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TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE I. FAILURE OF THE DIFFUSION APPROXIMATION

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GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE

I. FAILURE OF THE DIFFUSION APPROXIMATION

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September 1976

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TEST PARTICLE PROPAGATION IN MAGNETOSTATIC TURBULENCE

I. FAILURE OF THE DIFFUSION APPROXIMATION

Abstract

Test particle propagation in magnetostatic turbulence with a strong mean field component is considered. The equation which governs the quasi-linear approximation to the ensemble and gyro-phase averaged one-body probability distribution function is constructed from first principles. This derived equation (the quasi-linear diabatic equation) is subjected to a thorough investigation in order to calculate the possible limitations of the quasi-linear approximation. It is shown that the reduction of this equation to a standard diffusion equation in the Markovian limit can be accomplished through the application of the "adiabatic" approximation. It has been shown that this standard diffusion equation is identical to that obtained through the assumption of a Fokker-Planck equation for the probability distribution function. In the presence of the strong mean magnetic field, the reduction to the Markovian limit is shown to be invalid. Numerical solutions of the (integrodifferential) quasi-linear diabatic equation are obtained using a simple axisymmetric slab model of the turbulent field for (i) narrow parallel beam injection, (ii) broad parallel beam injection, and (iii) narrow cross-field beam injection. A numerical solution of the standard diffusion equation in

the Markovian limit is obtained for the narrow parallel beam injection. Comparison of the diabatic and adiabatic results explicitly demonstrates the failure of the Markovian description of the probability distribution function. This failure is discussed in terms of an appropriate mode ("Laplace-mode") expansion of these solutions. For parallel beam injection, the relaxation to isotropy is shown to proceed slowly according to ln(time)/time, in contrast to the familiar exponential relaxation usually associated with diffusive behavior. The relaxation of a parallel beam to isotropy in the Markovian approximation is shown to proceed with a zero exponential decay rate; i.e., isotropy is never reached. Through the use of a linear time-scale extension the failure of the adiabatic approximation, which leads to the Markovian limit, is shown to be due to mixing of the relaxation and interaction time scales in the presence of the strong mean field.

I. INTRODUCTION

This is the first in a series of papers in which the propagation of a charged test particle in magnetostatic turbulence is considered. The central issue in this work is the presence of a mean magnetic field which is strong enough to affect the collision process between the charged particle and the turbulent field. For the purpose of this study, the magnetic field is considered "strong" if the Larmor radius of the particle in the mean field is comparable to the two-point correlation length associated with the turbulent field. If electron-ion collisions in a plasma were being considered here, the field would be considered "strong" if the Larmor radius were comparable to the Debye length. In this situation some standard methods of kinetic theory which are suitable in studies of neutral gases or non-magnetized plasmas, become inapplicable. The failure of these methods will be demonstrated here, and alternatives will be presented.

Test particle propagation in magnetostatic turbulence can be considered an idealization of the behavior of a particle in a hot, turbulent MHD plasma in which the effects of particle-particle collisions are negligible compared to the effects of particle-wave collisions. If the speed of the particle is high compared to the MHD wave propagation speeds, then the motion of the particle is dominated by the magnetic field, and, as interesting as the effects of the electric field are, they are nevertheless, secondary. The full, self consistent, treatment of the plasma problem is not attempted here; if it were, then the work being presented here would be a necessary part of that attempt. Alternatively, sufficient

measurements of the field can be relied upon to test this portion of the complete treatment. The solar wind plasma which fills interplanetary space is an excellent candidate for this type of study; it is a dilute, high temperature, turbulent MHD plasma whose particle and field properties have been measured extensively.[1]

The interplanetary plasma has been observed to have an extremely long, power law tail in its energy distribution. The particles which make up this tail are called cosmic rays. They have their source predominantly at the Sun in the several tens of Mev/particle energy range, outside the solar system for higher energies, and perhaps, even outside the Milky Way galaxy at the highest observed energies ($\simeq 10^{20}$ eV/particle).[1,2] The speeds of these particles are usually high compared to the typical MHD wave speeds found in the interplanetary or interstellar plasma. Thus, the assumption of magnetostatic turbulence is a very good approximation for these particles. In addition, the energy density of these particles is so low, they almost always have negligible affect on the field through which they move in any physical system which has dimension less than that of an entire galaxy like the Milky Way.[3] As might be expected, there is a long history of the treatment of the cosmic ray propagation problem in the magnetostatic, test particle limit.[4]

The earliest attempts to describe the spatial transport of cosmic rays were invariably based on diffusion equations for the cosmic ray density with spatial and energy convection terms added when it was deemed necessary.[2,5] With the advent of space exploration, and in particular, with the advent of detailed

measurements of the interplanetary magnetic field, it became possible to investigate the theoretical foundations of these assumed diffusion equations. In the first efforts in this direction, kinetic theories which were known to be applicable in other situations were borrowed and assumed to apply to the cosmic ray probability distribution function. Klimas,[6] and independently, Gleeson and Axford[7] adopted the Boltzman equation and through moment expansions of the distribution function, constructed transport equations which were similar to those in use previously except that the transport coefficients all became interrelated through an effective mean free path for scattering on the magnetic inhomogenieties. This effective mean free path came from the Boltzman collision integral which was assumed to represent the wave-particle interaction. Application of these theories was accomplished through phenomenological adjustments of the effective mean free path. The relationship between this mean free path and the actual interaction mechanism remained vague.

Jokippi,[8] and independently Sturrock,[9] adopted the Fokker-Planck equation to describe the phase-space propagation of the cosmic ray probability distribution function. These theories were also convection-diffusion theories in phase-space, and through moment expansions, could be used to construct convection-diffusion transport theories in configuration space which were again similar to those in use previously. The major significance of these theories was that the transport coefficients which appeared in them could be directly related to measurable properties of the interplanetary field; specifically, the mean field and the two-point correlation tensor associated with the turbulent field.

These results were reassuring in a certain sense; no matter what kinetic basis was considered, a convection-diffusion transport theory seemed to result. However, what had really been accomplished was to make the assumption of a diffusion process somewhat less visible. Both the Boltzman, and the Fokker-Planck equations can be derived from first principles under appropriate conditions and in their respective realms of applications.[10] The adiabatic approximation, or its equivalent, is a necessary part of these derivations. But, the imposition of the adiabatic approximation is equivalent to the assumption of a Markov chain for the relevant collision process. If the adiabatic approximation can be applied, then the particle motion is well approximated by a random walk process, and the macroscopic transport description of the fluid in question is necessarily governed by a diffusion equation with convective phenomena possibly included.

The first attempt to derive the kinetic theory of test particle propagation in magnetostatic turbulence from first principles was made by Hall and Sturrock.[11] They managed to regain the Fokker-Planck equations which had been assumed by Jokipii and Sturrock, but, in doing so, they had to apply two approximations; first the quasi-linear approximation, and then, in fact, the adiabatic approximation. In the quasi-linear approximation, the impulse imparted to a particle during an interaction with the turbulent field is assumed small compared to the momentum of the particle. Thus, this impulse is calculated on the basis of an undeviated trajectory through the interaction. Implicit in this picture, is the assumption that the interaction between the particle and the turbulent field can be characterized

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as a weak "collision" of short duration so that the net impulse imparted to the particle remains small. In keeping with this assumed weak coupling of short duration, it seems reasonable to assume that the distribution function should evolve on a time scale which is much larger than the duration of a collision. Thus, it seems reasonable to make the adiabatic approximation, in which the evolution of the distribution function during a collision is ignored. This picture is correct if the mean magnetic field is not strong, or alternatively, if the particle energy is high.[12] However, Klimas and Sandri,[13] using a special isotropic model of the magnetostatic turbulence and a spherical harmonic expansion of the distribution function, showed that this picture becomes incorrect when the field becomes strong. Klimas and Sandri showed that, within their model, when the mean magnetic field is strong, then it is inconsistent to apply both the quasi-linear and adiabatic approximations. This inconsistency arises because in the presence of the strong mean magnetic field, the undeviated, or zero'th order trajectory, is a helix and therefore progress of the particle in space in the direction of the mean magnetic field is governed by the parallel component of its velocity which is a constant of the particle motion. If the parallel velocity is made arbitrarily small, then the particle becomes quasi-trapped in its interaction with the random component of the field, and one, or both, of the assumptions on the smallness of the impulse imparted to the particle during the interaction, as well as on the clear separation between the interaction and evolution times scales, breaks down.

Recently, Golstein, Klimas and Sandri[14] have shown that, within the quasi-linear adiabatic system of approximations (or equivalently, in the Fokker-Planck equation), when the parallel component of the particle velocity is zero, then with few exceptions, the calculated strength of the interaction with the random field becomes infinite. They have also shown that this divergence can be directly attributed to the physical phenomenon of mirroring. Clearly, if the impulse imparted to a particle through its interaction with the random field results in mirroring of the particle, then this impulse in the parallel direction cannot be considered small. In the approximation scheme being discussed here, in which the particle-wave interaction is considered asymptotically small, the finite impulse due to mirroring appears as a divergence in the theory.

One of the few exceptions to the appearance of a divergence in the wave-particle interaction strength as calculated in the quasi-linear adiabatic approximation, is found in the slab model of the random magnetic field which we consider in this series of papers. In this model of the field, the interaction strength is calculated to be zero at zero parallel velocity. Because first order mirroring is impossible in this model of the field, the divergence due to mirroring vanishes. However, the interaction and evolution time scales still remain mixed, and a more subtle failure of the quasi-linear adiabatic approximation scheme will be explicitly demonstrated with the use of a linear time-scale extension in Section IV. It is the general failure of the quasi-linear adiabatic approximation due to the mixing of the interaction and evolution time scales which is of concern

to us here, more than any specific field-model dependent manifestiation of this failure.

Various attempts have been made to construct non-linear theories of test particle propagation in which the undeviated particle trajectory is replaced by one which contains the effects of the wave-particle interaction being calculated.[15] In these theories, it is assumed that the particle propagation in the random field can be described through a diffusion process, and the undeviated trajectory is replaced by a diffusing trajectory. The amount of diffusion in the trajectory is computed so that the diffusion coefficient which is calculated using that trajectory is self consistently calculated. In this case, it becomes impossible for a particle trajectory to remain trapped, and with this modification of the quasi-linear approximation, the adiabatic approximation can be formally retained. The final result, then, is a Markovian, or diffusive, description of the particle transport which is consistent with the diffusive modification of the undeviated trajectory. In this method, the adiabatic approximation is retained, and the quasi-linear approximation is modified to allow that choice. The reasons for this choice are unclear. In particular, we believe that the predictions of the quasi-linear theory, with no further approximations, have never been calculated. It has not been clear what, if anything, is incorrect in the quasi-linear theory. If the quasi-linear theory does fail, then modifications of that theory should be made on the basis of that failure. With this approach, perhaps a kinetic theory can be constructed which is free from initial prejudices on the ultimate result of that construction.

In this series of papers, we determine the predictions of the quasi-linear theory in a very simple model of the turbulent magnetic field which will be fully described later. We find that the adiabatic approximation to the quasi-linear theory does not make sense. We further find an alternative kinetic approximation which works very well, and is not governed by a diffusion equation. We conclude that the quasi-linear theory does fail as a leading approximation in a systematic expansion scheme, and we demonstrate why this is so.

In this first paper, we construct an equation from first principles for the quasi-linear approximation to the ensemble and gyrophase averaged probability distribution function. This equation is a velocity space diffusion equation which is non-local in time; it is integrodifferential in time. We demonstrate that application of the adiabatic approximation to this equation leads to a Markovian description of the probability distribution function which is governed by an ordinary velocity space diffusion equation. This diffusion equation can also be obtained through the assumption of a Fokker-Planck equation for the probability distribution function.[13] Through both analytic and numerical considerations, we demonstrate the failure of the Markovian, or diffusive, description of the quasi-linear probability distribution function. Thus, we conclude that the presence of a strong mean magnetic field can preclude the application of a Fokker-Planck or diffusion equation to the propagation problem for charged particles in plasma turbulence.

We proceed with a study of the properties of the quasi-linear solutions with no further approximations in this paper. Numerical solutions of the quasi-linear,

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and quasi-linear adiabatic equations are obtained, and compared, for an axi-symmetric, slab model of the plasma turbulence with an exponential two-point correlation function. In this model, the random component of the field is orthogonal to the mean field direction, it is stationary in time, and a function of the spatial coordinate in the direction of the mean field only. In addition, the two-point correlation tensor associated with the random component of the field is assumed cylindrically symmetric about the mean field direction. With this model the quasi-linear adiabatic theory predicts that the parallel velocity of a particle can never reverse itself as a result of the interaction with the turbulent field. This prediction is in strong disagreement with the numerical solutions of the quasi-linear equation without the adiabatic approximation (the quasi-linear diabatic equation).

We conclude this paper with a discussion of a mode expansion of the solutions of the quasi-linear diabatic equation. We introduce a "Laplace-mode" analysis which is analogous to the standard modal analysis often used in plasma physics. We show that the long time evolution of the quasi-linear solutions can be understood in terms of these Laplace-modes. In addition, we find that the reduction of these solutions to their adiabatic limit is easily expressed with the Laplace-modes, and that the failure of this limit becomes apparent within this picture. For example, the injection of a beam of particles in the direction of the mean magnetic field is considered in some detail. As pointed out above, in the quasi-linear adiabatic approximation, no particles ever reverse their directions, and

so, the beam never relaxes to isotropy. In the quasi-linear diabatic approximation, the beam does relax to isotropy, but very slowly. From the Laplace-mode analysis we find that in this case the relaxation to isotropy goes like ln(time)/time, in contrast to the exponential relaxation usually associated with diffusive behavior. (In the adiabatic case, the diffusive relaxation of the beam is exponential, however, with a zero decay rate. Thus, the beam never relaxes.) In the adiabatic limit, we find that the Laplace-modes become doubly degenerate, and discontinuous functions of the parallel component of the particle velocity. The failure of the adiabatic limit follows from these discontinuities in the Laplace-modes. In the next paper in this series we introduce a new kinetic approximation which is constructed specifically to remove these discontinuities in the Laplace-modes.

