

THE DIFFUSION APPROXIMATION - AN APPLICATION

TO RADIATIVE TRANSFER IN CLOUDS

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INTRODUCTION

Water droplets and ice crystals, the constituents of clouds, are very nearly transparent (i.e. they absorb almost no radiation in the visual wavelengths (400 nm to 700 nm)). Clouds are also optically thick with optical depths for a one kilometer path ranging from 10 to 50, depending upon droplet size and number density of droplets. Therefore, visible light which enters a cloud is scattered many times before being absorbed or exiting the cloud. This type of process is well described by a diffusion model.

In this paper it is shown how the radiative transfer equation reduces to the diffusion equation. To keep the mathematics as simple as possible, the approximation is applied to a cylindrical cloud of radius R and height h . The diffusion equation separates in cylindrical coordinates and, in a sample calculation, the solution is evaluated for a range of cloud radii with cloud heights of 0.5 km and 1.0 km.

The simplicity of the method and the speed with which solutions are obtained give it potential as a tool with which to study the effects of finite-sized clouds on the albedo of the earth-atmosphere system.

THE DIFFUSION APPROXIMATION

The diffusion approximation has long been used in nuclear reactor theory (refs. 1-3) and has recently been applied to the transfer of visual radiation in snow (ref. 4). In the diffusion approximation, the radiation is assumed to have an almost isotropic angular distribution, so that the specific intensity at space point \underline{r} for radiation traveling in direction $\underline{\Omega}$ is

$$I(\underline{r}, \underline{\Omega}) \approx J(\underline{r}) - 3D\underline{\Omega} \cdot \underline{\nabla}J(\underline{r}) \quad (1)$$

$$J(\underline{r}) = \frac{1}{4\pi} \int I(\underline{r}, \underline{\Omega}) d\underline{\Omega} \quad (2)$$

is the mean intensity, D is the diffusion coefficient, and $\underline{\nabla}$ is the gradient operator. Because the net vector flux

$$\underline{\Phi} \equiv \int \underline{\Omega} I \, d\Omega \quad (3)$$

is proportional to the gradient of J :

$$\underline{\Phi} = - 4\pi D \underline{\nabla} J \quad (4)$$

it follows that J satisfies the diffusion equation

$$\nabla^2 J - L^{-2} J = 0 \quad (5)$$

where the diffusion length, L , is related to D according to

$$L^2 = D/\kappa \quad (6)$$

κ is the absorption coefficient.

The net flux through a surface with normal \hat{n} is

$$\Phi_n = \hat{n} \cdot \underline{\Phi} = - 4\pi D \frac{\partial J}{\partial n} \quad (7)$$

where $\partial/\partial n$ denotes the directional derivative. The flux in the direction of $+\hat{n}$ is

$$\Phi_n^{(+)} = \pi J - 2\pi D \frac{\partial J}{\partial n} \quad (8a)$$

and the flux in direction $-\hat{n}$ is

$$\Phi_n^{(-)} = \pi J + 2\pi D \frac{\partial J}{\partial n} \quad (8b)$$

The plus and minus signs indicate the positive and negative senses of \hat{n} .

The diffusion coefficient and the diffusion length are related to the extinction coefficient χ and the phase function of the cloud droplets according to

$$D = 1/[3\chi(1 - \omega g)] \quad (9)$$

and

$$L^2 = 1/[3\chi(1 - \omega)(1 - \omega g)] \quad (10)$$

In these equations, ω is the single scattering albedo (or fraction of light scattered in a single interaction with a cloud droplet) and g is the mean cosine of single scattering. Equations (9) and (10) are essentially approximations to the dispersion relation derived by Mika (ref. 5) for the largest singular eigenvalue in the singular eigenfunction solution for a plane-parallel atmosphere and expanded by van de Hulst (refs. 6 and 7) in his scaling laws. It may be noted that the diffusion equation (5) is identical to the Eddington approximation in a plane-parallel medium, in that for this geometry, the second moment of the radiation field, K , in a conservative atmosphere is equal to one-third of the mean intensity

$$K \equiv \frac{1}{2} \int_{-1}^1 \mu^2 I d\mu \approx \frac{1}{3} J \quad (11)$$

deep in the medium. Equation (11) uses the assumption of the plane-parallel atmosphere and describes the direction of propagation $\underline{\Omega}$ in terms of a polar coordinate system with a polar angle $\theta = \cos^{-1}\mu$ such that $\theta = 0$ is perpendicular to the plane of symmetry.

THE CYLINDRICAL CLOUD

Clouds in the atmosphere vary markedly in shape and size. To keep the mathematics as simple as possible, we have applied the diffusion approximation to a cylindrical cloud of radius R and height h . The cloud is illuminated from the top by a diffuse source, which is normalized to unit flux. It is assumed that none of the radiation that escapes the cloud returns.

