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16. Abstract

A complete derivation of the geopotential function and its gradient is presented. Also included, in an appendix, is the transformation of Laplace's equation from Cartesian to spherical coordinates. The analytic solution to Laplace's equation is obtained from the transformed version, in the classical manner of separating the variables. A cursory introduction to the method devised by Pines, using direction cosines to express the orientation of a point in space, is presented in an appendix together with sample computer program listings for computing the geopotential function and the components of its gradient. The use of the geopotential function is illustrated in figures.

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THE THEORY OF THE GRAVITATIONAL POTENTIAL APPLIED TO ORBIT PREDICTION

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THE THEORY OF THE GRAVITATIONAL POTENTIAL APPLIED TO ORBIT PREDICTION

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SUMMARY

Because of the significance of high-order-gravitational-potential terms in trajectory prediction, an analysis was performed to determine the magnitude of position and velocity vector errors associated with the geopotential function. The analysis included a complete derivation of the geopotential function and its gradient, the transformation of Laplace's equation from Cartesian to spherical coordinates, and the analytic solution to Laplace's equation from the transformed version obtained in the classical manner of separating the variables. The solution, therefore, expresses the gravitational potential at a point in space in terms of the magnitude of its position vector and its orientation in terms of its geocentric latitude and east longitude. The method devised by Pines, in which direction cosines are used to express the orientation of a point in space, also is considered in the solution.

The effect of varying the order and the degree of the potential function was demonstrated by plotting trajectory integration data on position and velocity vector differences for a single orbit and for 32 orbits. In the resulting curves, the data for lower order and lower degree potential functions are compared with data for an eighth-order model used as a standard. The short- and long-duration comparison studies were performed both including and excluding the effects of drag. Third-body perturbations considering the Sun and the Moon only were included in all studies performed.

INTRODUCTION

The question of the importance of high-order-gravitational-potential perturbations of satellite orbits has been raised many times. The question is not without merit, for the inclusion of high-order-potential logic greatly increases the storage and execution time of any numerical integration of trajectory equations of motion. Although the works of Pines (ref. 1) and Spencer (ref. 2) have made it possible to minimize the burden of storage and execution time for any trajectory integration considering high-order-potential terms, the doubting still persists as to the necessity for such terms. This indecision is unfortunate, since it is often important to include high-order-gravitational-potential effects for trajectory prediction. As a result, this report has been prepared to fully explain not only the potential function but also its gradient, since the gravitational force, being conservative in nature, is therefore derivable from the gradient of the potential.

To explain the potential function, it will be necessary to derive it fully. Once the potential function has been derived, its gradient will be obtained. The derivation will be performed in the classical manner; that is, from Laplace's equation in spherical coordinates by the method of separating the variables. As a result, the solution will require that the position of a point in space be expressed in terms of the magnitude of its position vector and that its orientation be expressed in terms of its geocentric latitude and east longitude. The work of Pines and Spencer (refs. 1 and 2) is essentially a reworking of the classical solution in terms of direction cosines to eliminate its isolated singularities over the poles and its costly trigonometric functions.

The effect of varying the order of the potential function is illustrated in the figures. One series of illustrations considers the position and velocity vector differences resulting from a trajectory integration over one single orbit; the results of lower order potential functions are compared with an eighth-order model as the standard. Another series of illustrations presents the same comparison study extended over 32 orbits. Both comparison studies are performed with and without inclusion of the effects of drag.

The transformation of Laplace's equation from Cartesian to spherical coordinates is performed in appendix A. A concise formulation of the method of Pines is given in appendix B.

SYMBOLS

A,B	arbitrary constants of power series
$\lvert A_{ij}\rvert$	determinant of identity matrix formed by excluding i-th row and j-th column
$A_{nm}(\mu)$	function defined in equation (B3)
a,b	constant coefficients of power series
a_{ij}	elements of the transformation matrix
a_1,a_2,a_3,a_4	coefficients of the gravitational potential gradient defined in equations (B29), (B30), (B31), and (B32), respectively
b_{ij}	elements of the inverse transformation matrix
C	arbitrary constant defined in equation (12)

C_{nm}, S_{nm}	harmonic coefficient constants
$D_{nm}(s,t)$	mass coefficient function defined in equation (B7)
$E_{nm}(s,t)$	mass coefficient function defined in equation (B11)
$F(a,b;c;d)$	hypergeometric series function defined in equation (44)
$F_{nm}(s,t)$	mass coefficient function defined in equation (B12)
g_E	gravitational parameter
h	variable defined in equation (86)
$I(\mu)$	function defined in equation (52)
i	positive integer counter used in equation (44); $0 \leq i \leq k - 1$
i	constant used in equation (92) and appendix B; $i = \sqrt{-1}$
i	positive integer used in equation (111); $i = 1, 2, 3$
$\underline{i}, \underline{j}, \underline{k}$	unit base vectors along Cartesian coordinate axes
k_1	integer used in equation (111); $k_1 = 0, 0, 1$
k_2	integer used in equation (111); $k_2 = 1, -1, 0$
ℓ	upper limit imposed to avoid negative factorials
m	arbitrary constant defined in equation (15)
n, m	positive integers denoting polynomial degree and order, respectively
$P_{nm}(\mu)$	Legendre polynomial of the first kind, defined in equation (81)
$P_{n0}(\mu)$	Legendre polynomial of the first kind, defined in equation (53)
$Q_{nm}(\mu)$	Legendre polynomial of the second kind, defined in equation (82)
$Q_{n0}(\mu)$	Legendre polynomial of the second kind, defined in equation (54)
q, k	positive integers
R, Φ, Λ	spherical coordinate functions

$R' = \partial R/\partial r$	
$R'' = \partial^2 R/\partial r^2$	
$\underline{R}$	unit vector defined in equation (B9)
R_E	equatorial radius of attracting body
$R_m(s,t), I_m(s,t)$	real and imaginary parts, respectively, of the complex variable $(s + it)^m$, where $i = \sqrt{-1}$
r	magnitude of position vector of reference point
S	arbitrary constant
s	direction cosine; $s = x/r$
T_{nmt}	coefficient defined in equation (103)
t	positive integer in text equations (51) and following
t	direction cosine in appendix B; $t = y/r$
V	gravitational potential
W^m	complex number defined in equation (B15)
x,y,z	component magnitudes of position vector along Cartesian coordinate axes
α	arbitrary constant used in equations (23), (84), and (91); $\alpha = \pm im$, where $i = \sqrt{-1}$
α,β	constant coefficients of power series
λ	east longitude of reference point
$M(\mu)$	function defined in equation (61)
μ	direction cosine; $\mu = z/r = \sin \varphi$
ρ_n	variable defined in equation (B4)
φ	geocentric latitude of reference point

Operator

∇ gradient vector operator

ANALYSIS

Laplace's Equation in Cartesian Coordinates

The derivation of the geopotential function is obtained as the solution of one of the most important partial differential equations of mathematical physics: Laplace's equation. Laplace's equation is defined as

$$\nabla^2 V = 0 \tag{1}$$

or

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2}$$

where $\nabla^2 V$ is defined as the Laplacian of the potential function $V = V(x,y,z)$; x, y, and z are the component magnitudes of the position vector along the Cartesian coordinate axes; and ∇ is the gradient vector operator

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k} \tag{3}$$

defined in Cartesian space, with $\underline{i}$, $\underline{j}$, and $\underline{k}$ constituting a set of unit base vectors along the coordinate axes. The scalar product of $\nabla \cdot \nabla$ operating on V gives Laplace's equation:

$$\begin{aligned} \nabla \cdot \nabla V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\ &= 0 \end{aligned} \tag{4}$$

The theory of the solution of Laplace's equation is called potential theory.

The Solution of Laplace's Equation in Spherical Coordinates

Laplace's equation in spherical coordinates, expressed in the form developed in appendix A, is

$$\nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi \frac{\partial V}{\partial\varphi}\right) + \frac{1}{\cos^2\varphi}\frac{\partial^2 V}{\partial\lambda^2}$$

$$= 0 \tag{5}$$

where r is the magnitude of the position vector to the point in question, φ is the geocentric latitude, and λ is the east longitude. This equation can be solved analytically by the technique of separating the variables, which is done by assuming a solution of the form

$$V(r,\varphi,\lambda) = R(r)\Phi(\varphi)\Lambda(\lambda) \tag{6}$$

Therefore,

$$\frac{\partial V}{\partial r} = \frac{\partial R}{\partial r}\Phi\Lambda \tag{7}$$

$$\frac{\partial V}{\partial\varphi} = R\frac{\partial\Phi}{\partial\varphi}\Lambda \tag{8}$$

$$\frac{\partial V}{\partial\lambda} = R\Phi\frac{\partial\Lambda}{\partial\lambda} \tag{9}$$

Substituting equations (7) to (9) in equation (5) and dividing by the product of the spherical coordinate functions $R\Phi\Lambda$ yields

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\Phi}{\partial\varphi}\right) + \frac{1}{\cos^2\varphi}\frac{1}{\Lambda}\frac{\partial^2\Lambda}{\partial\lambda^2} = 0 \tag{10}$$

or

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\left[\frac{1}{\Phi\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\Phi}{\partial\varphi}\right) + \frac{1}{\cos^2\varphi}\frac{1}{\Lambda}\frac{\partial^2\Lambda}{\partial\lambda^2}\right] \tag{11}$$

It is clear that the left-hand side of equation (11) is a function of r only. Therefore, if the equality is to hold, each component on the right must be equal to some arbitrary constant, say C. Under these conditions, equation (11) becomes

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = C \tag{12}$$

which leads to the differential equation

$$r^2R'' + 2rR' - CR = 0 \tag{13}$$

where $R' = \partial R/\partial r$ and $R'' = \partial^2 R/\partial r^2$. Solving for the arbitrary constant C in equation (11) gives

$$-C = \frac{1}{\Phi \cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\Phi}{\partial\varphi}\right) + \frac{1}{\cos^2\varphi}\frac{1}{\Lambda}\frac{\partial^2\Lambda}{\partial\lambda^2} \tag{14}$$

Equation (14) can be separated by multiplying both sides of the equation by $\cos^2\varphi$ and rearranging to yield

$$-C\cos^2\varphi - \frac{\cos\varphi}{\Phi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\Phi}{\partial\varphi}\right) = \frac{1}{\Lambda}\frac{\partial^2\Lambda}{\partial\lambda^2}$$

$$= -m^2 \tag{15}$$

Equation (15) implies that Λ''/Λ is equal to a constant $-m^2$, and this relationship establishes the differential equation

$$\Lambda'' + m^2\Lambda = 0 \tag{16}$$

Furthermore, equation (15) yields

$$-C\cos^2\varphi - \frac{\cos\varphi}{\Phi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\Phi}{\partial\varphi}\right) = -m^2 \tag{17}$$

Rearranging equation (17) yields the differential equation

$$\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial \Phi}{\partial \varphi}\right) + \left(C - \frac{m^2}{\cos^2 \varphi}\right) \Phi = 0 \tag{18}$$

Equation (18) can be written in a more concise form by letting $\mu = \sin \varphi$; then, $\cos \varphi = \partial\mu/\partial\varphi$, and $\partial\Phi/\partial\varphi$ can be written as

$$\frac{\partial \Phi}{\partial \varphi} = \frac{\partial \Phi}{\partial \mu} \frac{\partial \mu}{\partial \varphi}$$

$$= \cos \varphi \frac{\partial \Phi}{\partial \mu} \tag{19}$$

Substituting equation (19) in equation (18) yields

$$\frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) \frac{\partial \Phi}{\partial \mu}\right] + \left(C - \frac{m^2}{1 - \mu^2}\right) \Phi = 0 \tag{20}$$

Equation (20) leads to the differential equation

$$\left(1 - \mu^2\right)\Phi'' - 2\mu\Phi' + \left(C - \frac{m^2}{1 - \mu^2}\right) \Phi = 0 \tag{21}$$

Laplace's equation in spherical coordinates has now been separated into the following three second-order differential equations.

$$\left(1 - \mu^2\right)\Phi'' - 2\mu\Phi' + \left(C - \frac{m^2}{1 - \mu^2}\right) \Phi = 0 \tag{21}$$

$$r^2R'' + 2rR' - CR = 0 \tag{13}$$

$$\Lambda'' + m^2\Lambda = 0 \tag{16}$$

In these equations, C and m represent arbitrary constants. Thus, the solution to equation (5) will be formed from equation (6) when the functions $\Phi(\varphi)$, $R(r)$, and $\Lambda(\lambda)$ have been determined by solving equations (21), (13), and (16), respectively.

Equation (21) is known as Legendre's associated differential equation. The simpler form of this equation, which results when $m = 0$, is known as Legendre's differential equation. Equation (13) is of the form of Cauchy's (or Euler's) linear differential equation, and equation (16) is the differential equation of the undamped harmonic oscillator. As a result, solutions to equations (13) and (16) may be obtained by assuming solutions of the form

$$R(r) = Cr^h \tag{22}$$

$$\Lambda(\lambda) = Ae^{\alpha\lambda} \tag{23}$$

where h is a variable defined in equation (86), A is an arbitrary constant, and α is an arbitrary constant, and by substituting these functions in the appropriate differential equations. The resulting arbitrary constants are then evaluated from the initial and boundary conditions of the problem. Equations (13) and (16) will be solved after the solution of equation (21) has been completed.

The solution of Legendre's associated differential equation will be obtained from the solution of Legendre's differential equation, which will be obtained first. It will be shown that Legendre's differential equation has a solution in terms of two infinite series, each multiplied by one of the arbitrary constants of the solution. It will be shown further that, because of the boundary conditions of the problem $(-1 \leq \mu \leq 1)$, one of the arbitrary constants of the solution must be zero.

Legendre's associated differential equation with $m = 0$ reduces to the form

$$\left(1 - \mu^2\right)\Phi'' - 2\mu\Phi' + C\Phi = 0 \tag{24}$$

If the assumption is made that Legendre's differential equation, equation (24), has a power series solution of the form

$$\Phi = \sum_{k=0}^{\infty} \alpha_{2k}\mu^{q-2k} \tag{25}$$

where q and k are positive integers, then, when this expression is substituted in equation (24), the following expression results.

$$\left(1 - \mu^2\right) \sum_{k=0}^{\infty} (q - 2k)(q - 2k - 1)\alpha_{2k}\mu^{q-2k-2} - 2\mu \sum_{k=0}^{\infty} (q - 2k)\alpha_{2k}\mu^{q-2k-1} + C \sum_{k=0}^{\infty} \alpha_{2k}\mu^{q-2k} = 0 \tag{26}$$

Equation (26) reduces to the form

$$\sum_{k=0}^{\infty} (q - 2k)(q - 2k - 1)\alpha_{2k}\mu^{q-2k-2} - [(q - 2k)(q - 2k + 1) - C]\alpha_{2k}\mu^{q-2k} = 0 \tag{27}$$

Shifting the index k to k + 1 in the negative term in equation (27) and setting α_{-2} equal to zero since q is still to be determined gives

$$\sum_{k=0}^{\infty} (q - 2k)(q - 2k - 1)\alpha_{2k}\mu^{q-2k-2} - [(q - 2k - 2)(q - 2k - 1) - C]\alpha_{2k+2}\mu^{q-2k-2} = 0 \tag{28}$$

Solving for α_{2k} in equation (28) gives the recursive relation

$$\alpha_{2k} = \frac{(q - 2k - 2)(q - 2k - 1) - C}{(q - 2k)(q - 2k - 1)}\alpha_{2k+2} \tag{29}$$

Since the assumed power series solution was defined only for positive values of k (i.e., $0 \leq k \leq +\infty$), setting $k = -1$ in equation (29) causes $\alpha_{-2} = 0$. Under these conditions, the following relationship results.

$$q(q + 1) - C = 0 \tag{30}$$

If the constant C is set equa. to

$$C = n(n + 1) \tag{31}$$

where n is, of necessity, a positive integer greater than or equal to zero, then modifying the left-hand side of equation (30) by adding and subtracting the product nq permits factoring of this equation accordingly.

$$(q - n)(q + n + 1) = 0 \tag{32}$$

Equation (32) has two possible solutions.

$$q = n \tag{33}$$

$$q = -(n + 1) \tag{34}$$

The solution to equation (24) may now be written using A and B as the arbitrary constants of the solution.

$$\Phi = A \sum_{k=0}^{\infty} \alpha_{2k}\mu^{n-2k} + B \sum_{k=0}^{\infty} \beta_{2k}\mu^{-n-2k-1} \tag{35}$$

where the constants α_{2k} and β_{2k} can be computed from the recursive relations

$$\alpha_{2k+2} = -\frac{(n - 2k)(n - 2k - 1)}{2(k + 1)(2n - 2k - 1)}\alpha_{2k} \tag{36}$$

$$\beta_{2k+2} = \frac{(n + 2k + 1)(n + 2k + 2)}{2(k + 1)(2n + 2k + 3)}\beta_{2k} \tag{37}$$

If α_0 and β_0 are the arbitrary constants of the solution, then, for $k = 0$,

$$\alpha_2 = -\frac{n(n-1)}{1\cdot 2(2n-1)}\alpha_0 \tag{38}$$

$$\beta_2 = \frac{(n+1)(n+2)}{1\cdot 2(2n+3)}\beta_0 \tag{39}$$

When $k = 1$, equations (36) and (37) give

$$\begin{aligned}\alpha_4 &= -\frac{(n-2)(n-3)}{2\cdot 2(2n-3)}\alpha_2 \\ &= \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 4(2n-1)(2n-3)}\alpha_0\end{aligned} \tag{40}$$

$$\begin{aligned}\beta_4 &= \frac{(n+3)(n+4)}{2\cdot 2(2n+5)}\beta_2 \\ &= \frac{(n+1)(n+2)(n+3)(n+4)}{1\cdot 2\cdot 4(2n+3)(2n+5)}\beta_0\end{aligned} \tag{41}$$

Equation (35) can now be written in terms of equations (40) and (41), with α_0 and β_0 equal to A and B within an arbitrary constant.

$$\begin{aligned}\Phi = A&\left[\mu^n - \frac{n(n-1)}{1\cdot 2(2n-1)}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 4(2n-1)(2n-3)}\mu^{n-4} - \dots\right] \\ &+ B\left[\frac{1}{\mu^{n+1}} + \frac{(n+1)(n+2)}{1\cdot 2(2n+3)}\frac{1}{\mu^{n+3}} + \frac{(n+1)(n+2)(n+3)(n+4)}{1\cdot 2\cdot 4(2n+3)(2n+5)}\frac{1}{\mu^{n+5}} + \dots\right]\end{aligned} \tag{42}$$

Hobson (ref. 3) points out that equation (42) can be written in the more compact form

$$\begin{aligned}\Phi = A\mu^n F&\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1-2n}{2}; \frac{1}{\mu^2}\right) \\ &+ B\frac{1}{\mu^{n+1}} F\left(\frac{n+1}{2}, \frac{n+2}{2}; \frac{2n+3}{2}; \frac{1}{\mu^2}\right)\end{aligned} \tag{43}$$

where the function $F(a,b;c;d)$ is the hypergeometric series defined as

$$F(a,b;c;d) = 1 + \frac{ab}{c} d + \frac{a(a+1)b(b+1)}{1 \cdot 2c(c+1)} d^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3c(c+1)(c+2)} d^3$$

$$+ \ldots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\prod_{i=0}^{k-1} (a+i)(b+i)}{k! \prod_{i=0}^{k-1} (c+i)} d^k \tag{44}$$

where the positive integer counter i has limits $0 \leq i \leq k - 1$.

Equations (42) to (44) are not very useful for computational purposes. Equation (35) may be written in a more useful form if α_0 and β_0 are written as

$$\alpha_0 = \frac{(2n)!}{2^n n! n!} \tag{45}$$

$$\beta_0 = \frac{2^n n! n!}{(2n+1)!} \tag{46}$$

Substituting equations (45) and (46) in equations (38) to (41) gives

$$\alpha_2 = - \frac{n(n-1)}{1 \cdot 2(2n-1)} \cdot \frac{2n(2n-1)(2n-2)!}{2^n n(n-1)(n-2)! n(n-1)!}$$

$$= - \frac{(2n-2)!}{2^n (n-1)(n-2)!} \tag{47}$$

$$\beta_2 = \frac{(n+1)(n+2)}{1 \cdot 2(2n+3)} \cdot \frac{2^n n! n! (2n+2)}{(2n+1)!(2n+2)}$$

$$= \frac{2^n (n+1)! (n+2)!}{(2n+3)!} \tag{48}$$

$$\alpha_4 = -\frac{(n-2)(n-3)}{4(2n-3)} \cdot -\frac{(2n-2)!}{2^n(n-1)(n-2)!}$$

$$= \frac{(2n-2)!}{2^n 2!\,(n-2)(n-4)!} \tag{49}$$

$$\beta_4 = \frac{(n+3)(n+4)}{4(2n+5)} \cdot \frac{2^n(n+1)!\,(n+2)!}{(2n+3)!}$$

$$= \frac{2^n(n+2)!\,(n+4)!}{2(2n+5)!} \tag{50}$$

Equation (42) can now be written as

$$\Phi(\mu) = A\,\frac{1}{2^n}\sum_{t=0}^{\ell} \frac{(-1)^t(2n-2t)!}{t!\,(n-t)!\,(n-2t)!}\,\mu^{n-2t}$$

$$+ B2^n \sum_{k=0}^{\infty} \frac{(n+k)!\,(n+2k)!}{k!\,(2n+2k+3)!}\,\frac{1}{\mu^{n+2k+1}} \tag{51}$$

where t is a positive integer and ℓ is the greatest integer equal to $n/2$ when n is an even integer and the greatest integer equal to $(n-1)/2$ when n is odd. This restriction must be imposed to avoid negative factorials.

It is clear from equation (51) that the first series will terminate or converge for all values of μ, including complex values. It is also immediately clear that the second series will diverge for all values of μ such that their absolute values are less than unity. Since for the case of the geopotential, the value of μ is bounded such that $-1 \leq \mu = \sin\varphi \leq 1$, the second series does not satisfy the boundary conditions and, as such, must be disregarded in the solution. Thus, the solution of equation (24) is written as

$$I(\mu) = AP_{n0}(\mu) + BQ_{n0}(\mu) \tag{52}$$

where $P_{n0}(\mu)$ and $Q_{n0}(\mu)$ are Legendre polynomials of the first and second kind, of degree n and order zero, respectively. These functions are defined as

$$P_{n0}(\mu) = \frac{1}{2^n} \sum_{t=0}^{\ell} \frac{(-1)^t (2n - 2t)!}{t!\,(n - t)!\,(n - 2t)!} \mu^{n-2t} \tag{53}$$

$$Q_{n0}(\mu) = 2^n \sum_{k=0}^{\infty} \frac{(n + k)!\,(n + 2k)!}{k!\,(2n + 2k + 3)!} \frac{1}{\mu^{n+2k+1}} \tag{54}$$

Therefore, the value of the arbitrary constant B in equation (52) must be zero to satisfy the boundary conditions. It should be said, in passing, that the conspicuous factors 2^{-n} and 2^n are there for the purpose of making the value of the polynomials equal to unity when $n \neq 0$ and $\mu = \pm 1$.

