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THE RELATIVISTIC EQUATIONS OF STELLAR STRUCTURE AND EVOLUTION\*

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## ABSTRACT

The general relativistic equations of stellar structure and evolution are reformulated in a notation which makes easy contact with Newtonian theory. Also, a general relativistic version of the mixing-length formalism for convection is presented. Finally, it is argued that in previous work on spherical systems general relativity theorists have identified the wrong quantity as "total mass-energy inside radius  $r$ ."

Subject headings: interiors: stellar, relativity.

## I. INTRODUCTION

The general relativistic equations of stellar structure for zero-temperature stars (neutron stars) were first presented in their modern form by Oppenheimer and Volkoff (1939). Two decades later the discovery of discrete, galactic X-ray sources (Giacconi et al. 1962, Gursky et al. 1963) motivated theoretical studies of hot, relativistic neutron stars by Chiu and Salpeter (1964), Morton (1964), and Tsuruta (1964); and the huge energy requirements of strong radio sources motivated Hoyle and Fowler (1963a,b) to develop the theory of hot, efficiently convective supermassive stars in which, it was soon realized, relativistic effects can be important (Feynman 1964, Chandrasekhar 1964, Fowler 1964). In response to these developments, and others, Bardeen (1965), Misner and Sharp (1965), and Lindquist (1966) developed the theory of diffusive heat transfer in relativistic stars, and Bondi (1964), Chandrasekhar (1965) and Thorne (1966a) elucidated the relativistic version of the Schwarzschild criterion for convection. All of these pieces of relativistic stellar theory were put together and combined with relativistic equations for nuclear energy generation by Hämeen-Anttila and Anttila (1966) and by Thorne (1966b, 1967) to give the currently standard version of the relativistic equations of stellar structure and evolution.

Recently Anna Żytkow and I began analyzing the structure of red supergiant stars with degenerate neutron cores (see the following paper and references cited therein). For this purpose the standard relativistic stellar equations are unsatisfactory in two ways: (i) they do not make easy contact with the standard Newtonian equations; and (ii) they do not include a mixing-length formalism for convective energy transport. The purpose of this paper is to remedy these defects by (i) translating the relativistic

equations into a new notation, and (ii) presenting a straightforward relativistic generalization of the standard Newtonian mixing-length equations. No detailed derivations will be given because the translation from the old notation (Thorne 1966b, 1967) to the new is straightforward; and the derivation of the relativistic mixing-length theory is identical to the Newtonian derivation, if one works in the proper reference frame of a static relativistic observer.

Throughout the paper c.g.s. units will be used; the speed of light  $c$  and Newton's gravitation constant  $G$  will not be set equal to unity.

## II. FUNDAMENTAL VARIABLES

As our independent thermodynamic variables we choose the following — all of which are determined by measurements using standard, physical rods and clocks, in the mean local rest frame of the baryons. After the symbol for each quantity, we indicate its units in brackets.

$$\rho \left[ \frac{\text{g}}{\text{cm}^3} \right] \equiv (\text{density of "rest mass"}) \equiv \left( \begin{array}{l} \text{mass of one hydrogen} \\ \text{atom in its ground state} \end{array} \right) \times \left( \begin{array}{l} \text{number density} \\ \text{of baryons} \end{array} \right); \quad (1a)$$

$$T[^\circ\text{K}] \equiv (\text{temperature}); \quad (1b)$$

$$X_i \equiv \left( \begin{array}{l} \text{fractional abundance of nuclear species } i, \\ \text{by rest mass or equivalently by baryon number} \end{array} \right). \quad (1c)$$

As our independent radial and time variables we choose

$$M_r[\text{g}] \equiv \left( \begin{array}{l} \text{"rest mass" inside} \\ \text{"radius } r" \end{array} \right) \equiv \left( \begin{array}{l} \text{mass of one hydrogen} \\ \text{atom in its ground state} \end{array} \right) \times \left( \begin{array}{l} \text{total number of baryons} \\ \text{inside radius } r \end{array} \right); \quad (2a)$$

$$t[\text{sec}] \equiv \left( \begin{array}{l} \text{"Schwarzschild time"} \\ \text{coordinate"} \end{array} \right) = \left( \begin{array}{l} \text{time coordinate such that } [\partial/\partial t]_r \text{ is} \\ \text{the time-translation Killing vector and} \\ t \text{ is proper time at radial infinity} \end{array} \right). \quad (2b)$$

