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PROPAGATION OF A LASER BEAM IN A PLASMA

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ABSTRACT

This paper shows that for a nonabsorbing medium with a prescribed index of refraction, the effects of beam stability, line focusing, and beam distortion can all be predicted from simple ray optics. When the paraxial approximation is used, diffraction effects are examined for Gaussian, Lorentzian, and square beams. Most importantly, it is shown that for a Gaussian beam, diffraction effects can be included simply by adding imaginary solutions to the paraxial ray equations. Also presented are several procedures to extend the paraxial approximation so that the solution will have a domain of validity of greater extent.

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PROPAGATION OF A LASER BEAM IN A PLASMA

1.) Introduction

The process of heating a plasma in a solenoid using a laser beam is difficult to describe analytically when one wants to examine in a self-consistent manner, the coupled behavior of plasma heating and beam propagation. Some insight into the beam behavior may be obtained however, when the ion density is arbitrarily prescribed. Consequently, heating has only been partially accounted for by allowing the ion density to change without an accompanying loss of energy from the laser beam. This also implies that in the plasma, only the electron 'fluid' is excitable.

The impressed magnetic field along the solenoid is assumed to be zero since its effect upon beam propagation is small. A constant magnetic field in the direction of beam propagation effects right and left circularly polarized light differently so that Faraday rotation is observed for a plane polarized wave. However, for frequencies much larger than the cyclotron and plasma frequencies, the plane of polarization is essentially fixed and the impressed magnetic field can be ignored [1]. In the case of interest, the ratio of the cyclotron frequency to the exciting laser frequency is $O(10^{-2})$ for a magnetic field of $100\ kG$ and a laser wavelength of $10.6\ \mu$.

The cold plasma approximation requires that

$$\frac{1}{C}\sqrt{\frac{KT_e}{m_e}} << 1$$

where C is the speed of light, K is the Boltzmann constant, and m_e is the mass of an electron. [illegible] electron temperatures of $10\ kev$ the above ratio is $O(10^{-1})$.

The above assumptions motivates the study of beam propagation through a cold, collisionless and field-free plasma. The dimensionless parameters that are pertinent to the discription of the interaction are obtained by nondimensionalizing Maxwell's equations and the momentum equation for an electron fluid.

These parameters are then determined to be η , M , and ε ; the first measures a typical electron (ion) density n_o to the critical electron density n_{ec} , the second measures the electron quiver velocity V_o to the speed fo light C , and the third parameter compares the laser wavelength λ to the cylinder circumference $2\pi b$. The parameter η is also given through the ratio of the plasma frequency ω_p to the exciting laser frequency ω . We thus have

$$\eta \equiv \frac{n_o}{n_{ec}} = \frac{\omega_p^2}{\omega^2} \quad M = \frac{V_o}{c} \quad \varepsilon = \frac{\lambda}{2\pi b}$$

Here $\omega_p^2 \equiv \frac{n_o e^2}{\varepsilon_o m_e}$ and $V_o = \frac{eE_o}{m_e \omega}$ where the magnitude of the

electric field E_0 is determined from the beam intensity (c and ϵ_0 are physical constants).

If the laser intensity is of the order of 10^{12} watts/cm^2 and the wavelength of the laser beam is $10.6\ \mu$ then $M = O(10^{-2})$. Also, for ion densities of $10^{16}/cm^3$ and a cylinder radius of 1 cm we find that $\eta = O(10^{-3})$ and $\epsilon = O(10^{-3})$.

The effect that M has on beam propagation may be understood by considering a plane wave traveling through a homogeneous, cold, and field-free plasma. This analysis is straightforward and will not be presented here. We find that the transverse components of the electric and magnetic fields are unchanged to $O(M^2)$ and that sinusoidal charge separation occurs with a magnitude $O(M^4)$ at twice the exciting frequency. Consequently, its effects upon beam propagation are small when $M = O(10^{-2})$ and will be ignored here.

The index of refraction n is then easily obtained using the above simplifications. We find that

$$n = (1 - \eta\tilde{n}_i)^{1/2} \tag{1.1}$$

where $\tilde{n}_i$ is the nondimensionalized ion density. In this paper we are going to analyze the case when n satisfies the relation

$$n^2 = k^2(x) - \epsilon^2\mu^2(x)n^2 \tag{1.2}$$

Here $k(z)$ and $\mu(z)$ are determined from the ion density and are arbitrary functions of the parameter z, the axial distance along the cylinder nondimensionalized by the cylinder's radius b. The coordinate r is the nondimensional radial distance from the cylinder axis. From Eq. (1.2) we see that the ion density is always parabolic but is allowed to vary with axial distance. Time variation of the ion density involving the hydrodynamic time scale will be ignored in this presentation.

The parabolic assumption is useful when the beam boundary is contained well inside a favorable density profile so that 'beam trapping' occurs. This approximation has been discussed by Steinhauer [2] and Mani, et al. [3] and is coupled with the additional approximation of extending the effective boundary conditions at the cylinder wall to infinity.