By returning to Legendre's associated differential equation (eq. (21)) and replacing the arbitrary constant C by its more convenient form $n(n + 1)$, discovered in the previous solution, the following expression is obtained.

$$\left(1 - \mu^2\right)\Phi'' - 2\mu\Phi' + \left[n(n + 1) - \frac{m^2}{1 - \mu^2}\right]\Phi = 0 \tag{55}$$

Equation (55) is Legendre's associated differential equation of degree n and order m. A power series solution of equation (55) is inconvenient because of the $(1 - \mu^2)$ term appearing in the denominator. A more successful approach is to consider a solution of the form

$$\Phi(\mu) = \left(1 - \mu^2\right)^{m/2} M(\mu) \tag{56}$$

Substituting equation (56) in equation (55) gives a differential equation in $M(\mu)$, with $(1 - \mu^2)^{m/2}$ as a common factor. The functions Φ' and Φ'' are computed from equation (56) to be

$$\Phi'(\mu) = \left(1 - \mu^2\right)^{m/2}\left[-\mu m\left(1 - \mu^2\right)^{-1} M(\mu) + M'(\mu)\right] \tag{57}$$

$$\Phi''(\mu) = \left(1 - \mu^2\right)^{m/2}\left\{\left[\mu^2 m(m-2)\left(1-\mu^2\right)^{-1} - m\left(1-\mu^2\right)^{-1}\right]M(\mu) - 2\mu m\left(1-\mu^2\right)^{-1}M'(\mu) + M''(\mu)\right\} \tag{58}$$

Substituting equations (56) to (58) in equation (55) gives

$$\left(1-\mu^2\right)M''(\mu) - 2\mu(m+1)M'(\mu) + \left[m\mu^2(m-2)\left(1-\mu^2\right)^{-1} - m + 2m\mu^2\left(1-\mu^2\right)^{-1} + n(n+1) - m^2\left(1-\mu^2\right)^{-1}\right]M(\mu) = 0 \tag{59}$$

Simplifying equation (59) yields

$$\left.\begin{aligned} &\left(1-\mu^2\right)M''(\mu) - 2\mu(m+1)M'(\mu) + [n(n+1) - m(m+1)]M(\mu) = 0 \\ &\left(1-\mu^2\right)M''(\mu) - 2\mu(m+1)M'(\mu) + (n-m)(n+m+1)M(\mu) = 0 \end{aligned}\right\} \tag{60}$$

It is possible to obtain a power series solution of equation (60) by assuming the solution

$$M(\mu) = \sum_{k=0}^{\infty} a_{2k}\mu^{q-2k} \tag{61}$$

wherein a_{2k} is a constant coefficient. Substituting equation (61) in equation (60) gives

$$\left(1-\mu^2\right)\sum_{k=0}^{\infty}(q-2k)(q-2k-1)a_{2k}\mu^{q-2k-2} - 2\mu(m+1)\sum_{k=0}^{\infty}(q-2k)a_{2k}\mu^{q-2k-1} + (n-m)(n+m+1)\sum_{k=0}^{\infty}a_{2k}\mu^{q-2k} = 0 \tag{62}$$

$$\sum_{k=0}^{\infty} (q - 2k)(q - 2k - 1)a_{2k}\mu^{q-2k-2} - \sum_{k=0}^{\infty} [(q - 2k)(q - 2k + 2m + 1) - (n - m)(n + m + 1)]\, a_{2k}\mu^{q-2k} = 0 \tag{63}$$

Shifting the index in the second summation term from k to k + 1 and again requiring $a_{-2} = 0$ gives

$$\sum_{k=0}^{\infty} (q - 2k)(q - 2k - 1)a_{2k}\mu^{q-2k-2} - \sum_{k=0}^{\infty} [(q - 2k - 2)(q - 2k + 2m - 1) - (n - m)(n + m + 1)]\, a_{2k+2}\mu^{q-2k-2} = 0 \tag{64}$$

The recursive relation may now be obtained by solving for a_{2k} in equation (64). Thus,

$$a_{2k} = \frac{(q - 2k - 2)(q - 2k + 2m - 1) - (n - m)(n + m + 1)}{(q - 2k)(q - 2k - 1)}\, a_{2k+2} \tag{65}$$

The numerator of equation (65) can be factored as follows.

$$a_{2k} = \frac{[(q - 2k - 2) - (n - m)][(q - 2k - 2) + (n + m + 1)]}{(q - 2k)(q - 2k - 1)}\, a_{2k+2} \tag{66}$$

When $k = -1$, $a_{-2} = 0$ and the following solutions are possible.

$$q_1 = n - m \tag{67}$$

$$q_2 = -(n + m + 1) \tag{68}$$

Since for a power series solution, q must be constrained to positive integer values, equation (68) must be associated with a solution to be disregarded. From equation (67), it is seen that the relation between n and m must always be such that

$$n \geq m \tag{69}$$

The solution of equation (55) may now be written with A and B as the arbitrary constants of the solution as

$$\Phi(\mu) = \left(1 - \mu^2\right)\left(A \sum_{k=0}^{\infty} a_{2k}\mu^{n-m-2k} + B \sum_{k=0}^{\infty} b_{2k}\mu^{-n-m-2k-1}\right) \tag{70}$$

where the coefficients a_{2k} and b_{2k} are obtained from the recursive relations

$$a_{2k+2} = -\frac{(n - m - 2k)(n - m - 2k - 1)}{2(k + 1)(2n - 2k - 1)} a_{2k} \tag{71}$$

$$b_{2k+2} = \frac{(n + m + 2k + 1)(n + m + 2k + 2)}{2(k + 1)(2n + 2k + 3)} b_{2k} \tag{72}$$

Equations (71) and (72) are identical to equations (36) and (37), respectively, when m is equal to zero. It is interesting to note that m appears only in the numerator, bearing opposite signs in each recursive relation. Taking for a_0 and b_0 the following values

$$a_0 = \frac{(2n)!}{2^n n!(n - m)!} \tag{73}$$

$$b_0 = \frac{2^n n!(n + m)!}{(2n + 1)!} \tag{74}$$

then for $k = 0$,

$$a_2 = -\frac{(n - m)(n - m - 1)}{2(2n - 1)} a_0$$

$$= -\frac{(n - m)(n - m - 1)}{2(2n - 1)} \cdot \frac{(2n)!}{2^n n!\,(n - m)!}$$

$$= -\frac{(2n - 2)!}{2^n (n - 1)!\,(n - m - 2)!} \tag{75}$$

$$b_2 = \frac{(n + m + 1)(n + m + 2)}{2(2n + 3)} b_0$$

$$= \frac{(n + m + 1)(n + m + 2)}{2(2n + 3)} \cdot \frac{2^n n!\,(n + m)!}{(2n + 1)!}$$

$$= \frac{2^n (n + 1)!\,(n + m + 2)!}{(2n + 3)!} \tag{76}$$

When $k = 1$,

$$a_4 = -\frac{(n - m - 2)(n - m - 3)}{2 \cdot 2(2n - 3)} a_2$$

$$= -\frac{(n - m - 2)(n - m - 3)}{2 \cdot 2(2n - 3)} \cdot -\frac{(2n - 2)!}{2^n (n - 1)!\,(n - m - 2)!}$$

$$= \frac{(2n - 4)!}{2^n 2 (n - 2)!\,(n - m - 4)!} \tag{77}$$

$$b_4 = \frac{(n + m + 3)(n + m + 4)}{2 \cdot 2(2n + 5)} b_2$$

$$= \frac{(n + m + 3)(n + m + 4)}{2 \cdot 2(2n + 5)} \cdot \frac{2^n (n + 1)!\,(n + m + 2)!}{(2n + 3)!} \; \frac{(2n + 4)}{(2n + 4)}$$

$$= \frac{2^n (n + 2)!\,(n + m + 4)!}{2(2n + 5)!} \tag{78}$$

The solution for $\Phi(\mu)$ can now be written as

$$\Phi(\mu) = \left(1 - \mu^2\right)^{\frac{m}{2}} A \frac{1}{2^n} \sum_{t=0}^{\ell} \frac{(-1)^t (2n - 2t)!}{t!\,(n - t)!\,(n - m - 2t)!} \mu^{n-m-2t}$$

$$+ \left(1 - \mu^2\right)^{\frac{m}{2}} B 2^n \sum_{k=0}^{\infty} \frac{(n + k)!\,(n + m + 2k)!}{k!\,(2n + 2k + 3)!} \frac{1}{\mu^{n+m+}} \tag{79}$$

where ℓ is the greatest integer equal to $(n - m)/2$ when $(n - m)$ is an even integer and the greatest integer equal to $(n - m - 1)/2$ when $(n - m)$ is odd. This restriction is imposed to avoid negative factorials.

Equation (79) reduces to equation (51) when $m = 0$. As in the case of equation (51), the first series terminates or converges for all positive values of n and m such that $n \geq m$ regardless of the value of μ. The second series converges only for values of μ with a magnitude greater than unity. As a result, the series must be disregarded in this solution, as the value of μ is bounded such that $-1 \leq \mu = \sin\varphi \leq +1$. Therefore, the solution to Legendre's associated differential equation may be written as

$$\Phi(\mu) = AP_{nm}(\mu) + BQ_{nm}(\mu) \tag{80}$$

where $P_{nm}(\mu)$ and $Q_{nm}(\mu)$ are the associated Legendre functions of the first and second kind, respectively, each of degree n and order m, defined as

$$P_{nm}(\mu) = \left(1 - \mu^2\right)^{\frac{m}{2}} \frac{1}{2^n} \sum_{t=0}^{\ell} \frac{(-1)^t (2n - 2t)!}{t!\,(n - t)!\,(n - m - 2t)!} \mu^{n-m-2t} \tag{81}$$

$$Q_{nm}(\mu) = \left(1 - \mu^2\right)^{\frac{m}{2}} 2^n \sum_{k=0}^{\infty} \frac{(n + k)!\,(n + m + 2k)!}{k!\,(2n + 2k + 3)!} \frac{1}{\mu^{n+m+2k+1}} \tag{82}$$

The arbitrary constant B in equation (80) must be set equal to zero to satisfy the boundary conditions of the geopotential.

Equation (13) written with $C = n(n + 1)$ becomes

$$r^2R'' + 2rR' - n(n + 1)R = 0 \tag{83}$$

Substituting equation (22) in equation (83) gives

$$\alpha h(h - 1)r^2r^{h-2} + 2\alpha hrr^{h-1} - n(n + 1)\alpha r^h = 0 \tag{84}$$

The characteristic equation that results from equation (84) is

$$\left.\begin{aligned} &h(h - 1) + 2h - n(n + 1) = 0 \\ &h^2 + h - n(n + 1) = 0 \end{aligned}\right\} \tag{85}$$

By applying the quadratic formula, the values of h become

$$h = -\frac{1}{2} \pm \left(n + \frac{1}{2}\right) \tag{86}$$

or

$$h_1 = n \tag{87}$$

$$h_2 = -(n + 1) \tag{88}$$

The solution of equation (83) becomes

$$R = C_1r^n + C_2r^{-(n+1)} \tag{89}$$

Since the potential function V is expected to vanish as $r \to \infty$, the arbitrary constant C_1 must be set equal to zero. The solution reduces to the form

$$R = Cr^{-(n+1)} \tag{90}$$

The solution to equation (16) is obtained by substituting equation (23) in equation (16). This gives

$$Ae^{\alpha\lambda}\left(\alpha^2 + m^2\right) = 0 \tag{91}$$

which gives, for the solution of the characteristic equation, $\alpha = \pm im$, where $i = \sqrt{-1}$. The solution of equation (16) becomes

$$\Lambda(\lambda) = Ce^{im\lambda} + Se^{-im\lambda} \tag{92}$$

where C and S are arbitrary constants of the solution. By ignoring the imaginary terms, equation (92) can be simplified to

$$\Lambda(\lambda) = C_m \cos(m\lambda) + S_m \sin(m\lambda) \tag{93}$$

After substituting equations (80), (89), and (93) in equation (6), the complete solution to Laplace's equation in spherical coordinates becomes

$$V_{nm} = \left[C_1 r^n + C_2 r^{-(n+1)}\right]\left[AP_{nm}(\sin\varphi) + BQ_{nm}(\sin\varphi)\right]\left[C_m \cos(m\lambda) + S_m \sin(m\lambda)\right] \tag{94}$$

By setting C_1 and B equal to zero, as discussed earlier, the equation for the geopotential function reduces to

$$V(r,\varphi,\lambda) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{P_{nm}(\sin\varphi)}{r^{n+1}}\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right] \tag{95}$$

The constants C_m and S_m have been written as C_{nm} and S_{nm} to incorporate all the constants of the solution. These values are determined experimentally and may be found in normalized form in reference 4 and in unnormalized form in reference 5.

Equation (95) has units of reciprocal length [illegible] be written to reflect units of kinetic energy squared so that its gradient [illegible] of acceleration by introducing the gravitational parameter g_E, which has units of length cubed per time squared, and the normalizing parameter R_E, usually the equatorial radius of the body. Thus,

$$V(r,\varphi,\lambda) = \sum_{n=0}^{\infty} \frac{g_E}{r}\left(\frac{R_E}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\varphi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right] \tag{96}$$

The harmonic coefficient constants C_{nm} and S_{nm} require additional discussion. The impossibility of determining values for S_{n0} produced when $m = 0$ is of no consequence since $\sin(0 \cdot \lambda) = 0$. The values of C_{n0} are a special set, known as the zonal harmonic coefficients, and are entirely dependent on latitude. For values of n and m greater than zero but such that $n > m$, these constants are given the name "tesseral harmonic coefficients" and are dependent on both latitude and longitude. When $n = m$, these coefficients are given the name "sectoral harmonic coefficients" and are entirely dependent on longitude.

The Gradient of the Geopotential Function

The gravitational force, being conservative in nature, can be obtained from the gradient of the geopotential function. To obtain the gradient of the geopotential function, equation (96) can be written as

$$V(r,\varphi,\lambda) = \frac{g_E}{r} C_{00} + \frac{g_E}{r}\sum_{n=1}^{\infty}\left(\frac{R_E}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\varphi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right] \tag{97}$$

since $\sin(0 \cdot \lambda) = 0$ and $(R_E/r)^0 = \cos(0 \cdot \lambda) = P_{00} = 1$, as may be seen from equation (81). The term $(g_E C_{00})/r$ is the gravitational potential associated with an unperturbed inverse square law of force, and the constant $C_{00} = \pm 1$, depending on whether the force is one of attraction or repulsion, respectively. The summation terms form that part of the potential associated with the disturbing acceleration. From equation (97),

$$\nabla V = \left[\nabla \frac{g_E}{r}\right]\left\{C_{00} + \sum_{n=1}^{\infty}\left(\frac{R_E}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\varphi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right\}$$

$$+ \frac{g_E}{r}\sum_{n=1}^{\infty}\left\{\left[\nabla\left(\frac{R_E}{r}\right)^n\right]\sum_{m=0}^{n} P_{nm}(\sin\varphi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right.$$

$$+ \left(\frac{R_E}{r}\right)^n \sum_{m=0}^{n}\left(\left[\nabla P_{nm}(\sin\varphi)\right]\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right.$$

$$\left.\left. + P_{nm}(\sin\varphi)\left[\nabla\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right]\right)\right\} \tag{98}$$

where

$$\nabla \frac{g_E}{r} = -\frac{g_E}{r}\,\frac{x\underline{i} + y\underline{j} + z\underline{k}}{r^2} \tag{99}$$

$$\nabla\left(\frac{R_E}{r}\right)^n = -n\left(\frac{R_E}{r}\right)^n \frac{x\underline{i} + y\underline{j} + z\underline{k}}{r^2} \tag{100}$$

$$\sin\varphi = z/r \tag{101}$$

$$\nabla P_{nm}(\sin\varphi) = \left\{\nabla\left[1 - \left(\frac{z}{r}\right)^2\right]^{\frac{m}{2}}\right\}\sum_{t=0}^{\ell} T_{nmt}\left(\frac{z}{r}\right)^{n-m-2t}$$

$$+ \left[1 - \left(\frac{z}{r}\right)^2\right]\sum_{t=0}^{\ell} T_{nmt}\nabla\left[\left(\frac{z}{r}\right)^{n-m-2t}\right] \tag{102}$$

$$T_{nmt} = \frac{(-1)^t (2n - 2t)!}{2^n t!\,(n - t)!\,(n - m - 2t)!} \tag{103}$$

$$\nabla\left[\left(\frac{z}{r}\right)^{n-m-2t}\right] = (n - m - 2t)\left(\frac{z}{r}\right)^{n-m-2t-1} \nabla\left(\frac{z}{r}\right) \tag{104}$$

$$\nabla\left(\frac{z}{r}\right) = \frac{1}{r}\left(\underline{k} - z\,\frac{x\underline{i} + y\underline{j} + z\underline{k}}{r^2}\right) \tag{105}$$

$$\nabla\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right] = m\left[-C_{nm}\sin(m\lambda) + S_{nm}\cos(m\lambda)\right]\nabla\lambda \tag{106}$$

$$\lambda = \text{arc}\tan\left(\frac{y}{x}\right) \tag{107}$$

$$\nabla\lambda = \frac{\partial\lambda}{\partial x}\,\underline{i} + \frac{\partial\lambda}{\partial y}\,\underline{j} + \frac{\partial\lambda}{\partial z}\,\underline{k} \tag{108}$$

$$d\lambda = \frac{x\,dy - y\,dx}{x^2 + y^2} \tag{109}$$

Substituting equation (109) in equation (108) gives

$$\begin{aligned} \nabla\lambda &= -\frac{y}{x^2 + y^2}\,\underline{i} + \frac{x}{x^2 + y^2}\,\underline{j} \\ &= \frac{x\underline{j} - y\underline{i}}{x^2 + y^2} \end{aligned} \tag{110}$$

Therefore, by writing $x_1 = x$, $x_2 = y$, and $x_3 = z$,

$$
\left.\begin{aligned}
\nabla V &= \frac{\partial V}{\partial x_1}\,\underline{i} + \frac{\partial V}{\partial x_2}\,\underline{j} + \frac{\partial V}{\partial x_3}\,\underline{k} \\
\frac{\partial V}{\partial x_i} &= -\frac{g_E}{r}\left(\frac{x_i}{r^2}\right)\Bigg\{C_{00} \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{R_E}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\varphi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\Bigg\} \\
&\quad + \frac{g_E}{r}\sum_{n=1}^{\infty}\Bigg\{-n\left(\frac{R_E}{r}\right)^n \frac{x_i}{r^2}\sum_{m=0}^{n} P_{nm}(\sin\varphi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right] \\
&\quad + \left(\frac{R_E}{r}\right)^n\Bigg\{\Bigg(\frac{m}{2}\left[1-\left(\frac{z}{r}\right)^2\right]^{\frac{m}{2}-1}\left[-2\,\frac{z}{r^2}\left(k_2 - z\,\frac{x_i}{r^2}\right)\right]\sum_{t=0}^{\ell} T_{nmt}\left(\frac{z}{r}\right)^{n-m-2t} \\
&\quad + \frac{1}{r}\left(k_1 - z\,\frac{x_i}{r^2}\right)\left[1-\left(\frac{z}{r}\right)^2\right]^{\frac{m}{2}}\sum_{t=0}^{\ell} T_{nmt}(n-m-2t)\left(\frac{z}{r}\right)^{n-m-2t-1}\Bigg) \\
&\quad \cdot\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right] - \frac{k_2 x_{i\pm1}}{x^2+y^2}\sum_{m=0}^{n} P_{nm}(\sin\varphi) \\
&\quad \cdot m\left[-C_{nm}\sin(m\lambda) + S_{nm}\cos(m\lambda)\right]\Bigg\}\Bigg\}
\end{aligned}\right\} \quad (111)
$$

where $i = 1, 2, 3$, $k_1 = 0, 0, 1$, and $k_2 = 1, -1, 0$, respectively, and where ℓ equals $(n - m)$ when $(n - m)$ is an even integer and equals $(n - m - 1)/2$ when $(n - m)$ is odd.

Equation (111) is useful in computing the gradient components, but its vector form can be written concisely as follows.

$$\nabla V = \sum_{n=0}^{\infty} \frac{g_E}{r}\left(\frac{R_E}{r}\right)^n \left\{ -\frac{(n+1)}{r}\left[\frac{x\underline{i} + y\underline{j} + z\underline{k}}{r^2}\right] \sum_{m=0}^{n} P_{nm}(\sin\varphi)\left[C_{nm}\cos(m\lambda)\right.\right.$$

$$\left. + S_{nm}\sin(m\lambda)\right] + \sum_{m=0}^{n}\left\{\nabla\left[P_{nm}(\sin\varphi)\right]\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right]\right.$$

$$\left.\left. + mP_{nm}(\sin\varphi)\left[-C_{nm}\sin(m\lambda) + S_{nm}\cos(m\lambda)\right]\frac{x\underline{j} - y\underline{i}}{x^2 + y^2}\right\}\right\} \tag{112}$$

It may be seen at once that, in the neighborhood of the poles (that is, where $\varphi = \pm 90°$), λ becomes indeterminate and the gradient of the gravitational potential obtained from equation (112) possesses two isolated singularities. These singularities are eliminated by Pines (ref. 1) and Spencer (ref. 2). A condensed form of the work of Pines is given in appendix B.

RESULTS AND DISCUSSION

Both the classical method, equation (112), and the Pines method, equation (B33), have been programed and evaluated on the basis of execution time, storage requirements, and accuracy. The accuracy study was conducted for an integration time of approximately 2 days. It was found that, except in the polar regions, both methods are identical in accuracy. However, the Pines method was approximately 15 times faster and required only one-third of the storage requirements. For this reason, the Pines method has been adopted as the formulation to use in any integration requiring gravitational potential perturbations, especially if both the gravitational potential and its gradient are required.

A small study was conducted to show the importance of gravitational potential perturbations, especially those of higher orders. In this study, the position and velocity vector differences for various potential models of varying order and degree were each computed with the use of an eighth-order model as the standard. These differences have been plotted in figures 1 to 4. Figures 1(a) to 1(f) and 2(a) to 2(f) present the differences obtained over the initial orbit at 32 equidistant time steps in the orbit. Figures 3(a) to 3(f) and 4(a) to 4(f) present differences resulting over 32 equally spaced time increments equal to the period of Keplerian orbit of the initial state. The initial state of the orbit is as follows.

Semimajor axis	6629.656565 kilometers
Eccentricity	0.01
Inclination	0.7854 radian
Longitude of the node	0.7854 radian
Argument of periapsis	0.7854 radian
Eccentric anomaly	0.7854 radian
Julian date	2442332.17
Keplerian period	5372.137432 seconds

The comparison was conducted both by considering potential perturbations only and by considering potential perturbations together with drag and third-body perturbation from the Sun and the Moon. The atmospheric model in reference 6 was used to compute density in the drag formulation, and NASA Jet Propulsion Laboratory ephemeris tape data were used for the third-body and atmospheric-model perturbations.

The legend for figures 1 to 4 is as follows. Each figure page includes six curves, which are designated with numbers ranging from 2 to 7. Curves marked with a "2" are curves in which only second-order or second-degree terms were considered. Curves marked with a "3" are curves in which both second- and third-order or second- and third-degree terms were included. In short, the higher the number used to designate the curve, the more complete was the extent of the potential function used. As a result, seventh-order curves show the lowest errors because they approximate the eighth-order reference more closely.

The results of the study are as follows. High-order-gravitational-potential models are essential to the accuracy of a trajectory integration. This statement is true whether the integration is to be performed over a relatively short period, such as over one orbit (fig. 1(b)), or (and especially so) over longer periods (fig. 3(b)). These curves verify the fact that nothing is gained in accuracy by increasing the degree of the potential model without adding the corresponding tesseral terms. These facts are also substantiated by the velocity error curves. In fact, the position error curves, as expected, are virtually identical to the velocity error curves. Only the scale along the ordinate is different. As a result, this discussion of results will be confined to the position error curves because the same conclusions apply for the velocity error curves.

As was stated earlier, comparisons were made with and without the effects of drag. In both instances, the shapes of the curves were identical except that the magnitude of the errors for both long- and short-duration runs was slightly higher when drag effects were not considered. Since the retarding force of the drag was absent and the effects of the Sun and the Moon are negligible, this result was expected. The curves show additional interesting effects.

It may be said, intuitively, that the higher the order of the model, the closer the agreement between the results and the higher order reference would be. The results of this study show this theory to be true only for the very lowest and the highest order comparison curves. For the one-orbit case, although the second-order model did prove to be the worst of all models compared, the fourth-order model proved worse than the third-order model and the fifth-, sixth-, and seventh-order models varied continuously. In the long-duration case, the second- and seventh-order models were clearly the worst and best, respectively, with the seventh-order model maintaining almost a constant difference with the reference model much unlike the rest of the comparison curves, which showed a definite secular trend. However, the fourth- to sixth-order curves were fairly well grouped, with the sixth-order curve showing a clearer and more distinct pattern.