The gravitational field is characterized by three fundamental variables which are functions of  $M_r$  and  $t$ :

$$r[\text{cm}] \equiv \left( \begin{array}{l} \text{"radius"} \end{array} \right) \equiv (1/2\pi) \times (\text{circumference around center of star}); \quad (3a)$$

$$M_{tr}[\text{g}] \equiv \left( \begin{array}{l} \text{"total mass inside radius } r \text{"} - \text{including contributions from rest} \\ \text{mass, nuclear binding energy, internal energy, and gravity} \end{array} \right); \quad (3b)$$

$$\Phi \left[ \frac{\text{cm}^2}{\text{sec}^2} \right] \equiv \left( \begin{array}{l} \text{"gravitational potential"} \end{array} \right) \equiv \frac{1}{2} c^2 \ln \left| \left( \frac{\partial}{\partial t} \right)_r \cdot \left( \frac{\partial}{\partial t} \right)_r \right|. \quad (3c)$$

Energy transport through the star is characterized by three quantities, each of which is determined by measurements using standard, physical rods and clocks in the mean local rest frame of the baryons at radius  $r$ :

$$L_r \left[ \frac{\text{erg}}{\text{sec}} \right] \equiv \left( \begin{array}{l} \text{"local luminosity"} \end{array} \right) \equiv \left( \begin{array}{l} \text{non-neutrino energy being transported across} \\ \text{the sphere at radius } r, \text{ per unit time} \end{array} \right); \quad (4a)$$

$$L_r^{nv} \left[ \frac{\text{erg}}{\text{sec}} \right] \equiv \left( \begin{array}{l} \text{"neutrino luminosity"} \\ \text{from nuclear burning"} \end{array} \right) \equiv \left( \begin{array}{l} \text{Same as above, but for neutrino energy} \\ \text{produced in thermonuclear reaction} \\ \text{cycles which change the abundances } X_i \end{array} \right); \quad (4b)$$

$$L_r^{ov} \left[ \frac{\text{erg}}{\text{sec}} \right] \equiv \left( \begin{array}{l} \text{"neutrino luminosity not"} \\ \text{from nuclear burning"} \end{array} \right) \equiv \left( \begin{array}{l} \text{same as above, but for neutrino} \\ \text{energy produced by processes} \\ \text{which, in time-average, do not} \\ \text{change the abundances } X_i \end{array} \right). \quad (4c)$$

The complete stellar structure and evolution are characterized by the functions  $\rho(M_r, t)$ ,  $T(M_r, t)$ ,  $X_i(M_r, t)$ ,  $r(M_r, t)$ ,  $M_{tr}(M_r, t)$ ,  $\Phi(M_r, t)$ ,  $L_r(M_r, t)$ ,  $L_r^{nv}(M_r, t)$ , and  $L_r^{ov}(M_r, t)$ .

III. AUXILIARY VARIABLES

a) Thermodynamic, Nuclear Burning, and Opacity Variables

The following auxiliary variables are algebraic functions of the fundamental variables; and like the fundamental variables they are determined by measurements using standard, physical rods and clocks in the mean local rest frame of the baryons

$$P(\rho, T, X_i) \left[ \frac{\text{dynes}}{\text{cm}^2} \right] \equiv (\text{total pressure}); \quad (5a)$$

$$B(X_i) \left[ \frac{\text{erg}}{\text{g}} \right] \equiv \left( \begin{array}{l} \text{binding energy of nuclei, per unit} \\ \text{rest mass, relative to hydrogen} \end{array} \right) \equiv \left( 1 - \sum_i \frac{m_i X_i}{m_H A_i} \right) c^2, \quad (5b)$$

where  $m_i$  is the mass of atomic species  $i$  in its ground state,  $m_H$  is the mass of atomic hydrogen,  $A_i$  is the number of baryons in atomic species  $i$ , and  $c$  is the speed of light;

$$\begin{aligned} \Pi(\rho, T, X_i) \left[ \frac{\text{erg}}{\text{g}} \right] &\equiv (\text{"specific internal energy"}) \\ &\equiv \frac{\left( \begin{array}{l} \text{total mass-energy of a sample of} \\ \text{stellar material, in energy units} \end{array} \right)}{\left( \begin{array}{l} \text{total rest mass of the sample} \end{array} \right)} + B(X_i) - c^2; \end{aligned} \quad (5c)$$

