

INCREMENTAL ANALYSIS OF LARGE ELASTIC
DEFORMATION OF A ROTATING CYLINDER

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INTRODUCTION

The effect of finite deformation upon a rotating, orthotropic cylinder was investigated by Sandman (ref. 1). He was able to predict the influence of finite deformations and relate his results to the degree of orthotropy. In this study an attempt has been made to study the same problem using a general incremental theory.

The incremental equations of motion are developed using the variational principle discussed by Washizu (ref. 2). A more than adequate development of the governing equations has been given by Atluri (ref. 3). Although his intention is to implement a finite element scheme to solve boundary value problems, the equations are given in general tensor notation. Hofmeister, Greenbaum, and Evensen (ref. 4) have presented an excellent discussion of the use of an incremental analysis; again, their goal is the application of a finite element analysis. The governing equations are also developed in the treatise by Biot (ref. 5), using both a geometrical viewpoint and a variational method. The governing equations are rederived here, in somewhat less detail, using the principle of virtual work for a body with initial stress (ref. 2).

The governing equations are reduced to those for the title problem and a numerical solution is obtained using finite difference approximations. Since the problem is defined in terms of one independent space coordinate, the finite difference grid can be modified as the incremental deformation occurs without serious numerical difficulties. The nonlinear problem is solved incrementally by totaling a series of linear solutions. This method was used to solve the same problem discussed in ref. 1 and gave identical results.

GOVERNING EQUATIONS

The derivation of the governing equations is based upon an incremental variational principle (ref. 2). The body is assumed to be in equilibrium at some arbitrary reference state along the load path. Let

$$\vec{x} = \vec{a} + \vec{u} \tag{1}$$

be the transformation of a particle at point \vec{a} to point \vec{x} in the same space, then \vec{u} is the displacement of the particle. At the beginning of some increment of load, \vec{a} is the initial coordinate and \vec{x} is the current coordinate, and the

two are identical. Let initial stresses $\underline{\sigma}^0$, initial surface tractions \vec{t}^0 , and initial body forces \vec{f}^0 act on the body before the addition of the load increment. These stresses and loads are with respect to the initial coordinate axis and are referred to a unit area before the loading increment is applied; hence, they are referred to an undeformed area and volume.

Assuming the initial stress system is in equilibrium, it follows that,

$$\text{div } \underline{\sigma}^0 + \vec{f}^0 = 0 \quad (2)$$

$$\underline{\sigma}^0 \vec{n} = \vec{t}^0 \quad (3)$$

where \vec{n} is a unit normal vector. If the body is then loaded with some increment of surface traction or body force, the total stresses at the end of that increment of load are the sum of the initial stresses and incremental stresses.

In order to formulate the principle of virtual work, first define a non-linear strain tensor, such as,

$$\underline{D} = \underline{E} + \underline{N} \quad (4)$$

where

$$\underline{E} = (\nabla \underline{u} + \nabla \underline{u}^T) \quad (5)$$

$$\underline{N} = (\nabla \underline{u}^T \nabla \underline{u}) \quad (6)$$

where \vec{u} is the displacement field corresponding to \underline{D} and \underline{D} is referred to as Green's strain tensor (ref. 6). The notation is basically the direct notation used by Gurtin (ref. 7), although some symbols are different.

Introduce a virtual displacement $\delta \vec{u}$ and incremental stresses, body forces, and surface tractions, $\underline{\sigma}$, \vec{f} , and \vec{t} , respectively. The principle of virtual work for a body with initial stress may be written,

$$\int_V \{(\underline{\sigma}^0 + \underline{\sigma}) \cdot \delta \underline{D} - (\vec{f}^0 + \vec{f}) \cdot \delta \vec{u}\} dv - \int_{S_1} (\vec{t}^0 + \vec{t}) \cdot \delta \vec{u} dS = 0 \quad (7)$$

where S_1 corresponds to the surface on which stresses are specified. Substituting equations (5) and (6) into (7) and noting that $\underline{\sigma}^0$ and $\underline{\sigma}$ are symmetric yields

$$\int_V \{ \underline{\sigma}^0 \cdot \delta \nabla \underline{u} + \underline{\sigma}^0 \cdot \nabla \underline{u}^T \delta \nabla \underline{u} + \underline{\sigma} \cdot \delta \nabla \underline{u} + \underline{\sigma} \cdot \nabla \underline{u}^T \delta \nabla \underline{u} \} dv - \int_V (\vec{f}^0 \cdot \delta \vec{u} + \vec{f} \cdot \delta \vec{u}) dv - \int_{S_1} (\vec{t}^0 \cdot \delta \vec{u} + \vec{t} \cdot \delta \vec{u}) dS = 0 \quad (8)$$