The preceding discussion thus motivates the study of the following reduced wave equation for the transverse component of the polarized laser beam.

$$\{\Delta_c + \frac{k^2-\varepsilon^2\mu^2r^2}{\varepsilon^2}\} E_x = 0 \tag{1.3}$$

The symbol Δ_c represents the Laplacian in cylindrical coordinates. Time has been nondimensionalized w.r.t. the exciting frequency ω so that E_x is understood to include the factor e^{-it}.

An asymptotic representation for E_x is obtained when the intensity of the incident beam is negligible in regions where the ion density is close to or greater than the critical ion density (this condition in fact motivates the parabolic approximation). This asymptotic representation is easily obtained from the exact solution of Eq. (1.3) when k and μ in Eq. (1.2) are constants. This procedure will be developed in Section 2. Alternatively, the asymptotic representation for E_x can be obtained directly from the differential Eq. (1.3) and will be presented in Section 3.

When k and μ are constants the usual asymptotic representation predicts a periodic structure for the beam propagation (as does the standard paraxial approximation) and thus does not include 'washout' effects of the beam's intensity distribution. The term washout means that the original intensity distribution (at $z = 0$) can never be regenerated for any value of z and that it continually changes for increasingly larger values of the variable z. Ray optics predicts this distortion effect from the result that rays with different initial displacements from the axis have different focal lengths in the parabolic focusing medium [4]. It is because of this property that line focuses appear with monotonically increasing lengths.

In Section 2 we will examine the 'beam boundary' describing the propagation of Gaussian beams. Some results are also given for beams that have nonGaussian distributions; in particular we will look at a beam that is initially Lorentzian and a beam that is initially square.

In Section 3, of prime importance, is the development of a less complicated equation which describes the beam boundary trajectories for Gaussian beams. Analysis here will be mostly devoted to the case when k and μ are 'slowly' varying functions of z. However, other special situations will be examined. In particular, we will see that instabilities can occur even in a 'favorable' density profile.

In Section 4 we will see how the normally neglected washout effects of the beam's intensity distribution may be included in the asymptotic representation.

Section 5 includes for completeness, the analysis of the ray optics equations when k and μ are slowly varying functions. The idea here is that a uniformly valid solution can be obtained in regions away from the line focuses. Consequently, washout effects have been obtained at the expense of neglecting diffraction.

Section 6 considers higher order terms for a Gaussian beam. The small secular effects of beam distortion and line focusing are then examined.

2.) The Asymptotic Development from the Exact Solution: Fixed Parabolic Profiles

The solution for the transverse component of the electric field E_x can be represented through an infinite sum of Laguerre polynomials. To accomplish this introduce the complex field amplitude ψ expressed in transformed variables ρ and ξ so that

$$E_x = e^{\frac{i\kappa z}{\varepsilon}} \psi(\rho, \xi, \varepsilon)$$

where

$$\rho = \mu r^2$$

$$\xi = \frac{\mu \varepsilon z}{k} \tag{2.1}$$

Here k and μ are constants.

The differential equation for $\psi_{(\rho, \xi)}$ is obtained from Eq. (1.3). Thus

$$\rho\psi_{\rho\rho} + \psi_\rho - \frac{\rho\psi}{4} + \frac{i}{2} \psi_\xi + \frac{\mu\varepsilon^2}{4k^2} \psi_{\xi\xi} = 0 \tag{2.2}$$

Separation of variables and the physical requirement of boundedness of ψ as ρ approaches infinity gives rise to the exponential Laguerre polynomials as eigenfunctions of Eq. (2.2). In terms of these eigenfunctions ψ can be expressed as

$$\psi(\rho, \xi, \varepsilon) = e^{-\rho/2} \sum_0^\infty \Gamma_n(\xi, \varepsilon) L_n(\rho) \tag{2.3}$$

where $L_n(\rho)$ is the Laguerre polynomial of order n and

$$\Gamma_n(\xi,\varepsilon) = \int_0^\infty e^{-\rho/2} L_n(\rho)\, \psi(\rho,\xi)\, d\rho \tag{2.4}$$

Applying the transform defined by Eq. (2.4) to Eq. (2.2) gives the differential equation for $\Gamma_n(\xi,\varepsilon)$. Thus

$$\frac{\mu\varepsilon^2}{k^2}\Gamma_n'' + 2i\,\Gamma_n' - 4(n+\tfrac{1}{2})\,\Gamma_n = 0 \tag{2.5}$$

The prime denotes differentiation w.r.t. ξ .

The roots $\alpha_{n_{1,2}}$ of the characteristic equation corresponding to Eq. (2.5) are

$$\alpha_{n_{1,2}} = -\frac{k^2 i}{\mu\varepsilon^2}\left(1 \mp \sqrt{1 - 4(n+\tfrac{1}{2})\frac{\mu\varepsilon^2}{k^2}}\right) \tag{2.6}$$

The choice of the minus or plus sign corresponds to right or left propagating waves respectively. Since we are interested in beam propagation down a solenoid the second root can be ignored.

Several observations considering the exact solution may now be made:

(1) Different eigenfunction solutions have different periods in ξ so that a solution representing a sum of such terms cannot be strictly periodic.

(2) If the initial distribution can be represented by a sum of eigenfunctions of relatively low order then the solution will be nearly periodic. This will be the basis of the asymptotic analysis.