$$\rho_t(\rho, T, X_i) \left[ \frac{\text{g}}{\text{cm}^3} \right] \equiv \left( \begin{array}{l} \text{density of total non-gravitational} \\ \text{mass-energy, in mass units} \end{array} \right) \equiv \rho(1 - B/c^2 + \Pi/c^2); \quad (5d)$$

$$\kappa(\rho, T, X_i) \left[ \frac{\text{cm}^2}{\text{g}} \right] \equiv (\text{opacity}) \equiv (\text{Rosseland mean opacity}); \quad (5e)$$

$$\epsilon_{\text{nuc}}(\rho, T, X_i) \left[ \frac{\text{erg}}{\text{g sec}} \right] \equiv \left( \begin{array}{l} \text{rate, per unit rest mass, at which nuclear} \\ \text{burning creates non-neutrino energy} \end{array} \right); \quad (5f)$$

$$\epsilon_{\text{nv}}(\rho, T, X_i) \left[ \frac{\text{erg}}{\text{g sec}} \right] \equiv \left( \begin{array}{l} \text{rate, per unit rest mass, at which nuclear} \\ \text{burning creates neutrino energy} \end{array} \right); \quad (5g)$$

$$\epsilon_{\text{ov}}(\rho, T, X_i) \left[ \frac{\text{erg}}{\text{g sec}} \right] \equiv \left( \begin{array}{l} \text{rate, per unit rest mass, at which non-nuclear-} \\ \text{burning processes [processes with no change} \\ \text{in } X_i \text{] create neutrino energy} \end{array} \right); \quad (5h)$$

$$\alpha_i(\rho, T, X_i) \left[ \text{sec}^{-1} \right] \equiv \left( \begin{array}{l} \text{rate at which the abundance } X_i \text{ of species } i \\ \text{changes due to nuclear burning} \end{array} \right). \quad (5i)$$

### b) Relativistic Correction Functions

The above auxiliary variables (except  $B$  and  $\rho_t$ ) are all familiar from the Newtonian theory of stellar interiors. In the relativistic theory it is useful to introduce the following additional auxiliary variables, each of which is dimensionless and is unity in the Newtonian limit

$$\mathcal{R} \equiv (\text{"redshift correction factor"}) \equiv \exp(\phi/c^2); \quad (6a)$$

$$\mathcal{V} \equiv (\text{"volume correction factor"}) \equiv (1 - 2GM_{\text{tr}}/c^2 r)^{-1/2}; \quad (6b)$$

$$\mathcal{G} \equiv (\text{"gravitational-acceleration correction factor"}) \equiv \frac{M_{\text{tr}} + 4\pi r^3 P/c^2}{M_r}; \quad (6c)$$

$$\mathcal{E} \equiv (\text{"energy correction factor"}) \equiv 1 + (\Pi - B)/c^2 = \rho_t/\rho; \quad (6d)$$

$$\mathcal{H} \equiv (\text{"enthalpy correction factor"}) \equiv 1 + (\Pi - B + p/\rho)/c^2. \quad (6e)$$

In terms of these variables, the general relativistic metric for spacetime inside and around the star is

$$ds^2 = -\mathcal{R}^2 c^2 dt^2 + \mathcal{V}^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (7)$$

c) Mixing-Length Variables

The Newtonian mixing-length theory of convective energy transport is readily generalized to general relativity. One need only introduce the local proper reference frame of an observer at rest at radius  $r$ , and in that reference frame analyze, in a manner identical to Newtonian theory, the buoyant forces on convective cells and the heat exchange between convective cells and their surroundings. The auxiliary variables that enter into such an analysis, patterned after Paczyński's (1969) Newtonian variant, are

$$g \left[ \frac{\text{cm}}{\text{sec}^2} \right] \equiv \left( \begin{array}{c} \text{local acceleration} \\ \text{of gravity} \end{array} \right) = \frac{GM_r}{r^2} \mathcal{G} \gamma ; \quad (8a)$$

$$H_p \text{ [cm]} \equiv (\text{pressure scale height}) = (P/\rho g)^{1/2} ; \quad (8b)$$

