RESEARCH STUDY ON STABILIZATION AND CONTROL

MODERN SAMPLED-DATA CONTROL THEORY

SYSTEMS RESEARCH LABORATORY

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FINAL REPORT

RESEARCH STUDY ON IPS DIGITAL CONTROLLER DESIGN

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1. Analysis of the Continuous-Data Instrument Pointing System

The objective of this chapter is to investigate the performance of the simplified continuous-data model of the Instrument Pointing System (IPS). Although the ultimate objective is to study the digital model of the system, knowledge on the performance of the continuous-data model is important in the sense that the characteristics of the digital system should approach those of the continuous-data system as the sampling period approaches zero.

The planar equations which describe the motion of the Spacelab using Inside-Out Gimbal (IOG) for the pointing base are differential equations of the fourteenth order. A total of seven degrees of freedom are represented by these equations. Three of these degrees of freedom \((X_s, Z_s, \phi)\) belong to the orbitor, and three degrees of freedom \((X_m, Z_m, \psi)\) are for the mount, and the scientific instrument (SI) has one degree of freedom in \(e\).

A simplified model of the IPS model is obtained by assuming that all but motion about two of the seven degrees of freedom axes are negligible. The simplified IPS model consists of only the motion about the scientific instrument axis, \(e\), and the mount rotation \(\psi\). The block diagram of the simplified linear IPS control system is shown in Figure 1-1.

In this chapter we shall investigate only the performance of the continuous-data IPS control system. The signal flow graph of the continuous-data IPS control system in Figure 1-1 is shown in Figure 1-2.

The characteristic equation of the continuous-data IPS is obtained from Figure 1-2.

\[
\Delta = (1 - K_4 K_6) s^5 + (K_2 + K_1 K_7 - K_1 K_4 K_7) s^4 + (K_3 - K_0 K_4 K_7 + K_0 K_7 + K_1 K_2 K_7) s^3
\]
Compensator = 1 for rigid-body study

Figure 1-1. Block diagram of simplified linear IPS control system.
Figure 1-2. Signal flow graph of the simplified linear continuous-data IPS control system.
\[ + (K_1 K_7 - K_1 K_4 K_7 + K_1 K_3 K_7 + K_0 K_2 K_7) s^2 + (K_1 K_2 K_7 + K_0 K_3 K_7) s + K_1 K_3 K_7 = 0 \]

\[(1-1)\]

The system parameters are:

\[ K_0 = 8 \times 10^5 \text{ n-m} \]
\[ K_1 = 6 \times 10^4 \text{ n-m/sec} \]
\[ K_2 = 0.0012528 \]
\[ K_3 = 0.0036846 \]
\[ K_4 = \text{variable} \]
\[ K_5 = 0.80059 \]
\[ K_6 = 10798.49 \]
\[ K_7 = 0.0000926 \]

Equation (1-1) is simplified to

\[ s^5 + 16.704 s^4 + 222.63 s^3 + (1.706 + 27.816 \times 10^{-5} K_1) s^2 \]
\[ + (4.11 + 174.7 \times 10^{-8} K_1) s + 513.855 \times 10^{-8} K_1 = 0 \]

\[(1-2)\]

The root locus plot of Eq. (1-2) when \( K_1 \) varies is shown in Figure 1-3.

It is of interest to notice that two of the root loci of the fifth-order IPS control system are very close to the origin in the left-half of the s-plane, and these two loci are very insensitive to the variation of \( K_1 \).

The characteristic equation roots for various values of \( K_1 \) are tabulated in the following:

<table>
<thead>
<tr>
<th>( K_1 )</th>
<th>ROOTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0, -0.00314+0.13587, -8.3488+j12.3614</td>
</tr>
<tr>
<td>( 10^4 )</td>
<td>-0.0125 -0.0031398+j0.135873, -8.3426+j12.3571</td>
</tr>
<tr>
<td>( 10^5 )</td>
<td>-0.1262 -0.003142+j0.135879, -8.28577+j12.3192</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>-1.38146 -0.0031398+j0.135882, -7.65813+j11.9457</td>
</tr>
</tbody>
</table>
The continuous-data IPS control system is asymptotically stable for $K$ less than $1.5 \times 10^7$.

It is of interest to investigate the response of the IPS control system due to its own initial condition. The transfer function between and its initial condition $\varepsilon_0$ for $K = 5 \times 10^6$ is

$$
\frac{\varepsilon(s)}{\varepsilon_o(s)} = \frac{s^2(s^3 + 16.7s^2 - 893s - 5581.77)}{s^5 + 16.7s^4 + 222.63s^3 + 1392.5s^2 + 12.845s + 25.693}
$$

where we have considered that $\varepsilon(0) = \varepsilon_o$ is a unit-step function input applied at $t = 0$; i.e., $\varepsilon_o(s) = 1/s$.

It is interesting to note that the response of $\varepsilon$ due to its own initial condition is overwhelmingly governed by the poles near the origin. In this case, the transfer function has zeros at $s = 80.5$, $-12.375 + j23.75$. The zero at $s = 80.5$ causes the response of $\varepsilon$ to go negative first before eventually reaching zero in the steady-state, as shown in Figure 1-4. The first overshoot is due to the complex poles at $s = -3.8134 + j11.7806$. Therefore, the eigenvalues of the closed-loop system at $s = -3.8134 + j11.7806$ controls only the transient response of $\varepsilon(t)$ near $t = 0$, and the time response of $\varepsilon(t)$ has a long duration.
Figure 13. Responses of $e(t)$ due to an initial value of $e_0 = e(0) = 1$.
Figure 1-5. Response of $e(t)$ due to an initial value of $e_0 = 1$, $K_1 = 10^5$. 

$e_0$ (Initial Condition)

TIME (seconds)
period of oscillation. The eigenvalues at \( s = -0.00313 + j0.1358 \) due to
the isolator dynamics give rise to an oscillation which takes several
minutes to damp out. The conditional frequency is 0.1358 rad/sec, so the
period of oscillation is approximately 46 seconds. Figure 1-5 shows the
response of \( \varepsilon(t) \) for \( \varepsilon_o = 1 \) rad on a different time scale over 100 seconds.
Although the initial condition of 1 radian far exceeds the limitation under
which the linear approximation of the system model is valid, however, for
linear analysis, the response will have exactly the same characteristics
but with proportionally smaller amplitude if the value of \( \varepsilon_o \) is reduced.

Since the transient response of the system is dominated by the
isolator dynamics with eigenvalues very close to the origin, changing
the value of \( K_1 \) within the stability bounds would only affect the time
response for the first second as shown in Figure 1-4, the oscillatory
and slow decay characteristics of the response would not be affected in
any significant way.
2. The Simplified Linear Digital IPS Control System

In this chapter the model of the simplified digital IPS control system will be described, and the dynamic performance of the system will be analyzed.

The block diagram of the digital IPS is shown in Figure 2-1, where the element S/H represents sample-and-hold. The system parameters are identical to those defined in Chapter 1. An equivalent signal flow graph of the block diagram in Figure 2-1 is drawn as shown in Figure 2-2.

Applying Mason's gain formula to Figure 2-2 with $\Theta_A(s)$ and $\Omega_A(s)$ as outputs, $\Theta_A^*(s)$ and $\Omega_A^*(s)$ as inputs, we have

$$
\Omega_A(s) = -K_1 K_T \frac{G_h(s)}{\Delta_1 - K_4} \Omega_A^*(s) - (K_0 + \frac{K_T}{z-1}) K_7 \frac{G_h(s)}{\Delta_1 - K_4} \Theta_A^*(s) \quad (2-1)
$$

$$
\Theta_A(s) = -K_1 K_T \frac{G_h(s)}{\Delta_1 - K_4} \Omega_A^*(s) - (K_0 + \frac{K_T}{z-1}) K_7 \frac{G_h(s)}{\Delta_2} (\Delta_1 - K_4) \Theta_A^*(s) \quad (2-2)
$$

where

$$
\Delta_1 = 1 + K_2 s^{-1} + K_3 s^{-2} \quad (2-3)
$$

$$
\Delta = 1 - K_4 K_6 + K_2 s^{-1} + K_3 s^{-2} \quad (2-4)
$$

$$
G_h(s) = \frac{1 - e^{-Ts}}{s} \quad (2-5)
$$

Letting

$$
G_1(s) = K_7 \frac{G_h(s)}{\Delta_1 - K_4} \frac{K_2 (1 - e^{-Ts}) ((1 - K_4) s^2 + K_2 s + K_3)}{s^2 ((1 - K_4 K_6) s^2 + K_2 s + K_3)} \quad (2-6)
$$

$$
G_2(s) = K_7 \frac{G_h(s)}{\Delta_2} (\Delta_1 - K_4) \frac{G_1(s)}{s} \quad (2-7)
$$

and taking the z-transform on both sides of Eqs. (2-1) and (2-2), we have

$$
\Omega_A(z) = -K_1 G_1(z) \Omega_A(z) - (K_0 + \frac{K_T}{z-1}) G_1(z) \Theta_A(z) \quad (2-8)
$$

$$
\Theta_A(z) = -K_1 G_2(z) \Omega_A(z) - (K_0 + \frac{K_T}{z-1}) G_2(z) \Theta_A(z) \quad (2-9)
$$
Figure 2-1. Block diagram of simplified linear digital IPS control system.
Figure 2-2. Signal flow graph of the simplified linear digital IPS control system.
From these two equations the characteristic equation of the digital system is found to be

\[ \Delta(z) = 1 + K_1 G_1(z) + \left( K_0 + \frac{K_1 T}{z - 1} \right) G_2(z) = 0 \] (2-10)

where

\[ G_1(z) = \phi(G_1(s)) = K_7 (1 - z^{-1})^2 \left( \frac{1 - K_4 s^2 + K_2 s + K_3}{s^2 [(1 - K_4 K_6) s^2 + K_2 s + K_3]} \right) \] (2-11)

\[ G_2(z) = \phi(G_1(s)/s) \] (2-12)

The characteristic equation of the digital IPS control system is of the fifth order. However, the values of the system parameters are such that two of the characteristic equation roots are very close to the \( z = 1 \) point in the \( z \)-plane. These two roots are inside the unit circle and they are relatively insensitive to the values of \( K_1 \) and \( T \), so long as \( T \) is not very large. The sampling period appears in terms such as \( e^{-0.0094T} \) which is approximately one unless \( T \) is very large.

Since the values of \( K_2 \) and \( K_3 \) are relatively small,

\[ G_1(z) = K_7 (1 - z^{-1})^2 \left( \frac{1 - K_4}{1 - K_4 K_6} s^2 \right) = \frac{K_7 (1 - K_4)}{1 - K_4 K_6} \frac{T}{z - 1} = 2.79 \times 10^{-4} \frac{T}{z - 1} \] (2-13)

\[ G_2(z) = \frac{K_7 (1 - K_4)}{1 - K_4 K_6} (1 - z^{-1})^2 \left( \frac{1}{s^3} \right) = 2.79 \times 10^{-4} \frac{T^2 (z + 1)}{2(z - 1)^2} \] (2-14)

Substituting \( G_1(z) \) and \( G_2(z) \) into Eq. (2-10), the characteristic equation of the digital IPS is approximated by the following third-order equation:

\[
\begin{align*}
    z^3 &+ \left( K_0 K_1 T^2 \right) z^2 + \left( K_1 K_1 T^3 \right) z + \left( \frac{K_0 K_1 T^3}{2} - 2 K_1 K_1 T + 3 \right) = 0
\end{align*}
\] (2-15)
where \( K_p = 2.79 \times 10^{-4} \), and it is understood that two other characteristic equation roots are at \( z = 1 \). It can be shown that in the limit as \( T \) approaches zero, the three roots of Eq. (2-15) approaches to the roots of the characteristic equation of the continuous-data IPS control system, and the two roots near \( z = 1 \) also approach to near \( s = 0 \), as shown by the root locus diagram in Figure 1-3.

Substituting the values of the system parameters into Eq. (2-15), we have

\[
\Delta(z) = z^3 + (111.6T^2 + 16.74T - 3)z^2 + (3 - 33.48T + 1.395 \times 10^{-4}K_IT^3)z
+ (1.395 \times 10^{-4}K_IT^3 + 16.74T - 111.6T^2 - 1) = 0 \quad (2-16)
\]

Applying Jury's test on stability to the last equation, we have

(1) \( \Delta(1) > 0 \) or \( K_pK_IT^3 > 0 \)

Thus, \( K_I > 0 \), since \( T > 0 \).

(2) \( \Delta(-1) < 0 \) or \( -8 + 66.96T < 0 \)

Thus, \( T < 0.12 \) sec.

(3) Also, \( |a_0| < a_3 \), or

\[
|1.395 \times 10^{-4}K_IT^3 + 16.74T - 111.6T^2 - 1| < 1 \quad (2-20)
\]

The relation between \( T \) and \( K_I \) for the satisfaction of Eq. (2-20) is plotted as shown in Fig. 2-3.

(4) The last criterion that must be met for stability is

\[
|b_0| > b_2 \quad (2-21)
\]

where

\[
b_0 = a_0^2 - a_3^2
\]
\[
b_2 = a_0a_2 - a_1a_3
\]
\[
a_0 = 1.395 \times 10^{-4}K_IT^3 + 16.74T - 111.6T^2 - 1
\]
Figure 2-3. Stability regions of the digital IPS.
\[ a_1 = 3 - 33.48T + 1.395 \times 10^{-4}K_I T^3 \]
\[ a_2 = 111.6T^2 + 16.74T - 3 \]
\[ a_3 = 1 \]

The relation between \( T \) and \( K_I \) for the satisfaction of Eq. (2-21) is plotted as shown in Figure 2-3. It turns out that the inequality condition of Eq. (2-21) is the more stringent one for stability. Notice also that as the sampling period \( T \) approaches zero, the maximum value of \( K_I \) for stability is slightly over \( 10^7 \), as indicated by the root locus plot of the continuous-data IPS.

