

STABILITY OF A VISCOUS FLUID IN A RECTANGULAR CAVITY

IN THE PRESENCE OF A MAGNETIC FIELD

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SUMMARY

The stability of an electrically conducting fluid subjected to two dimensional disturbance was investigated. The physical system consists of two parallel infinite vertical plates which are thermally insulated. Applied normal to the plates is an external magnetic field of constant strength. The fluid is heated from below so that a steady temperature gradient is maintained in the fluid. The governing equations were derived by perturbation technique, and solutions were obtained by a modified Galerkin method. It was found that the presence of the magnetic field increases the stability of the physical system, and instability can occur in the form of neutral or oscillatory instability.

INTRODUCTION

The Bérnard problem of stability of a viscous, stationary fluid heated from below has been a subject of many investigations (ref. 1). It is found that when the temperature gradient exceeds a critical value, instability sets in as stationary cellular convection. For the case of an electrically conducting fluid in the presence of a magnetic field, however, the physical phenomenon becomes more complicated. The motion of the fluid crossing the magnetic field causes electrical currents to be generated, and the current carrying fluid elements traversing the magnetic field gives rise to an additional body force (the Lorentz force) which tends to retard the fluid motion. Consequently, the stability of the system is greatly increased. Furthermore, Chandrasekhar (ref. 1) proved that depending on the relationship between the electric resistivity and the thermal diffusivity of the fluid, instability can manifest itself in the form of stationary or oscillatory motion.

Exact solution of instability due to small two dimensional disturbance of a viscous fluid bounded by two parallel vertical planes was obtained by Yih (ref. 2). The bounding planes are thermally insulated, and an upward temperature gradient is applied to the fluid. Yih showed that for disturbances

periodic in the vertical direction, the most unstable modes are associated with the wave number zero. It is also shown that instabilities due to anti-symmetric disturbances with respect to a median vertical plane are more easily excited than those due to symmetric disturbances of the same wave number.

In this investigation, it is proposed to extend Yih's finding by incorporating the effect of a magnetic field applied normal to the bounding walls. The governing equations were simplified by using Boussinesq approximation. The onset of neutral as well as oscillatory instabilities was studied.

SYMBOLS

d	distance between the vertical planes
g	acceleration due to gravity
H_0	intensity of applied magnetic field of constant strength
h	perturbation quantity of magnetic intensity
n	wave number
P	pressure
Pr_1	Prandtl number, ν/κ
Pr_2	magnetic Prandtl number, ν/η
Pr_2'	parameter defined as η/κ
Q	electromagnetic number, $(\mu H^2 d^2)/(4\pi\rho\nu\eta)$
Q'	parameter defined as $(\mu H^2 d^2)/(4\pi\rho\kappa^2)$
R	Rayleigh number, $(g\alpha\beta d^4)/(\kappa\nu)$
u,w	perturbation velocity quantity in x and z direction
x,z	space co-ordinates
α	coefficient denoting change of fluid density per degree rise in temperature
β	temperature gradients in fluid
η	electric resistivity
θ	fluid temperature

κ	thermal diffusivity
μ	magnetic permeability
ν	kinematic viscosity
ρ	density
ω_c	critical wave speed
$\bar{\delta\omega}$	parameter defined as $(\delta P/\rho) + (\mu H_o h_x)/(4\pi\rho)$
∇^2	Laplace operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$

Superscript:

* dimensionless

Subscripts:

x,z x and z component, respectively

c critical

FORMULATION

The physical system under study consists of two infinite and thermally insulated vertical planes placed at a distance d apart. Applied normal to the bounding planes is an external magnetic field of constant strength H_o . The electrically conducting fluid is heated from below so that a steady temperature gradient is maintained in the fluid. Figure 1 shows the schematic diagram of the physical system and the Cartesian co-ordinate adopted for this study.

