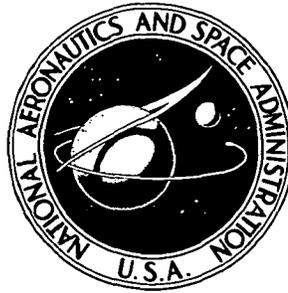


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**SIMPLIFIED SOLUTION FOR
POINT CONTACT DEFORMATION
BETWEEN TWO ELASTIC SOLIDS**

David E. Brewe and Bernard J. Hamrock

Lewis Research Center

and U.S. Army Air Mobility R&D Laboratory

Cleveland, Ohio 44135



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16. Abstract <p>A linear regression by the method of least squares is made on the geometric variables that occur in the equation for point contact deformation. The ellipticity and the complete elliptic integrals of the first and second kind are expressed as a function of the x, y-plane principal radii. The ellipticity was varied from 1 (circular contact) to 10 (a configuration approaching line contact). The procedure for solving for these variables without the use of charts or a high-speed computer would be quite tedious. These simplified equations enable one to calculate easily the point-contact deformation to within 3 percent without resorting to charts or numerical methods.</p>			
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SIMPLIFIED SOLUTION FOR POINT CONTACT DEFORMATION BETWEEN TWO ELASTIC SOLIDS

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Lewis Research Center and
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SUMMARY

A linear regression by the method of least squares is made on the geometric variables that occur in the equation for point contact deformation. The ellipticity and the complete elliptic integrals of the first and second kind are expressed as a function of the x, y -plane principal radii. The ellipticity was varied from 1 (circular contact) to 10 (a configuration approaching line contact). The procedure for solving for these variables without the use of charts or a high-speed computer would be quite tedious. These simplified equations enable one to calculate easily the point-contact deformation to within 3 percent accuracy without resorting to charts or numerical methods.

INTRODUCTION

The classical Hertz solution requires the calculation of k , the ellipticity, and \mathcal{F} and \mathcal{E} , the complete elliptic integrals of the first and second kind, respectively. Previously this entailed finding a solution to a transcendental equation that related k , \mathcal{F} , and \mathcal{E} to the geometry (ref. 1). This was usually accomplished by some iterative numerical procedure (ref. 2) or with the aid of charts (refs. 1 and 3).

In this report a linear regression by the method of least squares is made on these variables. The ellipticity and the complete elliptic integrals of the first and second kind are expressed as functions of the x, y -plane principal radii. The ellipticity was varied from 1 (circular contact) to 10 (a configuration approaching line contact). The resulting simplified equations enable one to easily calculate the point-contact deformation to within 3 percent accuracy without resorting to charts or numerical methods.

SYMBOLS

A	solid A
B	solid B
E	modulus of elasticity
E'	$2 / \left(\frac{1 - \nu_A^2}{E_A} + \frac{1 - \nu_B^2}{E_B} \right)$
\mathcal{E}	complete elliptic integral of the first kind
$\overline{\mathcal{E}}$	\mathcal{E} expressed by method of least squares
F	applied load
\mathcal{F}	complete elliptic integral of the second kind
$\overline{\mathcal{F}}$	\mathcal{F} expressed by method of least squares
k	ellipticity (ratio of semimajor to semiminor axis)
\overline{k}	k expressed by method of least squares
R_x	effective radius of curvature in the principal x-plane
R_y	effective radius of curvature in the principal y-plane
r_{Ax}, r_{Ay}	principal radii of solid A
r_{Bx}, r_{By}	principal radii of solid B
Γ	curvature difference
δ_P	point-contact deformation at the center of contact
$\overline{\delta_P}$	point-contact deformation calculated using curve fit data
$1/R$	$(1/R_x + 1/R_y)$, curvature sum