For quick reference, the maximum values of \( K_I \) for stability corresponding to various \( T \) are tabulated as follows:

<table>
<thead>
<tr>
<th>( T )</th>
<th>Max. ( K_I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>( 10^7 )</td>
</tr>
<tr>
<td>0.08</td>
<td>( 7.5 \times 10^6 )</td>
</tr>
<tr>
<td>0.1</td>
<td>( 6.9 \times 10^6 )</td>
</tr>
</tbody>
</table>

When \( T = 0.05 \) sec, the characteristic equation is factored as

\[
(z - 1)(z^2 - 0.884z + 0.442 - 0.1744 \times 10^{-7}K_I) = 0
\]

(2-22)

Thus, there is always a root at \( z = 1 \) for all values of \( K_I \), and the system is not asymptotically stable.

The root locus plot for \( T = 0.01, 0.08, \) and 0.1 second when \( K_I \) varies are shown in Figures 2-4, 2-5, and 2-6, respectively. The two roots which are near \( z = 1 \) and are not sensitive to the values of \( T \) and \( K_I \) are also included in these plots. The root locations on the root loci are tabulated as follows:
\( T = 0.01 \text{ sec} \)

<table>
<thead>
<tr>
<th>( K_1 )</th>
<th>( r )</th>
<th>ROOTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00000</td>
<td>0.91072 ± j0.119788</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>0.9864</td>
<td>0.917509 ± j0.115822</td>
</tr>
<tr>
<td>( 5 \times 10^6 )</td>
<td>0.907357</td>
<td>0.957041 ± j0.114946</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>0.853393</td>
<td>0.984024 ± j0.137023</td>
</tr>
<tr>
<td>( 5 \times 10^7 )</td>
<td>0.734201</td>
<td>1.04362 ± j0.224901</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>0.672306</td>
<td>1.07457 ± j0.282100</td>
</tr>
</tbody>
</table>

\( T = 0.08 \)

<table>
<thead>
<tr>
<th>( K_1 )</th>
<th>( r )</th>
<th>ROOTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.999972</td>
<td>-0.0267061 ± j0.611798</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>0.888688</td>
<td>0.0289362 ± j0.583743</td>
</tr>
<tr>
<td>( 5 \times 10^6 )</td>
<td>0.0273573</td>
<td>0.459601 ± j0.665086</td>
</tr>
<tr>
<td>( 7.5 \times 10^6 )</td>
<td>-0.156092</td>
<td>0.551326 ± j0.851724</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>-0.253228</td>
<td>0.599894 ± j0.989842</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>-0.770222</td>
<td>0.850391 ± j2.83716</td>
</tr>
</tbody>
</table>
There are two loci near $z = 1$ that are very insensitive to the variation of $K_1$.

Figure 2-5. Root locus plot of simplified digital I^P^S with $T = 0.08$ sec and $K_1$ varies.
There are two root loci near \( z = 1 \) that are very insensitive to the variation of \( K_1 \).

Figure 2-6. Root locus plot of simplified digital IPS with \( T = 0.1 \) sec and \( K_1 \) varies.
\[ T = 0.1 \]

<table>
<thead>
<tr>
<th>( K_1 )</th>
<th>ROOTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 0.990937 + j0.522871 )</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>( 0.860648 + j0.495624 )</td>
</tr>
<tr>
<td>( 5 \times 10^6 )</td>
<td>( -0.417697 + j0.716370 )</td>
</tr>
<tr>
<td>( 7 \times 10^6 )</td>
<td>( -0.526164 + j0.938274 )</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>( -0.613794 + j1.17600 )</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>( -0.922282 + j3.784949 )</td>
</tr>
</tbody>
</table>

The linear digital IPS control system shown by the block diagram of Figure 2-1, with the system parameters specified in Chapter 1, was simulated on a digital computer. The time response of \( e(t) \) when the initial value of \( e(t) \) is one, is obtained. Figure 2-7 illustrates the responses of \( e(t) \) for \( T = 0.01 \), \( K_1 = 5 \times 10^6 \), and \( T = 0.1 \), \( K_1 = 10^5 \) and \( K_1 = 10^6 \). Similar to the continuous-data IPS control system analyzed in Chapter 1, the time response of the digital IPS is controlled by the closed-loop poles which are very near the \( z = 1 \) point in the \( z \)-plane. Therefore, when the sampling period \( T \) and the value of \( K_1 \) change within the stable limits, the system response will again be characterized by the long time in reaching zero as time approaches infinity.

The results show that as far as the linear simplified model is concerned, the sampling period can be as large as 0.1 second, and the digital IPS system is still stable for \( K_1 \) less than \( 10^7 \).
Figure 2-7: Responses of $e(t)$ of simplified digital IRS control system.
3. Analysis of Continuous-Data IPS Control System With Wire Cable Torque Nonlinearity

In this chapter we will conduct a stability analysis of the simplified IPS control system with the nonlinear characteristics of the torques caused by the combined effect of the flex pivot of the gimbal and by wire cables, for transmitting power, etc., across the gimbal to the experiment, as shown in Figure 3-1.

The flex pivot torque disturbance has been modeled as a linear spring with slope $K_{FP}$, as shown in Figure 3-2. The wire torque disturbance is modeled as a nonlinear spring, with a slope of $K_{WT}$, as shown in Figure 3-3. The total torque disturbance associated with the flex pivot and wire cables is summed up as shown in Figure 3-4. The continuous-data IPS control system with the nonlinear torque characteristics is shown in Figure 3-5.

The objective of the analysis in this chapter is to study the condition of sustained oscillation of the nonlinear IPS control system shown in Figure 3-5.

The Describing Function of the Wire-Cable Nonlinearity

It was pointed out in the above discussion that the wire cable nonlinearity can be modeled by either the arrangement shown in Figure 3-4(a) or Figure 3-4(b). For the model of Figure 3-4(a), a relay characteristic is present between $\varepsilon$ and $T_C$, and, in addition, a linear gain of $K_{WT} + K_{FP}$ exists between $\varepsilon$ and $T_C$, where $K_{FP}$ denotes the gain constant due to the flex pivot torque. Using this model, one can conduct a describing function analysis with the relay nonlinearity, but the linear system model is altered by the addition of the branch with the gain of $K_{WT} + K_{FP}$. However, careful examination of the block diagram of Figure 3-5 reveals that the branch with
Figure 3-1. Experiment with wire cables
Figure 3-2. Flex-pivot torque characteristic.

Figure 3-3. Wire cable torque characteristic.
Figure 3-4. Combined flex pivot and wire cables torque characteristics.
Figure 3-5. IPS control system with the nonlinear flex pivot and wire cable torque characteristics.
the gain of $K_{WT} + K_{FP}$ is parallel to the branch with the gain $K_0$ which has a magnitude of $8 \times 10^5$ N-m/rad. Since the value of $K_{WT}$ lies between 0.25 and 25 N-m/rad, and the maximum value of $K_{FP}$ is in the order of several hundred, it is apparent that the value of $K_0$ will be predominant on the system performance. This means that the linear transfer function will not change appreciably by the variation of the values of $K_{WT}$ and $K_{FP}$.

Prediction of Self-Sustained Oscillations With the Describing Function Method

From Figure 3-5 the equivalent characteristic equation of the nonlinear system is written as

$$1 + N(\dot{\epsilon})G_{\text{eq}}(s) = 0$$

(3-1)

where $N(\dot{\epsilon})$ is the describing function of the relay characteristic shown in Figure 3-4(a), and is given by

$$N(\dot{\epsilon}) = \frac{4H_{WT}}{\pi \dot{\epsilon}}$$

(3-2)

The transfer function $G_{\text{eq}}(s)$ is derived from Figure 3-5,

$$G_{\text{eq}}(s) = \frac{0.0013946(s^4 + 0.0012528s^3 + 0.0036846s^2)}{\Delta(s)}$$

(3-3)

where

$$\Delta(s) = s^5 + 16.704s^4 + 222.6s^3 + (2.781 \times 10^{-4}K_1 + 1.706)s^2$$

$$+ (4.11 + 1.747 \times 10^{-6}K_1)s + 513.855 \times 10^{-8}K_1$$

(3-4)

A necessary condition of self-sustained oscillation is

$$G_{\text{eq}}(s) = -1/N(\dot{\epsilon})$$

(3-5)

Figure 3-6 shows the frequency response plots of $G_{\text{eq}}(s)$ for $K_1 = 10^5$, $10^6$, and $10^7$, as a function of $\omega$, and the trajectory of $-1/N(\dot{\epsilon}) = -\frac{\pi \dot{\epsilon}}{4H_{WT}}$. 
Figure 3-6. Frequency response and describing function plots for self-sustained oscillations analysis.
the latter lies on the \(-180^\circ\) axis for all combinations of magnitudes of \(\varepsilon\) and \(H_{WT}\). Figure 3-6 shows that for each value of \(K_1\) there are two equilibrium points, one stable and the other unstable. For instance, for \(K_1 = 10^6\), the \(G_{eq}(s)\) curve intersects the \(-180^\circ\) axis at \(\omega = 0.16\) rad/sec and \(\omega = 0.05\) rad/sec. The equilibrium point that corresponds to \(\omega = 0.16\) rad/sec is a stable equilibrium point, whereas \(\omega = 0.05\) rad/sec represents an unstable equilibrium point. The stable solutions of the sustained oscillations for \(K_1 = 10^5, 10^6,\) and \(10^7\) are tabulated below:

<table>
<thead>
<tr>
<th>(K_1)</th>
<th>(\dot{\varepsilon}/H_{WT}) (rad/sec)</th>
<th>(\varepsilon/H_{WT}) (rad) (arc-sec)</th>
<th>(\omega) (rad/sec)</th>
<th>(T_e) (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^7)</td>
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The conclusion is that self-sustained-oscillations may exist in the nonlinear continuous-data control system.

Since the value of \(K_o\) is very large, the effect of using various values of \(K_{WT}\) and \(K_{FP}\) within their normal ranges would not be noticeable. Changing the value of \(H_{WT}\) has a one-to-one effect on the amplitudes of oscillations of \(\varepsilon\) and \(\dot{\varepsilon}\).

Digital Computer Simulation of the Continuous-Data Nonlinear IPS Control System

Since the transient time duration of the IPS control system is exceedingly long, it is extremely time consuming and expensive to verify the self-sustained oscillation by digital computer simulation. Several digital computer simulation runs indicated that the transient response of the IPS system does not die out after many minutes of real time simulation. It should be pointed out that the
describing function solution simply gives a sufficient condition for self-sustained oscillations to occur. The solutions imply that there is a certain set of initial conditions which will induce the indicated self-sustained oscillations. However, in general, it may be impractical to look for this set of initial conditions, especially if the set is very small. It is entirely possible that a large number of simulation runs will result in a totally stable situations.

Figure 3-7 illustrates a section of the time response of $\varepsilon(t)$ from $t = 612$ sec to 692 sec. The initial conditions are $\varepsilon(0) = 10^{-4}$ and $\varepsilon(0) = 10^{-5}$, and all other initial conditions are set to zero; $K_1 = 10^5$, $H_{WT} = 1$, and $K_{WT} + K_{FP} = 100$. It was mentioned earlier that the system is not sensitive to the values of $K_{WT}$ and $K_{FP}$. From Figure 3-7 it is seen that the period of the oscillation is 48 seconds, or 0.13 rad/sec, which is very close to the predicted value.
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**Figure 3-7.** A section of the response of $\varepsilon(t)$, $K_i = 105$. 
4. Analysis of the Digital IPS Control System With Wire Cable Torque Nonlinearity

In this chapter, we will conduct a stability analysis of the simplified digital IPS control system with nonlinear characteristics of the torques caused by the combined effect of the flex pivot of the gimbal and wire cables.

The analysis used here is the discrete describing function which will give sufficient conditions on self-sustained oscillations in nonlinear digital systems. The block diagram of the nonlinear digital IPS system is shown in Figure 4-1. For mathematical convenience, a sample-and-hold is inserted at the input of the nonlinear element.

A signal flow graph of the system in Figure 4-1 is drawn in Figure 4-2. The z-transforms of the variables in Figure 4-2 are written,

\[ \Theta_A(z) = -G_2(z)T_c(z) \]  (4-1)

\[ \Omega_A(z) = -G_1(z)T_c(z) \]  (4-2)

\[ \dot{\epsilon}(z) = -G_h(z)T_c(z) \]  (4-3)

\[ T_c(z) = K_i\Omega_A(z) + (K_o + \frac{K_iT}{z - 1})\Theta_A(z) - (K_{WT} + K_{FP})G_3(z)T_c(z) + N(z)\dot{\epsilon}(z) \]  (4-4)

where

\[ G_1(z) = \frac{3}{6} \left[ K_7 \frac{G_h(s)}{s} (\Delta_1 - K_4) \right] \]

\[ G_2(z) = \frac{3}{6} \left[ K_7 \frac{G_h(s)}{s^2} (\Delta_1 - K_4) \right] \]

\[ G_3(z) = \frac{3}{6} \left[ K_7 \frac{G_h(s)}{s^2} \Delta_1 \right] \]

\[ G_h(s) = \frac{1 - e^{-Ts}}{s} \]

\[ \Delta_1 = 1 + K_2s^{-1} + K_3s^{-2} \]
Figure 4-1. Digital IPS control system with the nonlinear flex pivot and wire cable torque characteristics.
Figure 4.2. Signal flowgraph of the digital IPS control system.
\[ A = (1 - K_4 K_6) + K_2 s^{-1} + K_3 s^{-2} \]

\[ N(z) = \text{discrete describing function of ideal relay } (+H_{WT}, 0, -H_{WT}) \]

Since \( K_2 \) and \( K_3 \) are very small, approximations lead to

\[ G_1(z) \approx \frac{2.79 \times 10^{-4} T}{z - 1} \]
\[ G_2(z) \approx \frac{2.79 \times 10^{-4} T^2 (z + 1)}{2(z - 1)^2} \]
\[ G_3(z) \approx \frac{0.000697 T^2 (z + 1)}{2(z - 1)^2} \]
\[ G_4(z) \approx \frac{0.0001394 T}{z - 1} \]

Equations (4-1) through (4-4) lead to the sampled flow graph of Figure 4-3, from which we have the characteristic equation,

\[ \Delta(z) = 1 + K_1 G_1(z) + \left( K_0 + \frac{K_1 T}{z-1} \right) G_2(z) + (K_{WT} + K_{FP}) G_3(z) + G_4(z) N(z) \quad (4-5) \]

Equating \( \Delta(z) \) to zero, the equivalent transfer function \( G_{eq}(z) \) is obtained,

\[ G_{eq}(z) = \frac{G_4(z)}{1 + K_1 G_1(z) + \left( K_0 + \frac{K_1 T}{z-1} \right) G_2(z) + (K_{WT} + K_{FP}) G_3(z)} \quad (4-6) \]

The intersect between \( G_{eq}(z) \) and \(-1/N(z)\) gives the condition of self-sustained oscillations in the digital IPS control system.