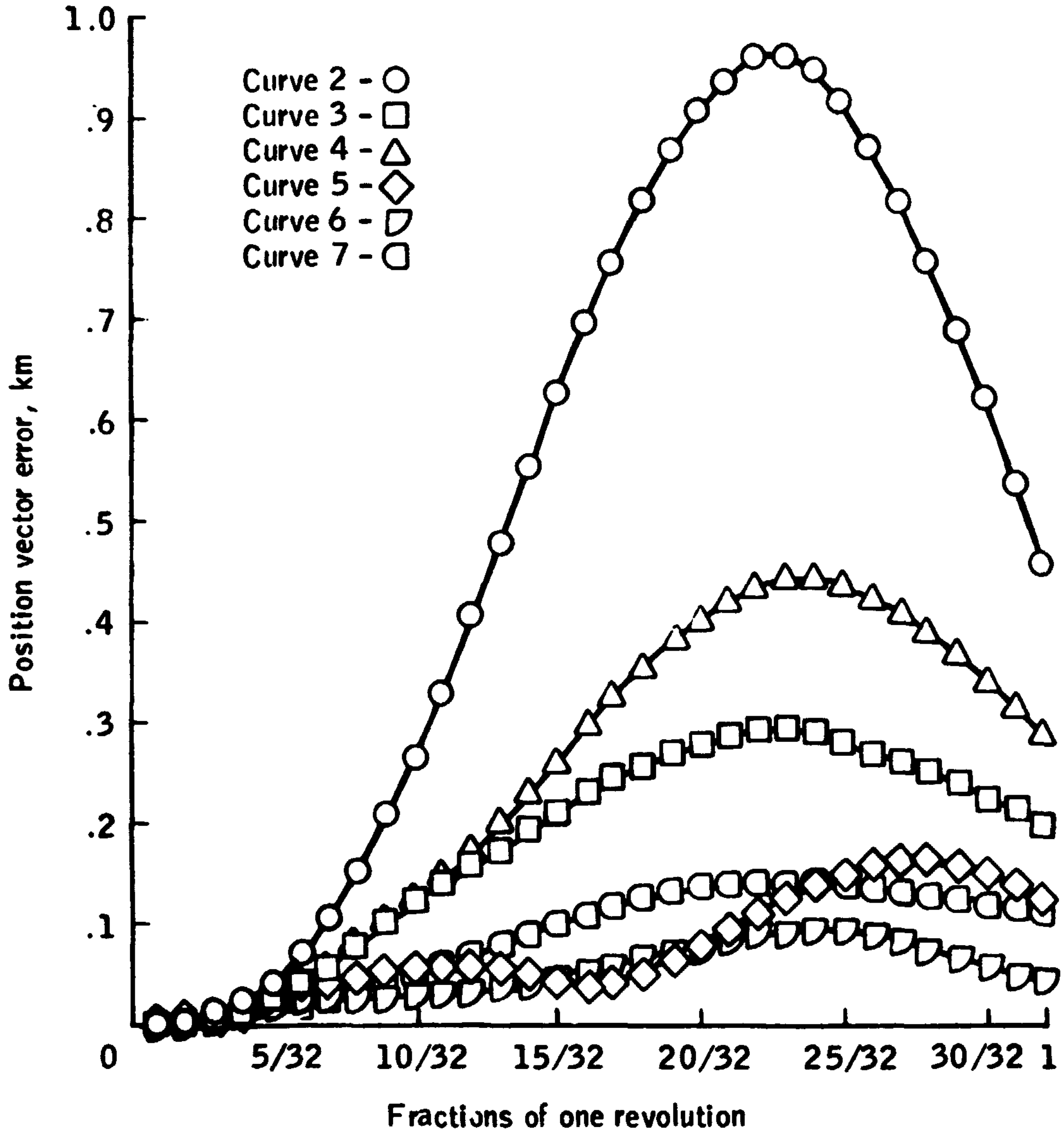
When the tesseral terms were removed (figs. 1(b) and 3(b)), all curves of fourth degree and higher showed the same amount of error. However, when drag was removed, the short-duration run (one-orbit case) seemed little affected as compared with the case when drag was present. In the long-duration run, all curves, regardless of the order, showed the same error, which amounted to approximately 576 kilometers after 32 orbits.

No comparison runs were made with the tesseral terms removed from the reference model. However, from the results concerning the error produced when the tesseral terms were removed, it may be concluded that beyond the fourth degree, there is little to be gained in adding additional zonal terms without the benefit of their corresponding tesseral terms.

CONCLUDING REMARKS

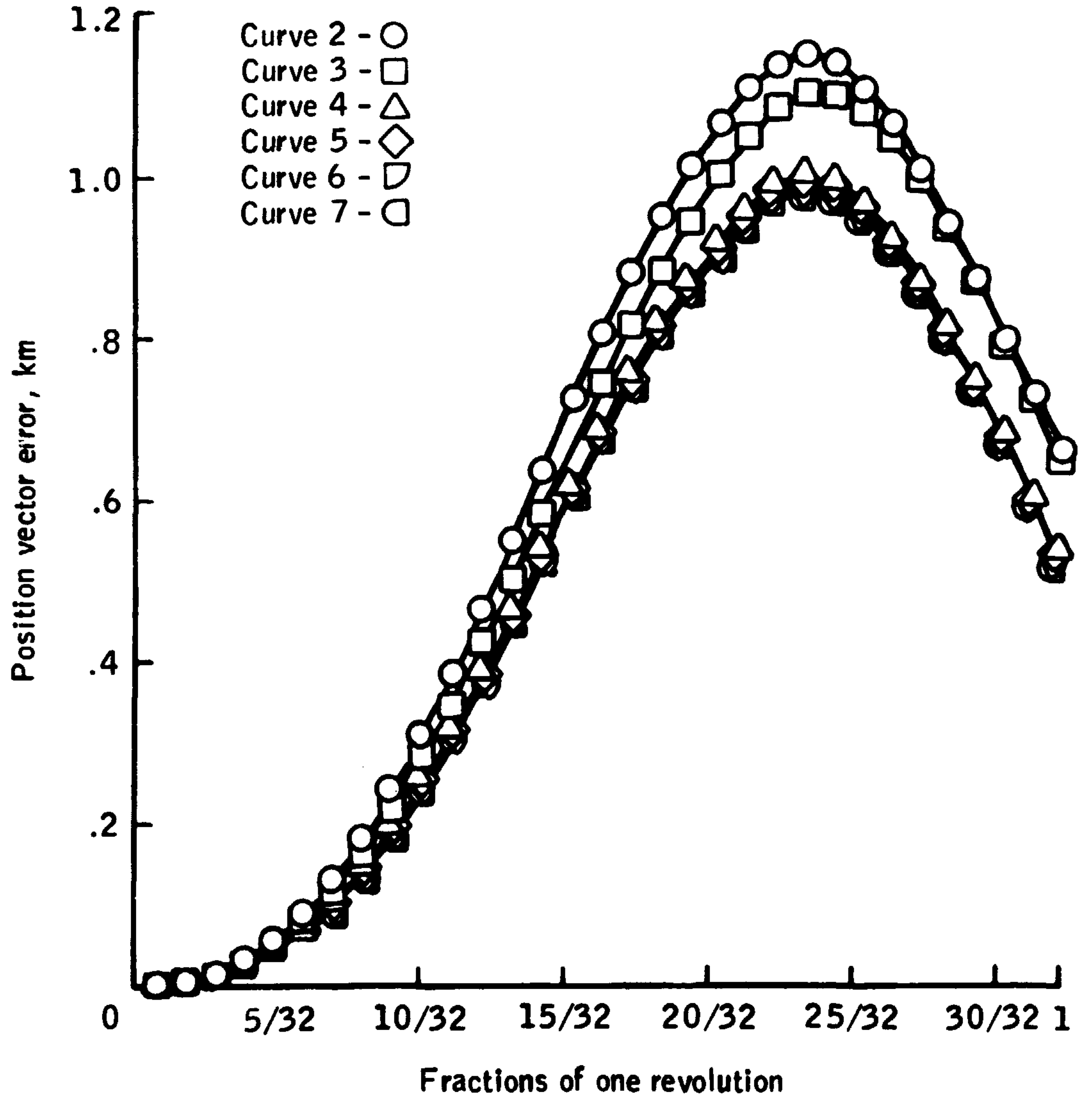
It may be concluded from the results of this study that, for any trajectory integration that extends for any appreciable length of time (especially for more than one orbit), the inclusion of high-order-gravitational-potential models in the equations of motion is an absolute requirement if any degree of accuracy is to be achieved. Errors of 25 kilometers and more are not uncommon if the gravitational potential terms are excluded.

Lyndon B. Johnson Space Center
National Aeronautics and Space Administration
Houston, Texas, July 31, 1976
986-16-00-00-72



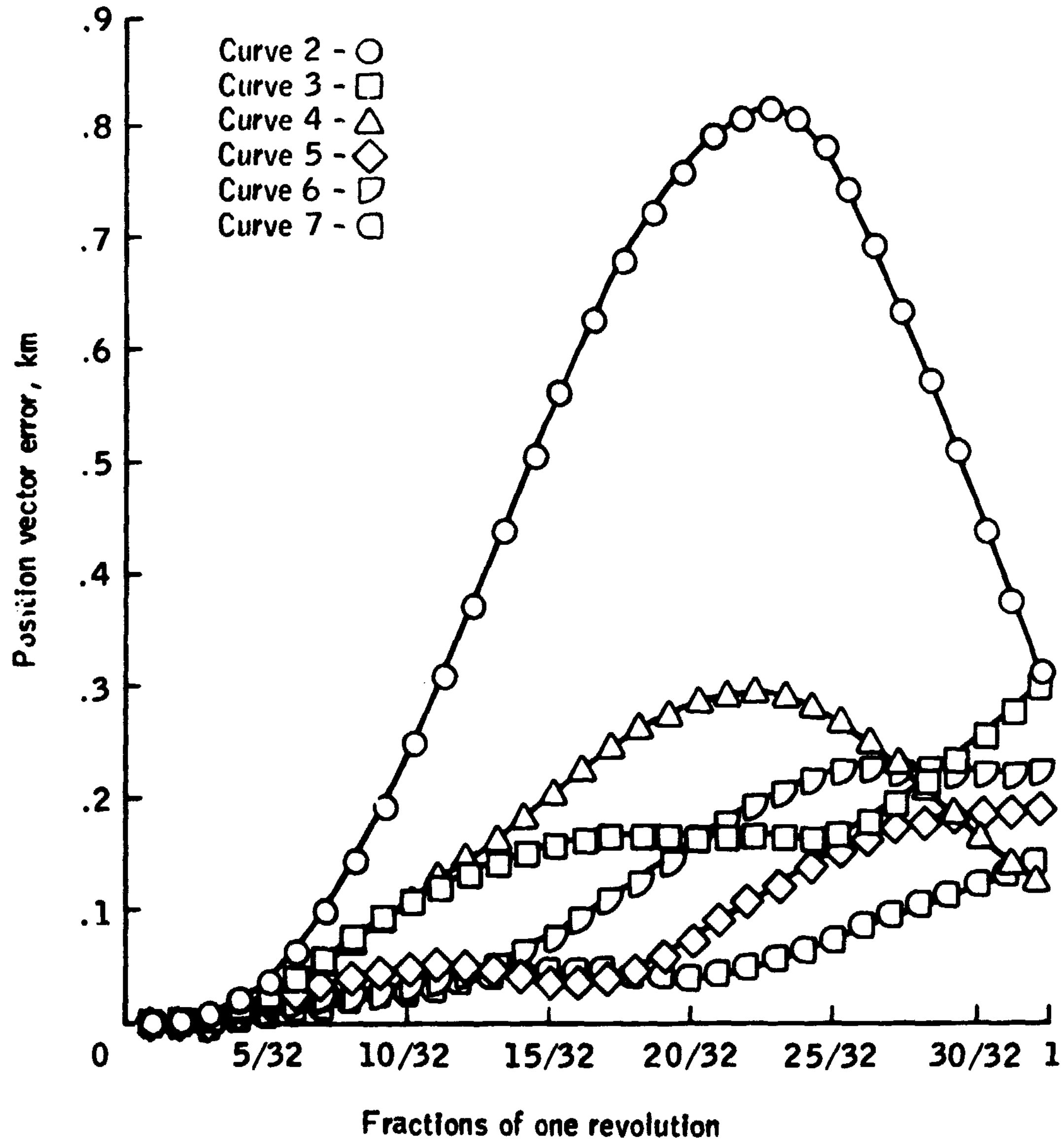
(a) Eighth-order model reference with drag; lower order models with drag.

Figure 1.- Position vector differences obtained over the initial orbit. The curves include the following order or degree terms: (2) second only, (3) second and third, (4) second to fourth, (5) second to fifth, (6) second to sixth, and (7) second to seventh.



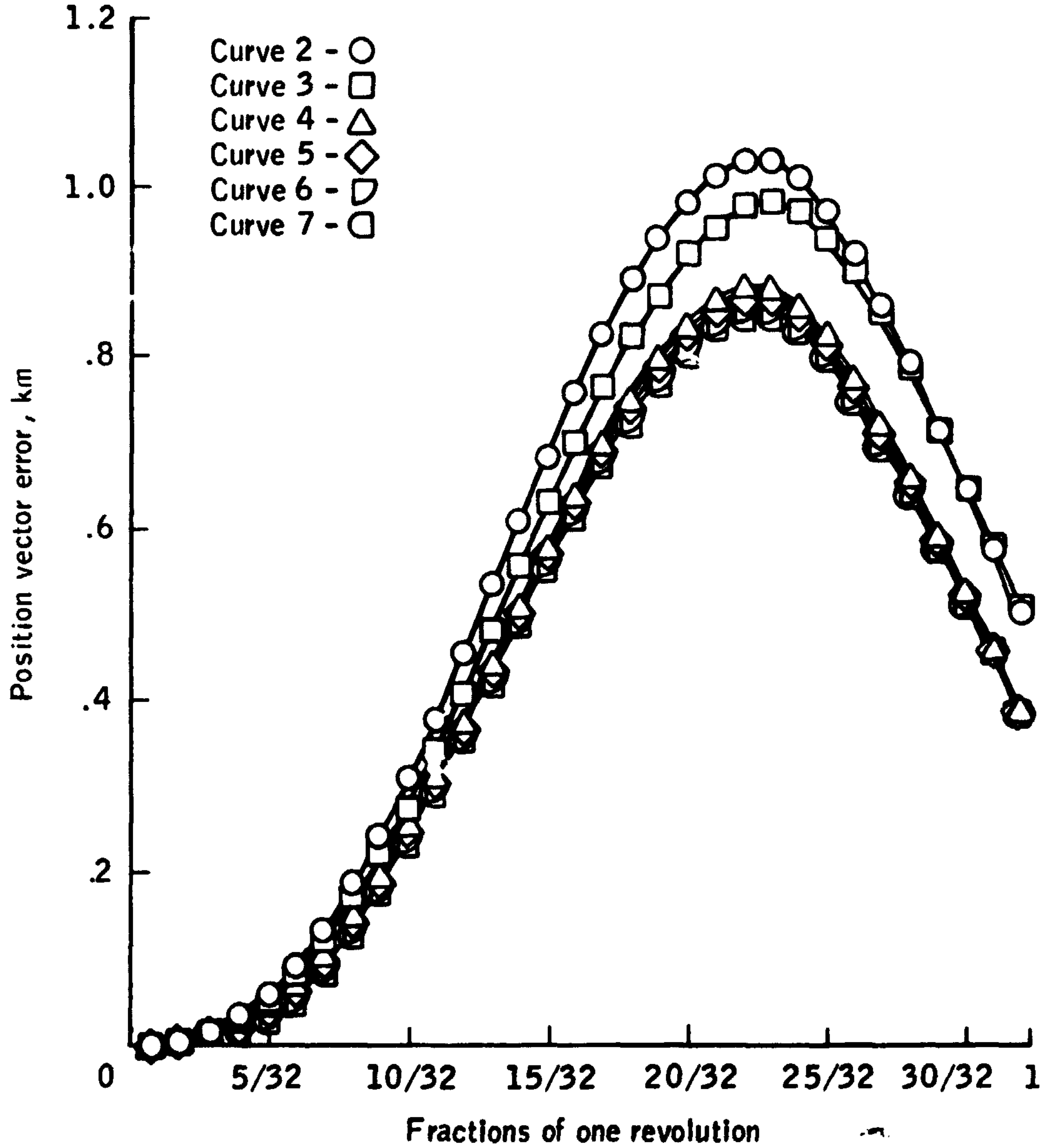
(b) Eighth-order model reference with drag; lower degree models with drag.

Figure 1.- Continued.



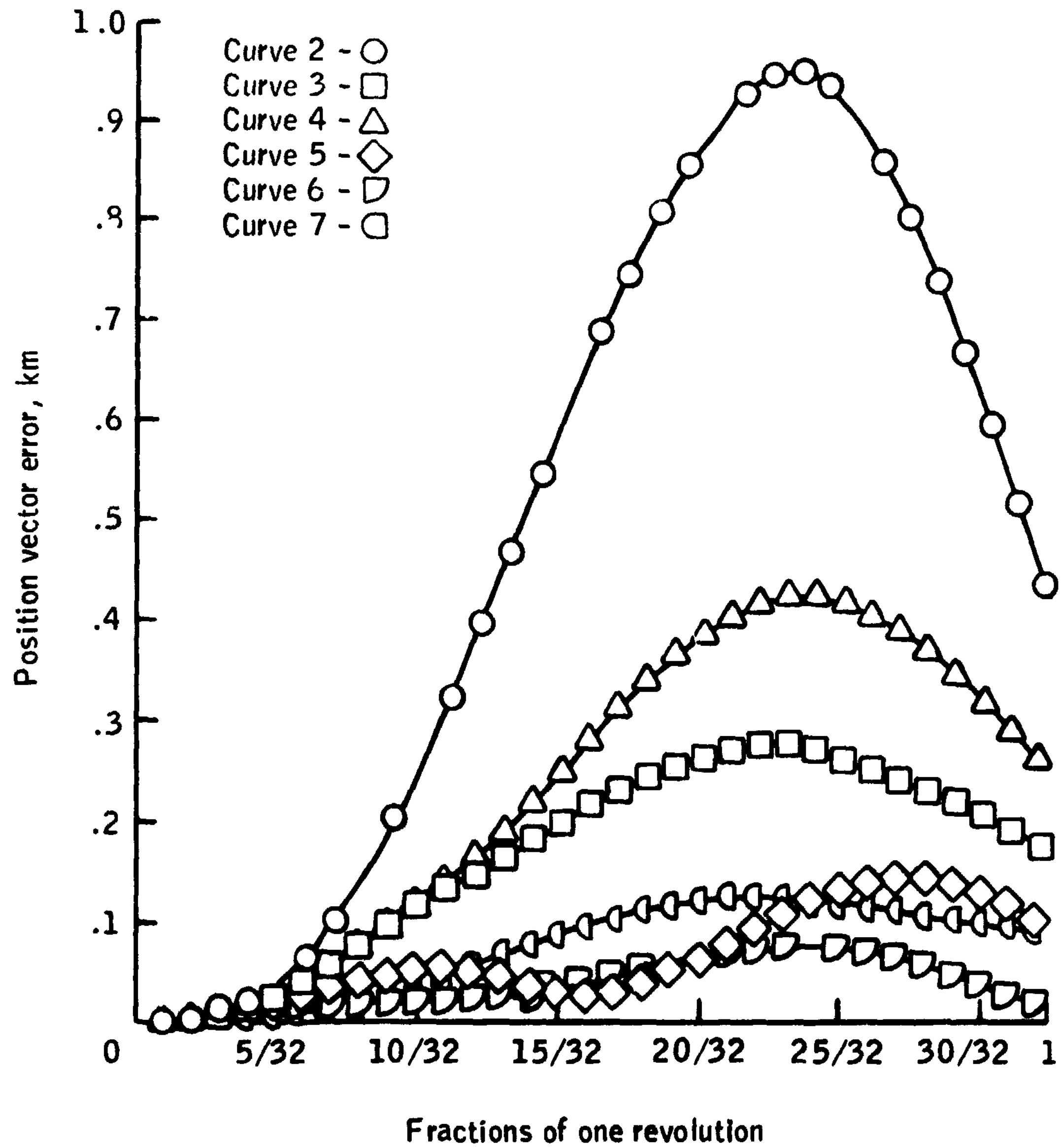
(c) Eighth-order model reference with drag; lower order models without drag.

Figure 1.- Continued.



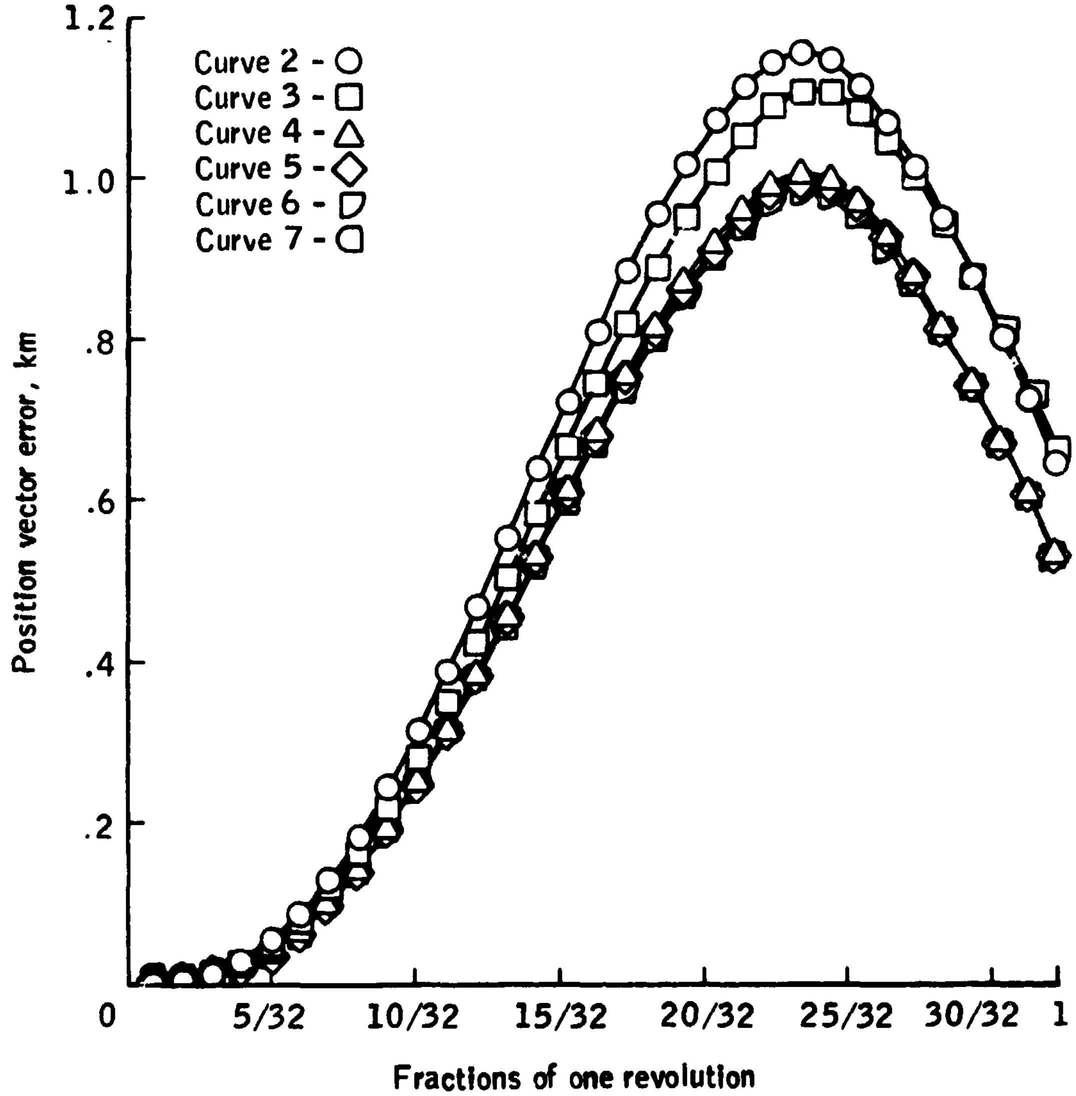
(d) Eighth-order model reference with drag; lower degree models without drag.