A detailed development of the governing differential equations and the perturbation equations are presented in chapters 2 and 4 in reference 1. For the present problem, the perturbation equations are given below:

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (1)$$

Momentum equation

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} (\bar{\delta\omega}) + \nu \nabla^2 u + \frac{\mu H_o}{4\pi\rho} \frac{\partial h_x}{\partial x} \quad (2)$$

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial z} (\delta\bar{\omega}) + g\alpha\theta + v\nabla^2 w + \frac{\mu H_0}{4\pi\rho} \frac{\partial h_z}{\partial x} \quad (3)$$

Energy equation

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa \nabla^2 \theta \quad (4)$$

Equations for E-M field

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_z}{\partial z} = 0 \quad (5)$$

$$\frac{\partial h_x}{\partial t} = H_0 \frac{\partial u}{\partial x} + \eta \nabla^2 h_x \quad (6)$$

$$\frac{\partial h_z}{\partial t} = H_0 \frac{\partial w}{\partial x} + \eta \nabla^2 h_z \quad (7)$$

The physical variables used in the above equations are defined in the SYMBOLS. In order to reduce the above equation to dimensionless form, the following dimensionless quantities are introduced:

$$\begin{aligned} u^* &= \frac{ud}{\kappa}, & w^* &= \frac{wd}{\kappa}, & x^* &= \frac{x}{d}, \\ z^* &= \frac{z}{d}, & h_x^* &= \frac{h_x}{H_0}, & h_z^* &= \frac{h_z}{H_0}, \\ \theta^* &= \frac{\theta}{\beta d}, & t^* &= \frac{\kappa t}{d^2}, & \delta\bar{\omega}^* &= \frac{d^2}{\kappa} \delta\bar{\omega} \end{aligned}$$

By employing these dimensionless quantities, eqs. (1) to (7) can be expressed in non-dimension form as in the following:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial z^*} = 0 \quad (8)$$

$$\frac{\partial u^*}{\partial t^*} = -\frac{\partial}{\partial x^*} (\delta\bar{\omega}^*) + Pr_1 \nabla^{*2} u^* + Q' \frac{\partial h_x^*}{\partial x^*} \quad (9)$$

$$\frac{\partial w^*}{\partial t^*} = -\frac{\partial}{\partial z^*} (\delta\bar{\omega}^*) + Pr_1 R \theta^* + Pr_1 \nabla^{*2} w^* + Q' \frac{\partial h_z^*}{\partial x^*} \quad (10)$$

$$\frac{\partial \theta^*}{\partial t^*} = w^* + \nabla^{*2} \theta^* \quad (11)$$

$$\frac{\partial h_x^*}{\partial x^*} + \frac{\partial h_z^*}{\partial z^*} = 0 \quad (12)$$

$$\frac{\partial h_x^*}{\partial t^*} = \frac{\partial u^*}{\partial x^*} + Pr_2' \nabla^{*2} h_x^* \quad (13)$$

$$\frac{\partial h_z^*}{\partial t^*} = \frac{\partial w^*}{\partial x^*} + Pr_2' \nabla^{*2} h_z^* \quad (14)$$

Elimination of $\bar{\omega}^*$ from eqs. (9) and (10) yields

$$\begin{aligned} \left[\frac{\partial}{\partial t^*} - Pr_1 \nabla^{*2} \right] \left(\frac{\partial u^*}{\partial z^*} - \frac{\partial w^*}{\partial x^*} \right) &= Pr_1 R \frac{\partial \theta^*}{\partial x^*} \\ &+ Q' \frac{\partial}{\partial x^*} \left[\frac{\partial h_x^*}{\partial z^*} - \frac{\partial h_z^*}{\partial x^*} \right] \end{aligned} \quad (15)$$

The system of equations can be simplified by introducing the following stream functions ψ and ϕ which satisfy eqs. (8) and (12) automatically,

$$\begin{aligned} u^* &= - \frac{\partial \psi}{\partial z^*}, \quad w^* = \frac{\partial \psi}{\partial x^*} \\ h_x^* &= - \frac{\partial \phi}{\partial z^*}, \quad h_z^* = \frac{\partial \phi}{\partial x^*} \end{aligned} \quad (16)$$