Figure 4-4 shows the \( G_{eq}(z) \) plots for various values of \( N > 2 \) for \( K_1 = 10^5 \). In this case, it has been assumed that the periods of oscillations are integral multiples of the sampling period \( T \). Therefore, if \( T_c \) denotes the period of oscillation, \( T_c = NT \), where \( N \) is a positive integer \( \geq 2 \).

From Figure 4-4 it is seen that when \( N \) is very large and \( T \) is very small, \( G_{eq}(z) \) approaches \( G_{eq}(s) \). However, for relatively large sampling periods,
Figure 4-3. Sampled signal flow graph of the digital IPS control system.

$G_{eq}(z)$ does not approach to $G_{eq}(s)$ even for very large $N$. This indicates the fact that digital simulation of the continuous-data IPS can only be carried out accurately by using extremely-small sampling periods.

The critical regions for $-1/N(z)$ of an ideal relay are a family of cylinders in the gain-phase coordinates, as shown in Figure 4-5. These regions extend from $-\infty$ db to $+\infty$ db, since the dead zone of the ideal relay is zero. For $N = 2$, the critical region is a straight line which lies on the $-180^0$ axis. The widest region is for $N = 4$, which extends from $-225^0$ to $-135^0$. Therefore, any portions of the $G_{eq}(z)$ loci which do not lie in the critical regions will correspond to stable operations.

It is interesting to note from Figure 4-4 that the digital IPS system has the tendency to oscillate at very low and very high sampling periods, but there is a range of sampling periods for which self-sustained oscillations can be completely avoided.
Figure 4-4. Frequency response plots of $G_{eq}(s)$ for various values of $N$ for $K = 10^5$. The plots illustrate the range of frequencies and phase shifts for different values of $N$. The graph shows the relationship between the frequency response and the number of poles $N$. The phase and magnitude plots are used to analyze the system's stability and behavior over different frequencies.
Referring to Figure 4-4 it is noticed that when the sampling period $T$ is very small, ($T < 10^{-2}$ sec approximately), the loci of $G_{eq}(z)$ for $N = 2, 3, \text{ and } 4$ lie in their respective critical regions, and self-sustained oscillations characterized by these modes are possible. However, since the actual period of these oscillations are so small, being 2, 3, or 4 times the sampling period which is itself less than 0.01 sec, the steady-state oscillations are practically impossible to observe on a digital computer simulation unless the print-out interval is made very small. This may explain why it was difficult to pick up a self-sustained oscillation in the digital computer simulation of the continuous-data IPS system, since a digital computer simulation is essentially a sampled-data analysis.

When the sampling period is large ($T > 9$ sec approximately), the digital IPS system may again exhibit self-sustained oscillations, as shown by the $G_{eq}(z)$ loci converging toward the $-180^\circ$ axis as $T$ increases. However, the $G_{eq}(z)$ loci of Figure 4-4 show that there is a midrange of $T$ for which the digital system is stable. The $G_{eq}(z)$ locus for $N = 2000$ actually represents the locus for all large $N$. Therefore, for $T = 0.1$, the $G_{eq}(z)$ locus point will be outside of the critical regions, since as $N$ increases, the widths of the critical regions decrease according to

$N = \text{even}$ \hspace{1cm} width of critical region $= 2\pi/N$

$N = \text{odd}$ \hspace{1cm} width of critical region $= \pi/N$

Therefore, from the standpoint of avoiding self-sustained oscillations in the simplified digital IPS control system model, the sampling period may be chosen to lie approximately in the range of 0.01 to 1 second.
Digital Computer Simulation of the Digital Nonlinear IPS

Control System

The digital IPS control system with wire-cable nonlinearity, as shown by the block diagram of Figure 4-1, has similar characteristics as the continuous-data system, especially when the sampling period is small. The digital IPS control system of Fig. 4-1, without the sample-and-hold in front of the nonlinear element, was simulated on the IBM 360/75 digital computer. With initial conditions set for $c$ and $\dot{c}$, typical responses showed that the nonlinear digital IPS system exhibited a long oscillatory transient period. Figure 4-6 shows the beginning portion of the response of $\epsilon(t)$ for $\epsilon(0) = 10^{-3}$, $\dot{\epsilon}(0) = 10^{-4}$, $K_i = 10^5$, $T = 0.1$ sec, $H_{WT} = 1$, $K_{WT} + K_{FP} = 100$. Figure 4-7 shows the same response over the period of 765 sec to 1015 seconds. Figure 4-8 gives the response of $\epsilon(t)$ for the time duration of 1530 sec to 1725 sec, and it shows that the response eventually settles to nonoscillatory and finally should be stable.
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Figure 4.6. Beginning portion of \( e(t) \) of digital I.P. system.
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Figure 4-7. Response of $\epsilon(t)$ of digital IPS control system. $T = 0.1$ sec, $H_M = 1$, $K_I = 10^5$, $K_{Df} + K_{SP} = 100$.  

$\epsilon(t)$ is the error signal.
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Figure 4.8: Response of $\varepsilon(t)$ of digital IFS control system.

$T = 0.1$ sec, $K_I = 1$, $K_T = 10^5$, $K_{IF} + K_{FP} = 100$. 
5. Gross Quantization Error Study of the Digital IPS Control System

This chapter is devoted to the study of the effect of gross quantization in the linear digital IPS control system. The nonlinear characteristics of the torques caused by the combined effect of the flex pivot of the gimbal and wire cables are neglected.

Since the quantization error has a maximum bound of +h/2, with h as the quantization level, the "worst" error due to quantization in a digital system can be studied by replacing the quantizers in the system by an external noise source with a signal magnitude of ±h/2.

Figure 5-1 shows the simplified digital IPS control system with the quantizers shown to be associated with the sample-and-hold operations. The quantizers in the displacement Ω_A, rate Ω_A, and torque T_c, channels are denoted by Q_o, Q_i, and Q_T, respectively. The quantization levels are represented by h_o, h_i, and h_T, respectively.

Figure 5-2 shows the signal flow graph of the digital IPS system with the quantizers represented as operators on digital signals. Treating the quantizers as noise sources with constant amplitudes of ±h_o/2, ±h_i/2, and ±h_T/2, we can predict the maximum errors in the system due to the effect of quantization. The following equations are written from Figure 5-2. Since the noise signals are constants, the z-transform relations include the factor z/(z − 1).

\[ T_c(z) = K_1Ω_A(z) + K_1\left(\frac{h_1}{2} \frac{z}{z - 1}\right) + \left(\frac{h_0}{2} \frac{z}{z - 1}\right) + Ω_A(z) \] (5-1)

\[ Ω_A(z) = \mathcal{Z}\left\{\frac{-K_iG_h(s)Δ_1(s)}{sΔ(s)} + \frac{K_iG_h(s)}{sΔ(s)}\right\}\left[T_c(z) + \frac{h_T}{2} \frac{z}{z - 1}\right] \] (5-2)

\[ Ω_A(z) = \mathcal{Z}\left\{\frac{-K_iG_h(s)Δ_1(s)}{s^2Δ(s)} + \frac{K_iG_h(s)}{s^2Δ(s)}\right\}\left[T_c(z) + \frac{h_T}{2} \frac{z}{z - 1}\right] \] (5-3)
Figure 5-1. Linear simplified digital IPS control system with quantization.
Figure 5-2. Signal flow graph of simplified digital IPS system with quantization.
The last two equations are written as

\[ \Omega_A(z) = \mathcal{G}_1(z) \left[ \Delta(z) + \frac{h_T}{2} \left( \frac{z}{z - 1} \right) \right] \]  \hspace{1cm} (5-4)

\[ \Theta_A(z) = \mathcal{G}_2(z) \left[ \Delta(z) + \frac{h_T}{2} \left( \frac{z}{z - 1} \right) \right] \]  \hspace{1cm} (5-5)

where

\[ \Delta_1(s) = 1 + K_2s^{-1} + K_2s^{-2} \]  \hspace{1cm} (5-6)

\[ \Delta(s) = 1 - K_4K_6 + K_2s^{-1} + K_3s^{-2} \]  \hspace{1cm} (5-7)

\[ \mathcal{G}_1(z) = \frac{2.78 \times 10^{-4} T}{z - 1} \]  \hspace{1cm} (5-8)

\[ \mathcal{G}_2(z) = \frac{2.78 \times 10^{-4} T^2(z + 1)}{2(z - 1)^2} \]  \hspace{1cm} (5-9)

Figure 5-3 gives the digital signal flow graph representation of Eqs. (5-1), (5-4), and (5-5). Using Figure 5-3 we can analyze the worst-case errors due to quantization in the steady state at any point of the IPS system.

As derived in previous chapters, the transfer functions \( \mathcal{G}_1(z) \) and \( \mathcal{G}_2(z) \) are given in Eqs. (5-8) and (5-9). The characteristic equation of the system is obtained from Figure 5-3:

\[ \Delta(z) = 1 + K_4 \mathcal{G}_1(z) + \left( K_5 + \frac{K_T}{z - 1} \right) \mathcal{G}_2(z) \]  \hspace{1cm} (5-10)

We shall now evaluate the maximum steady-state quantization errors for \( \Theta_A, \Omega_A, \) and \( T_c \) in terms of the quantization levels \( h_0, h_1, \) and \( h_T. \)

From Figure 5-3, \( T_c(z) \) is written

\[ T_c(z) = \frac{1}{\Delta} \left( h_o \left( K_0 + \frac{K_T}{z - 1} \right) + h_1 \frac{1}{2} K_1 + \frac{h_T}{2} \left( K_0 + \frac{K_T}{z - 1} \right) \mathcal{G}_2(z) \right) \frac{z}{z - 1} \]  \hspace{1cm} (5-11)

The steady-state value of \( T_c(kT) \) is given by the final-value theorem,

\[ \lim_{k \to \infty} T_c(kT) = \lim_{z \to 1} (1 - z^{-1}) T_c(z) \]  \hspace{1cm} (5-12)

Substitution of Eq. (5-11) into Eq. (5-12), we have
\[
\lim_{k \to \infty} T_c(kT) = \pm \frac{h_T}{2}
\]  \hspace{1cm} (5-13)

Similarly,
\[
\Theta_A(z) = \frac{1}{\Delta} \left\{ \frac{h_o}{2} \left( K_o + \frac{K_1 T}{z - 1} \right) + \frac{h_1}{2} K_1 + \frac{h_T}{2} \right\} G_2(z) \frac{z}{z - 1}
\]  \hspace{1cm} (5-14)

Then,
\[
\lim_{k \to \infty} \Theta_A(kT) = \lim_{z \to 1} (1 - z^{-1}) \Theta_A(z) = \pm \frac{h_o}{2}
\]  \hspace{1cm} (5-15)

\[
\Omega_A(z) = \frac{1}{\Delta} \left\{ \frac{h_o}{2} \left( K_o + \frac{K_1 T}{z - 1} \right) + \frac{h_1}{2} K_1 + \frac{h_T}{2} \right\} G_1(z) \frac{z}{z - 1}
\]  \hspace{1cm} (5-16)

\[
\lim_{k \to \infty} \Omega_A(kT) = \lim_{z \to 1} (1 - z^{-1}) \Omega_A(z) = 0
\]  \hspace{1cm} (5-17)

Therefore, we conclude that the maximum error in \( T_c \) due to quantization is \( \pm \frac{h_T}{2} \) and is not affected by the other two quantizers. The quantizer in the \( \Theta_A \) channel affects only \( \Theta_A(z) \) in a one-to-one relation. The quantizer in the \( \Omega_A \) path does not affect either \( \Theta_A, \Omega_A \), or \( T_c \).

Figure 5-3. Digital signal flow graph of the simplified digital IPS with quantization.
6. Describing Function Analysis of the Quantization Effects of the Digital IPS Control System

In this section the effects of quantization in the digital IPS control system are investigated with respect to self-sustained oscillations.

Since the quantizers represent nonlinear characteristics, it is natural to expect that the level of quantization together with the selection of the sampling period may cause the system to enter into undesirable self-sustained oscillations.

The digital IPS control system with quantizers located in the $\Theta_A$, $\Omega_A$, and $T_c$ channels is shown in Figure 5-1. We shall consider the effects of only one quantizer at a time, since the discrete describing function method is used.

With reference to the signal flow graph of Figure 5-2, which contains all the quantizers, we can find the equivalent characteristic equation of the system when each quantizer is operating alone. Then, the equivalent linear transfer function that each quantizer sees is derived for use in the discrete describing function analysis.

