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1. Introduction.

Let $\pi_1,\ldots,\pi_m$ be populations whose multivariate observations in $\mathbb{R}^n$ are distributed with respective normal density functions

$$p_i(x) = \frac{1}{(2\pi)^{n/2}|\Sigma_i^o|^{1/2}} e^{-\frac{1}{2}(x-\mu_i^o)^T\Sigma_i^{o-1}(x-\mu_i^o)}, \quad i = 1,\ldots,m.$$

If π_o is a given mixture of members of these populations, then observations on π_o are distributed in $\mathbb{R}^n$ with density function

$$p(x) = \sum_{i=1}^{m} \alpha_i^o p_i(x)$$

for an appropriate set of proportions $\{\alpha_i^o\}_{i=1,-,m}$. These proportions necessarily satisfy $\sum_{i=1}^{m} \alpha_i^o = 1$ and $\alpha_i^o \geq 0$, $i = 1,-,m$. In this note, we also assume that each α_i^o is strictly positive.

We address here the problem of numerically approximating the maximum-likelihood estimates of the parameters $\{\alpha_i^o,\mu_i^o,\Sigma_i^o\}_{i=1,\ldots,m}$ determined by samples of two types. Samples of both types consist of sets $\{x_{ik}\}_{k=1,\ldots,N_i}$

of independent observations on π_i, $i = 0,\ldots,m$. (The sets $\{x_{ik}\}_{k=1,\ldots,N_i}$, $i = 1,\ldots,m$, comprise the <u>identified observations</u> of such samples, and such samples are said to be <u>partially identified</u>.) We distinguish samples of the two types according to whether the numbers N_i of identified observations contain information about the proportions α_i^o, $i = 1,\ldots,m$. If the numbers of identified observations contain no information about the proportions, then the sample is of the first type; otherwise, the sample is of the second type. The following are examples of how samples of the first and second types, respectively, might be obtained:

(1) For $i = 0,\ldots m$, numbers N_i are arbitrarily choosen and independent observations $\{x_{ik}\}_{k=1,-,N_i}$ are obtained from π_i.

(2) A number K_o of observations are obtained from π_o. For some $N_o < K_o$, N_o of these observations are left unidentified, while the remaining $K_o - N_o$ observations are identified. For $i = 1,\ldots,m$, a subset $\{x_{ik}\}_{k=1,\ldots,N_i}$ of the identified observations is determined whose member observations come from π_i.

In the following, we consider likelihood equations determined by the two types of samples which are necessary conditions for a maximum-likelihood estimate. These equations, which were derived by Coberly [1], suggest certain successive-approximations iterative procedures for obtaining maximum-likelihood estimates. These procedures, which are generalized steepest ascent (deflected gradient) procedures, contain those of Hosmer [2] as a special case. Using arguments that parallel those of [3], we show that, with probability 1 as

N_o approaches infinity (regardless of the relative sizes of N_o and N_i, $i = 1,\ldots,m$), these procedures converge locally to the strongly consistent maximum-likelihood estimates* whenever the step-size is between 0 and 2. Furthermore, the value of the step-size which yields optimal local convergence rates is bounded from below by a number which always lies between 1 and 2.

2. Samples of the first type.

We first assume that numbers $\{N_i\}_{i=0,\ldots,m}$ are given and that, for $i = 0,\ldots,m$, N_i independent observations $\{x_{ik}\}_{k=1,\ldots,N_i}$ are drawn on π_i. The log-likelihood function for a sample of this type is

$$L_1(\Theta) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} \log p_i(x_{ik}) + \sum_{k=1}^{N_o} \log p(x_{ok}) \ .$$

In this expression, the parameter vector Θ (with components α_i, μ_i, Σ_i, $i = 1,\ldots,m$) belongs to the vector space $\mathcal{A} \oplus \mathcal{M} \oplus \mathcal{S}$ defined in [3], and the density functions on the right-hand side are evaluated with the true parameter vector Θ^o (with components α_i^o, μ_i^o, Σ_i^o, $i = 1,\ldots,m$) replaced by Θ.