With this choice of geometry and boundary conditions, the diffusion equation (5) separates. The solution for the mean intensity may be found as an expansion of standard mathematical functions in the form

$$J(r, z) = \sum_{m=0}^{\infty} [A_m \exp(\alpha_m z) + B_m \exp(-\alpha_m z)] J_0(\beta_m r) \quad (12)$$

with $J_0(\beta_m r)$ being the zeroth order Bessel function of the first kind. The separation constants α_m and β_m are related through the diffusion length in the following manner:

$$\alpha_m^2 = \beta_m^2 + L^{-2} \quad (13)$$

β_m is the solution of the transcendental equation that results from the application of the boundary condition at the sides:

$$\beta_m R J_1(\beta_m R) - C J_0(\beta_m R) = 0 \quad (14)$$

where

$$C = \frac{R}{2D} \quad (15)$$

Equation (14) is solved numerically using a Newton-Raphson method of root finding. The application of the boundary conditions at the top and bottom yields expressions for the expansion coefficients

$$B_m = \frac{2}{\pi \beta_m R J_1(\beta_m R)} \frac{1+2D\alpha_m}{[(1+2D\alpha_m)^2 - (1-2D\alpha_m)^2 \exp(-2\alpha_m h)]} \cdot \frac{1}{[1 + (\frac{\beta_m R}{C})^2]} \quad (16)$$

and

$$A_m = \frac{(2D\alpha_m - 1)}{(2D\alpha_m + 1)} \exp(-2\alpha_m h) B_m \quad (17)$$

Our primary interest in this study is in the fate of the energy incident on the cloud. As one might expect, the energy may be reflected back out the top, may be transmitted out the bottom, escape out the sides, or be absorbed within the cloud. The power incident on the cloud top is

$$F_{TOP}^{\downarrow} = \int_0^{2\pi} d\phi \int_0^R r \phi_z^{(-)}(z=0) dr = \pi R^2 \quad (18)$$

If we normalize the reflected and absorbed power by this amount, we find that they may be expressed in terms of the expansion coefficients and geometrical properties of the cloud in the form

$$E_{\text{TOP}}^{\uparrow} = 2\pi \sum_{m=0}^{\infty} [A_m (1+2D\alpha_m) + B_m (1-2D\alpha_m)] J_1(\beta_m R) / \beta_m R \quad (19)$$

$$E_{\text{SIDE}} = 2\pi \sum_{m=0}^{\infty} [A_m (\exp(\alpha_m h) - 1) + B_m (1 - \exp(-\alpha_m h))] [J_0(\beta_m R) + 2D\beta_m J_1(\beta_m R)] / \alpha_m R \quad (20)$$

$$E_{\text{BOTTOM}}^{\downarrow} = 2\pi \sum_{m=0}^{\infty} [A_m \exp(\alpha_m h) (1-2D\alpha_m) + B_m \exp(-\alpha_m h) (1+2D\alpha_m)] J_1(\beta_m R) / \beta_m R \quad (21)$$

and

$$E_{\text{ABS}} = 8\pi\kappa \sum_{m=0}^{\infty} [A_m (\exp(\alpha_m h) - 1) + B_m (1 - \exp(-\alpha_m h))] \cdot J_1(\beta_m R) / \alpha_m \beta_m R \quad (22)$$

A SAMPLE CALCULATION

In a sample calculation the above expressions were evaluated for clouds of heights 0.5 and 1.0 km and a range of cloud radii from 0.5 to 10.0 km. The cloud droplet radius was assumed to be 10 microns with a number density of 100 cm^{-3} . The mean cosine g was assumed to be 0.8516 and the single scattering albedo ω was chosen to represent nearly conservative scattering ($1-\omega = 10^{-8}$) and non-conservative scattering ($1-\omega = 10^{-2}$). These values are typical of cumulus clouds as seen in Deirmendjian's model C1 (ref. 8).

The effect of the finite radius is marked, as can be seen in Figs. 1 and 2. The albedo of an isolated cloud, 1 km thick, may be reduced by 5 percent or more if the radius is less than 5 km. This reduction in albedo is due to the leakage of energy out the sides of the cloud. As the radius becomes smaller, the escape out the sides becomes more and more important. Clouds

whose radii are about equal to their height lose nearly as much energy out the sides as is reflected back out the top.

As the radius increases we expect the results to closely approach those from a plane-parallel treatment. Table 1 shows the difference between the results using the diffusion approximation for a cylindrical cloud and those using the Eddington approximation for a plane-parallel layer. The difference is quite small for clouds whose horizontal extent is much larger than the vertical.

It is generally known that Monte Carlo techniques, which are currently being used to study the effects of finite-sized clouds (refs. 9 and 10), consume great amounts of computer time. The above results using the diffusion approximation required less than 30 seconds execution time on a CDC CYBER 175 computer. The simplicity of the approximation and the speed with which results are obtained give the diffusion approximation potential as a tool to study the effects of finite-sized clouds on the earth-atmosphere system.

