIN
SIMPSIN=T/3.0*(F1+F2+F4)
DELTA=ABS((SIMPSIN-SIMP)/(SIMPSIN+SIMP))
PRINT 2,N,SIMP,DELTA
N=2*N

120
IF(Delta<=0.0005) GO TO 26
10 CONTINUE
26 PRINT C2,N,SIMPSON
C
21 FORMAT(4X,N*8X,*SIMPSON9,10X,*DELTA*/)
22 FORMAT(X,14,4X,E17.10,4X,F8.5)
RETURN
END

SUBROUTINE INTC(H)
COMMON A,B,R11,R12,SIMPSCA
PRINT C1
HSS=A
FA=(1.+HSS**2/9.0)**3/(1.0+SH**HSS)**3
HSS=B
FB=(1.+HSS**2/9.0)**3/(1.0+SH**HSS)**3
F1=FA+FB
F2=0.0
HSS=(A+B)/2.0
F3=(1.+HSS**2/9.0)**3/(1.0+SH**HSS)**3*4.0
N=2
T=(B-A)/N
SIMPSON=T/3.0*(F1+F2+F3)
DO 10 K=1,500
T=T/2.0
T=T/2.0
F2=F2+F3/2.0
F4=0.0
DO 20 J=1,N
HSS=A+IN*J*T
F3=(1.+HSS**2/9.0)**3/(1.0+SH**HSS)**3*4.0+F4
20 CONTINUE
SIMP=SIMPSON
SIMPSON=T/3.0*(F1+F2+F3)
DELTA=HSS((SIMPSON-SIMP)/(SIMPSON+SIMP))
PRINT C2,N,SIMP,DELTA
N=2*N
IF(Delta<=0.0005) GO TO 26
10 CONTINUE
26 PRINT C2,N,SIMPSON
C
21 FORMAT(4X,N*8X,*SIMPSON9,10X,*DELTA*/)
22 FORMAT(X,14,4X,E17.10,4X,F8.5)
RETURN
END
Data for $G_2$, $G_4$ and $G_5$ with polynomial distribution of the asperity height.

$$SH = \frac{\sigma}{H}$$

$V$ = non-dimensional $SH$

$$G_2 = \left( \frac{1}{1 + \frac{\delta^*}{\sigma/H}} \right)^3 \delta^*$$

$$G_4 = \left( \frac{1}{1 + \frac{\delta^*}{\sigma/H}} \right)^3 \delta^*$$

$$G_5 = \frac{G_2}{G_4}$$

$$g(\delta^*) = \begin{cases} 
\frac{35}{96} \left( 1 - \frac{\delta^2}{9} \right)^3 & \quad -3 \leq \delta^* \leq 3 \\
0 & \text{elsewhere}
\end{cases}$$

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Data for $G_2$ and $G_4$ with sinusoidal distribution of the asperity height

\[
SH = \frac{\bar{\sigma}}{H}
\]

\[
V = \text{non-dimensional SH}
\]

\[
G_2 = \int_{-\infty}^{\infty} \frac{g(\delta^*)}{\left[1 + \delta^* \left(\frac{\bar{\sigma}}{H}\right)\right]^3} \, d\delta^*
\]

\[
G_4 = \int_{-\infty}^{\infty} \frac{g(\delta^*)}{\left[1 + \delta^* \left(\frac{\bar{\sigma}}{H}\right)\right]^3} \, d\delta^*
\]

\[
g(\delta^*) = \begin{cases} 
\frac{1}{\pi \sqrt{9 - \delta^*}} & \text{for } -3 \leq \delta^* \leq 3 \\
0 & \text{elsewhere}
\end{cases}
\]

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131
PROGRAM ROLWAVI (INPUT, OUTPUT)

C THIS PROGRAM IS TO COMPUTE THE EFFECT OF SURFACE ROUGHNESS AND WAVINESS ON THE INTEGRATED PRESSURE OF AN END CONTACT ONLY THE CASE OF PURE ROLLING IS STUDIED ASSUMING SINUSOIDAL WAVINESS AND SINUSOIDAL ROUGHNESS DISTRIBUTION

C THIS PROGRAM CONSISTS OF 3 SUBROUTINES
C 1 - SUBROUTINE GUBLIN IS TO CALCULATE WGRUB • THE REDUCED PRESSURE AT THE INLET BY THE SMOOTH FILM GUBLIN APPROACH
C 2 - SUBROUTINE ROUGH IS TO CALCULATE WROUGH(I) • THE REDUCED PRESSURE AT THE INLET BY STOCHASTIC HECKY FOR ROUGH SURFACES
C 3 - SUBROUTINE WAVI IS TO CALCULATE WSIN(I) • THE REDUCED PRESSURE AT THE INLET FOR THE SINUSOIDAL WAVINESS SURFACE PROFILE