Substituting eq. (16) into eqs. (15), (11) and (14), the governing eqs. become

$$\left[\frac{\partial}{\partial t^*} - Pr_1 \nabla^{*2} \right] \nabla^{*2} \psi = Pr_1 R \frac{\partial \theta^*}{\partial x^*} + Q' \frac{\partial}{\partial x^*} (\nabla^{*2} \phi) \quad (17)$$

$$\left[\frac{\partial}{\partial t^*} - \nabla^{*2} \right] \theta^* = \frac{\partial \psi}{\partial x^*} \quad (18)$$

$$\left[\frac{\partial}{\partial t^*} - Pr_2' \nabla^{*2} \right] \frac{\partial \phi}{\partial x^*} = \frac{\partial^2 \psi}{\partial x^{*2}} \quad (19)$$

Eqs. (17), (18), and (19) are subjected to the boundary conditions

$$\text{at } x^* = \pm \frac{1}{2}, \quad \psi = D\psi = D\theta^* = \phi = 0 \quad (20)$$

SOLUTION

Solutions were sought by using Galerkin method modified by Finlayson (ref. 3). Assuming disturbances of the following form

$$\psi = A(t) \psi(x) e^{inz} \quad (21)$$

$$\theta^* = B(t) \theta(x) e^{inz} \quad (22)$$

$$\phi = C(t) \Phi(t) e^{inz} \quad (23)$$

In selecting an appropriate trial function for Ψ , a disturbance symmetrical with respect to the medium plane was chosen as

$$\psi_i = \frac{\text{Cosh } \lambda_i X^*}{\text{Cosh } (\lambda_i/2)} - \frac{\text{Cos } \lambda_i X^*}{\text{Cos } (\lambda_i/2)} \quad (24)$$

where the characteristic values λ_i are given by

$$\tanh (\lambda_i/2) + \tan (\lambda_i/2) = 0 \quad (25)$$

The assumed function for θ and s are

$$\theta = \sin \alpha_i x^*, \quad s = \sin \beta_i x^* \quad (26)$$

where $\alpha_i = (2j-1)\pi$, $\beta_i = 2j\pi$ and $j = 1, 2, 3$, etc.

Substituting the trial functions given by eqs. (24) and (26) into the governing eqs. (17), (18) and (19), and after diagonalization, the governing differential equations are reduced to

$$\begin{aligned} [(C_j/C_i)'' - \eta^2(C_j/C_i)] \frac{dA}{dt} = \text{Pr}_1 [(C_j/C_i)^{iv} - 2\eta^2(C_j/C_i)'' \\ + \eta^4(C_j/C_i)] A + \text{Pr}_1 R\alpha_i (C_j/\text{Cos}\alpha_i X^*) B + \\ Q' [-\beta_i(\beta_i^2 + \eta^2)(C_j/\text{Cos}\beta_i X^*)] C \end{aligned} \quad (27)$$

$$(\sin\alpha_j/\sin\alpha_i) \frac{dB}{dt} = (\sin\alpha_j/C_i') A + [-(\alpha_j^2 + \eta^2)(\sin\alpha_j/\sin\alpha_i)] B \quad (28)$$

$$(\sin\beta_j/\sin\beta_i) \frac{dC}{dt} = (\sin\beta_j/C_i') A + \text{Pr}_2 [(\beta_j^2 + \eta^2)(\sin\beta_j/\sin\beta_i)] C \quad (29)$$

Where the inner products are defined in the following manner

$$(C_j/C_i)'' = \int_{-\frac{1}{2}}^{\frac{1}{2}} C_j D^2 C_i dx^* \quad (30)$$

The method for determining the stability criteria for eqs. (28), (29) and (30) is presented in reference 3.

DISCUSSION OF RESULTS

Numerical results were obtained by using a digital computer. The critical Rayleigh number was obtained for different combinations of the Prandtl number

Pr_1 , the electromagnetic Prandtl number Pr_2 , the electromagnetic number Q and wave number n . The results were presented in graphical form from figures 2 to 5.

For the case when $Q = 0$, the present problem was reduced to the Yih's problem. Examination of the results presented in Table 1 shows that the critical Rayleigh number converges rapidly to Yih's exact solution with the second approximation, thus confirming the validity of the results.