Figure 1.- Continued.



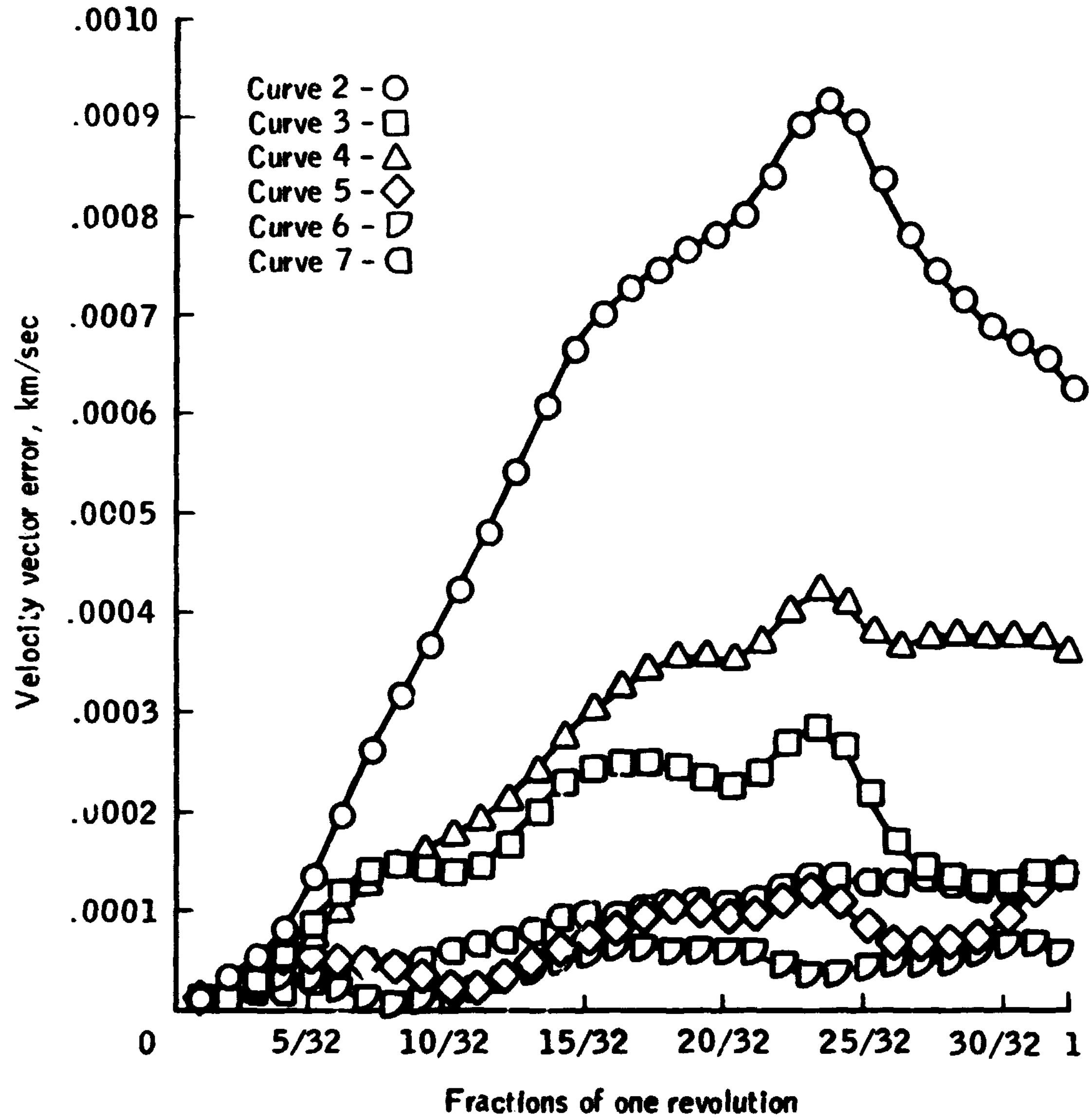
(e) Eighth-order model reference without drag; lower order models without drag.

Figure 1.- Continued.



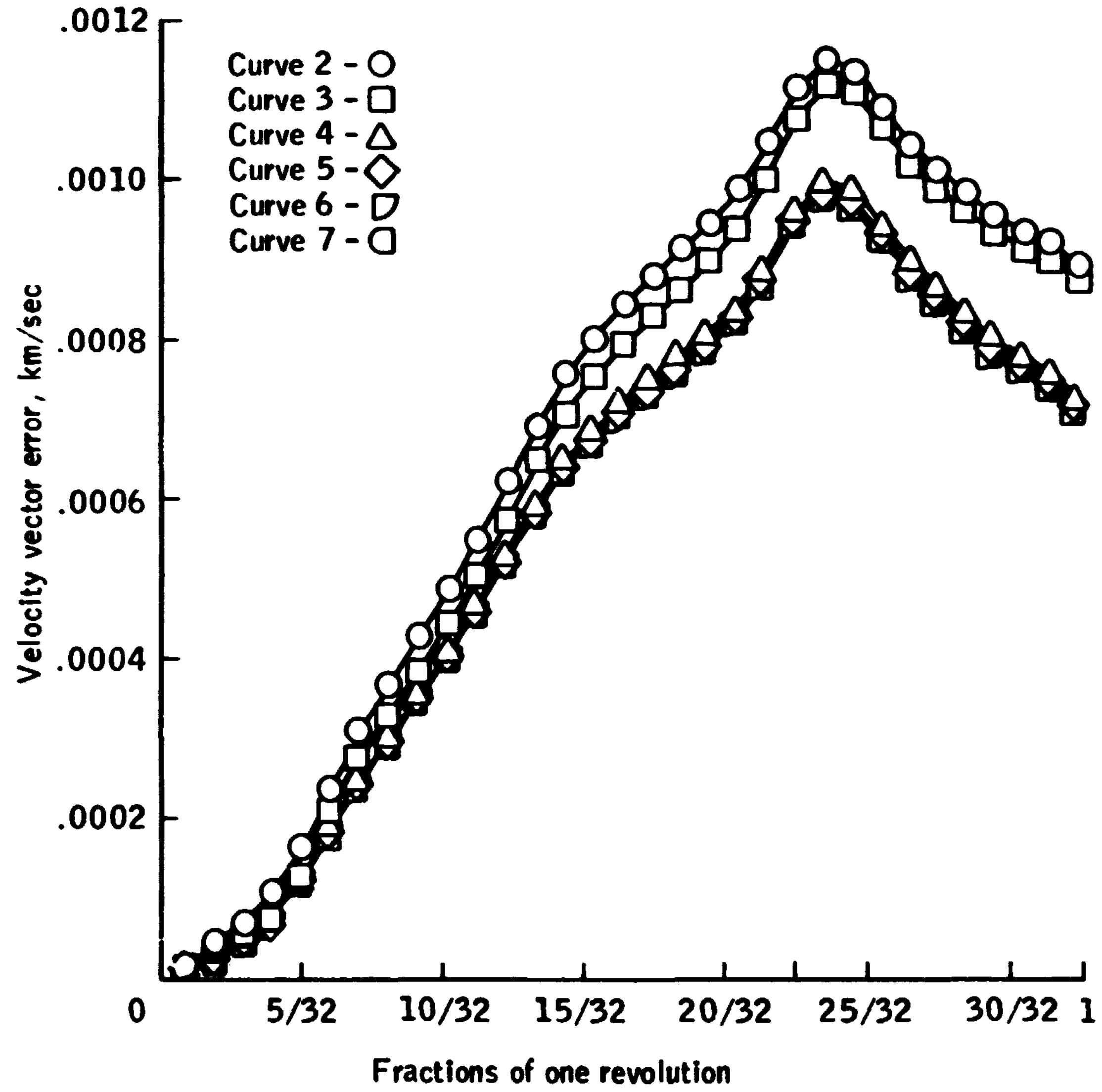
(f) Eighth-order model reference without drag; lower degree models without drag.

Figure 1.- Concluded.



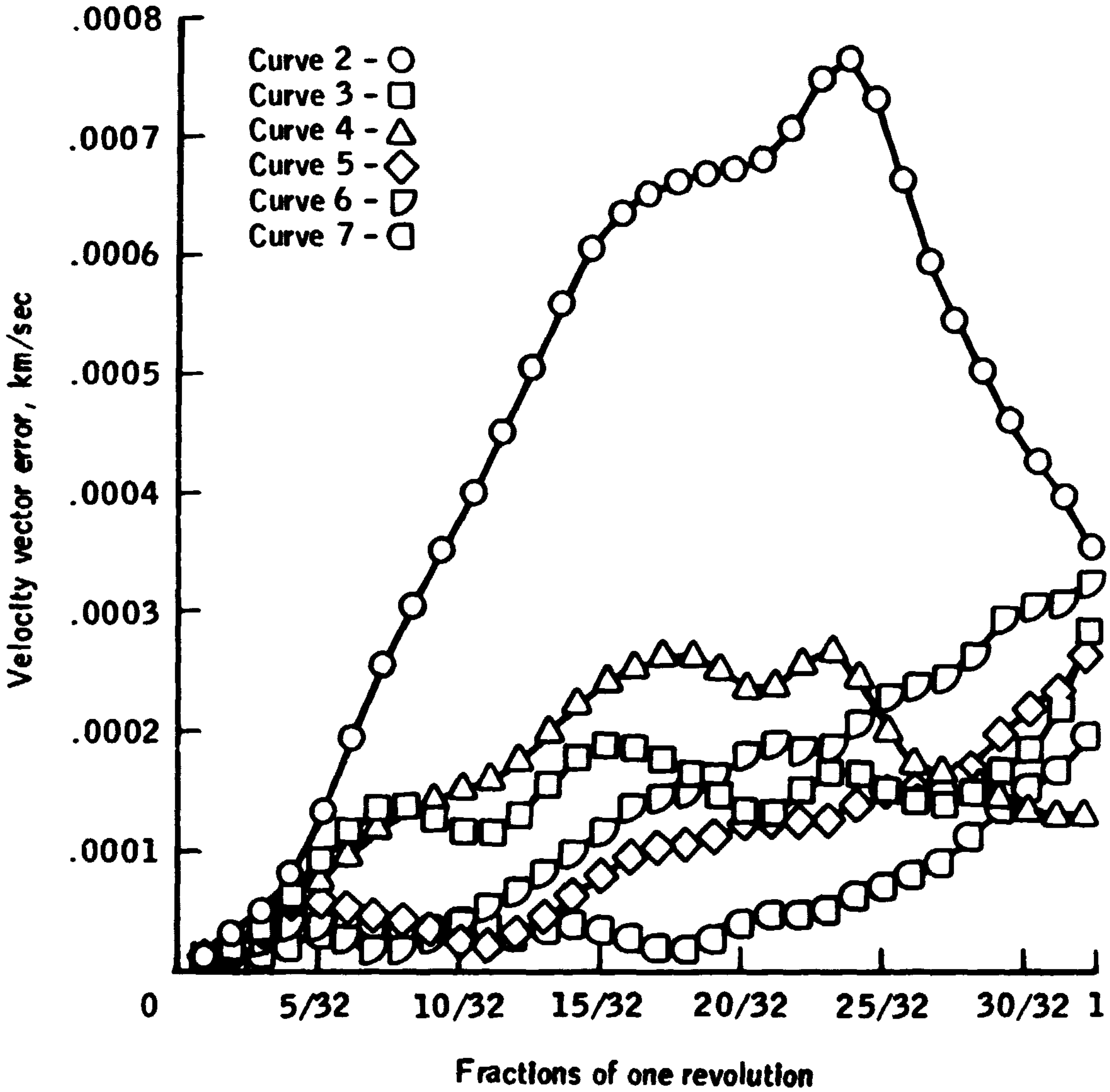
(a) Eighth-order model reference with drag; lower order models with drag.

Figure 2.- Velocity vector differences obtained over the initial orbit. The curves include the following order or degree terms: (2) second only, (3) second and third, (4) second to fourth, (5) second to fifth, (6) second to sixth, and (7) second to seventh.



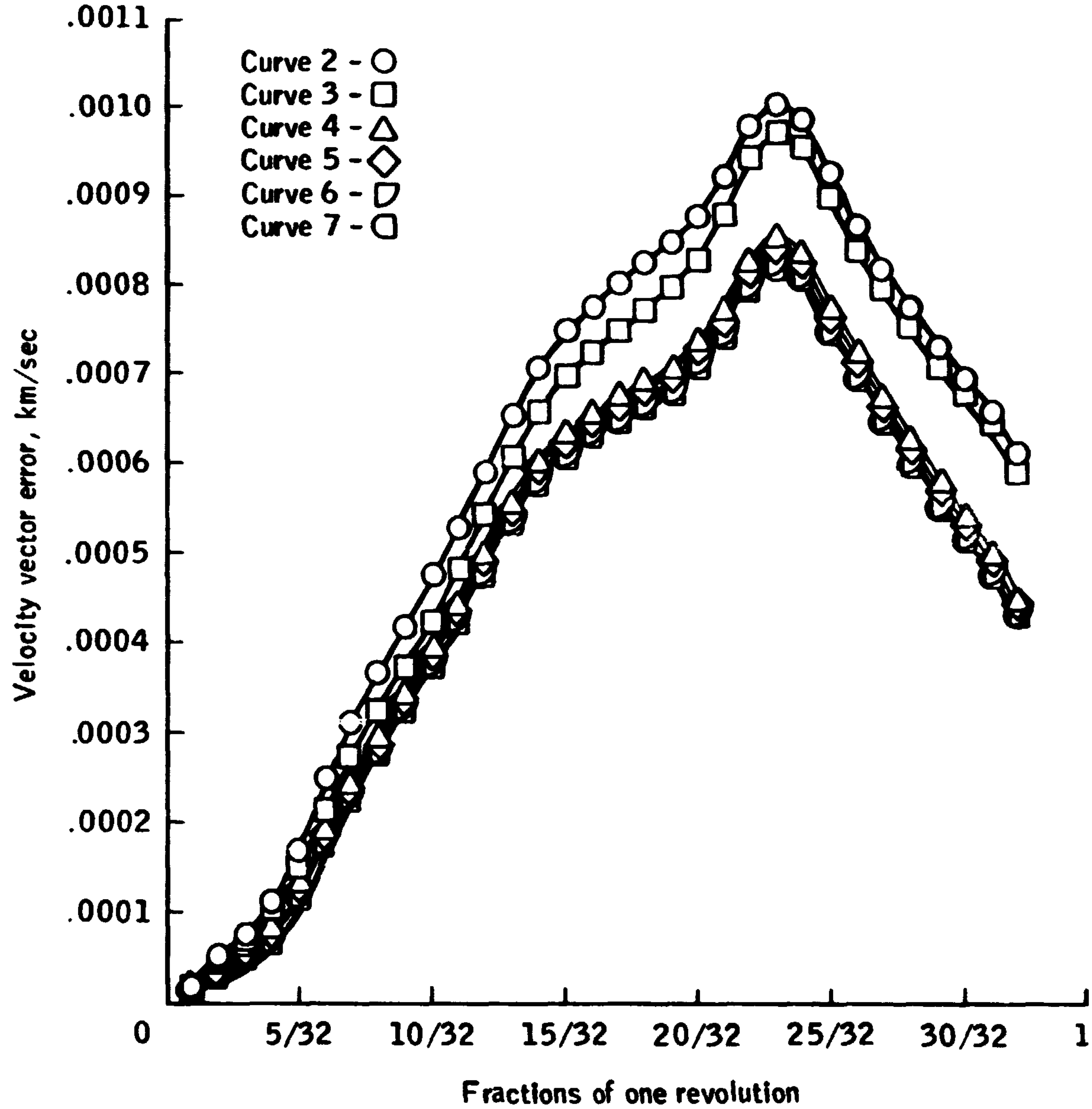
(b) Eighth-order model reference with drag; lower degree models with drag.

Figure 2.- Continued.



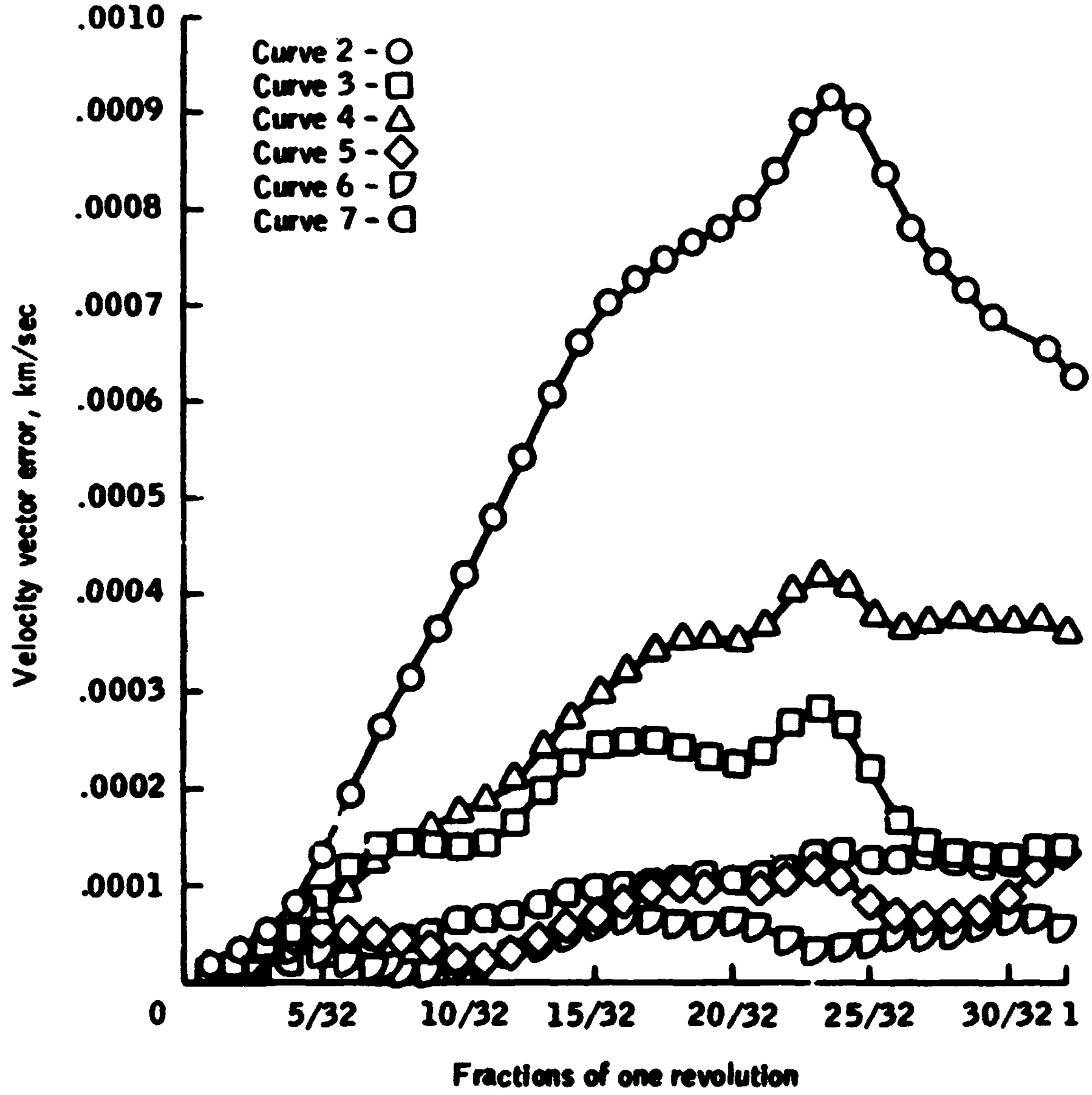
(c) Eighth-order model reference with drag; lower order models without drag.

Figure 2.- Continued.



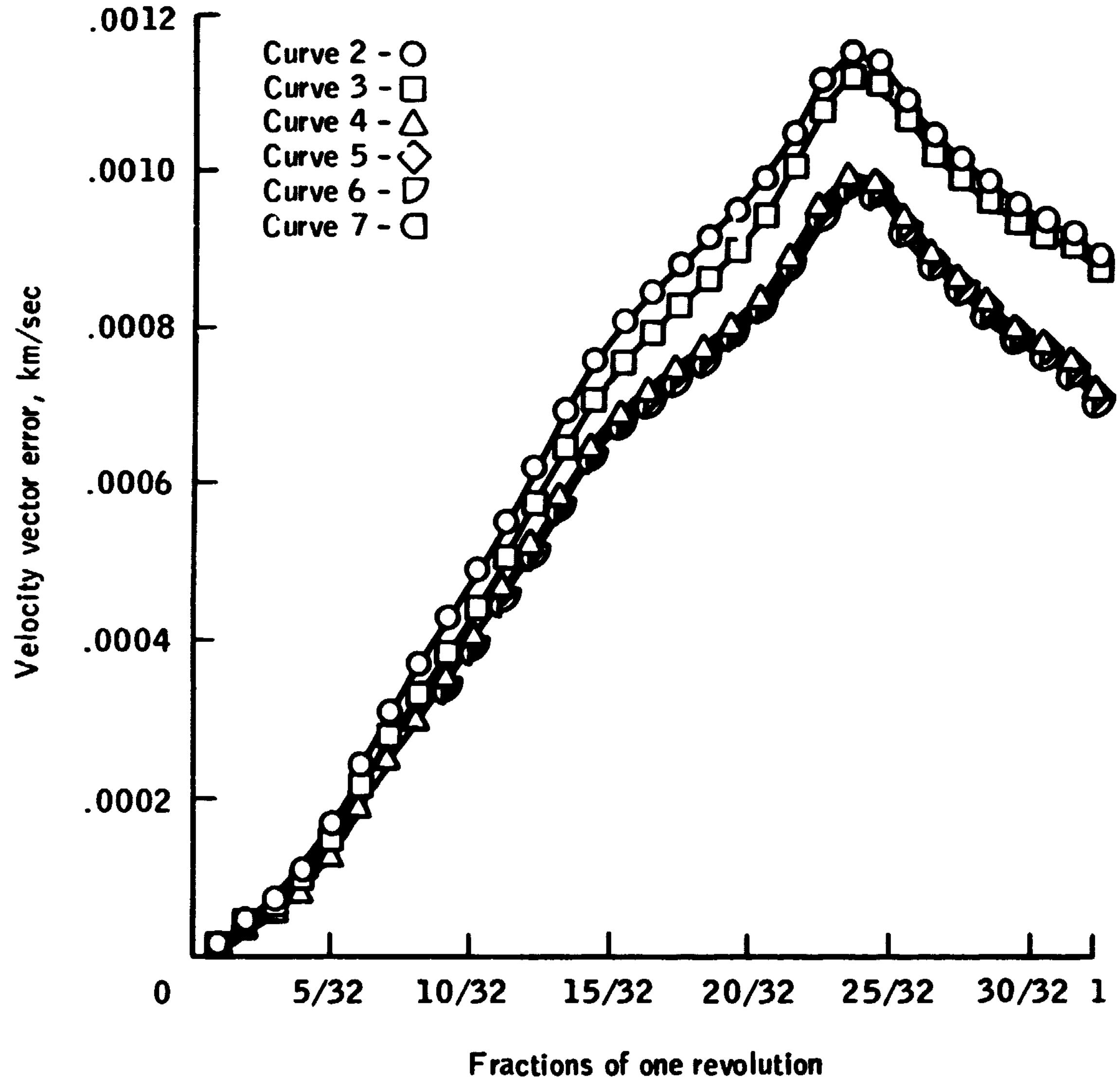
(d) Eighth-order model reference with drag; lower degree models without drag.