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*As in [3], one can show that, given any sufficiently small neighborhood of the true parameters, there is, with probability 1 as N_o approaches infinity (regardless of the relative sizes of N_o and N_i, $i = 1,\ldots,m$), a unique solution of the likelihood equations for either type of sample in that neighborhood, and this solution is a maximum-likelihood estimate.

Differentiating $L_1(\Theta)$ and setting its partial derivatives to zero gives the likelihood equations

(1.a) $\alpha_i = A_i(\Theta) \equiv \frac{\alpha_i}{N_o} \sum_{k=1}^{N_o} \frac{p_i(x_{ok})}{p(x_{ok})}$

(1.b) $\mu_i = M_i(\Theta) \equiv \{ \sum_{k=1}^{N_i} x_{ik} + \sum_{k=1}^{N_o} x_{ok} \frac{\alpha_i p_i(x_{ok})}{p(x_{ok})} \} \Big/ \{ N_i + \sum_{k=1}^{N_o} \frac{\alpha_i p_i(x_{ok})}{p(x_{ok})} \}$

(1.c) $\Sigma_i = S_i(\Theta) \equiv \{ \sum_{k=1}^{N_i} (x_{ik}-\mu_i)(x_{ik}-\mu_i)^T + \sum_{k=1}^{N_o} (x_{ok}-\mu_i)(x_{ok}-\mu_i)^T \frac{\alpha_i p_i(x_{ok})}{p(x_{ok})} \} \Big/$

$$\{ N_i + \sum_{k=1}^{N_o} \frac{\alpha_i p_i(x_{ok})}{p(x_{ok})} \}$$

for $i = 1,\ldots,m$.

We set

$$A(\Theta) = \begin{pmatrix} A_1(\Theta) \\ \cdot \\ \cdot \\ \cdot \\ A_m(\Theta) \end{pmatrix} , \quad M(\Theta) = \begin{pmatrix} M_1(\Theta) \\ \cdot \\ \cdot \\ \cdot \\ M_m(\Theta) \end{pmatrix} , \quad S(\Theta) = \begin{pmatrix} S_1(\Theta) \\ \cdot \\ \cdot \\ \cdot \\ S_m(\Theta) \end{pmatrix}$$

and define an operator Φ_ϵ on $\mathcal{A} \oplus \mathcal{M} \oplus \mathcal{S}$ by

$$\Phi_\epsilon(\Theta) = (1-\epsilon)\Theta + \epsilon \begin{pmatrix} A(\Theta) \\ M(\Theta) \\ S(\Theta) \end{pmatrix} .$$

Clearly, for any non-zero ϵ, the likelihood equations are satisfied by a vector $\Theta \in \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{S}$ if and only if $\Theta = \Phi_\epsilon(\Theta)$.

We consider the following iterative procedure: Beginning with some starting value $\Theta^{(1)}$, define successive iterates inductively by

(2) $$\Theta^{(j+1)} = \Phi_\epsilon(\Theta^{(j)})$$

for $j = 1, 2, 3, \ldots$. Our local convergence result for this iterative procedure, as stated in the introduction, follows immediately from the theorem below.

Theorem 1: With probability 1 as N_0 approaches infinity, Φ_ϵ is a locally contractive operator (in some norm on $\mathcal{A} \oplus \mathcal{M} \oplus \mathcal{S}$) near the strongly consistent maximum-likelihood estimate whenever $0 < \epsilon < 2$.

In saying that Φ_ϵ is a locally contractive operator near a point $\Theta \in \mathcal{A} \oplus \mathcal{M} \oplus \mathcal{S}$, we mean that there is a vector norm $|| \quad ||$ on $\mathcal{A} \oplus \mathcal{M} \oplus \mathcal{S}$ and a number λ, $0 \le \lambda < 1$, such that

$$||\Phi_\epsilon(\Theta') - \Theta|| \le \lambda ||\Theta' - \Theta||$$

whenever Θ' lies sufficiently near Θ.