$$l_t \text{ [cm]} \equiv (\text{mixing length [normally chosen equal to } H_p]); \quad (8c)$$

$$\omega \equiv (\text{optical thickness of one scale height}) = \kappa \rho l_t ; \quad (8d)$$

$$C_p \equiv (\text{specific heat at constant pressure}) \equiv \left( \frac{\partial \Pi}{\partial T} \right)_{P, X_i} - \frac{P}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_{P, X_i} ; \quad (8e)$$

$$\gamma_o \left[ \frac{\text{sec}}{\text{cm}} \right] \equiv (\text{coefficient of heat exchange}) = \frac{C_p \rho}{8\sigma T^3} \frac{1 + \omega^2/3}{\omega} \quad (8f)$$

where  $\sigma \equiv ac/4$  is the Stephan-Boltzmann constant; and

$$Q \equiv - \left( \frac{\partial}{\partial} \frac{\ln \rho}{\ln T} \right)_{P, X_i} . \quad (8g)$$

In terms of these auxiliary variables, the basic algebraic equations of the

mixing-length theory are these: (i) An equation which defines the "radiative gradient"

$$\nabla_{\text{rad}} \equiv \left( \begin{array}{l} \text{value that } (\partial \ln T / \partial M_{\text{r}})_t (\partial \ln P / \partial M_{\text{r}})_t^{-1} \equiv d \ln T / d \ln P \\ \text{would have if the material were non-convective} \end{array} \right);$$

the equation for  $\nabla_{\text{rad}}$  follows from equations (3.11-3) and (3.11-7a) of Thorne (1966, 1967) by straightforward change of notation:

$$\nabla_{\text{rad}} = \frac{3}{64\pi} \frac{\kappa_{\text{r}} L_{\text{r}} P}{GM_{\text{r}} \sigma T^4} \frac{1}{\mathcal{N} \mathcal{G} \mathcal{V}} + \left( 1 - \frac{\mathcal{E}}{\mathcal{N}} \right). \quad (9a)$$

(ii) The usual equation for the "adiabatic gradient"

$$\nabla_{\text{rad}} \equiv \left( \frac{\partial \ln T}{\partial \ln P} \right)_{\text{entropy}, X_i} \equiv \frac{\Gamma_2 - 1}{\Gamma_2} \quad (9b)$$

where  $\Gamma_2$  is the adiabatic index of the second kind. (iii) A set of four coupled algebraic equations which determine the energy flux carried by convection  $F_{\text{conv}}$ , the mean velocity of a convective cell ("turbulent velocity")  $v_{\text{t}}$ , the gradient associated with a convective cell  $\nabla'$ , and the actual gradient averaged over all convective cells and over the medium through which they move,  $\nabla$ :

$$F_{\text{conv}} = \frac{16\sigma T^4}{3\kappa\rho H_{\text{P}}} (\nabla_{\text{rad}} - \nabla), \quad (10a)$$

$$F_{\text{conv}} = \frac{1}{2} C_{\text{P}} \rho T v_{\text{t}} (\ell_{\text{t}} / H_{\text{P}}) (\nabla - \nabla'), \quad (10b)$$

$$v_{\text{t}}^2 = \frac{1}{8} g \ell_{\text{t}} (\ell_{\text{t}} / H_{\text{P}}) Q (\nabla - \nabla'), \quad (10c)$$

$$(\nabla - \nabla') / (\nabla' - \nabla_{\text{ad}}) = \gamma_0 v_{\text{t}}. \quad (10d)$$

Equations (10b)-(10d) have identically the same form as in Newtonian theory because their derivation in the proper reference frame of a static observer

is identical to that of Newtonian theory. Equation (10a) also has standard Newtonian form. It follows from equation (9a) with  $L_r$  rewritten as  $4\pi r^2(F_{\text{conv}} + F_{\text{rad}})$ , and from the analogous equation for the actual gradient  $\nabla$  in terms of the radiative flux  $F_{\text{rad}}$

$$\nabla = \frac{3}{64\pi} \frac{\kappa (4\pi r^2 F_{\text{rad}})^2 P}{GM_r \sigma T^4} \frac{1}{\kappa \epsilon \nu} + \left(1 - \frac{\epsilon}{\kappa}\right).$$

Because equations (10) all have the same form as in Newtonian theory, one can use the standard technique [eqs. (22)-(27) of Paczyński (1969)] to solve them for the four unknowns  $F_{\text{conv}}$ ,  $v_t$ ,  $\nabla'$ , and  $\nabla$ .