(3) To each eigenfunction there is an associated bundle of rays located at a particular initial radius from the axis. This is due to the fact that ray optics predicts different periods for rays with different initial displacements. For larger values of n we obtain larger initial displacements.

(4) By the analogy of (3) we see that a damped high order eigenfunction solution corresponds to a ray entering the medium with an initial off axis displacement large enough so that the ion density exceeds the critical density.

We now seek an asymptotic expression for $\psi_{(\rho,\xi,\varepsilon)}$ in the following way. Let the infinite sum in Eq. (2.3) be replaced by a finite sum of the first N terms. If N is large enough this sum will converge to the exact solution. Now let ε be small enough so that

$$4(N + \frac{1}{2}) \frac{\mu\varepsilon^2}{k^2} < 1 .$$

The square root in Eq. (2.6) can now be expanded. We find that

$$a_{n1} = i(2n+1) + O(\varepsilon^2) \qquad \forall\, n < N \tag{2.7}$$

The expansion for $\Gamma_n(\xi,\varepsilon)$ is then represented by the series

$$\Gamma_n(\xi,\varepsilon) = c_n\, e^{-i(2n+1)\xi} \{1 - i(2n+1)^2 \frac{\mu\varepsilon^2}{2k^2}\, \xi + \ldots\} \tag{2.8}$$

We also obtain a corresponding series for $\psi(\rho,\xi,\varepsilon)$. Thus

$$\psi(\rho,\xi,\varepsilon) \doteq e^{-\rho/2} \sum_{0}^{N} c_n \, e^{-i(2n+1)\xi} \, L_n(\rho)$$

$$- \frac{i\mu\varepsilon^2\xi}{2k^2} \sum_{0}^{N} C_n (2n+1)^2 \, e^{-i(2n+1)\xi} \, L_n(\rho) + \ldots \tag{2.9}$$

The coefficients c_n are to be determined from the initial conditions.

Since N is a measure of the degree of accuracy by which one can replace an infinite sum by a finite sum, we expect that the first sum in Eq. (2.9) may be considered in the limit as $N \to \infty$.

Let $\psi_0(\rho,\xi)$ be defined by this infinite sum. Thus

$$\psi_0(\rho,\xi) = e^{-\rho/2} \sum_{0}^{\infty} c_n \, e^{-i(2n+1)\xi} \, L_n(\rho) \tag{2.10}$$

Consequently, the solution for $\psi(\rho,\xi,\varepsilon)$ can be expressed asymptotically by the series

$$\psi(\rho,\xi,\varepsilon) \sim \psi_0 + \frac{i\mu\varepsilon^2\xi}{2k^2} \frac{\partial^2\psi_0}{\partial\xi^2} + \ldots \tag{2.11}$$

We see that the representation is useful when both the secular term $\mu\varepsilon^2\xi$ and the curvature term $\left|\frac{\partial^2\psi_0}{\partial\xi^2}\right|$ are small. When the latter condition is fulfilled a 'multi-variable' procedure [5] can be used to extend the domain of validity of the asymptotic representation.

It is evident that ψ_0 in periodic in ξ and that downstream 'washout' effects are not included. If $\mu\varepsilon^2 L << 1$ where L is the extent of the domain of ξ that is of interest, then the distortion effects can be considered by including the next term in Eq. (2.11). However, if $\mu\varepsilon^2 L = O(1)$ then one must resort to the multi-variable procedure of Section 4.

To obtain ψ_0 in a 'summed' form we must consider the coefficients c_n given by the relation

$$c_n = \int_0^\infty e^{-\rho/2} L_n(\rho)\, \psi(\rho, 0)\, d\rho \tag{2.12}$$

If we introduce this expression into Eq. (2.10) and interchange summation and integration, the following expression can be obtained

$$\psi_0(\rho, \xi) = \frac{e^{i\,\rho/2 \cot \xi}}{2i \sin \xi} \int_0^\infty e^{i\,\eta/2 \cot \xi}\, \psi(\eta, 0)\, J_0\left[\frac{(\rho\eta)^{1/2}}{\sin \xi}\right] d\eta \tag{2.13}$$

We will consider three cases of beam propagation that involve an infinite sum of Laguerre polynomials. These cases may be designated by their initial field distribution. The designations are

i) Gaussian $\quad \psi(\rho, 0) = e^{-\frac{\rho}{2\mu a_0^2}}$

ii) Lorentzian $\psi(\rho,0) = \dfrac{e^{-\nu/2\,\rho}}{1+c\rho}$

iii) Square $\psi(\rho,0) = \begin{cases} 1 & \rho < \mu d^2 \\ 0 & \rho > \mu d^2 \end{cases}$ (2.14)

Here a_0 , c , and d are constants defined through the given intensity distribution and ν is considered to be small.