Proof of Theorem 1: Let

$$\Theta = \begin{pmatrix} \overline{\alpha} \\ \\ \overline{\mu} \\ \\ \overline{\Sigma} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \\ \vdots \\ \mu_1 \\ \vdots \\ \mu_m \\ \\ \Sigma_1 \\ \vdots \\ \Sigma_m \end{pmatrix}$$

be the strongly consistent maximum-likelihood estimate. We assume that

$\alpha_i \neq 0$, $i = 1,\ldots,m$. (As N_o approaches infinity, the probability is 1 that this is the case.) As in [3], it suffices to show that, with probability 1, $\nabla\Phi_\epsilon(\Theta)$ converges to an operator which has operator norm less than 1 with respect to a suitable vector norm on $\mathcal{A} \oplus \mathcal{M} \oplus \mathcal{S}$.

Now

$$\nabla\Phi_\epsilon(\Theta) = (1-\epsilon)I + \epsilon\,\nabla\begin{pmatrix} A(\Theta) \\ M(\Theta) \\ S(\Theta) \end{pmatrix} ,$$

and we write

$$\nabla\begin{pmatrix} A \\ M \\ S \end{pmatrix} = \begin{pmatrix} \nabla_{\overline{\alpha}}A & \nabla_{\overline{\mu}}A & \nabla_{\overline{\Sigma}}A \\ \nabla_{\overline{\alpha}}M & \nabla_{\overline{\mu}}M & \nabla_{\overline{\Sigma}}M \\ \nabla_{\overline{\alpha}}S & \nabla_{\overline{\mu}}S & \nabla_{\overline{\Sigma}}S \end{pmatrix} .$$

Define inner products $\langle\, ,\, \rangle_i'$ on $\mathcal{M}$, $\langle\, ,\, \rangle_i''$ on $\mathcal{S}$, and $\langle\, ,\, \rangle$ on $\mathcal{A} \oplus \mathcal{M} \oplus \mathcal{S}$ as in [3]. Setting

$$\beta_i(x) = \frac{p_i(x)}{p(x)},\ \gamma_i(x) = (x - \mu_i),\ \delta_i(x) = [\Sigma_i^{-1}(x-\mu_i)(x-\mu_i)^T - I], K_i = N_i + \alpha_i N_o$$

for $i = 1,\ldots,m$, one calculates

$$\nabla_{\overline{\alpha}}A(\Theta) = I - (\mathrm{diag}\ \alpha_i)\ \frac{1}{N_o}\sum_1^{N_o} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}^T$$

$$\nabla_{\overline{\mu}}A(\Theta) = -\ (\mathrm{diag}\ \alpha_i)\ \frac{1}{N_o}\sum_1^{N_o} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \begin{pmatrix} \langle\beta_1\gamma_1, \cdot\rangle_1' \\ \vdots \\ \langle\beta_m\gamma_m, \cdot\rangle_m' \end{pmatrix}^T$$

$$\nabla_{\overline{\Sigma}}A(\Theta) = -\ (\mathrm{diag}\ \alpha_i)\ \frac{1}{N_o}\sum_1^{N_o} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \begin{pmatrix} \langle\beta_1\delta_1, \cdot\rangle_1'' \\ \vdots \\ \langle\beta_m\delta_m, \cdot\rangle_m'' \end{pmatrix}^T$$

$$\nabla_{\alpha} M(\Theta) = \left(\mathrm{diag}\ \frac{1}{K_i} \sum_1^{N_O} \beta_i \gamma_i\right) - \left(\mathrm{diag}\ \frac{\alpha_i}{K_i}\right) \left\{ \sum_1^{N_O} \begin{pmatrix} \beta_1\gamma_1 \\ \vdots \\ \beta_m\gamma_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}^T \right\}$$

$$\nabla_{\mu} M(\Theta) = \left(\mathrm{diag}\ \frac{\alpha_i}{K_i} \sum_1^{N_O} \gamma_i \gamma_i^T \Sigma_i^{-1} \beta_i\right) - \left(\mathrm{diag}\ \frac{\alpha_i}{K_i}\right) \left\{ \sum_1^{N_O} \begin{pmatrix} \beta_1\gamma_1 \\ \vdots \\ \beta_m\gamma_m \end{pmatrix} \begin{pmatrix} \langle\beta_1\gamma_1, \cdot\rangle'_1 \\ \vdots \\ \langle\beta_m\gamma_m, \cdot\rangle'_m \end{pmatrix}^T \right\}$$