Making use of 18(1) (ref. 7), equation (8) can be rewritten as

$$\begin{aligned} \int_V \delta \vec{u} \cdot [\text{div } \underline{\sigma} + \text{div}(\underline{\sigma}^0 \nabla \underline{u}^T) + \text{div}(\underline{\sigma} \nabla \underline{u}^T) + \vec{f}] dv \\ - \int_{S_1} \delta \vec{u} \cdot [\underline{\sigma} \vec{n} + (\underline{\sigma}^0 \nabla \underline{u}^T)^T \vec{n} + (\underline{\sigma} \nabla \underline{u}^T)^T \vec{n} - \vec{t}] dS = \\ \int_{S_1} \delta \vec{u} \cdot [\underline{\sigma}^0 \vec{n} - \vec{t}^0] dS - \int_V \delta \vec{u} \cdot [\text{div } \underline{\sigma}^0 + \vec{f}^0] dv \end{aligned} \quad (9)$$

According to equations (2) and (3) the righthand side of equation (9) should be zero; therefore, the equations of equilibrium become

$$\text{div } \underline{\sigma} + \text{div}[(\underline{\sigma}^0 + \underline{\sigma})\nabla \underline{u}^T] + \underline{\vec{f}} = 0 \quad (10)$$

and the boundary condition is

$$\underline{\sigma} \underline{\vec{n}} + [(\underline{\sigma}^0 + \underline{\sigma})\nabla \underline{u}^T]^T \underline{\vec{n}} = \underline{\vec{t}} \quad (11)$$

The assumption that the incremental strains are small implies that \underline{u} is small incrementally and

$$\underline{D} = \underline{E}, \text{ i.e. } N = 0 \text{ in equation (4)}. \quad (12)$$

The initial stress may not be small; hence, we retain $\underline{\sigma}^0$ terms in equations (10) and (11). It follows from equation (12) that for a linear incremental stress-strain relation the incremental stress will be small. Therefore products of $\underline{\sigma}\nabla \underline{u}^T$ can be neglected and the governing equations become

$$\text{div } \underline{\sigma} + \text{div}(\underline{\sigma}^0 \nabla \underline{u}^T) + \underline{\vec{f}} = 0 \quad (13)$$

$$\underline{\sigma} \underline{\vec{n}} + (\underline{\sigma}^0 \nabla \underline{u}^T)^T \underline{\vec{n}} = \underline{\vec{t}} \quad (14)$$

Equations (2) and (3) serve as an error check and can be used at any increment to determine the equilibrium status of the initial stress system.

The total stress $\underline{\sigma}$ at the end of any load increment becomes the initial stress $\underline{\sigma}^0$ for the next load increment. Then, $\underline{\sigma}$ must be referred to the initial coordinates $\underline{\vec{a}}$ and the deformed area in order to become $\underline{\sigma}^0$. The transformation has been given by Fung (ref. 6) and can be rewritten as

$$\underline{\sigma}^0 = (\rho/\rho_0) \nabla_{\underline{\vec{a}}} \underline{\sigma} \nabla_{\underline{\vec{a}}}^T \quad (15)$$

where ρ/ρ_0 is the ratio of final mass to initial mass and $\nabla_{\underline{\vec{a}}}$ indicates that the operator is with respect to the initial coordinates $\underline{\vec{a}}$. It follows from equation (1) that

$$\nabla_{\underline{\vec{a}}} \underline{x} = \nabla_{\underline{\vec{a}}} (\underline{a} + \underline{u}) = \underline{\delta} + \nabla_{\underline{\vec{a}}} \underline{u} \quad (16)$$

where $\underline{\delta}$ is a unit tensor. For an incremental theory equation (16) may be written

$$\underline{\delta} + \nabla_{\underline{\vec{a}}} \underline{u} = \underline{\delta} + \nabla \underline{u} = \underline{J} \quad (17)$$

It follows that

$$\rho/\rho_0 = \det |\nabla \underline{a}| \approx 1 - \text{tr}(\nabla \underline{u}) \quad (18)$$

where $\text{tr}(\)$ represents the trace of a tensor. Combining equations (15) through (18) gives the transformation

$$\underline{\sigma}^0 = [1 - \text{tr}(\nabla \underline{u})] \underline{J} \underline{\sigma} \underline{J}^T \quad (19)$$

where

$$\underline{J}^T = (\underline{\delta} + \nabla \underline{u})^T = \underline{\delta} + \nabla \underline{u}^T \quad (20)$$

GOVERNING EQUATIONS FOR A ROTATING CYLINDER

The general equations can be reduced to plane cylindrical coordinates in order to implement the analysis of a rotating cylinder. The problem is one of axisymmetric plane strain; hence, the displacement vector \vec{u} reduces to u_r , the radial component, which will be referred to as u .