When the beam is initially Gaussian the integral in Eq. (2.13) can be easily evaluated. We find that

$$\psi_0(\rho,\xi) = \frac{1}{h(\xi)} \exp\left(\frac{i\rho}{2h}\frac{dh}{d\xi}\right)$$

$$h(\xi) = \cos\xi + i\left(\frac{1}{\mu a_0^{\,2}}\right)\sin\xi \qquad (2.15)$$

We may also represent the complex function $h(\xi)$ by its amplitude $a(\xi)/a_0$ and phase $\phi(\xi)$ so that

$$h(\xi) = \frac{a(\xi)}{a_0}\, e^{i\phi}$$

and $$a_0 = a(0) \qquad (2.16)$$

In terms of these real functions we find that

$$\psi_0(\rho,\xi) = \frac{a_0}{a(\xi)}\, e^{-i\phi} \exp\left\{\frac{i\rho}{2a}\frac{da}{d\xi} - \frac{\rho}{2\mu a^2}\right\}$$

with $$a^2(\xi) = a_0^2\left[\cos^2\xi + \left(\frac{1}{\mu a_0^2}\right)^2 \sin^2\xi\right]$$

and $$\phi(\xi) = \tan^{-1}\left[\frac{1}{\mu a_0^2}\tan\xi\right] \tag{2.17}$$

We see from Eqs. (2.17) that $a(\xi)$ is the so-called 'beam boundary' because it gives the value of r at which the intensity is down by a factor of $1/e$ as compared to its value along the axis. Moreover, we find that the ray paths for E_x are simply described by some constant times $a(\xi)$. These results concerning Gaussian beams have been obtained previously by various authors [3,6].

The integral in Eq. (2.13) is harder to evaluate for non-Gaussian beams. There may be special circumstances however, when one might use to advantage the complex convolution theorem concerning Laplace transforms. To given an example of this procedure we will consider the Lorentizian beam.

Let $F(s) = L[f]$ denote the Laplace transform of a function $f(\rho)$. For the Lorentzian beam we have

$$L\left[\frac{1}{1+c\rho}\right] = \frac{1}{c}\, e^{s/c}\, E_1(s/c)$$

$$L\left[J_0(2\sqrt{b\rho})\right] = \frac{1}{s}\, e^{-b/s} \tag{2.18}$$

Here $E_1(z)$ is the exponential integral written as

$$E_1(z) = \int_z^\infty \frac{e^{-v}}{v}\, dv \tag{2.19}$$

We are interested in the case when

$$s = \frac{1}{2}(\nu - i \cot \xi) \tag{2.20}$$

Since

$$L[f_1(\rho), f_2(\rho)] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F_1(\omega) \quad F_2(s-\omega)\, d\omega \tag{2.21}$$

the integral in Eq. (2.13) can be expressed in convoluted form. Let $I(s)$ represent this integral. We find that the contour of $I(s)$ may be closed to the right so that

$$I(s) = \frac{1}{2\pi i} \oint \frac{e^{\frac{b}{\omega - s}}}{\omega - s} \left[\frac{1}{c}\, e^{\omega/c}\, E_1(\omega/c)\right] d\omega \tag{2.22}$$

There is an essential singularity at the point $\omega = S$. Consequently, the integral can be evaluated by Cauchy's integral theorem of complex analysis to give

$$I(s) = \sum_0^\infty \frac{b^n}{(n!)^2} \frac{d^n}{dS^n} \left[\frac{1}{c}\, e^{s/c}\, E_1(s/c)\right] \tag{2.23}$$

Needless to say, this form is also difficult to interpret.

It may be of use however, in that it may converge faster than the corresponding infinite series of Laguerre polynomials. Of course, Eq. (2.22) is of interest in itself in that it represents an alternate expression for the integral in Eq. (2.13) and that powerful techniques of analysis are available to examine its behavior.

We next ask if there are any special values for ξ in which the integral of Eq. (2.13) can be simply evaluated. We see that the focal point $\xi = \pi/2$ is such a point and we obtain the following results:

i) Lorentzian $\quad \psi_0(\rho, \frac{\pi}{2}) = -\frac{i}{c} K_0(\sqrt{\rho/c}$

ii) Square $\quad \psi_0(\rho, \frac{\pi}{2}) = -id\sqrt{\mu/\rho}\, J_1\,(d\sqrt{\mu\rho}) \qquad (2.24)$

Here K_0 is the modified Bessel function of order zero and J_1 is the standard Bessel function of order one. Diffraction rings at the focal point are apparent for the square beam because of the zeros of J_1. However, when $\xi = \pi$ these fade away and the solution regenerates its initial distribution. This is just the periodic behavior of $\psi_0(\rho, \xi)$.

An alternate approach for the description of beam propagation can be accomplished with use of higher order Gaussian modes since any beam can be considered to be an infinite sum of such modes. The advantage of such an approach is

that the ray paths of each mode can be easily determined; the disadvantage is that the solution still involves an infinite sum.

3.) Changing Parabolic Profile

The asymptotic representation for E_x can be obtained directly from the differential equation (1.3) when k and μ in Eq. (1.2) are arbitrary functions of the scaled variable $z_1 = \varepsilon z$. Thus

$$k(z,\varepsilon) = k(z_1)$$

and

$$\mu(z,\varepsilon) = \mu(z_1) \tag{3.1}$$

Physically, this has the interpretation that the ion density is allowed to vary on an axial scale length comparable to or slower than the focal length of the parabolic focusing medium.