POINT CONTACT DEFORMATION

A widely used method of describing the geometry of two ellipsoidal solids, A and B, in contact (fig. 1) is to express it in terms of the effective curvatures, that is,

$$\frac{1}{R} = \frac{1}{R_x} + \frac{1}{R_y} \quad (1)$$

where

$$\frac{1}{R_x} = \frac{1}{r_{Ax}} + \frac{1}{r_{Bx}} \quad (2)$$

$$\frac{1}{R_y} = \frac{1}{r_{Ay}} + \frac{1}{r_{By}} \quad (3)$$

The variables R_x and R_y represent the effective radius of curvature in the principal x and y planes, respectively. From reference 1 an auxiliary equation relating the curvature difference and the elliptic integrals can be written as

$$k = \sqrt{\frac{2\mathcal{F} - \mathcal{E}(1 + \Gamma)}{\mathcal{E}(1 - \Gamma)}} \quad (4)$$

where

$$\Gamma = R \left(\frac{1}{R_x} - \frac{1}{R_y} \right) \quad (5)$$

Since \mathcal{F} and \mathcal{E} are functions only of k , equation (4) indicates that the ellipticity k can be expressed strictly in terms of the principal plane curvatures.

A numerical one-point iteration procedure (ref. 4) was used to generate k_i for a given $(R_x, R_y)_i$ for $i = 1, 2, \dots, 39$. The corresponding elliptic integrals were evaluated numerically using a Landen transformation described by Bulirsh (ref. 5).

Reference 6 indicates that k has been approximated as $k = (R_y/R_x)^{2/3}$. Thus for a

given set of pairs of data, $\left\{ \left[k_i, (R_y/R_x)_i \right], i = 1, 2, \dots, 39 \right\}$, a power fit using a

linear regression by the method of least squares resulted in the following equation:

$$\bar{k} = 1.0339 \left(\frac{R_y}{R_x} \right)^{0.6360} \quad (6)$$

The asymptotic behavior of \mathcal{E} and \mathcal{F} (ref. 7) was suggestive of the type of functional dependence that $\bar{\mathcal{E}}$ and $\bar{\mathcal{F}}$ might follow. As a result, an inverse and logarithmic curve fit was tried for \mathcal{E} and \mathcal{F} , respectively. The following expressions provided excellent curve fits:

$$\bar{\mathcal{E}} = 1.0003 + \frac{0.5968}{R_y/R_x} \quad (7)$$

and

$$\bar{\mathcal{F}} = 1.5277 + 0.6023 \ln \frac{R_y}{R_x} \quad (8)$$

Values of \bar{k} , $\bar{\mathcal{E}}$, and $\bar{\mathcal{F}}$ are presented in table I and compared with the numerically determined values of k , \mathcal{E} , and \mathcal{F} . All of the values of R_y/R_x in table I (except $R_y/R_x = 1$) were included in the least squares fit. An indication that the curve fits were excellent was that the coefficient of determination was not less than 0.9997 for any of these expressions. A value of 1.000 would indicate a perfect fit, while a value of zero would be the worst possible fit.

From reference 1 the point contact deformation at the center of contact is

$$\delta_P^* = \frac{\delta_P^*}{2R} \left[\frac{3FR}{2} \left(\frac{1 - \nu_A^2}{E_A} + \frac{1 - \nu_B^2}{E_B} \right) \right]^{2/3} \quad (9)$$

where

$$\delta_P^* = \frac{2\mathcal{F}}{\pi} \left(\frac{\pi}{2k^2\mathcal{E}} \right)^{1/3} \quad (10)$$

If solid A is of the same material as solid B, then equations (9) and (10) can be combined to give

$$\delta_P = \left(\frac{9F^2\mathcal{F}^3}{2\pi^2 E'^2 k^2 \mathcal{E} R} \right)^{1/3} \quad (11)$$

The point contact deformation can now be calculated in terms of the curve fit equations.

$$\bar{\delta}_P = \left(\frac{9F^2 \bar{\mathcal{F}}^3}{2\pi^2 E'^2 \bar{k}^2 \bar{\mathcal{E}} R} \right)^{1/3} \quad (12)$$

Consequently, we see that for a given load and modulus, the deformation can be expressed entirely in terms of R_x and R_y . Equation (11) can be determined also by using the charts by Jones (ref. 3). The methods of this report provide an alternative, simplified solution for the point-contact deformation, whereas Jones' charts are not always convenient or available.