**Quantizer in the $\Theta_A$ Channel**

Let the quantizer in the $\Theta_A$ channel be denoted by $Q_0$, as shown in Figure 5-2, and neglect the effects of the other quantizers. Let the discrete describing function of $Q_0$ be denoted by $Q_0(z)$. From Figure 5-2 the following equations are written:

\[
T_c(z) = \left[ K_0 + \frac{K_1 T}{z - 1} \right] Q_0(z)\Theta_A(z) + K_1 \Omega_A(z) \tag{6-1}
\]

\[
\Theta_A(z) = 2^\frac{-\Delta_1 G_h(s)}{s^2 \Delta} + \frac{K_4 K_2 G_h(s)}{s^2 \Delta} T_c(z)
\]

\[
= G_2(z) T_c(z) \tag{6-2}
\]
\[ \Omega_A(z) = \left( -\frac{K_2 G_1(s) \Delta_1(s)}{s \Delta} + \frac{K_4 K_6 G_1(s)}{s \Delta} \right) T_c(z) \]
\[ = -G_1(z) T_c(z) \]  
(6-3)

where
\[ \Delta_1(s) = 1 + K_2 s^{-1} + K_3 s^{-2} \]  
(6-4)
\[ \Delta(s) = 1 - K_4 K_6 + K_2 s^{-1} + K_3 s^{-2} \]  
(6-5)
\[ G_1(z) = \frac{2.78 \times 10^{-4} T}{z - 1} \]  
(6-6)
\[ G_2(z) = \frac{2.78 \times 10^{-4} T^2 (z + 1)}{2(z - 1)^2} \]  
(6-7)

A digital signal flow graph portraying Eqs. (6-1), (6-2), and (6-3) is shown in Figure 6-1. The characteristic equation of the system is written directly from Figure 6-1.

\[ \Delta(z) = 1 + G_2(z) \left( K_0 + \frac{K_1 T}{z - 1} \right) Q_0(z) + K_1 G_1(z) = 0 \]  
(6-8)

To obtain the equivalent transfer function that \( Q_0(z) \) sees, we divide both sides of Eq. (6-8) by the terms that do not contain \( Q_0(z) \). We have

\[ 1 + \frac{G_2(z) \left( K_0 + \frac{K_1 T}{z - 1} \right) Q_0(z)}{1 + K_1 G_1(z)} = 0 \]  
(6-9)

Thus,

\[ G_{eqo}(z) = \frac{G_2(z) \left( K_0 + \frac{K_1 T}{z - 1} \right)}{1 + K_1 G_1(z)} \]  
(6-10)

Quantizer in the \( \Omega_A \) Channel

Using the same method as described in the last section, let \( Q_1(z) \) denote the discrete describing function of the quantizer \( Q_1 \). When only \( Q_1 \) is considered effective, the following equations are written directly from Figure 5-2.

\[ \Theta_A(z) = -G_2(z) T_c(z) \]  
(6-11)
\[ \Omega_A(z) = -G_1(z) T_c(z) \]  
(6-12)
\[ T_c(z) = \left( K_0 + \frac{K_1 T}{z - 1} \right) Q_A(z) + K_1 Q_1(z) A(z) \]  
(6-13)
Figure 6-1. Digital signal flow graph of IPS when $Q_o$ is in effect.

Figure 6-2. Digital signal flow graph of IPS with $Q_1$ if effect.
The digital signal flow graph for these equations is drawn as shown in Figure 6-2. The characteristic equation of the system is
\[ \Delta(z) = 1 + K_1 G_1(z) Q_1(z) + G_2(z) \left( K_o + \frac{K_1 T}{z - 1} \right) = 0 \]  
(6-14)

Thus, the linear transfer function \( Q(z) \) sees is
\[ G_{eq1}(z) = \frac{K_1 G_1(z)}{1 + G_2(z) \left( K_o + \frac{K_1 T}{z - 1} \right)} \]  
(6-15)

**Quantizer in the \( T_c \) Channel**

If \( Q_T \) is the only quantizer in effect, the following equations are written from Figure 5-2.
\[ \Theta_A(z) = -G_2(z) Q_T(z) T_c(z) \]  
(6-16)
\[ \Omega_A(z) = -G_1(z) Q_T(z) T_c(z) \]  
(6-17)

\[ T_c(z) = \left( K_o + \frac{K_1 T}{z - 1} \right) \Theta_A(z) + K_1 \Omega_A(z) \]  
(6-18)

The digital signal flow graph for these equations is drawn in Figure 6-3. The characteristic equation of the system is
\[ \Delta(z) = 1 + K_1 G_1(z) Q_T(z) + G_2(z) Q_T(z) \left( K_o + \frac{K_1 T}{z - 1} \right) = 0 \]  
(6-19)

The linear transfer function seen by \( Q_T(z) \) is
\[ G_{eqT}(z) = K_1 G_1(z) + G_2(z) \left( K_o + \frac{K_1 T}{z - 1} \right) \]  
(6-20)

The discrete describing function of quantizers has been derived in a previous report. By investigating the trajectories of the linear equivalent transfer function of Eqs. (6-10), (6-15), and (6-20) against the critical regions of the discrete describing function of the quantizer, the possibility of self-sustained oscillations due to quantization in the digital IPS system may be determined.
Let $T_c$ denote the period of the self-sustained oscillation, and

$$T_c = NT$$

where $N$ is a positive integer $> 2$. $T$ represents the sampling period in seconds.

When $N = 2$, the periodic output of the quantizer can have an amplitude of $kh$, where $k$ is a positive integer and $h$ is the quantization level. The critical region of the quantizer for $N = 2$ is shown in Figure 6-4. For $N = 3$, the periodic output of the quantizer is a pulse train which can be described by the mode $(k_0, k_1, k_2)$, where $k_0h$, $k_1h$, and $k_2h$ are the magnitudes of the output pulses during one period. Similarly, for $N = 4$, the modes are described by $(k_0, k_1)$.

The critical regions of the quantizer for $N = 3$ and $N = 4$ are shown in Figures 6-5 and 6-6, respectively. Figure 6-7 illustrates the frequency loci of $G_{eqo}(z)$, $G_{eq1}(z)$, and $G_{eqT}(z)$ of Eqs. (6-10), (6-15), and (6-20), respectively, together with the corresponding critical region of the quantizer. The value of $K_1$ is equal to $10^5$ in this case. It so happens that the frequency loci of these transfer functions are almost identical. Notice that the frequency loci for the range of $0.06 \leq T \leq 0.18$ sec overlap with the critical region. Therefore,
self-sustained oscillations of the mode $N = 2$ are possible for the sampling period range of $0.06 \leq T \leq 0.18$ sec. It turns out that the frequency loci for $N = 2$ are not sensitive to the value of $K_I$, so that the same plots of Figure 6-7 and the same conclusions apply to $K_I = 10^6$ and $K_I = 10^7$.

Figures 6-8, 6-9, and 6-10 illustrate the frequency loci of $G_{eqo}(z)$, $G_{eql}(z)$, and $G_{eqT}(z)$ for $N = 3$ and for $K_I = 10^5$, $10^6$, and $10^7$, respectively. From Figure 6-8 we notice that for $K_I = 10^5$ the frequency loci do not intersect with the critical region for any sampling period. Thus, for $K_I = 10^5$ the $N = 3$ mode of oscillations cannot occur.

For $K_I = 10^6$, Figure 6-9 shows that sustained oscillations for $N = 3$ would not occur for $T < 0.075$ sec and large values of $T$. For $K_I = 10^7$, Figure 6-10 shows that the critical value of $T$ is increased to approximately 0.085 sec.

For $N = 4$, Figures 6-11, 6-12, and 6-13 illustrate the critical region and the frequency loci for $K_I = 10^5$, $10^6$, and $10^7$, respectively. In this case, self-sustained oscillations are absent for $K_I = 10^5$ for any sampling period. For $K_I = 10^6$, the critical sampling period is approximately 0.048 sec for quantizers $Q_1$ and $Q_T$, whereas for $Q_o$ the critical $T$ is 0.07 sec. The stability condition is improved when $K_I$ is increased to $10^7$, for $Q_1$ and $Q_T$, the critical values of $T$ are 0.05 sec and 0.055 sec, respectively; for $Q_o$ it is 0.09 sec.

As $N$ increases, the critical regions shrinks toward the negative real axis of the complex plane, and at the same time the frequency loci move away from the negative real axis. Thus, the $N = 2$, 3, and 4 cases represent the worst possible conditions of self-sustained oscillations in the system.

The conclusion of this analysis is that the sampling period of the IPS system should be less than 0.048 second, in order to avoid self-sustained
oscillations excited by the quantizer nonlinearities described in these sections.

Figure 6-4. Critical region of $-1/Q(z)$ of quantizer for $N = 2$. 
Figure 6-5. Critical regions of a quantizer for $N = 3$.
Figure 6-6: Critical regions of a quantizer for N = 4.
Figure 6-8. Frequency loci $G_{eq}(z)$, $G_{eqT}(z)$, $G_{eqT}(z) \text{ and critical region for } N = 3 \text{ and } K = 10^5$. 

Critical region of $-1/Q(z)$ for $N = 3$. 

Legend: 
- $G_{eq}(z)$ 
- $G_{eqT}(z)$ 
- $G_{eqT}(z)$
Critical regions for $-1/0(z)$ for $N = 3$ and $K_1 = 10^6$.
Figure 6-10: Frequency loci of $g_{eq}(z)$, $g_{eq1}(z)$, and $g_{eq2}(z)$ and critical region of $-1/Q(z)$ of quantizer for $N=3$ and $K=107$. 

Critical region of $-1/Q(z)$ for $N = 3$.
Critical region of \(-I/0(z)\) for \(N = 4\).
Figure 5-12. Frequency loci $G_{eq}(z)$, $G_{eq}(z)$, $G_{eq}(z)$, and the critical region of $\frac{1}{Q(z)}$ for $N = 4$ and $K = 10$.
Figure 6.13. Frequency loci of $G_{eq}(z)$, $G_{eq}(z)$, $G_{eq}(z)$, and the critical region of $\frac{1}{Q(z)}$ of quantizer for $N = 4$ and $K = 10$
7. Digital Computer Simulation of the Digital IPS Control System With Quantization

The digital IPS control system with quantizers has been studied in Chapter 5 and 6 using the gross quantization error and the describing function methods. In this chapter the effect of quantization in the digital IPS is studied through digital computer simulation. The main purpose of the analysis is to support the results obtained in the last two chapters.

The linear IPS control system with quantization is modeled by the block diagram of Figure 5-1, and it is not repeated here. The quantizers are assumed to be located in the $T_c$, $\Theta_A$, and $\Omega_A$ channels. From the gross quantization error analysis it was concluded that the maximum error due to quantization at each of the locations is equal to the quantization level at the point, and it is not affected by the other two quantizers. In Chapter 6 it is found that the quantizer in the $T_c$ channel seems to be the dominant one as far as self-sustained oscillations are concerned. It was also found that the self-sustained oscillations due to quantization may not occur for a sampling period or approximately 0.05 sec or less.

A large number of computer simulation runs were conducted with the quantizer located at the $T_c$ channel. However, it was difficult to induce any periodic oscillation in the system due to quantization alone. It appears that the signal at the output of the quantizer due to an arbitrary initial condition will eventually vary between $+\frac{h_T}{2}$ and $-\frac{h_T}{2}$ indefinitely in a random fashion. This points to the fact that the system the quantizer sees is not a low-pass filter so that the describing function method becomes inaccurate. However, the results still substantiates the results obtained in Chapter 5; i.e., the quantization error is $+\frac{h_T}{2}$. Figure 7-1 and 7-2 show typical responses of the simulation runs for $K_1 = 10^5$ and $T = 0.08$ sec. When $K_1 = 10^7$ the system is unstable.
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Figure 7-1. Response of output of quantizer at $T_c = \frac{h_T}{2} = 0.001$, $K_l = 10^5$, $T = 0.08$ sec.

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*Figure 7-2 (continued).*
8. Modeling of the Continuous-Data IPS Control System With Wire Cable Torque and Flex Pivot Nonlinearities

In this chapter the mathematical model of the IPS Control System is investigated when the nonlinear characteristics of the torques caused by the wire cables and the friction at the flex pivot of the gimbal are considered. In Chapter 3 the IPS model includes the wire cable disturbance which is modeled as a nonlinear spring (Figure 3-3). The combined effect of the wire cable and flex pivot is also modeled as a nonlinear spring characteristics as shown in Figure 3-4.

In this chapter the Dahl model is used to represent the ball bearing friction torque at the flex pivot of the gimbal, together with the nonlinear characteristics of the wire cables.

Figure 8-1 shows the block diagram of the combined flex pivot and wire cables torque characteristics, where it is assumed that the disturbance torque at the flex pivot is described by the Dahl dry friction model. The combined torque is designated $T_N$.

Figure 8-1. Block diagram of combined nonlinear torque characteristics of flex pivot and wire cables of LST.
Figure 8-2 shows the simplified IPS control system with the nonlinear flex pivot and wire cable torque characteristics.

The nonlinear spring torque characteristics of the wire cable are described by the following relations:

\[ T_{WC}(\varepsilon) = H_{WT} SGN(\dot{\varepsilon}) + K_{WT} \varepsilon \]  
(8-1)

where \( H_{WT} \) is in N-m, \( K_{WT} \) in N-m/rad, \( \varepsilon \) is in rad, and \( T_{WC}(\varepsilon) \) in N-m.

Equation (8-1) is also equivalent to

\[ T^+(\varepsilon) = H + K \]  
(8-2)
\[ T^-_{WC}(\varepsilon) = -H_{WT} + K_{WT} \varepsilon \]  
(8-3)

It has been established that the solid rolling friction characteristic can be approximated by the nonlinear relation,

\[ \frac{dT_{FP}(\varepsilon)}{d\varepsilon} = \gamma(T_{FP1} - T_{FP0})^i \]  
(8-4)

where

\[ i = \text{positive number} \]
\[ \gamma = \text{positive constant} \]
\[ T_{FP1} = T_{FP} SGN(\dot{\varepsilon}) \]
\[ T_{FP0} = \text{saturation level of } T_{FP} \]

For \( i = 2 \), Equation (8-4) is integrated to give

\[ \varepsilon + C_1 = \frac{-1}{\gamma(T_{FP1}^+ - T_{FP0})} \]  
(8-5)
\[ \varepsilon + C_2 = \frac{-1}{\gamma(T_{FP1}^- + T_{FP0})} \]  
(8-6)

where \( C_1 \) and \( C_2 \) are constants of integration, and

\[ T_{FP}^+ = T_{FP} \]  
(8-7)
\[ T_{FP}^- = T_{FP} \]  
(8-8)

The constants of integration are determined at the initial point where
Figure 8-2. IPS Control System with the nonlinear flex pivot and wire cable torque characteristics using the Dahl solid rolling friction model.
\[ C_i = \text{initial value of } C \]
\[ T_{FPi} = \text{initial value of } T_{FP} \]

Then:
\[ C_1 = -\varepsilon_i - \frac{1}{Y(T_{FPi} - T_{FP0})} \quad \varepsilon > 0 \quad (8-9) \]
\[ C_2 = -\varepsilon_i - \frac{1}{Y(T_{FPi} + T_{FP0})} \quad \varepsilon < 0 \quad (8-10) \]

The main objective is to investigate the behavior of the nonlinear elements under a sinusoidal excitation, so that the describing function analysis can be conducted.