We now obtain the equation for the complex field amplitude ψ through the transformations

$$E_x = \psi(r, z_1, \varepsilon)\, e^{i\frac{z_0(z)}{\varepsilon}}$$

where

$$z_1 = \varepsilon z$$

and

$$\frac{dz_0}{dz} = k(z_1) \tag{3.2}$$

The differential equation for ψ is then given as

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + 2ik\frac{\partial}{\partial z_1} + i\frac{\partial k}{\partial z_1} - \mu^2 r^2 + \varepsilon^2\frac{\partial^2}{\partial z_1^2}\right\}\psi = 0 \tag{3.3}$$

We may now neglect the last term in Eq. (3.3) whenever the curvature term $|\frac{\partial^2\psi}{\partial z^2}|$ is small. From our previous analysis we see that this is just the paraxial approximation and consequently washout and line focusing effects are neglected.

When the beam is initially Gaussian, then the equation for the beam boundary $a(z_1)$ as expressed by Tien, et al. [6] can be obtained through the following transformation. Assume

$$\psi(r,z,\epsilon) = \sqrt{\frac{k(0)}{k(z_1)}}\,\frac{a(0)}{a(z)}\,\exp\{i(\frac{kr^2}{2a}\frac{da}{dz_1} - \phi) - \frac{r^2}{2a^2}\} \tag{3.4}$$

Substitution of this equation into Eq. (3.3) gives the differential equations for the transformed functions $a(z_1)$ and $\phi(z_1)$. Thus

$$k^2(z_1)\frac{d^2a}{dz_1^2} + \mu^2(z_1)a + kk'a - \frac{1}{a^3} = 0 \tag{3.5}$$

$$k(z_1)\frac{d\phi}{dz_1} = \frac{1}{a^2} \tag{3.6}$$

These equations can be written in simpler form if we introduce the variable z_1^+ where

$$\frac{dz_1^+}{dz_1} = \frac{1}{k(z_1)} \tag{3.7}$$

$$z_1^+(0) = 0 \tag{3.8}$$

With this transformation Eqs. (3.5) and (3.6) become

$$\frac{d^2a}{dz_1^{+2}} + \mu^2(z_1)a - \frac{1}{a^3} = 0 \tag{3.9}$$

$$\frac{d\phi}{dz_1^{+}} = \frac{1}{a^2} \tag{3.10}$$

The function $\mu(z_1)$ is to be considered as an implicit function of z_1^{+} obtained through use of Eq. (3.7).

Obviously the analysis of Eq. (3.5) or (3.9) is difficult because of the nonlinear term. One can obtain however, a linear equation that describes the beam boundary trajectories. To do this we assume that

$$\psi(r,z_1,\varepsilon) = \sqrt{\frac{k(0)}{k(z_1)}}\,\frac{1}{h(z_1^{+})}\; e^{\frac{ir^2}{2}\left(\frac{1}{h}\frac{dh}{dz_1^{+}}\right)} \tag{3.11}$$

The differential equation for $h(z_1^{+})$ is then easily determined to be

$$\frac{d^2h}{dz_1^{+2}} + \mu^2(z_1)k = 0 \tag{3.12}$$

This is exactly Eq. (3.9) without the nonlinear term.

Comparing Eqs. (3.11) and (3.4) gives the following relationships

$$h(z_1^{+}) = \frac{a(z_1^{+})}{a(0)}\; e^{i\phi} \tag{3.13}$$

$$\frac{1}{h}\frac{dh}{dz_1^+} = \frac{1}{a}\frac{da}{dz_1^+} + \frac{i}{a^2} \tag{3.14}$$

From Eq. (3.13) we see that $\frac{a(z_1^+)}{a(0)}$ is just the amplitude of the complex function $h(z_1^+)$ and $\phi(z_1^+)$ is the phase.

To determine the solution of Eq. (3.12), initial conditions must be specified. These can be obtained from Eqs. (3.13) and (3.14) when the initial conditions on the beam boundary are known. When $a(0) = a_0$, $\frac{da}{dz_1}(0) = d_0$ and $\phi(0) = 0$ we obtain

$$h(0) = 1$$

$$\frac{dh}{dz_1^+}(0) = \frac{k_0 d_0}{a_0} + \frac{i}{a_0^2}$$

when

$$k_0 = k(0) \tag{3.15}$$

If the initial conditions on k were arbitrarily specified to be real then Eq. (3.12) describes the trajectories of the beam's center [6]. Consequently, by allowing complex values for $h(z_1^+)$, we are able to describe the trajectories of the beam's boundary. Moreover, since the real and imaginary parts of $h(z_1^+)$ satisfy the same equation, stability or instability of the beam's center occurs together with the stability or instability of the beam's boundary.

The solution of Eq. (3.12) is readily determined when k and μ are slowly varying. We measure this slowness through the small parameter ν so that

$$k(x_1,\nu) = k(x_2)$$

$$\mu(x_1,\nu) = \mu(x_2)$$

where

$$x_2 = \nu x_1 \tag{3.16}$$

We now represent $h(x_1^+,\nu)$ asymptotically and assume this to depend upon two axial scales Z and x_2 so that

$$h(x_1^+,\nu) \sim h^{(0)}(Z,x_2) + \nu h^{(1)}(Z,x_2) + \ldots \tag{3.17}$$

where Z is defined by

$$\frac{dZ}{dx_1^+} = \mu(x_2) \; ; \; Z(0) = 0 \tag{3.18}$$

The derivative term in Eq. (3.12) can be obtained through the relation

$$\frac{d}{dx_1^+} = \mu(x_2)\frac{\partial}{\partial Z} + \nu k(x_2)\frac{\partial}{\partial x_2} \tag{3.19}$$

Notice must also be made of initial conditions and we obtain these from (3.15).