Figure 2.- Continued.



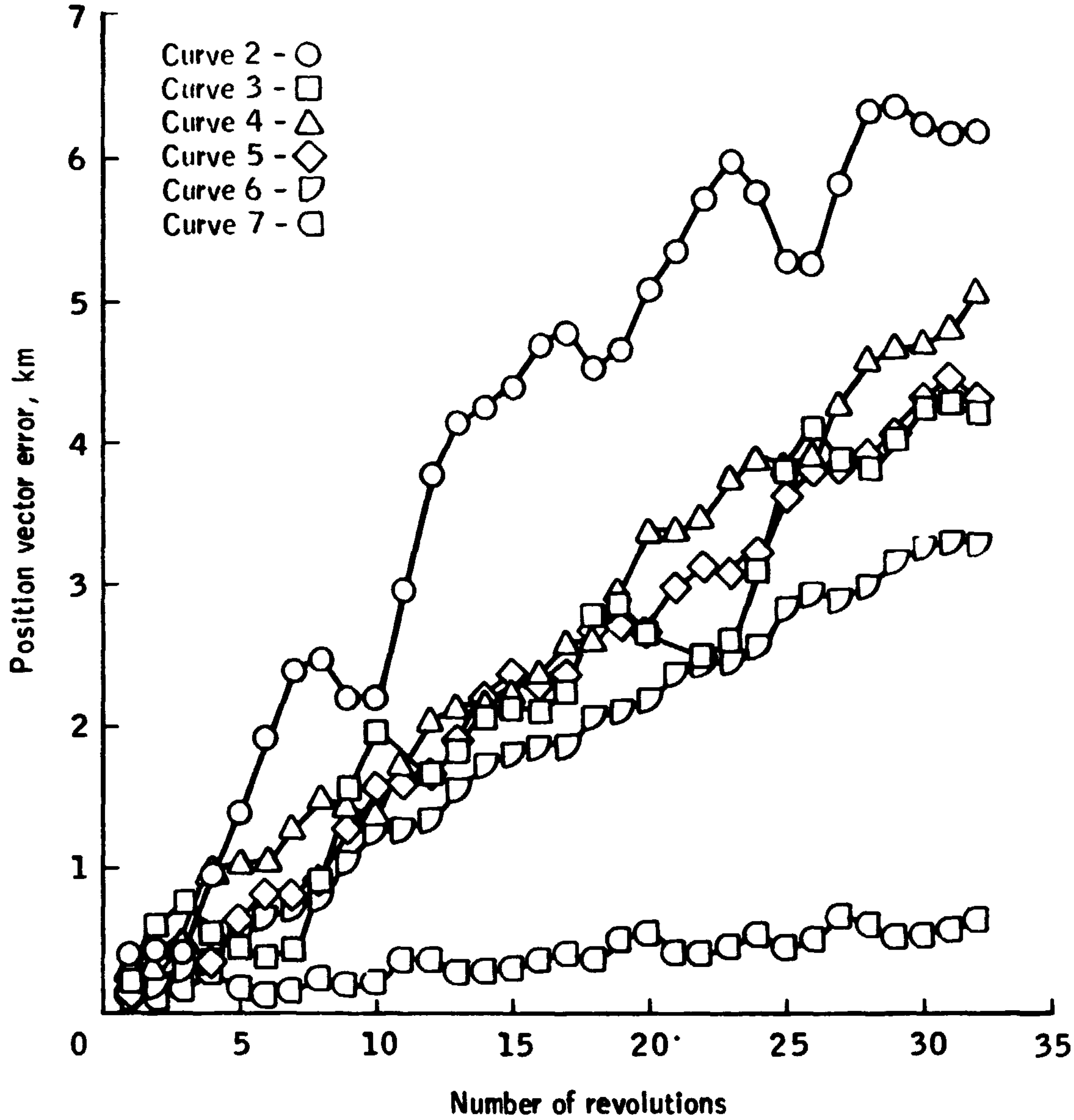
(e) Eighth-order model reference without drag; lower order models without drag.

Figure 2.- Continued.



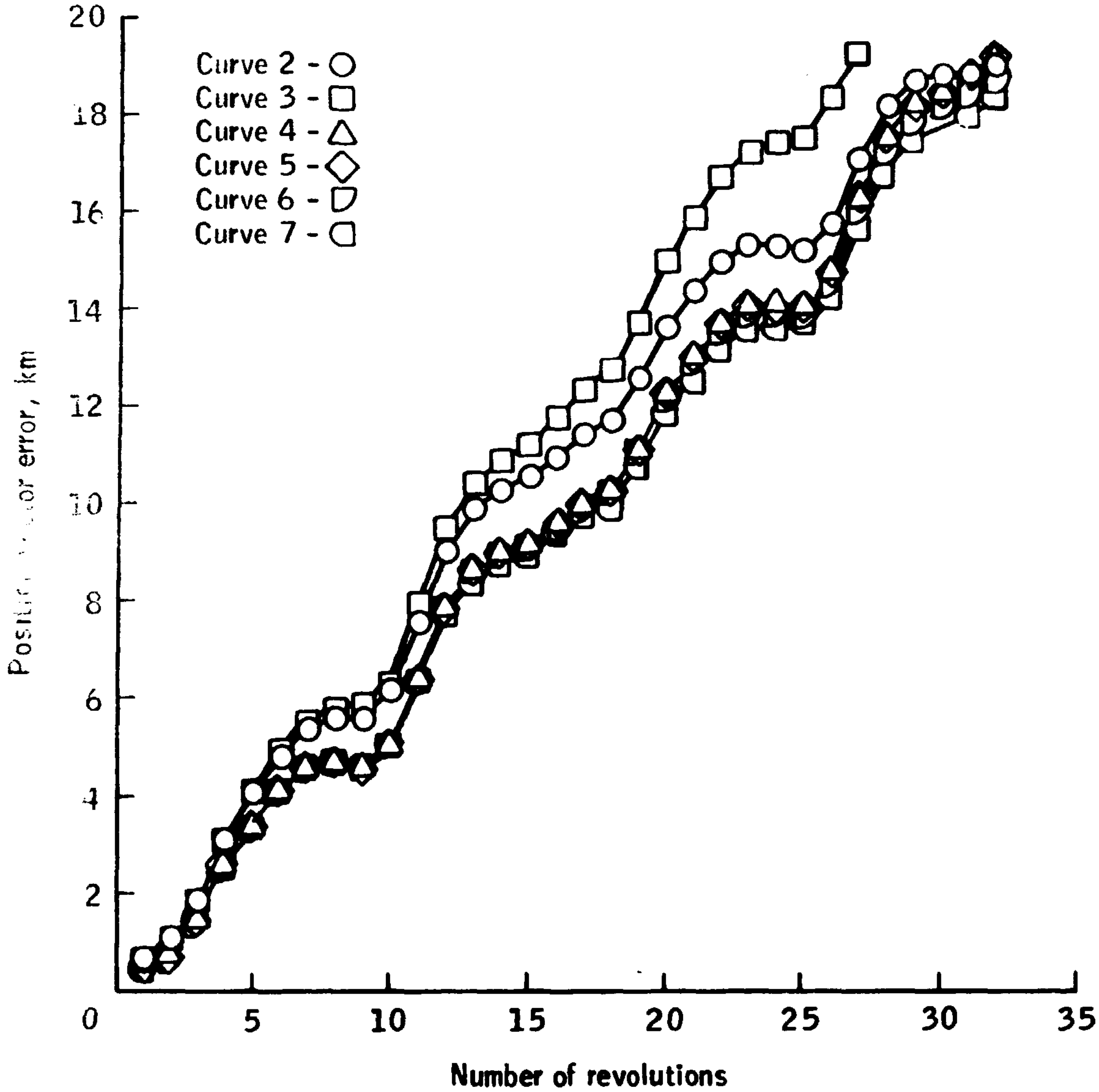
(f) Eighth-order model reference without drag; lower degree models without drag.

Figure 2.- Concluded.



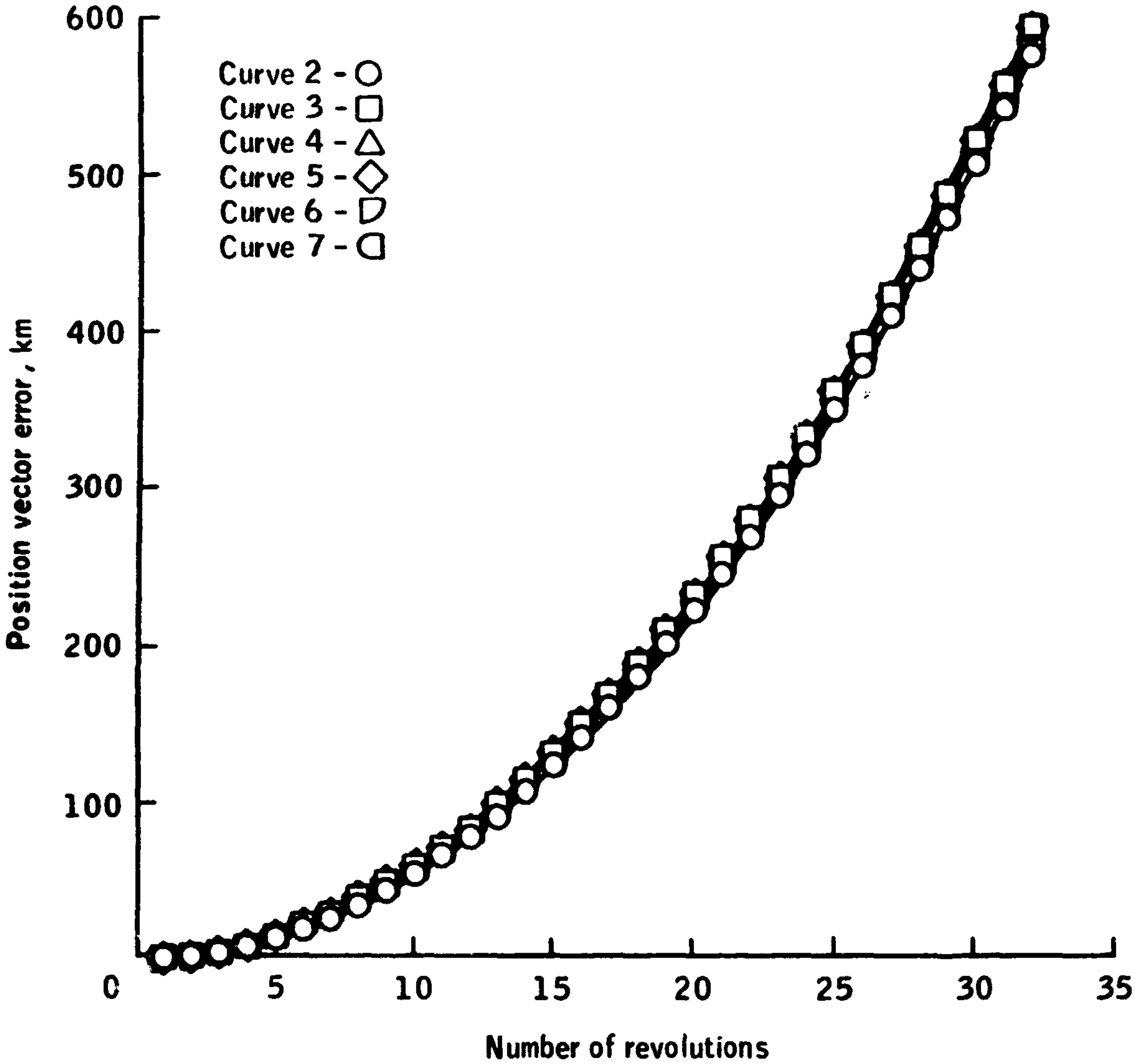
(a) Eighth-order model reference with drag; lower order models with drag.

Figure 3.- Position vector differences obtained over 32 revolutions. The curves include the following order or degree terms: (2) second only, (3) second and third, (4) second to fourth, (5) second to fifth, (6) second to sixth, and (7) second to seventh.



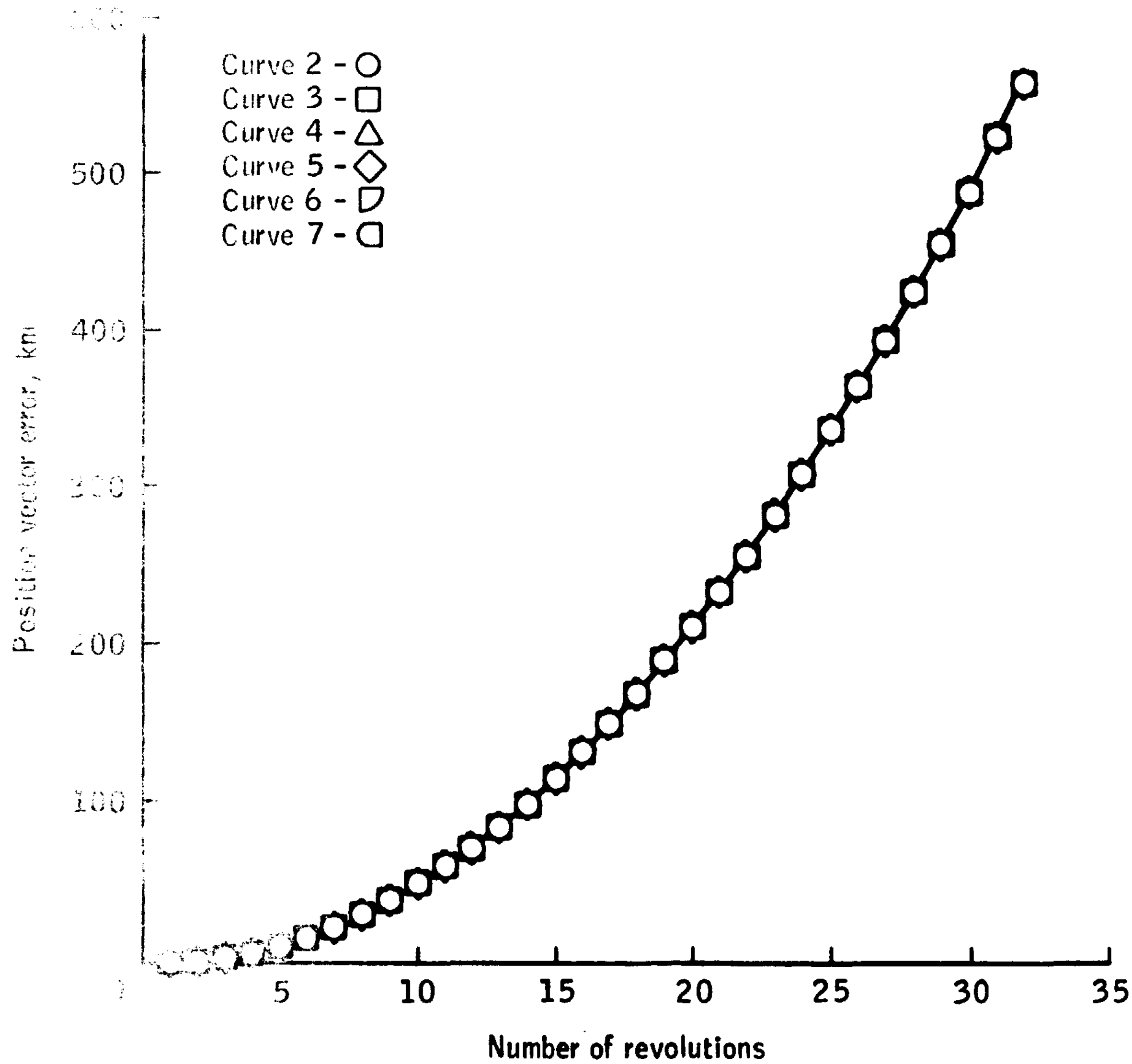
(b) Eighth-order model reference with drag; lower degree models with drag.

Figure 3.- Continued.



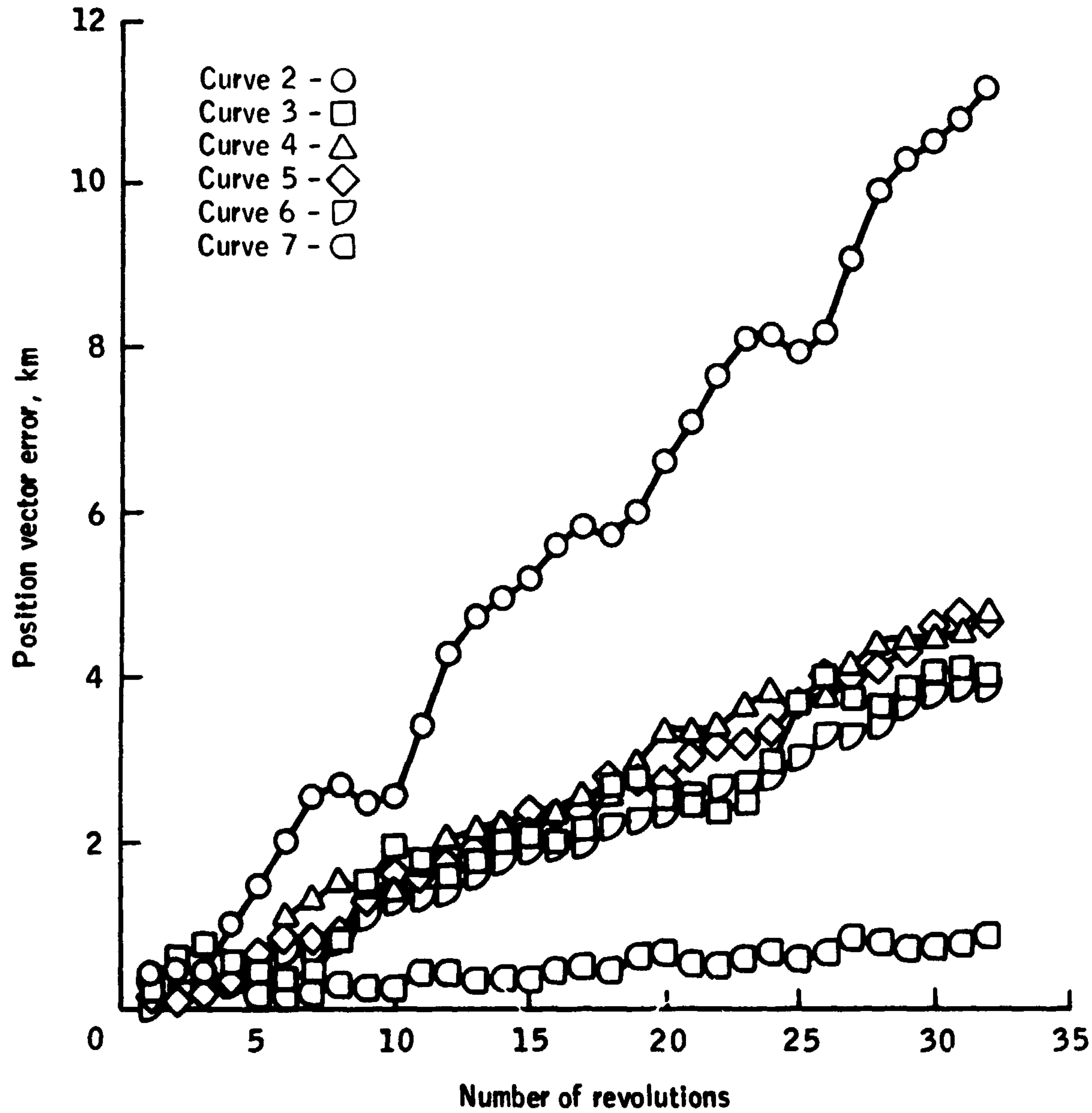
(c) Eighth-order model reference with drag; lower order models without drag.

Figure 3.- Continued.



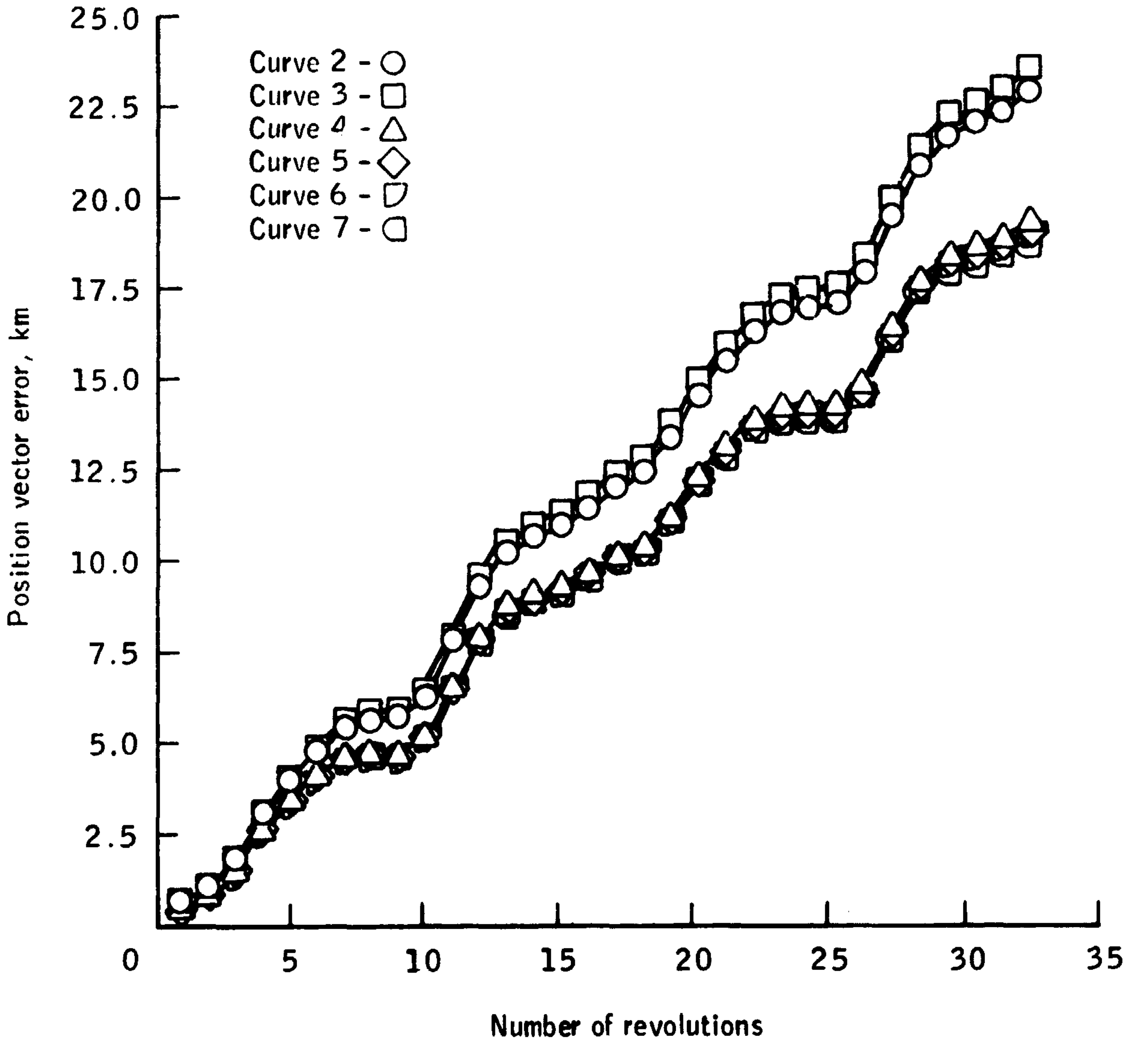
(c) Eighth-order model reference with drag; lower degree models without drag.

Figure 3.- Continued.