The numerical method will be applied to the equation of equilibrium (13), which in plane cylindrical coordinates may be written

$$\begin{aligned} \sigma_r' + (\sigma_r - \sigma_\theta)/r + \sigma_r^{\circ'} u' + \sigma_r^\circ(u'' + u'/r) - \sigma_\theta^\circ u/r^2 + \\ \rho\omega^2(r + u) = 0 \end{aligned} \quad (21)$$

where $f = \rho(r+u)\omega^2$ the inertia force, σ_r and σ_θ are the radial and tangential stresses, respectively, and the prime denotes differentiation with respect to r .

Equation (12) is represented by the linear strains

$$E_r = u' \quad \text{and} \quad E_\theta = u/r \quad (22)$$

Following Sandman (ref. 1) we assume a linear anisotropic stress-strain relation

$$\sigma_r = C_{11} u' + C_{12} u/r \quad (23)$$

$$\sigma_\theta = C_{22} u/r + C_{12} u/r \quad (24)$$

Substituting equations (23) and (24) into equation (21) yields the incremental governing equation

$$\begin{aligned} u'' + u'/r - \alpha u/r^2 + \sigma_r^\circ u''/C_{11} + u'(\sigma_r^{\circ'} + \sigma_r^\circ/r)/C_{11} \\ - \sigma_\theta^\circ u/C_{11}r^2 + \rho\omega^2(r+u)/C_{11} = 0 \end{aligned} \quad (25)$$

where

$$\alpha = C_{22}/C_{11} \quad \text{and} \quad \beta = C_{12}/C_{11} \quad (26)$$

The boundary condition, equation (14), becomes

$$u'(1 + \sigma_r^\circ/C_{11}) + \beta u/r = 0 \quad (27)$$

The linearized incremental stress transformation, equation (19), becomes

$$\sigma_r^\circ = \sigma_r(1 + u' - u/r) \quad (28)$$

$$\sigma_{\theta}^0 = \sigma_{\theta}(1 - u' + u/r) \quad (29)$$

NUMERICAL ANALYSIS

The governing equation (25) was solved using a finite difference technique. The primary constraint to be dealt with is the magnitude of each increment of strain. It must be small enough to insure that equation (12) is not violated. After each increment of displacement is calculated, the finite difference grid must be updated; hence, the finite difference equations must be reformulated after each incremental solution. The difference operations may be derived as follows

$$(du/dr)_i \approx (u_{i+1} - u_i)/\Delta r_2 \approx (u_i - u_{i-1})/\Delta r_1 \approx (u_{i+1} - u_{i-1})/(\Delta r_1 + \Delta r_2) \quad (30)$$

$$(d^2u/dr^2)_i \approx \frac{2u_{i-1}}{\Delta r_1(\Delta r_1 + \Delta r_2)} - \frac{2u_i}{\Delta r_1 \Delta r_2} + \frac{2u_{i+1}}{\Delta r_2(\Delta r_1 + \Delta r_2)} \quad (31)$$

The first incremental solution is merely the linear solution for the first increment of body force. Before the second incremental solution is determined, the initial stresses are assumed to be equal to the stresses obtained for the first increment. These stresses are transformed according to equations (28) and (29). The incremental displacement associated with each finite difference node is added to the coordinate of that node; hence, a new initial stress problem is formulated. The nonlinear analysis for the equation developed by Sandman (ref. 1) was obtained by transposing all nonlinear terms to the right. The displacements for the previous analysis were used to evaluate the nonlinear terms, and a solution for u is obtained. The calculated displacements are then used to calculate new nonlinear terms, and the solution is repeated. This process continues until the two sets of displacements agree to within some tolerance. This method was used to verify the results obtained by Sandman (ref. 1) and appears to be accurate and efficient.

Equations (2) and (3) can be used at any increment to determine if the initial stress system is still in equilibrium. If the initial stress system is not in equilibrium, the solution can be corrected by including equation (2) in the governing equation (25).

NUMERICAL RESULTS

Solutions were obtained for three different materials. These material parameters were assumed to approximate the behavior of steel, aluminum, and a composite epoxy-fiber orthotropic material. The maximum radial and tangential stresses are shown in figure 1 as a function of ω^2 . The cylinder was assumed to have an outside radius of 0.127 m (5 inches) and inside radius of 0.254 m (10 inches). The maximum radial stress occurs approximately halfway between the inside and outside, while σ_{θ} is maximum at the inside radius.