II. THE BASIC EQUATIONS

We imagine an ensemble of stochastic, stationary magnetic fields. A mean field, $\langle \mathbf{B} \rangle$, which is an ensemble average is assumed, and the field in any ensemble representative is represented by $\mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{B}'$. The "random field," $\mathbf{B}'$, is assumed to obey $\langle \mathbf{B}' \rangle = 0$, or equivalently, it is assumed that $\langle\langle \mathbf{B} \rangle\rangle = \langle \mathbf{B} \rangle$. In the absence of two-body or higher order particle correlations the one-body probability distribution function, $F(\mathbf{x}, \mathbf{p}, t)$, in any one of the ensemble representatives obeys the Liouville equation,

$$\frac{\partial F}{\partial \tau} + KF + \mathcal{L}F + \eta \mathcal{L}' F = 0 \qquad \text{I.1}$$

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The dimensionless version of the Liouville equation given here has been obtained by setting $\tau = \omega_0 t$, where ω_0 is the Larmor frequency in the mean field strength, and by measuring lengths in units of the gyro-radius, r_0, in the mean field strength. The parameter, η, is a measure of the strength of the random field compared to the strength of the mean field; it is defined by,

$$\eta \equiv \frac{B'_{rms}}{\langle B \rangle} \qquad \text{I.2}$$

and will be assumed small. The linear, first order partial differential operators, K, $\mathfrak{L}$, and, $\mathfrak{L}'$, are given by,

$$K = \hat{\mathbf{p}} \cdot \frac{\partial}{\partial (\mathbf{x}/r_g)} \qquad \text{I.3}$$

$$\mathfrak{L} = (\hat{\mathbf{p}} \times \hat{\boldsymbol{\beta}}) \cdot \frac{\partial}{\partial \hat{\mathbf{p}}} = - \hat{\mathbf{p}} \cdot \boldsymbol{\Omega} \cdot \frac{\partial}{\partial \hat{\mathbf{p}}} \qquad \text{I.4}$$

and,

$$\mathfrak{L}' = (\hat{\mathbf{p}} \times \boldsymbol{\beta}') \cdot \frac{\partial}{\partial \hat{\mathbf{p}}} = - \hat{\mathbf{p}} \cdot \boldsymbol{\Omega}' \cdot \frac{\partial}{\partial \hat{\mathbf{p}}} \qquad \text{I.5}$$

where $\hat{\boldsymbol{\beta}}$ is a unit vector in the direction of the mean field, $\hat{\mathbf{p}}$ is a unit vector in the direction of the particle momentum, $\boldsymbol{\beta}' = \mathbf{B}'/B'_{rms}$, and $\partial/\partial\hat{p}$ is defined by

$$\left(\frac{\partial}{\partial \hat{\mathbf{p}}}\right)_i = p(\delta_{ij} - \hat{p}_i \hat{p}_j) \frac{\partial}{\partial p_j}. \qquad \text{I.6}$$

The skew-symmetric tensors, $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$, are defined by $\Omega_{ij} = \epsilon_{ijk}\beta_k$ and $\Omega'_{ij} = \epsilon_{ijk}\beta'_k$. Later, the integral or correlation length associated with the two-point correlation function for the random field will be introduced. In this work

this length will be assumed comparable to the particle gyro-radius, and as a matter of convenience we set

$$\epsilon \equiv \frac{\lambda_c}{r_g} = 1. \qquad \text{I.7}$$

We also introduce an ensemble averaged probability distribution function, $f(\mathbf{p}, \tau) = \langle F \rangle$, which is assumed independent of position, $\mathbf{x}$, and the random probability distribution function through $F(\mathbf{x}, \mathbf{p}, \tau) = f(\mathbf{p}, \tau) + F'(\mathbf{x}, \mathbf{p}, \tau)$. We further assume $\langle F' \rangle = 0$, or $\langle f \rangle = f$. Klimas and Sandri,[16] using a technique developed by Kaufman,[17] have shown that, if $F'(\tau = 0) = 0$, then

$$\frac{\partial f}{\partial \tau} + \mathcal{L} f = \eta^2 \left[1 - \eta \langle \mathcal{L}' G \rangle\right]^{-1} \langle \mathcal{L}' G \mathcal{L}' \rangle \, f \qquad \text{I.8}$$

where, $G \equiv [\partial/\partial\tau + \mathcal{H}]^{-1}$, is the Green's integral operator, with the total Hamiltonian operator, $\mathcal{H} = K + \mathcal{L} + \eta \mathcal{L}'$, for the generator of the particle motion. The Green's integral operator can be written,

$$G = [1 + \eta G_0 \mathcal{L}']^{-1} G_0 \qquad \text{I.9}$$

where, $G_0 = [\partial/\partial\tau + \mathcal{H}_0]^{-1}$, with the zero'th order Hamiltonian operator, $\mathcal{H}_0 = \mathcal{L} + K$, for the generator of the particle motion. (We assume that the mean field is uniform in space, and then, $\mathcal{H}_0$ generates the well known helical particle trajectories in a uniform magnetic field.) With the use of equation I.9, we expand equation I.8 in powers of $\eta G_0 \mathcal{L}'$ to obtain,

$$\frac{\partial f}{\partial \tau} + \mathcal{L} f = \eta^2 \langle \mathcal{L}' G_0 \mathcal{L}' \rangle \, f + O(\eta^3) \qquad \text{I.10}$$

If this expansion is valid, then we say that we are in the "weak coupling" regime, or that we have made the weak coupling assumption. A truncation of this expansion beyond the second order term which is exhibited explicitly on the right hand side gives an equation for the quasi-linear approximation to the ensemble averaged probability distribution function. A major purpose of this series of papers is to present a class of situations in which the formal order of the terms which have been dropped in this truncation can be shown to be incorrect when the quasi-linear approximation to the probability distribution function is assumed a valid approximation and is used to evaluate the actual order of the correction terms in this approximation scheme.

III. THE QUASI-LINEAR TRUNCATION IN THE AXISYMMETRIC SLAB FIELD MODEL

We adopt the quasi-linear truncation of equation I.10, and develop the specific form that that equation takes on in an "axisymmetric slab" model of the random magnetic field. In this model, the random field is orthogonal to the mean field, and a function only of the spatial coordinate which lies along the direction of the mean field. In addition, the random field takes on any direction in a plane orthogonal to the mean field with equal probability from ensemble representative to representative.

The particle energy is a constant of the motion in the static magnetic field. Since we have already assumed that the ensemble averaged probability distribution function is independent of position, we see that the phase space relevant to

our problem is the surface of a sphere at constant energy in the particle momentum space. Actually, through our dimensional analysis which lead to the dimensionless Liouville equation, we have reduced our problem to that of studying the motion of the particles on the unit sphere. Thus, it is most convenient to introduce a polar coordinate system, (θ, ϕ) in the particle momentum space, in which the polar coordinate, θ, is the pitch angle of the particle relative to the mean field, and ϕ is the gyro-phase angle of the particle measured about the mean field. In addition, we introduce, $\mu = \cos\theta$. The differential operators, $\mathcal{L}$ and $\mathcal{L}'$, are given in terms of these variables by,

$$\mathcal{L} = -\frac{\partial}{\partial\phi} \qquad \text{I.11}$$

$$\mathcal{L}' = (\hat{\mathbf{p}} \cdot \Omega \cdot \boldsymbol{\beta}')\frac{\partial}{\partial\mu} + \left(\frac{\mu}{1-\mu^2}\right)(\hat{\mathbf{p}} \cdot \boldsymbol{\beta}')\frac{\partial}{\partial\phi}$$

$$= \frac{\partial}{\partial\mu}(\hat{\mathbf{p}} \cdot \Omega \cdot \boldsymbol{\beta}') + \frac{\partial}{\partial\phi}\left(\frac{\mu}{1-\mu^2}\right)(\hat{\mathbf{p}} \cdot \boldsymbol{\beta}') \qquad \text{I.12}$$

The gyro-phase average of any quantity, $Q(\phi)$, is defined by,

$$\overline{Q} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\, Q(\phi) \qquad \text{I.13}$$

Through the substitution of equations I.11 and I.12 into the truncated equation I.10, and further, through gyro-phase averaging the entire equation, we find,

$$\frac{\partial \bar{f}(\mu, \tau)}{\partial \tau} =$$

$$= -\eta^2 \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\tau} d\lambda \frac{\partial}{\partial \mu} \hat{\mathbf{p}} \cdot \boldsymbol{\Omega} \cdot \left\langle \boldsymbol{\beta}'(z)\, e^{-\mathcal{H}_0 \lambda}\, \boldsymbol{\beta}'(z) \right\rangle \cdot \boldsymbol{\Omega} \cdot \hat{\mathbf{p}} \frac{\partial f(\mu, \phi, \tau - \lambda)}{\partial \mu}$$

I.14

$$+ \eta^2 \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\tau} d\lambda \frac{\partial}{\partial \mu} \hat{\mathbf{p}} \cdot \boldsymbol{\Omega} \cdot \left\langle \boldsymbol{\beta}'(z)\, e^{-\mathcal{H}_0 \lambda}\, \boldsymbol{\beta}'(z) \right\rangle \cdot \hat{\mathbf{p}} \left(\frac{\mu}{1 - \mu^2} \right) \frac{\partial f(\mu, \phi, \tau - \lambda)}{\partial \phi}$$

where we have adopted the symbol, z, for the spatial coordinate along the mean field, and where the zero'th order streaming operator, $\exp(-\mathcal{H}_0 \lambda)$ acts on any function, $A(z, \hat{\mathbf{p}})$, as follows:

$$e^{-\mathcal{H}_0 \lambda} A(z, \hat{\mathbf{p}}) = A(z - \mu\lambda)\, \mathbf{C}(\lambda) \cdot \hat{\mathbf{p}}) \qquad \text{I.15}$$

with,

$$\mathbf{C}(\lambda) = \mathbf{P} + \mathbf{N} \cos \lambda - \boldsymbol{\Omega} \sin \lambda \qquad \text{I.16}$$

in which,

$$P_{ij} = \hat{\beta}_i \hat{\beta}_j \qquad \text{I.17}$$

$$N_{ij} = \delta_{ij} - \hat{\beta}_i \hat{\beta}_j \qquad \text{I.18}$$

and $\boldsymbol{\Omega}$ is as defined previously. We assume that the magnetostatic turbulence is homogeneous and introduce the two-point correlation tensor and its axisymmetric slab model through,

$$\left\langle \beta_i'(z)\, \beta_j'(z + \zeta) \right\rangle \equiv R_{ij}(\zeta) = N_{ij} R(\zeta), \qquad \text{I.19}$$

with the correlation function, $R(\zeta)$, an even function of ζ. Through the substitution of equations I.15 thru I.18 into equation I.14, we find the second term on the right side of that equation is identically zero, and the remainder of that equation reduces to,

$$\frac{\partial \bar{f}(\mu, \tau)}{\partial \tau} = \eta^2 \frac{\partial}{\partial \mu} (1 - \mu^2) \int_0^\tau d\lambda \, \mathcal{K}(\mu, \lambda) \frac{\partial \bar{f}(\mu, \tau - \lambda)}{\partial \mu} . \qquad \text{I.20}$$

where the kernel function, $\mathcal{K}(\mu, \lambda)$, is given by,

$$\mathcal{K}(\mu, \lambda) = R(\mu\lambda) \cos \lambda. \qquad \text{I.21}$$

Thus, in the axisymmetric turbulence, the gyrotropic part of the probability distribution function evolves independently.

Equation I.20 can be written in the renewal form,

$$\bar{f}(\mu, \tau) = \bar{f}(\mu, 0) + \eta^2 \frac{\partial}{\partial \mu} (1 - \mu^2) \int_0^\tau d\lambda \int_0^\lambda ds \, \mathcal{K}(\mu, s) \frac{\partial \bar{f}(\mu, \tau - \lambda)}{\partial \mu} \qquad \text{I.22}$$

and, with the introduction of

$$N(\mu, \tau) = \int_{-1}^{\mu} d\mu' \, \bar{f}(\mu', \tau) \qquad \text{I.23}$$

the equivalent equations,

$$\frac{\partial N(\mu, \tau)}{\partial \tau} = \eta^2 (1 - \mu^2) \int_0^\tau d\lambda \, \mathcal{K}(\mu, \lambda) \frac{\partial^2 N(\mu, \tau - \lambda)}{\partial \mu^2} \qquad \text{I.24}$$

and,

$$N(\mu, \tau) = N(\mu, 0) + \eta^2 (1 - \mu^2) \int_0^\tau d\lambda \int_0^\lambda ds \, \mathcal{K}(\mu, s) \frac{\partial^2 N(\mu, \tau - \lambda)}{\partial \mu^2} \qquad \text{I.25}$$

can be constructed. All of these equations will be used in the following to understand the behavior of $\bar{f}(\mu, \tau)$.

IV. THE MARKOVIAN LIMIT

Equation I.20 can be characterized as a non-local diffusion equation; it is integrodifferential in time. This equation can be further reduced to a standard diffusion equation through the application of the "adiabatic" approximation. This diffusion equation is identical to the diffusion equation which follows from the assumption that the ensemble averaged probability distribution function obeys a Fokker-Planck equation.[14] Since the Fokker-Planck equation has a Markovian process as a fundamental assumption,[18] the adiabatic approximation reduces the non-local diffusion equation, which has an obvious memory of the past, and is non-Markovian, to its Markovian approximation.

We will demonstrate in this section that a necessary requirement for the reduction of the non-local equation to the local one, is a clear cut separation between the presumed fast time scale during which a "collision" with the turbulent magnetic field takes place, and the much slower time scale over which the distribution function is assumed to evolve. In fact, this separation of the two time scales does not exist, and the Markovian approximation can not consistently be made. We will first present a brief intuitive derivation of the adiabatic approximation and then a more rigorous derivation based on the time-scale extension method.[10] We will see that, although the extension method is able to produce a uniform expansion of $\overline{f}(\mu, \tau)$ in time, the expansion is still non-uniform in μ. The non-uniformity in μ can be traced directly to the mixing of the interaction and evolution time scales in the μ-domain in which the expansion is non-uniform.