Table I compares $\bar{\delta}_P$ (determined from the curve fit equation) with δ_P (determined numerically). The ellipticity R was varied from 1 (a ball on a plate) to 10 (a configuration approaching line contact). For convenience the regression coefficients for \bar{k} , $\bar{\mathcal{E}}$, and $\bar{\mathcal{F}}$ were rounded off to two places to the right of the decimal. Consequently, a slight round-off error may add to or subtract from the percent error (percent given in table I). The percent error was determined as follows:

$$e = \frac{(\bar{X} - X)100}{X} \quad (14)$$

where

$$X = \{k, \mathcal{E}, \mathcal{F}, \delta_P\}$$

$$\bar{X} = \{\bar{k}, \bar{\mathcal{E}}, \bar{\mathcal{F}}, \bar{\delta}_P\}$$

Note that for the case of a ball on a plate, which results in a circular contact, $R_y/R_x = 1.0$ and $k = 1.00$. Further, for this case the elliptic integrals \mathcal{E} and \mathcal{F} reduce to $\pi/2$. Consequently, δ_P is determined directly without the need to determine k , \mathcal{E} , and \mathcal{F} via curve fitting, charts, or the numerical methods described in references 5 and 6. The largest error (-5 percent) in determining $\bar{\delta}_P$ was for the circular contact case. If one eliminates the circular contact case, the largest error in determining the contact deformation using the curve fit values becomes less than 3 percent.

Lewis Research Center,
National Aeronautics and Space Administration,
and
U. S. Army Air Mobility R&D Laboratory,
Cleveland, Ohio, May 18, 1976,
505-04.

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TABLE I. - COMPARISON OF THE NUMERICALLY DETERMINED VALUES WITH THE CURVE FIT
VALUES FOR THE GEOMETRICALLY DEPENDENT VARIABLES

Radius of curvature ratio, R_y/R_x	Ellipticity			Complete elliptic integral of first kind			Complete elliptic integral of second kind			Point-contact deformation at center of contact, δ_p		
	k	\bar{k}	Percent error, e	\mathcal{E}	$\bar{\mathcal{E}}$	Percent error, e	\mathcal{F}	$\bar{\mathcal{F}}$	Percent error, e	Determined numerically, δ_p	Determined from curve fit equation, $\bar{\delta}_p$	Percent error, e
^a 1.000	1.00	1.03	3.00	1.57	1.60	1.91	1.57	1.53	-2.55	1.230×10^{-4}	1.168×10^{-4}	-5.04
2.820	1.99	2.00	.50	1.21	1.21	0	2.15	2.15	0	1.020	1.017	-.29
5.314	3.01	3.00	-.33	1.11	1.11	↓	2.53	2.53	0	.897	.899	.22
8.330	4.01	4.00	-.25	1.07	1.07		2.80	2.80	0	.814	.816	.25
11.805	4.99	5.00	.20	1.05	1.05		3.02	3.01	-.29	.756	.752	-.53
15.697	5.97	6.00	.50	1.04	1.04		3.19	3.18	-.25	.706	.701	-.71
19.971	6.92	7.00	1.16	1.03	1.03		3.33	3.33	0	.667	.662	-.75
24.605	7.87	8.00	1.65	1.02	1.02		3.46	3.45	-.24	.636	.628	-1.26
29.576	8.80	9.00	2.27	1.02	1.02		3.57	3.56	-.22	.608	.598	-1.64
34.869	9.72	10.00	2.88	1.02	1.02		3.67	3.66	-.25	.584	.571	-2.23

^a $R_y/R_x = 1.000$, then $\mathcal{E} = \mathcal{F} = \pi/2$ and $k = 1.00$. There is no need to use the curve fit values for this case.

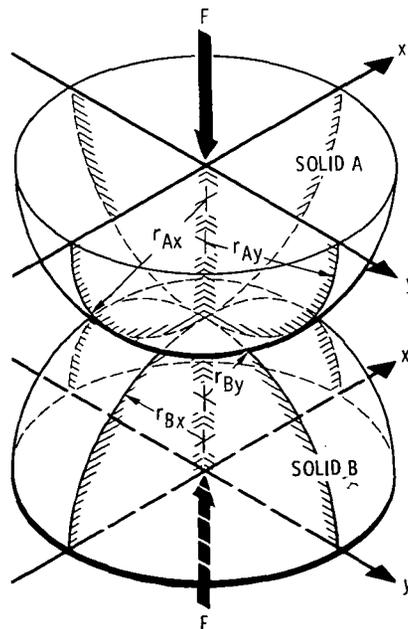


Figure 1. - Geometry of contacting elastic solids.



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