The percent deviation of the nonlinear solutions above the linear is illustrated in figures 2 and 3. The increase in stress using the equations of reference 1 appear to be almost linear in every case. The radial stress increase, using the incremental theory, is similar for both steel and aluminum and reflects a nonlinear behavior. The increase for the composite appears to become constant. The nonlinear tangential stress deviation increases and then tends to decrease for both isotropic materials; however, this behavior is not demonstrated for the composite.

In all cases the increase in stress level does not appear to be significant for stresses in the elastic range. The analysis presented herein should be extended to include nonlinear material behavior.

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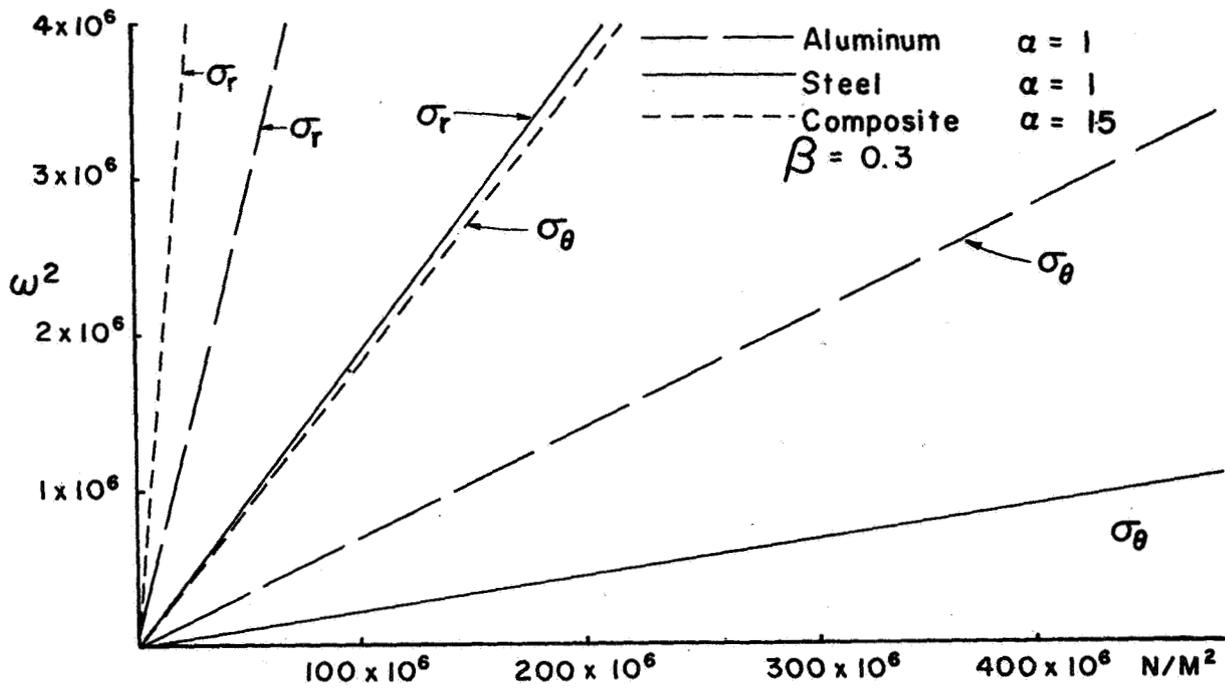


Figure 1.- Linear solution for maximum stress.