$$\nabla_{\Sigma} M(\Theta) = \left(\mathrm{diag}\ \frac{1}{K_i} \sum_1^{N_O} \beta_i \gamma_i \langle\delta_i, \cdot\rangle''_i\right) - \left(\mathrm{diag}\ \frac{\alpha_i}{K_i}\right) \left\{ \sum_1^{N_O} \begin{pmatrix} \beta_1\gamma_1 \\ \vdots \\ \beta_m\gamma_m \end{pmatrix} \begin{pmatrix} \langle\beta_1\delta_1, \cdot\rangle''_1 \\ \vdots \\ \langle\beta_m\delta_m, \cdot\rangle''_m \end{pmatrix}^T \right\}$$

$$\nabla_{\alpha} S(\Theta) = \left(\mathrm{diag}\ \frac{\Sigma_i}{K_i} \sum_1^{N_O} \beta_i \delta_i\right) - \left(\mathrm{diag}\ \frac{\alpha_i \Sigma_i}{K_i}\right) \left\{ \sum_1^{N_O} \begin{pmatrix} \beta_1\delta_1 \\ \vdots \\ \beta_m\delta_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}^T \right\}$$

$$\nabla_{\mu} S(\Theta) = \left(\mathrm{diag}\ \frac{1}{K_i} \left\{-\sum_1^{N_i} [(\cdot)\gamma_i^T + \gamma_i(\cdot)^T] - \alpha_i \sum_1^{N_O} [(\cdot)\gamma_i^T + \gamma_i(\cdot)^T]\beta_i + \Sigma_i \sum_1^{N_O} \delta_i \langle\beta_i\gamma_i, \cdot\rangle'_i\right\}\right) -$$

$$- \left(\mathrm{diag}\ \frac{\alpha_i \Sigma_i}{K_i}\right) \left\{ \sum_1^{N_O} \begin{pmatrix} \beta_1\delta_1 \\ \vdots \\ \beta_m\delta_m \end{pmatrix} \begin{pmatrix} \langle\beta_1\gamma_1, \cdot\rangle'_1 \\ \vdots \\ \langle\beta_m\gamma_m, \cdot\rangle'_m \end{pmatrix}^T \right\}$$

$$\nabla_{\Sigma} S(\Theta) = \left(\mathrm{diag}\ \frac{\Sigma_i}{K_i} \sum_1^{N_O} \beta_i \delta_i \langle\delta_i, \cdot\rangle''_i\right) - \left(\mathrm{diag}\ \frac{\alpha_i \Sigma_i}{K_i}\right) \left\{ \sum_1^{N_O} \begin{pmatrix} \beta_1\delta_1 \\ \vdots \\ \beta_m\delta_m \end{pmatrix} \begin{pmatrix} \langle\beta_1\delta_1, \cdot\rangle \\ \vdots \\ \langle\beta_m\delta_m, \cdot\rangle \end{pmatrix}^T \right\}$$

Here, the arguments of β_i, γ_i and δ_i can be determined from the indices of summation, e.g.,

$$\sum_1^{N_O} \beta_i \gamma_i = \sum_{k=1}^{N_O} \beta_i(x_{ok}) \gamma_i(x_{ok}) .$$

Setting

$$V = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \beta_1\gamma_1 \\ \vdots \\ \beta_m\gamma_m \\ \beta_1\delta_1 \\ \vdots \\ \beta_m\delta_m \end{pmatrix}$$

one obtains at Θ

$$\nabla \begin{pmatrix} A \\ M \\ S \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} - \begin{pmatrix} (\operatorname{diag} \frac{\alpha_i}{N_O}) & 0 & 0 \\ 0 & (\operatorname{diag} \frac{\alpha_i}{K_i}) & 0 \\ 0 & 0 & (\operatorname{diag} \frac{\alpha_i \Sigma_i}{K_i}) \end{pmatrix} \{\sum_1^{N_O} V(x_{ok}) \langle V(x_{ok}), \cdot \rangle\} ,$$

where

$$B_{21} = (\operatorname{diag} \frac{1}{K_i} \sum_1^{N_O} \beta_i\gamma_i)$$

$$B_{22} = (\operatorname{diag} \frac{\alpha_i}{K_i} \sum_1^{N_O} \gamma_i\gamma_i^T\Sigma_i^{-1}\beta_i)$$