Standard multiple scale techniques then yield

$$h^{(0)}(Z, z_2) = \sqrt{\frac{\mu_o}{\mu(z_2)}}\left\{\cos Z + \left(\frac{k_o d_o}{\mu_o a_o}\right)\sin Z + i\left(\frac{1}{\mu_o a_o^2}\right)\sin Z\right\} \tag{3.20}$$

Observe that the real part of this solution is just what one would obtain from ray optics for a paraxial ray. The addition of an imaginary term satisfying special initial conditions then improves the optics approximation by including the effects of diffraction. We have also remarked that a uniformly valid solution could be obtained by extending the paraxial approximation. If thus seems feasible that this may be accomplished by allowing complex rays where the real part satisfies the exact ray optics equation and where the imaginary part satisfies special initial conditions. This has not yet been verified.

A case exhibiting various regions of stability and instability can be examined by studying Mathieu's equation.

$$\frac{d^2 h}{dX^2} + (A + 2Q\cos 2X)h = 0 \tag{3.21}$$

Eq. (3.12) can be placed in this form when k is constant a and μ^2 varies sinusoidally. When A is zero the medium exhibits periodic focusing and the solution is first stable for values of Q less than 0.86 [6]. Regions of stability

and instability also occur for positive A and are again determined by the values of Q. Consequently, instabilities can occur even in a 'favorable' density profile.

The case when k has small sinusoidal variations and μ is constant has also been examined by Tien, et al. [6]. A more general stability analysis is needed for the case when both k and μ are varying sinusoidally w.r.t. the fast variable z_1 but are also allowed to vary slowly w.r.t. the parameter $z_2 = \nu z_1$.

4.) Extended Paraxial Approximation

In Section 2 we obtained an approximation that was valid as long as the secular term $\mu\varepsilon^2\xi << 1$. If we define $\Delta\tilde{n}$ through the relation

$$\eta\,\Delta\tilde{n} = \varepsilon^2\mu^2 \tag{4.1}$$

so that $\Delta\tilde{n}$ represents the normalized change in the ion density at the cylinder's edge compared to the axis when the distribution has the assumed parabolic profile, then the secular criterion can be expressed as $\eta\Delta\tilde{n}\,\varepsilon z << 1$. Since z has been nondimensionalized by a dimension on the order of $1\ cm$ and $\varepsilon = O(10^{-3})$ we find that the paraxial approximation is valid for distances less than $\frac{10\ km}{\Delta\tilde{n}}$. If $\Delta\tilde{n}$ is small, that is if the density distribution is fairly flat, then the approximation will be valid over the domain of interest for fusion reactors; whereas if $\Delta\tilde{n} = O(10^2)$ then clearly we need to extend the domain of validity of the approximation. Note that if we require the second term in Eq. (2.11) to be less than 10% of the first term, then $\Delta\tilde{n} = O(10)$ would require us to examine the extended asymptotic representation.

The method of analysis will be to introduce a new 'slow variable' $\tilde{\tau} = \frac{\mu\varepsilon^2\xi}{k^2}$ and to obtain the necessary functional dependence of the approximation on this variable by considering

both the Laguerre series solution and the differential equations obtained directly from the reduced wave equation (1.3). For simplicity we will consider here only the case when k and μ in Eq. (1.2) are constants.

From the expansion of (2.6) we can easily obtain the dependence of ψ upon $\tilde{\tau}$. Thus

$$\psi = e^{-\rho/2} \sum_{0}^{\infty} c_n \, e^{-i(2n+1)\xi - i(2n+1)^2 \frac{\tilde{\tau}}{4}} L_n(\rho) + O(\varepsilon^4 \xi) \tag{4.2}$$

When the beam is initially Guassian, it is hoped that we can sum the above series. This summation may be best accomplished however, if we examine the partial differential equations directly.

To do this we introduce the transformations

$$\frac{\partial}{\partial \xi} \to \frac{\partial}{\partial \tilde{\xi}} + \frac{2\varepsilon^2 \mu}{k^2} \frac{\partial}{\partial \tilde{\tau}}$$

$$\psi(\rho, \xi, \varepsilon) \to \psi_0(\rho, \xi, \tilde{\tau}) + O(\varepsilon^4) \tag{4.3}$$

The Eq. (2.2) is then transformed into the pair of equations

$$\left\{\rho \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial \rho} - \frac{\rho}{4} + \frac{i}{2} \frac{\partial}{\partial \xi}\right\} \psi_0 = 0 \tag{4.4}$$

and

$$\left\{i \frac{\partial}{\partial \tilde{\tau}} + \frac{1}{4} \frac{\partial^2}{\partial \xi^2}\right\} \psi_0 = 0 \tag{4.5}$$