(a) Preliminary Construction

The intuitive argument which is often used to justify the adiabatic approximation proceeds as follows: One can see from equation I.20 that the time rate of change of $\bar{f}(\mu, \tau)$ is small if η is small. Therefore replace $\bar{f}(\mu, \tau-\lambda)$ in the integrand of that equation by $\bar{f}(\mu, \tau)$ when τ is large and λ is not. Rely on the presumed short range of the kernel to prevent contributions to the integral for large λ; i.e., assume that $\mathcal{K}(\mu, \lambda) = 0$, if $\lambda >> 1$. Then, for $\tau >> 1$,

$$\frac{\partial \bar{f}(\mu, \tau)}{\partial \tau} \approx \eta^2 \frac{\partial}{\partial \mu} \mathcal{D}_0(\mu) \frac{\partial \bar{f}(\mu, \tau)}{\partial \mu} \qquad \text{I.26}$$

where,

$$\mathcal{D}_0(\mu) = (1 - \mu^2) \int_0^\infty d\lambda\, \mathcal{K}(\mu, \lambda). \qquad \text{I.27}$$

Equation I.26 gives the Markovian, approximation to $\bar{f}(\mu, \tau)$. This equation has been obtained by assuming that $\bar{f}(\mu, \tau)$ does not evolve in time over the short period of time during which $\mathcal{K}(\mu, \lambda) \to 0$ with increasing λ. From equation I.21, the short range of the kernel should be provided by the short range of the correlation function in the random field, but the range of the correlation function in λ can be made arbitrarily large as μ is allowed to take on values arbitrarily close to zero. It is this mixing of the evolution and interaction time scales for those particle trajectories which are quasi-trapped in the interaction with the random field due to their small parallel velocity, μ, which leads us to suspect the Markovian description of $\bar{f}(\mu, \tau)$.

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(b) Linear Time-Scale Construction

Through the use of a linear time scale extension[10] we are able to obtain equation I.26 as a leading result in a systematic expansion procedure which also yields a higher order correction to $\overline{f}(\mu,\tau)$. We will demonstrate here that this correction becomes unbounded for large times when $\mu = 0$, thus providing evidence for the failure of the adiabatic, or Markovian, approximation to $\overline{f}(\mu,\tau)$.

The extension of equation I.20 which we consider here is given by,

$$\left(\frac{\partial}{\partial\tau_0} + \eta^2 \frac{\partial}{\partial\tau_2}\right) \mathfrak{F}(\mu, \tau_0, \tau_2) = \eta^2 \frac{\partial}{\partial\mu} (1-\mu^2) \int_0^{\tau_0} d\lambda\, \mathcal{K}(\mu, \lambda) \frac{\partial}{\partial\mu} \mathfrak{F}(\mu, \tau_0 - \lambda, \tau_2 - \eta^2\lambda) \quad \text{I.28}$$

The "restricted trajectory" is characterized by $\tau_0 = \tau$ and $\tau_2 = \eta^2\tau$, and on this restricted trajectory, we require

$$\mathfrak{F}(\mu, \tau, \eta^2\tau) = \overline{f}(\mu, \tau) \quad \text{I.29}$$

Thus, on the restricted trajectory, equation I.28 reduces to equation I.20. We further introduce,

$$\mathfrak{F}(\mu, \tau_0, \tau_2) = \mathfrak{F}_0(\mu, \tau_0, \tau_2) + \eta^2 \mathfrak{F}_2(\mu, \tau_0, \tau_2) + O(\eta^4) \quad \text{I.30}$$

and expand equation I.28 in powers of η^2. By equating coefficients of powers of η^2 we find,

$$\frac{\partial\mathfrak{F}_0(\mu, \tau_0, \tau_2)}{\partial\tau_0} = 0 \quad \text{I.31}$$

and

$$\frac{\partial\mathfrak{F}_2(\mu, \tau_0, \tau_2)}{\partial\tau_0} + \frac{\partial\mathfrak{F}_0(\mu, \tau_0, \tau_2)}{\partial\tau_2} = \frac{\partial}{\partial\mu} (1-\mu^2) \int_0^{\tau_0} d\tau\, \mathcal{K}(\mu, \lambda) \frac{\partial\mathfrak{F}_0(\mu, \tau_0 - \lambda, \tau_2)}{\partial\mu} \quad \text{I.32}$$

From equation I.31,

$$\mathfrak{J}_0(\mu, \tau_0, \tau_2) = \mathfrak{J}_0(\mu, 0, \tau_2) \qquad \text{I.33}$$

and then, from equation I.32,

$$\mathfrak{J}_2(\mu, \tau_0, \tau_2) = \mathfrak{J}_2(\mu, 0, \tau_2) - \tau_0 \frac{\partial \mathfrak{J}_0(\mu, 0, \tau_2)}{\partial \tau_2}$$

$$+ \tau_0 \frac{\partial}{\partial \mu}(1 - \mu^2) \int_0^\infty d\lambda\, \mathcal{K}(\mu, \lambda) \frac{\partial \mathfrak{J}_0(\mu, 0, \tau_2)}{\partial \mu}$$

$$- \frac{\partial}{\partial \mu}(1 - \mu^2) \int_0^{\tau_0} d\lambda \int_\lambda^\infty ds\, \mathcal{K}(\mu, s) \frac{\partial \mathfrak{J}_0(\mu, 0, \tau_2)}{\partial \mu} \qquad \text{I.34}$$

We remove the secular growth of $\mathfrak{J}_2$ by setting

$$\frac{\partial \mathfrak{J}_0(\mu, 0, \tau_2)}{\partial \tau_2} = \frac{\partial}{\partial \mu}(1 - \mu^2) \int_0^\infty d\lambda\, \mathcal{K}(\mu, \lambda) \frac{\partial \mathfrak{J}_0(\mu, 0, \tau_2)}{\partial \mu} \qquad \text{I.35}$$

and then, along the restricted trajectory, we find,

$$\frac{\partial \overline{f}_0(\mu, \tau)}{\partial \tau} = \eta^2 \frac{\partial}{\partial \mu} \mathcal{D}_0(\mu) \frac{\partial \overline{f}_0(\mu, \lambda)}{\partial \mu} \qquad \text{I.36}$$

and,

$$\overline{f}_2(\mu, \tau) = \overline{f}_2(\mu, 0) - \frac{\partial}{\partial \mu}(1 - \mu^2) \int_0^\tau d\lambda \int_\lambda^\infty ds\, \mathcal{K}(\mu, s) \frac{\partial \overline{f}_0(\mu, \tau)}{\partial \mu} \qquad \text{I.37}$$

The behavior of $\overline{f}_0(\mu, \tau)$, as $\tau \to \infty$, depends on the properties of $\mathcal{D}_0(\mu) = (1 - \mu^2)\, \mathfrak{M}_0(\mu)$, where $\mathfrak{M}_0(\mu)$ is the zeroth moment of the kernel. With the introduction of the power spectral density,

$$P(\omega) \equiv \int_0^\infty ds\, R(s) \cos \omega s \qquad \text{I.38}$$

we find that,

$$\mathfrak{M}_0(\mu) = \frac{1}{|\mu|} P\left(\frac{1}{|\mu|}\right) \qquad \text{I.39}$$

We will assume the best possible case; i.e., we will assume that $P(\omega)$ is non-negative, and has no zeroes in the range $1 \leq \omega < \infty$, but, of course $P(\infty) = 0$. In fact we must have,

$$P(\omega) = o\left(\frac{1}{\omega}\right) \quad (\omega \to \infty) \qquad \text{I.40}$$

so that the total power in the random field remains finite. Thus, it is very generally true that

$$\mathfrak{M}_0(\mu) \to 0 \quad (\mu \to 0) \qquad \text{I.41}$$

and, in the "best possible case" being considered here, $\mathfrak{M}_0(\mu)$ has no other zeroes. Thus, $\mathfrak{D}_0(\mu)$ is non-negative, and has zeroes at $\mu = 0, \pm 1$.

The consequences of the zero in $\mathfrak{D}_0(\mu)$ at $\mu = 0$ are developed thoroughly in Appendix B, but the discussion there depends on the normal mode expansion of the probability distribution function which will be constructed in section VI. In the following paragraph we give a brief description of the results of Appendix B and discuss the consequences of these results in the expansion being attempted here.

The solutions to equation I.36 must approach a final steady state in which $\bar{f}_0(\mu, \tau)$ becomes independent of μ everywhere that $\mathcal{D}_0(\mu) \neq 0$. Because of the zeroes in $\mathcal{D}_0$ at $\mu = \pm 1$, equation I.36 conserves the total probability contained in the domain, $-1 \leq \mu \leq 1$. Because of the zero in $\mathcal{D}_0(\mu)$ at $\mu = 0$, we show in Appendix B that the total probabilities in the half-domains, $-1 \leq \mu \leq 0$, and $0 \leq \mu \leq 1$, are also individually conserved. Generally speaking, $\bar{f}_0(\mu, \tau)$ approaches a final state in which a discontinuity at $\mu = 0$ appears. Thus,

$$\frac{\partial \bar{f}_0(0, \tau)}{\partial \mu} \sim \infty \quad (\tau \to \infty) \qquad \text{I.42}$$

From equation I.37, we see therefore, that $\bar{f}_2(\mu, \tau)$ must also become unbounded at $\mu = 0$ with increasing time. The inability of equation I.36 to allow the propagation of probability through $\mu = 0$, leads to a discontinuous $\bar{f}_0(\mu, \tau)$ at $\mu = 0$, which in turn, leads to divergences in higher order corrections to $\bar{f}_0(\mu, \tau)$ that invalidate the entire expansion procedure.

The usual Markovian approximation to the non-local diffusion equation, equation I.20, cannot be made here. The solution of equation I.20 is not adequately described by the ordinary diffusion equation, equation I.36. The failure of the Markovian approximation can be traced to mixing of the time scale over which the probability distribution function evolves with the time scale during which an interaction with the random field occurs. This mixing is a fundamental problem in the quasi-linear expansion scheme in the presence of a "strong" mean magnetic field. In this expansion scheme the particles are carried through an

interaction with the random field along the undisturbed helical trajectories in the uniform mean field. Those particle trajectories which have small components of velocity along the mean field direction become effectively trapped in the interaction with the random field. The interaction time, in these cases, becomes arbitrarily long in contrast with our assumption that it is short compared to the time over which $\overline{f}(\mu,\tau)$ evolves. Thus, we come to the important conclusion, that in the presence of a strong mean magnetic field, the use of the quasi-linear expansion scheme can preclude the Markovian, or diffusive, description of the probability distribution function. Of equal importance is the converse, that the Markovian, or diffusive, description of the probability distribution function cannot be used to judge the validity of the quasi-linear expansion scheme itself.

V. NUMERICAL SOLUTIONS

In this section we present the results of numerical integration of equation I.20 for several important initial conditions. The Markovian approximation to the quasi-linear solution, which was discussed in the previous section, has also been obtained numerically. A comparison of these two solutions, with the same initial conditions, will be given here.

Numerical integration of equation I.20 was found to be of limited use due to the large amount of computer time and core necessary to compute the convolution integral in that equation at each integration step. With the introduction of a special form for the correlation function,

$$R(\zeta) = e^{-|\zeta|} \qquad \text{I.43}$$

we found that equation I.24 could be considerably reduced to a system of three coupled partial differential equations (I.46-I.48) so that numerical integration became feasible. (We have investigated the consequences of using a double exponential correlation function of the form suggested by Chernov[19] in order to satisfy the requirement introduced by Khintchine[20] that the correlation function have a zero first derivative at its origin, but have not found any qualitative differences.)

With the exponential correlation function given by equation I.43, equation I.24 is equivalent to,

$$\frac{\partial N}{\partial \tau} = \eta^2 g \qquad \text{I.44}$$

$$\frac{\partial g}{\partial \tau} + |\mu|\, g = (1 - \mu^2) \frac{\partial^2 N}{\partial \mu^2} - h \qquad \text{I.45}$$

and,

$$\frac{\partial h}{\partial \tau} + |\mu|\, h = g \qquad \text{I.46}$$

where,

$$g(\mu, \tau) \equiv (1 - \mu^2) \int_0^\tau d\lambda\, e^{-|\mu|\lambda} \cos \lambda \frac{\partial^2 N(\mu, \tau - \lambda)}{\partial \mu^2} \qquad \text{I.47}$$

and,

$$h(\mu, \tau) \equiv (1 - \mu^2) \int_0^\tau d\lambda\, e^{-|\mu|\lambda} \sin \lambda \frac{\partial^2 N(\mu, \tau - \lambda)}{\partial \mu^2}, \qquad \text{I.48}$$

with the following boundary-initial conditions:

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$$g(\mu, 0) = 0 \quad h(\mu, 0) = 0$$

$$g(\pm 1, \tau) = 0 \quad h(\pm 1, \tau) = 0 \qquad \text{I.49}$$

We treat $N(\mu,0)$ as a given function of μ, and notice from equation I.24 that $N(\pm 1,\tau)$ is independent of time. From equation I.23, we see that $N(-1,\tau) = 0$, and, since equation I.24 is homogeneous, we choose $N(+1,\tau) = 1$ without loss of generality. With the exponential correlation function, equation I.24 is also equivalent to,

$$\frac{\partial}{\partial \tau}\left[\frac{\partial^2 N}{\partial \tau^2} + 2|\mu| \frac{\partial N}{\partial \tau} + (1 + \mu^2) N - \eta^2 (1 - \mu^2) \frac{\partial^2 N}{\partial \mu^2}\right]$$

$$= \eta^2 |\mu| (1 - \mu^2) \frac{\partial^2 N}{\partial \mu^2} \qquad \text{I.50}$$

with,

$$\frac{\partial N(\mu, 0)}{\partial \tau} = 0 \quad \frac{\partial^2 N(\mu, 0)}{\partial \tau^2} = \eta^2 (1 - \mu^2) \frac{\partial^2 N(\mu, 0)}{\partial \mu^2} \qquad \text{I.51}$$

It is instructive to consider the structure of equation I.50 in order to establish some contact with more standard equations of mathematical physics, and to suggest some qualitative features of its solutions. For this purpose, it is convenient to rewrite equation I.50 as the equivalent pair,

$$-V^2(\mu) \frac{\partial^2 N}{\partial \mu^2} + \frac{\partial^2 N}{\partial \tau^2} + \nu(\mu) \frac{\partial N}{\partial \tau} + \chi^2(\mu) N = C(\mu, \tau) \qquad \text{I.52a}$$

$$\frac{\partial C}{\partial \tau} = \beta(\mu) \frac{\partial^2 N}{\partial \mu^2} \qquad \text{I.52b}$$

where,

$$V^2(\mu) = \eta^2(1 - \mu^2) \qquad \beta(\mu) = \frac{1}{2} V(\mu)\, \nu(\mu) \tag{I.53a}$$

$$\nu(\mu) = 2\,|\mu| \qquad \chi^2(\mu) = 1 + \mu^2 \tag{I.53b}$$

If $C(\mu,\tau)$ is set to zero in equation I.52a, it reduces to a general dissipative Klein-Gordon wave equation.[21] The coupling between equations I.52a and I.52b, when $C(\mu,\tau) \neq 0$, exhibits an interaction between hyperbolic (wave-like) and parabolic (diffusive-like) behaviors in the single equation I.50. There are four possible special cases of the homogeneous equation which depend on there being special relationships between the coefficients in equations I.53:

(a) The scalar wave equation

$$\nu = \chi = 0; \quad V = (\rho\epsilon)^{1/2}$$

(b) The dissipative scalar wave equation

$$\chi = 0;\ V = (\rho\epsilon)^{1/2}; \quad \nu = \sigma/\epsilon \tag{I.54}$$

(c) The Klein-Gordon equation

$$V = c; \quad \nu = 0$$

(d) The telegraph equation

$$\nu = 2(\alpha + \delta); \quad \chi^2 = 4\alpha\delta; \quad (\alpha,\ \delta)\ \text{real}$$

In case (d) the solution $N(\mu,\tau)$, is analogous to the propagation of a voltage along a cable, where $V = (LC)^{-1}$, $\alpha = R$, $\delta = G/2C$, R is the resistance, L is the self conductance, C is the capacitance, G is the leakage conductance, σ is the conductivity, ρ is the permitivity, and ϵ is the dielectric.