#### IV. DIFFERENTIAL EQUATIONS OF STELLAR STRUCTURE

There are  $8+N$  (where  $N$  is the number of nuclear species) differential equations of stellar structure for the  $8+N$  fundamental variables  $\rho$ ,  $T$ ,  $X_i$ ,  $r$ ,  $M_{\text{tr}}$ ,  $\phi$ ,  $L_r$ ,  $L_r^{\text{nv}}$ , and  $L_r^{\text{ov}}$  as functions of  $M_r$  and  $t$ . In these differential equations  $\partial/\partial M_r$  acts at fixed  $t$ , and  $\partial/\partial t$  acts at fixed  $M_r$ . Each equation is a translation of the indicated combination of equations from Thorne (1966b, 1967).

The equation for  $M_r$  as a proper volume integral of  $\rho$ ; translation of equation (3.11-1):

$$\partial r / \partial M_r = (4\pi r^2 \rho)^{-1}. \quad (11a)$$

The equation for total mass-energy inside radius  $r$ ; translation of equations (3.11-2) and (3.11-1):

$$\partial M_{\text{tr}} / \partial M_r = \epsilon / \nu. \quad (11b)$$

The source equation for the gravitational potential  $\Phi$ ; translation of equations (3.11-4) and (3.11-1):

$$\frac{\partial \psi}{\partial M_r} = \frac{GM_r}{4\pi r^4 \rho} \mathcal{D} \nu. \quad (11c)$$

The equation of energy generation; translation of equations (3.11-5), (3.11-6), and (3.11-1):

$$\frac{1}{R^2} \frac{\partial(L_r R^2)}{\partial M_r} = \epsilon_{\text{nuc}} - \epsilon_{\text{ov}} - \frac{1}{R} \frac{\partial \Pi}{\partial t} + \frac{P}{\rho} \frac{1}{R} \frac{\partial \rho}{\partial t}. \quad (11d)$$

The equation for neutrino losses due to nuclear burning; translation of equations (3.11-6) and (3.11-1), specialized to nuclear-burning neutrinos

$$\frac{1}{R^2} \frac{\partial(L_r^{\text{nv}} R^2)}{\partial M_r} = \epsilon_{\text{nv}}. \quad (11e)$$

The equation for non-nuclear-burning neutrino losses; translation of equations (3.11-6) and 3.11-1), specialized to non-nuclear-burning neutrinos

$$\frac{1}{R^2} \frac{\partial(L_r^{\text{ov}} R^2)}{\partial M_r} = \epsilon_{\text{ov}}. \quad (11f)$$

The equation for changes of nuclear abundances due to nuclear burning; translation of equation (3.8)

$$R^{-1} \frac{\partial X_i}{\partial t} = \alpha_i. \quad (11g)$$

The equation of energy transport; follows directly from the definition of  $\nabla$ ; translation of the mixing-length-generalized version of equations (3.11-7)

$$\begin{aligned} \frac{\partial \ln T}{\partial M_r} &= \nabla_{\text{rad}} \frac{\partial \ln P}{\partial M_r} & \text{if } \nabla_{\text{rad}} \leq \nabla_{\text{ad}}, \\ \frac{\partial \ln T}{\partial M_r} &= \nabla \frac{\partial \ln P}{\partial M_r} & \text{if } \nabla_{\text{rad}} > \nabla_{\text{ad}}. \end{aligned} \quad (11h)$$

The Oppenheimer-Volkoff equation of hydrostatic equilibrium; translation of equations (3.11-3) and (3.11-1)

$$\frac{\partial P}{\partial M_r} = - \frac{GM_r}{4\pi r^4} \mathcal{K}V. \quad (11i)$$

This equation must be combined with the equation of state  $P(\rho, T, X_i)$  and with equations (11g, h) for  $\partial T/\partial M_r$  and  $\partial X_i/\partial M_r$  to yield  $\partial \rho/\partial M_r$ .

#### V. BOUNDARY CONDITIONS

Corresponding to each different derivative with respect to  $M_r$  in the equations of stellar structure there is a radial boundary condition. The obvious boundary conditions at the star's center are

$$r = M_{tr} = L_r = L_r^{nv} = L_r^{ov} = 0 \quad \text{at} \quad M_r = 0 \quad (12a)$$

(translation of [3.38a]).