$$B_{23} = (\operatorname{diag} \frac{1}{K_i} \sum_1^{N_O} \beta_i\gamma_i \langle\delta_i, \cdot\rangle''_i)$$

$$B_{31} = (\operatorname{diag} \frac{\Sigma_i}{K_i} \sum_1^{N_O} \beta_i\delta_i)$$

$$B_{32} = (\operatorname{diag} \frac{1}{K_i}\{- \sum_1^{N_i} [(\cdot)\gamma_i^T+\gamma_i(\cdot)^T] - \alpha_i \sum_1^{N_O} [(\cdot)\gamma_i^T+\gamma_i(\cdot)^T]\beta_i + \Sigma_i \sum_1^{N_O} \delta_i \langle\beta_i\gamma_i, \cdot\rangle'_i\})$$

$$B_{33} = (\operatorname{diag} \frac{\Sigma_i}{K_i} \sum_1^{N_O} \beta_i\delta_i \langle\delta_i, \cdot\rangle''_i) .$$

We have assumed that Θ is the strongly consistent maximum-likelihood estimate. Then, regardless of the relative sizes of N_i and N_o, one can show as in [3] that, with probability 1, $\{\nabla\Phi_\epsilon(\Theta) - E(\nabla\Phi_\epsilon(\Theta^o))\}$ converges to zero as N_o approaches infinity. Now

$$E\left(\nabla \begin{pmatrix} A(\Theta^o) \\ M(\Theta^o) \\ S(\Theta^o) \end{pmatrix}\right) = \begin{pmatrix} I & 0 & 0 \\ 0 & (\mathrm{diag}\ \frac{\alpha_i^o N_o}{K_i}\ I) & 0 \\ 0 & 0 & (\mathrm{diag}\ \frac{\alpha_i N_o}{K_i}\ I) \end{pmatrix} -$$

$$- \begin{pmatrix} (\mathrm{diag}\ \alpha_i^o) & 0 & 0 \\ 0 & (\mathrm{diag}\ \frac{\alpha_i^o N_o}{K_i}\ I) & 0 \\ 0 & 0 & (\mathrm{diag}\ \frac{\alpha_i^o N_o}{K_i}\ \Sigma_i^o) \end{pmatrix} \{ \int_{\mathbb{R}^n} V(x) \langle V(x), \cdot \rangle p(x) dx \}$$

$$= B(I - QR),$$

where

$$B = \begin{matrix} I & 0 & 0 \\ 0 & (\mathrm{diag}\ \frac{\alpha_i^o N_o}{K_i}\ I) & 0 \\ 0 & 0 & (\mathrm{diag}\ \frac{\alpha_i^o N_o}{K_i}\ I) \end{matrix}$$

$$Q = \begin{matrix} (\mathrm{diag}\ \alpha_i^o) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (\mathrm{diag}\ \Sigma_i^o) \end{matrix}$$

$$R = \int_{\mathbb{R}^n} V(x) \langle V(x), \cdot \rangle p(x) dx .$$

It was shown in [3] that QR is positive-definite and symmetric with operator norm less than 1 with respect to the inner product $\langle\cdot,Q^{-1}\cdot\rangle$ on $\mathcal{A}\oplus\mathcal{M}\oplus\mathcal{S}$. It follows that $I-QR$ is positive-definite and symmetric with norm less than 1 with respect to $\langle\cdot,Q^{-1}\cdot\rangle$. Since B and Q commute, $\langle\cdot,Q^{-1}B^{-1}\cdot\rangle$ is an inner product on $\mathcal{A}\oplus\mathcal{M}\oplus\mathcal{S}$, and one sees that $\langle W,Q^{-1}W\rangle \le \langle W,Q^{-1}B^{-1}W\rangle$ for $W \in \mathcal{A}\oplus\mathcal{M}\oplus\mathcal{S}$. Consequently, $B(I-QR)$ is positive-definite and symmetric with norm less than 1 with respect to the inner product $\langle\cdot,Q^{-1}B^{-1}\cdot\rangle$. One concludes that

$$E(\nabla\Phi_{\epsilon}(\Theta^{0})) = (1-\epsilon)I + \epsilon\, E(\nabla \begin{pmatrix} A(\Theta^{0}) \\ M(\Theta^{0}) \\ S(\Theta^{0}) \end{pmatrix})$$

has norm less than 1 with respect to $\langle\cdot,Q^{-1}B^{-1}\cdot\rangle$ whenever $0 < \epsilon < 2$. This completes the proof of the theorem.