Table 1: COMPARISON OF R_c FOR $Q = 0$

	n	0	1	2	4
R_c	Yih's soln.	500.8	592.0	856.0	2952
	1st approx.	514.6	597.1	885.5	3100
	2nd approx.	501.0	578.8	857.0	2927

For the case of neutral stability, the critical Rayleigh number is only a function of the wave number n and Q which is a measure of the strength of the applied magnetic field (see figs. 2 and 3). The Prandtl numbers Pr_1 and Pr_2 have no effect on the stability of the system. This is expected since for the case of neutral stability, the terms involving the time derivative in the governing equations (eqs. 28 to 30) are dropped, and Pr_1 and Pr_2 would be cancelled from the resulting equations. Similar situations arise in the Bérnard problem.

For the present physical system, overstability can occur. Figure 4 shows the variation of R_c as a function of Q , Pr_1 and Pr_2 . The results show that an increase in either Pr_1 or Pr_2 tend to decrease R_c . This finding is in agreement with the results obtained for Bérnard Problem (Chapter 4, ref. 1).

With the trial functions used in this study, the convergence of the numerical results are fairly satisfactory. This is especially so in the case of neutral stability; an average discrepancy of about 5% was observed between the first and second approximation. For the case of overstability, greater deviation is observed. A difference of 25% was observed between the first and second approximations. Higher order approximation would be needed to obtain more accurate results.

CONCLUDING REMARKS

The following conclusions can be drawn from this study:

1. The presence of a magnetic field increases the stability.

2. In the presence of a magnetic field, instability can occur in the form of neutral or oscillatory instability. When oscillatory instability occurs, the effect of Pr_1 and Pr_2 becomes important. An increase in either one or both parameters tend to render the system less stable.

3. The most unstable modes are associated with the wave number zero.

REFERENCES

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2. Yih, C. S.: Thermal Instability of Viscous Fluid. Quart. Appl. Math., Vol. 17, pp. 25-42, 1959.
3. Finlayson, B. A.: The Galerkin Method Applied to Convective Instability Problems. J. of Fluid Mechanics, Vol. 33, Part 1, pp. 201-208, 1968.

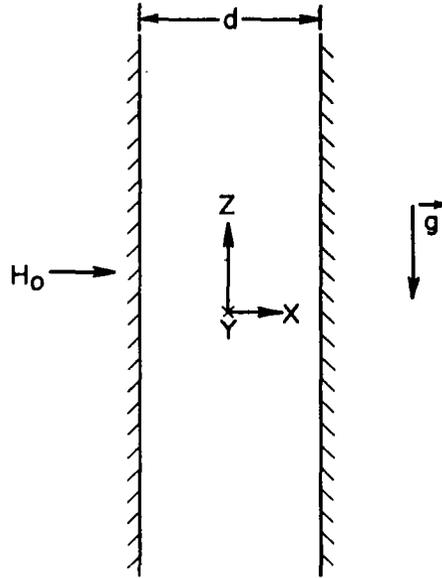


Figure 1.- Schematic diagram of physical system.