We shall denote the surface values of rest mass, total mass, radius, and the total luminosities by

$$M = M_r, M_t = M_{tr}, R = r, L = L_r, L^{nv} = L_r^{nv}, L^{ov} = L_r^{ov} \quad \text{at surface.} \quad (13)$$

At the surface the star's spacetime geometry (7) must match onto the external Schwarzschild geometry

$$ds^2 = -(1 - 2GM_t/c^2 r) c^2 dt^2 + (1 - 2GM_t/c^2 r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (14)$$

Smoothness of the match ("continuity of intrinsic geometry of surface") requires that  $\phi$  satisfy the surface boundary condition

$$\phi = \frac{1}{2} c^2 (1 - 2GM_t/c^2 R) \quad \text{at} \quad M_r = M \quad (\text{surface of star}). \quad (12b)$$

Note that the luminosities as measured far from the star — which we denote  $\mathcal{L}$ ,  $\mathcal{L}^{nv}$ , and  $\mathcal{L}^{ov}$  — are not the same as the surface luminosities  $L$ ,  $L^{nv}$ , and  $L^{ov}$ . Rather, they are the surface luminosities corrected for gravitational redshift

$$\mathcal{L}/L = \mathcal{L}^{nv}/L^{nv} = \mathcal{L}^{ov}/L^{ov} = (1 - 2GM_t/c^2R). \quad (15)$$

In addition to the boundary conditions (12a,b) one must also impose surface boundary conditions on pressure  $P$  and temperature  $T$ . If moderate errors near the surface are allowable, one can impose the "zero boundary conditions"

$$P = T = 0 \quad \text{at} \quad M_r = M \quad (12c)$$

(translation of eqs. [3.38c,d]). If higher accuracy is desired one can impose the boundary conditions of the relativistic version of the Eddington approximation

$$L = 4\pi R^2 \sigma T^4, \quad \kappa P = \frac{2}{3} (GM_t/R^2) \gamma \quad \text{at} \quad M_r = M \quad (12c')$$

(translation of eqs. [3.38c',d']). For still higher accuracy one can join onto a model stellar atmosphere. If the atmosphere is thin compared to the stellar radius  $R$ , then it can be constructed in the standard Newtonian manner using a surface gravity of

$$g_s = (GM_t/r^2) \gamma \quad \text{at} \quad M_r = M, \quad (16)$$

a surface luminosity equal to  $L$ , and radial and time coordinates  $r'$  and  $t'$  related to  $r$  and  $t$  by

$$r' = (r - R)\gamma, \quad t' = t\mathcal{R}; \quad \mathcal{R} = \gamma^{-1} = (1 - 2GM_t/c^2R)^{1/2}. \quad (17)$$

If the atmosphere is not thin compared to  $R$ , one can construct it using the formalism of general relativistic radiative transfer theory, which is reviewed in §2.6 of Novikov and Thorne (1973). In the case of a thin atmosphere all spectral features as observed by a distant observer are redshifted relative to their rest wavelengths by

$$\Delta\lambda/\lambda = (1 - 2GM_t/c^2R)^{-\frac{1}{2}} - 1. \quad (18)$$

## VI. SOME USEFUL RELATIONS

In this section we list several useful relations among the stellar-interior variables.

The sum of the fractional abundances  $X_i$  must be unity at all times; and consequently, the sum of their rates of change must vanish:

$$\sum_i X_i = 1, \quad \sum_i \alpha_i = 0. \quad (19)$$

The total rate of energy release by nuclear burning must equal the rate of change of nuclear binding energy

$$\epsilon_{\text{nuc}} + \epsilon_{\text{nv}} = \frac{1}{\mathcal{R}} \frac{\partial B}{\partial t} = - \sum_i \frac{m_i c^2}{m_H A_i} \alpha_i ; \quad (20)$$

see equations (5b) and (11g).