Let \( \varepsilon(t) \) be described by a cosinusoidal function,
\[ \varepsilon(t) = A\cos\omega t \quad (8-11) \]

Then,
\[ \dot{\varepsilon}(t) = -A\omega\sin\omega t \quad (8-12) \]

Thus,
\[ \varepsilon_i = -A \quad \varepsilon > 0 \quad (8-13) \]
\[ \varepsilon_i = A \quad \varepsilon < 0 \quad (8-14) \]

The constants of integration in Eqs. (8-9) and (8-10) become
\[ C_1 = A - \frac{1}{Y(T_{FPi} - T_{FP0})} \quad (8-15) \]
\[ C_2 = -A - \frac{1}{Y(T_{FPi} + T_{FP0})} \quad (8-16) \]

Substitution of Eqs. (8-11) and (8-15) in Eq. (8-5) and simplifying, the solution of \( T_{FP}^+ \) is
\[ \frac{T_{FP}^+}{T_{FP0}} = \frac{R}{R-1} + \frac{a^2(1 - \cos\omega t)}{2(1 - \cos\omega t) + \frac{1}{R-1}} \quad (8-17) \]

which is valid for \( \varepsilon > 0 \) or \( (2k+1)\pi \leq \omega t \leq (2k+2)\pi, \ k = 0, 1, 2, \ldots \)
\[ a = 2\gamma AT_{FP0} \quad (8-18) \]
\[ R = -\frac{1}{a} + \sqrt{\frac{a^2 + 1}{a^2}} = \frac{T_{FPi}}{T_{FP0}} \quad (8-19) \]

Similarly, for \( \varepsilon \leq 0 \), using Eqs. (8-11) and (8-16) in Eq. (8-6), we have
\[
\frac{T_{FP}^-}{T_{FP}} = \frac{R}{R+1} - \frac{a}{2}(1 - \cos \omega t)
\]

which is valid for \(2k\pi \leq \omega t \leq (2k+1)\pi\), \(k = 0, 1, 2, \ldots\).

The expressions for \(T_{FP}^+\) and \(T_{FP}^-\) obtained in Eqs. (8-17) and (8-20) together with those of \(T_{WC}(\varepsilon)\) in Eqs. (8-2) and (8-3) are useful for the derivation of the describing function of the combined nonlinearity of the wire cable and the flex pivot characteristics.

The torque disturbance due to the two non-linearities is modeled by

\[
T_N^+ = T_{WC}^+ + T_{FP}^+
\]

\[
= H_{WT} + K_{WT}\cos \omega t + T_{FP0} \frac{R}{R-1} + \frac{a}{2}(1 - \cos \omega t)
\]

\(2k\pi \leq \omega t \leq (2k+2)\pi\), \(k = 0, 1, 2, \ldots\)

\[
T_N^- = T_{WC}^- + T_{FP}^-
\]

\[
= -H_{WT} + K_{WT}\cos \omega t + T_{FP0} \frac{R}{R+1} - \frac{a}{2}(1 - \cos \omega t)
\]

Figure 8-3 shows the \(T_{FP}/T_{FP0}\) versus \(\varepsilon/A\) characteristics for several values of \(A\) when the input is the cosinusoidal function of Eq. (8-11).

Figure 8-4 shows the normalized \((3T_{FP} + T_{WC})/(3T_{FP0} + K_{WT} + H_{WT})\) versus \(\varepsilon/A\) for several typical combinations of \(A, H_{WT}\), and \(K_{WT}\).
Figure 8-3. Normalized flex pivot torque (Dahl model) versus $\varepsilon/A$ for IPS with cosine function input.
Figure 8-4. Normalized flex pivot plus wire cable torques versus $\epsilon/A$ for IPS with cosine function input.
9. Describing Function of the Combined Wire Cable and Flex Pivot Nonlinearities

Figure 8-1 shows that the disturbance torques due to the wire cable and the gimbal flex pivot are additive. Thus,

\[ T_N = T_{WC} + T_{FP} \quad (9-1). \]

For the cosinusoidal input of Eq. (8-11), let the describing function of the wire cable nonlinearity be designated as \( N_{WC}(A) \) and that of the flex pivot nonlinearity be \( N_{FP}(A) \). Then, in the frequency domain, the total disturbance torque is

\[ T_N(\omega) = N_{FP}(A)\epsilon(\omega) + N_{WC}(A)\epsilon(\omega) + (N_{FP}(A) + N_{WC}(A))\epsilon(\omega) \quad (9-2) \]

Thus, let \( N(A) \) be the describing function of the combined flex pivot and wire cable nonlinear characteristics,

\[ N(A) = N_{FP}(A) + N_{WC}(A) \quad (9-3) \]

The describing function of the Dahl solid friction nonlinearity has been derived elsewhere\(^6\) for the cosinusoidal input. The results is

\[ N_{FP}(A) = \frac{B_1 - jA_1}{A} \quad (9-4) \]

where

\[ A_1 = -\frac{4}{\pi} T_{FP0} + \frac{2}{\pi A Y} \ln \left( \frac{C_1 + A}{C_1 - A} \right) \quad (9-5) \]

\[ B_1 = \frac{2}{\gamma A} \left( \frac{C_1}{C_1^2 - A^2} - 1 \right) \quad (9-6) \]

The describing function of the wire cable nonlinearity is derived as follows.

For a cosinusoidal input the input-output waveform relations are shown in Figure 9-1. The wire cable torque due to the cosinusoidal input, over one period, is

\[ T_{WC}(t) = K_{WT} \cos \omega t - H_{WT} \quad 0 \leq \omega t < \pi \]
Figure 9-1. Input-output characteristics of wire cable torque nonlinearity.
The fundamental component of the Fourier series representation of \( T_{WC}(t) \) is

\[
T_{WC}(t) = A_1 \sin wt + B_1 \cos wt
\]

\[
= \sqrt{A_1^2 + B_1^2} \cos (wt - \phi)
\]

\[
\phi = \tan^{-1} \frac{A_1}{B_1}
\]

The coefficients \( A_1 \) and \( B_1 \) are derived as follows:

\[
A_1 = \frac{1}{\pi} \int_0^{2\pi} T_{WC}(t) \sin wt \, dt
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} (K_{WT} \cos wt - H_{WT}) \sin wt \, dt
\]

\[
= -\frac{4H_{WT}}{\pi}
\]

\[
B_1 = \frac{1}{\pi} \int_0^{2\pi} T_{WC}(t) \cos wt \, dt
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} (K_{WT} \cos wt - H_{WT}) \cos wt \, dt
\]

\[
= K_{WT} A
\]

Then,

\[
\sqrt{A_1^2 + B_1^2} = \sqrt{\left(\frac{4H_{WT}}{\pi}\right)^2 + (K_{WT} A)^2}
\]

\[
\phi = \tan^{-1} \left(\frac{-4H_{WT}}{\pi K_{WT} A}\right)
\]
The describing function of the wire cable nonlinearity is written as

\[ N_{WT}(A) = \frac{B_1 - jA_1}{A} = \frac{\sqrt{A_1^2 + B_1^2}}{A} \phi \]

\[ = \sqrt{\left(\frac{4H_{WT}}{\pi A}\right)^2 + K_{WT}^2} \quad \text{tan}^{-1} \left( \frac{4H_{WT}}{\pi K_{WT} A} \right) \]  

(9-16)

For the combined nonlinearity, the describing function is the sum of the two describing functions. However, since there are three ball bearings on the flex pivot, the final expression is

\[ N(A) = N_{WT}(A) + 3N_{FP}(A) = N_R(A) + jN_I(A) \]  

(9-17)

where

\[ N_R(A) = 3 \frac{2}{\gamma A^2} \left( \frac{C_1}{C_2 - A^2} - 1 \right) + K_{WT} \]  

(9-18)

\[ N_I(A) = 3 \left( \frac{1}{\pi A} T_{FPO} - \frac{2}{\pi A^2} \ln \left( \frac{C_1 + A}{C_2 - A} \right) \right) + \frac{4H_{WT}}{\pi A} \]  

(9-19)

Asymptotic Behavior of \(-1/N(A)\) for Very Small Values of A

The asymptotic behavior of \(-1/N(A)\) for very small values of A can be derived analytically. It can be shown that

\[ \lim_{A \to 0} N_R(A) = \gamma T_{FPO}^2 + K_{WT} \]  

(9-20)

and

\[ \lim_{A \to 0} N_I(A) = \lim_{A \to 0} \frac{4H_{WT}}{\pi A} \]  

(9-21)

Therefore,

\[ \lim_{A \to 0} \frac{-1}{N(A)} = \lim_{A \to 0} \frac{-1}{N_R(A) + jN_I(A)} \]

\[ = \frac{-1}{4H_{WT}} = 0/-270^\circ \]  

(9-22)
Asymptotic Behavior of \(-1/N(A)\) for Very Large Values of \(A\)

For very large values of \(A\) it can be shown that

\[
\lim_{A \to \infty} N_r(A) = K_{WT} \tag{9-23}
\]

and

\[
\lim_{A \to \infty} N_i(A) = 0 \tag{9-24}
\]

Then,

\[
\lim_{A \to \infty} -1/N(A) = \frac{1}{K_{WT}} \left( -180^\circ \right) \tag{9-25}
\]

Magnitude Versus Phase Plots of \(-1/N(A)\) of the Combined Nonlinearities

A digital computer program for the computation of \(N(A)\) and \(-1/N(A)\) is listed in Table 9-1. The constant \(A\) is designated as \(E\) in this program.

The parameters of the nonlinearities are:

\[
T_{FPO} = 0.00225 \text{ N-m} \\
\gamma = 9.2444 \times 10^4 \text{ (N-m-rad)}^{-1} \\
K_{WT} = 0.25 \text{ to } 100 \text{ N-m/rad} \\
H_{WT} = 0.01 \text{ to } 1 \text{ N-m}
\]

Figure 9-2 shows the plots of \(-1/N(A)\) for the nonlinearities in magnitude (db) versus phase for varies combinations of \(K_{WT}\) and \(H_{WT}\). It is seen that varying the value of \(H_{WT}\) between the range of 0.01 to 1 does not affect the curves appreciably.

Prediction of Self-Sustained Oscillations in the IPS System With the Combined Nonlinearity By Means of the Describing Function Method

The characteristic equation of the nonlinear IPS control system with the wire cable Dahl-model nonlinearities is determined from Figure 8-2.

\[
1 + N(A)G_{eq}(s) = 0 \tag{9-26}
\]

where \(N(A)\) is defined in Eq. (9-17), and
TABLE 9-1

IPS CONTINUOUS, DESCRIBING FUNCTION OF COMBINED NONLINEARITY


calculation for -1-n
complex: g, h, g, h
real*8 p, 20, pi, rad, to, gamma, estart, e, aa, p, tgfi, tgf,
1tgfp, c1, c2, a1, a1, b1, b1
real*8 ac, c
pi=3.1415926
rad=180.0*rad
hti=0.0022510
wht=1.0
z=10.0
kmt=100.0
gamma=5.2444d3
estart=1.0-10
np=5
nd=12
write(5, 100)
write(5, 101)
do 1 j=1, nd
do 1 i=1, np
e=estart*dfloat(i)*(1.0-10.0*(j-1))
aa=2.0+gamma*e*to
r=1.0*(aa)+dfloat((aa+aa+1.0).*(aa+aa))
tgfi=r*to
tgf=tgfi
tgfp=ttgfi
c1=e-1.0*(gamma*(tgfp-to))
c2=e-1.0*(gamma*(tgfn+to))
a1=1.0-4.0*to*rad+
(1.0-4.0*rad+gamma+e*to)*dfloat((c1+e)*(c2-e))
1.0*(c1-e)*(c2+e))
a2=dfloat((c1+e)*(c2-e)/(c1-e)*(c2+e))
a1=1.0-4.0*to*rad+(aa*pi+gamma+e)
b1=1.0-4.0*to*(rad+(aa*pi+gamma+e))
b1=1.0-4.0-4.0*to*(rad+(aa*pi+gamma+e))
b1=1.0-4.0-4.0*to*(rad+(aa*pi+gamma+e))
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\[ G_{eq}(s) = \frac{0.0013946(s^3 + 0.0012528s^2 + 0.0036846s)}{\Delta(s)} \]  

(9-27)

where \( \Delta(s) \) is defined in Eq. (3-4). Notice that \( G_{eq}(s) \) in Eq. (9-27) is equal to \( s^{-1} \) times the \( G_{eq}(s) \) in Eq. (3-3), since in Chapter 3 the input to \( N \) is \( \epsilon \) whereas now it is \( \varepsilon \). The frequency plots of \( G_{eq}(s) \) of Eq. (9-27) are plotted in Figure 9-2 for \( K_I = 10^5 \) and \( 10^6 \). Similar to the curves in Figure 3-6, these frequency loci for \( G_{eq}(s) \) have two equilibrium points for each curve, one stable and the other unstable. For instance, for \( K_I = 10^5 \), the \( G_{eq}(s) \) curve intersects the \(-1/N(A)\) loci at \( \omega = 0.138 \) rad/sec and \( \omega = 0.055 \) rad/sec. The equilibrium point that corresponds to \( \omega = 0.138 \) rad/sec is a stable equilibrium point, whereas \( \omega = 0.055 \) rad/sec represents an unstable equilibrium point. These results are very close to those obtained in Chapter 3 where the flex pivot is presented as a spring. Therefore, the impact of using the Dahl solid friction model is not great although all the loci are substantially different. For \( K_I = 10^6 \), the stable equilibrium point is at \( \omega = 0.16 \) rad/sec, and the unstable equilibrium point is at \( \omega = 0.05 \) rad/sec.