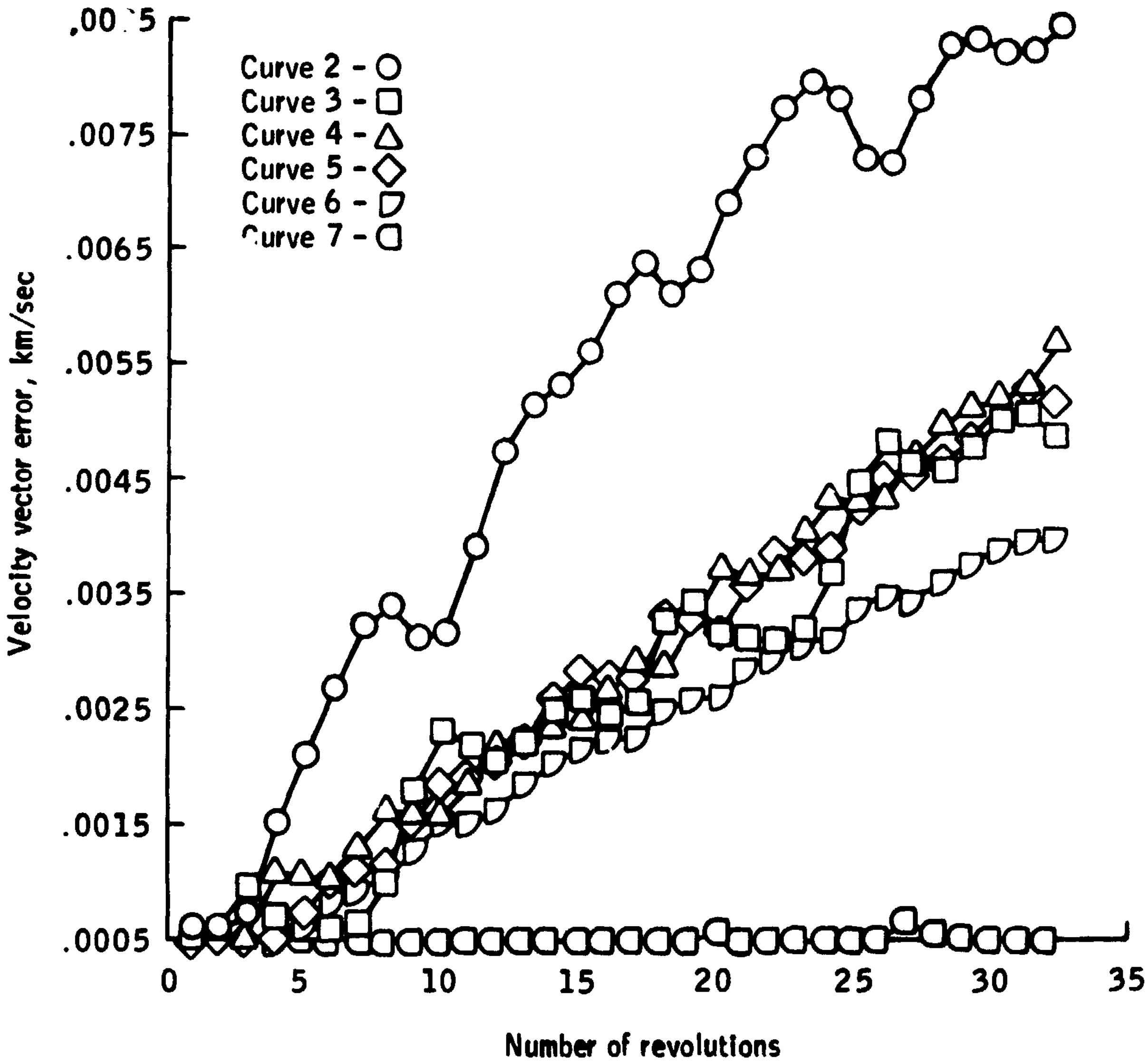
(e) Eighth-order model reference without drag; lower order models without drag.

Figure 3.- Continued.



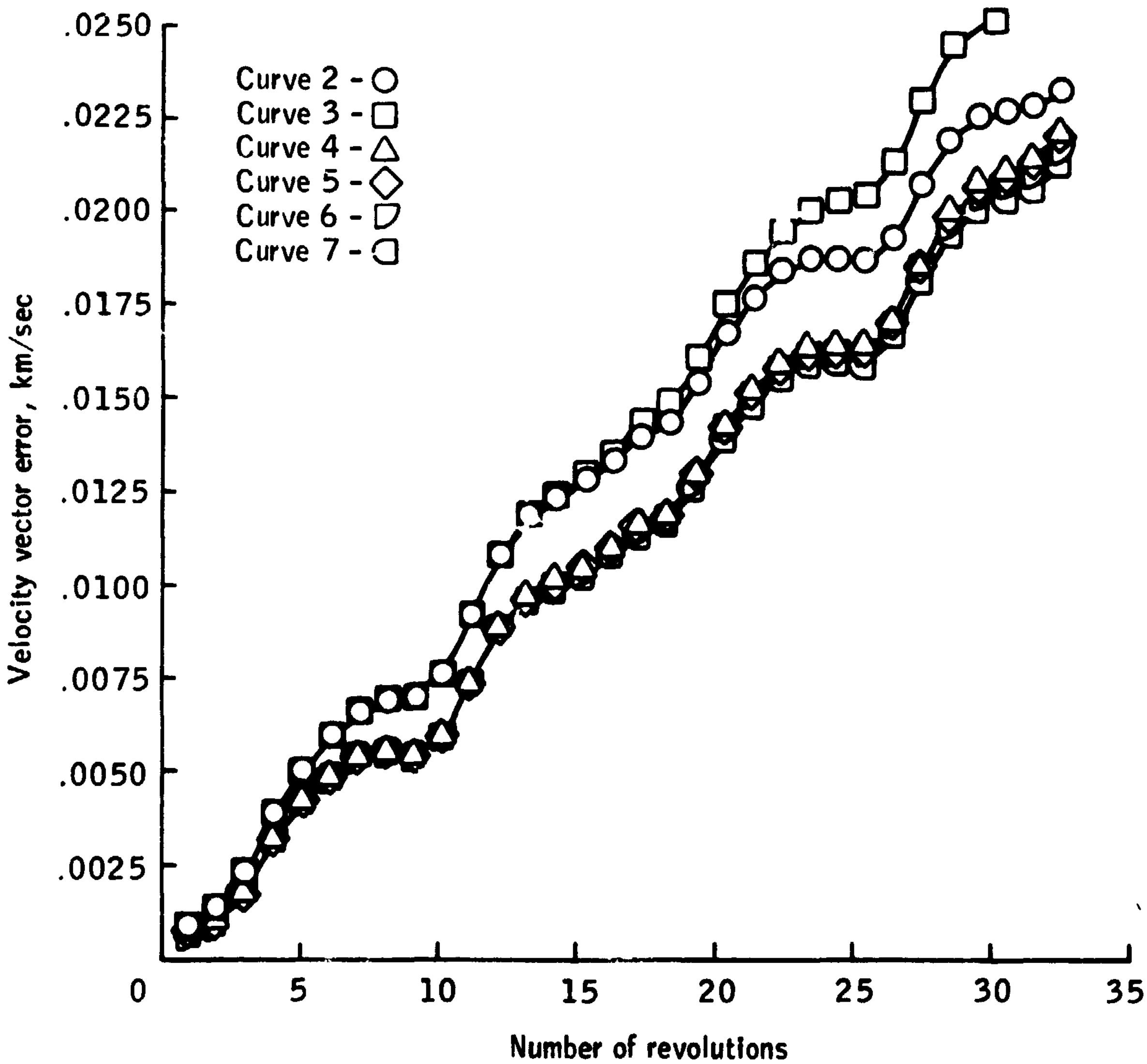
(f) Eighth-order model reference without drag; lower degree models without drag.

Figure 3.- Concluded.



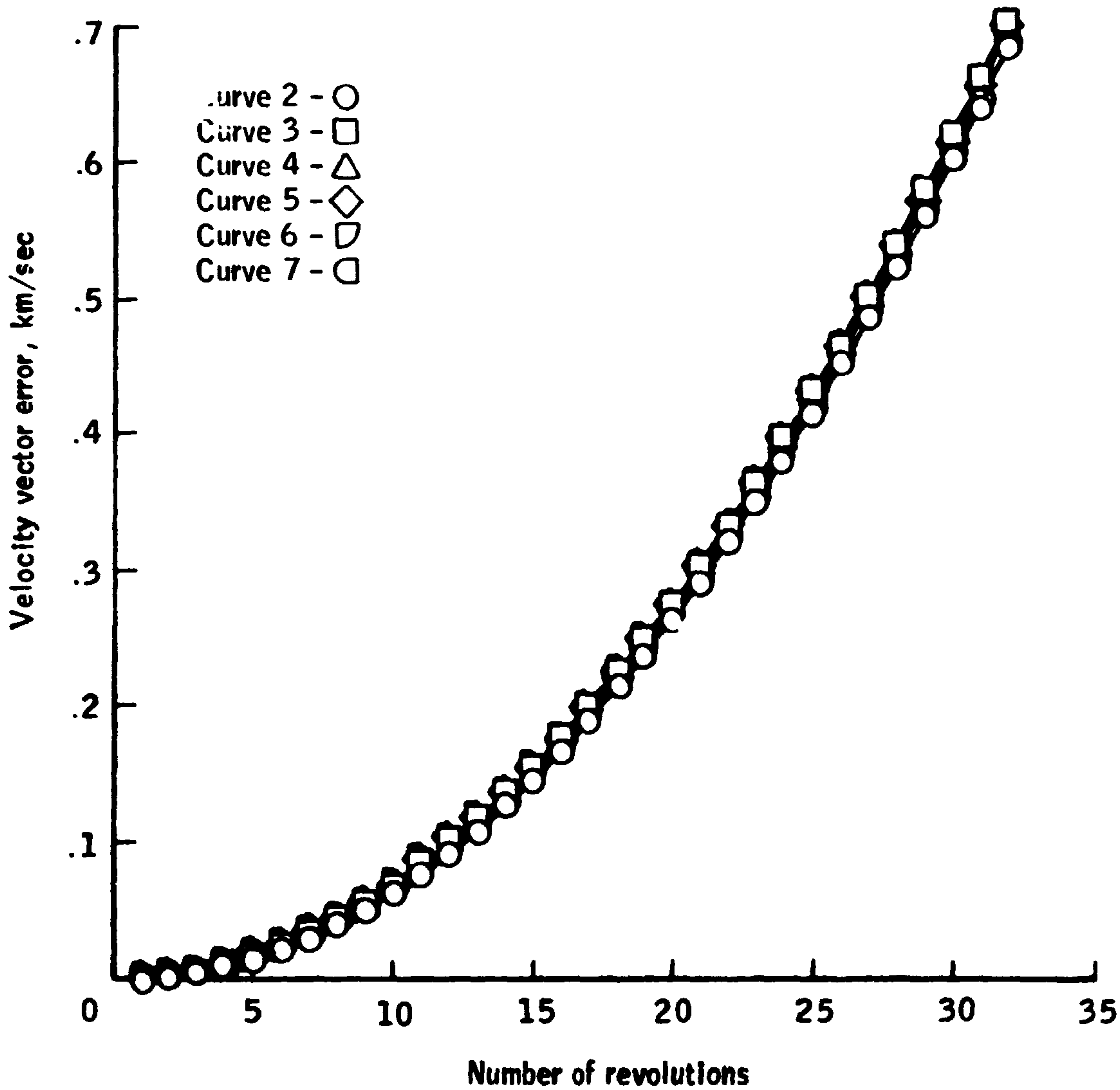
(a) Eighth-order model reference with drag; lower order models with drag.

Figure 4.- Velocity vector differences obtained over 32 revolutions. The curves include the following order or degree terms: (2) second only, (3) second and third, (4) second to fourth, (5) second to fifth, (6) second to sixth, and (7) second to seventh.



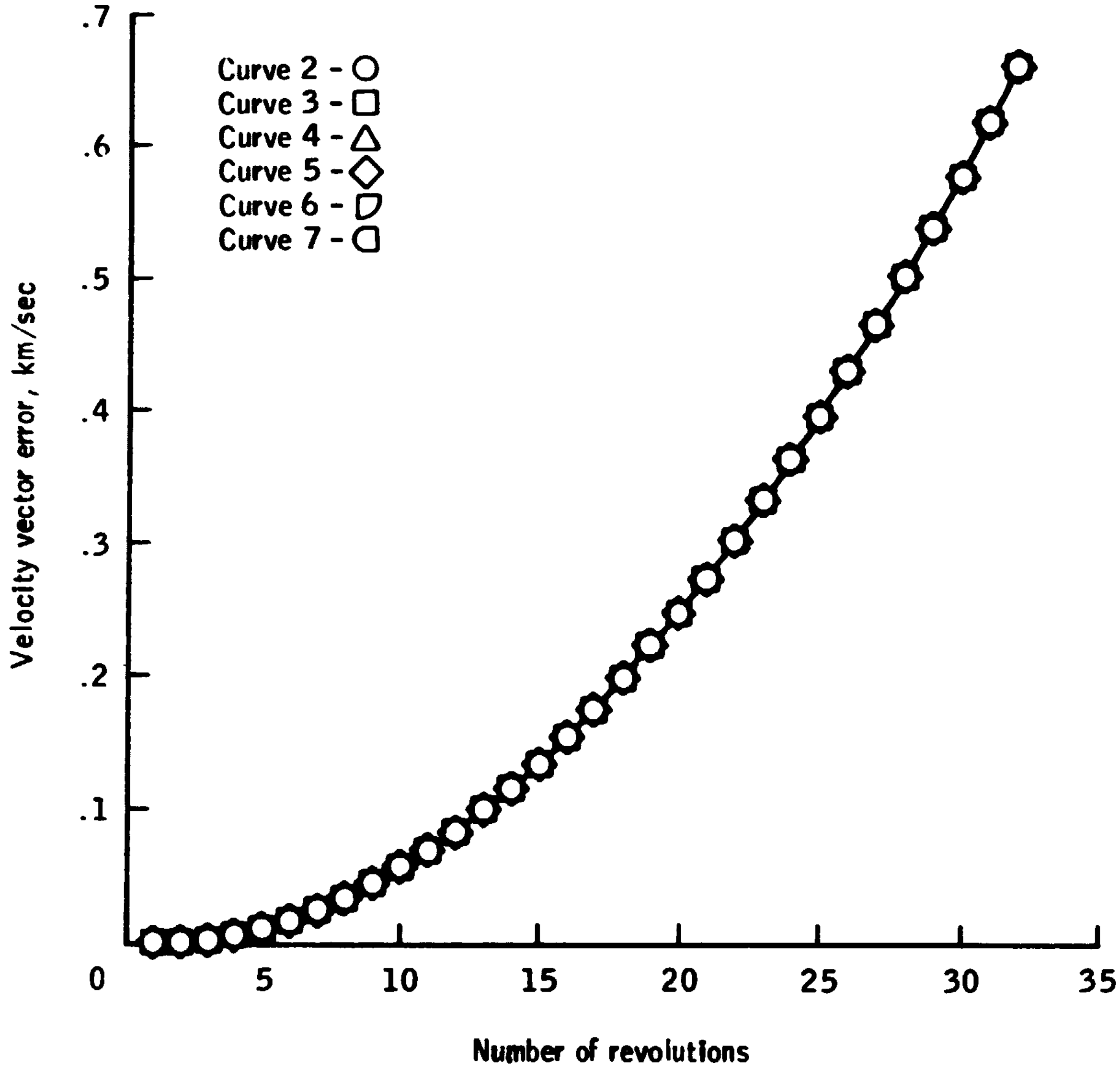
(b) Eighth-order model reference with drag; lower degree models with drag.

Figure 4.- Continued.



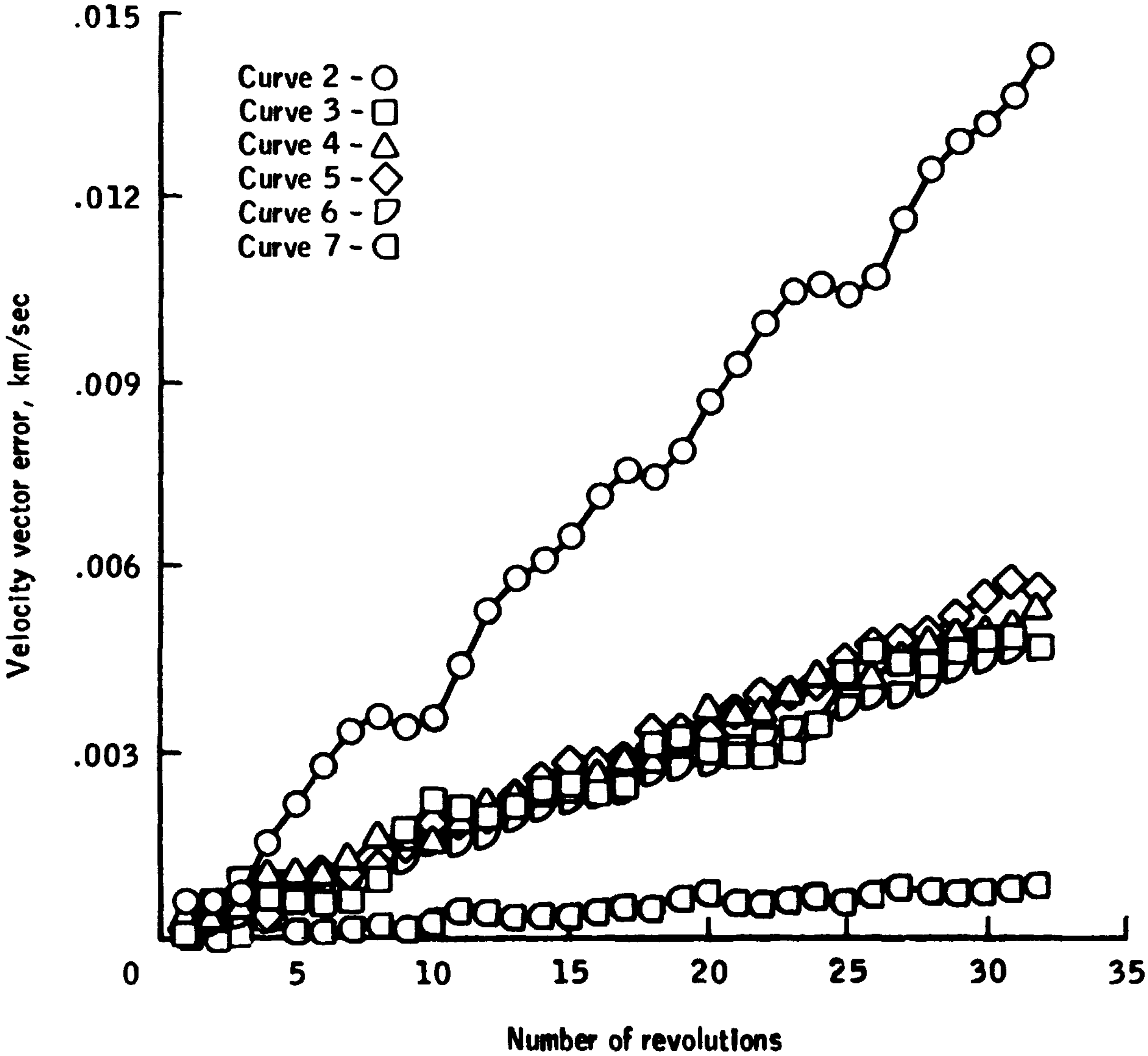
(c) Eighth-order model reference with drag; lower order models without drag.

Figure 4.- Continued.



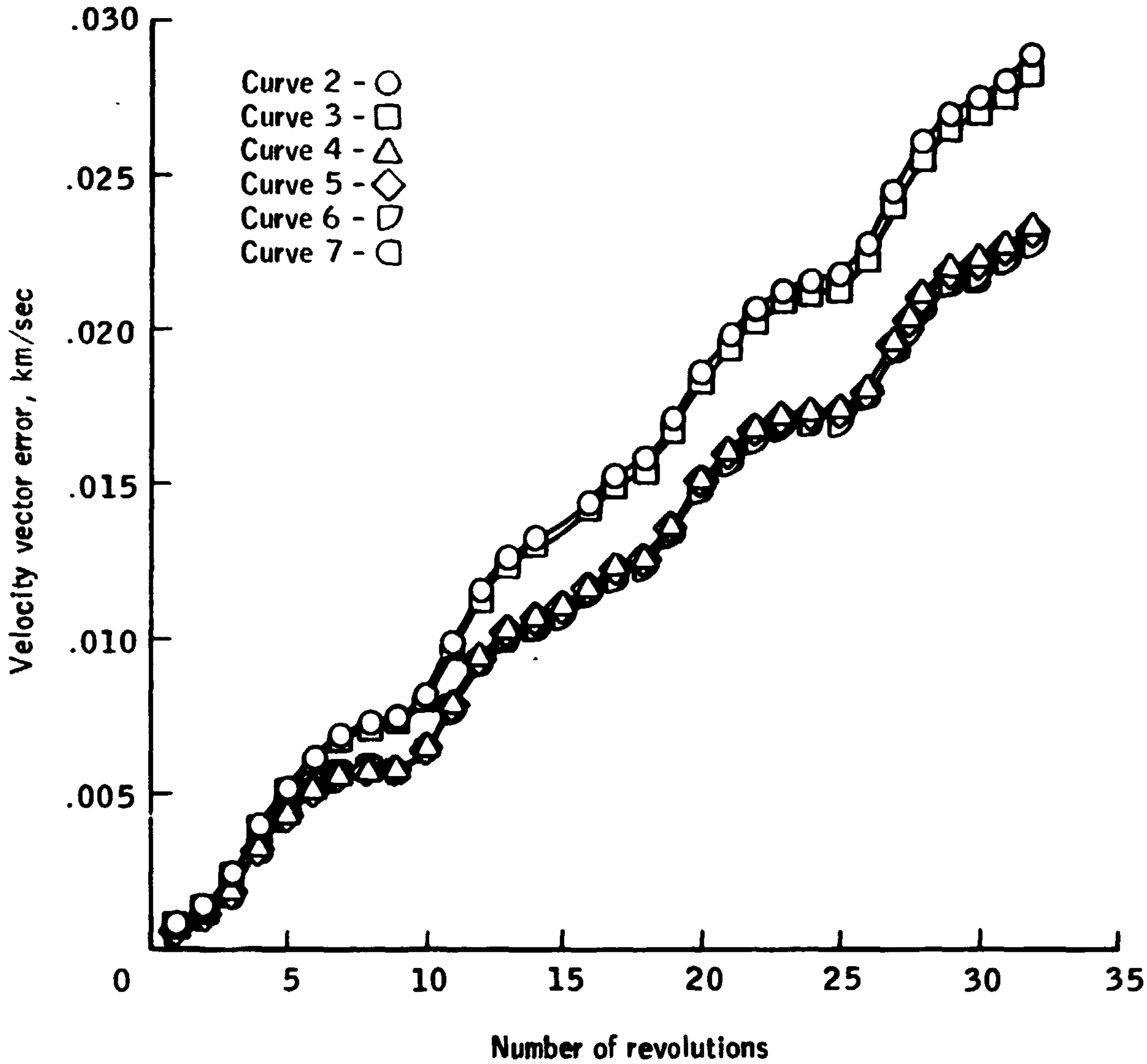
(d) Eighth-order model reference with drag; lower degree models without drag.

Figure 4.- Continued.



(e) Eighth-order model reference without drag; lower order models without drag.

Figure 4.- Continued.



(f) Eighth-order model reference without drag; lower degree models without drag.

Figure 4.- Concluded.