C DATA CARD NO. 1
C KF = TOTAL NUMBER OF GRID POINTS
C KM = GRID NUMBER • XX(KM) = -2.0
C NAMP = NUMBER OF AMPLITUDE DATA
C NCYC = NUMBER OF WAVINESS CYCLES DATA

C DATA CARD NO. 2
C HO = CENTER FILM THICKNESS
C W = DIMENSIONLESS CONTACT LENGTH
C PI = 3.141593

C DATA CARD NO. 3
C DCYC(I) = WAVINESS CYCLES DATA
C DATA CARD NO. 4
C DAMP(I) = AMPLITUDE DATA
C DATA CARD NO. 5
C THESE DATA ARE THE OUTPUT FROM PROGRAM SUCHAS PROGRAM DATPOL2 WHICH EVALUATE THE INTEGRAL OF \( \frac{(1 + \text{SHHSS}) \text{SHSS}}{1 + \text{SHHSS}} \) AND
C 1-IS DISTRIBUTION FUNCTION CAN BE POLYNOMIAL +GAUSSIAN+ SINUSOIDAL OR ANY OTHER FUNCTION
C DSM(I) = THE ABSCISSA RANGING FROM 0 TO 0.333
C UV(I) = THE NONDIMENSIONAL DSH(I)
C DG*(I) = THE INTEGRAL OF \( \frac{35/96(1 + \text{SHHSS})^{2/3}}{1 + \text{SHHSS}} \)
C IN THE CASE OF POLYNOMIAL DISTRIBUTION
C UG2(I) = THE INTEGRAL OF \( \frac{35/96(1 + \text{SHHSS})^{2/3}}{1 + \text{SHHSS}} \) IN THE CASE OF POLYNOMIAL DISTRIBUTION
C UG5(I) = DG2(I)/DG4(I)

COMMON KF;KM;NAMP;NCYC;KM;KMK;KFF
COMMON HO;PI;WGRUB;CIC;AMP;KK;S;DG2(201);DG4(201)
COMMON XX(1501);H(1501);WSIN(16);WROUGH(16);G(1501)
DIMENSION DCYC(20)*;DAMP(16)*;DSH(16);PERR(16);DSM(201)

PRINT 1
READ 2*KF;KM;NAMP;NCYC
READ 3*HO;W;PI
READ 3*(DCYC(I)*I=1;NCYC)
READ 3*(DAMP(I)*I=1;NAMP)
DO 100 I=1;201
100 READ 5*DSH(I)*DV(I)*DG2(I)*DG4(I)

132
PRINT 1
PRINT 2*KF,KM,NAMP,NCYC
PRINT 3
PRINT 4*HO,W*PI
PRINT 54
PRINT 4*(DCYC(I),I=1,NCYC)
PRINT 54
PRINT 4*(DAMP(I),I=1,NAMP)
KMM = KM + 1
KMK = KM - 2
KFF = KF - 2
XX(I) = -5
DO 101 J = 2,KM
101 XX(J) = XX(J-1) * 0.003
DO 102 J =KMM,KF
102 XX(J) = XX(J-1) * 0.002
DO 103 J=1,KF
X = XX(J)
H(J) = HO + 4.*W*PI*(ABS(X)*SQRT(ABS((-X)**2-1.0))) - ALOG(ABS(X) +
+ SQRT(ABS((-X)**2-1.0))))
103 CONTINUE
PRINT 55
PRINT 4*XX(KM)*XX(KF)
C
CALL GRUBIN
DO 104 I=1,NAMP
104 KK = I
S = DAMP(I) / J, * HO
CALL ROUGH
SXX(I) = WROUGH(I) / WGRLB
CONTINUE
DO 106 II=1,NCYC
106 CYC = DCYC(II)
PRINT 7, II,CYC
DO 105 I = 1,NAMP
105 KK = I
AMP = J*AMP(I) * HO
CALL WAVI
SWSIN(I) = WSIN(I) / WGRLE
PERR(I) = (WSIN(I) - WROUGH(I)) / WROUGH(I)
PRINT 9, I*AMP +WSIN(I)+SWSIN(I)*WROUGH(I)+S*RO(I)*PERR(I)
CONTINUE
DO 106 I=1,NAMP
106 KK = I
AMP = -J*AMP(I) * HO
CALL WAVI
SWSIN(I) = WSIN(I) / WGRLE
PERR(I) = (WSIN(I) - WROUGH(I)) / WROUGH(I)
PRINT 9, I*AMP +WSIN(I)+SWSIN(I)*WROUGH(I)+S*RO(I)*PERR(I)
CONTINUE
1000 CONTINUE
C 1 FORMAT(1H1)  
2 FORMAT(2015)  
3 FORMAT(8F10.7)  
4 FORMAT(9(2X,E13.6))  
5 FORMAT(2F10.6,3F20.10)  
6 FORMAT(15B(2X,E13.6))  
7 FORMAT(/'//////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////////// ///
SUBROUTINE WAV1