Figure 9-2 shows that for the system parameters used, the intersections between \( G_{eq}(s) \) and \(-1/N(A)\) all fall on the portion of the \(-1/N(A)\) loci that lie on the \(-270^\circ\) axis. This means that as we vary the values of \( K_{WT} \) and \( H_{WT} \) of the wire cable nonlinearity characteristics within the stipulated ranges, only the amplitude of oscillation, \( A \), will be varied. Equations (9-21) and (9-22) further show that for small values of \(-1/N(A)\), which correspond to the range of intersections in the present case, the amplitude of oscillation is not sensitive to the values of \( \gamma \), \( T_{FP0} \), and \( K_{WT} \). However, the amplitude of oscillation, \( A \), is directly proportional to the value of \( H_{WT} \). Typical results of the stable sustained oscillations are tabulated in Table 9-2.
Figure 9-2. Frequency response plots and describing function loci of IPS system with flex pivot and wire cable nonlinearities (Dahl model).
<table>
<thead>
<tr>
<th>$K_I$</th>
<th>$K_{WT}$</th>
<th>$H_{WT}$</th>
<th>$A$ (rad)</th>
<th>(arc-sec)</th>
<th>$\omega$ (rad/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^5$</td>
<td>0.25</td>
<td>0.01</td>
<td>$10^{-6}$</td>
<td>0.2</td>
<td>0.138</td>
</tr>
<tr>
<td>$10^5$</td>
<td>0.25</td>
<td>0.1</td>
<td>$10^{-5}$</td>
<td>2.0</td>
<td>0.138</td>
</tr>
<tr>
<td>$10^5$</td>
<td>1.00</td>
<td>0.01</td>
<td>$10^{-6}$</td>
<td>0.2</td>
<td>0.138</td>
</tr>
<tr>
<td>$10^5$</td>
<td>1.00</td>
<td>1.00</td>
<td>$10^{-4}$</td>
<td>20.0</td>
<td>0.138</td>
</tr>
<tr>
<td>$10^5$</td>
<td>5.00</td>
<td>0.01</td>
<td>$10^{-6}$</td>
<td>0.2</td>
<td>0.138</td>
</tr>
<tr>
<td>$10^5$</td>
<td>5.00</td>
<td>1.00</td>
<td>$10^{-4}$</td>
<td>20.0</td>
<td>0.138</td>
</tr>
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<td>0.01</td>
<td>$10^{-6}$</td>
<td>0.2</td>
<td>0.138</td>
</tr>
<tr>
<td>$10^5$</td>
<td>10.00</td>
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<td>$10^{-4}$</td>
<td>20.0</td>
<td>0.138</td>
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<tr>
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<td>$10^{-4}$</td>
<td>20.0</td>
<td>0.138</td>
</tr>
<tr>
<td>$10^5$</td>
<td>100.00</td>
<td>1.00</td>
<td>$10^{-4}$</td>
<td>20.0</td>
<td>0.138</td>
</tr>
<tr>
<td>$10^6$</td>
<td>0.25</td>
<td>0.01</td>
<td>$3 \times 10^{-8}$</td>
<td>0.006</td>
<td>0.16</td>
</tr>
<tr>
<td>$10^6$</td>
<td>0.25</td>
<td>0.1</td>
<td>$3 \times 10^{-7}$</td>
<td>0.06</td>
<td>0.16</td>
</tr>
<tr>
<td>$10^6$</td>
<td>1.00</td>
<td>0.01</td>
<td>$3 \times 10^{-8}$</td>
<td>0.006</td>
<td>0.16</td>
</tr>
<tr>
<td>$10^6$</td>
<td>1.00</td>
<td>1.00</td>
<td>$3 \times 10^{-6}$</td>
<td>0.6</td>
<td>0.16</td>
</tr>
<tr>
<td>$10^6$</td>
<td>5.00</td>
<td>0.01</td>
<td>$3 \times 10^{-8}$</td>
<td>0.006</td>
<td>0.16</td>
</tr>
<tr>
<td>$10^6$</td>
<td>5.00</td>
<td>1.00</td>
<td>$3 \times 10^{-6}$</td>
<td>0.6</td>
<td>0.16</td>
</tr>
<tr>
<td>$10^6$</td>
<td>10.00</td>
<td>0.01</td>
<td>$3 \times 10^{-8}$</td>
<td>0.006</td>
<td>0.16</td>
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<tr>
<td>$10^6$</td>
<td>10.00</td>
<td>1.00</td>
<td>$3 \times 10^{-6}$</td>
<td>0.6</td>
<td>0.16</td>
</tr>
<tr>
<td>$10^6$</td>
<td>25.00</td>
<td>1.00</td>
<td>$3 \times 10^{-6}$</td>
<td>0.6</td>
<td>0.16</td>
</tr>
<tr>
<td>$10^6$</td>
<td>100.00</td>
<td>1.00</td>
<td>$3 \times 10^{-6}$</td>
<td>0.6</td>
<td>0.16</td>
</tr>
</tbody>
</table>
It is of interest to compare these results with those obtained in Chapter 3. Using the results tabulated on page 30, the following comparisons are obtained.

\[ H_{WT} = 1, \quad K_{WT} = \text{arbitrary} \]

<table>
<thead>
<tr>
<th>( K_I )</th>
<th>Results in Chapter 3</th>
<th>Dahl model results</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^6 )</td>
<td>( 3.17 \times 10^{-6} ), 0.16</td>
<td>( 3 \times 10^{-6} ), 0.16</td>
</tr>
<tr>
<td>( 10^5 )</td>
<td>( 8.19 \times 10^{-5} ), 0.14</td>
<td>( 10^{-4} ), 0.138</td>
</tr>
</tbody>
</table>

Therefore, we see that for all practical purposes these results are identical.
10. Modeling of the Solid Rolling Friction by the First-Order Dahl Model

It has been established that the solid rolling friction characteristics can be approximated by the nonlinear relation

\[ \frac{dT_{FP}(\varepsilon)}{d\varepsilon} = \gamma (T_{FP1} - T_{FP0})^i \]  

(10-1)

where \( i \) = positive number
\( \gamma = \) constant
\( T_{FP1} = T_{FP} \text{sgn}(\dot{s}) \)
\( T_{FP}(\varepsilon) = \) friction torque
\( T_{FP0} = \) saturation level of \( T_{FP} \)
\( \varepsilon = \) angular displacement

The describing function of the friction nonlinearity for \( n = 2 \) has been derived. In this chapter the input-output relationship will be obtained by solving Eq. (10-1) with \( i = 1 \). The describing function for the \( i = 1 \) case is derived in the next chapter.

Let the angular displacement \( \varepsilon \) be a cosinusoidal function,

\[ \varepsilon(t) = A \cos \omega t \]  

(10-2)

Then, we can write

\[ \frac{dT_{FP}(\varepsilon)}{dt} = \frac{dT_{FP}(\varepsilon)}{d\varepsilon} \cdot \varepsilon = -\gamma A \omega \sin \omega t (T_{FP1} - T_{FP0}) \]  

(10-3)

where \( i \) has been set to 1.

Since \( T_{FP1} = T_{FP} \text{sgn}(\dot{s}) \), Eq. (10-3) is written

\[ \frac{dT_{FP}(\varepsilon)}{dt} = -\gamma A \omega \sin \omega t (T_{FP} - T_{FP0}) \] 

\[ \varepsilon \geq 0 \]  

\[ = \gamma A \omega \sin \omega t (T_{FP} + T_{FP0}) \] 

\[ \varepsilon \leq 0 \]  

(10-4)

For \( \dot{\varepsilon} \leq 0 \), \( 2\pi k \leq \omega t \leq (2k + 1)\pi \), \( k = 0,1,2,\ldots \).
Equation (10-4) is integrated on both sides to give

\[
\int_{T_{FP}[\epsilon(0)]}^{T_{FP}[\epsilon(t)]} \frac{dT_{FP}[\epsilon]}{(T_{FP} + T_{FP0})} = \int_{0}^{\omega t} -\gamma A \sin \omega t \, dt
\]  
(10-5)

Thus,

\[
\ln \left( \frac{T_{FP}[\epsilon(t)]}{T_{FP}[\epsilon(0)]} \right) = -\gamma A \cos \omega t \bigg|_{0}^{\omega t}
\]  
(10-6)

Or,

\[
\ln \left( T_{FP}(\epsilon) + T_{FP0} \right) - \ln \left( T_{FP}(0) + T_{FP0} \right) = -\gamma A (\cos \omega t - 1)
\]  
(10-7)

\[
\ln \left( \frac{T_{FP}(\epsilon) + T_{FP0}}{T_{FP}(0) + T_{FP0}} \right) = -\gamma A (\cos \omega t - 1)
\]  
(10-8)

Since for the cosinusoidal input \( \dot{\epsilon}(0) = A \) and \( \dot{\epsilon} \leq 0 \) for \( 2\pi k \leq \omega t \leq (2k + 1)\pi, \) \( k = 0,1,2,..., \) \( T_{FP}(0) = T_{FP1} \geq 0. \) This is because at \( \epsilon = 0, \) \( \dot{\epsilon} \) is decreasing, and \( T_{FP} \) acts in the direction opposite to the motion; thus \( T_{FP1} \geq 0. \) Equation (10-8) is written as

\[
\ln \left( \frac{T_{FP}(\epsilon) + T_{FP0}}{T_{FP1} + T_{FP0}} \right) = -\gamma A (\cos \omega t - 1)
\]  
(10-9)

Or,

\[
\frac{T_{FP}(\epsilon)}{T_{FP0}} = -1 + \left( \frac{T_{FP1}}{T_{FP0}} + 1 \right) e^{-\gamma A (\cos \omega t - 1)}
\]  
(10-10)

For \( \dot{\epsilon} > 0, (2k + 1)\pi \leq \omega t \leq (2k + 2)\pi, \) \( k = 0,1,2,..., \)

Equation (10-4) is integrated on both sides to give

\[
\int_{T_{FP}[\epsilon(t)]}^{T_{FP}(2\pi)} \frac{dT_{FP}[\epsilon]}{(T_{FP} - T_{FP0})} = \int_{\omega t}^{2\pi} -\gamma A \sin \omega t \, d\omega t
\]  
(10-11)
Carrying out the integration, we get

\[
\ln \left( \frac{T_{FP}(\varepsilon) - T_{FP0}}{T_{FP}(2\pi) - T_{FP0}} \right) = -\gamma A(1 - \cos \omega t) \tag{10-12}
\]

Now at \( \omega t = 2\pi \), \( T_{FP}(2\pi) = T_{FP1} > 0 \). Equation (10-12) leads to

\[
\frac{T_{FP}(\varepsilon)}{T_{FP0}} = 1 + \left( \frac{T_{FP1}}{T_{FP0}} - 1 \right)e^{\gamma A(\cos \omega t - 1)} \tag{10-13}
\]

Let

\[
R = \frac{T_{FP1}}{T_{FP0}} \tag{10-14}
\]

Then, Eqs. (10-10) and (10-13) become

\[
\frac{T_{FP}(\varepsilon)}{T_{FP0}} = -1 + (R + 1)e^{-\gamma A(\cos \omega t - 1)} \tag{10-15}
\]

\[
\frac{T_{FP}(\varepsilon)}{T_{FP0}} = 1 + (R - 1)e^{\gamma A(\cos \omega t - 1)} \tag{10-16}
\]

respectively.

In order to evaluate \( R \), we equate the last two equations at \( \omega t = \pi \).

Then,

\[-1 + (R + 1)e^{2\gamma A} = 1 + (R - 1)e^{-2\gamma A} \tag{10-17}\]

The solution is

\[
R = \frac{2}{e^{2\gamma A} - e^{-2\gamma A}} \tag{10-18}
\]

or

\[
R = \text{csch} (2\gamma A) - \cot h (2\gamma A) \tag{10-19}
\]

Since \( \varepsilon = A \cos \omega t \) and \( \varepsilon_i = \varepsilon(0) = A \), Eqs. (10-15) and (10-16) are written as

\[
\frac{T_{FP}(\varepsilon)}{T_{FP0}} = -1 + (R + 1)e^{-\gamma (\varepsilon - \varepsilon_i)} \quad \varepsilon \leq 0 \tag{10-19}
\]
\[
\frac{T_{FP}(\varepsilon)}{T_{FP0}} = 1 + (R - 1)e^{\frac{\gamma(\varepsilon - \varepsilon_f)}{e}} \quad \varepsilon \geq 0
\]

(10-20)

Figure 10-1 shows the \(\frac{T_{FP}}{T_{FP0}}\) versus \(\varepsilon/A\) characteristics for several values of \(A\) when the input is the cosinusoidal function of Eq. (10-2).