APPENDIX A

THE TRANSFORMATION OF LAPLACE'S EQUATION FROM CARTESIAN TO SPHERICAL COORDINATES

To accomplish the transformation of Laplace's equation from Cartesian to spherical coordinates, it is first necessary to establish certain differential relations that exist between Cartesian and spherical coordinates. These relationships are established as follows.

$$\left.\begin{aligned} x &= r \cos\varphi \cos\lambda \\ y &= r \cos\varphi \sin\lambda \\ z &= r \sin\varphi \end{aligned}\right\} \tag{A1}$$

where r is the magnitude of the position vector to the point in question, φ is the geocentric latitude, λ is the east longitude, and x, y, and z are the component magnitudes of the position vector along the Cartesian coordinate axes. Taking differentials of equation (A1) gives

$$\left.\begin{aligned} dx &= \cos\varphi \cos\lambda \, dr - r \sin\varphi \cos\lambda \, d\varphi - r \cos\varphi \sin\lambda \, d\lambda \\ dy &= \cos\varphi \sin\lambda \, dr - r \sin\varphi \sin\lambda \, d\varphi + r \cos\varphi \cos\lambda \, d\lambda \\ dz &= \sin\varphi \, dr + r \cos\varphi \, d\varphi \end{aligned}\right\} \tag{A2a}$$

Writing equation (A2a) in matrix form gives

$$\begin{vmatrix} dx \\ dy \\ dz \end{vmatrix} = \begin{vmatrix} \cos\varphi \cos\lambda & -r \sin\varphi \cos\lambda & -r \cos\varphi \sin\lambda \\ \cos\varphi \sin\lambda & -r \sin\varphi \sin\lambda & r \cos\varphi \cos\lambda \\ \sin\varphi & r \cos\varphi & 0 \end{vmatrix} \begin{vmatrix} dr \\ d\varphi \\ d\lambda \end{vmatrix} \tag{A2b}$$

or, by solving for dr, $d\varphi$, and $d\lambda$,

$$\begin{vmatrix} dr \\ d\varphi \\ d\lambda \end{vmatrix} = \begin{vmatrix} \cos\varphi\cos\lambda & -r\sin\varphi\cos\lambda & -r\cos\varphi\sin\lambda \\ \cos\varphi\sin\lambda & -r\sin\varphi\sin\lambda & r\cos\varphi\cos\lambda \\ \sin\varphi & r\cos\varphi & 0 \end{vmatrix}^{-1} \begin{vmatrix} dx \\ dy \\ dz \end{vmatrix} \qquad \text{(A3)}$$

To compute the inverse of the preceding matrix, its determinant is first computed by expanding by minors along the third row

$$\begin{aligned} |\det| &= \sin\varphi\left(-r^2\cos^2\lambda\sin\varphi\cos\varphi - r^2\sin^2\lambda\sin\varphi\cos\varphi\right) \\ &\quad - r\cos\varphi\left(r\cos^2\varphi\cos^2\lambda + r^2\cos^2\lambda\sin^2\lambda\right) \\ &= -r^2\sin^2\varphi\cos\varphi - r^2\cos^2\varphi\cos\varphi \\ &= -r^2\cos\varphi \end{aligned} \qquad \text{(A4)}$$

making use of the trigonometric identity $\sin^2\varphi + \cos^2\varphi = 1$. The same identity applies in the case of λ. The inverse may now be obtained by writing the transpose of the matrix of cofactors (i.e., the adjoint of the matrix) and dividing each term by the determinant of the matrix. The cofactor of the element a_{ij} is given by $(-1)^{i+j}|A_{ij}|$, where $|A_{ij}|$ is the determinant of the matrix formed by excluding the i-th row and j-th column of the original matrix. If b_{ij} represents the elements of the inverse matrix, then

$$
\left.\begin{aligned}
b_{11} &= \left(-r^2\cos^2\varphi \cos\lambda\right)\Big/\left(-r^2\cos\varphi\right) \\
&= \cos\varphi \cos\lambda \\
b_{12} &= -\left(r^2\cos^2\varphi \sin\lambda\right)\Big/\left(-r^2\cos\varphi\right) \\
&= \cos\varphi \sin\lambda \\
b_{13} &= \left(-r^2\cos^2\lambda \sin\varphi \cos\lambda - r^2\sin^2\lambda \sin\varphi \cos\lambda\right)\Big/\left(-r^2\cos\varphi\right) \\
&= \sin\varphi \\
b_{21} &= -(-r \sin\varphi \cos\varphi \cos\lambda)\Big/\left(-r^2\cos\varphi\right) \\
&= -(1/r)\sin\varphi \cos\lambda \\
b_{22} &= (r \sin\varphi \cos\varphi \sin\lambda)\Big/\left(-r^2\cos\varphi\right) \\
&= -(1/r)\sin\varphi \sin\lambda \\
b_{23} &= -\left(r \cos^2\varphi \cos^2\lambda + r \cos^2\varphi \sin^2\lambda\right)\Big/\left(-r^2\cos\varphi\right) \\
&= (1/r)\cos\varphi \\
b_{31} &= \left(r \cos^2\varphi \sin\lambda + r \sin^2\varphi \sin\lambda\right)\Big/\left(-r^2\cos\varphi\right) \\
&= -[(\sin\lambda)/(r\cos\varphi)] \\
b_{32} &= -\left(r \cos^2\varphi \cos\lambda + r \sin^2\varphi \cos\lambda\right)\Big/\left(-r^2\cos\varphi\right) \\
&= (\cos\lambda)/(r\cos\varphi) \\
b_{33} &= (-r \sin\varphi \cos\varphi \sin\lambda \cos\lambda + r \sin\varphi \cos\varphi \sin\lambda \cos\lambda) \\
&= 0
\end{aligned}\right\} \quad \text{(A5a)}
$$

Therefore, with the elements of the inverse matrix thus established, it is possible to write equation (A3) as follows.

$$\begin{vmatrix} dr \\ d\varphi \\ d\lambda \end{vmatrix} = \begin{vmatrix} \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \\ -\frac{1}{r}\sin\varphi\cos\lambda & -\frac{1}{r}\sin\varphi\sin\lambda & \frac{1}{r}\cos\varphi \\ -\frac{\sin\lambda}{r\cos\varphi} & \frac{\cos\lambda}{r\cos\varphi} & 0 \end{vmatrix} \begin{vmatrix} dx \\ dy \\ dz \end{vmatrix} \qquad \text{(A5b)}$$

The required derivative relationships can now be obtained from equation (A5b). These derivatives will be presented with use of the following shorthand notations.

$$r_x = \frac{\partial r}{\partial x};\ r_{xr} = \frac{\partial^2 r}{\partial x\,\partial r} \qquad \text{(A6)}$$

$$\left.\begin{array}{lll}
r_x = \cos\varphi\cos\lambda & r_y = \cos\varphi\sin\lambda & r_z = \sin\varphi \\[4pt]
\varphi_x = -\frac{1}{r}\sin\varphi\cos\lambda & \varphi_y = -\frac{1}{r}\sin\varphi\sin\lambda & \varphi_z = \frac{1}{r}\cos\varphi \\[4pt]
\lambda_x = -\frac{\sin\lambda}{r\cos\varphi} & \lambda_y = \frac{\cos\lambda}{r\cos\varphi} & \lambda_z = 0 \\[4pt]
r_{xr} = 0 & r_{x\varphi} = -\sin\varphi\cos\lambda & r_{x\lambda} = -\cos\varphi\sin\lambda \\[4pt]
r_{yr} = 0 & r_{y\varphi} = -\sin\varphi\sin\lambda & r_{y\lambda} = \cos\varphi\cos\lambda \\[4pt]
r_{zr} = 0 & r_{z\varphi} = \cos\varphi & r_{z\lambda} = 0 \\[4pt]
\varphi_{xr}\ \frac{1}{r^2}\sin\varphi\cos\lambda & \varphi_{x\varphi} = -\frac{1}{r}\cos\varphi\cos\lambda & \varphi_{x\lambda} = \frac{1}{r}\sin\varphi\sin\lambda \\[4pt]
\varphi_{yr} = \frac{1}{r^2}\sin\varphi\sin\lambda & \varphi_{y\varphi} = -\frac{1}{r}\cos\varphi\sin\lambda & \varphi_{y\lambda} = -\frac{1}{r}\sin\varphi\cos\lambda \\[4pt]
\varphi_{zr} = -\frac{1}{r^2}\cos\varphi & \varphi_{z\varphi} = -\frac{1}{r}\sin\varphi & \varphi_{z\lambda} = 0 \\[4pt]
\lambda_{xr} = \frac{\sin\lambda}{r^2\cos\varphi} & \lambda_{x\varphi} = -\frac{\sin\varphi\sin\lambda}{r\cos^2\varphi} & \lambda_{x\lambda} = -\frac{\cos\lambda}{r\cos\varphi} \\[4pt]
\lambda_{yr} = -\frac{\cos\lambda}{r^2\cos\varphi} & \lambda_{y\varphi} = \frac{\sin\varphi\cos\lambda}{r\cos^2\varphi} & \lambda_{y\lambda} = -\frac{\sin\lambda}{r\cos\varphi} \\[4pt]
\lambda_{zr} = 0 & \lambda_{z\varphi} = 0 & \lambda_{z\lambda} = 0
\end{array}\right\} \qquad \text{(A7)}$$

Using the relationships given in equation (A7), Laplace's equation can be transformed from Cartesian to spherical coordinates.

From equation (A1), it is seen that x, y, and z are functions of r, φ, and λ. Therefore, the potential function $V = V(x,y,z)$ is expressed in terms of r, φ, and λ.

$$\left.\begin{aligned} V_x &= V_r r_x + V_\varphi \varphi_x + V_\lambda \lambda_x \\ V_y &= V_r r_y + V_\varphi \varphi_y + V_\lambda \lambda_y \\ V_z &= V_r r_z + V_\varphi \varphi_z + V_\lambda \lambda_z \end{aligned}\right\} \tag{A8a}$$

$$\left.\begin{aligned} V_{xx} &= V_{xr} r_x + V_{x\varphi} \varphi_x + V_{x\lambda} \lambda_x \\ V_{yy} &= V_{yr} r_y + V_{y\varphi} \varphi_y + V_{y\lambda} \lambda_y \\ V_{zz} &= V_{zr} r_z + V_{z\varphi} \varphi_z + V_{z\lambda} \lambda_z \end{aligned}\right\} \tag{A8b}$$

Expanding equation (A8b) gives

$$\begin{aligned} V_{xx} = {} & \left(V_{rr} r_x + V_r r_{xr} + V_{\varphi r} \varphi_x + V_\varphi \varphi_{xr} + V_{\lambda r} \lambda_x + V_\lambda \lambda_{xr}\right) r_x \\ & + \left(V_{r\varphi} r_x + V_r r_{x\varphi} + V_{\varphi\varphi} \varphi_x + V_\varphi \varphi_{x\varphi} + V_{\lambda\varphi} \lambda_x + V_\lambda \lambda_{x\varphi}\right) \varphi_x \\ & + \left(V_{r\lambda} r_x + V_r r_{x\lambda} + V_{\varphi\lambda} \varphi_x + V_\varphi \varphi_{x\lambda} + V_{\lambda\lambda} \lambda_x + V_\lambda \lambda_{x\lambda}\right) \lambda_x \end{aligned} \tag{A9}$$

$$\begin{aligned} V_{yy} = {} & \left(V_{rr} r_y + V_r r_{yr} + V_{\varphi r} \varphi_y + V_\varphi \varphi_{yr} + V_{\lambda r} \lambda_y + V_\lambda \lambda_{yr}\right) r_y \\ & + \left(V_{r\varphi} r_y + V_r r_{y\varphi} + V_{\varphi\varphi} \varphi_y + V_\varphi \varphi_{y\varphi} + V_{\lambda\varphi} \lambda_y + V_\lambda \lambda_{y\varphi}\right) \varphi_y \\ & + \left(V_{r\lambda} r_y + V_r r_{y\lambda} + V_{\varphi\lambda} \varphi_y + V_\varphi \varphi_{y\lambda} + V_{\lambda\lambda} \lambda_y + V_\lambda \lambda_{y\lambda}\right) \lambda_y \end{aligned} \tag{A10}$$

$$\begin{aligned} V_{zz} = {} & \left(V_{rr} r_z + V_r r_{zr} + V_{\varphi r} \varphi_z + V_\varphi \varphi_{zr} + V_{\lambda r} \lambda_z + V_\lambda \lambda_{zr}\right) r_z \\ & + \left(V_{r\varphi} r_z + V_r r_{z\varphi} + V_{\varphi\varphi} \varphi_z + V_\varphi \varphi_{z\varphi} + V_{\lambda\varphi} \lambda_z + V_\lambda \lambda_{z\varphi}\right) \varphi_z \\ & + \left(V_{r\lambda} r_z + V_r r_{z\lambda} + V_{\varphi\lambda} \varphi_z + V_\varphi \varphi_{z\lambda} + V_{\lambda\lambda} \lambda_z + V_\lambda \lambda_{z\lambda}\right) \lambda_z \end{aligned} \tag{A11}$$

Collecting terms from equations (A9) to (A11) gives

$$\nabla^2 V = V_{rr}\left(r_x{}^2 + r_y{}^2 + r_z{}^2\right) + V_{\varphi\varphi}\left(\varphi_x{}^2 + \varphi_y{}^2 + \varphi_z{}^2\right) + V_{\lambda\lambda}\left(\lambda_x{}^2 + \lambda_y{}^2 + \lambda_z{}^2\right)$$
$$+ 2V_{r\varphi}\left(r_x\varphi_x + r_y\varphi_y + r_z\varphi_z\right) + 2V_{r\lambda}\left(r_x\lambda_x + r_y\lambda_y + r_z\lambda_z\right)$$
$$+ 2V_{\varphi\lambda}\left(\varphi_x\lambda_x + \varphi_y\lambda_y + \varphi_z\lambda_z\right) + V_r\left(r_{xr}r_x + r_{x\varphi}\varphi_x + r_{x\lambda}\lambda_x\right.$$
$$\left. + r_{yr}r_y + r_{y\varphi}\varphi_y + r_{y\lambda}\lambda_y + r_{zr}r_z + r_{z\varphi}\varphi_z + r_{z\lambda}\lambda_z\right)$$
$$+ V_\varphi\left(\varphi_{xr}r_x + \varphi_{x\varphi}\varphi_x + \varphi_{x\lambda}\lambda_x + \varphi_{yr}r_y + \varphi_{y\varphi}\varphi_y + \varphi_{y\lambda}\lambda_y\right.$$
$$\left. + \varphi_{zr}r_z + \varphi_{z\varphi}\varphi_z + \varphi_{z\lambda}\lambda_z\right) + V_\lambda\left(\lambda_{xr}r_x + \lambda_{x\varphi}\varphi_x + \lambda_{x\lambda}\lambda_x\right.$$
$$\left. + \lambda_{yr}r_y + \lambda_{y\varphi}\varphi_y + \lambda_{y\lambda}\lambda_y + \lambda_{zr}r_z + \lambda_{z\varphi}\varphi_z + \lambda_{z\lambda}\lambda_z\right) \tag{A12}$$

where ∇ is the gradient vector operator. Substituting the derivatives obtained from equation (A7) in equation (A12) gives

$$r_x{}^2 + r_y{}^2 + r_z{}^2 = (\cos\varphi\cos\lambda)^2 + (\cos\varphi\sin\lambda)^2 + (\sin\varphi)^2$$
$$= 1 \tag{A13}$$

$$\varphi_x{}^2 + \varphi_y{}^2 + \varphi_z{}^2 = \left(-\frac{1}{r}\sin\varphi\cos\lambda\right)^2 + \left(-\frac{1}{r}\sin\varphi\sin\lambda\right)^2 + \left(\frac{1}{r}\cos\varphi\right)^2$$
$$= \frac{1}{r^2} \tag{A14}$$

$$\lambda_x{}^2 + \lambda_y{}^2 + \lambda_z{}^2 = \left(-\frac{\sin\lambda}{r\cos\varphi}\right)^2 + \left(\frac{\cos\lambda}{r\cos\varphi}\right)^2 + (0)^2$$
$$= \frac{1}{r^2\cos^2\varphi} \tag{A15}$$

$$r_x\varphi_x + r_y\varphi_y + r_z\varphi_z = (\cos\varphi\cos\lambda)\left(-\frac{1}{r}\sin\varphi\cos\lambda\right) + (\cos\varphi\sin\lambda)\left(-\frac{1}{r}\sin\varphi\sin\lambda\right)$$
$$+ \sin\varphi\left(\frac{1}{r}\cos\varphi\right)$$
$$= 0 \tag{A16}$$

$$r_x\lambda_x + r_y\lambda_y + r_z\lambda_z = (\cos\varphi\cos\lambda)\left(-\frac{\sin\lambda}{r\cos\varphi}\right) + (\cos\varphi\sin\lambda)\left(\frac{\cos\lambda}{r\cos\varphi}\right)$$

$$+ \sin\varphi\,(0)$$

$$= 0 \tag{A17}$$

$$\varphi_x\lambda_x + \varphi_y\lambda_y + \varphi_z\lambda_z = \left(-\frac{1}{r}\sin\varphi\cos\lambda\right)\left(-\frac{\sin\lambda}{r\cos\varphi}\right)$$

$$+ \left(-\frac{1}{r}\sin\varphi\sin\lambda\right)\left(\frac{\cos\lambda}{r\cos\varphi}\right) + \frac{\cos\varphi}{r}\cdot 0$$

$$= 0 \tag{A18}$$

$$r_{xr}r_x + r_{x\varphi}\varphi_x + r_{x\lambda}\lambda_x = (0)(\cos\varphi\cos\lambda) + (-\sin\varphi\cos\lambda)\left(-\frac{1}{r}\sin\varphi\cos\lambda\right)$$

$$+ (-\cos\varphi\sin\lambda)\left(-\frac{\sin\lambda}{r\cos\varphi}\right)$$

$$= \frac{1}{r}\left(\sin^2\varphi\cos^2\lambda + \sin^2\lambda\right) \tag{A19}$$

$$r_{yr}r_y + r_{y\varphi}\varphi_y + r_{y\lambda}\lambda_y = (0)(\cos\varphi\sin\lambda) + (-\sin\varphi\sin\lambda)\left(-\frac{1}{r}\sin\varphi\sin\lambda\right)$$

$$+ (\cos\varphi\cos\lambda)\left(\frac{\cos\lambda}{r\cos\varphi}\right)$$

$$= \frac{1}{r}\left(\sin^2\varphi\sin^2\lambda + \cos^2\lambda\right) \tag{A20}$$

$$r_{zr}r_z + r_{z\varphi}\varphi_z + r_{z\lambda}\lambda_z = (0)(\sin\varphi) + (\cos\varphi)\left(\frac{1}{r}\cos\varphi\right) + (0)(0)$$

$$= \frac{1}{r}\cos^2\varphi \tag{A21}$$

$$r_{xr}r_x + r_{x\varphi}\varphi_x + r_{x\lambda}\lambda_x + r_{yr}r_y + r_{y\varphi}\varphi_y + r_{y\lambda}\lambda_y + r_{zr}r_z + r_{z\varphi}\varphi_z + r_{z\lambda}\lambda_z = \frac{2}{r} \tag{A22}$$

$$\varphi_{xr}r_x + \varphi_{x\varphi}\varphi_x + \varphi_{x\lambda}\lambda_x = \left(\frac{1}{r^2}\sin\varphi\cos\lambda\right)(\cos\varphi\cos\lambda)$$

$$+ \left(-\frac{1}{r}\cos\varphi\cos\lambda\right)\left(-\frac{1}{r}\sin\varphi\cos\lambda\right)$$

$$+ \left(\frac{1}{r}\sin\varphi\sin\lambda\right)\left(-\frac{\sin\lambda}{r\cos\varphi}\right) \qquad \text{(A23)}$$

$$\varphi_{yr}r_y + \varphi_{y\varphi}\varphi_y + \varphi_{y\lambda}\lambda_y = \left(\frac{1}{r^2}\sin\varphi\sin\lambda\right)(\cos\varphi\sin\lambda)$$

$$+ \left(-\frac{1}{r}\cos\varphi\sin\lambda\right)\left(-\frac{1}{r}\sin\varphi\sin\lambda\right)$$

$$+ \left(-\frac{1}{r}\sin\varphi\cos\lambda\right)\left(\frac{\cos\lambda}{r\cos\varphi}\right) \qquad \text{(A24)}$$

$$\varphi_{zr}r_z + \varphi_{z\varphi}\varphi_z + \varphi_{z\lambda}\lambda_z = \left(-\frac{1}{r^2}\cos\varphi\right)(\sin\varphi) + \left(-\frac{1}{r}\sin\varphi\right)\left(\frac{1}{r}\cos\varphi\right) + (0)(0)$$

$$= 0 \qquad \text{(A25)}$$

$$\varphi_{xr}r_x + \varphi_{x\varphi}\varphi_x + \varphi_{x\lambda}\lambda_x + \varphi_{yr}r_y + \varphi_{y\varphi}\varphi_y + \varphi_{y\lambda}\lambda_y + \varphi_{zr}r_z$$

$$+ \varphi_{z\varphi}\varphi_z + \varphi_{z\lambda}\lambda_z = -\frac{\sin\varphi}{r^2\cos\varphi}$$

$$\text{(A26)}$$

$$\lambda_{xr}r_x + \lambda_{x\varphi}\varphi_x + \lambda_{x\lambda}\lambda_x = \left(\frac{\sin\lambda}{r^2\cos\varphi}\right)(\cos\varphi\cos\lambda) + \left(-\frac{\sin\varphi\sin\lambda}{r\cos^2\varphi}\right)\left(-\frac{1}{r}\sin\varphi\cos\lambda\right)$$

$$+ \left(-\frac{\cos\lambda}{r\cos\varphi}\right)\left(-\frac{\sin\lambda}{r\cos\varphi}\right)$$

$$= \frac{\sin\lambda\cos\lambda\cos\varphi}{r^2\cos\varphi} + \frac{\sin\lambda\cos\lambda\sin^2\varphi}{r\cos^2\varphi}$$

$$+ \frac{\sin\lambda\cos\lambda}{r^2\cos^2\varphi} \qquad \text{(A27)}$$

$$\lambda_{yr}r_y + \lambda_{y\varphi}\varphi_y + \lambda_{y\lambda}\lambda_y = \left(-\frac{\cos\lambda}{r^2\cos\varphi}\right)(\cos\varphi\sin\lambda)$$

$$+\left(\frac{\sin\varphi\cos\lambda}{r\cos^2\varphi}\right)\left(-\frac{1}{r}\sin\varphi\sin\lambda\right) + \left(-\frac{\sin\lambda}{r\cos\varphi}\right)\left(\frac{\cos\lambda}{r\cos\varphi}\right)$$

$$= -\frac{\sin\lambda\cos\lambda\cos\varphi}{r^2\cos\varphi} - \frac{\sin\lambda\cos\lambda\sin^2\varphi}{r\cos^2\varphi} - \frac{\sin\lambda\cos\lambda}{r^2\cos^2\varphi} \tag{A28}$$

$$\lambda_{zr}r_z + \lambda_{z\varphi}\varphi_z + \lambda_{z\lambda}\lambda_z = (0)(\sin\varphi) + (0)\left(\frac{1}{r}\cos\varphi\right) + (0)(0)$$

$$= 0 \tag{A29}$$

$$\lambda_{xr}r_x + \lambda_{x\varphi}\varphi_x + \lambda_{x\lambda}\lambda_x + \lambda_{yr}r_y + \lambda_{y\varphi}\varphi_y + \lambda_{y\lambda}\lambda_y + \lambda_{zr}r_z + \lambda_{z\varphi}\varphi_z + \lambda_{z\lambda}\lambda_z = 0 \tag{A30}$$

Adding all resulting terms gives

$$\nabla^2 V = V_{rr}(1) + V_{\varphi\varphi}\left(\frac{1}{r^2}\right) + V_{\lambda\lambda}\left(\frac{1}{r^2\cos^2\varphi}\right) + 2V_{r\varphi}(0) + 2V_{r\lambda}(0)$$

$$+ 2V_{\varphi\lambda}(0) + V_r\left(\frac{2}{r}\right) + V_\varphi\left(-\frac{\sin\varphi}{r^2\cos\varphi}\right) + V_\lambda(0)$$

$$= \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial\varphi^2}\left(\frac{1}{r^2}\right) + \frac{\partial^2 V}{\partial\lambda^2}\left(\frac{1}{r^2\cos^2\varphi}\right) + \frac{\partial V}{\partial r}\left(\frac{2}{r}\right) + \frac{\partial V}{\partial\varphi}\left(-\frac{\sin\varphi}{r^2\cos\varphi}\right)$$

$$= 0 \tag{A31}$$

If equation (A31) is multiplied through by r^2, there results, after some simplification,

$$\nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi}\left(\cos \varphi \frac{\partial V}{\partial \varphi}\right) + \frac{1}{\cos^2 \varphi} \frac{\partial^2 V}{\partial \lambda^2}$$

$$= 0 \qquad (5)$$

The solution of equation (5) is discussed in the section entitled "Analysis."