Notice, if we were to neglect the first three terms of equation I.52a, so that $C(\mu,\tau) = \chi^2 \ (\mu) N(\mu,\tau)$, we would obtain,

$$\frac{\partial N}{\partial \tau} = \eta^2 \ \frac{|\mu| \ (1 - \mu^2)}{(1 + \mu^2)} \ \frac{\partial^2 N}{\partial \mu^2} \qquad \text{I.55}$$

which is just the diffusion equation which generates the adiabatic approximation to the probability distribution function in the special case being considered here. Thus, in making the adiabatic approximation we not only neglect higher order time derivatives, but we also neglect higher order crossed μ and time derivatives. We will see shortly, that, the solutions of equations I.52a and I.52b, tend to evolve slowly after some initial transients, but, they also tend to develop a large gradient in μ, in the vicinity of $\mu = 0$. This large gradient invalidates the neglect of the term containing the crossed derivatives. In the next paper in this series an alternate to equation I.55 will be developed in which the crossed derivative term is retained. We will see that this term plays an important role in approximating the solutions of equations I.52a and I.52b.

The numerical solutions of the system of partial differential equations (I.44-I.46) were obtained by an explicit "marching" method on a uniform μ-space mesh, but with a variable, self-adjusting time step. We write the system I.44-I.46 in a condensed vector notation as

$$\frac{\partial \mathbf{Q}}{\partial \tau} = \mathbf{K}(\mathbf{Q}) \qquad \text{I.56}$$

where,

$$\mathbf{Q} = \begin{pmatrix} N \\ g \\ h \end{pmatrix} \qquad \text{I.57}$$

and,

$$\mathbf{K}(\mathbf{Q}) = \begin{pmatrix} \eta^2 g \\ -|\mu|\ g - h + (1-\mu^2)\dfrac{\partial^2 N}{\partial \mu^2} \\ -|\mu|\ h + g \end{pmatrix} \qquad \text{I.58}$$

and then the shortest dynamical time scale in the system at the current time

$$\tau^n = \sum_{\ell=1}^{n} \Delta\tau^{\ell} \text{ is,}$$

$$\tau_D^n = \min_{\substack{i=1,3 \\ j=1,NPTS}} \left\{ \frac{{}^nQ_i^j}{\dfrac{\partial^n Q_i^j}{\partial \tau}} \right\} \qquad \text{I.59}$$

where the subscripts indicate components of the vector equation I.57; the left superscript indicates the index associated with the time marching, and the right superscript indicates the index associated with the μ-space mesh. τ_D^n is the shortest time over which "substantial" changes occur in any one of the integration variables, Q_i^j.

Since we sought an accurate evolution for $f(\mu, \tau) = \partial N/\partial\mu$, we augmented equation I.59 by introducing

$$\tau_f^n = \min_{j=1,NPTS} \left(\frac{f^j}{\dfrac{\partial f^j}{\partial \tau}} \right) \qquad \text{I.60}$$

and then by defining the shortest evolutionary time scale, τ_E^n = minimum (τ_f^n, τ_D^n). The quadrature time step $\Delta\tau_Q^n$, after n time steps, was then set to some convenient fraction of τ_E^n; for the quadratures discussed here, $\Delta\tau_Q^n = 0.1\,\tau_E^n$.

A threshold time step $\Delta\tau_{thresh}$ was introduced so that node formation in any of the Q_i, or initialization (equation I.49), could not cause $\Delta\tau_Q^n = 0$ for any n. The threshold time step was adjusted by investigating the fraction of time when it superseded the quadrature time step, and by the effects its size had on external quality figures of the quadrature. Thus, the final form of our time step algorithm is given by $\Delta\tau_Q^n$ = maximum $(\Delta\tau_Q^n, \Delta\tau_{Thresh})$.

It is clear from equations I.52a and I.52b, that at $\mu = 0$, $C(0, \tau)$ is a true constant of the evolving solution which by equation I.51 is given by, $C(0,\tau) = N(0,0)$. Thus,

$$\frac{\partial^2 N(0,\ \tau)}{\partial \tau^2} + N(0,\ \tau) - \eta^2 \frac{\partial^2 N(0,\ \tau)}{\partial \mu^2} = N(0,\ 0) \qquad \text{I.61}$$

This condition is not a part of the explicit marching algorithm but is checked after each integration step as a measure of the fidelity of the overall finite difference scheme.

In Figure 1, we show an example of a plot of the left hand side of equation I.61 for the solution shown in Figure 5. Also shown is the reference constant, $C(0,0)$. Note the departures of $C(0,\tau)$ from the analytic constant after approximately one Larmor period, coincident with the appearance of oscillations in the steep gradient in $f(\mu, \tau)$ near $\mu = 0$. Beyond this point the finite difference

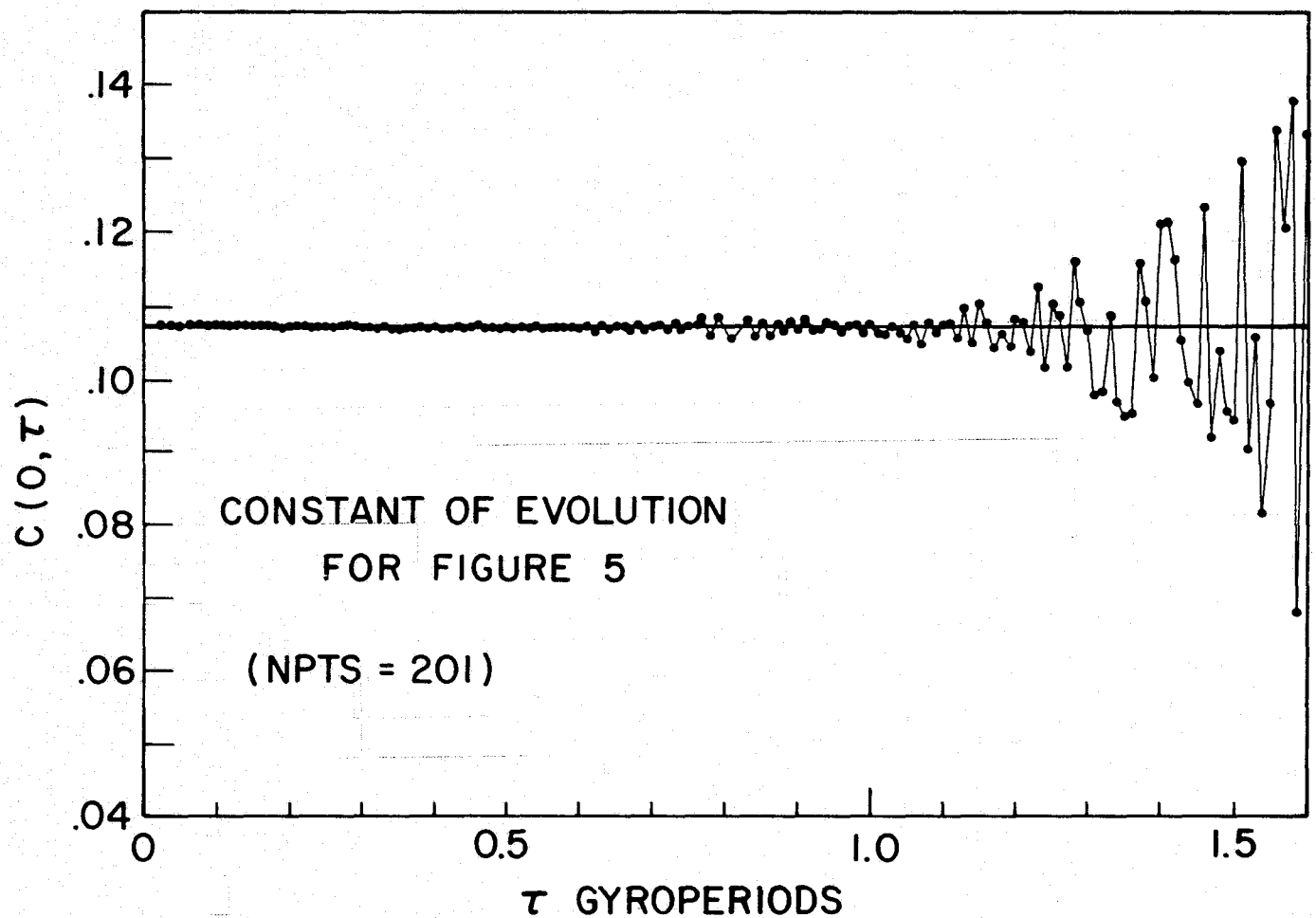

Figure 1. Evolution of $C(0,\tau)$ (see equation I.61) for the broad parallel beam injection shown in Figure 5. The failure of $C(0,\tau)$ to remain constant coincides with the development of the erratic oscillations in Figure 5 in the vicinity of $\mu = 0$.

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equations are unable to faithfully replicate the behavior of the continuum solutions, and the numerical solution is terminated. For all other numerical solutions presented in this paper, $C(0, \tau)$ has been monitored and has been found to remain very nearly constant as in the first Larmor period of Figure 1.

We have considered a variety of implementations of the Laplacian finite difference operator with no perceptible change in the results reported here. The solutions which will be discussed shortly have generally been checked by doubling the mesh density in μ and finding no significant variations in the numerical results.

A numerical quasi-linear adiabatic solution was obtained from equation I.55 by treating it as a standard diffusion equation with a spatially dependent diffusion coefficient. From equation I.55 it is clear that $N(0, \tau) = N(0, 0)$. The zero in the diffusion coefficient is equivalent to an impenetrable membrane at $\mu = 0$. By placing a μ-space mesh point at $\mu = 0$, we have prevented any "leakage" from one half-space to the other. The finite difference equation is implemented via the explicit method of DuFort and Frankel[22] which is unconditionally stable.

All solutions reported graphically in this paper depict ${}^{n}f^{j+1/2} = ({}^{n}N^{j+1} - {}^{n}N^{j})/(\mu^{j+1} - \mu^{j})$ in a connect-a-dot fashion. No smoothing has been made in order to extract ${}^{n}f^{j+1/2}$. The plots were made on a Calcomp plotter with 0.01" resolution with the distance between Larmor period tickmarks being 6.25 inches.

In Figure 2, we show an explicit numerical solution of the quasi-linear adiabatic equation for an anisotropic initial condition corresponding to a narrow beam of particles injected with Gaussian probability about the mean magnetic field direction.