Figure 10-2 shows the normalized \((3T_{FP} + T_{Wc})/(3T_{FP0} + K_{WT} + H_{WT})\) versus \(\varepsilon/A\) for several typical combinations of \(A, H_{WT},\) and \(K_{WT}\).
Figure 10-1. Normalized flex pivot torque (Dahl model $i = 1$) versus $\varepsilon/A$ for IPS with cosine function input.
Figure 10-2. Normalized flex pivot plus wire cable torque
versus $\varepsilon/A$ for IPS with cosine function input.
11. Describing Function of the First-Order Dahl Model Solid Rolling Friction

The mathematical description of the first-order Dahl model of the solid rolling friction is presented in the last chapter. The frictional torques for the two ranges of $\dot{\theta}$ for a cosinusoidal input displacement are given by Eqs. (10-19) and (10-20). These equations are rewritten in the following form:

$$T_{FP}(\varepsilon) = T_{FP1}e^{-\gamma A}(\cos \omega t - 1) + T_{FP0}(e^{-\gamma A}(\cos \omega t - 1) - 1) \quad (11-1)$$

$$T_{FP}(\varepsilon) = T_{FP1}e^{\gamma A}(\cos \omega t - 1) - T_{FP0}(e^{\gamma A}(\cos \omega t - 1) - 1) \quad (11-2)$$

For the cosinusoidal input, $\varepsilon(t) = A \cos \omega t$, let $T_{FP}(\varepsilon)$ be approximated by the fundamental component of its Fourier series representation, i.e.,

$$T_{FP}(\varepsilon) = A_1 \sin \omega t + B_1 \cos \omega t \quad (11-3)$$

The describing function of the friction nonlinearity ($i = 1$) is defined as

$$N_{FP}(A) = \frac{B_1 - jA_1}{A} \quad (11-4)$$

where

$$A_1 = \frac{1}{\pi} \int_0^{2\pi} T_{FP}(\varepsilon) \sin \omega t \, d\omega t$$

$$= \frac{1}{\pi} \int_0^{\pi} \left( (T_{FP1} + T_{FP0})e^{-\gamma A}(\cos \omega t - 1) - T_{FP0} \right) \sin \omega t \, d\omega t$$

$$+ \frac{1}{\pi} \int_{\pi}^{2\pi} \left( (T_{FP1} - T_{FP0})e^{\gamma A}(\cos \omega t - 1) + T_{FP0} \right) \sin \omega t \, d\omega t \quad (11-5)$$

Evaluating the integrals in the last equation, we have

$$A_1 = \frac{-4T_{FP0}}{\pi} + \frac{1}{\pi\gamma A} \left( T_{FP1}(e^{-2\gamma A} + e^{2\gamma A} - 2) + T_{FP0}(e^{2\gamma A} - e^{-2\gamma A}) \right) \quad (11-6)$$
\[ A_1 = \frac{-4T_{FP0}}{\pi} + \frac{2}{\pi \gamma A} \left( T_{FP1} (\cosh 2\gamma A - 1) + T_{FP0} (\sinh 2\gamma A) \right) \quad (11-7) \]

\[ B_1 = \frac{1}{\pi} \int_0^{\pi} \left( (T_{FP1} + T_{FP0}) e^{-\gamma A (\cos wt - 1)} - T_{FP0} \right) \cos wt \, dw \] 
\[ + \frac{1}{\pi} \int_{2\pi}^{\pi} \left( (T_{FP1} - T_{FP0}) e^{\gamma A (\cos wt - 1)} + T_{FP0} \right) \cos wt \, dw \quad (11-8) \]

In order to evaluate the integrals of \( B_1 \), let us represent \( e^{-\gamma A \cos wt} \) as a power series:

\[ e^{-\gamma A \cos wt} = 1 - \gamma A \cos wt + \frac{(\gamma A)^2}{2!} \cos^2 wt - \frac{(\gamma A)^3}{3!} \cos^3 wt + \ldots \quad (11-9) \]

Consider the integral

\[ I_{B1} = \int_0^{\pi} e^{-\gamma A \cos wt} \cos wt \, dw \]

\[ = \int_0^{\pi} \left( 1 - \gamma A \cos wt + \frac{(\gamma A)^2}{2!} \cos^2 wt - \frac{(\gamma A)^3}{3!} \cos^3 wt + \ldots \right) \cos wt \, dw \quad (11-10) \]

Since

\[ \int_0^{\pi} \cos^m wt \, dw = 0 \quad \text{for} \quad m = \text{odd integers} \quad (11-11) \]

Eq. (11-10) becomes

\[ I_{B1} = - \sum_{i=0}^{\infty} \frac{(\gamma A)^i}{i!} \int_0^{\pi} \cos^{i+1} wt \, dw \quad i = \text{odd integers} \quad (11-12) \]

Evaluating the integral, the result is

\[ I_{B1} = \frac{-\pi \gamma A}{2} \left[ 1 + \frac{(\gamma A)^2}{2!} \left( \frac{1}{4} \right) + \frac{(\gamma A)^4}{4!} \left( \frac{1}{6} \right) \left( \frac{3}{4} \right) + \frac{(\gamma A)^6}{6!} \left( \frac{1 \cdot 3 \cdot 5}{8 \cdot 6 \cdot 4} \right) \right. \]
\[ + \frac{(\gamma A)^8}{8!} \left( \frac{1 \cdot 3 \cdot 5 \cdot 7}{10 \cdot 8 \cdot 6 \cdot 4} \right) + \ldots \right] \quad (11-13) \]
Similarly, the integral

\[ I_{B2} = \int_{\pi}^{2\pi} e^{\gamma A \cos \omega t} \cos \omega t \, d\omega t \]

is evaluated, and the result is

\[ I_{B2} = -I_{B1} \quad (11-14) \]

Substituting the results of \( I_{B1} \) and \( I_{B2} \) into Eq. (11-8), we have

\[
B_1 = \frac{T_{FP1} + T_{FP0}}{\pi} e^{\gamma A} I_{B1} + \frac{T_{FP1} - T_{FP0}}{\pi} e^{-\gamma A} I_{B2}
\]

\[
= \frac{I_{B1}}{\pi} \left( T_{FP1} (e^{\gamma A} - e^{-\gamma A}) + T_{FP0} (e^{\gamma A} + e^{-\gamma A}) \right) \quad (11-15)
\]

For very small values of \( \gamma A \)

\[ I_{B1} \approx -\frac{\pi \gamma A}{2} \quad (11-16) \]

Equation (11-15) becomes

\[
B_1 \approx -\frac{\gamma A}{2} \left( T_{FP1} (e^{\gamma A} - e^{-\gamma A}) + T_{FP0} (e^{\gamma A} + e^{-\gamma A}) \right) \quad (11-17)
\]

or

\[
B_1 \approx -\gamma A \left( T_{FP1} \sinh (\gamma A) + T_{FP0} \cosh (\gamma A) \right) \quad (11-18)
\]

For large values of \( \gamma A \), \( I_{B1} \) becomes very large. However, we shall show that \( T_{FP1} \) approaches \(-T_{FP0}\) as \( \gamma A \) becomes very large, so that \( B_1 \) becomes zero.

We shall now investigate the limiting values of \( A_1/A \) and \( B_1/A \) when \( A \) approaches zero and infinity. Since

\[
\lim_{A \to 0} T_{FP1} = 0, \quad (11-19)
\]

\[
\lim_{A \to 0} \frac{A_1}{A} = \lim_{A \to 0} \left( \frac{-4T_{FP0} \gamma}{2\pi A \gamma} + \frac{4T_{FP0} \gamma}{2\pi A \gamma} \right) = 0 \quad (11-20)
\]
\[
\lim_{A \to 0} \frac{B_1}{A} = -\gamma T_{FP0} \quad (11-21)
\]

Therefore,
\[
\lim_{A \to 0} \left( -\frac{1}{N_{FP}(A)} \right) = \frac{1}{\gamma T_{FP0}} \quad (11-22)
\]

When \( A \) approaches infinity,
\[
\lim_{A \to \infty} T_{FP1} = -T_{FP0} \quad (11-23)
\]

\[
\lim_{A \to \infty} \frac{B_1}{A} = -\frac{\gamma}{2} \left( -T_{FP0} e^{\gamma A} + T_{FP0} e^{-\gamma A} \right) \left( \frac{1}{B_1} \right) = 0 \quad (11-24)
\]

The value of \( A_1/A \) also approaches zero as \( A \) becomes very large; however it decreases at a much slower rate than \( B_1/A \). Thus,
\[
\lim_{A \to \infty} \left( -\frac{1}{N_{FP}(A)} \right) = \lim_{A \to \infty} \frac{-1}{j\frac{A_1}{A}} = \infty \left( \frac{-270^\circ}{-270^\circ} \right)
\]

Figure 11-1 shows the plot of \(-1/N_{FP}(A)\) in the magnitude (db) versus phase coordinates for

\[
\gamma = 13429.75 \text{ (N-m-rad)}^{-1}
\]

\[
T_{FP0} = 0.0088 \text{ N-m}
\]

**Figure 11-1** shows the plots of \(-1/N_{FP}(A)\) in the magnitude versus phase coordinates for

\[
\gamma = 13429.75 \text{ (N-m-rad)}^{-1}
\]

\[
T_{FP0} = 0.0088 \text{ N-m}
\]

**Magnitude versus Phase Plots of \(-1/N_{FP}(A)\) of the Combined Nonlinearities**

For the combined nonlinearity of the Dahl friction model \((i = 1)\) and the wire cable, the nonlinear describing function is written as (Eq. (10-17))

\[
N(A) = N_{WT}(A) + 3N_{FP}(A) \quad (11-25)
\]

where \( N_{WT}(A) \) is given in Eq. (9-16).

Figure 11-2 shows the plots of \(-1/N(A)\) for the combined nonlinearity in magnitude versus phase for various combinations of \( K_{WT} \) and \( H_{WT} \). These
Figure 11.1: Magnitude versus phase plot of $-1/N(A)$ of Dahl solid rolling friction model with $i = 1$.
Figure 11-2. Frequency response plots and describing function loci of IPS system with flex pivot and wire cable nonlinearities (Dahl model i = 1).
curves are similar to the plots shown in Fig. 9-2 which are for \( i = 2 \) in the Dahl model, especially when the values of \( A \) are very small and very large.

The frequency loci of \( G_{eq}(s) \) of Eq. (9-27) are plotted in Fig. 11-2 for \( K_I = 10^5 \) and \( 10^6 \). Similar to the cases in Fig. 9-2, these frequency loci have two equilibrium points for each curve, one stable and the other unstable.

For \( K_I = 10^6 \) the frequency of oscillation at the stable equilibrium is approximately 0.16 rad/sec, and is rather independent on the values of \( K_{WT} \) and \( H_{WT} \). This result is identical to that obtained in Chapter 9 when \( i = 2 \) is used for the Dahl friction model.

Figure 11-2 shows that for \( K_I = 10^5 \) the frequency of oscillation at the stable equilibrium varies as a function of the values of \( K_{WT} \) and \( H_{WT} \). For the various combinations of \( K_{WT} \) and \( H_{WT} \) shown in Figure 11-2, the variation of frequency is not large, from 0.138 to 0.14 rad/sec. Of more importance is perhaps the fact that when \( i = 1 \), the amplitude of oscillation \( A \) is larger as compared with that for \( i = 2 \). For example, for \( K_{WT} = 100 \), \( H_{WT} = 1.0 \), \( K_I = 10^5 \), Figure 9-2 shows that the amplitude of \( A \) is approximately \( 10^{-5} \) for \( k = 2 \), whereas for the same set of parameters Figure 11-2 shows that \( A = 10^{-3} \) for \( i = 1 \).
12. Modeling of the Solid Rolling Friction by the ith-order Dahl Model and the Describing Function

In the previous sections the solid rolling friction was modelled by Eq. (10-1) with $i = 1$ and 2. In general, the exponent $i$ can be of any other value. In this section we shall derive the mathematical model of the solid rolling friction for $i \neq 1$.

Let the frictional characteristics be approximated by the nonlinear relation

$$\frac{dT_{FP}(\varepsilon)}{d\varepsilon} = \gamma (T_{FP1} - T_{FP0})^i$$  \hspace{1cm} (12-1)

where

- $i =$ positive number $\neq 1$
- $\gamma =$ constant
- $T_{FP1} = T_{FP} \text{ SGN}(\dot{\varepsilon})$ \hspace{1cm} (12-2)
- $T_{FP}(\varepsilon) =$ friction torque
- $T_{FP0} =$ saturation level of $T_{FP}$
- $\dot{\varepsilon} =$ angular displacement

Let the angular displacement $\varepsilon$ be a cosinusoidal function,

$$\varepsilon(t) = A \cos \omega t$$ \hspace{1cm} (12-3)

Then,

$$\dot{\varepsilon}(t) = -A\omega \sin \omega t$$ \hspace{1cm} (12-4)

We can write

$$\frac{dT_{FP}}{dt} = -\gamma A\omega \sin \omega t (T_{FP1} - T_{FP0})^i$$ \hspace{1cm} (12-5)

In view of Eq. (12-2), the last equation is written

$$\frac{dT_{FP}}{dt} = -\gamma A\omega \sin \omega t (T_{FP} - T_{FP0})^i$$ \hspace{1cm} (12-6)

If $\dot{\varepsilon} > 0$ for $2k\pi \leq \omega t \leq (2k+1)\pi$, $k = 0, 1, 2, \ldots$, Eq. (12-6) is integrated to give

$$T_{FP}(\varepsilon(t))_{dT_{FP}} (T_{FP} - T_{FP0})^i = -\gamma A \left[ \int_0^{\omega t} \sin u du = -\gamma A (-\cos u) \right]$$
The last equation becomes

\[ T_{\text{FP}} = \frac{(T_{\text{FP}} - T_{\text{FP}0})^{-(i-1)}}{(-i+1)} \]

\[ = \gamma A(\cos \omega t - 1) \]  

(12-8)

where \( T_{\text{FP}i} = T_{\text{FP}}(\theta(0)) \) and \( T_{\text{FP}} = T_{\text{FP}}(\theta(t)) \).

Equation (12-8) is further simplified to

\[ (T_{\text{FP}} - T_{\text{FP}0})^{-(i-1)} - (T_{\text{FP}i} - T_{\text{FP}0})^{-(i-1)} = -(i-1)\gamma A(\cos \omega t - 1) \]  

(12-9)

Defining

\[ R = \frac{T_{\text{FP}i}}{T_{\text{FP}0}} \]  

(12-10)

Eq. (12-9) leads to

\[ \frac{T_{\text{FP}}}{T_{\text{FP}0}} = 1 + \frac{R - 1}{\left\{1 - (i-1)\gamma A(T_{\text{FP}0}R-1)(\cos \omega t - 1)\right\}^{1/(i-1)}} \]  

(12-11)

If \( \theta \leq 0 \), for \((2k+1)\pi \leq \omega t \leq (2k+2)\pi\), \( k = 0, 1, 2, \ldots \) Eq. (12-6) is integrated to give

\[ \int_{T_{\text{FP}}(\theta(0))}^{T_{\text{FP}}(\theta(2\pi))} dT_{\text{FP}} = - \int_{\omega t}^{2\pi} \gamma A \sin u du = A(1 - \cos \omega t) \]  

(12-12)

Following the same steps as in Eqs. (12-8) through (12-11), we have

\[ \frac{T_{\text{FP}}}{T_{\text{FP}0}} = \frac{R + 1}{\left\{1 + (i-1)\gamma A(T_{\text{FP}0}(R+1))(\cos \omega t - 1)\right\}^{1/(i-1)}} - 1 \]  

(12-13)

In order to evaluate \( R \), we equate Eqs. (12-11) and (12-13) at \( \omega t = \pi \). After simplification, the result is

\[ 2 + R \left[ \frac{1}{\left\{1 + \hat{a}(R-1)(i-1)\right\}^{1/(i-1)}} - \frac{1}{\left\{1 - \hat{a}(-R+1)(i-1)\right\}^{1/(i-1)}} \right] \]

\[ = \left[ \frac{1}{\left\{1 + \hat{a}(R-1)(i-1)\right\}^{1/(i-1)}} + \frac{1}{\left\{1 - \hat{a}(-R+1)(i-1)\right\}^{1/(i-1)}} \right] \]  

(12-14)

where

\[ \hat{a} = 2(i - 1)\gamma AT_{\text{FP}0}^{(i-1)} \]  

(12-15)
Once the value of \( i \) \((i \neq 1) \) is specified, \( R \) can be solved from Eq. (12-14).