APPENDIX B

THE PINES FORMULATION FOR THE GEOPOTENTIAL FUNCTION AND ITS GRADIENT

The classical potential function (eq. (96)) and its derivatives (eq. (111) or (112)) require that the position of a point in space be expressed in terms of geocentric latitude φ, east longitude λ, and magnitude of the position vector r determined from the component magnitudes along the Cartesian coordinate axes x, y, and z. Because these orientation angles require for their determination the projection of the position vector on the equatorial reference plane, the longitude becomes undefined over the poles. What is theoretically two isolated singularities is actually, for computational purposes, an infinite number of singularities at each of two isolated regions in the immediate neighborhood of the poles, as the computer cannot distinguish from zero a computed noninteger number having a magnitude less than some specified tolerance characteristic of the computer in question. Although a polar orbit, or even a nearly polar orbit, is almost impossible to achieve because of a number of perturbing influences, the possibility of a singularity other than at the origin is eliminated in this formulation by expressing the orientation of a point in space in terms of its direction cosines instead of its latitude and longitude. The direction cosines are always clearly defined for any direction in space, and the elimination of all trigonometric functions from the formulation contributes greatly to decreasing execution time and storage requirements.

The classical expression for the gravitational potential V exerted at a point in space located at a distance r from the center of the attracting body having radius R_E and gravitational parameter g_E is given in equation (96) as

$$V(r,\varphi,\lambda) = \sum_{n=0}^{\infty} \frac{g_E}{r}\left(\frac{R_E}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\varphi)\left[C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right] \tag{96}$$

where C_{nm} and S_{nm} are the harmonic coefficients of the potential function and $P_{nm}(\sin\varphi)$ represents the associated Legendre functions of the first kind, of degree n and order m. Since $\sin\varphi = z/r = \mu$, where μ is a direction cosine, the associated Legendre functions may be expressed by using Rodrigues' Theorem, presented here without proof.

$$P_{nm}(\mu) = \frac{\cos^m\varphi}{2^n n!}\frac{d^{n+m}}{d\mu^{n+m}}\left(\mu^2 - 1\right)^n \tag{B1}$$

Equation (B1) is equivalent to equation (81). Equation (B1) is more useful for analytical purposes, whereas equation (81) is more useful for computational purposes.

Pines (ref. 1) points out that if equation (B1) is written as

$$P_{nm}(\mu) = \cos^m\varphi \, A_{nm}(\mu) \tag{B2}$$

where $A_{nm}(\mu)$ is defined in terms of the Legendre polynomials $P_n(\mu)$ obtained when $m = 0$ in equation (B1), the following expression results.

$$\begin{aligned} A_{nm}(\mu) &= \frac{1}{2^n n!} \frac{d^{n+m}}{d\mu^{n+m}} \left(\mu^2 - 1\right)^n \\ &= \frac{d^m}{d\mu^m}\left[P_n(\mu)\right] \end{aligned} \tag{B3}$$

Then, by writing

$$\begin{aligned} \rho_n &= \frac{g_E}{r}\left(\frac{R_E}{r}\right)^n \\ &= \frac{r}{R_E}\rho_{n+1} \end{aligned} \tag{B4}$$

equation (96) may be written as

$$V = \sum_{n=0}^{\infty} \rho_n \sum_{m=0}^{n} A_{nm}(\mu)\left[C_{nm}\cos(m\lambda)\cos^m\varphi + S_{nm}\sin(m\lambda)\cos^m\varphi\right] \tag{B5}$$

From equation (B5), Pines recognizes that the terms $\cos(m\lambda)\cos^m\varphi$ and $\sin(m\lambda)\cos^m\varphi$ are merely the real and imaginary parts of the complex number $(s + it)^m$, where s and t are the direction cosines (x/r) and (y/r), respectively, and $i = \sqrt{-1}$. Equation (96) may be written as

$$V = \sum_{n=0}^{\infty} \rho_n \sum_{m=0}^{n} A_{nm}(\mu) D_{nm}(s,t) \tag{B6}$$

where the mass coefficient function

$$D_{nm}(s,t) = C_{nm} R_m(s,t) + S_{nm} I_m(s,t) \tag{B7}$$

and $R_m(s,t)$ and $I_m(s,t)$ are the real and imaginary parts of the complex variable $(s + it)^m$ mentioned previously. That $s = \cos\lambda \cos\varphi$ and $t = \sin\lambda \cos\varphi$ may be established from the right spherical triangle that results from the definition of φ and λ.

Equation (B6) is an expression for the gravitational potential in terms of the magnitude of the position vector and its direction cosines. Its gradient may be written as

$$\nabla V = \left(\frac{\partial V}{\partial r} - \frac{s}{r}\frac{\partial V}{\partial s} - \frac{t}{r}\frac{\partial V}{\partial t} - \frac{\mu}{r}\frac{\partial V}{\partial \mu}\right)\underline{R} + \frac{1}{r}\frac{\partial V}{\partial s}\underline{i} + \frac{1}{r}\frac{\partial V}{\partial t}\underline{j} + \frac{1}{r}\frac{\partial V}{\partial \mu}\underline{k} \tag{B8}$$

where $\underline{i}$, $\underline{j}$, and $\underline{k}$ are the unit base vectors along the Cartesian coordinate axes, ∇ is the gradient vector operator, and

$$\underline{R} = s\underline{i} + t\underline{j} + \mu\underline{k} \tag{B9}$$

The comparable equation for the gradient of equation (96) is

$$\nabla V = \frac{\partial V}{\partial r} + \frac{\partial V}{\partial(\sin\varphi)}\nabla(\sin\varphi) + \frac{\partial V}{\partial r}\nabla\lambda \tag{B10}$$

Differentiation of equation (B6) requires differentiation of equation (B7), which gives

$$\frac{\partial D_{nm}(s,t)}{\partial s} = C_{nm} \frac{\partial R_m(s,t)}{\partial s} + S_{nm} \frac{\partial I_m(s,t)}{\partial s}$$

$$= m\left[C_{nm}R_{m-1}(s,t) + S_{nm}I_{m-1}(s,t)\right]$$

$$= mE_{nm}(s,t) \tag{B11}$$

$$\frac{\partial D_{nm}(s,t)}{\partial t} = C_{nm} \frac{\partial R_m(s,t)}{\partial t} + S_{nm} \frac{\partial I_m(s,t)}{\partial t}$$

$$= m\left[-C_{nm}I_{m-1}(s,t) + S_{nm}R_{m-1}(s,t)\right]$$

$$= mF_{nm}(s,t) \tag{B12}$$

since from the Cauchy-Riemann conditions,

$$\frac{\partial R_m(s,t)}{\partial s} = \frac{\partial I_m(s,t)}{\partial t}$$

$$= mR_{m-1}(s,t) \tag{B13}$$

$$\frac{\partial R_m(s,t)}{\partial t} = -\frac{\partial I_m(s,t)}{\partial s}$$

$$= mI_{m-1}(s,t) \tag{B14}$$

In equations (B11) and (B12), respectively, the functions $E_{nm}(s,t)$ and $F_{nm}(s,t)$ are mass coefficient functions. Equations (B13) and (B14) show another advantage to the Pines formulation in that these differentiations do not have to be performed because the derivatives can be obtained recursively from previous values of the real and imaginary parts. This relationship may be seen by writing the complex number

$$w^m = R_m(s,t) + iI_m(s,t) \tag{B15}$$

or

$$W^{m-1}W = \left[R_{m-1}(s,t) + iI_{m-1}(s,t)\right](s + it)$$
$$= \left[sR_{m-1}(s,t) - tI_{m-1}(s,t)\right]$$
$$+ i\left[sI_{m-1}(s,t) + tR_{m-1}(s,t)\right] \tag{B16}$$

so that

$$R_m(s,t) = sR_{m-1}(s,t) - tI_{m-1}(s,t) \tag{B17}$$

$$I_m(s,t) = sR_{m-1}(s,t) + tR_{m-1}(s,t) \tag{B18}$$

The recursive computation can easily be performed by setting $R_0(s,t) = 1$ and $I_0(s,t) = 0$.

Differentiation of equation (B6) also requires differentiation of $A_{nm}(\mu)$. Clearly,

$$\frac{\partial A_{nm}(\mu)}{\partial \mu} = A_{n,m+1}(\mu) \tag{B19}$$

To compute the matrix $A_{nm}(\mu)$ recursively, Pines makes use of the recursive relations satisfied by all Legendre polynomials. (Spencer (ref. 2) presents five of these recursive relations.)

$$\frac{d}{d\mu}\left[P_{n+1}(\mu)\right] - \mu \frac{d}{d\mu}\left[P_n(\mu)\right] = (n + 1)P_n(\mu) \tag{B20}$$

$$\mu \frac{d}{d\mu}\left[P_n(\mu)\right] - \frac{d}{d\mu}\left[P_{n-1}(\mu)\right] = nP_n(\mu) \tag{B21}$$

Writing equations (B20) and (B21) in terms of the $A_{nm}(\mu)$ notation gives

$$A_{n+1,1}(\mu) - \mu A_{n1}(\mu) = (n + 1)A_{n0}(\mu) \tag{B22}$$

$$\mu A_{n1}(\mu) - A_{n-1,1}(\mu) = nA_{n0}(\mu) \tag{B23}$$

Differentiating equations (B22) and (B23) m times gives

$$A_{n+1,m+1}(\mu) = (n + m + 1)A_{nm}(\mu) + \mu A_{n,m+1}(\mu) \tag{B24}$$

$$(n - m)A_{nm}(\mu) = \mu A_{n,m+1}(\mu) - A_{n-1,m+1}(\mu) \tag{B25}$$

Equation (B25) is the recursive relationship used by Pines in computing the matrix $A_{nm}(\mu)$. Spencer shows five different relations by which this computation can be accomplished. Both authors make use of the initial relationships

$$A_{nn}(\mu) = 1\cdot3\cdot5\cdot7\cdot\ldots\cdot(2n - 1) \tag{B26}$$

and

$$A_{n,m+1}(\mu) = \mu A_{nn}(\mu) \tag{B27}$$

Equation (B27) is obtained from equation (B19). Equation (B22) is useful in collecting terms in equation (B8). Equations (B25), (B26), and (B27) reveal another desirable feature of this formulation in that errors in the direction cosine μ are diminished by $1/(n - m)$. Equation (B8) may now be written as

$$\nabla V = \left(a_1 + sa_4\right)\underline{i} + \left(a_2 + ta_4\right)\underline{j} + \left(a_3 + \mu a_4\right)\underline{k} \tag{B28}$$

where the coefficients are defined as

$$a_1 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{R_E} \sum_{m=0}^{n} mA_{nm}(\mu)E_{nm}(s,t) \tag{B29}$$

$$a_2 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{R_E} \sum_{m=0}^{n} mA_{nm}(\mu)F_{nm}(s,t) \tag{B30}$$

$$a_3 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{R_E} \sum_{m=0}^{n} A_{n,m+1}(\mu)D_{nm}(s,t) \tag{B31}$$

$$a_4 = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{R_E} \sum_{m=0}^{n} A_{n+1,m+1}(\mu)D_{nm}(s,t) \tag{B32}$$

so that

$$\begin{aligned} \nabla V = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{R_E} \sum_{m=0}^{n} \Big\{ & mA_{nm}(\mu) \left[E_{nm}(s,t)\underline{i}\right. \\ & \left. + F_{nm}(s,t)\underline{j}\right] + A_{n,m+1}(\mu)D_{nm}(s,t)\underline{k} \\ & - A_{n+1,m+1}(\mu)D_{nm}(s,t)\underline{R} \Big\} \end{aligned} \tag{B33}$$

where, from equation (B4),

$$\frac{\partial \rho_n}{\partial r} = \frac{\rho_{n+1}}{R_E} \tag{B34}$$

Equation (B33) is suitable for obtaining higher derivatives of the potential function to be used in estimating errors realized from variations in the harmonic coefficients. Pines gives these relations in reference 1, but Spencer (ref. 2) shows this work in better form in that he corrects a few errors found in this part of reference 1. Spencer's paper is by far the most complete and accurate, as well as the most comprehensible, and should be preferred to reference 1.

Sample computer program listings for the computation of the gravitational potential function and the components of its gradient are provided in the following pages of this appendix. The listings are presented primarily to emphasize the brevity.

```
@ FOR,S GRVPLT,GRVPUT                                                          13 NOV 75              18: 3127.150
UNIVAC 1100 FORTRAN V EXEC II LEVEL 25A  (EXEC8 LEVEL 8128/1001DA)
THIS COMPILATION WAS DONE ON 13 NOV 75 AT 18:03:27

   SUBROUTINE GRVPUT      ENTRY POINT 000337

   STORAGE USED: CODE(1) 000360; DATA(0) 000322; BLANK COMMON(2) 000000

    COMMON BLOCKS:

    0003   EAST   000556

   EXTERNAL REFERENCES (BLOCK, NAME)

    0004   NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001     000024 120G       0001     000043 130G       0001     000007 140G       0001     000130 150G       0001     000171 173G
    0001     000262 5L         0000 D 000142 A           0000 D 000000 A9          0003 R 000000 C           0000 D 000206 DNR
    0000 D 000015 F1           0000 D 000017 F2          0000 D 000021 F3          0000 D 000023 F4          0000 I 000000 I
    0000   000267 INJP$        0000 I 000001 J           0000 I 000014 K           0000 I 000012 KK          0000 I 000026 N
    0003 I 000552 NMAB         0000 I 000213 N           0003 I 000554 NMAA        0000 L 000020 N           0003 D 000552 NL
    0000 D 000074 R1           0000 D 300030 RS          0000 D 000070 R1          0000 D 000002 RU          0000 D 000000 RU
    0003 R 000263 S            0003 R 000520 AJ          0003 D 000550 AMV         0000 D 000010 AE

 00101      1.          SUBROUTINE GRVPUT(PH,UR,RINV)
 00101      2.    C
 00103      3.          PARAMETER  NMAAUP = 15
 00103      4.    C
 00104      5.          PARAMETER  NMAAUP = NMAAUP + 1
 00105      6.    C
 00105      7.          PARAMETER  NTOTAL = 171
 00106      8.    C
 00106      9.          IMPLICIT DOUBLE PRECISION(A-H,O-Z)
 00107     10.    C
 00107     11.          REAL
 00107     12.         AC       ,S       ,AJ
 00107     13.    C
 00110     14.          DIMENSION
 00110     15.         APH(3)        ,UR(3)         ,R(NMAAUP)    ,RI(NMAAUP)   ,A(NMAAUP,4)
 00110     16.    C
 00111     17.          COMMON /EAST/
 00111     18.         &C(NTOTAL)    ,S(NTOTAL)     ,AJ(NMAAUP)   ,AMV    ,NL     ,NMAA   ,NMAA
 00111     19.    C
 00112     20.          DATA ((A(I,J),I=1,NMAAUP),J=1,2)/NMAAUP*0.0D0,NMAAUP*0.0D0/
 00112     21.    C
 00114     22.          DATA (R(I),I=1,NMAAUP)/1.0D0, NMAAUP*0.0D0/
 00114     23.    C
 00116     24.          DATA (RI(I),I=1,NMAAUP)/NMAAUP*0.0D0/
 00116     25.    C
```

ORIGINAL PAGE IS
OF POOR QUALITY

```
00120    260*          EQUIVALENCE
00120    270*         1(H(2) , HS) , (H(2) , HT) , (A(NMAXUP,1) , OMM)
00120    280*     C
00121    290*          HS = PH(1)*RINV
00122    300*          HT = PH(2)*RINV
00123    310*          HU = PH(3)*RINV
00124    320*          DO 6 I = 1,3
00127    330*        6 DH(I) = 0.0
00131    340*          A4 = 0.0
00132    350*          HO = RE*RINV
00133    360*          A6 = HU*AMU*RINV**2
00134    370*          KK = 0
00135    380*          DO 1 I = 1,NMAX
00140    390*          H(I+1) = HS*H(I) - HT*H(I)
00141    400*        1 H(I+1) = HS*H(I) + HT*H(I)
00143    410*          A(1,2) = 3.0*HU
00144    420*          A(2,2) = 3.0
00146    430*          DO 2 N = 2,NMAX
00150    440*          A(N+1,1) = 0.0
00151    450*          A(N+1,2) = (2*N+1)*A(N,2)
00152    460*          A(N,1) = A(N,2)
00153    470*          A(N,2) = HU*A(N+1,2)
00154    480*          K = 1
00156    490*          DO 3 I = N,2,-1
00160    500*          K = K + 1
00161    510*          A(I-1,1) = A(I-1,2)
00163    520*        3 A(I-1,2) = (HU*A(I,2) - A(I,1))/K
00164    530*          F1 = 0.0
00165    540*          F2 = 0.0
00166    550*          F3 = A(1,1)*AJ(N)
00167    560*          F4 = A(1,2)*AJ(N)
00170    570*          IF(N.GT.NMAX)GO TO 5
00172    580*          DO 4 M = 1,N
00175    590*          KK = KK + 1
00176    600*          F1 = F1 + (M*A(M,1))*(C(KK)*R(M) + S(KK)*RI(M))
00177    610*          F2 = F2 + (M*A(M,1))*(S(KK)*R(M) - C(KK)*RI(M))
00200    620*          OMM = C(KK)*R(M+1) + S(KK)*RI(M+1)
00201    630*          F3 = F3 + A(M+1,1)*OMM
00202    640*        4 F4 = F4 + A(M+1,2)*OMM
00204    650*        5 AA = RI*HO
00205    660*          DH(1) = DH(1) + AA*F1
00206    670*          DH(2) = DH(2) + AA*F2
00207    680*          DH(3) = DH(3) + AA*F3
00210    690*        2 A4 = A4 + AA*F4
00212    700*          DH(1) = DH(1) - HS*A4
00213    710*          DH(2) = DH(2) - HT*A4
00214    720*          DH(3) = DH(3) - HU*A4
00215    730*          RETURN
00216    740*          END

     END OF COMPILATION:          NO DIAGNOSTICS.
     GNVPUT           SYMBOLIC                         12 NOV 75  17:15:40   0  DI&40079   14   79  (DELETED)
     GNVPOI   CODE    RELOCATABLE                      12 NOV 75  17:15:40   1  DI&4211D   24    1  (DELETED)
                                                                             0  DI&4214U   14   40
```

```
@ FOR,* PUT,PUT
UNIVAC 1100 FORTRAN V EXEC II LEVEL 25A  *(EXEC8 LEVEL E1201B01DA)                    13 NOV 75          101 3129+D80
THIS COMPILATION WAS DONE ON 13 NOV 75 AT 18:03:29

   FUNCTION PUT          ENTRY POINT 000255

   STORAGE USED: CODE(1) 000270; DATA(0) 000312; BLANK COMMON(2) 000000

    COMMON BLOCKS:

    0003   LAST   000556

   EXTERNAL REFERENCES (BLOCK, NAME)

    0004   USQRT
    0005   NERR3$

   STORAGE ASSIGNMENT  (BLOCK, TYPE, RELATIVE LOCATION, NAME)

    0001   000051 132G      0001   000101 143G      0001   000156 157G      0001   000205 165G      0001   000223 6L
    0000 D 000002 A         0003 R 000000 C         0000 D 000132 FI        0000 I 000110 I         0000   000264 INJP$
    0000 I 000117 J         0000 I 000134 K         0000 I 000130 KK        0000 I 000135 N         0003 I 000555 MMAX
    0000 I 000131 M         0003 I 000557 MMAX      0000 D 000000 PUT       0000 D 000130 R         0003 D 000552 RL
    0000 D 000264 R1        0003 D 000120 RINV      0000 D 000130 RS        0000 D 000204 RT        0000 D 000122 RV
    0000 D 000129 R3        0003 R 000453 S         0003 R 000520 AJ        0003 D 000550 XNU       0000 D 000126 XX

      FUNCTION PUT(PR)
      PARAMETER  MMAXUP = 16
      PARAMETER  MMAXUP = MMAXUP + 1
      PARAMETER  NTOTAL = 171
      IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      REAL
     *C        ,S       ,AJ
      DIMENSION
     *PR(3)       ,N(MMAXUP)    ,NJ(MMAXUP)    ,R(MMAXUP,4)
      COMMON /LAST/
     *C(NTOTAL)    ,S(NTOTAL)    ,AJ(MMAXUP)   ,XNU  ,RL     ,MMAX  ,MMAX
      DATA ((R(I,J),I=1,MMAXUP),J=1,2)/MMAXUP*0.D00, MMAXUP*0.D00/
      DATA (N(I),I=1,MMAXUP)/1.D00, MMAXUP*0.D00/
      DATA (NJ(I),I=1,MMAXUP)/MMAXUP*0.D00/
```

```
00120     26*          EQUIVALENCE
00120     27*         1(R(1) , RS) , (RI(1) , RT)
00120     28*
00121     29*          POT = 0.0
00122     30*          RINV = 1.0/SQRT(PR(1)**2 + PR(2)**2 + PR(3)**2)
00123     31*          RS = PR(1)*RINV
00124     32*          RT = PR(2)*RINV
00125     33*          RU = PR(3)*RINV
00126     34*          RO = RE*RINV
00127     35*          AA = RO*AMU*RINV
00130     36*          KK = 0
00131     37*          DO 1 I = 1,NMAX
00134     38*          R(I+1) = RS*R(I) - RT*RI(I)
00136     39*        1 RI(I+1) = RS*RI(I) + RT*R(I)
00137     40*          A(1,2) = 3.0*RU
00140     41*          A(2,2) = 3.0
00141     42*          A(1,1) = 1.0
00142     43*          DO 2 N = 2,NMAX
00145     44*          FI = - (RU*A(1,2)-A(1,1))*(N+1)*J(N)
00146     45*          A(N+1,1) = 0.0
00147     46*          A(N+1,2) = (2*N+1)*A(N,2)
00150     47*          A(N,1) = A(N,2)
00151     48*          A(N,2) = RU*A(N+1,2)
00152     49*          K = 1
00153     50*          DO 3 I = N,2,-1
00156     51*          K = K + 1
00157     52*          A(I-1,1) = A(I-1,2)
00160     53*        3 A(I-1,2) = (RU*A(I,2) - A(I,1))/K
00162     54*          IF(N.GT.MMAX)GO TO 5
00164     55*          DO 4 M = 1,N
00167     56*          KK = KK + 1
00170     57*        4 FI = FI - A(N,1)*(C(KK)*R(M) + S(KK)*RI(M))
00172     58*        5 AA = AA*RO
00173     59*        2 POT = POT + AA*FI
00175     60*          RETURN
00176     61*          END

   END OF COMPILATION:         NO  DIAGNOSTICS.
 D.                 SYMBOLIC                              14 NOV 75  19:16:46   0  01ADBBU9   14   01  (DELETED)
 PO.       CODE     RELOCATABLE                           14 NOV 75  19:16:46   1  01AD7232   24    1  (DELETED)
                                                                                0  01AD7262   14   36
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