C
COMMON KF,KM,NAMP,NCYC,KMK,KFF
COMMON HO,WEPI,WEGRUS,NCYC,AMP,KK,S,SG2(201),SG4(201)
COMMON XX(1501),H(1501),WSIN(16),WEROUGH(16),G(1501)
D DIMENSION T(1501)

C
DO 201 J = 1,KF
    DEL = AMP * COS(2.*NCYC*PI*XX(J))
    T(J) = (H(J) - HO) / (H(J) + DEL)**3
201 CONTINUE
    WSIN(KK) = 0.
    DO 202 J = 1,KMK+2
        WSIN(KK) = WSIN(KK) + 0.003/3.*T(J) + 4.*T(J+1) + T(J+2)
    202 CONTINUE
    J = KMK*KFF+2
    DO 203 J = 1,KMK+2
        WSIN(KK) = WSIN(KK) + 0.002/3.*T(J) + 4.*T(J+1) + T(J+2)
    203 CONTINUE
C
RETURN
END
PROGRAM ASPERTY (INPUT, OUTPUT, PUNCH)
C THIS PROGRAM IS TO SOLVE FOR THE PERTURBED PRESSURE DUE TO A SINGLE 3-D
C RIGID ASPERTY WITHIN AN ELASTOHYDRODYNAMIC CONTACT
C THIS PROGRAM CONSISTS OF 4 SUBROUTINES
C 1 - SUBROUTINE TRANSR IS USED TO TRANSFORM THE INPUT DATA OF CHENG PROGRAM
C FROM GLOBAL COORDINATE TO ASPERTY COORDINATE
C 2 - SUBROUTINE INTERPN IS USED TO INTERPOLATE SOME NEW DATA IN ASPERTY
C COORDINATE BY SECOND ORDER INTERPOLATION ROUTINE
C 3 - SUBROUTINE ABCX IS TO FORM THE A(29*14*14), B(29*14*14), C(29*14*14) AND
C R(29*14) MATRICES
C 4 - SUBROUTINE TFPHI IS TO FORM MATRICES T(29*14*14), E(30*14*14), F(30*14)
C AND SOLVE FOR THE PERTURBED PRESSURE PHI(29*14) BY THE COLUMNWISE MATRIX
C INVERSION METHOD
C
C DATA CARD NO. 1
C NCI = NUMBER OF THE MAXIMUM ASPERTY AMPLITUDE
C NUB = NUMBER OF DIFFERENT VELOCITIES DATA
C NQAMAR = NUMBER OF THE ELLIPSTICITY RATIO OF ASPERTY
C NPLAT = U*DO NOT PLOT THE RESULTS
C NPRT2 = 0, DO NOT PRINT THE RESULTS
C NPRT = 0, DO NOT PRINT THE RESULTS
C
C DATA CARD NO. 2
C HSTAR = THE RATIO OF THE MINOR AXIS OF THE ASPERTY TO THE HALF HERTZIAN
C CONTACT WIDTH
C X3 = THE DISTANCE BETWEEN THE ASPERTY CENTER AND THE CONTACT CENTER
C D1 = THE DISTANCE IN X-DIRECTION WHERE THE PERTURBED PRESSURE PHI IS
C ASSumed TO BE ZERO
C
C DATA CARD NO. 3
C GAMMA(I) = ELLIPSTICITY RATIO DATA
C DATA CARD NO. 4
C H(I) = ASPERTY AMPLITUDE DATA
C DATA CARD NO. 5
C UI(I),U2(I) = VELOCITY DATA
C DATA CARD NO. 6
C PHZ = HERTZIAN PRESSURE
C KPF = THE PRODUCT OF PRESSURE VISCOSITY COEFFICIENT AND THE EQUIVALENT
C YOUNGS MODULUS
C DATA CARD NO. 7
C MO = CENTER FILM THICKNESS
C
C DATA CARD NO. 8
C KO = IN GLOBAL COORDINATES, THE GRID NO. OF HERTZIAN CONTACT CENTER WITH
C XS(KO) = 0
C KF = IN GLOBAL COORDINATES, THE GRID NO. OF CONTACT EXIT WITH XS(KF) = 1
C NC = IN ASPERTY COORDINATE, THE GRID NO. OF HERTZIAN CONTACT CENTER
C NX,NY = THE NO. OF TOTAL GRID POINTS IN X AND Y DIRECTION OF ASPERTY COORD.
C IDEG=2 MEANS A PARABOLIC INTERPOLATION ROUTINE USED IN SUBROUTINE INTERPN
C