Let the function $\psi_0(\rho,\xi,0)$ satisfy Eq. (4.4). When the beam is initially Gaussian, we can specify $\psi_0(\rho,\xi,0)$ through Eq. (2.15). Separation of variables of Eq. (4.5) leads us to suspect that

$$\psi_0(\rho,\xi,\tilde{\tau}) = \int_{-\infty}^{\infty} g(\xi,\eta,\tilde{\tau})\ \psi_0(\rho,\eta,0)\ d\eta \tag{4.6}$$

when

$$g(\xi,\eta,\tilde{\tau}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2i\lambda(\xi-\eta)-i\lambda^2\tilde{\tau}}\ d\lambda \tag{4.7}$$

The integral (4.8) is quickly evaluated to be

$$g(\xi,\eta,\tilde{\tau}) = \frac{e^{\frac{i(\eta-\xi)^2}{\tilde{\tau}} - \frac{i\pi}{4}}}{\sqrt{\pi\tilde{\tau}}} \tag{4.8}$$

so that Eq. (4.7) takes the form

$$\psi_0(\rho,\xi,\tilde{\tau}) = \frac{e^{-i\pi/4}}{\sqrt{\pi\tilde{\tau}}}\int_{-\infty}^{\infty} e^{\frac{i(\eta-\xi)^2}{\tilde{\tau}}}\ \psi_0(\rho,\eta,0)\ d\eta \tag{4.9}$$

Powerful techniques of complex analysis may now be used to study the behavior of $\psi_0(\rho,\xi,\tilde{\tau})$ once it has been verified that Eq. (4.9) is correct. We find that this is indeed so by substituting the Laguerre series representation for $\psi_0(\rho,\xi,0)$ from Eq. (2.10) into Eq. (4.9) to recover Eq. (4.2).

5.) Ray Optics and Slow Variations

A uniformly valid solution can be obtained in regions away from line focuses be neglecting diffraction. With this approximation, ray trajectories are simply obtained from the standard ray optics equation. Here we again consider a medium with an axially symmetric index of refraction and which is also allowed to vary slowly. We thus specify the index of refraction to be given by

$$n^2 = k^2(\tilde{z}) - \tilde{\mu}^2(\tilde{z})r^2$$

$$\tilde{z} = \tilde{\nu}z \tag{5.1}$$

so that the slowness is defined by the small parameter $\tilde{\nu}$. In terms of previous notation we have the following identities.

$$\tilde{\nu} = \varepsilon\nu$$

$$\tilde{\mu} = \varepsilon\mu$$

$$\tilde{z} = \nu z_1 \tag{5.2}$$

The ray equation for a medium with index of refraction (5.1) is simply

$$\frac{d}{dz}\left[\frac{n^2}{1 + (\frac{dr}{dz})^2}\right] = 2n\,\frac{\partial n}{\partial z} \tag{5.3}$$

Introduce a new function $P(z,\nu)$ through the relation

$$P(z,\nu) = \frac{n^2}{1 + (\frac{dr}{dz})^2} \tag{5.4}$$

Then tne above ray equation becomes a system of two equations for r and P .

$$(\frac{dr}{dz})^2 = \frac{n^2}{P} - 1 \tag{5.5}$$

$$\frac{dP}{dz} = 2n \frac{\partial n}{\partial z} \tag{5.6}$$

By differentiating Eq. (5.5) w.r.t. z and using Eq. (5.1) the above equations can be expressed in the following form

$$\frac{d^2r}{dz^{+2}} + \tilde{\mu}^2(\tilde{z})r = 0 \tag{5.7}$$

$$\frac{dP}{dz^+} = 2\nu[kk' - \tilde{\mu}\tilde{\mu}'r^2]P^{1/2} \tag{5.8}$$

where the new variable z^+ has been introduced by

$$\frac{dz^+}{dz} = P^{-1/2}$$

$$z^+(0) = 0 \tag{5.9}$$

The prime denotes differentiation w.r.t. the parameter $\tilde{z}$.

Again $\tilde{z}$ is interpreted to be an implicit function of z^+ obtained through use of Eq. (5.9).

We now assume $r(z^+,\nu)$ and $P(z^+,\nu)$ to have asymptotic expansions dependent upon two axial scales Z and $\tilde{z}$ so that

$$r(z^+,\nu) = r_0(Z,\tilde{z}) + \nu r_1(Z,\tilde{z}) + \ldots$$

$$P(z^+,\nu) = P_0(Z,\tilde{z}) + \nu P_1(Z,\tilde{z}) + \ldots \tag{5.10}$$

when the new variable Z is defined by

$$\frac{dZ}{dz^+} = \tilde{\mu}(\tilde{z}) \quad Z(0) = 0 \tag{5.11}$$

The derivative terms are obtained through the relation

$$\frac{d}{dz^+} = \tilde{\mu}(\tilde{z})\,\frac{\partial}{\partial Z} + \nu P^{1/2}\,\frac{\partial}{\partial \tilde{z}} \tag{5.12}$$

Usual multiple scale techniques then yield

$$r_0(Z,\tilde{z}) = r_m\sqrt{\frac{\tilde{\mu}(0)}{\tilde{\mu}(\tilde{z})}}\,\cos(Z+\phi_0)$$

$$P_0(Z,\tilde{z}) = k^2(\tilde{z}) - r_m{}^2\tilde{\mu}(0)\,\tilde{\mu}(\tilde{z}) \tag{5.13}$$

The constants r_m and ϕ_0 are determined from initial conditions.