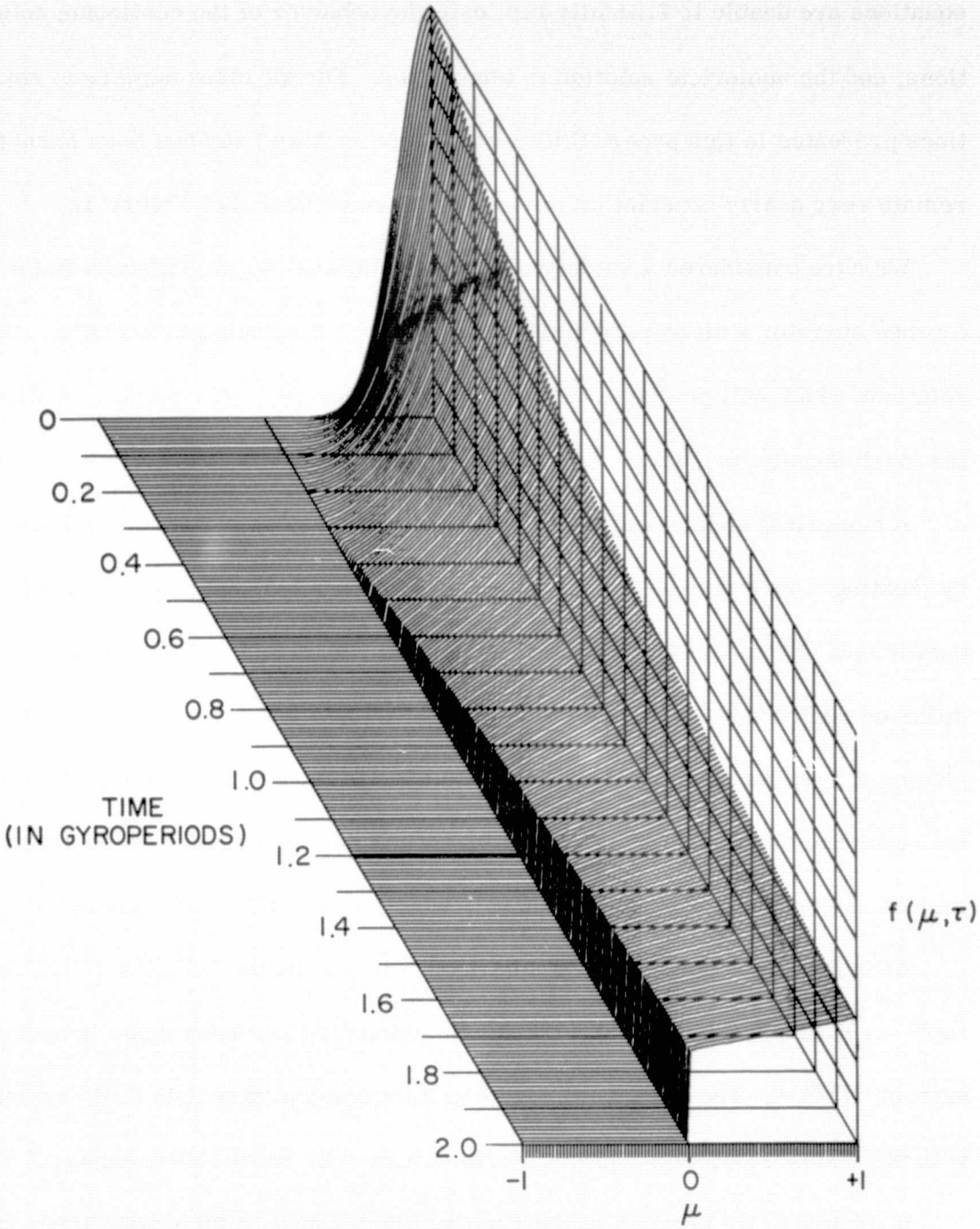

Figure 2. Evolution of a narrow beam injected parallel to the mean magnetic field in the quasi-linear adiabatic approximation to the gyro-phase averaged probability distribution function.

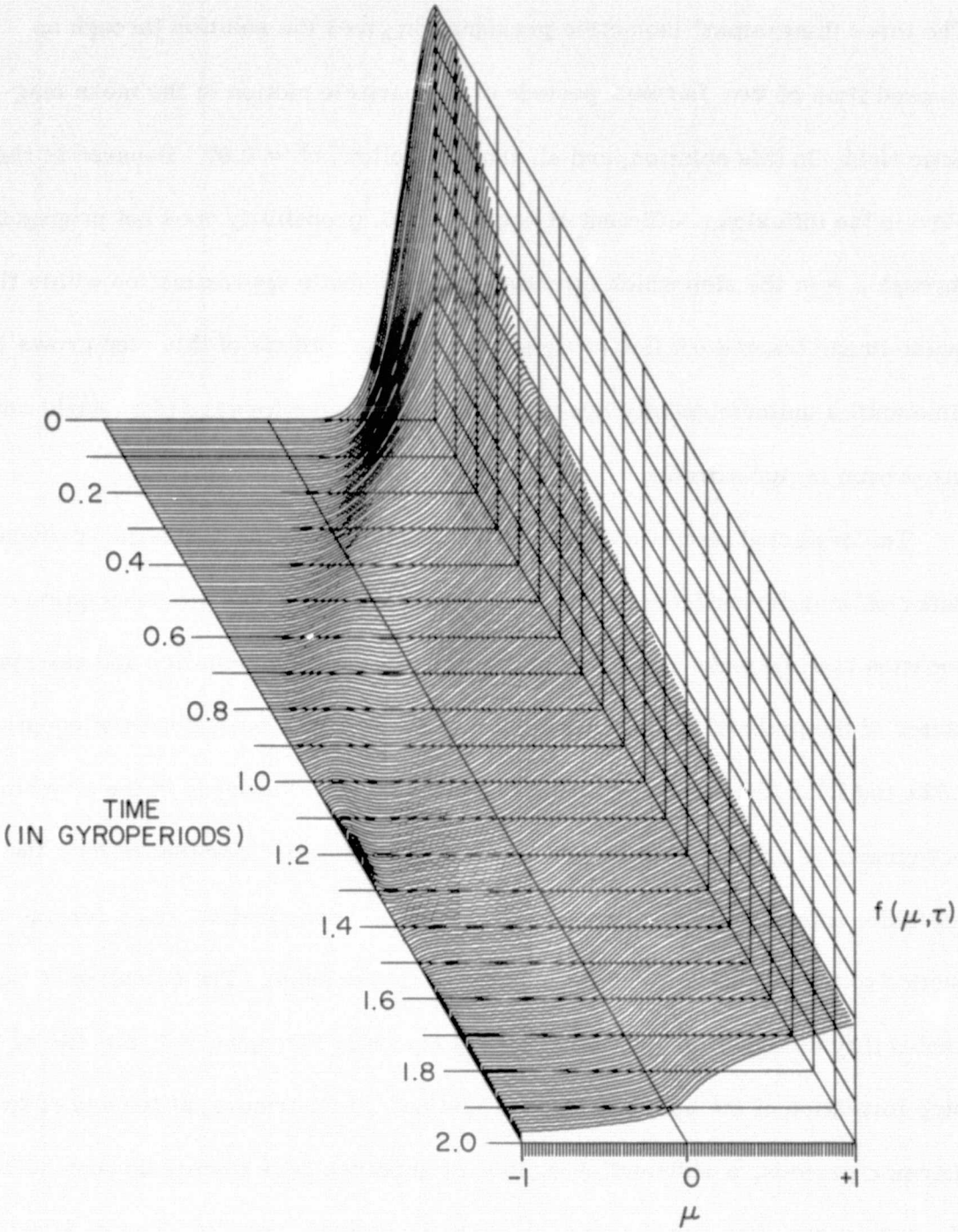

Figure 3. Evolution of a narrow beam injected parallel to the mean magnetic field in the quasi-linear diabatic approximation to the gyro-phase averaged probability distribution function.

The three dimensional isometric presentation gives the solution through an elapsed time of two Larmor periods of the particle motion in the mean magnetic field. In this solution, and all those to follow, $\eta^2 = 0.09$. Because of the zero in the diffusion coefficient of equation I.55, probability does not propagate through $\mu = 0$; the step which invalidates the adiabatic approximation within the quasi-linear framework is clearly shown. The magnitude of this step grows in time until a uniform density is established through the forward pitch angle cone (as proven in Appendix B).

The neglected terms in equation I.52a, which make the quasi-linear diabatic description different from the Markovian quasi-linear adiabatic description of equation I.55, do make important differences in the time evolution and asymptotic states of the probability distribution function. A numerical solution of equations I.52a and I.52b, for the same initial condition as that discussed in the previous paragraph, is presented in Figure 3. It is immediately clear that making the adiabatic approximation within the quasi-linear approximation, does serious injustice to the evolution which it purports to approximate. The evolution of the probability distribution function proceeds smoothly through $\mu = 0$, the site of the step formation of the adiabatic approximation. Furthermore, at the end of two Larmor periods, a substantial number of particles have passed through 90°. We notice that after two Larmor periods the diabatic solution (Figure 3) still contains a narrow strong-gradient transition region which is represented in the adiabatic approximation by the step. We expect this narrow transition region,

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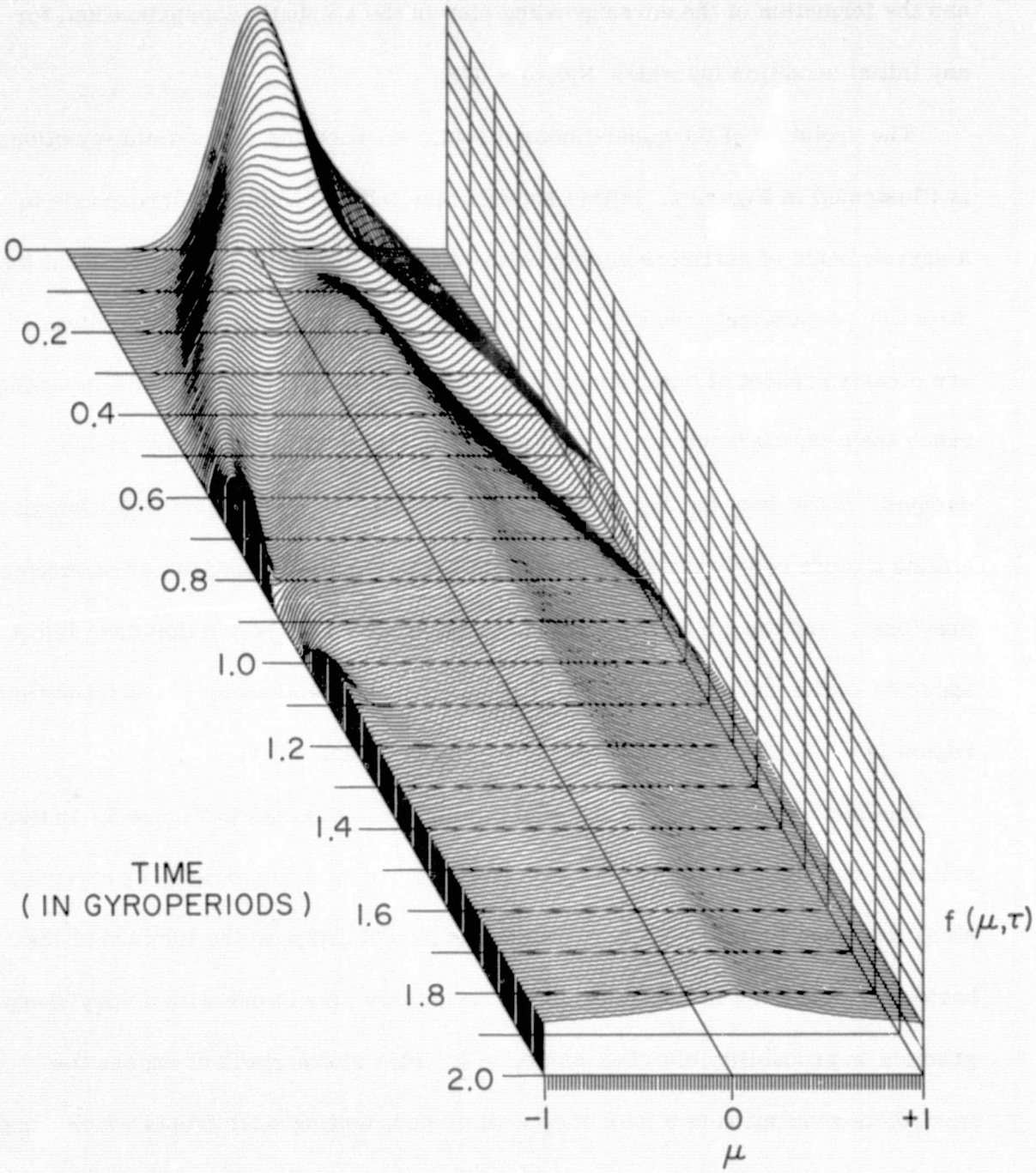

Figure 4. Evolution of a narrow beam injected across the mean magnetic field in the quasi-linear diabatic approximation to the gyro-phase averaged probability distribution function.

and the formation of the corresponding step in the adiabatic approximation, for any initial condition for which $N(0,0) \neq 1/2$.

The evolution of the quasi-linear diabatic solution for cross-field injection, is illustrated in Figure 4. In this solution, the initial condition corresponds to a narrow beam of particles injected with Gaussian probability centered about the direction perpendicular to the mean magnetic field. Low frequency oscillations are clearly present at early times near $\mu = 0$; the telegraph-like transients which reach the μ-space bounds at approximately 0.8 Larmor periods are rapidly damped. In the two Larmor periods shown, the probability distribution function attains a more nearly isotropic state than in the parallel injection case considered previously, and the narrow transition region which developed in that case is not apparent in this solution. As we will demonstrate, this absence of the transition region is due to the even (in μ) initial condition of this solution.