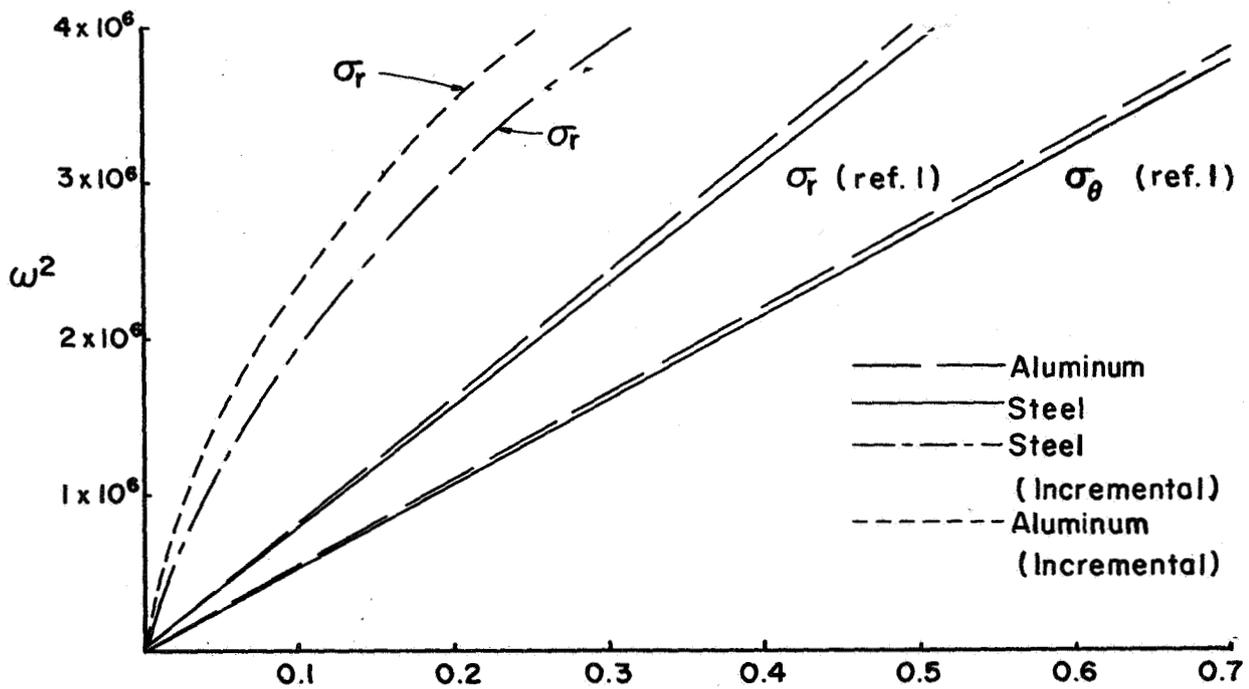


Figure 2.- Percent deviation from linear solution.

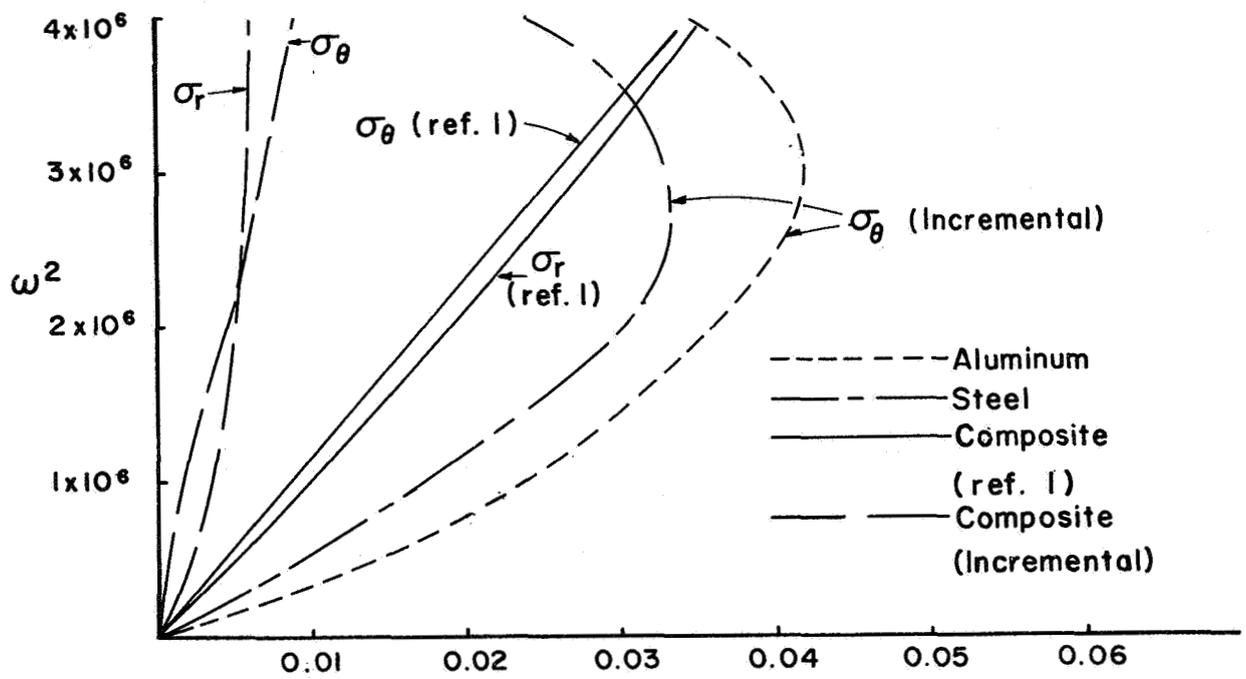


Figure 3.- Percent deviation from linear solution.