It can be seen from Eqs. (5.13) that slow on axis density variations effects only the period of each ray

and does not influence its slowly changing amplitude. This result is in agreement with Steinhauer [4].

6.) Higher Order Considerations

In Section 4, the multiple scale technique was used to extend the domain of the asymptotic representation so that $\mu\varepsilon^2\xi = O(1)$ is allowable. On the other hand, if we are interested in the higher order terms with $\mu\varepsilon^2\xi << 1$ then a particularly effective method of including the secular terms in the asymptotic formula can be obtained by 'straining' the coordinates ρ and ξ. The small effects of beam distortion and line focusing can then be examined under the stated restriction $\mu\varepsilon^2\xi << 1$.

The asymptotic expansion of $\psi(\rho,\xi,\varepsilon)$ was found to be

$$\psi(\rho,\xi,\varepsilon) \sim \psi_0 + \frac{i}{2}\frac{\mu\varepsilon^2\xi}{k^2}\frac{\partial^2\psi_0}{\partial\xi^2} + O(\varepsilon^2\xi)^2 \tag{2.11}$$

The secular term can be removed from this expansion by use of the strained coordinates R and z defined through the reciprocal transformations

$$\xi = z + \varepsilon^2 s_1(R,z) + \ldots$$

$$\rho = R + \varepsilon^2 t_1(R,z) + \ldots \tag{6.1}$$

From these equations one can determine z and R as functions of ξ and ρ. Thus

$$z = \xi - \varepsilon^2 s_1(\rho,\xi) + \ldots$$

$$R = \rho - \varepsilon^2 t_1(\rho,\xi) + \ldots \tag{6.2}$$

The functions s_1 and t_1 are to be so chosen that the secular term in (2.11) is eliminated. This can be accomplished by substituting (6.1) into (2.11) expanding and collecting terms of $O(\varepsilon^2)$. This term must be identically zero. Thus

$$s_1 \frac{\partial \psi_0}{\partial \xi}\Big|_{(R,z)} + t_1 \frac{\partial \psi_0}{\partial \rho}\Big|_{(R,z)} + \frac{i\mu z}{2k^2} \frac{\partial^2 \psi_0}{\partial \xi^2}\Big|_{(R,z)} = 0 \tag{6.3}$$

From this we see that we are at will to specify one of the straining coefficients. We thus set s_1 equal to zero for this choice allows readily interpretable results. We then must choose

$$t_1 = - \frac{i\mu z}{2k^2} \frac{\partial^2 \psi_0}{\partial \xi^2} \Big/ \frac{\partial \psi_0}{\partial \rho} \tag{6.4}$$

Performing the required differentiations on ψ_0 when the beam is Gaussian gives the result

$$\psi_0 = \frac{1}{h(\xi)} e^{\left\{\frac{i\rho h'}{2h} + \frac{i\varepsilon^2 \xi}{2k^2} \left[\frac{2(1-\beta^2)}{h^2} \left(1+i\rho \frac{h'}{h}\right) - \frac{\rho^2 (1-\beta^2)^2}{4h^4} - 1\right]\right\}} \tag{6.5}$$

We may express h'/h in terms of its real and imaginary parts through the relation

$$\frac{h'}{h} = \frac{c'}{a} + i \left(\frac{1}{\mu a^2}\right) \tag{6.6}$$

Also $$h = \frac{a}{a_0} e^{i\phi}$$

Using these expressions, the amplitude of ψ_o can be found. Thus

$$|\psi_o| = \frac{a}{a} e^{-\frac{\rho}{2\mu a^2}} \left\{ \exp\left(-\frac{\varepsilon^2 \mu \xi}{2k^2}\right) \left[2(1-\beta^2)\left(\frac{a_o}{a}\right)^2 \left[\rho \frac{a'}{a} \cos 2\phi - \left(1 - \frac{\rho}{\mu a^2}\right) \sin 2\phi\right] - \frac{\rho^2(1-\beta^2)^2}{4} \left(\frac{a_o}{a}\right)^4 \sin 4\phi \right] \right\} \tag{6.7}$$

The beam has its maximum and minimum amplitude when $\phi = n\pi$ and $\phi = \frac{\pi}{2}(2n+1)$ respectively. At these points the intensity distribution determined from (6.7) is identical to that which ignores the secular term. Consequently, the beam is undistorted at these points. Line focusing effects can also be determined from (6.7).

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This paper shows that for a nonabsorbing medium with a prescribed index of refraction, the effects of beam stability, line focusing, and beam distortion can be predicted from simple ray optics. When the paraxial approximation is used, diffraction effects are examined for Gaussian, Lorentzian, and square beams. Most importantly, it is shown that for a Gaussian beam, diffraction effects can be included simply by adding imaginary solutions to the paraxial ray equations. Also presented are several procedures to extend the paraxial approximation so that the solution will have a domain of validity of greater extent.

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