One further quasi-linear diabatic solution is illustrated in Figure 5. In this solution, the initial condition corresponds injection of a broad beam of particles along the mean field direction. The ratio of probabilities in the forward to the backward directions in this beam is approximately nine to one with a very sharp gradient in probability distribution at $\mu = 0$. This sharp gradient causes the immediate generation of a pair of oppositely propagating wave fronts which damp rapidly in the vicinities of $\mu = \pm 1$. For post transient times, this solution takes on a shape which is similar to that shown in Figure 3. In the next section of this paper, we will show that this similarity can be understood in terms of a

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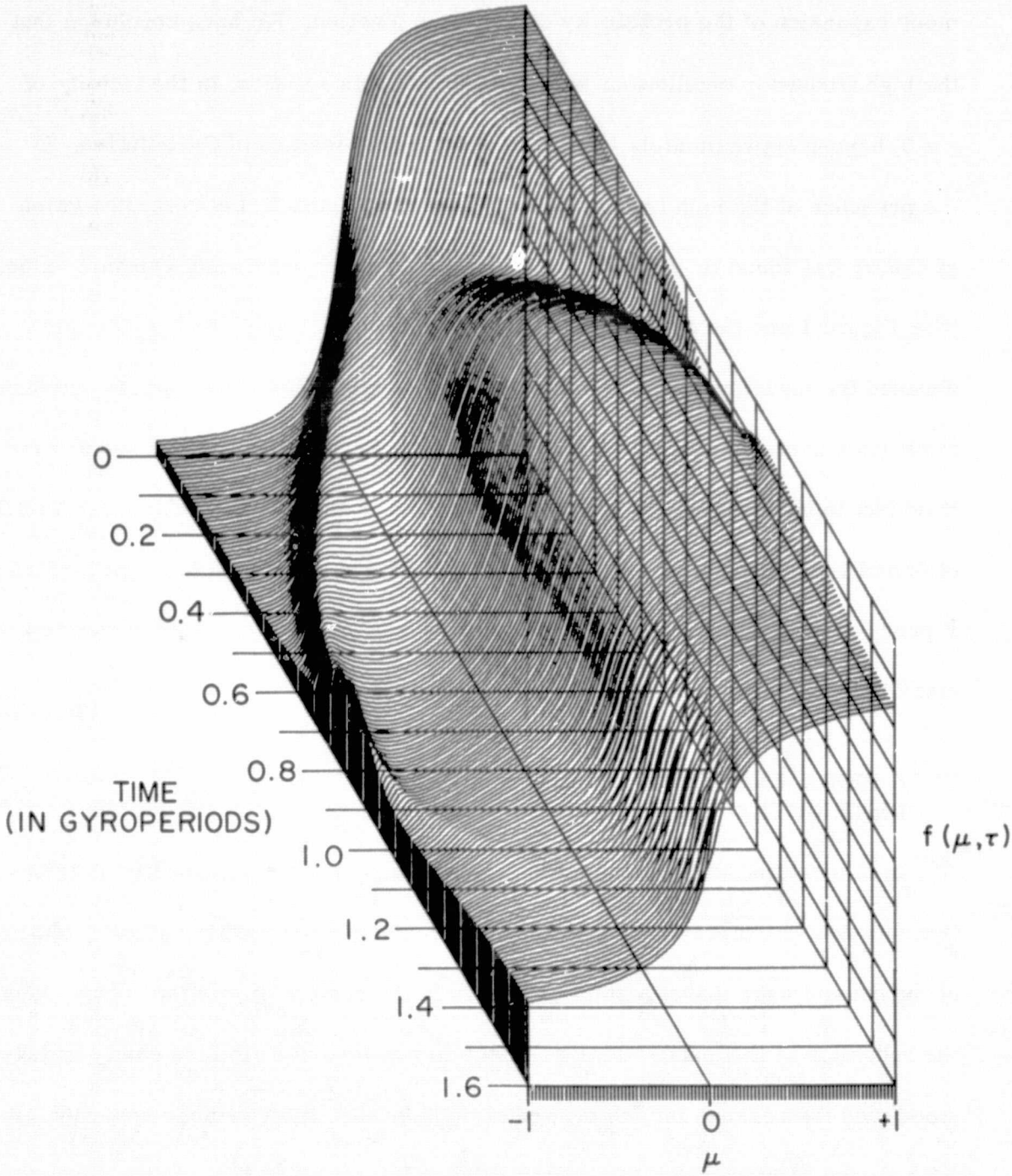

Figure 5. Evolution of a broad beam injected parallel to the mean magnetic field in the quasi-linear diabatic approximation to the gyro-phase averaged probability distribution function.

mode expansion of the probability distribution function. We have concluded that the high frequency oscillations which develop in this solution, in the vicinity of $\mu = 0$, beyond approximately $\tau = 1.0$, are not a real feature of the solution. In the presence of the high frequency oscillations in Figure 5, the computed value of $C(0,\tau)$ was found to oscillate considerably about its previously constant value. (See Figure 1 and the discussion surrounding equation I.61.) We have further checked the invariance of all the diabatic solutions which we have presented here to changes in the μ -space grid spacing. The high frequency oscillations are not invariant to changes in grid spacing, but instead, manifest themselves in a variety of forms which depend on the grid spacing. In the particular solution presented in Figure 5, the μ-domain has been divided into two hundred intervals; every tick mark in the figure corresponds to two grid points.

VI. A MODE EXPANSION OF THE PROBABILITY DISTRIBUTION FUNCTION

In this section we introduce a "mode" expansion of the probability distribution function which provides considerable insight into the post transient evolution of the quasi-linear diabatic solutions. Within the context of this mode expansion, the reduction of the quasi-linear solutions to the adiabatic limit is easily understood, and the reasons for the failure of the adiabatic limit become apparent. It is this point of view which has provided the primary motivation for the successful kinetic approximation which will be introduced in the next paper in this series.

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The modes which are introduced in this section are obtained by Laplace transforming the quasi-linear diabatic equation, equation I.20, and then finding the eigenfunctions generated by the Laplace transform of the integrodifferential operator on the right hand side of that equation. We call the eigenfunctions of the Laplace transformed operator "Laplace-modes." This procedure is analogous to the standard modal analysis of plasma physics.

The integral of the right side of equation I.20 is a convolution integral under Laplace transformation. Thus, the Laplace transform of equation I.20 is,

$$p\tilde{f}(\mu, p) - \bar{f}(\mu, 0) = \eta^2 \frac{\partial}{\partial\mu} \tilde{\mathfrak{D}}(\mu, p) \frac{\partial \tilde{f}(\mu, p)}{\partial\mu} \qquad \text{I.62}$$

where,

$$\tilde{\mathfrak{D}}(\mu, p) = (1 - \mu^2)\, \tilde{\mathfrak{K}}(\mu, p) \qquad \text{I.63}$$

and $\tilde{\mathfrak{K}}(\mu, p)$ is the Laplace transform of the kernel with p for the Laplace variable. Unless otherwise stated, only real, non-negative p will be considered here. The normal modes are introduced through

$$\frac{\partial}{\partial\mu} \tilde{\mathfrak{D}}(\mu, p) \frac{\partial\psi_m(\mu, p)}{\partial\mu} + \lambda_m(p)\, \psi_m(\mu, p) = 0 \qquad \text{I.64}$$

In Appendix A, we show that $\tilde{\mathfrak{K}}(\mu, p)$ is positive definite in μ for real, positive p. However, when $p = 0$, $\tilde{\mathfrak{K}}(\mu, 0)$ takes on a zero at $\mu = 0$. Thus, $\tilde{\mathfrak{D}}(\mu, p)$ is non-negative in μ, with isolated zeroes at $\mu = \pm 1$, and also at $\mu = 0$ when $p = 0$. The situation when $p = 0$ will be considered separately in a moment. When $p \neq 0$, the

eigenfunctions which are the solutions of equation I.64 are the solutions of the Stürm-Liouville problem.[23] Thus, these eigenfunctions form a complete, orthogonal set which can be used to form a mode expansion of $\tilde{f}(\mu, p)$.

$$\tilde{f}(\mu, p) = \sum_{m=0}^{\infty} f_m(p)\, \psi_m(\mu, p) \qquad \text{I.65}$$

By substituting this expression into equation I.62 and using the orthogonality of the eigenfunctions, we find,

$$f_m(p) = \frac{\int_{-1}^{1} d\mu\, \psi_m(\mu, p)\, \bar{f}(\mu, 0)}{E_m^2(p)\, [p + \eta^2 \lambda_m(p)]} \qquad \text{I.66}$$

where,

$$E_m^2(p) = \int_{-1}^{1} d\mu\, \psi_m^2(\mu, p) \qquad \text{I.67}$$

Equations I.65 and I.66 together form a representation of the exact solution of equation I.20.

One immediate solution of equation I.64 follows from setting $\lambda = 0$, and ψ = constant. This solution is p-independent. Furthermore, this eigenvalue has the minimum value allowed; all other eigenvalues are positive.[23] Thus, we set $\lambda_0 = 0$, and $\psi_0 = 1$. From equation I.66,

$$f_0(p) = \frac{1}{2p} \int_{-1}^{1} d\mu\, \bar{f}(\mu, 0). \qquad \text{I.68}$$

The isotropic (μ-independent, m = 0) part of the initial probability distribution function remains constant in time.

In the limit, $p = 0$, a different situation presents itself. Because, $\tilde{\mathfrak{D}}(\mu, 0)$ contains a zero at $\mu = 0$, the Stürm-Liouville method for the generation of the eigenfunctions cannot be applied as above. However, in the half domain, $-1 \leq \mu \leq 0$, where $\tilde{\mathfrak{D}}(\mu, 0) \neq 0$, except on the boundaries, the Stürm-Liouville method can be applied. Because equation I.64 is even in μ, this set of eigenfunctions is appropriate for the positive half domain, $0 \leq \mu \leq 1$, as well. For each half domain eigenfunction, two mutually orthogonal eigenfunctions in the full μ-domain can be constructed by taking even and odd combinations of the half domain eigenfunctions. With this choice of symmetry in μ, these eigenfunctions in the full μ-domain represent the limits, as $p \to 0$, of two of the $(p \neq 0)$ eigenfunctions discussed above. Thus, in the limit, $p = 0$, the eigenfunctions which are the solutions of equation I.64 become doubly degenerate; each eigenvalue corresponds to an even and odd pair of eigenfunctions.

Notice from equation I.62, if p is set to zero in $\tilde{\mathfrak{D}}(\mu, p)$, then it becomes the Laplace transform of equation I.26. Thus, if $\tilde{\mathfrak{D}}(\mu, p)$ is replaced by $\tilde{\mathfrak{D}}(\mu, 0)$ in equation I.62, the adiabatic, or Markovian, approximation to the probability distribution function is generated. The eigenbasis for these adiabatic solutions can be generated from equation I.64 by setting $p = 0$. Thus, the doubly degenerate eigenfunctions discussed above are the "adiabatic eigenfunctions"; they are reached by taking the limit, $p = 0$, in the "diabatic eigenfunctions" which are the solutions of equation I.64 for positive p. Otherwise, the adiabatic eigenfunctions can be constructed from the even and odd combinations of the half domain

eigenfunctions as discussed above. An important property of these eigenfunctions, which is apparent from this construction, is that the odd adiabatic eigenfunctions are discontinuous at $\mu = 0$.

As an example of the construction of the adiabatic eigenfunctions, consider the following simple case. Let $\lambda_0(0) = \lambda_0^{(A)} = 0$. The half domain eigenfunction which corresponds to this choice is $\psi_0^{(-)}(\mu) = 1$. The adiabatic eigenfunctions which can be constructed from this choice are,

$$\psi_0^{(A)}(\mu) = \psi_0(\mu, 0) = 1 \qquad \text{I.69}$$

and

$$\psi_1^{(A)}(\mu) = \psi_1(\mu, 0) = 1 \quad (\mu < 0)$$
$$= -1 \quad (\mu > 0) \qquad \text{I.70}$$

We have already seen that $\psi_0(\mu, p)$ is independent of p, so the choice given by equation I.69 is obvious. The second choice, equation I.70, follows from the considerations that $\psi_1(\mu, p)$ must be an odd function of μ with only one node in μ for any p; $\psi_1^{(A)}(\mu)$ is the only possible limit of $\psi_1(\mu, p)$. We will demonstrate that this choice for $\psi_1(\mu, 0)$ is also consistent with the monotonic increase of the eigenvalues with increasing order at a fixed value of p. Notice that $\psi_1(\mu, 0)$ is discontinuous at $\mu = 0$ as stated above.

We have found that $\lambda_0^{(A)} = 0$, and from the Stürm-Liouville method, we can assure ourselves that all other adiabatic eigenvalues are greater than zero. We have concluded that $\lambda_1(p) \to 0$ as $p \to 0$, but, we can now further conclude that all higher order eigenvalues reduce to the positive adiabatic eigenvalues as $p \to 0$.

With these results, we are able, in Appendix B, to explain the presence of the step in Figure 2, as well as to prove the failure of the adiabatic approximation to propagate probability through $\psi_1 = 0$.

In Figures 6 and 7, we present $\psi_1(\mu, p)$ and $\psi_2(\mu, p)$ for several values of p ranging from zero to infinity. These curves were obtained using a Runge-Kutta integration routine and the "shot-gun method" on equation I.64. The corresponding eigenvalues are listed in Table I. The adiabatic eigenfunctions were obtained by constructing the half-space eigenfunctions and then taking the appropriate combinations of these as discussed above. The approach to the adiabatic limit is demonstrated in these figures for several small values of p. All of these numerical results were obtained for the exponential correlation function introduced in equation I.43. In this case,

$$\tilde{\mathcal{K}}(\mu, p) = \frac{p + |\mu|}{1 + (p + |\mu|)^2} \qquad \text{I.71}$$

$$\underset{p \uparrow}{\sim} \frac{1}{p}$$

Since $\tilde{\mathcal{K}}(\mu, p)$ is essentially independent of μ for very large p, the eigenfunctions are well approximated by the Legendre polynomials in this case, and the eigenvalues are well approximated by,

$$\lambda_m(p) \underset{p \uparrow}{\sim} \frac{m(m+1)}{p} \qquad \text{I.72}$$

Table I

The First and Second Eigenvalues for Various Values of the Laplace Variable, p

p	$\lambda_1(p)$	$\lambda_2(p)$
∞	0	0
10^3	0.002	0.005994
10^2	0.01992	0.0596
10^1	0.1908	0.5593
1.0	0.940	2.66
10^{-1}	0.566	2.64
10^{-2}	0.275	2.36
10^{-3}	0.1705	2.317
10^{-4}	0.1232	2.312
0	0	2.250