We can show that when \( i = 2 \),

\[
\dot{a} = a = 2 \gamma AT_{FP0}
\]

which is identical to Eq. (8-18), and

\[
R = -\frac{1}{a} + \sqrt{\frac{a^2 + 1}{a^2}}
\]

which is the same result as in Eq. (8-19).

Once \( R \) is determined from Eq. (12-14), the torque relationships are expressed by Eqs. (12-11) for \( \dot{e} > 0 \) and Eq. (12-13) for \( \dot{e} \leq 0 \).

**Describing Function For the Dahl Model For \( i \neq 1 \)**

Let

\[
\beta = \dot{a}/2 = (i - 1)\gamma AT_{FP0}
\]

Equations (12-11) and (12-13) are simplified to

\[
\frac{T_{FP}}{T_{FP0}} = \frac{R - 1}{(1 - \beta(R - 1)(\cos \omega t - 1))^{1/(i-1)} + 1} \quad (\dot{e} > 0) \tag{12-19}
\]

\[
\frac{T_{FP}}{T_{FP0}} = \frac{R + 1}{(1 + \beta(-R + 1)(\cos \omega t - 1))^{1/(i-1)} - 1} \quad (\dot{e} \leq 0) \tag{12-20}
\]

For the cosinusoidal input of Eq. (12-3), let the output torque be represented by the fundamental components of its Fourier series; i.e.,

\[
T_{FP} = A_1 \sin \omega t + B_1 \cos \omega t
\]

where

\[
A_1 = \frac{1}{\pi} \int_0^{2\pi} T_{FP} \sin \omega t \, dt \tag{12-22}
\]

and

\[
B_1 = \frac{1}{\pi} \int_0^{2\pi} T_{FP} \cos \omega t \, dt \tag{12-23}
\]

Substitution of Eqs. (12-19) and (12-20) into Eq. (12-22), we get

\[
A_1 = \frac{T_{FP0}}{\pi} \left[ \int_0^{2\pi} \frac{R + 1}{(1 + \beta(-R + 1)(\cos \omega t - 1))^{1/(i-1)} - 1} \sin \omega t \, dt \right]
\]
The last equation is reduced to the following form:

$$A_1 = \frac{4T_{FP0}}{\pi} + \frac{T_{FP0}(R + 1)}{\pi} \int_0^\pi \frac{\sin \omega t \, d\omega t}{\sin \omega t \, d\omega t} \frac{1}{1/(i-1)}$$

Let

$$x = \beta(R - 1)(i-1) \cos \omega t$$

and

$$y = \beta(-(r + 1))(i-1) \cos \omega t$$

Then

$$dx = -\beta(R - 1)(i-1) \sin \omega t \, d\omega t$$

$$dy + \beta(R + 1)(i-1) \sin \omega t \, d\omega t$$

Equation (12-24) is written

$$A_1 = -\frac{4T_{FP0}}{\pi} + \frac{T_{FP0}(R + 1)}{\pi} \int_0^\pi \frac{\sin \omega t \, d\omega t}{\sin \omega t \, d\omega t} \frac{1}{1/(i-1)}$$

Let

$$\varphi = 1 + \beta(-(r + 1))(i-1) + y$$

$$\Delta = 1 + \beta(R - 1)(i-1)$$

Then

$$d\varphi = dy$$

$$d\Delta = dx$$

Substitution of the last four equations into Eq. (12-29) yields

$$A_1 = -\frac{4T_{FP0}}{\pi} + \frac{T_{FP0}(R + 1)}{\pi \beta(-(R+1))(i-1)} \int_0^\varphi \frac{1}{1+2\beta(R+1)(i-1)} \frac{d\varphi}{(\varphi)1/(i-1)}$$
The integral on the right-hand side of the last equation are now carried out, and after simplification, the result is

\[ A_1 = -\frac{4T_{FP0}}{\pi} + \frac{T_{FP0}(R + 1)}{\pi\beta(R + 1)(i-1)} \left\{ \frac{1 + 2\beta(-(R+1))(i-1)}{(i-2)/(i-1)} - 1 \right\} \]

The Fourier coefficient \( B_1 \) is determined as follows:

\[ B_1 = \frac{T_{FP0}}{\pi} \int_0^\pi \left\{ \frac{R + 1}{1 + (-\beta(R + 1))(i-1)} \right\} \cos wt \, dt \]

The last equation is simplified to

\[ B_1 = \frac{T_{FP0}}{\pi} \left\{ \frac{1}{\beta(-R+1)(i-1)} \right\} \int_0^\pi \frac{\beta(-R+1)(i-1)}{1 - \beta(-R+1)(i-1) + \beta(-R+1)(i-1)} \cos wt \, dt + \frac{(R - 1)}{\beta(R - 1)(i-1)} \int_0^{2\pi} \frac{\beta(R - 1)(i-1)}{1 + \beta(R - 1)(i-1) - \beta(R - 1)(i-1) \cos wt} \, dt \]  

Letting

\[ x = \beta(-R + 1)(i-1) \cos wt \]  
\[ y = \beta(R - 1)(i-1) \cos wt \]  
\[ dx = -\beta(-R + 1)(i-1) \sin wt \, dt \]  
\[ dy = -\beta(R - 1)(i-1) \sin wt \, dt \]  

Eq. (12-37) becomes
\[ B_1 = \frac{T_{FP0}}{\pi} \left\{ \frac{-(R + 1)}{\beta(-(R + 1))(i-1)} \int_{x(0)}^{x(\pi)} \cot \omega t \, dx \cdot \right. \\
\left. \frac{x(\pi)}{\{ 1 - \beta(-(R + 1))(i-1) + x \}^{1/(i-1)}} \right\} \\
- \frac{R - 1}{\beta(R - 1)(i-1)} \int_{y(\pi)}^{y(2\pi)} \cot \omega t \, dy \cdot \frac{x(\pi)}{\{ 1 + \beta(R - 1)(i-1) - y \}^{1/(i-1)}} \right\} \\
\text{For the first integral in the last equation,} \\
\cot \omega t = \sqrt{\frac{x}{\beta^2(-(R + 1))^2(i-1) - x^2}} \quad (12-43) \\
\text{and for the second integral,} \\
\cot \omega t = \sqrt{\frac{y}{\beta^2(R - 1)^2(i-1) - y^2}} \quad (12-44) \\
\text{Therefore,} \\
\begin{align*}
B_1 &= \frac{T_{FP0}}{\pi} \left\{ \frac{-(R + 1)}{\beta(-(R + 1))(i-1)} \int_{x(0)}^{x(\pi)} \frac{x}{\{ 1 - \beta(-(R+1))(i-1) + x \}^{1/(i-1)}} \\
&\quad - \frac{R - 1}{\beta(R - 1)(i-1)} \int_{y(\pi)}^{y(2\pi)} \frac{y}{\{ 1 + \beta(R - 1)(i-1) - y \}^{1/(i-1)}} \right\} \\
\text{These integrals can be carried out only if the value of } i \text{ is given } (i \neq 1). 
\end{align*} \
13. Digital Computer Simulation of the Continuous-Data Nonlinear IPS Control System With Dahl Model

The IPS control system with the nonlinear flex pivot torque modelled by the Dahl solid friction model is simulated on the digital computer for \( i = 1 \) and \( i = 2 \). The block diagram of the IPS system is shown in Fig. 8-2, and the nonlinearities are modelled by Fig. 8-1.

The main objective of the computer simulation is to verify the results on the sustained-oscillation predicted by the describing function method.

Dahl Model \( i = 1 \)

The computer program using the IBM 360 CSMP for \( i = 1 \) in the Dahl model is given in Table 13-1. The simulation runs were able to predict and verify the results obtained by the describing function method of Chapter 9. The difficulty with the long response time of the IPS system still exists in this case. Generally, it would be very time consuming and expensive to wait for the transient to settle completely in a digital computer simulation. Figure 13-1 shows the response of \( \epsilon(t) \) over a one-hundred second time interval, with \( \epsilon(0) = 10^{-5} \), \( \dot{\epsilon}(0) = 0 \), \( K_I = 10^5 \), \( K_{WT} = 100 \), \( H_{WT} = 1 \). For the Dahl model, \( i = 1 \), \( T_{FP} = 0.0088 \) N-m, and \( \gamma = 13429.75 \). The parameters of the linear portion of the system are tabulated on page 4 in Chapter 1. Figure 13-1 shows that the response is oscillatory with an increasing amplitude, and the period is 46 sec or 0.136 rad/sec. Since it would take a long time for the amplitude to settle to a final steady-state value, we selected another initial value \( \epsilon(0) \) and repeated the simulation. Figure 13-2 shows the response of \( \epsilon(t) \) with \( \epsilon(0) = 10^{-3} \) which is decreasing in amplitude as time increases. Therefore, the stable operating point should be at an amplitude between \( 10^{-3} \) and \( 10^{-5} \), and the frequency of oscillation is 0.136 rad/sec. In general it would be very difficult to find the initial state which corresponds to the steady-state oscillation exactly. The results predicted by Fig. 9-2 are very close for the amplitude
TABLE 13-1

INIT
PARAM K0=1.15, K1=6.4, K2=0.004258, K3=0.003504
PARAM K4=0.00459, K5=10/28.49, K6=1.1661, K7=5.0060926
PARAM K8=1, K9
PARAM NT=100, NW=1,1
PARAM NQ=3.4297524, TQF=0.000360
INIT 1 T0=1.0, T1=1.0, IN10=0.0
TAB1 1 T1=0
N2=0, T9=HYPERFO

PROF1 HR1=PR(A=1.010)

SINH (T1, T9)

SINH (TQF, TQF)

TQF=0.000360

K4A=K4*K4
K7A=K4*K4
K40=K8.0-46

NUSD1
ELAS (T1)=EO

K(T)=K1

MF1HON RASFX

SORT:

DYNAM

FKNUL1 TMC=WERC(NUT+NUT+C, EWNT)

SIGNET=1.0

IF (EM0L1, L1, 0) SIGNET=1.000

TMC=SIGNET*NUT1 NUT2 NUT3

ENDPRO

FKNUL1 T1=MAH1(TFO=T D=EO, ROMA, EO)

IF (T1, T1, 0) AND (EO AND T) GO TO 1

IF. 2

2 SIGNET=1.00

IF (FMA, T1, T9) SIGNET=1.00

B=SINH1 (E-ELAS1 (T1)) *SANF1

TQF=SANF1*(1.0+EXP(-1.0)*EXP(EO)*TFP0

IF (TPF, GT TQP) TP=TQP0

TFP=TPF

TFP=-TPF

TFP=TQP

IF (T1, T1, 0) AND T1 10 3

IF (T1, T1, 0) AND T1 10 2

E-TP=EO*TFP

EO=EO*EO+EO

ENDPRO

FNU01=K7*IC-K4*X1

NEWT=INURL/CD010-CD010

E=INURL(EO, EWNT)

C1=K4*EO/CD010

C2=K4*IC/CD010

C3=K4*EO/CD010

X1=1-C1-C2-C3

X2=INURL(EO, X1)

X3=INURL(EO, X2)

X4=X2+EO

X5=1-K3

X6=INURL(EO, X5)

TC=K1*EO+X6+EO+TFP+TFP

TEHIT

LIMIT 1 INIT=100, 0, O1T=1.0, C-3-OUTCL=7.0, EO=PRMEL=2.0

FRIPL1 E(EO, EWNT, TC)

FIN

STOP

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E(t) = 10^t, i = 1, Tp0 = 0.0008 s, y = 13426, 75.
Figure 13.2: The response of $e(t)$ for $K_i = 10^5$, $K_H = 100$, $N_{\text{hit}} = 1$, $T_f = 0.0088$ N$_{\text{hit}}$, $Y = 13.2975$. 

**Simulation Parameters**

- $K_i = 10^5$
- $K_H = 100$
- $N_{\text{hit}} = 1$
- $T_f = 0.0088$ N$_{\text{hit}}$
- $Y = 13.2975$

**Simulation Data**

- $e(t) = 10^{-3}$
- $i = 1$

---

**Notes:**

- The simulation data is for a control system with various parameters and responses over time.
- The output shows the simulation results for different time steps and corresponding control errors.
- The data is presented in a tabular format with time intervals and error values.
- The simulation is conducted using the CSMP/360 simulation software.
and $\omega = 0.138$ rad/sec.

**Dahl Model $i = 2$**

Table 13-2 gives the computer simulation program for the $i = 2$ case, with $T_{FP0} = 0.00225$ N-m and $\gamma = 92444$. All other system parameters are the same as the $i = 1$ case. Figure 13-3 shows a stable response when the initial state $\epsilon(0)$ is small, $10^{-10}$. As shown in Fig. 11-2, when the initial state is small the stable equilibrium point is $\epsilon = 0$. Figure 13-4 shows another stable response which would take longer time to die out, when $\epsilon(0) = 10^{-8}$. Figures 13-5 and 13-6 show a sustained oscillation solution with amplitude lying between $10^{-7}$ and $7 \times 10^{-7}$, and a period of 44 sec or 14.28 rad/sec. These results are again very close to those predicted in Fig. 11-2. The simulations for the $i = 2$ case are carried out with $K_{WT} = 0.25$, $H_{WT} = 0.1$, and $K_1 = 10^6$. 
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REPRODUCIBILITY OF THE ORIGINAL PAGE IS POOR.
Simulation of Nonlinear IFP Control System

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*** CSMP/360 SIMULATION DATA ***
### Simulation of Nonlinear IPS Control System

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**Figure 13.4**: Stable response of $e(t)$ for $i = 2$, $a(0) = 10^{-8}$. 
Figure 13-5. Time response of e(t) for \( i = 2 \), e(t) = 10^-7.
REFERENCES