CONCLUDING REMARKS

Clouds represent an optically thick medium for visible radiation in which the internal radiation field is very nearly isotropic. Such a medium is well-suited to a description by a diffusion model. Applying the diffusion approximation to a cloud of cylindrical geometry, the fraction of the incident energy emerging from each of the cloud's surfaces has been calculated. The amount of radiation escaping from the sides becomes significant when the cloud's horizontal extent is less than ten times its vertical extent. The speed and simplicity of the method argue for its use to study the effects of finite-sized clouds on the earth's albedo.

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TABLE I.-COMPARISON BETWEEN EDDINGTON (PLANE-PARALLEL) AND
DIFFUSION (CYLINDRICAL) APPROXIMATIONS
(Cloud Height = 1 km)

Plane-Parallel			Cylindrical				
			R = 10 km	5 km	2 km	1 km	0.5 km
$1-\omega = 10^{-8}$	Albedo	0.875	0.848	0.821	0.744	0.631	0.464
	Transmission	0.125	0.114	0.104	0.076	0.040	0.009
$1-\omega = 10^{-2}$	Albedo	0.549	0.538	0.528	0.498	0.450	0.361
	Transmission	0.009	0.009	0.008	0.007	0.005	0.002

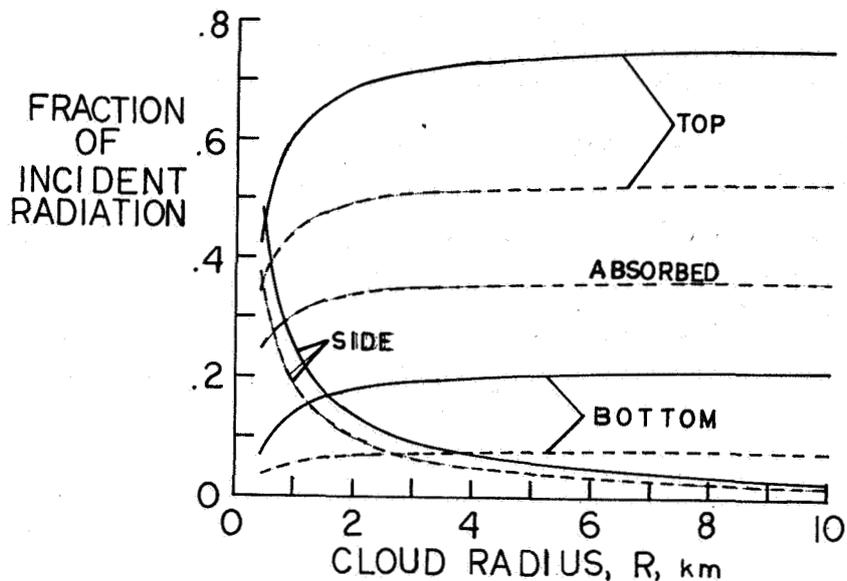


Figure 1.- Fate of energy incident upon a cylindrical cloud of height 0.5 km as a function of radius (solid line: $1-\omega = 10^{-8}$; broken line: $1-\omega = 10^{-2}$).

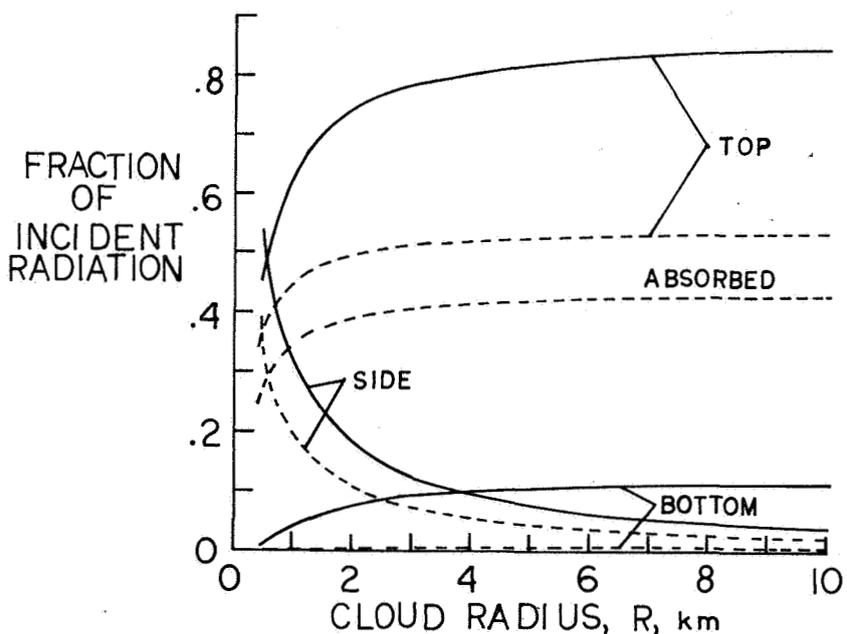


Figure 2.- Fate of energy incident upon a cylindrical cloud of height 1.0 km as a function of radius (solid line: $1-\omega = 10^{-8}$; broken line: $1-\omega = 10^{-2}$).