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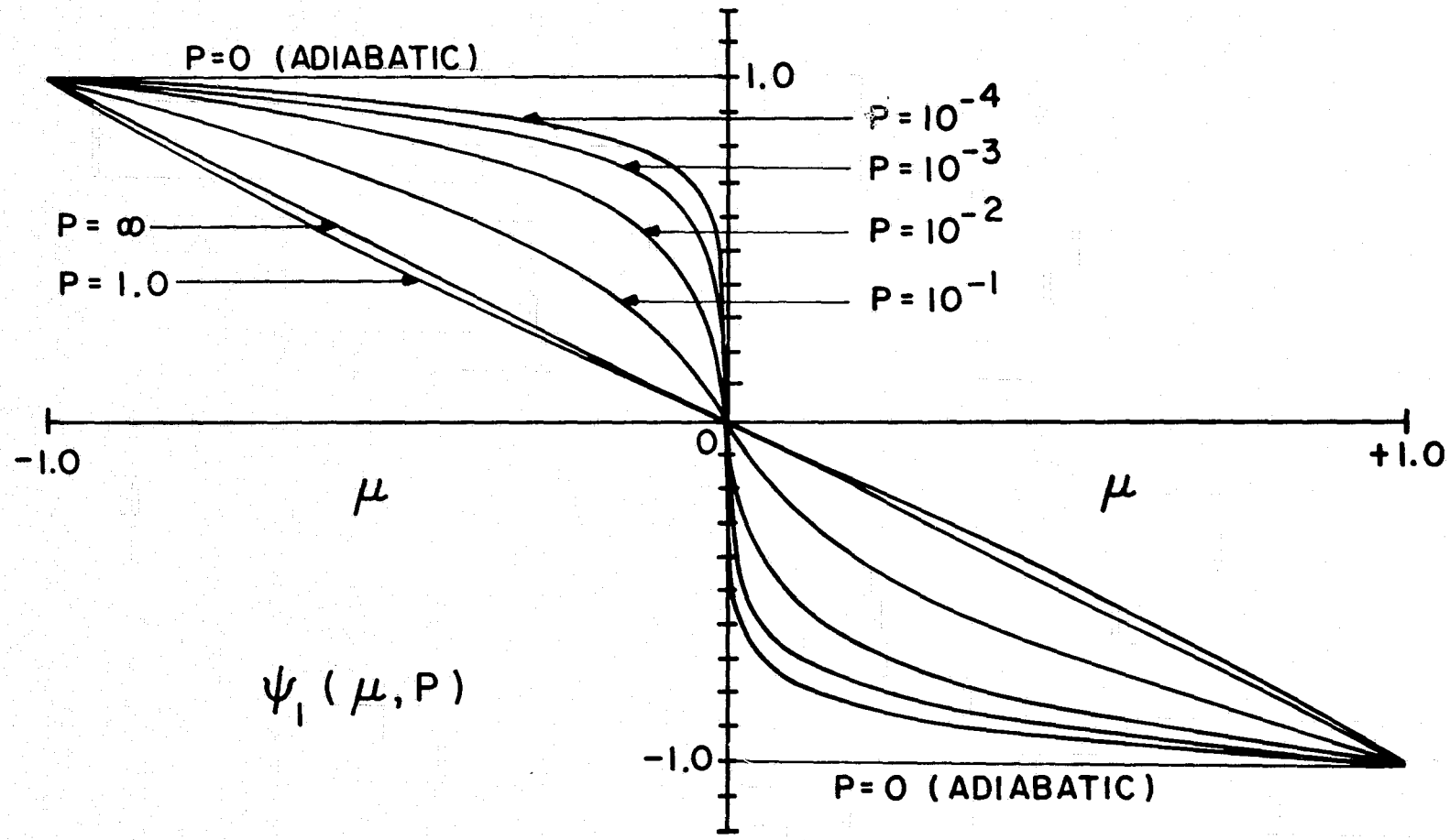

Figure 6. The first eigenmode for various values of the Laplace variable including the adiabatic limit.

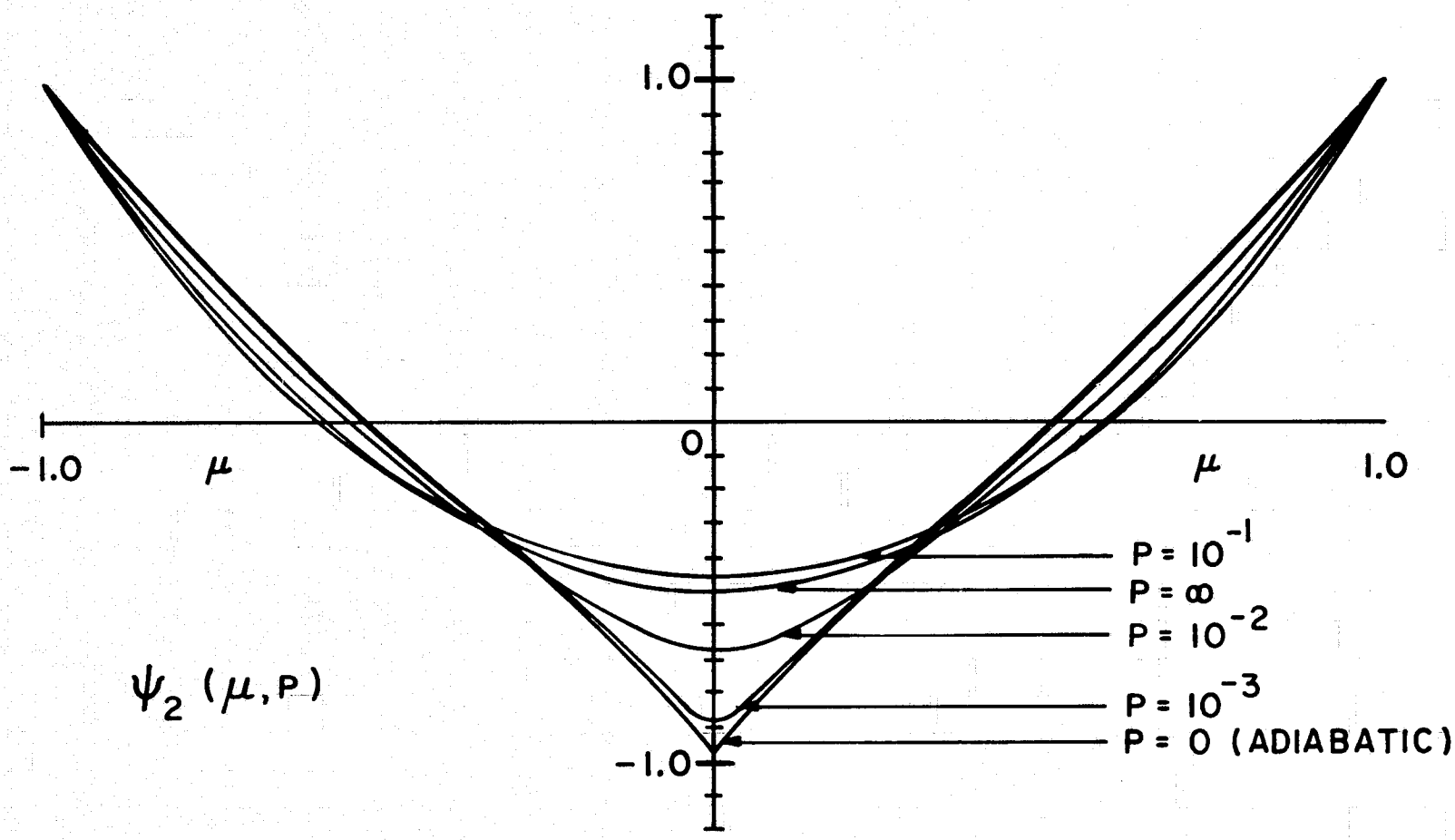

Figure 7. The second eigenmode for various values of the Laplace variable including the adiabatic limit.

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Thus, for non-zero m, we have, $\lambda_m(\infty) = 0$, and the eigenfunctions plotted in figures 6 and 7 for $p = \infty$ are just the Legendre polynomials with the appropriate normalization.

We have plotted $\lambda_1(p)$ and $\lambda_2(p)$ in Figure 8 for the values of p given in Table I. The solid lines in that figure give various approximations to the p-dependence of the eigenvalues. Equation I.72 has been used to obtain the solid line fits to the numerical results for large p. For small p, the solid line fits are given by

$$\lambda_m(p) \underset{\downarrow}{\tilde{p}} \lambda_m^{(A)} + \frac{C_m}{-\ln(p)} \qquad \text{I.73}$$

with $C_1 = 1.13$ and $C_2 = 0.571$. We have been unable to construct an argument for equation I.73 other than the observation that it works very well for small p.

A visual comparison of Figures 3 and 5 with Figure 6, and of Figure 4 with Figure 7, has given us the impression that, on top of the contribution of $\psi_0(\mu, p) = 1$ the long time behavior of the numerical solutions for $\overline{f}(\mu, \tau)$ is dominated by the lowest order eigenfunction which has a non-zero projection onto the initial conditions for the probability distribution function. In each case, we have the impression that $\overline{f}(\mu, \tau)$ evolves, with increasing τ, into distributions in μ which have the appearance of the appropriate eigenfunctions with decreasing values of p. If this impression is correct, then a kinetic approximation to the probability distribution function should be obtainable through any simplification of equation I.62 which provides an accurate description of the low order eigenfunctions and eigenvalues

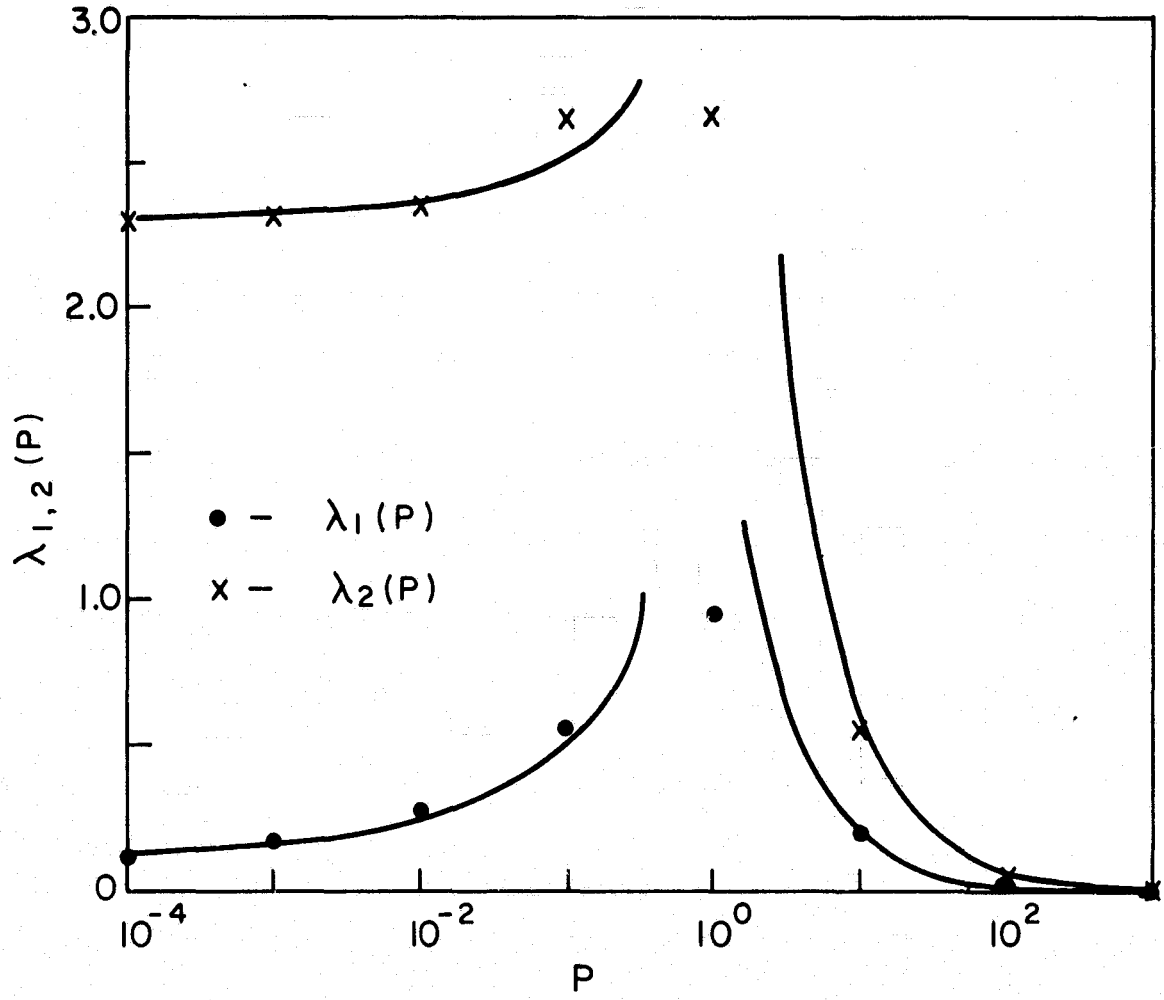

Figure 8. The first and second eigenvalues with analytic approximations for very large or very small values of the Laplace variable.

when p is small, but does not retain the full complexity of that equation when p is not small. In paper II of this series, we will construct a kinetic approximation using this guiding principle; here, we present a crude argument which supports our visual impression.

From Figures 6 and 7, we see that both $\psi_1(\mu, p)$ and $\psi_2(\mu, p)$ are relatively independent of p for small p and μ not too close to zero. We will assume that the large time behavior of $\bar{f}(\mu, \tau)$ is given by the low order eigenfunctions with small p. Therefore, we approximate equation I.65 by,

$$\tilde{f}(\mu, p) = \sum_{m=0}^{2} f_m(p)\, \psi_m(\mu, 0) \qquad \text{I.74}$$

with,

$$f_m(p) = \frac{\int_{-1}^{1} d\mu\, \psi_m(\mu, 0)\, \bar{f}(\mu, 0)}{E_m^2(0)\, [p + \eta^2 \lambda_m(p)]} \qquad \text{I.75}$$

and then,

$$f(\mu, \tau) = \sum_{m=0}^{2} f_m(\tau)\, \psi_m(\mu, 0). \qquad \text{I.76}$$

First, consider the case m = 2. Under our working assumption, and for any $\bar{f}(\mu, 0)$ which is even in μ, the dominant time-dependent part of $\bar{f}(\mu, \tau)$ for large τ, should be given by $f_2(\tau)\, \psi_2(\mu, 0)$. We obtain an approximation to $f_2(\tau)$ by using equation I.75 and also $\lambda_2(p) = \lambda_2^{(A)}$, which can be justified from equation I.73 for small p. After these approximations, we see that we expect $f_2(\tau) \propto \exp(-\eta^2 \lambda_2^{(A)} \tau)$; i.e., we expect the non-isotropic part of the distribution function to decay away exponentially with a predicted decay rate. To test this prediction, we have studied

the numerical solution for $\bar{f}(\mu, \tau)$ given in Figure 4. This solution has a Gaussian distribution in μ, centered at $\mu = 0$, for an initial condition; in particular, the initial condition is even in μ. The amplitude of $f_2(\tau)$ was determined by measuring the difference between the value of the numerical solution and the final isotropic level at $\mu \simeq \pm 0.3$, where, from Figure 7 we can see that $\psi_2(\mu, p)$ is nearly constant over a wide range in values of p. This amplitude is plotted in Figure 9 as a function of time, and compared to a strictly exponential decay with the predicted decay rate. (The amplitudes of both curves are arbitrary; they have simply been placed near each other for easy comparisons.) In spite of the level of this argument, the agreement between the numerical solution and the exponential decay predicted here is quite good. The fact that the agreement in decay rate does not seem to persist to very large times is due to our inability to accurately measure $f_2(\tau)$ once its amplitude has become very small.