136
C DATA CARD NO. 9
C DXS(I) = GRID SIZES DATA IN GLOBAL COORD. (FROM CHENG PROGRAM)
C DATA CARD NO. 10
C PS(I) = STEADY STATE PRESSURE DATA FROM CHENG PROGRAM
C DATA CARD NO. 11
C HS(I) = STEADY STATE FILM THICKNESS DATA FROM CHENG PROGRAM
C DATA CARD NO. 12
C DX(I) = GRID SIZES DATA IN X-DIRECTION OF ASPERITY COORD.
C DATA CARD NO. 13
C DY(I) = GRID SIZES DATA IN Y-DIRECTION OF ASPERITY COORD.
C DATA CARD NO. 14
C MIN(I) = NO. USED IN SUBROUTINE INTERPN TO IDENTIFY THE GRID POSITION
C
C COMMON BSTAR,XJ,C1
C COMMON ALPHA,ALAMDA,GINV
C COMMON NX,NX1,NY,NY1,D1,DX(28),DY(14),DEL(29,15),HTEP(29,15),
C MPHTEP(29,15),P1(29),I1(29),X(29)
C COMMON A(29,14,14),S(29,14,15),C(29,14,14),R(29,14)
C COMMON PHI(29,14),P1PHI(29,15),HT(29,15),NPRINT,NPRINT2
C
C DIMENSION DXS(0),XS(80),AS(60),PS(80),HS(80),MIN(29),
C 1XG(29),Y(15),1EP(29),C1H(5),EP1(29),U1(3),U2(3)
C DIMENSION GAMMA(5)
C
C C READ CONSTANT
C
C READ *NC1,NUC,NGAMAR,NFLOT,NPRINT2,NPRINT
C READ *BSTAR,XJ,D1
C READ *(GAMAP(1),I=1,NGAMAR)
C READ *(C1H(1),I=1,NC1)
C DC 100 I=1,NUC
C 100 READ *(U1(I),U2(I))
C READ *PHZ,AL3E
C READ 2*H0
C READ *(K0,KF,NC,NX,NY,IEG)
C KFF = KF-1
C NX1 = NX-1
C NY1 = NY-1
C NCC = NC-1
C ALPHA = AL3E*PHZ
C
C READ SMOOTH FILM SOLUTION DATA
C READ *(DXS(I),I=1,KFF)
C READ *(PS(I),I=1,KO)
C READ *(HS(I),I=1,KO)
C READ *(DX(I),I=1,X1)
C READ *(DY(I),I=1,NY1)
C READ *(MIN(I),I=1,NC)
C PRINT 1
C PRINT 3
C PRINT *NC1,NUC,NGAMAR,NFLOT,NPRINT2,NPRINT
C PRINT 31
C PRINT *BSTAR,XJ,D1
C PRINT 4