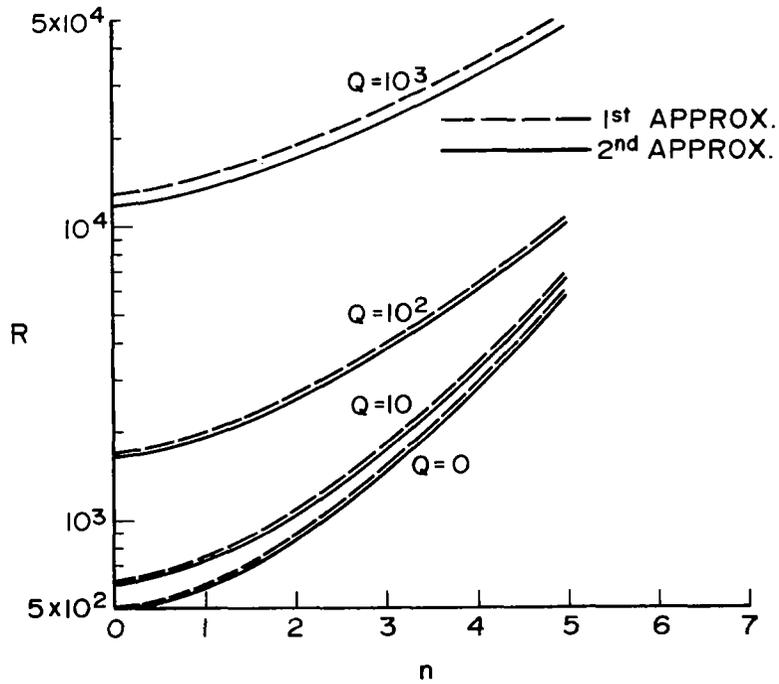


Figure 2.- Variation of R as a function of n and Q for onset of neutral stability.

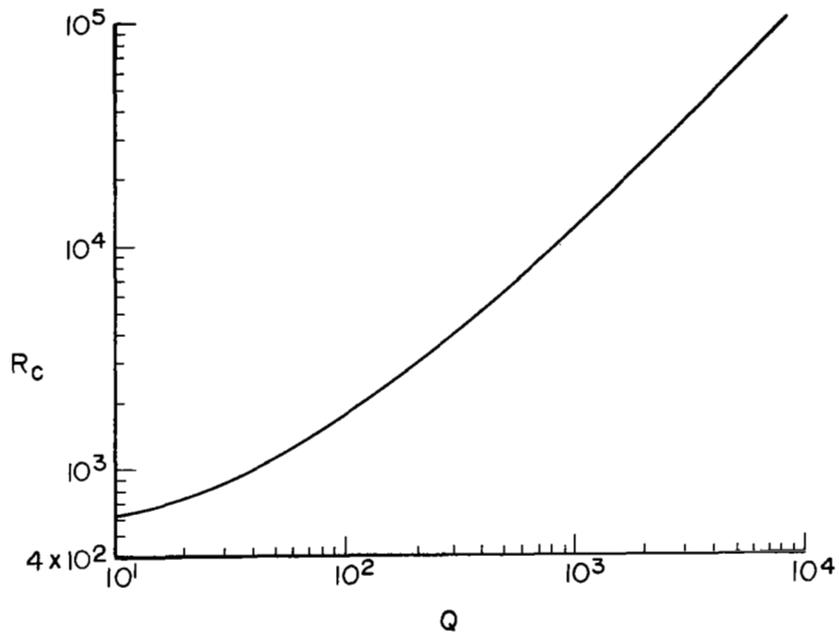


Figure 3.- Variation of R_c as a function of Q for onset of neutral stability ($Q = 0$).

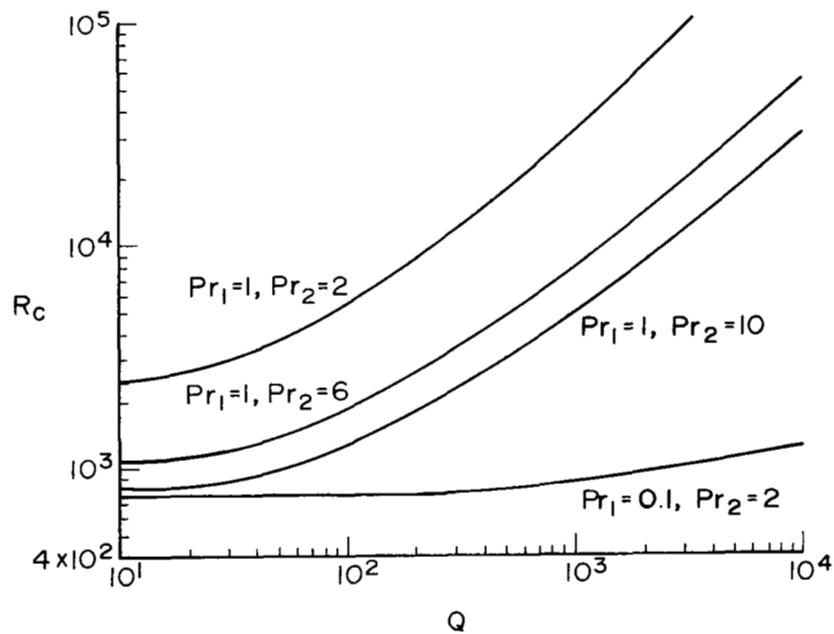


Figure 4.- Variation of R_c as a function of Q , Pr_1 and Pr_2 for onset of overstability.