We remark that, reasoning as in [3], one may determine a particular value of ϵ (the "optimal ϵ") which yields, with probability 1 as N_0 approaches infinity, the fastest asymptotic uniform rates of local convergence of the iterative procedure (2) near Θ. This optimal ϵ is given by

$$\epsilon = \frac{2}{2-(\tau+\rho)}$$

where ρ and τ are, respectively the largest and smallest eigenvalues of $B(I-QR)$ regarded as an operator on $\mathcal{E}\oplus\mathcal{M}\oplus\mathcal{S}$ ($\mathcal{E}$ is the subspace of $\mathcal{A}$ whose components sum to zero.) Since ρ and τ lie between zero and 1, one sees that the optimal ϵ is always greater than 1. If the component populations are "widely separated," then ρ and τ are near zero and, hence, the optimal ϵ is near 1. If two or more of the component populations are nearly indistinguishable and if N_o is large relative to the N_i's, then τ is near zero, and the optimal ϵ cannot be much smaller than 2.

3. Samples of the second type.

We now assume that K_o observations are obtained from the mixture population π_o, and that, for some $N_o < K_o$, N_o of these observations are left unidentified, while the remaining $K_o - N_o$ observations are identified. For $i = 1,\ldots,m$, let $\{x_{ik}\}_{k=1,\ldots,N_i}$ denote the subset of the identified observations which come from π_i, and let $\{x_{ok}\}_{k=1,\ldots,N_o}$ be the set of unidentified observations from π_o. The log-likelihood function for this sample is

$$L_2(\Theta) = \log\left\{\frac{(\sum_{i=1}^{m} N_i)!}{N_1!\ldots N_m!}\,\alpha_1^{N_1}\ldots\alpha_m^{N_m}\right\} + \sum_{i=1}^{m}\sum_{k=1}^{N_i}\log p_i(x_{ik}) + \sum_{k=1}^{N_o}\log p(x_{ok})$$

$$= \log\left\{\frac{(\sum_{i=1}^{m} N_i)!}{N_1!\ldots N_m!}\right\} + \sum_{i=1}^{m}\sum_{k=1}^{N_i}\log[\alpha_i p_i(x_{ik})] + \sum_{k=1}^{N_o}\log p(x_{ok}) .$$

Differentiating L_2 and setting its partial derivatives to zero gives the likelihood equations

$$\text{(3.a)} \qquad \alpha_i = \tilde{A}_i(\Theta) \equiv \frac{N_i}{K_o} + \frac{\alpha_i}{K_o}\sum_{k=1}^{N_o}\frac{p_i(x_{ok})}{p(x_{ok})}$$

$$\text{(3.b)} \qquad \mu_i = M_i(\Theta)$$

$$\text{(3.c)} \qquad \Sigma_i = S_i(\Theta)$$

for $i = 1,\dots,m$.

We set

$$\tilde{A}(\Theta) = \begin{pmatrix} \tilde{A}_1(\Theta) \\ \vdots \\ \tilde{A}_m(\Theta) \end{pmatrix}$$

and define an operator $\tilde{\Phi}_\epsilon$ on $\mathcal{A}\oplus\mathcal{M}\oplus\mathcal{S}$ by

$$\tilde{\Phi}_\epsilon(\Theta) = (1-\epsilon)\Theta + \epsilon \begin{pmatrix} A(\Theta) \\ M(\Theta) \\ S(\Theta) \end{pmatrix} .$$

Our iterative procedure is the following: Beginning with some starting value $\Theta^{(1)}$, define successive iterates inductively by

$$\Theta^{(j+1)} = \tilde{\Phi}_\epsilon(\Theta^{(j)}) \tag{4}$$

for $j = 1,2,3,\dots$. As before, the desired local convergence result for this iterative procedure follows from the theorem below.