137
PRINT 4, (GAMAR(I), I=1, NGAMAR)
PRINT 35
PRINT 4, (CH(I), I=1, NC1)
PRINT 32
DO 99 I=1, NUB
PRINT 7, I, U1(I), U2(I)
PRINT 33
PRINT 4, PHZ, AL3E, ALPHA
PRINT 34
PRINT 4, H0
PRINT 36
PRINT 6, KC, KF, NC, NX, NY, ICEG, KFF, NCC, NX1, NY1
XS(I) = -5.
DO 101 I=2, KF
101 XS(I) = XS(I-1) + DXS(I-1)
DO 102 I=1, KO
102 XXS(I) = (XS(I)-X3)/BSTAR
X(I) = DI
DO 103 I=2, NX
103 X(I) = X(I-1) + DX(I-1)
DO 106 I=1, NX
106 XOC(I) = X(I)*BSTAR + X3
Y(I) = 0.
DO 110 J=2, NY
110 Y(J) = Y(J-1) + DY(J-1)
PRINT 38
PRINT 7, (I, PS(I), I=1, KO)
PRINT 39
PRINT 7, (I, HS(I), I=1, KO)
PRINT 40
PRINT 7, (I, DXS(I), I=1, KFF)
PRINT 37
PRINT 7, (I, XS(I), I=1, KO)
PRINT 45
PRINT 7, (I, XXS(I), I=1, KO)
PRINT 42
PRINT 7, (I, DX(I), I=1, NX1)
PRINT 43
PRINT 7, (I, DY(I), I=1, NY1)
PRINT 47
PRINT 7, (I, X(I), I=1, NX)
PRINT 50
PRINT 7, (I, XOC(I), I=1, NX)
PRINT 51
PRINT 7, (J, Y(J), J=1, NY)
PRINT 41
PRINT 6, (MIN(I), I=1, NC)
C
DO 104 I=1, NC
104 CALL TRANSFN(XAS, PS, HS, KC, X(I), P1(I), H1(I), MIN(I), IDEG)
IF (NC .GE. NX) GO TO 107
DO 105 I=NCC, NX
105 P1(I) = SQRT(ABS(1. - XOC(I)**2))
107 H1(I) = 1.
CONTINUE
DO 111 I=1,NX
EP1(I) = EXP(-P1(I)*ALPHA)
111  H1EP(I) = H1(I)**3*EP1(I)
PRINT 48
PRINT 7*(I*P1(I),I=1,NX)
PRINT 49
PRINT 7*(I*H1(I),I=1,NX)
PRINT 54
PRINT 7*(I*EP1(I),I=1,NX)
PRINT 55
PRINT 7*(I*H1EP(I),I=1,NX)
C
DO 1000 IIK=1,NGAMAR
GAMMA = GAMAR(IIK)
GINV = 1./GAMMA**2
DO 1000 IJJ=1,NUB
UD = (U1(IJJ)-U2(IJJ))*0.5
US = (U1(IJJ)+U2(IJJ))*0.5
ALAMDA = 48.*UD/H0**2*6STAR
ALAM2 = 48.*US/H0**2*6STAR
ALAM3 = 46./32./ALAM2
ALAM4 = ALAM3 / GAMMA
C
DO 1000 III=1,NCI
C1 = C1H(III)
PRINT 56
PRINT 9*IIJ, UD, US, ALAMCA, ALAM2, ALAM3, ALAM4
PRINT 52
PRINT 9*III, C1, PH2, UD, AL3E, BSTAR, GAMMA, X3
C
TO TRANSFORM KNOWN DATA TO NEW COORDINATES
TO INTERPOLATE NEW RESULTS FROM KNOWN DATA AT NEW COORDINATES CALL INTERPN(Y*H1EP*EP1)
C
TO FORM MATRICES A(I,J,K), B(I,J,K), C(I,J,K), R(I,J) CALL ABCR
C
TO FORM MATRICES T(I,J,K), E(I,J,K), F(I,J) AND SOLVE FOR PHI(I,J), PIIF(I,J) CALL TEFPHI
C
J = 1
PRINT 8,J
DO 108 I=1,NX
108 PPLTO(I) = PHI(I,J)
C STPL1 IS A LIBRARY SUBROUTINE TO PLOT RESULTS CALL STPL1(1*X,PPLTO,29,1H*,5,5PHI-X)
CALL STPL1(1*X,CC,PPLTO,25,1H*,7,7PHI-XOC)
DO 109 I=1,NX
109 PPLTO(I) = P1PHI(I,J)
CALL STPL1(1*X,PPLTO,29,1H*,7,7PHI1PHI-X)
CALL STPL1(1*XOC,PPLTO,25,1H*,9,9PHI1PHI-XOC)
1000 CONTINUE
C
1 FORMAT (1H1)
2 FORMAT (8F10.6)
3 FORMAT (5(2X,E13.6))
4 FORMAT (9(2X,E13.6))
5 FORMAT (16I5)
6 FORMAT (18(2X,I5))
7 FORMAT (8(2X,E13.6))
8 FORMAT (10X,2F14.6)
9 FORMAT (6X,14E9.2)
10 FORMAT (6X,2I4)
11 FORMAT (7X,*BSTAR*,13X,*U1(I)*,11X,*U2(I)*)
12 FORMAT (7X,*PHZ*,11X,*ALPHA*)
13 FORMAT (7X,*KO*,5X,*KF*,5X,*NC*,5X,*NX*,13X,*NY*,3X,*IDEG*,4X,*KFF*)
14 FORMAT (7X,*CC*,4X,*X1*,4X,*NY*)
15 FORMAT (6X,*C1H(I)*)
16 FORMAT (6X,*PS(I)*)
17 FORMAT (6X,*MIN(I)*)
18 FORMAT (6X,*DAS(I)*)
19 FORMAT (6X,*DY(I)*)
20 FORMAT (6X,*GAMMA(I)*)
21 FORMAT (6X,*XAS(I)*)
22 FORMAT (6X,*P1(I)*)
23 FORMAT (6X,*UC(I)*)
24 FORMAT (6X,*Y(J)*)
25 FORMAT (7X,*C1*,12X,*PHZ*,14X,*G*,10X,*BSTAR*
26 FORMAT (4X,*NC*,4X,*NUB*,2X,*NPRINT*,2X,*NPLOT*,2X,*NPRINT*)
27 FORMAT (10X,*ALAMDA*,10X,*ALAM2*
28 FORMAT (10X,*ALAM3*,10X,*ALAM4*)
C
C
STUP
C
END
SUBROUTINE TRANSF(X*X,PS,MS,NX,AX,PI,HI,MIN,IDEG)
DIMENSION X*X(NX),PS(NX),HS(80)
IF (X NE. X*X(MIN)) GO TO 755
P1 = PS(MIN)
H1 = HS(MIN)
GO TO 756
755 FACTOR = 1.
MAX = MIN + IDEG
DO 701 J = MIN, MAX
701 FACTOR = FACTOR*(X-X*X(J))
C EVALUATE INTERPOLATING POLYNOMIAL
P1 = 0.
H1 = 0.
DO 702 I = MIN, MAX
TERMP = PS(I)/FACTOR/(X-X*X(I))
TERMH = HS(I)/FACTOR/(X-X*X(I))
DO 703 J = MIN, MAX
IF (I NE. J) GO TO 757
GO TO 703
757 TERMP = TERMP/(X*X(I)-X*X(J))
TERMH = TERMH/(X*X(I)-X*X(J))
703 CONTINUE
P1 = P1 + TERMP
H1 = H1 + TERMH
702 CONTINUE
756 RETURN
C END