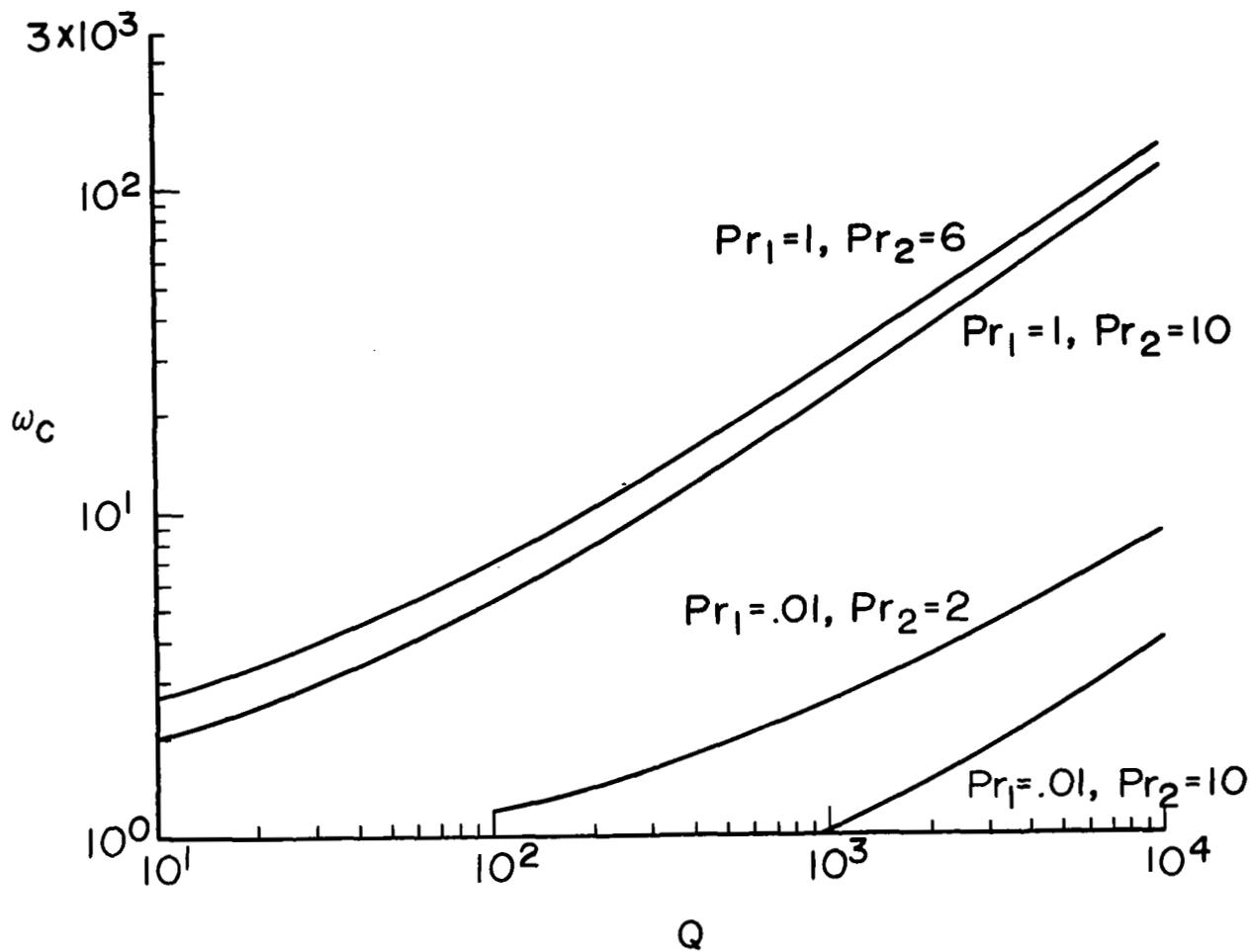


Figure 5.- Variation of critical wave speed ω_c for onset of overstability.