Theorem 2: With probability 1 as N_0 approaches infinity, $\tilde{\Phi}_\epsilon$ is a locally contractive operator (in some norm on $\mathcal{A}\oplus\mathcal{M}\oplus\mathcal{S}$) near the strongly consistent maximum-likelihood estimate whenever $0 < \epsilon < 2$.

Proof of Theorem 2: If Θ is the strongly consistent maximum-likelihood estimate, then, as before, it suffices to show that, with probability 1, $\nabla\tilde{\Phi}_\epsilon(\Theta)$ converges as N_0 approaches infinity to an operator which has operator norm less than 1 with respect to some vector norm on $\mathcal{A}\oplus\mathcal{M}\oplus\mathcal{S}$. Proceeding as before, one sees that

$$\nabla_{\overline{\alpha}}\tilde{A}(\Theta) = (\mathrm{diag}\ (1 - \frac{N_i}{\alpha_i K_o})) - (\mathrm{diag}\ \frac{\alpha_i}{K_o}) \left\{ \sum_{1}^{N_o} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}^T \right\}$$

$$\nabla_{\overline{\mu}}\tilde{A}(\Theta) = -(\mathrm{diag}\ \frac{\alpha_i}{K_o}) \left\{ \sum_{1}^{N_o} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \begin{pmatrix} \langle \beta_1\gamma_1, \cdot \rangle_1' \\ \vdots \\ \langle \beta_m\gamma_m, \cdot \rangle_m' \end{pmatrix}^T \right\}$$

$$\nabla_{\overline{\Sigma}}\tilde{A}(\Theta) = -(\mathrm{diag}\ \frac{\alpha_i}{K_o}) \left\{ \sum_{1}^{N_o} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \begin{pmatrix} \langle \beta_1\delta_1, \cdot \rangle_1'' \\ \vdots \\ \langle \beta_m\delta_m, \cdot \rangle_m'' \end{pmatrix}^T \right\}$$

The remaining Fréchet derivatives, i.e., the derivatives at Θ of M and S with respect to $\overline{\alpha}$, $\overline{\mu}$, and $\overline{\Sigma}$, are unchanged, except that K_i must be replaced by $\alpha_i K_o$ wherever it appears.

One obtains at Θ

$$(4) \qquad \nabla \begin{pmatrix} \tilde{A} \\ M \\ S \end{pmatrix} = \begin{pmatrix} (\mathrm{diag}(1 - \frac{N_i}{\alpha_i K_o})) & 0 & 0 \\ \tilde{B}_{21} & \tilde{B}_{22} & \tilde{B}_{23} \\ \tilde{B}_{31} & \tilde{B}_{32} & \tilde{B}_{33} \end{pmatrix} -$$

$$\begin{pmatrix} (\mathrm{diag}\ \frac{\alpha_i}{K_o}) & 0 & 0 \\ 0 & \frac{1}{K_o} I & 0 \\ 0 & 0 & (\mathrm{diag}\ \frac{\Sigma_i}{K_o}) \end{pmatrix} \left\{ \sum_{k=1}^{N_o} V(x_{ok}) \langle V(x_{o,k}), \cdot \right\}$$

In this expression, each $\tilde{B}_{jk}$ is the same as the corresponding B_{jk} defined previously, except that each K_i in the latter is replaced by $\alpha_i K_o$ in the former. One verifies that, with probability 1 as N_o approaches infinity, (4) has the same limit as $\tilde{B}(I-QR)$, where Q and R are as before and $\tilde{B} = \frac{N_o}{K_o} I$. Repeating our earlier reasoning, one verifies that $\tilde{B}(I-QR)$ is positive-definite and symmetric with norm less than 1 with respect to the inner product $\langle \cdot, Q^{-1}\tilde{B}^{-1} \cdot \rangle$. Hence

$$\nabla\tilde{\Phi}_\epsilon(\Theta) = (1 - \epsilon) + \epsilon\nabla \begin{pmatrix} \tilde{A}(\Theta) \\ M(\Theta) \\ S(\Theta) \end{pmatrix}$$

converges to an operator which has norm less than 1 with respect to $\langle \cdot, Q^{-1}\tilde{B}^{-1} \cdot \rangle$ whenever $0 < \epsilon < 2$. This completes the proof of the theorem.

The remarks concerning the "optimal ϵ" at the conclusion of the preceding section are valid here verbatim.

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