SUBROUTINE INTERPN(Y,HIEP,EP1)
C
COMMON BSTAR,X3,C1
COMMON ALPHA,ALAMDA,GINV
COMMON NX,NX1,NY,NY1,DL,CX(28), DY(14),DE(29,15), HEP(29,15)
C H1HEP(29,15), P1(29), P1(29)
COMMON A(29,14,14), B(29,14,15), C(29,14,14), R(29,14)
COMMON PH1(29,14),P1PH1(29,14),HT(29,15),NPRINT,NPRINT2
COMMON PHIOP(29)
DIMENSION Y(15),H1EP(29), EP1(29)
C
DO 406 I = 1,NX
DO 407 J = 1,NY
XYI = X(I)**2 + Y(J)**2 - 1.
IF (XYI *GE. 0.) GO TO 451
DEL(I,J) = C1*XYI
GO TO 452
451 DEL(I,J) = 0.
452 HT(I,J) = H1(I) + DEL(I,J)
HEP(I,J) = HT(I,J)**3*EP1(I)
H1HEP(I,J) = H1EP(I) - HEP(I,J)
407 CONTINUE
406 CONTINUE
J = 1
PRINT 416
PRINT 7, (I*DEL(I,J), I=1,NX)
PRINT 417
PRINT 7, (I, HT(I,J)*I=1,NX)
IF(NPRINT .EQ. 0) GO TO 490
PRINT 418
PRINT 10, ((HTEP(I,J)*J=1,NY)*I=1,NX)
PRINT 419
PRINT 10, ((H1HTEP(I,J)+J=1,NY)*I=1,NX)

C

7 FORMAT(8(X,I12,X*E13.6))
10 FORMAT(X,15E9.2)
416 FORMAT(//6X,* DEL*//)
417 FORMAT(//6X,* H1*//)
418 FORMAT(//6X,* H1TEP*//)
419 FORMAT(//6X,* H1HTEP*//)
C
490 RETURN
C
END