The rate of change of the total mass-energy inside radius  $r$ , as measured by an observer there, must be equal to the rate at which matter carries mass-energy inward minus the rate at which luminosity carries it outward:

$$\mathcal{V}\mathcal{R}^{-1} \left( \frac{\partial M_{\text{tr}}}{\partial t} \right)_r = \mathcal{R}^{-1} \left( \frac{\partial M_r}{\partial t} \right)_r \mathcal{K} - \frac{1}{c^2} (L_r + L_r^{\text{nv}} + L_r^{\text{ov}}). \quad (21)$$

This mass-energy conservation law requires some discussion: (i) The time derivatives here are taken at fixed radius  $r$ , whereas all previous time derivatives were taken at fixed rest mass  $M_r$ ; the two types of time derivatives are related by

$$(\partial/\partial t)_r = (\partial/\partial t)_{M_r} + (\partial M_r/\partial t)_r (\partial/\partial M_r)_t. \quad (22)$$

(ii) The operator  $\mathcal{R}^{-1}(\partial/\partial t)_r$  is derivative with respect to the proper time of an observer who sits at rest at radius  $r$ ; see equation (7). (iii)  $\mathcal{R}^{-1}(\partial M_r/\partial t)_r$  is the locally measured rate at which rest mass flows inward across radius  $r$ ; and  $\mathcal{R}^{-1}(\partial M_r/\partial t)_r \mathcal{K}$  is the rate of inflow of rest mass plus enthalpy in mass units. Enthalpy appears in the conservation law rather than energy ( $\mathcal{K}$  rather than  $\mathcal{E}$ ) for the same reason as it appears in the Bernoulli equation: in moving matter, pressure (the difference between  $\rho \mathcal{K}$  and  $\rho \mathcal{E}$ ) transports energy

$$(\text{energy flux}) = (\text{pressure}) \times (\text{velocity}).$$

(iv)  $(1/c^2)(L_r + L_r^{nv} + L_r^{ov})$  is the locally measured rate at which mass-energy is transported outward by neutrinos, photons, and diffusive heat flow. (v) Since  $M_{tr}$  is the total mass-energy inside radius  $r$ , one would have expected the left side of equation (21) to read  $\mathcal{R}^{-1}(\partial M_{tr}/\partial t)_r$  — i.e., one would have expected the  $\gamma$  to be absent. The presence of  $\gamma$  suggests to me that relativity theorists such as Misner, Thorne, and Wheeler (1973) should not have given the name "total mass-energy inside radius  $r$ " to  $M_{tr}$ . Rather, the quantity

$$\mathfrak{M}_{tr} \equiv (c^2/G)r (1 - \gamma^{-1}) = (c^2/G)r \left[ 1 - (1 - 2GM_{tr}/c^2r)^{\frac{1}{2}} \right] \quad (23)$$

$$\simeq M_{tr} + \frac{1}{2} GM_{tr}^2/c^2r \quad \text{in Newtonian limit}$$

should have been identified as total mass-energy inside radius r because it satisfies

$$\mathcal{R}^{-1} \left( \frac{\partial \mathcal{M}_{tr}}{\partial t} \right)_r = \mathcal{V} \mathcal{R}^{-1} \left( \frac{\partial \mathcal{M}_{tr}}{\partial t} \right)_r = \mathcal{R}^{-1} \left( \frac{\partial \mathcal{M}_r}{\partial t} \right)_r \mathcal{K} - \frac{1}{c^2} (L_r + L_r^{nv} + L_r^{ov}). \quad (21')$$

Out of deference to established convention I suggest that people retain the name "total mass-energy inside radius r" for  $\mathcal{M}_{tr}$ , but keep in mind that it is a misnomer.

The equation of mass-energy conservation (21) can be derived from the equations of stellar structure by first deriving the relation

$$\frac{\partial}{\partial r} \left[ \frac{1}{c^2} (L_r + L_r^{nv} + L_r^{ov}) \mathcal{R}^2 - \left( \frac{\partial \mathcal{M}_r}{\partial t} \right)_r \mathcal{K} \mathcal{R} + \left( \frac{\partial \mathcal{M}_{tr}}{\partial t} \right)_r \mathcal{V} \mathcal{R} \right] = 0, \quad (24)$$

where  $\partial/\partial r$  acts at fixed time  $t$ , and by then invoking the boundary conditions

$$L_r = L_r^{nv} = L_r^{ov} = M_r = M_{tr} = 0 \quad \text{at} \quad r = 0.$$

A derivation of equation (24) proceeds as follows: (i) By combining equations (11a,d,e,f), (20), and (6d) derive the relation

$$\frac{\partial}{\partial r} \left[ \frac{1}{c^2} (L_r + L_r^{nv} + L_r^{ov}) \mathcal{R}^2 \right] = -4\pi r^2 \rho \mathcal{V} \mathcal{R} \left[ \left( \frac{\partial \mathcal{E}}{\partial t} \right)_{M_r} - \frac{P/\rho^2}{c^2} \left( \frac{\partial \rho}{\partial t} \right)_{M_r} \right].$$