If the initial probability distribution function is not even in μ, then we expect the long time distribution in μ to be dominated by the $m = 1$ term in equation I.76. We do not predict an exponential decay in this case, however, since we cannot approximate $\lambda_1(p)$ by a constant value for small p. Instead, we have $\lambda_1(p) \tilde{p}$ $C_1/(-\ln(p))$ and $f_1(p) \propto -\ln(p)$ for small p. Erdelyi[24] has constructed an Abelian theorem for Laplace transforms which states, if a function of time behaves like

$$(\ln \tau)^{\alpha} \tau^{\lambda - 1} \quad \tau \to \infty \qquad \text{I.77}$$

then its Laplace transform behaves like,

$$(-\ln p)^{\alpha} p^{-\lambda} \quad p \to 0^{+} \qquad \text{I.78}$$

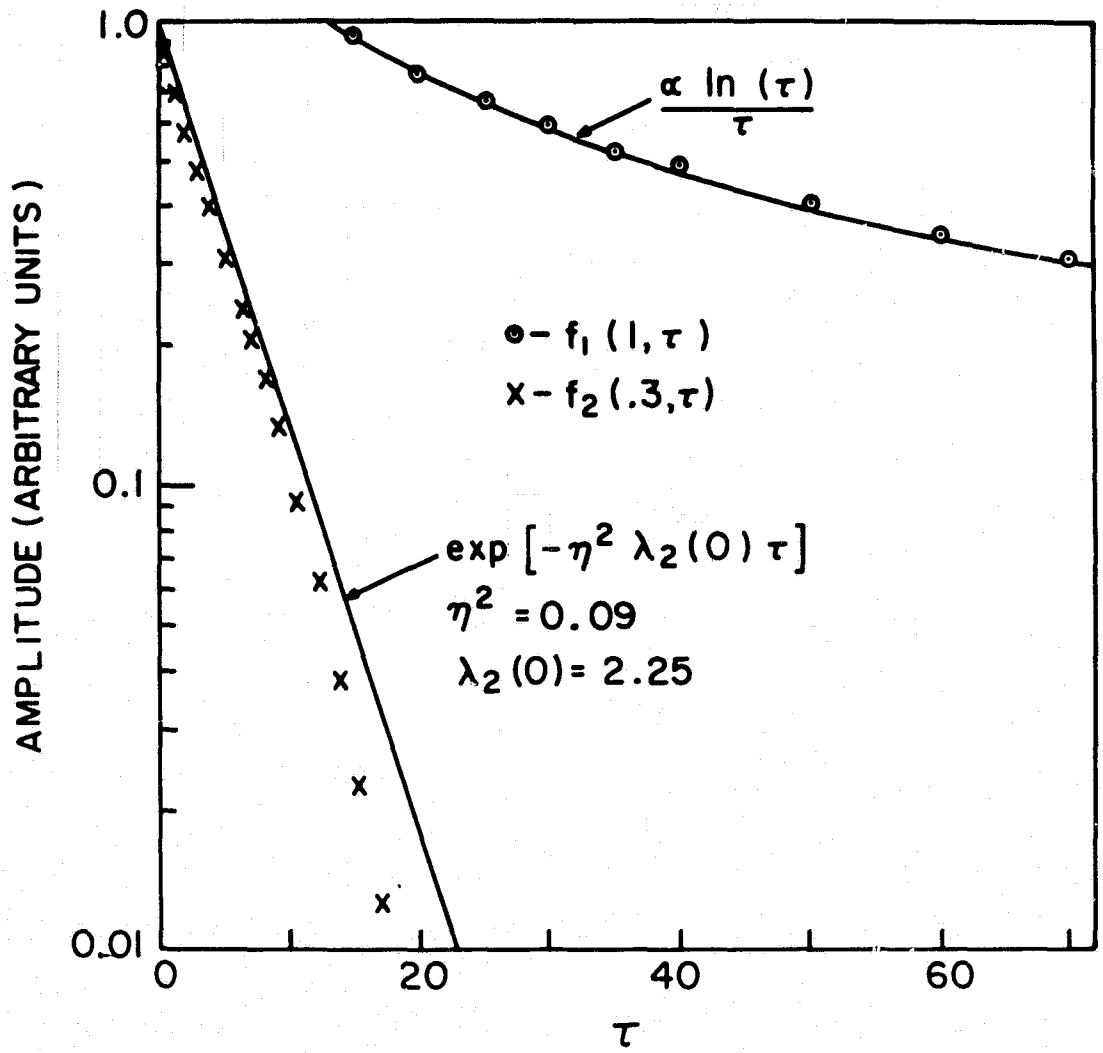

Figure 9. The large time decay of the first and second eigenmodes compared to their Abelian predictions.

where $\alpha = 1$ is included, but λ should be greater than zero. We will apply this theorem here for $\lambda = 0$ anyway, and find that it apparently makes some sense even in this limit. We have determined $f_1(\tau)$ in a manner similar to that outlined above for $f_2(\tau)$, but we have used the numerical solution presented in Figure 3, and have measured the amplitude of the non-isotropic part of $\overline{f}(\mu,\tau)$ at $\mu = \pm 1$. The amplitude of $f_1(\tau)$, so determined, is plotted in Figure 9 along with a fit to these results which is proportional to $\ln(\tau)/\tau$. Once again, the fit between the time dependence of the measured amplitude and the predicted behavior is quite good.

Although this argument is not conclusive, we feel that the evidence weighs in the direction of our original supposition. The broad features of $\overline{f}(\mu,\tau)$ seem to be dominated for large τ by the low order eigenfunctions. Since $f_0(\tau)$ is a constant in time, and $f_1(\tau)$ decays in time so slowly compared to the exponential decay of the higher order modes, we expect that a truncation of the Laplace-mode expansion beyond $m = 1$ would give a reasonable description of the average behavior of $\overline{f}(\mu,\tau)$ for large τ. Of course, if $\overline{f}(\mu,0)$ happened to be an even function of μ, then the $m = 2$ term in the mode expansion should also be included.

VII. DISCUSSION

One of the major purposes of this series of papers is to determine whether or not the quasi-linear approximation to the probability distribution function can be considered a leading approximation in a systematic expansion procedure. We have concluded in this paper that the presence of a strong mean magnetic field

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can preclude the application of the adiabatic approximation to the quasi-linear probability distribution function. Thus, the Markovian limit, in which the propagation of a charged particle in magnetostatic turbulence is governed by a Fokker-Planck equation in velocity space, cannot be reached. As a corollary to the above conclusion, we further conclude that the quasi-linear adiabatic approximation to the distribution function cannot be used to judge the success or failure of the proposed quasi-linear expansion scheme. We will show in paper III of this series that the quasi-linear approximation does contain an intrinsic non-uniformity in μ. This defect is not related to the failure of the adiabatic approximation.

We have proceeded with a study of the properties of the quasi-linear solutions with no further approximations in this paper. Numerical solutions of the quasi-linear diabatic equation have been obtained for an axisymmetric slab model of the plasma turbulence with an exponential two-point correlation function. Striking new wave phenomena have been discovered in the diabatic solutions. In these solutions, propagation through $\mu = 0$ has been found, in contrast to the adiabatic solutions. On the other hand, this propagation through $\mu = 0$ has been found to be very slow. For the case of a beam injection along the mean magnetic field direction, the relaxation to isotropy has been found to proceed like $\ln(\tau)/(\tau)$ in contrast to the familiar exponential relaxation associated with diffusive behavior.

A Laplace-mode expansion of the probability distribution function has been introduced. Because we have dealt in this paper with the quasi-linear diabatic equation which is integrodifferential in time, rather than differential with constant

coefficients, many of the familiar properties of the analogous modal analysis have not been readily apparent, but have been shown to still exist. In particular, it does seem that the long time evolution of the probability distribution function is dominated by the lowest order Laplace-mode which has a non-zero projection on the initial distribution function. It also seems that the long time evolution is determined by the low order Laplace-modes from the vicinity of the Laplace space origin. We have shown that at the origin (the adiabatic limit) the Laplace-modes become doubly degenerate and discontinuous functions of μ, thus leading to the failure of the adiabatic approximation. In the next paper of this series, we construct a new kinetic approximation to the quasi-linear solutions by considering the vicinity of the Laplace space origin, rather than just the origin, thereby removing the degeneracies and discontinuities in the Laplace-modes.

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APPENDIX A

BASIC PROPERTIES OF THE LAPLACE TRANSFORMED KERNEL FUNCTION

The quantity, $\tilde{\mathcal{K}}(\mu, p)$ is defined through,

$$\mathcal{K}(\mu, p) = \int_0^\infty d\tau \, e^{-p\tau} R(\mu, \tau) \cos \tau \qquad \text{A.1}$$

When $p = 0$, $\tilde{\mathcal{K}}(\mu, 0) = M_0(\mu)$, where $M_0(\mu)$ is the zero'th moment of the kernel whose properties have been discussed in the main text of this paper following equation I.39. In particular, we have seen in the text that $M_0(\mu)$ is positive definite except at $\mu = 0$, where it is zero. In the following we consider the case $p > 0$.

We introduce the Fourier transform of the correlation function (see equation I.10) through,

$$R(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, e^{ikr} \, \overline{R}(k) \qquad \text{A.2}$$

By introducing this expression into equation A.1, and then inverting the order of integration, we find,

$$\tilde{\mathcal{K}}(\mu, p) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \, \overline{R}(k) \left[\frac{p}{p^2 + (k\mu - 1)^2} + \frac{p}{p^2 + (k\mu + 1)^2} \right] \qquad \text{A.3}$$

If R(r) were a correlation in space or time which was obtained through space or time averaging in the following sense:

$$R(r) = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} dz\, B'(z)\, B'(z + r) \qquad \text{A.4}$$

then, by Cramer's law, $\overline{R}(k)$ would be non-negative for real k.[25] In the case being considered in this paper, in which the correlation function is defined through ensemble averaging, we assume that $\overline{R}(k)$ is non-negative in order to preserve the properties of the more realistic correlation functions. Then, from equation A.3, we see that when $p > 0$, the integrand which appears in that equation is non-negative for any μ and k. Thus, we conclude that $\tilde{\mathcal{K}}(\mu, p)$ is positive definite for $p > 0$ and for any μ unless $\overline{R}(k)$ is zero for all k; i.e., unless the random field is set to zero everywhere.

APPENDIX B

FAILURE OF THE ADIABATIC APPROXIMATION TO PROPAGATE PARTICLES THROUGH $\mu = 0$

The adiabatic eigenfunctions which were introduced in the main text in the discussion preceding equation I.69, make up a complete set of orthogonal eigenfunctions which can be used to construct the adiabatic approximation to the probability distribution function. Using the method which lead to equations I.65 and I.66, we find,

$$\tilde{f}(\mu, p) = \sum_{m=0}^{\infty} f_m(p)\, \psi_m^{(A)}(\mu) \qquad \text{B.1}$$

in which,

$$f_m(p) = \frac{\int_{-1}^{1} d\mu\, \psi_m^{(A)}(\mu)\, \overline{f}(\mu, 0)}{E_m^2 [p + \eta^2 \lambda_m^{(A)}]} \qquad \text{B.2}$$

where,

$$E_m^2 = \int_{-1}^{1} d\mu\, [\psi_m^{(A)}]^2 \qquad \text{B.3}$$

The minimum eigenvalue which is available to us, from this construction, is $\lambda_0^{(A)} = 0$, and the corresponding eigenfunctions (doubly degenerate) are $\psi_0^{(A)} = 1$, and $\psi_1^{(A)} = 1$ for $\mu < 0$ and $\psi_1^{(A)} = -1$ for $0 < \mu$. All other eigenvalues are positive, and part of a discrete spectrum.

Because of the simple dependence of these equations on the Laplace variable, p, the Laplace inversion can be done. We find

$$f(\mu, \tau) = \sum_{m=0}^{\infty} f_m(\tau)\, \psi_m^{(A)}(\mu) \qquad \text{B.4}$$

in which,

$$f_m(\tau) = f_m(0)\, e^{-\eta^2 \lambda_m^{(A)} \tau} \qquad \text{B.5}$$

where,

$$f_m(0) = \frac{1}{E_m^2} \int_{-1}^{1} d\mu\, \psi_m^{(A)}(\mu)\, \overline{f}(\mu, 0) \qquad \text{B.6}$$

Clearly,

$$f(\mu, \tau) \underset{\tau \to \infty}{\sim} f_0(0)\, \psi_0^{(A)}(\mu) + f_1(0)\, \psi_1^{(A)}(\mu) \qquad \text{B.7}$$

from which, we generally expect the adiabatic approximation to the probability distribution function to approach a final state which contains a discontinuous step at $\mu = 0$.

We have also seen, in the discussion leading to equation I.69, that the adiabatic eigenfunctions can be constructed from the half-domain eigenfunctions which form a complete, orthogonal set in either of the half-domains, $-1 \leq \mu \leq 0$, or $0 \leq \mu \leq 1$. Consider the half-domain, $-1 \leq \mu \leq 0$. In this half-domain, both members of each degenerate pair of adiabatic eigenfunctions are identical, and equal to one of the half-domain eigenfunctions. Thus, each member of each pair of adiabatic eigenfunctions is orthogonal, in the half-domain, to all other eigenfunctions which are members of other pairs. Therefore, from equation B.4,

$$N(0, \tau) = \int_{-1}^{0} d\mu' \; f(\mu', \tau) = f_0(0) + f_1(0)$$

$$= N(0, 0) \qquad \text{B.8}$$

Thus, the total probability in each of the half-domains is conserved in the adiabatic approximation; propagation through $\mu = 0$ is impossible. It is important to note that the conclusions stated in this appendix depend only on the following properties of the power spectrum (see equations I.38 through I.41); $P(\omega)$ must be non-zero for $1 \leq \omega \leq \infty$, and

$$P(\omega) = o\left(\frac{1}{\omega}\right) \quad (\omega \to \infty). \qquad \text{B.9}$$