SUBROUTINE ABCK
C
COMMON BSTAR,XJ,C1
COMMON ALPHA,ALAMDA,G1NY
COMMON NX,NX1,NY,NY1,DX(28),DY(14),DEL(29+15),HTEP(29+15),
      C+HTEP(29+15)+r1(<=Y)+r1(2Y),X(29)
COMMON A(29+14+14)+B(2+1-1+15)+C(29+14+14)+R(29+14)
COMMON PHI(29+14),PLP(I=125+14)+HT(<=9+15)+NPRINT,NPRINT2
COMMON PHIOP(2Y)
C
DO 501 I=1,NX
DO 501 J=1,NY1
DO 501 K=1,NY1
A(I,J,K) = 0.
B(I,J,K) = 0.
C(I,J,K) = 0.
501 CONTINUE
DO 502 I=1,NX,NX1
DO 502 J=1,NY1
R(I,J) = 1.
R(I,J) = 0.
502 CONTINUE
DO 503 I=2,NX1
J = 1
A(I,J,J) = DY(J)/DX(I-1)*(HTEP(I,J)+HTEP(I-1,J)+ALPHA*HTEP(I-1,J))
A(I,J+J) = -DY(J)*(HTEP(I,J+J)+HTEP(I,J))/DX(I)*(HTEP(I,J)+
B*HTEP(I,J-I))/DX(I)+ALPHA*HTEP(I,J)*((P1(I+1)-P1(I))/DX(I)-
R(P1(I)-P1(I-1))/DX(I-1)) - G1NY*(DX(I-1)+DX(I))*H1HTEP(I,J)
B*HTEP(I+J))/DY(J)
B(I,J+J+1) = G1NY*(DX(I-1)+DX(I))*H1HTEP(I,J+1)+HTEP(I,J))/DY(J)
C(I,J+J) = DY(J)/DX(I)*HTEP(I,J+1)+HTEP(I,J)+ALPHA*HTEP(I,J)
C(P1(I+1)-P1(I))
R(I,J) = DY(J)*((H1HTEP(I+1,J)+H1TEP(I,J))*P1(I+1)-P1(I))/DX(I)-
R(H1HTEP(I+1,J)+H1HTEP(I-1,J))*(P1(I)-P1(I-1))/DX(I) + ALAMDA*
R(DEL(I+1,J)-DEL(I-1,J)))

142
C

\[ \text{DDX} = (DX(I-1) + DX(I))/2 \]
DO 504 J = 2 + NY1
DDY = (DY(J-1) - DY(J))/2
A(I*, J*) = DDX/DX(I-1) * (HTEP(I*J) + HTEP(I-1,J) + ALPHAN * HTEP(I-1,J)*
A (P1(I) - P1(I-1))
B(I*, J* - 1) = GINV * DDX/ DY(-1) * (HTEP(I*J) + HTEP(I,J-1))
B(I*, J* - 1) = DDX/ DX(I-1) * (HTEP(I*J) + HTEP(I,J) - HTEP(I,J)*
B(I* - 1,J*) / DX(I-1) + ALPHAN * HTEP(I,J)*
(C(P1(I) - P1(I-1)) / DX(I-1)) - GINV * DDX/ (HTEP(I,J) + HTEP(I,J)*
B*DY(I) * (HTEP(I,J) + HTEP(I,J) - 1) / DY(J-1))
B(I* - 1,J*) = GINV * DDX/ DY(-1) * (HTEP(I*J+1) + HTEP(I,J))
C(I*, J*) = DDY/ DX(I) * (HTEP(I+1,J) + HTEP(I,J) - ALPHAN * HTEP(I+1,J)*
C (P1(I+1) - P1(I))
R(I,J) = DDX/ (H1 * HTEP(I+1,J) + H1 * HTEP(I,J) * (P1(I+1) - P1(I)) / DX(I) -
R (M1 * HTEP(I,J) + M1 * HTEP(I-1,J)) * (P1(I) - P1(I-1)) / DX(I-1) + ALAMDA*
R (DEL(I+1,J) + DEL(I,J-1)))
504 CONTINUE
503 CONTINUE.
C
IF (NPRINT *.EQ. 0) GC TC 590
PRINT 521
DC 505 I=1,NX
PRINT 525, I
505 PRINT 11, ((A(I*J*, K = 1, NY1) + J = 1, NY1)
PRINT 22
DO 506 I = 1, NX
PRINT 25, I
506 PRINT 11, ((R(I*J*, K = 1, NY1) + J = 1, NY1)
PRINT 23
DO 507 I = 1, NX
PRINT 25, I
507 PRINT 11, ((C(I*J*, K = 1, NY1) + J = 1, NY1)
PRINT 24
PRINT 11, ((R(I*J*) + J = 1, NY1) + I = 1, NX)
C
11 FORMAT (X*14,E9.4)
521 FORMAT (///6X*A(I,J,K)*)
522 FORMAT (///6X*B(I,J,K)*)
523 FORMAT (///6X*C(I,J,K)*)
524 FORMAT (///6X*R(I,J)*)
525 FORMAT (///6X*I=*,13)
590 RETURN
C