(ii) Use equations (22) to convert from time derivatives at fixed  $M_r$  to time derivatives at fixed  $r$ ; and then use equations (11a,c,i) and (6a,d,e) to obtain

$$\begin{aligned} \frac{\partial}{\partial r} \left[ \frac{1}{c^2} (L_r + L_r^{nv} + L_r^{ov}) \mathcal{R}^2 \right] &= -4\pi r^2 \rho \mathcal{V} \mathcal{R} \left[ \left( \frac{\partial \mathcal{E}}{\partial t} \right)_r - \frac{P/\rho^2}{c^2} \left( \frac{\partial \rho}{\partial t} \right)_r \right] \\ &+ \left( \frac{\partial \mathcal{M}_r}{\partial t} \right)_r \frac{\partial}{\partial r} (\mathcal{K} \mathcal{R}). \end{aligned} \quad (25a)$$

(iii) Use equation (11a) to derive the relation

$$\frac{\partial}{\partial r} \left[ \left( \frac{\partial M_r}{\partial t} \right)_r \kappa R \right] = \left( \frac{\partial M_r}{\partial t} \right)_r \frac{\partial}{\partial r} (\kappa R) + 4\pi r^2 \kappa R \left[ \frac{\partial(\rho \nu)}{\partial t} \right]_r . \quad (25b)$$

(iv) Use equations (6a,b,c,d,e) and (11a,b,c) to derive the relation

$$(\partial/\partial r)(\nu R) = 4\pi(G/c^2)r\rho\kappa\nu^3 R$$

and then use equations (11a,b) and (6b) to obtain

$$\frac{\partial}{\partial r} \left[ \left( \frac{\partial M_{tr}}{\partial t} \right)_r \nu R \right] = 4\pi r^2 R \left\{ \rho \kappa \left[ \frac{\partial \nu}{\partial t} \right]_r + \nu \left[ \frac{\partial(\rho \mathcal{E})}{\partial t} \right]_r \right\} . \quad (25c)$$

(v) Finally, combine equations (25a,b,c) and (6d,e) to obtain equation (24).

Note that the equation of mass-energy conservation (21), when evaluated far outside the star, just says that if the rest mass of the star is held fixed then its total mass-energy decreases at a rate given by the photon and neutrino mass-energy losses

$$dM_t/dt = - (1/c^2)(\mathcal{L} + \mathcal{L}^{nv} + \mathcal{L}^{ov}). \quad (26)$$

## VII. SUMMARY

Coordinates  $M_r$ ,  $t$  for the stellar interior are defined in equations (2a,b). The star's structure is described by  $8+N$  (where  $N$  is the number of nuclear species) fundamental variables  $\rho$ ,  $T$ ,  $X_i$ ,  $r$ ,  $M_{tr}$ ,  $\phi$ ,  $L_r$ ,  $L_r^{nv}$ , and  $L_r^{ov}$ , which are functions of  $M_r$  and  $t$ , and which are defined in equations (1), (3), and (4). These  $8+N$  variables satisfy the  $8+N$  differential equations of stellar structure (11), subject to the radial boundary conditions (12). The differential equations (11) contain a number of auxiliary variables, which are algebraic functions of the fundamental variables, and which are defined in

equations (5)-(10). Quantities which characterize the surface of the star, its external gravitational field, and the radiation which leaves the star are described by equations (13)-(18). Several useful relations among the stellar variables are given in equations (19)-(21) and (26).

This version of the equations of stellar structure and evolution reduces to the standard Newtonian version when one sets the following relativistic correction factors to unity:  $\mathcal{R}$ ,  $\mathcal{V}$ ,  $\mathcal{S}$ ,  $\mathcal{E}$ ,  $\mathcal{K}$  in the interior;  $(1 - 2GM_t/c^2R)$  and  $(1 - 2GM_t/c^2r)$  at the surface and in the exterior.

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