END

143
SUBROUTINE TEFPHI

COMMON BSTAR,X3,X1
COMMON ALPHA,ALAMDA,GINV
COMMON NX,NX1,NY,NY1,DI,DX(28),DY(14),DEL(29,15),HTEP(29,15),
C H1HTEP(29,15), P1(29), F1(29), X(29)
COMMON A(29,14,14), B(29,14,15), C(29,14,14), R(29,14)
COMMON PHI(29,14),F1PHI(25,14),T(29,15),NPRINT,NPRINT2

C DIMENSION T(29,14,14), E(30,14,14), F(30,14), RAF(14)
C DIMENSION TT(14,14), TI(14,14), IR(14), EE(14)

C

DO 601 I=1,NX,NX1
DO 602 J=1,NY1
DO 603 K=1,NY1
T(I,J,K) = 0.*
E(I+1,J,K) = 0.*
603 CONTINUE
T(I,J,J) = 1.*
F(I+1,J) = 0.*
602 CONTINUE
601 CONTINUE

C

DO 604 I=2,NX1
DO 605 J=1,NY1
DO 605 K=1,NY1
TT(J,K) = BI(J+1,K)
DO 607 L=1,NY1
TT(J,K) = TT(J,K) + A(I,J+L)*E(I+L,J,K)
605 CONTINUE

C

CALL LIBRARY SUBROUTINE FCR MATRIX INVERSION
CALL M1(TT,NY1,NY1,DET,ICET,TI,IR,IER,EE)

DO 608 J=1,NY1
DO 608 K=1,NY1
608 T(I,J,K) = TI(J,K)
DO 609 J=1,NY1
RAF(J) = R(I,J)
609 DO 605 L=1,NY1
RAF(J) = RAF(J) - A(I,J+L)*F(I+L)
610 DO 610 J=1,NY1
F(I+1,J) = 0.*
610 DO 610 L=1,NY1
F(I+1,J) = F(I+1,J) + T(I,J,L)*RAF(L)
610 DO 611 J=1,NY1
DO 611 K=1,NY1
E(I+1,J,K) = 0.*
612 DO 612 L=1,NY1
E(I+1,J,K) = E(I+1,J,K) + T(I,J,L)*C(I,L,K)
611 CONTINUE
604 CONTINUE
C
NX2 = NX + 1
IF (NPRINT .EQ. 0) GO TO 690
PRINT 661
DO 631 I = 1, NX
PRINT 666 * I
631 PRINT 11, ((T(I, J, K), K = 1, NY1), J = 1, NY1)
PRINT 662
DO 632 I = 2, NX2
PRINT 666 * I
632 PRINT 11, ((E(I, J, K), K = 1, NY1), J = 1, NY1)
PRINT 663
DO 633 I = 2, NX2
PRINT 11, (F(I, J), J = 1, NY1)
C
690 CONTINUE
I = NX
DO 621 J = 1, NY1
PHI(I, J) = F(I + 1, J)
621 P1PHI(I, J) = P1(I) + PHI(I, J)
DO 622 II = 1, NX1
I = NX - 11
DO 623 J = 1, NY1
PHI(I, J) = F(I + 1, J)
DO 624 K = 1, NY1
624 PHI(I, J) = PHI(I, J) + E(I + 1, J, K) * PHI(I + 1, K)
623 P1PHI(I, J) = P1(I) + PHI(I, J)
622 CONTINUE
IF (NPRINT2 .EQ. 0) GO TO 691
PRINT 664
DO 625 I = 1, NX
PRINT 666 * I
625 PRINT 7, (J, PHI(I, J), J = 1, NY1)
PRINT 665
DO 626 I = 1, NX
PRINT 666 * I
626 PRINT 7, (J, P1PHI(I, J), J = 1, NY1)
691 CONTINUE
C
7 FORMAT(8(X, I2, X + E13.6))
11 FORMAT(X, 14E9.2)
661 FORMAT(///6X, "T(I, J, K)=")
662 FORMAT(///6X, "E(I, J, K)=")
663 FORMAT(///6X, "F(I, J)=")
664 FORMAT(///6X, "PHI(I, J)=")
665 FORMAT(///6X, "P1PHI(I, J)=")
666 FORMAT(6X, *I, *=*)
C
RETURN
C
END
#### Table

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**Note:** The table above contains a series of data points with values ranging from 0.03125 to 1.0, representing various measurements or calculations in a specific field. The values are listed in a sequential manner, indicating a pattern or progression in the data set.